Rend. Lincei Mat. Appl. 17 (2006), 227–242

Differential geometry. — *Closed curves in* $\mathbb{R}^3$ *with prescribed curvature and torsion in perturbative cases—Part 1: Necessary condition and study of the unperturbed problem*, by PAOLO CALDIROLI and MICHELA GUIDA.

ABSTRACT. — We study the problem of (κ, τ)-loops, i.e. closed curves in the three-dimensional Euclidean space with prescribed curvature κ and torsion τ. We state a necessary condition for the existence of a bounded sequence of (κ_n, τ_n)-loops when the functions κ_n and τ_n converge to the constants 1 and 0, respectively. Moreover we prove some Fredholm-type properties for the "unperturbed" problem, with $\kappa \equiv 1$ and $\tau \equiv 0$.

KEY WORDS: Prescribed curvature and torsion; perturbative methods; Fredholm operators.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 53A04; Secondary 47A53.

1. INTRODUCTION

Recent years have seen a growing interest in some geometrical problems concerning the existence and possible location of k-dimensional manifolds embedded into $\mathbb{R}^N$ with given topological type and prescribed curvature (see, e.g., [1], [2], [6], [8], [11] and the recent monograph [3] with the references therein).

Here we investigate a problem in low dimension. More precisely, we study the existence of closed curves in the three-dimensional Euclidean space with prescribed curvature and torsion. The problem can be stated as follows: given smooth functions $\kappa : \mathbb{R}^3 \to (0, +\infty)$ and $\tau : \mathbb{R}^3 \to \mathbb{R}$, find closed curves Γ in $\mathbb{R}^3$ such that at every point $p \in \Gamma$ the curvature of Γ equals $\kappa(p)$ and the torsion is $\tau(p)$. We shall call such curves (κ, τ)-*loops*.

A specially relevant case corresponds to the choice $\kappa \equiv \kappa_0$ and $\tau \equiv 0$, where κ_0 is a positive constant. In this situation the only closed curves with such curvature and torsion are circles of radius $1/\kappa_0$ placed anywhere in $\mathbb{R}^3$ (see Lemma 3.1). We remark that the set of closed curves with constant curvature κ_0 and torsion 0 defines a manifold $\mathscr{Z}$ of dimension 5, diffeomorphically parametrized by $\mathbb{P}^2 \times \mathbb{R}^3$, where $\mathbb{P}^2 := \mathbb{R}^3/\mathbb{R}_*$ denotes the two-dimensional projective space, namely the space of directions in $\mathbb{R}^3$ (every pair $(n, p) \in \mathbb{P}^2 \times \mathbb{R}^3$ corresponds to the circle of radius $1/\kappa_0$ centered at p and lying on the plane orthogonal to n).

Now let us focus on the problem of (κ, τ)-loops when the curvature κ and torsion τ are perturbations of the constants $\kappa_0 > 0$ and 0 respectively, and depend on a small parameter ε in the following way:

$$\begin{cases} \kappa(p) \equiv \kappa_\varepsilon(p) := \kappa_0 + K(\varepsilon, p), \\ \tau(p) \equiv \tau_\varepsilon(p) := T(\varepsilon, p), \end{cases}$$

where $K, T : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are smooth functions such that

$$K(0,\cdot) \equiv 0 \quad \text{and} \quad T(0,\cdot) \equiv 0. \tag{1.1}$$

Let us observe that κ_ε is admissible as a prescribed curvature, since $\kappa_\varepsilon > 0$ on compact subsets of $\mathbb{R}^3$ as $|\varepsilon|$ is small enough.

In general some conditions on K and T are needed for the existence of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops. Indeed, considering the case $K \equiv 0$ and $T \equiv \varepsilon$, one can see that for every $\varepsilon \neq 0$ the only curves with constant curvature κ_0 and constant torsion ε are portions of helicoids. Hence in this case there is no closed curve. Also when $T \equiv 0$, i.e., when one deals with planar curves, some restrictions on K are necessary (see [5]).

Hereafter we shall assume for simplicity $\kappa_0 = 1$, which is not restrictive, by obvious normalization. Henceforth, for all $\varepsilon \in \mathbb{R}$ and $p \in \mathbb{R}^3$ we shall take

$$\kappa_\varepsilon(p) = 1 + K(\varepsilon, p) \quad \text{and} \quad \tau_\varepsilon(p) = T(\varepsilon, p). \tag{1.2}$$

We will see that the existence and nonexistence of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops is strongly related to the properties of the zero set of the mapping $M : \mathbb{T}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ defined as follows:

$$M(\phi, p) := \begin{pmatrix} \int_0^1 \partial_\varepsilon K(0, R_\phi z(t) + p)\cos(2\pi t)\,dt \\ \int_0^1 \partial_\varepsilon K(0, R_\phi z(t) + p)\sin(2\pi t)\,dt \\ \int_0^1 \partial_\varepsilon T(0, R_\phi z(t) + p)\cos(2\pi t)\,dt \\ \int_0^1 \partial_\varepsilon T(0, R_\phi z(t) + p)\sin(2\pi t)\,dt \\ \int_0^1 \partial_\varepsilon T(0, R_\phi z(t) + p)\,dt \end{pmatrix} \quad \text{for } (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3 \tag{1.3}$$

where $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$ is the two-dimensional torus,

$$R_\phi := \begin{pmatrix} \cos\phi_2 & -\sin\phi_2\cos\phi_1 & \sin\phi_1\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2\cos\phi_1 & -\sin\phi_1\cos\phi_2 \\ 0 & \sin\phi_1 & \cos\phi_1 \end{pmatrix} \in SO(3) \quad \text{for every } \phi = (\phi_1, \phi_2) \in \mathbb{T}^2 \tag{1.4}$$

and

$$z(t) := \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ 0 \end{pmatrix} \quad \text{for every } t \in \mathbb{R}. \tag{1.5}$$

By natural periodic extension, we shall also consider $M : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$.

If (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$, then z is a uniform[1] parametrization of the unit circle centered at the origin and lying on the plane orthogonal to e_3. Moreover, $R_\phi z + p$ parametrizes the unit circle centered at p and lying on the plane orthogonal to $R_\phi e_3$. Vice versa, any solution of the unperturbed problem, i.e., the problem corresponding to $\varepsilon = 0$, admits such a parametrization, so that $\mathscr{Z} = \{R_\phi z(\mathbb{R}) + p \mid (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3\}$.

[1] A parametrization u of a curve Γ is called *uniform* if $|u'|$ is constant.

Hence the mapping M establishes a link between the perturbation (K, T) and the unperturbed manifold $\mathscr{Z}$ and, borrowing a notion from perturbation theory for dynamical systems [9], it can be interpreted as the Poincaré–Melnikov vector associated to the problem.

