Rend. Lincei Mat. Appl. 17 (2006), 291–307

Differential geometry. — *Closed curves in $\mathbb{R}^3$ with prescribed curvature and torsion in perturbative cases—Part 2: Sufficient conditions,* by PAOLO CALDIROLI and MICHELA GUIDA.

ABSTRACT. — We investigate the problem of (κ, τ)-loops, that is, closed curves in the three-dimensional Euclidean space with prescribed curvature κ and torsion τ. In particular we focus on some perturbative cases, taking $\kappa = \kappa_\varepsilon(p)$ 291- and $\tau = \tau_\varepsilon(p)$ with κ_ε and τ_ε converging to the constants 1 and 0, respectively, as $\varepsilon \to 0$. We prove existence of branches of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops (for small $|\varepsilon|$) emanating from circles which correspond to stable zeroes of a suitable vector field $M : \mathbb{T}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$.

KEY WORDS: Prescribed curvature and torsion; perturbative methods; Fredholm operators.

MATHEMATICS SUBJECT CLASSIFICATION (2000): Primary 53A04; Secondary 47A53.

1. STATEMENT OF THE MAIN RESULTS

In this paper we study the existence of closed curves in the three-dimensional Euclidean space with prescribed curvature and torsion. The problem can be stated as follows: given smooth functions $\kappa : \mathbb{R}^3 \to (0, +\infty)$ and $\tau : \mathbb{R}^3 \to \mathbb{R}$, find closed curves Γ in $\mathbb{R}^3$ such that at every point $p \in \Gamma$ the curvature of Γ equals $\kappa(p)$ and the torsion is $\tau(p)$. We shall call such curves (κ, τ)-*loops*.

In particular we will focus on some perturbative cases, taking

$$\begin{cases} \kappa(p) \equiv \kappa_\varepsilon(p) := 1 + K(\varepsilon, p), \\ \tau(p) \equiv \tau_\varepsilon(p) := T(\varepsilon, p), \end{cases} \tag{1.1}$$

where $K, T : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are smooth functions such that

$$K(0, \cdot) \equiv 0 \quad \text{and} \quad T(0, \cdot) \equiv 0\,. \tag{1.2}$$

Let us observe that κ_ε is admissible as a prescribed curvature, since $\kappa_\varepsilon > 0$ on compact sets of $\mathbb{R}^3$ for $|\varepsilon|$ small enough.

As already noted in the first part [7] of our study, a key role for the existence and nonexistence of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops is played by a mapping $M : \mathbb{T}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ (see (3.22) for its definition) which can be viewed as the Poincaré–Melnikov vector associated to the problem (see [12]). Here $\mathbb{T}^2$ is the two-dimensional torus.

We point out that $\mathbb{T}^2 \times \mathbb{R}^3$ parametrizes the manifold $\mathscr{Z}$ of loops corresponding to the "unperturbed" problem, i.e. the problem with $\varepsilon = 0$. Indeed, for $\varepsilon = 0$ one has $\kappa_0 \equiv 1$ and $\tau_0 \equiv 0$. Moreover closed curves with constant curvature 1 and constant torsion 0 are unit circles placed anywhere in $\mathbb{R}^3$ and one can write

$$\mathscr{Z} = \{R_\phi z(\mathbb{R}) + p \mid (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3\}$$

where

$$R_\phi := \begin{pmatrix} \cos\phi_2 & -\sin\phi_2\cos\phi_1 & \sin\phi_1\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2\cos\phi_1 & -\sin\phi_1\cos\phi_2 \\ 0 & \sin\phi_1 & \cos\phi_1 \end{pmatrix} \in SO(3) \tag{1.3}$$

for every $\phi = (\phi_1, \phi_2) \in \mathbb{T}^2$ and

$$z(t) := \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ 0 \end{pmatrix} \quad \text{for every } t \in \mathbb{R}. \tag{1.4}$$

In [7] we proved that the fact that the mapping M vanishes somewhere is a necessary condition for the existence of a bounded sequence of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops as $\varepsilon \to 0$.

In this paper we show that the presence of "stable" zeroes for M is a sufficient condition for the existence of a branch of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops with small $|\varepsilon|$.

Before stating our main results, let us introduce the following definitions: let $\{\Gamma_\delta\}_{\delta\in I}$ be a family of closed curves depending on a parameter δ varying in an open interval I of $\mathbb{R}$; for $\delta_0 \in I$ fixed, we say that $\Gamma_\delta \to \Gamma_{\delta_0}$ in C^k as $\delta \to \delta_0$ if for every $\delta \in I$ the curve Γ_δ admits a 1-periodic, uniform parametrization $u_\delta \in C^3 \cap C^k(\mathbb{R}, \mathbb{R}^3)$ and $\|u_\delta - u_{\delta_0}\|_{C^k} \to 0$ as $\delta \to \delta_0$. Moreover we say that the mapping $\delta \mapsto \Gamma_\delta$ is of class $C^1(I, C^k)$ if the mapping $\delta \mapsto u_\delta$ belongs to $C^1(I, C^k([0, 1], \mathbb{R}^3))$.

In a first result, we consider the case of "topologically stable" zeroes for M, where the notion of "stable" zero is expressed by means of the topological degree, as follows.

