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**Partial differential equations.** — *Steklov eigenvalues for the $\infty$-Laplacian*, by JESUS GARCIA-AZORERO, JUAN J. MANFREDI, IRENEO PERAL and JULIO D. ROSSI.

ABSTRACT. — We study the Steklov eigenvalue problem for the $\infty$-Laplacian. To this end we consider the limit as $p \to \infty$ of solutions of $-\Delta_p u_p = 0$ in a domain $\Omega$ with $|\nabla u_p|^{p-2}\partial u_p/\partial\nu = \lambda|u|^{p-2}u$ on $\partial\Omega$. We obtain a limit problem that is satisfied in the viscosity sense and a geometric characterization of the second eigenvalue.

KEY WORDS: Quasilinear elliptic equations; viscosity solutions; Neumann boundary conditions; eigenvalue.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35J65, 35J50, 35J55.

## 1. INTRODUCTION

Let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ be the *p-Laplacian*. The limit operator $\lim_{p\to\infty} \Delta_p = \Delta_\infty$ is the $\infty$*-Laplacian* given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_i}\frac{\partial u}{\partial x_i}$$

in the viscosity sense (see [5], [6] and [10]). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function $f$ (see [1], [2], and [11]).

Our concern in this paper is the study of the Steklov eigenvalue problem for the $\infty$-Laplacian. To this end we consider the $\infty$-Laplacian in a bounded smooth domain as limit of the $p$-Laplacian as $p \to \infty$. Therefore our aim is to analyze the limit as $p \to \infty$ for the Steklov eigenvalue problem

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\partial/\partial\nu$ is the outer normal derivative. Steklov eigenvalues have been introduced in [16] for $p = 2$. For the existence of a sequence of variational eigenvalues see [16] for $p = 2$ and [7] for general $p$. As happens for the eigenvalues for the Dirichlet problem for the $p$-Laplacian, in general, it is not known if this sequence constitutes the whole spectrum. Note that the first eigenvalue of (1.1) is $\lambda_{1,p} = 0$ with eigenfunction $u_{1,p} \equiv 1$. Hence we can trivially pass to the limit and obtain $\lambda_{1,\infty} = 0$ with eigenfunction $u_{1,\infty} \equiv 1$. Our main result in this paper shows that we can pass to the limit in the variational eigenvalues defined in [7]. Since the first eigenvalue is isolated [15], there exists a second eigenvalue that has a variational characterization [8].

We can pass to the limit in this second eigenvalue and obtain a geometric characterization of the second Steklov eigenvalue for the $\infty$-Laplacian. Moreover we obtain a uniform limit of the sequence of eigenfunctions (along subsequences) and we find a limit eigenvalue problem that is satisfied in a viscosity sense which involves the $\infty$-Laplacian together with a boundary condition with the normal derivative $\partial u/\partial \nu$.

THEOREM 1.1. *For the first eigenvalue of* (1.1) *we have*

$$\lim_{p\to\infty} \lambda_{1,p}^{1/p} = \lambda_{1,\infty} = 0,$$

*with eigenfunction given by* $u_{1,\infty} = 1$. *For the second eigenvalue,*

$$\lim_{p\to\infty} \lambda_{2,p}^{1/p} = \lambda_{2,\infty} = \frac{2}{\operatorname{diam}(\Omega)}.$$

*Moreover, given eigenfunctions* $u_{2,p}$ *of* (1.1) *with eigenvalues* $\lambda_{2,p}$ *normalized by* $\|u_{2,p}\|_{L^\infty(\partial\Omega)} = 1$, *there exists a sequence* $p_i \to \infty$ *such that* $u_{2,p_i} \to u_{2,\infty}$ *in* $C^\alpha(\overline{\Omega})$. *The limit* $u_{2,\infty}$ *is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ \Lambda(x,u,\nabla u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

*in the viscosity sense, where*

$$\Lambda(x,u,\nabla u) \equiv \begin{cases} \min\{|\nabla u| - \lambda_{2,\infty}|u|, \partial u/\partial\nu\} & \text{if } u > 0, \\ \max\{\lambda_{2,\infty}|u| - |\nabla u|, \partial u/\partial\nu\} & \text{if } u < 0, \\ \partial u/\partial\nu & \text{if } u = 0. \end{cases}$$