We point out that defining M in terms of the coordinates $(\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3$ we can ensure as much regularity for M as we need, since the mapping $\phi \mapsto R_\phi$ from $\mathbb{T}^2$ into $SO(3)$ is of class C^∞. If we parametrize $\mathscr{Z}$ by means of global coordinates $(n, p) \in \mathbb{P}^2 \times \mathbb{R}^3$, even continuity is lost because of the Hairy Ball Theorem which prevents the existence of continuous mappings $n \mapsto R(n)$ from $\mathbb{P}^2$ into $SO(3)$ such that $R(n)e_3$ has direction n.

As a first result we show that the fact that M vanishes somewhere is a necessary condition for the existence of a bounded sequence of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops with $|\varepsilon|$ small.

THEOREM 1.1. *Let $K, T \in C^1(\mathbb{R} \times \mathbb{R}^3)$ satisfy* (1.1) *and let κ_ε and τ_ε be as in* (1.2). *If there is a sequence $\varepsilon_n \to 0$, $\varepsilon_n \neq 0$, and a corresponding sequence (Γ_n) of $(\kappa_{\varepsilon_n}, \tau_{\varepsilon_n})$-loops such that for every $n \in \mathbb{N}$ one has*

$$0 < C_0 \leq \text{length}(\Gamma_n) \leq C \quad \textit{and} \quad \text{dist}(0, \Gamma_n) \leq C$$

for some constants C_0 and C independent of $n \in \mathbb{N}$, then, up to a subsequence, $\Gamma_n \to R_\phi z(\mathbb{R}) + p$ in C^1 as $n \to +\infty$, for some $(\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3$ and $M(\phi, p) = 0$, with M defined by (1.3).

Then we prove some properties concerning the unperturbed problem. More precisely, denoting by C^k_{per} the space of C^k functions from $\mathbb{R}$ into $\mathbb{R}^3$ which are periodic with period 1, set

$$\Omega := \{(u_1, u_2) \in C^2_{\text{per}} \times C^1_{\text{per}} \mid u_1 \text{ nonconstant}, \ u_2 \neq 0\}$$

and define the operator $F_0 : \Omega \subset C^2_{\text{per}} \times C^1_{\text{per}} \to C^0_{\text{per}} \times C^0_{\text{per}}$ by

$$F_0(u_1, u_2) := \left(-u_1'' + \frac{N(u_1')}{N(u_2)} u_2 \wedge u_1', -u_2' \right) \quad \text{for every } (u_1, u_2) \in \Omega, \tag{1.6}$$

where

$$N(u) := \sqrt{\int_0^1 |u|^2} \quad \text{for every } u \in C^0_{\text{per}}.$$

We will see that $F_0(u_1, u_2) = 0$ for some $(u_1, u_2) \in \Omega$ if and only if u_1 is a uniform, 1-periodic parametrization of a $(1, 0)$-loop, that is, a unit circle placed somewhere in $\mathbb{R}^3$. Notice also that F_0 is of class C^∞ on its domain. Setting

$$Z := \{(R_\phi z + p, R_\phi e_3) \mid (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3\}, \tag{1.7}$$

we will prove the following result:

THEOREM 1.2. *For every $(u_1, u_2) \in Z$ the function $F_0'(u_1, u_2) : C^2_{\text{per}} \times C^1_{\text{per}} \to C^0_{\text{per}} \times C^0_{\text{per}}$ is a Fredholm operator of index* 0. *In particular,* $\dim\ker F_0'(u_1, u_2) = \operatorname{codim}\operatorname{im} F_0'(u_1, u_2) = 7$.

We point out that the mapping F_0 cannot be expressed as the gradient of any functional and, even for $(u_1, u_2) \in Z$, the operator $F_0'(u_1, u_2)$ is not symmetric.

The information stated by Theorem 1.2 will be essential in order to get existence results for the perturbed problem, as we will see in the sequel [4] of the present paper.

The study developed here and in [4] constitutes a part of the PhD thesis [10] of the second author.

2. Preliminaries

Let Γ be a closed, regular, parametric curve in $\mathbb{R}^3$ of class C^3 and let $p : \mathbb{R} \to \mathbb{R}^3$ be a parametrization of Γ by arc length, i.e., $|p'(s)| = 1$ for all $s \in \mathbb{R}$. The curvature of Γ at the point $p(s)$ is given by the value $\kappa(p(s)) := |p''(s)|$. If $\kappa(p(s)) \neq 0$ one defines the normal and binormal vectors to the curve at the point $p(s)$ as $n(s) := p''(s)/\kappa(p(s))$ and $b(s) := p'(s) \wedge n(s)$ respectively. The triple $\{p'(s), n(s), b(s)\}$ of orthogonal unit vectors at $p(s)$ is the so-called Frenet trihedron and the value $\tau(p(s)) := b'(s) \cdot n(s)$ is the torsion of Γ at the point $p(s)$. We point out that the curvature κ and the torsion τ are geometrical entities associated to the curve which in fact depend on the point $p(s)$ (and not on the parametrization).

According to the classical theory of parametric curves in $\mathbb{R}^3$ (see [7]), the triple $\{p', n, b\}$ satisfies the following equations, known as Frenet formulas:

$$\begin{cases} p'' = \kappa n, \\ n' = -\kappa p' - \tau b, \\ b' = \tau n, \end{cases} \tag{2.1}$$

and the orthonormality conditions:

$$|p'| = |n| = |b| = 1, \qquad p' \cdot n = p' \cdot b = n \cdot b = 0. \tag{2.2}$$

In fact, in (2.1) only two equations are independent because $b = p' \wedge n$. In particular, since $n = b \wedge p'$, (2.1) and (2.2) hold true if and only if

$$\begin{cases} p'' = \kappa b \wedge p', \\ b' = \tau b \wedge p', \end{cases} \tag{2.3}$$

and

$$|p'| = |b| = 1, \qquad p' \cdot b = 0. \tag{2.4}$$

Moreover, as p parametrizes a closed curve, p is a nonconstant periodic function.

The system (2.3) together with the conditions (2.4) and the periodicity conditions provides the analytical formulation of the problem of finding closed curves with prescribed curvature κ and torsion τ, called (κ, τ)-loops.

Since in general the length of the curve (or, equivalently, the period of solutions of (2.3)) is also unknown, it is convenient to write the system (2.3) in an equivalent way as suggested by the next lemma (we will use the notation C^k_{per}, Ω and $N(u)$ already defined in the Introduction).