THEOREM 1.1. *Let $K, T \in C^1(\mathbb{R} \times \mathbb{R}^3)$ satisfy* (1.2)*, let κ_ε and τ_ε be as in* (1.1)*, and M as in* (3.22)*. If there is a nonempty bounded open set $\mathscr{O}$ in $\mathbb{R}^5$ such that* $\deg(M, \mathscr{O}, 0) \neq 0$ *then for $|\varepsilon|$ small enough there exists a simple $(\kappa_\varepsilon, \tau_\varepsilon)$-loop Γ_ε. Moreover every sequence $\varepsilon_n \to 0$ admits a subsequence, still denoted (ε_n), such that $\Gamma_{\varepsilon_n} \to \Gamma_0$ in C^2 as $n \to +\infty$, where $\Gamma_0 = R_\phi z(\mathbb{R}) + p$ for some $(\phi, p) \in \mathscr{O}$ such that $M(\phi, p) = 0$.*

In the presence of nondegenerate zeroes of M we gain regularity on the branch $\varepsilon \mapsto \Gamma_\varepsilon$ of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops, and the following result holds.

THEOREM 1.2. *Let $K, T \in C^2(\mathbb{R} \times \mathbb{R}^3)$ satisfy* (1.2)*, let κ_ε and τ_ε be as in* (1.1)*, and M as in* (3.22)*. If $(\phi, p) \in \mathbb{R}^2 \times \mathbb{R}^3$ is a nondegenerate zero of M (i.e. $M(\phi, p) = 0$ and $DM(\phi, p)$ is invertible), then there is $\bar\varepsilon > 0$ and, for $|\varepsilon| < \bar\varepsilon$, a simple $(\kappa_\varepsilon, \tau_\varepsilon)$-loop Γ_ε of class C^4. Moreover the mapping $\varepsilon \mapsto \Gamma_\varepsilon$ is of class $C^1((-\bar\varepsilon, \bar\varepsilon), C^2)$ and $\Gamma_0 = R_\phi z(\mathbb{R}) + p$.*

Hence, if the function M admits only nondegenerate zeroes, as soon as one switches on the perturbation (K, T), for small $|\varepsilon|$, C^1 branches of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops come out from the manifold $\mathscr{Z}$, emanating exactly from those circles in $\mathscr{Z}$ corresponding to zeroes of M.

Let us sketch the argument used to prove Theorems 1.1 and 1.2. As a first step we introduce the analytical statement of the problem, by identifying curves with corresponding uniform parametrizations. In this way looking for closed curves with prescribed curvature and torsion turns out to be equivalent to the study of the existence of nonconstant periodic solutions of the Frenet system (see Section 2).

Then we introduce a functional setting in such a way that periodic solutions of the Frenet system can be found as zeroes of a pair of suitable nonlinear operators (F_ε, J). More precisely, the operator F_ε acts between spaces of periodic functions and it is naturally defined through the Frenet equations; the operator J takes account of the orthonormality conditions for the Frenet trihedron.

The operator F_ε can be written as the sum of the "unperturbed" operator F_0, which corresponds to the problem with constant curvature 1 and constant torsion 0, and the perturbation operator $G(\varepsilon, \cdot)$, which exhibits a dependence on $K(\varepsilon, \cdot)$ and $T(\varepsilon, \cdot)$.

We point out that F_0 vanishes on a manifold Z formed by injective, uniform parametrizations of the circles in $\mathscr{L}$. Moreover, as proved in [7], the differential of F_0 at any point of Z is a Fredholm operator of index zero. With this information we tackle the perturbed problem making a finite-dimensional reduction according to the Lyapunov-Schmidt method (see [15], [16], [19]), based on the Implicit Function Theorem. In this way we construct a function $f_\varepsilon : \mathbb{T}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$ whose zeroes correspond, according to a suitable procedure, to zeroes of the pair (F_ε, J) and thus to parametrizations of $(\kappa_\varepsilon, \tau_\varepsilon)$-loops. This is developed in Section 3.

The final step consists in showing that the mapping f_ε admits zeroes. This is obtained by studying the asymptotic behaviour of f_ε at $\varepsilon = 0$. The Poincaré–Melnikov function M defined in (3.22) is essentially the first order term in the Taylor expansion of f_ε with respect to the smallness parameter ε. Thus the existence of zeroes for f_ε is related to the presence of stable zeroes of M. This part is discussed in Section 4 where we also provide a couple of examples to which our general results apply.

We conclude by observing that the techniques used for this problem, as well as the kind of results stated in the above theorems are common to a wide class of perturbative problems in different contexts, like Hamiltonian systems (see, e.g., [1], [14], [6]), nonlinear Schrödinger type equations (see [5] and the references therein), Yamabe's problem ([3], [18]), H-bubbles [9], and other problems in conformal geometry (see, e.g., [2] and [11], for the scalar curvature problem for the standard sphere). See also the recent monograph [4].

However, as a technically relevant difference with respect to the above listed problems, here we deal with a nonvariational problem, even in the unperturbed case. Moreover, as a consequence of the nonvariational character of our problem, in general, the Poincaré–Melnikov function M cannot be expressed as a potential. In fact this holds true for the problem of closed curves in the plane, with prescribed curvature; for this case we refer to [8].

The study presented here and in [7] constitutes a part of the PhD thesis of the second author ([13]).

2. NOTATION AND PRELIMINARIES

In this section we introduce some notation and we recall some preliminary results already discussed and proved in [7].

Let κ_ε and τ_ε be as in (1.1). Looking for $(\kappa_\varepsilon, \tau_\varepsilon)$-loops is equivalent to finding periodic solutions of a system of nonlinear ode's (see [10]). More precisely, we can state the problem as follows.

