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A remark on the bifurcation diagrams of superlinear elliptic equations

Dedicated to Antonio Ambrosetti on his sixtieth birthday

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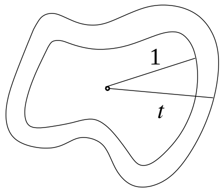
Abstract. We prove a formula relating the index of a solution and the rotation number of a certain complex vector along bifurcation diagrams.

We consider a deformation Ω_t of domains via uniform dilation. For the sake of simplicity, we will consider only the case of starshaped domains.



On Ω_t, we consider the partial differential equation

$$\begin{cases} -\Delta u = g(u), \\ u|_{\partial\Omega_t} = 0. \end{cases} \tag{1}$$

where $g(u)$ is "superlinear" and "subcritical", i.e., $g : \mathbb{R} \to \mathbb{R}$ and

$$\lim_{|s|\to\infty} \frac{g(s)}{s} = +\infty, \qquad |g(s)| \le C(1 + |s|^q) \quad \text{with } q < \frac{n+2}{n-2} \quad (n \ge 3).$$

We assume that g is C^∞ for the sake of simplicity.

For a generic shape of domains Ω_1, we may assume that the solution set (t, u_t), $t \in (0, \infty)$, is a one-dimensional manifold having possibly infinitely many connected components.

A natural question is: Does every connected component span over $t \in (0, \infty)$? Are there infinitely many components in the solution set spanning over $(0, \infty)$?

Both questions are reformulations of the following conjecture:

Conjecture. For any given t_0, (1) has infinitely many solutions.

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A related problem is the following. Let $0 < a < b$ be given. Are the connected components for $t \in [a, b]$ compact? i.e., assuming that we are considering a branch of solutions u_t, $t \in [a, b]$, of

$$\begin{cases} -\Delta u_t = g(u_t), \\ u_t|_{\partial\Omega_t} = 0, \end{cases}$$

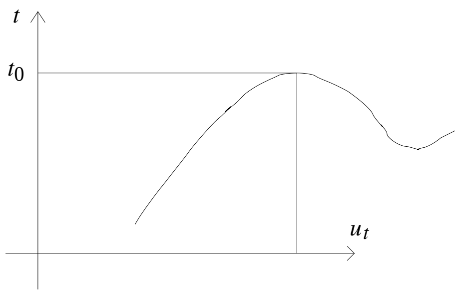
is the Morse index of u_t bounded on a given connected component for $t \in [a, b]$?

Indeed, by the results of X. F. Yang [2] and Harrabi–Rebhi–Selmi [1], a bound on the Morse index of u_t is equivalent to a bound on $\|u_t\|_\infty$ for $t \in [a, b]$ under the additional assumptions:

(i) $g(u) \underset{|u|\to\infty}{\sim} c_+(u^+)^{p_+} - c_-(u^-)^{p_-}$, $1 < p_+, p_- < (n+2)/(n-2)$,

(ii) $g'(u) \underset{|u|\to\infty}{\sim} p_+c_+(u^+)^{p_+-1} - p_-c_-(u^-)^{p_--1}$.

Let us consider such a connected component:



For values of t such as $t = t_0$, (1) degenerates at u_{t_0} and the Morse index of u_t changes.

Picking up two points (t_1, u_{t_1}) and (t_2, u_{t_2}) on $\mathcal{C}$, we would like to relate the Morse index of u_{t_2} to the Morse index of u_{t_1}.

We introduce the vector ($\mathcal{C}$ is parametrized by s):

$$V(s) = \int_{\Omega_{t(s)}} |\nabla u^s_{t(s)}|^2 + i\int_{\Omega_{t(s)}} G(u^s_{t(s)}) \quad \text{with} \quad G(u) = \int_0^u g(x)dx.$$

We claim that:

Theorem 1. *$\dot V(s)$ is never zero on $\mathcal{C}$ generically on Ω_1 and*

$$\textit{Morse index } (u_{t_2}) - \textit{ Morse index } (u_{t_1}) \\ = \textit{algebraic number of times } \dot V(s) \textit{ crosses the } y\textit{-axis}.$$