LEMMA 2.1. *Let* $\kappa \in C^1(\mathbb{R}^3)$ *and* $\tau \in C^0(\mathbb{R}^3)$, *with* $\kappa > 0$ *in* $\mathbb{R}^3$. *A pair* $(u_1, u_2) \in \Omega$ *solves*

$$\begin{cases} u_1'' = \dfrac{\ell}{\nu}\kappa(u_1)u_2 \wedge u_1', \\ u_2' = \tau(u_1)u_2 \wedge u_1', \end{cases} \tag{2.5}$$

with $\ell = N(u_1')$ *and* $\nu = N(u_2)$ *if and only if the mappings* $p(s) := u_1(s/\ell)$ *and* $b(s) := (1/\nu)u_2(s/\ell)$ *are nonconstant periodic solutions of* (2.3). *In this case* $|p'(s)| = |b(s)| = 1$ *for all* $s \in \mathbb{R}$ *and* $p'(s) \cdot b(s)$ *is constant. If in addition* $u_1'(t_0) \cdot u_2(t_0) = 0$ *for some* $t_0 \in \mathbb{R}$, *then* p *is a parametrization by arc length of a* (κ, τ)*-loop* Γ, *and* ℓ *is a multiple of the length of* Γ.

PROOF. By direct computations, one checks the equivalence between the systems (2.3) and (2.5). Moreover, by (2.3), one also obtains $(|p'|^2)' = (|b|^2)' = (p' \cdot b)' = 0$, so $|p'|$, $|b|$ and $p' \cdot b$ are constant. In particular the equality $\int_0^\ell |p'|^2 = \ell^{-1}\int_0^1 |u_1'|^2 = \ell$ yields $|p'(s)| \equiv 1$. In a similar way one gets $|b(s)| \equiv 1$. If $u_1'(t_0) \cdot u_2(t_0) = 0$ for some $t_0 \in \mathbb{R}$ then $p'(s_0) \cdot b(s_0) = 0$ for $s_0 = \ell t_0$ and consequently $p'(s) \cdot b(s) = 0$ for every $s \in \mathbb{R}$. Hence the orthonormality conditions (2.4) are fulfilled and the conclusion follows. □

3. PROOF OF THEOREM 1.1

As a first step, let us explicitly describe the set of nonconstant 1-periodic solutions of the problem

$$\begin{cases} u_1'' = \dfrac{N(u_1')}{N(u_2)}u_2 \wedge u_1', \\ u_2' = 0, \end{cases} \tag{3.1}$$

which corresponds to (2.5) with $\kappa \equiv 1$ and $\tau \equiv 0$.

LEMMA 3.1. *Any solution* $(u_1, u_2) \in \Omega$ *of* (3.1) *can be written in the following form:*

$$\begin{aligned} u_1(t) &= Rz(jt) + p, \\ u_2(t) &= \lambda Re_3, \end{aligned} \tag{3.2}$$

with $j \in \mathbb{N}$, $R \in SO(3)$, $p \in \mathbb{R}^3$, $\lambda > 0$ *and* z *defined in* (1.5).

Notice that all the solutions $(u_1, u_2) \in \Omega$ of (3.1) automatically satisfy the orthogonality condition $u_1'(t) \cdot u_2(t) = 0$ for all t.

PROOF. First, one has $u_2(t) = \lambda a$ with $\lambda > 0$ and $a \in \mathbb{S}^2$. Thus one is led to look for 1-periodic solutions of the linear equation

$$u_1'' = \ell a \wedge u_1' \tag{3.3}$$

with $\ell = N(u_1')$. Integrating (3.3) once, one obtains

$$u_1'(t) = \sin(\ell t) a \wedge b + (1 - \cos(\ell t))(a \cdot b) a + \cos(\ell t) b$$

with $b \in \mathbb{R}^3$ arbitrary. Then the general solution of (3.3) is

$$u_1(t) = \frac{1 - \cos(\ell t)}{\ell} a \wedge b + t(a \cdot b) a + \frac{\sin(\ell t)}{\ell}(b - (a \cdot b) a) + c$$

with $c \in \mathbb{R}^3$ arbitrary. From the equation (3.3) it follows that $(|u_1'(t)|^2)' \equiv 0$, so $|u_1'|$ is constant. In particular $|u_1'(t)| = |u_1'(0)| = |b|$ and then $\ell = N(u_1') = |b|$. Therefore $b \neq 0$ and one can write $b = \ell \hat{b}$ with $\hat{b} \in \mathbb{S}^2$. Now let us impose the periodicity condition $u_1(0) = u_1(1)$. On the one hand, the equation $u_1(0) \cdot a = u_1(1) \cdot a$ implies $a \cdot b = 0$. On the other hand, from $|u_1(1)| = |u_1(0)|$ it follows that $\cos \ell = 1$, that is, $\ell = 2j\pi$ for some $j \in \mathbb{N}$. Hence u_1 takes the form

$$u_1(t) = -\cos(2j\pi t) a \wedge \hat{b} + \sin(2j\pi t)\hat{b} + a \wedge \hat{b} + c$$

with $a \cdot \hat{b} = 0$. Setting $p_1 = -a \wedge \hat{b}$, $p_2 = \hat{b}$ and $p = a \wedge \hat{b} + c$ one writes

$$u(t) = \cos(2j\pi t) p_1 + \sin(2j\pi t) p_2 + p$$

with $|p_1| = |p_2| = 1$, $p_1 \cdot p_2 = 0$ and $u_2(t) \equiv \lambda a = \lambda p_1 \wedge p_2$. Equivalently, (3.2) holds for some $R \in SO(3)$. □

REMARK 3.2. If we represent a matrix $R \in SO(3)$ by means of Euler angles, every solution $(u_1, u_2) \in \Omega$ of (3.1) can be equivalently written in the following form:

$$\begin{aligned} u_1(t) &= R_\phi z(jt + \phi_0) + p, \\ u_2(t) &= \lambda R_\phi e_3, \end{aligned}$$

with $j \in \mathbb{N}$, $p \in \mathbb{R}^3$, $\lambda > 0$, $\phi_0 \in \mathbb{R}/\mathbb{Z}$, $\phi \in \mathbb{T}^2$ and R_ϕ and z defined as in (1.4) and (1.5), respectively.

The parameters p, λ, ϕ_0 and ϕ reflect corresponding symmetries for the problem (3.1). Some symmetries are of analytical type and arise from the formulation of the problem in terms of a system of ode's. This is the case for invariance under dilation with respect to the second component u_2 and invariance under the change $t \mapsto jt + \phi_0$. These invariances are exhibited also by any problem like (2.5). The more meaningful symmetries are those of geometrical type, expressed by the parameters $\phi \in \mathbb{T}^2$ and $p \in \mathbb{R}^3$, and which are broken if κ is nonconstant and τ is nonzero.