Let C^k_{per} denote the space of C^k functions from $\mathbb{R}$ into $\mathbb{R}^3$ which are periodic with period 1, let

$$X := C^2_{\mathrm{per}} \times C^1_{\mathrm{per}}, \qquad Y := C^0_{\mathrm{per}} \times C^0_{\mathrm{per}}$$

be the Banach spaces endowed with their own standard norms, and let

$$\Omega := \{(u_1, u_2) \in C^2_{\mathrm{per}} \times C^1_{\mathrm{per}} \mid u_1 \text{ nonconstant}, \ u_2 \neq 0\}.$$

We look for pairs $(u_1, u_2) \in \Omega$ solving the system

$$(P)_\varepsilon \qquad \begin{cases} u_1'' = \dfrac{N(u_1')}{N(u_2)}(1 + K(\varepsilon, u_1))u_2 \wedge u_1', \\ u_2' = T(\varepsilon, u_1)u_2 \wedge u_1', \end{cases}$$

where $K, T \in C^1(\mathbb{R} \times \mathbb{R}^3)$ satisfy (1.2) and

$$N(u) := \sqrt{\int_0^1 |u|^2} \quad \text{for every } u \in C^0_{\mathrm{per}}.$$

It turns out that a solution $(u_1, u_2) \in \Omega$ of $(P)_\varepsilon$ which also satisfies the condition $u_1' \cdot u_2 \equiv 0$ determines a $(\kappa_\varepsilon, \tau_\varepsilon)$-loop, and vice versa. In fact, thanks to $(P)_\varepsilon$, the orthogonality condition $u_1' \cdot u_2 \equiv 0$ is equivalent to

$$u_1'(0) \cdot u_2(0) = 0. \tag{2.1}$$

We remark that $(P)_\varepsilon$ and (2.1) are homogeneous with respect to u_2 and are invariant with respect to translation in t. Notice also that if K and T are of class C^k and $(u_1, u_2) \in \Omega$ solves $(P)_\varepsilon$, then $u_1 \in C^{k+2}_{\mathrm{per}}$ and $u_2 \in C^{k+1}_{\mathrm{per}}$.

Now let us introduce the following operators:

- $F_0 : \Omega \subset X \to Y$, defined as follows:

$$F_0(u_1, u_2) := \left(-u_1'' + \frac{N(u_1')}{N(u_2)} u_2 \wedge u_1', -u_2'\right) \quad \text{for every } (u_1, u_2) \in \Omega,$$

- $G : \mathbb{R} \times \Omega \to Y$, given by

$$G(\varepsilon; u_1, u_2) = \left(\frac{N(u_1')}{N(u_2)} K(\varepsilon, u_1)u_2 \wedge u_1', T(\varepsilon, u_1)u_2 \wedge u_1'\right)$$

  for $\varepsilon \in \mathbb{R}$ and $(u_1, u_2) \in \Omega$,
- $F_\varepsilon : \Omega \subset X \to Y$, defined by

$$F_\varepsilon(u_1, u_2) := F_0(u_1, u_2) + G(\varepsilon; u_1, u_2) \quad \text{for every } (u_1, u_2) \in \Omega.$$

We point out that $(u_1, u_2) \in \Omega$ solves $(P)_\varepsilon$ if and only if

$$F_\varepsilon(u_1, u_2) = 0. \tag{2.2}$$

Let us also introduce the following notation:

$$\theta := (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3 \quad \text{and} \quad \omega_\theta := (R_\phi z + p, R_\phi e_3).$$

Observe that $F_0(\omega_\theta) = 0$ for every $\theta \in \mathbb{T}^2 \times \mathbb{R}^3$ and the mapping $\theta \mapsto \omega_\theta$ is of class C^∞ from $\mathbb{T}^2 \times \mathbb{R}^3$ into X.

In [7] we proved that $F_0'(\omega_\theta)$ is a Fredholm operator of index zero from X into Y and the following decompositions hold:

$$X = \ker F_0'(\omega_\theta) \oplus (\ker F_0'(\omega_\theta))^\perp, \tag{2.3}$$

$$Y = \operatorname{im} F_0'(\omega_\theta) \oplus (\operatorname{im} F_0'(\omega_\theta))^\perp, \tag{2.4}$$

where all the subspaces are closed and $\dim \ker F_0'(\omega_\theta) = \dim (\operatorname{im} F_0'(\omega_\theta))^\perp = 7$. The orthogonality in (2.3) and (2.4) is meant with respect to the inner product

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle = \int_0^1 (u_1 \cdot v_1 + u_2 \cdot v_2).$$

As orthonormal bases for $\ker F_0'(\omega_\theta)$ and $(\operatorname{im} F_0'(\omega_\theta))^\perp$ one can take $\{\zeta_1(\theta), \dots, \zeta_7(\theta)\} \subset X$ and $\{\xi_1(\theta), \dots, \xi_7(\theta)\} \subset Y$, respectively, defined as follows:

$$\zeta_i(\theta) = R_\phi \bar{\zeta}_i \quad \text{and} \quad \xi_i(\theta) = R_\phi \bar{\xi}_i \quad \text{for } i = 1, \dots, 7 \text{ and } \theta = (\phi, p),$$

where

$$\begin{aligned} &\bar{\zeta}_1 = (0, e_3), && \bar{\xi}_1 = (e_3 \wedge z, 0), \\ &\bar{\zeta}_{1+i} = (e_i, 0), && \bar{\xi}_{1+i} = (0, e_i) \quad \text{for } i = 1, 2, 3, \\ &\bar{\zeta}_{4+i} = \sqrt{\frac{2}{3}}(e_i \wedge z, e_i \wedge e_3), && \bar{\xi}_{4+i} = \frac{1}{\sqrt{1 + 2\pi^2}}(e_i, 2\pi e_i \wedge z) \quad \text{for } i = 1, 2, \\ &\bar{\zeta}_7 = (e_3 \wedge z, 0), && \bar{\xi}_7 = \frac{1}{\sqrt{1 + 4\pi^2}}(e_3, 2\pi e_3 \wedge z). \end{aligned}$$