Proof. Let us differentiate (1) with respect to s. We derive

$$\begin{cases} -\Delta h = g'(u)h, \\ h + \dot t r(\sigma)\dfrac{\partial u}{\partial r}(\sigma, tr(\sigma))|_{\partial\Omega_t} = 0. \end{cases} \tag{$*$}$$

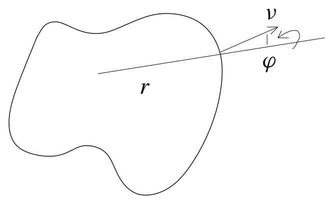
with $\partial\Omega_1$ parametrized by $(\sigma, r(\sigma))$, $\sigma \in S^{n-1}$.

Indeed, the Dirichlet boundary condition reads $u_t(\sigma, tr(\sigma)) = 0$ and we derive our boundary condition after differentiation.

The Morse index changes only when $\dot{t}$ vanishes, so that we have

$$\begin{cases} -\Delta h = g'(u)h, \\ h|_{\partial\Omega} = 0. \end{cases}$$

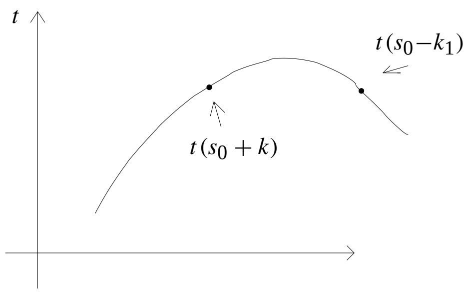
Observe that, with $I_t(u) = \frac{1}{2}\int_{\Omega_t} |\nabla u|^2 - \int_{\Omega_t} G(u)$, we find



$$\begin{aligned}
&\frac{d}{ds} I_{t(s)}(u^s) \\
&= \int_{\Omega_t} \nabla u^s \nabla h - \int_{\Omega_t} g(u^s)h + \frac{d}{ds}\left(\int_0^t \left(\int_{\partial\Omega_x} \frac{|\nabla u^x|^2}{2}\, d\sigma_x\right) \cos\varphi(\sigma) r(\sigma)\, dy\right) \\
&= \int_{\partial\Omega_t} \frac{\partial u^s}{\partial \nu} h + \frac{\dot{t}}{2}\int_{\partial\Omega_t} |\nabla u^s|^2 r(\sigma)\cos\varphi(\sigma)\, d\sigma_t \\
&= -\dot{t}\int_{\partial\Omega_t} \frac{\partial u^s}{\partial \nu}\frac{\partial u^s}{\partial r} r\, d\sigma_t + \frac{\dot{t}}{2}\int_{\partial\Omega_{t(s)}} \left|\frac{\partial u}{\partial \nu}\right|^2 \cos\varphi(\sigma) r(\sigma)\, d\sigma_t \\
&= -\frac{\dot{t}}{2}\int_{\partial\Omega_t} |\nabla u^s|^2 r(\sigma) r(\sigma)\cos\varphi(\sigma)\, d\sigma_t.
\end{aligned}$$

On the other hand, if $\dot{t}(s_0) = 0$, we compare $I_t(u_+)$ and $I_t(u_-)$, where u_+ and u_- are solutions for $s_0 + k$, $k > 0$ small, and $s_0 - k_1$, $k_1 > 0$ small, with

$$t(s_0 + k) = t(s_0 - k_1).$$



This will tell us how the Morse index changes as s increases because whichever of $I_{t(s_0+k)}(u(s_0 + k))$ or $I_{t(s_0-k_1)}(u(s_0 - k_1))$ is larger will correspond to the larger index:

when an elimination of a pair of critical points occurs in a variational problem, the highest index critical point is above the lowest one.