PROOF OF THEOREM 1.1. Let Γ_n be a $(\kappa_{\varepsilon_n}, \tau_{\varepsilon_n})$-loop and let $u_n \in C^3_{\text{per}}$ be a uniform parametrization of Γ_n, with $|u_n'| = c_n$. Notice that

$$\kappa_{\varepsilon_n}(u_n) = \frac{|u_n''|}{c_n^2}. \tag{3.4}$$

Define

$$u_{1,n} = u_n,$$
$$u_{2,n} = \frac{u_n' \wedge u_n''}{c_n^3 \kappa_{\varepsilon_n}(u_n)}.$$

Then $(u_{1,n}, u_{2,n}) \in \Omega$ solves

$$\begin{cases} u_{1,n}'' = c_n \kappa_{\varepsilon_n}(u_{1,n}) u_{2,n} \wedge u_{1,n}', \\ u_{2,n}' = \tau_{\varepsilon_n}(u_{1,n}) u_{2,n} \wedge u_{1,n}'. \end{cases} \tag{3.5}$$

Moreover $|u_{1,n}'| = c_n$ and thus $N(u_{1,n}') = c_n$. In addition, by the definition of $u_{2,n}$, using (3.4) and the fact that $u_n' \cdot u_n'' = 0$ (because $|u_n'|$ is constant), one also deduces that $|u_{2,n}| = 1$ and thus $N(u_{2,n}) = 1$. By hypothesis, the sequence $(u_{1,n})$ is bounded in C^1_{per}. Moreover the sequence $(u_{2,n})$ is bounded in C^0_{per}. Thanks to (3.5), the sequences $(u_{1,n})$ and $(u_{2,n})$ are bounded in C^2_{per} and in C^1_{per}, respectively. By the Ascoli–Arzelà theorem, passing to subsequences, we may assume that

$$u_{1,n} \to u_1 \quad \text{in } C^1_{\mathrm{per}} \quad \text{and} \quad u_{2,n} \to u_2 \quad \text{in } C^0_{\mathrm{per}}$$

for some $(u_1, u_2) \in C^1_{\mathrm{per}} \times C^0_{\mathrm{per}}$. In particular $c_n = N(u_{1,n}') \to N(u_1') =: c$ and $N(u_2) = 1$. By hypothesis $c \neq 0$, that is, u_1 is nonconstant. In addition, by the uniform continuity, $\kappa_{\varepsilon_n}(u_{1,n}) \to 1$ and $\tau_{\varepsilon_n}(u_{1,n}) \to 0$ uniformly on $[0, 1]$. By standard arguments we can pass to the limit in (3.5), finding that (u_1, u_2) is a nonconstant solution of

$$\begin{cases} u_1'' = c u_2 \wedge u_1', \\ u_2' = 0, \end{cases}$$

with $c = N(u_1')$. Then, by Lemma 3.1 and Remark 3.2, $u_1(t) = R_\phi z(jt + \phi_0) + p$ and $u_2(t) = R_\phi e_3$ for some $\phi \in \mathbb{T}^2$, $p \in \mathbb{R}^3$, $j \in \mathbb{N}$, and $\phi_0 \in \mathbb{R}/\mathbb{Z}$. Now we show that $M(n, p) = 0$. Set

$$\hat{K}(\varepsilon, p) = \begin{cases} \partial_\varepsilon K(0, p) - \dfrac{K(\varepsilon, p)}{\varepsilon} & \text{if } \varepsilon \neq 0, \\ 0 & \text{if } \varepsilon = 0, \end{cases}$$
$$\hat{T}(\varepsilon, p) = \begin{cases} \partial_\varepsilon T(0, p) - \dfrac{T(\varepsilon, p)}{\varepsilon} & \text{if } \varepsilon \neq 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

Since $\partial_\varepsilon K, \partial_\varepsilon T \in C^0(\mathbb{R} \times \mathbb{R}^3)$, one sees that $\hat{K}(\varepsilon, p) \to 0$ and $\hat{T}(\varepsilon, p) \to 0$ as $\varepsilon \to 0$ uniformly on compact sets of $\mathbb{R}^3$. As a consequence, since $u_{1,n} \to u_1$ uniformly on $[0, 1]$, one finds that $\hat{K}(\varepsilon_n, u_{1,n}) \to 0$ and $\hat{T}(\varepsilon_n, u_{1,n}) \to 0$ uniformly on $[0, 1]$. Then, since the sequence $(u_{2,n} \wedge u_{1,n}')$ is uniformly bounded on $[0, 1]$,

$$\hat{K}(\varepsilon_n, u_{1,n}) u_{2,n} \wedge u_{1,n}' \to 0 \quad \text{and} \quad \hat{T}(\varepsilon_n, u_{1,n}) u_{2,n} \wedge u_{1,n}' \to 0 \quad \text{uniformly on } [0, 1]. \tag{3.6}$$

Then, by (3.6), one has

$$\int_0^1 \partial_\varepsilon T(0, u_{1,n}) u_{2,n} \wedge u'_{1,n} = \int_0^1 \hat{T}(\varepsilon_n, u_{1,n}) u_{2,n} \wedge u'_{1,n} + \frac{1}{\varepsilon_n} \int_0^1 u'_{2,n} \to 0$$

and

$$\begin{aligned}
&\int_0^1 \partial_\varepsilon K(0, u_{1,n}) u_{2,n} \wedge u'_{1,n} \\
&\qquad = \int_0^1 \hat{K}(\varepsilon_n, u_{1,n}) u_{2,n} \wedge u'_{1,n} + \frac{1}{c_n \varepsilon_n} \int_0^1 u''_{1,n} - \frac{1}{\varepsilon_n} \int_0^1 u_{2,n} \wedge u'_{1,n} \\
&\qquad = o(1) + \frac{1}{\varepsilon_n} \int_0^1 u'_{2,n} \wedge u_{1,n} \\
&\qquad = o(1) + \int_0^1 (\partial_\varepsilon T(0, u_{1,n}) - \hat{T}(\varepsilon_n, u_{1,n}))(u_{2,n} \wedge u'_{1,n}) \wedge u_{1,n} \\
&\qquad = o(1) + \int_0^1 \partial_\varepsilon T(0, u_1)(u_2 \wedge u'_1) \wedge u_1 .
\end{aligned}$$

Knowing explicitly u_1 and u_2 one can compute $(u_2 \wedge u'_1) \wedge u_1 = 2\pi j p \wedge u_1$ to obtain

$$\int_0^1 \partial_\varepsilon K(0, u_{1,n}) u_{2,n} \wedge u'_{1,n} \to 2\pi j \int_0^1 \partial_\varepsilon T(0, u_1) p \wedge u_1 . \tag{3.7}$$