Here and in the following for every pair $(p_1, p_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ and for every $R \in SO(3)$ we set $R(p_1, p_2) = (Rp_1, Rp_2)$.

3. LYAPUNOV–SCHMIDT REDUCTION FOR THE PERTURBED PROBLEM

Our goal is to find a regular mapping $(\varepsilon, \theta) \mapsto \eta_\varepsilon(\theta) \in X$, defined for $|\varepsilon|$ small and θ in a compact set, such that $F_\varepsilon(\omega_\theta + \eta_\varepsilon(\theta)) \in (\operatorname{im} F_0'(\omega_\theta))^\perp$. As a consequence, the problem of searching for solutions of $(P)_\varepsilon$ is essentially reduced to the finite-dimensional problem of looking for zeroes of the mapping $(\varepsilon, \theta) \mapsto F_\varepsilon(\omega_\theta + \eta_\varepsilon(\theta))$, i.e. to the study of a system of seven equations in five unknowns, having fixed a basis in $(\operatorname{im} F_0'(\omega_\theta))^\perp$. This goal will be achieved through a reduction procedure in the spirit of the Lyapunov–Schmidt method (see [15], [16], [19]).

The following lemma, which gives the finite-dimensional reduction of the problem, constitutes the main result of this section and the first key step for the proof of Theorems 1.1 and 1.2.

LEMMA 3.1. *Let $K, T \in C^1(\mathbb{R} \times \mathbb{R}^3)$ satisfy (1.2). For every $r > 0$ there exist a value $\varepsilon_r > 0$ and unique mappings $\varepsilon \mapsto \eta^\varepsilon \in C^0(\mathbb{T}^2 \times \overline{B}_r, X)$ and $\varepsilon \mapsto \mu^\varepsilon \in C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^7)$ of class C^1 from $(-\varepsilon_r, \varepsilon_r)$ into their target spaces, such that for every $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$ and for every $\theta \in \mathbb{T}^2 \times \overline{B}_r$ one has:*

$$\|\eta^\varepsilon(\theta)\|_X < 2\pi, \tag{3.1}$$

$$\eta^0(\theta) = 0 \quad \textit{and} \quad \mu^0(\theta) = 0, \tag{3.2}$$

$$F_\varepsilon(\omega_\theta + \eta^\varepsilon(\theta)) = \sum_{i=1}^{7} \mu_i^\varepsilon(\theta)\xi_i(\theta), \tag{3.3}$$

$$\langle \eta^\varepsilon(\theta), \zeta_i(\theta)\rangle = 0 \quad \textit{for } i = 1, \ldots, 7. \tag{3.4}$$

Moreover for every $\theta \in \mathbb{T}^2 \times \overline{B}_r$,

$$\left.\frac{d\eta^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}(\theta) = F_0'(\omega_\theta)^{-1}\Big(\sum_{i=1}^{7}\langle \partial_\varepsilon G(0;\omega_\theta), \xi_i(\theta)\rangle \xi_i(\theta) - \partial_\varepsilon G(0;\omega_\theta)\Big), \tag{3.5}$$

$$\left.\frac{d\mu_i^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}(\theta) = \langle \partial_\varepsilon G(0;\omega_\theta), \xi_i(\theta)\rangle \quad \textit{for } i = 1, \ldots, 7. \tag{3.6}$$

In addition, if $\mu^\varepsilon(\theta) = 0$ for some $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$ and $\theta \in \mathbb{T}^2 \times \overline{B}_r$ then $(u_1^\varepsilon, u_2^\varepsilon) := \omega_\theta + \eta^\varepsilon(\theta)$ belongs to Ω and it solves problem $(P)_\varepsilon$. If, furthermore, we assume $K, T \in C^2(\mathbb{R} \times \mathbb{R}^3)$, then the mappings $\varepsilon \mapsto \eta^\varepsilon$ and $\varepsilon \mapsto \mu^\varepsilon$ belong to $C^2((-\varepsilon_r, \varepsilon_r),$ $C^0(\mathbb{T}^2 \times \overline{B}_r, X)) \cap C^1((-\varepsilon_r, \varepsilon_r), C^1(\mathbb{T}^2 \times \overline{B}_r, X))$ and to $C^2((-\varepsilon_r, \varepsilon_r), C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^7))$ $\cap\, C^1((-\varepsilon_r, \varepsilon_r), C^1(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^7))$, respectively.