*For the* $k$*-th eigenvalue, if* $\lambda_{k,p}$ *is the* $k$*-th variational eigenvalue of* (1.1) *with eigenfunction* $u_{k,p}$ *normalized by* $\|u_{k,p}\|_{L^\infty(\partial\Omega)} = 1$, *then every sequence tending to infinity has a subsequence* $p_i$ *such that*

$$\lim_{i\to\infty} \lambda_{k,p_i}^{1/p_i} = \lambda_{*,\infty}$$

*and* $u_{k,p_i} \to u_{*,\infty}$ *in* $C^\alpha(\overline{\Omega})$, *where* $(u_{*,\infty}, \lambda_{*,\infty})$ *is a solution of* (1.2).

We thus have a simple geometrical characterization of $\lambda_{2,\infty}$ as $2/\operatorname{diam}(\Omega)$. From this characterization and the convergence of the eigenfunctions we conclude that the second Steklov eigenfunction in an annulus or a ball is not radial. Also we find that the domain that maximizes $\lambda_{2,\infty}$ among domains with fixed volume is a ball.

We end the introduction with a brief comment on the Dirichlet case. Eigenvalues of the $p$-Laplacian, $-\Delta_p u = \lambda|u|^{p-2}u$, with Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, have been extensively studied since [9]. The limit as $p \to \infty$ was studied in [13], [12]. In these papers the authors prove results similar to ours. However our proofs are necessarily different due to the presence of the Neumann boundary condition. An anisotropic version of the Dirichlet problem was studied in [4].

## 2. The Steklov eigenvalue problem

First, let us recall some well known results concerning the Steklov eigenvalue problem for the $p$-Laplacian. To this end, we introduce a topological tool, the *genus* (see [14]).

DEFINITION 2.1. *Given a Banach space $X$, we consider the class $\Sigma = \{A \subset X : A$ is closed, $A = -A\}$. Over this class we define the* genus, $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$, *as*

$$\gamma(A) = \min\{k \in \mathbb{N} : \text{there exists } \varphi \in C(A, \mathbb{R}^k - \{0\}) \text{ such that } \varphi(x) = -\varphi(-x) \text{ for all } x \in A\}.$$

We have the following result whose proof can be obtained following [7]; we omit the details.

THEOREM 2.1. *There exists a sequence of eigenvalues $\lambda_n$ of* (1.1) *such that $\lambda_n \to \infty$ as $n \to \infty$. The so-called variational eigenvalues $\lambda_k$ can be characterized by*

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}}, \tag{2.1}$$

*where $C_k = \{C \subset W^{1,p}(\Omega) : C$ is compact, symmetric and $\gamma(C) \geq k\}$ and $\gamma$ is the genus.*

There exists a second eigenvalue for (1.1) and it coincides with the second variational eigenvalue $\lambda_{2,p}$ (see [8]). Moreover, the following characterization of the second eigenvalue $\lambda_{2,p}$ holds:

$$\lambda_{2,p} = \inf_{C \in A} \sup_{u \in C} \left\{ \frac{\int_\Omega |\nabla u|^p\, dx}{\int_{\partial\Omega} |u|^p\, d\sigma} \right\},$$

where $A = \{C \subset W^{1,p}(\Omega) : C$ is compact, symmetric and $\gamma(C) \geq 2\}$. Observe that every eigenfunction associated with $\lambda_2$ changes sign on $\partial\Omega$ (see [15]).

Following [3] let us recall the definition of viscosity solution taking into account general boundary conditions.