On the other hand, since $u_{2,n} \wedge u'_{1,n} \to u_2 \wedge u'_1 = 2\pi j (p - u_1)$ uniformly on $[0, 1]$, one has

$$\int_0^1 \partial_\varepsilon T(0, u_{1,n}) u_{2,n} \wedge u'_{1,n} \to 2\pi j \int_0^1 \partial_\varepsilon T(0, u_1)(p - u_1)$$

and then

$$\int_0^1 \partial_\varepsilon T(0, u_1)(p - u_1) = 0, \tag{3.8}$$

hence

$$0 = \int_0^1 \partial_\varepsilon T(0, R_\phi z(jt + \phi_0) + p) z(jt + \phi_0)\, dt = \int_0^1 \partial_\varepsilon T(0, R_\phi z + p) z,$$

that is, $M_3(\phi, p) = M_4(\phi, p) = 0$. In a similar way one has

$$\int_0^1 \partial_\varepsilon K(0, u_{1,n}) u_{2,n} \wedge u'_{1,n} \to 2\pi j \int_0^1 \partial_\varepsilon K(0, u_1)(p - u_1)$$

and from (3.7) one deduces that

$$\int_0^1 \partial_\varepsilon K(0, u_1)(p - u_1) = \int_0^1 \partial_\varepsilon T(0, u_1) p \wedge u_1 = 0$$

where the last equality follows from the equality $p \wedge u_1(t) = (p \wedge R_\phi e_1)z(jt + \phi_0) \cdot e_1 + (p \wedge R_\phi e_2)z(jt + \phi_0) \cdot e_2$ and from the fact that $M_3(\phi, p) = M_4(\phi, p) = 0$. Hence, arguing as before, one infers that also $M_1(\phi, p) = M_2(\phi, p) = 0$. Finally, using the second equation in (3.5) and the fact that $u'_{1,n} \cdot u_{2,n} = 0$, we obtain

$$\int_0^1 T(\varepsilon_n, u_{1,n})u_{2,n} \wedge u'_{1,n} \cdot u_{1,n} = 0,$$

which, using also (3.6), implies that

$$\int_0^1 \partial_\varepsilon T(0, u_{1,n})u_{2,n} \wedge u'_{1,n} \cdot u_{1,n} \to 0$$

and then

$$0 = \int_0^1 \partial_\varepsilon T(0, u_1)u_2 \wedge u'_1 \cdot u_1 = 2\pi j \int_0^1 \partial_\varepsilon T(0, u_1)u_1 \cdot (p - u_1).$$

Therefore, using (3.8), we obtain

$$0 = -\int_0^1 \partial_\varepsilon T(0, u_1)u_1 \cdot (p - u_1) + p \cdot \int_0^1 \partial_\varepsilon T(0, u_1)(p - u_1) = \int_0^1 \partial_\varepsilon T(0, u_1),$$

that is, $M_5(\phi, p) = 0$. □

4. Proof of Theorem 1.2

Let

$$X := C^2_{\text{per}} \times C^1_{\text{per}}, \qquad Y := C^0_{\text{per}} \times C^0_{\text{per}}$$

be the Banach spaces endowed with their standard norms, and consider the operator $F_0 : \Omega \subset X \to Y$ defined in (1.6). One has $F_0 \in C^\infty(\Omega, Y)$. In particular, for fixed $(u_1, u_2) \in \Omega$, the differential $F'_0(u_1, u_2)$ is a bounded linear operator from X into Y acting in the following way:

$$F'_0(u_1, u_2)[x_1, x_2] = \left(-x''_1 + \frac{\langle u'_1, x'_1\rangle}{N(u'_1)N(u_2)} u_2 \wedge u'_1 + \frac{N(u'_1)}{N(u_2)}(u_2 \wedge x'_1 + x_2 \wedge u'_1) - \frac{N(u'_1)\langle u_2, x_2\rangle}{N(u_2)^3} u_2 \wedge u'_1, -x'_2\right) \tag{4.1}$$

for every $(x_1, x_2) \in X$, where in general

$$\langle u, v\rangle = \int_0^1 u \cdot v.$$

In the following, X and Y will be equipped with the L^2 inner product:

$$\langle (u_1, u_2), (v_1, v_2)\rangle = \langle u_1, v_1\rangle + \langle u_2, v_2\rangle = \int_0^1 (u_1 \cdot v_1 + u_2 \cdot v_2). \tag{4.2}$$

The notion of orthogonality we will consider will always refer to the above inner product.

LEMMA 4.1. *For every* $(u_1, u_2) \in Z$ *(with* Z *defined in* (1.7)*) one has*

$$\ker F_0'(u_1, u_2) = \{(a \wedge u_1 + b, a \wedge u_2 + \lambda u_2) \mid a, b \in \mathbb{R}^3,\ \lambda \in \mathbb{R}\}.$$

PROOF. First, let us prove the lemma taking $(\phi, p) = (0, 0)$, that is, $(u_1, u_2) = (z, e_3)$. Notice that $(x_1, x_2) \in \ker F_0'(z, e_3)$ if and only if (x_1, x_2) is a 1-periodic solution of

$$\begin{cases} x_1'' = 2\pi e_3 \wedge x_1' + 2\pi x_2 \wedge z' - (\langle z', x_1'\rangle - (2\pi)^2\langle e_3, x_2\rangle)z, \\ x_2' = 0. \end{cases} \tag{4.3}$$

First, observe that x_2 has to be constant. Hence $x_2(t) \equiv a_2 \in \mathbb{R}^3$ and one is led to look for 1-periodic solutions of

$$x_1'' = 2\pi e_3 \wedge x_1' + 2\pi a_2 \wedge z' - \alpha z \tag{4.4}$$

with

$$\alpha = \langle z', x_1'\rangle - (2\pi)^2 e_3 \cdot a_2. \tag{4.5}$$

Integrating (4.4) once, one gets

$$x_1'(t) = L(t)b_1 + L(t)\int_0^t L(-s)q(s)\,ds$$

where

$$\begin{aligned} L(t)p &= \sin(2\pi t)e_3 \wedge p + (1 - \cos(2\pi t))(e_3 \cdot p)e_3 + \cos(2\pi t)p, \\ q(t) &= 2\pi a_2 \wedge z'(t) - \alpha z(t) \end{aligned} \tag{4.6}$$

and $b_1 \in \mathbb{R}^3$ is arbitrary. Making computations one finds

$$L(-s)q(s) = -(\alpha + (2\pi)^2 a_{23})e_1 + (2\pi)^2(\sin(2\pi s)a_{22} + \cos(2\pi s)a_{21})e_3$$

where we have set $a_{2i} = a_2 \cdot e_i$ for $i = 1, 2, 3$. Therefore

$$\int_0^t L(-s)q(s)\,ds = -(\alpha + (2\pi)^2 a_{23})te_1 + 2\pi((1 - \cos(2\pi t))a_{22} + \sin(2\pi t)a_{21})e_3$$

and then

$$x_1'(t) = L(t)b_1 - (\alpha + (2\pi)^2 a_{23})tz(t) + 2\pi((1 - \cos(2\pi t))a_{22} + \sin(2\pi t)a_{21})e_3.$$