REMARK 3.2. (i) For every $\theta \in \mathbb{T}^2 \times \mathbb{R}^3$ the operator $F_0'(\omega_\theta)$ is bijective from $(\ker F_0'(\omega_\theta))^\perp$ onto $\operatorname{im} F_0'(\omega_\theta)$ and $F_0'(\omega_\theta)^{-1} : \operatorname{im} F_0'(\omega_\theta) \to X$ denotes the inverse operator. Hence the right hand side in (3.5) is well defined because, according to (2.4), $\sum_{i=1}^7 \langle (y_1, y_2), \xi_i(\theta)\rangle \xi_i(\theta) - (y_1, y_2)$ belongs to $\operatorname{im} F_0'(\omega_\theta)$ for any $(y_1, y_2) \in Y$.
(ii) For every $\theta = (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3$ and for every $\zeta \in X$ one has $F_0'(\omega_\theta)[R_\phi \zeta] = R_\phi(F_0'(\omega)[\zeta])$, where $\omega = (z, e_3)$ (see formula (4.9) in [7]). Hence, by the above remark, one also has

$$F_0'(\omega_\theta)^{-1} = R_\phi F_0'(\omega)^{-1} R_\phi^{-1}. \tag{3.7}$$

(iii) The mapping μ^ε can be expressed in terms of η^ε since by (3.3) one has $\mu_i^\varepsilon(\theta) = \langle F_\varepsilon(\omega_\theta + \eta^\varepsilon(\theta)), \xi_i(\theta)\rangle$ for all $i = 1, \ldots, 7$ and $\theta \in \mathbb{T}^2 \times \overline{B}_r$.

PROOF. For fixed $r > 0$, set $U_r := \mathbb{T}^2 \times B_r$ and introduce the Banach spaces

$$\mathscr{X}_r^0 := C^0(\overline{U}_r, X), \quad \mathscr{Y}_r^0 := C^0(\overline{U}_r, Y), \quad \mathscr{R}_r^0 := C^0(\overline{U}_r, \mathbb{R}^7)$$

endowed with their standard norms. Clearly the maps $\theta \mapsto \omega_\theta$ and $\zeta_1, \ldots, \zeta_7$ belong to $\mathscr{X}_r^0$ whereas $\xi_1, \ldots, \xi_7 \in \mathscr{Y}_r^0$. Moreover, introduce the open subset of $\mathscr{X}_r^0$ given by

$$\mathscr{B} := \{\eta \in \mathscr{X}_r^0 \mid \|\eta\|_{\mathscr{X}_r^0} < 2\pi\}$$

and the function $\mathscr{F} : \mathbb{R} \times \mathscr{B} \times \mathscr{R}_r^0 \to \mathscr{Y}_r^0 \times \mathscr{R}_r^0$ defined as follows:

$$\mathscr{F}(\varepsilon, \eta, \mu)(\theta) := \Big(F_\varepsilon(\omega_\theta + \eta(\theta)) - \sum_{i=1}^{7} \mu_i(\theta)\xi_i(\theta), \langle \eta(\theta), \zeta_1(\theta)\rangle, \dots, \langle \eta(\theta), \zeta_7(\theta)\rangle\Big)$$

for all $\theta \in \overline{U}_r$ and for every $(\varepsilon, \eta, \mu) \in \mathbb{R} \times \mathscr{B} \times \mathscr{R}_r^0$. Observe that $\omega_\theta + \eta(\theta) \in \Omega$ for every $\theta \in \overline{U}_r$ since $\eta \in \mathscr{B}$. Hence $\mathscr{F}$ is well defined on $\mathbb{R} \times \mathscr{B} \times \mathscr{R}_r^0$. Our goal is to apply the Implicit Function Theorem to $\mathscr{F}$ at the point $(0, 0, 0)$ in order to find mappings $\varepsilon \mapsto \eta^\varepsilon$ and $\varepsilon \mapsto \mu^\varepsilon$ such that $\mathscr{F}(\varepsilon, \eta^\varepsilon, \mu^\varepsilon) = (0, 0)$, that is, (3.3) and (3.4) hold.

Regularity of $\mathscr{F}$. One has $\mathscr{F}(\varepsilon, \eta, \mu) = \mathscr{F}(0, \eta, \mu) + \mathscr{G}(\varepsilon, \eta)$ where

$$\mathscr{G}(\varepsilon, \eta)(\theta) := (G(\varepsilon; \omega_\theta + \eta(\theta)), 0, \dots, 0). \tag{3.8}$$

One can easily check that the mapping $(\eta, \mu) \mapsto \mathscr{F}(0, \eta, \mu)$ is of class C^∞ because F_0 is. As concerns the regularity of $\mathscr{G}$, by the definition of G, it follows from the regularity of the Nemytskiĭ operators associated to K and T. More precisely, as K, T are of class C^1, the mappings $\mathscr{K}, \mathscr{T} : \mathbb{R} \times C^0(\overline{U}_r, C^2_{\mathrm{per}}) \to C^0(\overline{U}_r, C^0_{\mathrm{per}})$ defined by

$$\mathscr{K}(\varepsilon, \chi)(\theta) := K(\varepsilon, \chi(\theta)) \quad \text{and} \quad \mathscr{T}(\varepsilon, \chi)(\theta) := T(\varepsilon, \chi(\theta)) \tag{3.9}$$

are of class C^1 (see [13] for the details). As a consequence, also $\mathscr{G}$ turns out to be of class C^1 from $\mathbb{R} \times \mathscr{B} \subset \mathbb{R} \times \mathscr{X}_r^0$ into $\mathscr{Y}_r^0$.