DEFINITION 2.2. *Consider the boundary value problem*

$$\begin{cases} F(x, \nabla u, D^2 u) = 0 & \text{in } \Omega, \\ B(x, u, \nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

(1) *A lower semicontinuous function $u$ is a* viscosity supersolution *of* (2.2) *if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ then*

$$\max\{B(x_0, \phi(x_0), \nabla\phi(x_0)), F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \geq 0,$$

*and if $x_0 \in \Omega$ then*

$$F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(2) *An upper semicontinuous function $u$ is a* viscosity subsolution *of* (2.2) *if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict maximum at $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ then*

$$\min\{B(x_0, \phi(x_0), \nabla\phi(x_0)), F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \le 0,$$

*and if $x_0 \in \Omega$ then*

$$F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \le 0.$$

(3) *Finally, $u$ is a* viscosity solution *if it is a viscosity super- and subsolution.*

In our case for the Steklov problem for the $p$-Laplacian we have

$$F_p(\eta, X) \equiv -\operatorname{Trace}(A_p(\eta)X),$$

where

$$A_p(\eta) = \operatorname{Id} + (p-2)\frac{\eta \otimes \eta}{|\eta|^2} \quad \text{if } \eta \neq 0, \qquad A_p(0) = I_N,$$

and

$$B_p(x, u, \eta) \equiv |\eta|^{p-2}\langle \eta, \nu(x)\rangle - \lambda |u|^{p-2}u. \tag{2.3}$$

With this notation we have

$$\Delta_p u = F_p(\nabla u, D^2 u) \equiv -\left\{ \frac{|\nabla\phi(x_0)|^2 \Delta\phi(x_0)}{p-2} + \Delta_\infty \phi(x_0) \right\}.$$

REMARK 2.1. If $B_p$ is monotone in the variable $\partial u/\partial \nu$, Definition 2.2 takes a simpler form (see [3]). This is indeed the case for (2.3). More concretely, if $u$ is a supersolution of (1.1) and $\phi \in C^2(\overline{\Omega})$ is such that $u - \phi$ has a strict minimum at $x_0$ with $u(x_0) = \phi(x_0)$, then:

(1) if $x_0 \in \Omega$, then

$$-\left\{ \frac{|\nabla\phi(x_0)|^2 \Delta\phi(x_0)}{p-2} + \Delta_\infty \phi(x_0) \right\} \ge 0,$$

(2) if $x_0 \in \partial\Omega$, then

$$|\nabla\phi(x_0)|^{p-2}\langle \nabla\phi(x_0), \nu(x_0)\rangle \ \ge \lambda |\phi(x_0)|^{p-2}\phi(x_0).$$

Let us state a lemma that says that weak solutions of (1.1) are viscosity solutions.

LEMMA 2.1. *A continuous weak solution of* (1.1) *is a viscosity solution.*

PROOF. Let $x_0 \in \Omega$ and let $\phi$ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at $x_0$. We want to show that

$$-(p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x_0) - |\nabla\phi|^{p-2}\Delta\phi(x_0) \ge 0.$$

Assume that this is not the case; then there exists a radius $r > 0$ such that

$$-(p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x) - |\nabla\phi|^{p-2}\Delta\phi(x) < 0$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u - \phi)(x)$ and let $\psi(x) = \phi(x) + m/2$. This function $\psi$ satisfies $\psi(x_0) > u(x_0)$ and

$$-\text{div}(|\nabla\psi|^{p-2}\nabla\psi) < 0.$$

Multiplying by $(\psi - u)^+$ extended by zero outside $B(x_0, r)$ we get

$$\int_{\{\psi>u\}} |\nabla\psi|^{p-2}\nabla\psi\nabla(\psi - u) < 0.$$

Taking $(\psi - u)^+$ as a test function in the weak form we get

$$\int_{\{\psi>u\}} |\nabla u|^{p-2}\nabla u\nabla(\psi - u) = 0.$$

Hence,

$$C(N, p)\int_{\{\psi>u\}} |\nabla\psi - \nabla u|^p \le \int_{\{\psi>u\}} \langle|\nabla\psi|^{p-2}\nabla\psi - |\nabla u|^{p-2}\nabla u, \nabla(\psi - u)\rangle < 0,$$

a contradiction.