We renormalize $\Omega_{t(s)}$ near $s = s_0$ so that we will be considering only one $\Omega_{t(s_0)} = \Omega_0$ with a functional

$$\widetilde{I}_{t(s)} = t(s)^{n-2}\bar{I}_{t(s)}\left(u\left(\frac{x}{t(s)}\right)\right) \quad (t(s_0) = 1 \text{ for example}).$$

Our critical points $u(s_0+k)$ and $u(s_0-k_1)$ change into $\tilde{u}(s_0+k)$ and $\tilde{u}(s_0-k_1)$. We know that $\dot{t}(s_0) = 0$.

The branch $(t(s), \tilde{u}(s))$ is differentiable. With $\dot{\tilde{u}}(s_0) = h$, the direction of degeneracy, we have

$$\begin{cases} \tilde{u}(s_0+k) = u(s_0) + kh + O(k^2), \\ \tilde{u}(s_0-k_1) = u(s_0) - k_1 h + O(k_1^2), \\ t(s_0+k) = t(s_0-k_1). \end{cases}$$

Let $w = \tilde{u}(s_0+k) - \tilde{u}(s_0-k_1)$. We expand

$$\begin{aligned}
\Delta &= \widetilde{I}_{t(s_0+k)}(\tilde{u}(s_0+k)) - \widetilde{I}_{t(s_0-k_1)}(\tilde{u}(s_0-k_1)) \\
&= t(s_0-k_1)^{n-2}(\bar{I}_{t(s_0+k)}(\tilde{u}(s_0+k)) - \bar{I}_{t(s_0-k_1)}(\tilde{u}(s_0-k_1))) = t(s_0-k_1)^{n-2}\bar{\Delta}, \\
\bar{\Delta} &= \bar{I}_{t(s_0+k)}(\tilde{u}(s_0+k)) - \bar{I}_{t(s_0-k_1)}(\tilde{u}(s_0-k_1)) = \tfrac{1}{2}\bar{I}''_{t(s_0+k)}(u(s_0-k_1)) \cdot w \cdot w \\
&\quad + \tfrac{1}{6}\bar{I}^{(3)}_{t(s_0+k)}(u(s_0-k_1)) \cdot w \cdot w \cdot w + \tfrac{1}{4}\bar{I}^{(4)}(u(s_0-k_1))w \cdot w \cdot w \cdot w + O(|w|^5_{H_0^1}).
\end{aligned}$$

We know that

$$w = (k+k_1)h + O(k^2+k_1^2) = (k+k_1)h + O((k+k_1)^2).$$

Thus,

$$\begin{aligned}
\bar{\Delta} &= \tfrac{1}{2}\bar{I}''_{t(s_0+k)}(u(s_0-k_1)) \cdot h \cdot h(k+k_1)^2 \\
&\quad + \tfrac{1}{2}\bar{I}''_{t(s_0+k)}(u(s_0-k_1)) \cdot h \cdot O((k+k_1)^2)(k+k_1) \\
&\quad + O((k+k_1)^4) + \tfrac{1}{6}\bar{I}^{(3)}_{t(s_0+k)}(u(s_0-k_1)) \cdot h \cdot h \cdot h(k+k_1)^3 \\
&= \tfrac{1}{2}\bar{I}''_{t(s_0)}(u(s_0)) \cdot h \cdot h(k+k_1)^2 + \frac{\dot{t}(s_0)}{2}k\frac{\partial}{\partial t}\bar{I}''_t(u(s_0)) \cdot h \cdot h(k+k_1)^2|_{t=t(s_0)} \\
&\quad + O((k+k_1)^4) + \tfrac{1}{2}(\bar{I}''_{t(s_0+k)}(u(s_0-k_1)) - \bar{I}''_{t(s_0+k)}(u(s_0))) \cdot h \cdot h(k+k_1)^2 \\
&\quad + \tfrac{1}{6}\bar{I}^{(3)}_{t(s_0)}(u(s_0)) \cdot h \cdot h \cdot h(k+k_1)^3 + O((k+k_1)^4) \\
&= \tfrac{1}{2}\bar{I}^{(3)}_{t(s_0)}(u(s_0)) \cdot h \cdot h \cdot h(k+k_1)^2 \cdot (-k_1) \\
&\quad + \tfrac{1}{6}\bar{I}^{(3)}_{t(s_0)}(u(s_0)) \cdot h \cdot h \cdot h(k+k_1)^3 + O((k+k_1)^4).
\end{aligned}$$