Now let us study the linearized problem for $\mathscr{F}$ at $(0, 0, 0)$. Clearly $\mathscr{F}(0, 0, 0) = (F_0(\omega_\theta), 0) = (0, 0)$ because $G(0; \cdot) = 0$. Moreover, considering the bounded linear operator

$$\mathscr{L} := \frac{\partial \mathscr{F}(0, 0, 0)}{\partial(\eta, \mu)} : \mathscr{X}_r^0 \times \mathscr{R}_r^0 \to \mathscr{Y}_r^0 \times \mathscr{R}_r^0,$$

for every $(\varphi, \nu) \in \mathscr{X}_r^0 \times \mathscr{R}_r^0$ we have

$$\mathscr{L}(\varphi, \nu)(\theta) = \Big(F_0'(\omega_\theta)[\varphi(\theta)] - \sum_{i=1}^{7} \nu_i(\theta)\xi_i(\theta), \langle \varphi(\theta), \zeta_1(\theta)\rangle, \dots, \langle \varphi(\theta), \zeta_7(\theta)\rangle\Big).$$

We will show that $\mathscr{L}$ is bijective from $\mathscr{X}_r^0 \times \mathscr{R}_r^0$ onto $\mathscr{Y}_r^0 \times \mathscr{R}_r^0$.

Injectivity. Let $(\varphi, \nu) \in \mathscr{X}_r^0 \times \mathscr{R}_r^0$ be such that $\mathscr{L}(\varphi, \nu) = 0$, that is,

$$\begin{cases} F_0'(\omega_\theta)[\varphi(\theta)] = \sum_{i=1}^{7} \nu_i(\theta)\xi_i(\theta), \\ \langle \varphi(\theta), \zeta_i(\theta)\rangle = 0, \quad i = 1, \dots, 7. \end{cases}$$

Since $\xi_i(\theta) \in (\operatorname{im} F_0'(\omega_\theta))^\perp$, the first equation implies $\nu_i(\theta) = 0$ for every $i = 1, \dots, 7$ and thus $\varphi(\theta) \in \ker F_0'(\omega_\theta)$. On the other hand, the second equation means that $\varphi(\theta) \in (\ker F_0'(\omega_\theta))^\perp$. Hence $\varphi(\theta) = 0$. As θ is arbitrary, we have injectivity.

Surjectivity. Given $(\psi,\rho)\in\mathscr{Y}_r^0\times\mathscr{R}_r^0$ we have to find $(\varphi,\nu)\in\mathscr{X}_r^0\times\mathscr{R}_r^0$ such that $\mathscr{L}(\varphi,\nu)=(\psi,\rho)$, that is, for every $\theta\in\overline{U}_r$,

$$F_0'(\omega_\theta)[\varphi(\theta)]-\sum_{i=1}^{7}\nu_i(\theta)\xi_i(\theta)=\psi(\theta), \tag{3.10}$$

$$\langle\varphi(\theta),\zeta_i(\theta)\rangle=\rho_i(\theta),\quad i=1,\ldots,7. \tag{3.11}$$

Fix $\theta\in\overline{U}_r$. According to the decomposition (2.3)–(2.4) and thanks to the orthonormality of the sets $\{\zeta_i(\theta)\}_{i=1}^7$ and $\{\xi_i(\theta)\}_{i=1}^7$ we can write

$$\varphi(\theta)=\sum_{i=1}^{7}\langle\varphi(\theta),\zeta_i(\theta)\rangle\zeta_i(\theta)+\bar\varphi(\theta),$$
$$\psi(\theta)=\sum_{i=1}^{7}\langle\psi(\theta),\xi_i(\theta)\rangle\xi_i(\theta)+\bar\psi(\theta),$$

with $\bar\varphi(\theta)\in(\ker F_0'(\omega_\theta))^\perp$ and $\bar\psi(\theta)\in\operatorname{im}F_0'(\omega_\theta)$. Then (3.11) gives

$$\varphi(\theta)=\sum_{i=1}^{7}\rho_i(\theta)\zeta_i(\theta)+\bar\varphi(\theta). \tag{3.12}$$

Moreover, by (3.10) and since $\xi_i(\theta)\in(\operatorname{im}F_0'(\omega_\theta))^\perp$ we see that

$$\nu_i(\theta)=-\langle\psi(\theta),\xi_i(\theta)\rangle,\quad i=1,\ldots,7. \tag{3.13}$$

In addition

$$F_0'(\omega_\theta)[\bar\varphi(\theta)]=\bar\psi(\theta)$$

because $\zeta_i(\theta)\in\ker F_0'(\omega_\theta)$. According to Remark 3.2(i), we can take

$$\bar\varphi(\theta)=F_0'(\omega_\theta)^{-1}[\bar\psi(\theta)]. \tag{3.14}$$

As θ varies, the expression (3.13) defines a continuous mapping from $\overline{U}_r$ into $\mathbb{R}^7$, that is, $\nu\in\mathscr{R}_r^0$. As concerns the regularity of the function $\theta\mapsto\varphi(\theta)$ defined by (3.12) and (3.14), we observe that by (3.7),

$$\bar\varphi(\theta)=R_\phi F_0'(z,e_3)^{-1}R_\phi^{-1}[\bar\psi(\theta)].$$

This shows that $\bar\varphi\in\mathscr{X}_r^0$ and then, by (3.12), also $\varphi\in\mathscr{X}_r^0$. Thus the surjectivity is proved.

Hence we can apply the Implicit Function Theorem, and since $\mathscr{F}(0,0,0)=(0,0)$, (3.2)–(3.4) follow.