If $x_0 \in \partial\Omega$ we want to prove

$$\begin{aligned}\max\{&|\nabla\phi(x_0)|^{p-2}\langle\nabla\phi(x_0), \nu(x_0)\rangle - \lambda|\phi(x_0)|^{p-2}\phi(x_0),\\ &- (p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x_0) - |\nabla\phi|^{p-2}\Delta\phi(x_0)\} \ge 0.\end{aligned}$$

Assume that this is not the case. We proceed as before to obtain

$$\int_{\{\psi>u\}} |\nabla\psi|^{p-2}\nabla\psi\nabla(\psi - u) < \int_{\partial\Omega\cap\{\psi>u\}} \lambda|u|^{p-2}u(\psi - u),$$

and

$$\int_{\{\psi>u\}} |\nabla u|^{p-2}\nabla u\nabla(\psi - u) \ge \int_{\partial\Omega\cap\{\psi>u\}} \lambda|u|^{p-2}u(\psi - u).$$

Therefore,

$$C(N, p)\int_{\{\psi>u\}} |\nabla\psi - \nabla u|^p \le \int_{\{\psi>u\}} \langle|\nabla\psi|^{p-2}\nabla\psi - |\nabla u|^{p-2}\nabla u, \nabla(\psi - u)\rangle < 0,$$

again a contradiction. This proves that $u$ is a viscosity supersolution. The proof that $u$ is a viscosity subsolution runs as above; we omit the details. □

With all these preliminaries we are ready to pass to the limit as $p \to \infty$ in the eigenvalue problem.

Since $u_{1,p} \equiv 1$ is the first eigenfunction of (1.1) associated to $\lambda_{1,p} = 0$ we can trivially pass to the limit and obtain

$$\lim_{p\to\infty} \lambda_{1,p}^{1/p} = 0 = \lambda_{1,\infty}$$

and

$$\lim_{p\to\infty} u_{1,p} = 1 = u_{1,\infty}.$$

Now let us prove a geometrical characterization of the second Steklov eigenvalue for the $\infty$-Laplacian, defined by

$$\lambda_{2,\infty} = \inf_{C\in A_0} \sup_{u\in C} \left\{ \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}} \right\}, \tag{2.4}$$

where $A_0 = \{C \subset W^{1,\infty}(\Omega) : C$ is compact, symmetric and $\gamma(C) \geq 2\}$. We have

LEMMA 2.2. *$\lambda_{2,\infty}$ has the following geometrical characterization:*

$$\lambda_{2,\infty} = \frac{2}{\operatorname{diam}(\Omega)}.$$

PROOF. Let

$$R = \sup\{r : \exists x_0, x_1 \in \overline{\Omega} \text{ with } B(x_0,r)\cap B(x_1,r) = \emptyset\} = \frac{\operatorname{diam}(\Omega)}{2}.$$

We can take as test functions in (2.4) normalized linear combinations of two cones centered at $x_0$ and $x_1$ with radius $R$, that is, if

$$C_0(x) = \left(1 - \frac{|x-x_0|}{R}\right)_+, \qquad C_1(x) = \left(1 - \frac{|x-x_1|}{R}\right)_+,$$

we consider

$$\phi(x) = \alpha C_0(x) + \beta C_1(x) \quad \text{ with } \|\phi\|_{L^\infty(\partial\Omega)} = 1.$$

We obtain (as in [13])

$$\lambda_{2,\infty} \leq \frac{1}{R} = \frac{2}{\operatorname{diam}(\Omega)}.$$

To prove the reverse inequality, take a function $u$ in $W^{1,\infty}(\Omega)$ that changes sign and is such that

$$\lambda_{2,\infty} \geq \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}} - \varepsilon.$$

Now $u^+$ and $u^-$ have disjoint supports and we may normalize so that $\|u^+\|_{L^\infty(\partial\Omega)} = \|u^-\|_{L^\infty(\partial\Omega)} = 1$; then

$$\|\nabla u^+\|_{L^\infty(\Omega)} \geq 1/R \quad \text{or} \quad \|\nabla u^-\|_{L^\infty(\Omega)} \geq 1/R.$$

Therefore,

$$\lambda_{2,\infty} \geq \frac{1}{R} - \varepsilon = \frac{2}{\operatorname{diam}(\Omega)} - \varepsilon.$$

Since this holds for every $\varepsilon$, the proof is complete. □

LEMMA 2.3.