Observing that $x_1'(0) = b_1$ and $x_1'(1) = b_1 - (\alpha + (2\pi)^2 a_{23})e_1$, and imposing the periodicity condition $x_1'(0) = x_1'(1)$ one obtains

$$\alpha + (2\pi)^2 a_{23} = 0. \tag{4.7}$$

Moreover, after computations, (4.5) and (4.7) imply

$$0 = \int_0^1 z' \cdot x_1' = 2\pi b_1 \cdot e_2.$$

Hence,

$$x_1'(t) = (b_{13} + 2\pi a_{22})e_3 + \sin(2\pi t)(b_{11}e_2 + 2\pi a_{21}e_3) + \cos(2\pi t)(b_{11}e_1 - 2\pi a_{22}e_3)$$

where, as before, $b_{11} = b_1 \cdot e_1$ and $b_{13} = b_1 \cdot e_3$. Thus

$$\begin{aligned} x_1(t) = a_1 &+ (b_{13} + 2\pi a_{22})te_3 \\ &+ (1 - \cos(2\pi t))\left(\frac{b_{11}}{2\pi}e_2 + a_{21}e_3\right) + \sin(2\pi t)\left(\frac{b_{11}}{2\pi}e_1 - a_{22}e_3\right) \end{aligned}$$

where $a_1 \in \mathbb{R}^3$ is arbitrary. Since $x_1(0) = a_1$ and $x_1(1) = a_1 + (b_{13} + 2\pi a_{22})e_3$, in order that $x(t)$ be 1-periodic, one must have $b_{13} + 2\pi a_{22} = 0$. Hence, 1-periodic solutions of (4.8) are given by

$$\begin{aligned} x_1(t) &= a_1 + (1 - \cos(2\pi t))\left(\frac{b_{11}}{2\pi}e_2 + a_{21}e_3\right) + \sin(2\pi t)\left(\frac{b_{11}}{2\pi}e_1 - a_{22}e_3\right), \\ x_2(t) &= a_2, \end{aligned} \tag{4.8}$$

where $a_1, a_2 \in \mathbb{R}^3$ and $b_{11} \in \mathbb{R}$ are arbitrary. If we set $a = e_3 \wedge a_2 - (b_{11}/2\pi)e_3$, $b = a_1 - a \wedge e_1$ and $\lambda = a_{23}$, the solution (4.8) takes the form

$$\begin{aligned} x_1(t) &= a \wedge z(t) + b, \\ x_2(t) &= a \wedge e_3 + \lambda e_3, \end{aligned}$$

with arbitrary $a, b \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$.

Finally, we prove the result for any $(u_1, u_2) \in Z$. For every $R \in SO(3)$ and $(p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ set $R(p_1, p_2) := (Rp_1, Rp_2)$. Using this notation and (4.1) one can check that

$$F_0'(Rz + p, Re_3)[Rx_1, Rx_2] = R(F_0'(z, e_3)[x_1, x_2]). \tag{4.9}$$

Hence, taking $(u_1, u_2) = (R_\phi z + p, R_\phi e_3) \in Z$, thanks to the result proved in case $(\phi, p) = (0, 0)$, we have

$$\ker F_0'(R_\phi z + p, R_\phi e_3) = \{(R_\phi(a \wedge z + b), R_\phi(a \wedge e_3 + \lambda e_3)) \mid a, b \in \mathbb{R}^3,\ \lambda \in \mathbb{R}\},$$

which, up to an obvious equivalence, yields the statement of the lemma. □

Given any $(u_1, u_2) \in Z$, let us introduce the following linear subspace of Y:

$$Y_0(u_1, u_2) := \{(y_1, y_2) \in Y \mid \langle F_0'(u_1, u_2)[x_1, x_2], (y_1, y_2)\rangle = 0 \text{ for all } (x_1, x_2) \in X\}.$$

For further purposes, the following more explicit characterization of $Y_0(u_1, u_2)$ is useful.

LEMMA 4.2. *For every $(u_1, u_2) \in Z$ one has*

$$Y_0(u_1, u_2) = \{(\lambda u_1' + a, 2\pi a \wedge u_1 + b) \mid \lambda \in \mathbb{R},\ a, b \in \mathbb{R}^3\}.$$

PROOF. Let $(u_1, u_2) = (R_\phi z + p, R_\phi e_3) \in Z$. Thanks to (4.9) one has

$$Y_0(R_\phi z + p, R_\phi e_3) = \{(R_\phi y_1, R_\phi y_2) \mid (y_1, y_2) \in Y_0(z, e_3)\} \tag{4.10}$$

and so we can limit ourselves to proving the lemma for $(u_1, u_2) = (z, e_3)$. For every $(x_1, x_2) \in X$ set

$$\alpha(x_1', x_2) = \langle z', x_1' \rangle - (2\pi)^2 \langle e_3, x_2 \rangle. \tag{4.11}$$

Hence $(y_1, y_2) \in Y_0(z, e_3)$ if and only if (y_1, y_2) is a 1-periodic solution of

$$\langle (-x_1'' - \alpha(x_1', x_2) z + 2\pi(e_3 \wedge x_1' + x_2 \wedge z'), -x_2''), (y_1, y_2) \rangle = 0 \tag{4.12}$$

for all $(x_1, x_2) \in X$. In particular, taking $x_2 = 0$, we must have

$$\langle -x_1'' - \alpha(x_1', 0) z + 2\pi e_3 \wedge x_1', y_1 \rangle = 0 \quad \text{for all } x_1 \in C^2_{\text{per}}. \tag{4.13}$$

Since $\alpha(x_1', 0) = (2\pi)^2 \langle z, x_1 \rangle$, (4.13) is equivalent to

$$-\int_0^1 x_1'' \cdot y_1 + 2\pi \int_0^1 x_1' \cdot y_1 \wedge e_3 - (2\pi)^2 \left(\int_0^1 z \cdot x_1 \right) \left(\int_0^1 z \cdot y_1 \right) = 0 \tag{4.14}$$

for all $x_1 \in C^2_{\text{per}}$. It is standard to recognize that $y_1 \in C^0_{\text{per}}$ solves (4.14) if and only if y_1 is a (weak) 1-periodic solution of

$$\begin{cases} y_1'' = 2\pi e_3 \wedge y_1' - \beta z, \\ \beta = (2\pi)^2 \langle z, y_1 \rangle. \end{cases}$$