As regards (3.5) and (3.6), we recall that

$$\left(\left.\frac{d\eta^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0},\left.\frac{d\mu^\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}\right)=-\mathscr{L}^{-1}\frac{\partial\mathscr{F}(0,0,0)}{\partial\varepsilon}. \tag{3.15}$$

Moreover, according to the proof of the surjectivity, we have

$$\begin{aligned}(3.16)\qquad &\mathscr{L}^{-1}(\psi,\rho)(\theta)\\ &=\Big(\sum_{i=1}^{7}\rho_i(\theta)\zeta_i(\theta)+F_0'(\omega_\theta)^{-1}\Big(\psi(\theta)-\sum_{i=1}^{7}\langle\psi(\theta),\xi_i(\theta)\rangle\xi_i(\theta)\Big),-\langle\psi(\theta),\xi_i(\theta)\rangle\Big)\end{aligned}$$

for every $(\psi,\rho)\in\mathscr{Y}_r^0\times\mathscr{R}_r^0$. In addition

$$\frac{\partial\mathscr{F}(0,0,0)}{\partial\varepsilon}(\theta)=(\partial_\varepsilon G(0;\omega_\theta),0). \tag{3.17}$$

In conclusion, (3.5) and (3.6) follow from (3.15)–(3.17).

The fact that if $\mu^\varepsilon(\theta)=0$ then $\omega_\theta+\eta^\varepsilon(\theta)$ provides a solution of $(P)_\varepsilon$ immediately follows from (3.3) and from the definition of F_ε.

Finally, let us discuss the last part of the lemma, concerning the regularity of the mappings $\varepsilon\mapsto\eta^\varepsilon$ and $\varepsilon\mapsto\mu^\varepsilon$ when $K,T\in C^2(\mathbb{R}\times\mathbb{R}^3)$. We just give a sketch, referring again to [13] for a more detailed proof. Since K,T are of class C^2, the operator $\mathscr{G}:\mathbb{R}\times\mathscr{B}\to\mathscr{Y}_r^0$ defined in (3.8) turns out to be of class C^2 and then one readily finds that the mappings $\varepsilon\mapsto\eta^\varepsilon$ and $\varepsilon\mapsto\mu^\varepsilon$ are of class C^2 from $(-\varepsilon_r,\varepsilon_r)$ into $C^0(\overline{U}_r,X)$ and $C^0(\overline{U}_r,\mathbb{R}^7)$, respectively. The further regularity is accomplished by repeating the same argument of the C^1 regularity, but making a different choice of spaces. More precisely, instead of $\mathscr{X}_r^0$, $\mathscr{Y}_r^0$ and $\mathscr{R}_r^0$ we take

$$\mathscr{X}_r^1:=C^1(\overline{U}_r,X),\quad \mathscr{Y}_r^1:=C^1(\overline{U}_r,Y),\quad \mathscr{R}_r^1:=C^1(\overline{U}_r,\mathbb{R}^7),$$

endowed with their standard norms. Clearly, in this case, the set $\mathscr{B}$ is given by $\{\eta\in\mathscr{X}_r^1\mid\|\eta\|_{\mathscr{X}_r^1}<2\pi\}$. The proof goes exactly as before without substantial differences. The only remark concerns the proof of the regularity of the mapping $\mathscr{F}$ and, in particular, the regularity of $\mathscr{G}$. Although the Nemytskiĭ operators $\mathscr{K}$ and $\mathscr{T}$ defined by (3.9) are of class C^2 from $\mathbb{R}\times C^0(\overline{U}_r,C^2_{\rm per})$ into $C^0(\overline{U}_r,C^0_{\rm per})$, the operator $\mathscr{G}$ defined in (3.8) is just C^1 (as needed in order to apply the Implicit Function Theorem) because the existence of the differential $d\mathscr{G}(\varepsilon,\eta)$ as a bounded linear operator from $\mathbb{R}\times\mathscr{X}_r^1$ into $\mathscr{Y}_r^1$, as well as its continuous dependence on (ε,η), involves the second order partial derivatives of K and T. Hence we can find C^1 functions $\varepsilon\mapsto\eta^\varepsilon\in C^1(\overline{U}_r,X)$ and $\varepsilon\mapsto\mu^\varepsilon\in C^1(\overline{U}_r,\mathbb{R}^7)$ satisfying $\mathscr{F}(\varepsilon,\eta^\varepsilon,\mu^\varepsilon)=0$ and the conclusion follows. □

In fact we are interested in solutions $(u_1,u_2)\in\Omega$ of (2.2) satisfying the additional condition $u_1'\cdot u_2\equiv 0$ or equivalently (2.1) which guarantees that (u_1,u_2) determines a $(\kappa_\varepsilon,\tau_\varepsilon)$-loop (see Section 2).

For this purpose let us introduce the functional $J:X\to\mathbb{R}$ defined as follows:

$$J(u_1,u_2):=u_1'(0)\cdot u_2(0)\quad\text{for every }(u_1,u_2)\in X. \tag{3.18}$$

Hence, a pair $(u_1,u_2)\in\Omega$ determines a $(\kappa_\varepsilon,\tau_\varepsilon)$-loop if and only if

$$\begin{cases}F_\varepsilon(u_1,u_2)=0,\\ J(u_1,u_2)=0.\end{cases} \tag{3.19}$$

Observe that $J \in C^\infty(X)$ and in particular

$$J'(u_1, u_2)[x_1, x_2] = u_1'(0) \cdot x_2(0) + u_2(0) \cdot x_1'(0) \tag{3.20}$$

for every $(u_1, u_2), (x_1, x_2) \in X$.