$$\limsup_{p\to\infty} \lambda_{2,p}^{1/p} \leq \lambda_{2,\infty}.$$

PROOF. As above, let $C_0(x)$ and $C_1(x)$ be two cones centered at $x_0$ and $x_1$ of radius $R$ as above, that is,

$$C_0(x) = \left(1 - \frac{|x - x_0|}{R}\right)_+, \qquad C_1(x) = -\left(1 - \frac{|x - x_1|}{R}\right)_+.$$

Let us normalize a function $v = aC_0 - bC_1$ $(a, b > 0)$ by $\|v\|_{L^\infty(\partial\Omega)} = 1$; then

$$\lambda_{2,p}^{1/p} \leq \frac{(\int_\Omega |\nabla v|^p)^{1/p}}{(\int_{\partial\Omega} |v|^p)^{1/p}}.$$

Hence

$$\limsup \lambda_{2,p}^{1/p} \leq \frac{1}{R} = \lambda_{2,\infty},$$

as we wanted to prove. □

LEMMA 2.4. *Given eigenfunctions $u_{2,p}$ of* (1.1) *with eigenvalues $\lambda_{2,p}$ normalized by $\|u_{2,p}\|_{L^\infty(\partial\Omega)} = 1$, there exists a sequence $p_i \to \infty$ such that*

$$u_{2,p_i} \to u_{2,\infty} \quad \text{in } C^\alpha(\overline{\Omega}).$$

*The limit $u_{2,\infty}$ satisfies $\|u_{2,\infty}\|_{L^\infty(\partial\Omega)} = 1$ and it changes sign on $\partial\Omega$. Moreover it is a minimizer of* (2.4) *and*

$$\lim_{p\to\infty} \lambda_{2,p}^{1/p} = \lambda_{2,\infty}.$$

PROOF. If $q < p$, then

$$\left(\int_\Omega |\nabla u_{2,p}|^q\right)^{1/q} \leq |\Omega|^{1/q-1/p}\left(\int_\Omega |\nabla u_{2,p}|^p\right)^{1/p} = \lambda_{2,p}^{1/p}|\Omega|^{1/q-1/p}\left(\int_{\partial\Omega} |u_{2,p}|^p\right)^{1/p} \leq \lambda_{2,p}^{1/p}|\Omega|^{1/q-1/p}|\partial\Omega|^{1/p}. \tag{2.5}$$

Therefore, by Lemma 2.3, there exists a constant $C$ independent of $p$ such that

$$\left(\int_\Omega |\nabla u_{2,p}|^q\right)^{1/q} \leq C. \tag{2.6}$$

Hence, as $u_{2,p}$ are uniformly bounded in $W^{1,q}(\Omega)$ we can take a subsequence which converges weakly in $W^{1,q}(\Omega)$ (and hence in $C^\alpha(\overline{\Omega})$ if $q > N$) to a limit $u_{2,\infty}$. Since this can be done for any $q$ we infer that $u_{2,\infty} \in W^{1,\infty}(\Omega)$. Indeed, from (2.5), we get

$$\left(\int_\Omega |\nabla u_{2,\infty}|^q\right)^{1/q} \leq \limsup_{p_i\to\infty}\left(\int_\Omega |\nabla u_{2,p_i}|^q\right)^{1/q} \leq \lambda_{2,\infty}|\Omega|^{1/q}. \tag{2.7}$$

Hence, letting $q \to \infty$ in (2.7) we get

$$\|\nabla u_{2,\infty}\|_{L^\infty(\Omega)} \leq \liminf_{p\to\infty} \lambda_{2,p}^{1/p} \leq \lambda_{2,\infty}. \tag{2.8}$$

From the convergence in $C^\alpha(\overline{\Omega})$ of the sequence $u_{2,p_i}$ and the normalization $\|u_{2,p_i}\|_{L^\infty(\partial\Omega)} = 1$ we obtain

$$\|u_{2,\infty}\|_{L^\infty(\partial\Omega)} = 1. \tag{2.9}$$

To end the proof we need to check that $u_{2,\infty}$ changes sign. Assume that $u_{2,\infty} \geq 0$. Hence $u^-_{2,p_i}$ converges uniformly to zero in $\overline{\Omega}$. From (2.9) there exists $x_0 \in \partial\Omega$ such that $u_{2,\infty}(x_0) = 1$. At level $p$ we have