On the other hand,

$$t(s_0+k) = t(s_0) + \tfrac{1}{2}t''(s_0)k^2 + O(k^3), \qquad t(s_0-k_1) = t(s_0) + \tfrac{1}{2}t''(s_0)k_1^2 + O(k_1^3),$$

so that, since $t(s_0+k)=t(s_0-k_1)$,

$$k=k_1(1+o(1)).$$

Thus

$$\bar{\Delta}=-\frac{1}{12}\bar{I}^{(3)}_{t(s_0)}(u(s_0))\cdot h\cdot h\cdot h(k+k_1)^3+O((k+k_1)^4).$$

We set $t(s_0)=1$ so that

$$\bar{\Delta}=\frac{1}{12}\int g''(u(s_0))h^3(k+k_1)^3+O((k+k_1)^4).$$

Differentiating $(*)$, we derive (at s_0)

$$\begin{cases} -\Delta\dot{h}-g'(u)\dot{h}=g''(u)h^2, \\ \dot{h}+r(\sigma)\ddot{t}(s_0)\dfrac{\partial u_t}{\partial r}(\sigma,tr(\sigma))|_{\partial\Omega_{t(s_0)}}=0. \end{cases}$$

Thus,

$$\begin{aligned} \int g''(u)h^3 &= \int_{\Omega_{t(s_0)}}(-\Delta\dot{h}-g'(u)\dot{h})h=\int\nabla\dot{h}\nabla h-\int g'(u)h\dot{h} \\ &= \int_{\partial\Omega_{t(s_0)}}\dot{h}\frac{\partial h}{\partial\nu}-\int_{\Omega}(\Delta h+g'(u)h)\dot{h}=\int_{\partial\Omega_{t(s_0)}}\dot{h}\frac{\partial h}{\partial\nu} \\ &= -\ddot{t}(s_0)\int_{\partial\Omega_{t(s_0)}}\frac{\partial u_t}{\partial r}\frac{\partial h}{\partial\nu}r(\sigma)\,d\sigma_t=-\ddot{t}(s_0)\int_{\partial\Omega_{t(s_0)}}\frac{\partial u_{t(s_0)}}{\partial\nu}\frac{\partial h}{\partial\nu}x\cdot\nu\,d\sigma_t. \end{aligned}$$

On the other hand, at every t,

$$\int_{\partial\Omega_{t(s_0)}}\left|\frac{\partial u_t}{\partial\nu}\right|^2x\cdot\nu\,d\sigma_t=c_n\left(\int_{\Omega_{t(s_0)}}g(u)u-\frac{n-2}{2n}G(u)\right).$$

Differentiating and applying at $s=s_0$, we find ($\dot{t}(s_0)=0$)

$$2\int_{\partial\Omega_{t(s_0)}}\frac{\partial h}{\partial\nu}\frac{\partial u_t}{\partial\nu}x\cdot\nu\,d\sigma_t=c_n\int_{\Omega_t}\left(\frac{n+2}{2n}g(u)h+g'(u)uh\right)=\bar{c}_n\int_{\Omega_{t(s_0)}}g(u)h.$$

Thus, at s_0,

$$\int g''(u)h^3=-\frac{\bar{c}_n}{2}\ddot{t}(s_0)\int_{\Omega_{t(s_0)}}g(u)h.$$

We see that the sign of $\int g''(u)h^3$ depends on $\ddot{t}(s_0)$ and on $\int_{\Omega_{t(s_0)}}g(u)h$. Thus, the change of the Morse index at the crossing of $t(s_0)$ depends on the convexity of $t(s)$ and on the sign of $\int_{\Omega_{t(s_0)}}g(u)h$. This is directly related to the rotation of $\dot{t}(s)+i\int_{\Omega_{t(s)}}g(u)h$, which in turn relates directly to $\dot{I}+i\dot{\overline{\int G}}$, hence to $\dot{\overline{\int|\nabla u|^2+i\int G}}$. Theorem 1 follows. □

References

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