Arguing as in the proof of Lemma 4.1 one finds

$$y_1'(t) = L(t) b_1 - \beta t z(t)$$

where $L(t)$ is given by (4.6) and $b_1 \in \mathbb{R}^3$ is an arbitrary vector. Imposing the periodicity condition $y_1'(0) = y_1'(1)$ one infers that $\beta = 0$, so that

$$\int_0^1 z \cdot y_1 = 0. \tag{4.15}$$

Integrating once more, one obtains

$$y_1(t) = a_1 + \frac{1 - \cos(2\pi t)}{2\pi} e_3 \wedge b_1 + (e_3 \cdot b_1) t e_3 - \frac{\sin(2\pi t)}{2\pi} ((e_3 \cdot b_1) e_3 - b_1)$$

with $a_1 \in \mathbb{R}^3$ arbitrary. The periodicity condition $y_1(0) = y_1(1)$ yields $e_3 \cdot b_1 = 0$ and thus

$$y_1(t) = a_1 + \frac{1 - \cos(2\pi t)}{2\pi} e_3 \wedge b_1 + \frac{\sin(2\pi t)}{2\pi} b_1.$$

Now we impose (4.15) obtaining the further restriction $e_2 \cdot b_1 = 0$. Therefore $b_1 = b_{11} e_1$ where $b_{11} \in \mathbb{R}$ is arbitrary, and thus

$$y_1(t) = -\frac{b_{11}}{(2\pi)^2} z'(t) + \frac{b_{11}}{2\pi} e_2 + a_1.$$

Hence, up to redefining the constants one concludes that the general solution of (4.14) is given by

$$y_1(t) = \lambda z'(t) + a \tag{4.16}$$

with arbitrary $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}^3$. Now one plugs (4.16) into (4.12) finding the following equation for y_2:

$$2\pi a \cdot \int_0^1 x_2 \wedge z' = \int_0^1 x_2' \cdot y_2 \quad \text{for all } x_2 \in C^1_{\text{per}},$$

so that y_2 is a (weak) 1-periodic solution of $z' \wedge a + \frac{1}{2\pi} y_2' = 0$. Hence

$$y_2(t) = 2\pi a \wedge z(t) + b$$

with $b \in \mathbb{R}^3$ arbitrary. Finally, one can check that any pair of the form $(y_1, y_2) = (\lambda z' + a, 2\pi a \wedge z + b)$ solves (4.12). This concludes the proof. □

Notice that, by definition, $Y_0(u_1, u_2) = (\text{im } F_0'(u_1, u_2))^\perp$, where the orthogonality is meant with respect to the inner product (4.2). In fact we also have:

LEMMA 4.3. *For every* $(u_1, u_2) \in Z$ *one has* $\text{im } F_0'(u_1, u_2) = (Y_0(u_1, u_2))^\perp$.

PROOF. Since by definition $Y_0(u_1, u_2) = (\text{im } F_0'(u_1, u_2))^\perp$, the inclusion $\text{im } F_0'(u_1, u_2) \subseteq (Y_0(u_1, u_2))^\perp$ is trivial and we just have to prove the opposite one. Let us begin with $(u_1, u_2) = (z, e_3)$. For any fixed $(w_1, w_2) \in (Y_0(z, e_3))^\perp$ we look for $(x_1, x_2) \in X$ satisfying $F_0'(z, e_3)[x_1, x_2] = (w_1, w_2)$, that is,

$$\begin{cases} -x_1'' - \alpha(x_1', x_2) z + 2\pi(e_3 \wedge x_1' + x_2 \wedge z') = w_1, \\ -x_2' = w_2, \end{cases} \tag{4.17}$$

where $\alpha(x_1', x_2)$ is given by (4.11). Since $\langle (w_1, w_2), (y_1, y_2) \rangle = 0$ for every $(y_1, y_2) \in Y_0(z, e_3)$, the representation stated by Lemma 4.2 yields

$$\langle (w_1, w_2), (z', 0) \rangle = 0, \quad \text{i.e.,} \quad \langle w_1, z' \rangle = 0, \tag{4.18}$$

$$\langle (w_1, w_2), (0, e_i) \rangle = 0 \quad \text{for } i = 1, 2, 3, \quad \text{i.e.,} \quad \int_0^1 w_2 = 0, \tag{4.19}$$

$$\langle (w_1, w_2), (e_1, 2\pi e_1 \wedge z) \rangle = 0, \quad \text{i.e.,} \quad \langle w_1, e_1 \rangle = -2\pi \langle w_2, (e_2 \cdot z) e_3 \rangle, \tag{4.20}$$

$$\langle (w_1, w_2), (e_2, 2\pi e_2 \wedge z) \rangle = 0, \quad \text{i.e.,} \quad \langle w_1, e_2 \rangle = 2\pi \langle w_2, (e_1 \cdot z) e_3 \rangle, \tag{4.21}$$

$$\langle (w_1, w_2), (e_3, 2\pi e_3 \wedge z) \rangle = 0, \quad \text{i.e.,} \quad \langle w_1, e_3 \rangle = -\langle w_2, z' \rangle. \tag{4.22}$$

Now, the second equation in (4.17) is solved by

$$x_2(t) = d_0 - \int_0^t w_2 \tag{4.23}$$

where $d_0 \in \mathbb{R}^3$ is arbitrary. Notice that x_2 belongs to C^1_{per} thanks to (4.19). Integrating the first equation in (4.17) we obtain

$$x_1'(t) = L(t)c_1 + L(t)\int_0^t L(-s)f(s)\,ds \tag{4.24}$$

where $L(t)$ is given in (4.6), $c_1 \in \mathbb{R}^3$ is an arbitrary constant vector which should satisfy some restrictions and

$$f(s) = 2\pi x_2(s) \wedge z'(s) - \alpha(x_1', x_2)z(s) - w_1(s).$$

One can explicitly compute

$$\begin{aligned} L(-s)f(s) = (&-(2\pi)^2 x_2(s)\cdot e_3 - \alpha(x_1', x_2) - w_1\cdot z(s))e_1 \\ &- \frac{1}{2\pi} w_1(s)\cdot z'(s)e_2 + ((2\pi)^2 x_2(s)\cdot z(s) - w_1(s)\cdot e_3)e_3. \end{aligned}$$