$$\int_{\partial\Omega} |u|^{p-2}u = 0,$$

so

$$\int_{\partial\Omega} (u^+)^{p-1} = \int_{\partial\Omega} (u^-)^{p-1}.$$

Therefore,

$$\begin{aligned} |\partial\Omega|^{1/r-1/(p_i-1)} \left( \int_{\partial\Omega} |u^+_{2,p_i}|^r \right)^{1/r} &\leq \left( \int_{\partial\Omega} |u^+_{2,p_i}|^{p_i-1} \right)^{1/(p_i-1)} \\ &= \left( \int_{\partial\Omega} |u^-_{2,p_i}|^{p_i-1} \right)^{1/(p_i-1)} \leq |\partial\Omega|^{1/(p_i-1)} \|u^-_{2,p_i}\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{2.10}$$

From the uniform convergence of $u_{2,p_i}$ and (2.8), letting $p_i \to \infty$ we get

$$|\partial\Omega|^{1/r} \left( \int_{\partial\Omega} |u^+_{2,\infty}|^r \right)^{1/r} \leq 0. \tag{2.11}$$

A contradiction. This proves that $u_{2,\infty}$ changes sign and satisfies (2.8) and (2.9). Hence, from the definition of $\lambda_{2,\infty}$ we obtain

$$\lambda_{2,\infty} \leq \liminf_{p\to\infty} \lambda_{2,p}^{1/p}.$$

This fact and Lemma 2.3 end the proof. $\square$

Now let us analyze the equation satisfied by $u_{2,\infty}$. Let

$$\Lambda(x,u,\eta) \equiv \begin{cases} \min\{|\eta| - \lambda_{2,\infty}|u|, \langle \eta, \nu(x)\rangle\} & \text{if } u > 0, \\ \max\{\lambda_{2,\infty}|u| - |\eta|, \langle \eta, \nu(x)\rangle\} & \text{if } u > 0, \\ \langle \eta, \nu(x)\rangle & \text{if } u = 0. \end{cases}$$

LEMMA 2.5. *The limit $u_{2,\infty}$ is a viscosity solution of*

$$\begin{cases} \Delta_\infty u = 0 & \textit{in } \Omega, \\ \Lambda(x,u,\nabla u) = 0 & \textit{on } \partial\Omega. \end{cases} \tag{2.12}$$

PROOF. First, let us check that $-\Delta_\infty u_{2,\infty} = 0$ in the viscosity sense in $\Omega$. Let us recall the standard proof. Let $\phi$ be a smooth test function such that $u_{2,\infty} - \phi$ has a strict maximum at $x_0 \in \Omega$. Since $u_{2,p_i}$ converges uniformly to $u_{2,\infty}$ we see that $u_{2,p_i} - \phi$ has a maximum at some point $x_i \in \Omega$ with $x_i \to x_0$. Now we use the fact that $u_{2,p_i}$ is a viscosity solution of

$$-\Delta_p u = 0$$

to obtain

$$-(p_i - 2)|\nabla\phi|^{p_i-4}\Delta_\infty\phi(x_i) - |\nabla\phi|^{p_i-2}\Delta\phi(x_i) \le 0. \tag{2.13}$$

If $\nabla\phi(x_0) = 0$ we get $-\Delta_\infty\phi(x_0) \le 0$. If this is not the case, we find that $\nabla\phi(x_i) \ne 0$ for large $i$ and then

$$-\Delta_\infty\phi(x_i) \le \frac{1}{p_i - 2}|\nabla\phi|^2\Delta\phi(x_i) \to 0 \quad \text{as } i \to \infty.$$

We conclude that

$$-\Delta_\infty\phi(x_0) \le 0.$$

That is, $u_{2,\infty}$ is a viscosity subsolution of $-\Delta_\infty u = 0$.

Now we check the boundary condition.

Assume that $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ such that $u_{2,\infty}(x_0) = \phi(x_0) > 0$. Using the uniform convergence of $u_{2,p_i}$ to $u_{2,\infty}$ we deduce that $u_{2,p_i} - \phi$ has a minimum at some $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many $i$, we can argue as before to obtain

$$-\Delta_\infty\phi(x_0) \ge 0.$$

On the other hand, if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \ge \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0) = 0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) = 0.$$

If $\nabla\phi(x_0) \ne 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \ge \lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

Since $\lambda_{2,p}^{1/(p-1)} \to \lambda_{2,\infty}$ as $p \to \infty$ we conclude that

$$\frac{\lambda_{2,\infty}|\phi|}{|\nabla\phi|}(x_0) \le 1.$$

Moreover,

$$\frac{\partial\phi}{\partial\nu}(x_0) \ge 0.$$

Hence, if $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $\phi(x_0) = u_{2,\infty}(x_0) > 0$, we have

$$\max\left\{\min\left\{(-\lambda_{2,\infty}|\phi| + |\nabla\phi|)(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\right\}, -\Delta_\infty\phi(x_0)\right\} \geq 0. \tag{2.14}$$

Now assume that $u_{2,\infty} - \phi$ has a strict maximum at $x_0 \in \partial\Omega$ with $u_{2,\infty}(x_0) = \phi(x_0) > 0$. The uniform convergence of $u_{2,p_i}$ to $u_{2,\infty}$ implies that $u_{2,p_i} - \phi$ has a maximum at some $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many $i$, as before we obtain

$$-\Delta_\infty u_{2,\infty}(x_0) \leq 0.$$

On the other hand, if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \leq \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0) = 0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) = 0.$$

If $\nabla\phi(x_0) \neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \leq \lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

If $\lambda_{2,\infty}|\phi|(x_0) < |\nabla\phi|(x_0)$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) \leq 0.$$

Hence,

$$\min\left\{\min\left\{(-\lambda_{2,\infty}|\phi| + |\nabla\phi|)(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\right\}, -\Delta_\infty\phi(x_0)\right\} \leq 0. \tag{2.15}$$

Inequalities (2.14) and (2.15) give the boundary condition, in the viscosity sense, in the region $u > 0$. The rest of the cases are handled similarly; we just state the final results:

If $u_{2,\infty} - \phi$ has a strict maximum at $x_0$ such that $u_{2,\infty}(x_0) = \phi(x_0) < 0$, then we get

$$\min\left\{\max\left\{(\lambda_{2,\infty}|\phi| - |\nabla\phi|)(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\right\}, -\Delta_\infty\phi(x_0)\right\} \leq 0. \tag{2.16}$$

If $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $u_{2,\infty}(x_0) = \phi(x_0) < 0$, then we get

$$\max\left\{\max\left\{(\lambda_{2,\infty}|\phi| - |\nabla\phi|)(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\right\}, -\Delta_\infty\phi(x_0)\right\} \geq 0. \tag{2.17}$$

Ift $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ such that $u_{2,\infty}(x_0) = \phi(x_0) = 0$ we get

$$\max\left\{\frac{\partial\phi}{\partial\nu}(x_0), -\Delta_\infty\phi(x_0)\right\} \geq 0. \tag{2.18}$$

Finally, if $u_{2,\infty} - \phi$ has a strict maximum at $x_0$ with $u_{2,\infty}(x_0) = \phi(x_0) = 0$ it follows that

$$\min\left\{\frac{\partial\phi}{\partial\nu}(x_0), -\Delta_\infty\phi(x_0)\right\} \le 0. \tag{2.19}$$

Inequalities (2.14)–(2.19) prove the result. □

Using the same ideas we can prove the following lemma.

LEMMA 2.6. *Let $\lambda_{k,p}$ be the $k$-th variational eigenvalue of* (1.1) *with eigenfunction $u_{k,p}$ normalized by $\|u_{k,p}\|_{L^\infty(\partial\Omega)} = 1$. Then every sequence tending to infinity has a subsequence $p_i$ such that*

$$\lim_{i\to\infty} \lambda_{k,p_i}^{1/p_i} = \lambda_{*,\infty}, \qquad u_{k,p_i} \to u_{*,\infty} \quad \text{in } C^\alpha(\overline{\Omega}),$$

*where $(u_{*,\infty}, \lambda_{*,\infty})$ is a solution of* (1.2).

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