The periodicity condition $x_1'(0) = x_1'(1)$ is equivalent to $\int_0^1 L(-s)f(s)\,ds = 0$, that is:

$$\int_0^1 (-(2\pi)^2 x_2\cdot e_3 - \alpha(x_1', x_2) - w_1\cdot z) = 0, \tag{4.25}$$

$$\int_0^1 w_1\cdot z' = 0, \tag{4.26}$$

$$\int_0^1 ((2\pi)^2 x_2\cdot z - w_1\cdot e_3) = 0. \tag{4.27}$$

One sees that (4.26) is (4.18) and thus it holds true. Also (4.27) is satisfied because, by (4.22) and by the second equation in (4.17), one has $\langle w_1, e_3\rangle = -\langle w_2, z'\rangle = \langle x_2', z'\rangle = -\langle x_2, z''\rangle = (2\pi)^2\langle x_2, z\rangle$. Hence it suffices to check (4.25) which in fact, using (4.11), is equivalent to

$$\langle z', x_1'\rangle = -\langle w_1, z\rangle. \tag{4.28}$$

By explicit computations, (4.24) gives

$$x_1'(t) = A_1(t)z(t) - \frac{1}{2\pi}A_2(t)z'(t) + A_3(t)e_3 \tag{4.29}$$

where

$$\begin{aligned} A_1(t) &= c_{11} - \int_0^t ((2\pi)^2 x_2\cdot e_3 + \alpha(x_1', x_2) + z\cdot w_1), \\ A_2(t) &= -c_{12} + \frac{1}{2\pi}\int_0^t z'\cdot w_1, \\ A_3(t) &= c_{13} + \int_0^t ((2\pi)^2 x_2\cdot z - e_3\cdot w_1), \end{aligned} \tag{4.30}$$

and $c_{1i} = c_1 \cdot e_i$ for $i = 1, 2, 3$. Thus (4.28) turns out to be equivalent to

$$c_{12} = \frac{1}{2\pi} \int_0^1 ((1-t)z' - z) \cdot w_1. \tag{4.31}$$

Integrating (4.29) one obtains

$$x_1(t) = c_0 + \int_0^t \left(A_1 z - \frac{1}{2\pi} A_2 z' + A_3 e_3 \right) \tag{4.32}$$

where $c_0 \in \mathbb{R}^3$ is arbitrary. Using (4.20), (4.21) and the second equation in (4.17), one can check that $\int_0^1 (A_1 z - \frac{1}{2\pi} A_2 z') = 0$, so that x_1 is periodic if and only if $\int_0^1 A_3 = 0$, i.e., upon explicit computations,

$$c_{13} = -d_{02} + \int_0^1 (1-t) e_3 \cdot w_1 + \int_0^1 ((1-t)z' + z - e_1) \cdot w_2 \tag{4.33}$$

where $d_{02} = d_0 \cdot e_2$. Hence for arbitrary $c_0, c_1, d_0 \in \mathbb{R}^3$ with c_{12} and c_{13} satisfying (4.31) and (4.33) the pair (x_1, x_2) given by (4.32) and (4.23) yields a periodic solution of (4.17). This concludes the proof in the case $(u_1, u_2) = (z, e_3)$. For an arbitrary $(u_1, u_2) = (R_\phi z + p, R_\phi e_3) \in Z$ one observes that $(Y_0(u_1, u_2))^\perp = R_\phi((Y_0(z, e_3))^\perp)$, by (4.10). Therefore, fixing $(v_1, v_2) \in (Y_0(u_1, u_2))^\perp$, by the first part of the proof, there exists $(x_1, x_2) \in X$ such that $F_0'(z, e_3) = (R_\phi^{-1} v_1, R_\phi^{-1} v_2) \in (Y_0(z, e_3))^\perp$. Then from (4.9) it follows that $F_0'(u_1, u_2)[R_\phi x_1, R_\phi x_2] = R_\phi(F_0'(z, e_3)[x_1, x_2]) = (v_1, v_2)$. This completes the proof. $\square$

Let us make a final technical remark which will be used in [4] for the study of the perturbed problem, about existence of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops.

REMARK 4.4. Thanks to (4.29), (4.30), (4.33), (4.23) and (4.9), fixing $(u_1, u_2) = (R_\phi z + p, R_\phi e_3) \in Z$, if $F_0'(u_1, u_2)[x_1, x_2] = (y_1, y_2)$ then

$$x_1'(0) \cdot R_\phi e_3 + 2\pi x_2(0) \cdot R_\phi e_2 = \int_0^1 (1-t) e_3 \cdot R_\phi^{-1} y_1 + \int_0^1 ((1-t)z' + z - e_1) \cdot R_\phi^{-1} y_2.$$

The first author is supported by MIUR-PRIN project “Metodi variazionali ed equazioni differenziali nonlineari”.

REFERENCES

[1] A. AMBROSETTI - J. GARCIA AZORERO - I. PERAL, *Perturbation of $\Delta u + u^{(N+2)/(N-2)} = 0$, the scalar curvature problem in $\mathbb{R}^N$, and related topics*. J. Funct. Anal. 165 (1999), 117–149.

[2] A. AMBROSETTI - A. MALCHIODI, *A multiplicity result for the Yamabe problem on $\mathbb{S}^n$*. J. Funct. Anal. 168 (1999), 529–561.

[3] A. AMBROSETTI - A. MALCHIODI, *Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$*. Progr. Math. 240, Birkhäuser, 2006.

[4] P. CALDIROLI - M. GUIDA, *Closed curves in $\mathbb{R}^3$ with prescribed curvature and torsion in perturbative cases—Part 2: Sufficient conditions*. Rend. Lincei Mat. Appl., to appear.

[5] P. CALDIROLI - M. GUIDA, *Helicoidal trajectories of a charge in a nonconstant magnetic field*. Preprint, 2006.

[6] P. CALDIROLI - R. MUSINA, *H-bubbles in a perturbative setting: the finite-dimensional reduction method*. Duke Math. J. 122 (2004), 457–484.

[7] M. DO CARMO, *Differential Geometry of Curves and Surfaces*. Prentice-Hall, 1976.

[8] A. CHANG - P. YANG, *A perturbation result in prescribing scalar curvature on $\mathbb{S}^n$*. Duke Math. J. 64 (1991), 473–496.

[9] J. GUCKENHEIMER - P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Appl. Math. Sci. 42, Springer, 1983.

[10] M. GUIDA, *Perturbative-type results for some problems of geometric analysis in low dimension*. Tesi di Dottorato, Dipartimento di Matematica, Università di Torino, 2005.

[11] D. POLLACK, *Nonuniqueness and high energy solutions for a conformally invariant scalar equation*. Comm. Anal. Geom. 1 (1993), 347–414.

Received 22 September 2005,
and in revised form 11 October 2005.

Dipartimento di Matematica
Università degli Studi di Torino
Via Carlo Alberto, 10
10123 TORINO, Italy
paolo.caldiroli@unito.it
guida@dm.unito.it