Now, fixing $r > 0$ let $\varepsilon_r > 0$, η^ε and μ^ε be given according to Lemma 3.1. For every $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$ let us introduce the mapping $\bar{f}_\varepsilon : \mathbb{T}^2 \times \overline{B}_r \to \mathbb{R}^7 \times \mathbb{R}$ given by

$$\bar{f}_\varepsilon(\theta) := (\mu^\varepsilon(\theta), J(\omega_\theta + \eta^\varepsilon(\theta))). \tag{3.21}$$

Our goal is to look for zeroes of $\bar{f}_\varepsilon$ since, by the last part of Lemma 3.1 and by the above discussion, if $\bar{f}_\varepsilon(\theta) = 0$ (for some ε and θ) then $\omega_\theta + \eta^\varepsilon(\theta)$ is a nonconstant periodic solution of $(P)_\varepsilon$ and satisfies (2.1), so it corresponds to a $(\kappa_\varepsilon, \tau_\varepsilon)$-loop.

Notice that, according to Lemma 3.1, $\bar{f}_\varepsilon \in C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8)$ for every $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$, $\bar{f}_0 \equiv 0$ and the mapping $\varepsilon \mapsto \bar{f}_\varepsilon$ is of class C^1 from $(-\varepsilon_r, \varepsilon_r)$ into $C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8)$. If $K, T \in C^2(\mathbb{R} \times \mathbb{R}^3)$ (and not just of class C^1) then the mapping $\varepsilon \mapsto \bar{f}_\varepsilon$ belongs to both $C^2((-\varepsilon_r, \varepsilon_r), C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8))$ and $C^1((-\varepsilon_r, \varepsilon_r), C^1(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8))$.

Now let us consider the mapping $M : \mathbb{T}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$, already introduced in [7] and defined as follows:

$$M(\phi, p) := \begin{pmatrix} \int_0^1 \partial_\varepsilon K(0, R_\phi z(t) + p) \cos(2\pi t)\, dt \\ \int_0^1 \partial_\varepsilon K(0, R_\phi z(t) + p) \sin(2\pi t)\, dt \\ \int_0^1 \partial_\varepsilon T(0, R_\phi z(t) + p) \cos(2\pi t)\, dt \\ \int_0^1 \partial_\varepsilon T(0, R_\phi z(t) + p) \sin(2\pi t)\, dt \\ \int_0^1 \partial_\varepsilon T(0, R_\phi z(t) + p)\, dt \end{pmatrix} \quad \text{for } (\phi, p) \in \mathbb{T}^2 \times \mathbb{R}^3. \tag{3.22}$$

By natural periodic extension, we shall also consider $M : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^5$.

The next result makes clear the relationship between the function M and the first order term in the expansion of $\bar{f}_\varepsilon$ with respect to ε.

LEMMA 3.3. *There exist $\Phi \in GL(8, \mathbb{R})$ and $\Psi \in GL(5, \mathbb{R})$ such that, setting $\Phi \bar{f}_\varepsilon =: (f_\varepsilon, \tilde{f}_\varepsilon) : \mathbb{T}^2 \times \overline{B}_r \to \mathbb{R}^5 \times \mathbb{R}^3$, as $\varepsilon \to 0$ one has:*

$$\begin{aligned} f_\varepsilon &= \varepsilon \Psi M + o(\varepsilon) && \text{in } C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^5), \\ \tilde{f}_\varepsilon &= o(\varepsilon) && \text{in } C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^3), \end{aligned}$$

where M is defined in (3.22)*. The convergences hold in $C^1(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^5)$ and in $C^1(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^3)$ respectively, if $K, T \in C^2(\mathbb{R} \times \mathbb{R}^3)$. Moreover if $f_\varepsilon(\theta) = 0$ for some $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$ and $\theta \in \mathbb{T}^2 \times \overline{B}_r$, then $\bar{f}_\varepsilon(\theta) = 0$.*

PROOF. By (3.2), (3.5) and (3.6), the first order Taylor expansion of the map $\varepsilon \mapsto \bar{f}_\varepsilon \in C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8)$ at $\varepsilon = 0$ is

$$\bar{f}_\varepsilon(\theta) = \varepsilon \Big(\langle \partial_\varepsilon G(0; \omega_\theta), \xi_i(\theta) \rangle, \\ J'(\omega_\theta) F_0'(\omega_\theta)^{-1} \Big(\sum_{i=1}^{7} \langle \partial_\varepsilon G(0; \omega_\theta), \xi_i(\theta) \rangle \xi_i(\theta) - \partial_\varepsilon G(0; \omega_\theta) \Big) \Big) + o(\varepsilon)$$

where $o(\varepsilon)/\varepsilon \to 0$ in $C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8)$. According to the last part of Lemma 3.1, if $K, T \in C^2(\mathbb{R} \times \mathbb{R}^3)$ one can take $C^1(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8)$ instead of $C^0(\mathbb{T}^2 \times \overline{B}_r, \mathbb{R}^8)$. Since

$$\partial_\varepsilon G(0; \omega_\theta) = -2\pi(2\pi\partial_\varepsilon K(0, R_\phi z + p)R_\phi z, \partial_\varepsilon T(0, R_\phi z + p)R_\phi z)$$

for any $\theta = (\phi, p) \in \mathbb{T}^2 \times \overline{B}_r$, we readily get

$$\langle \partial_\varepsilon G(0; \omega_\theta), \xi_i(\theta)\rangle = 0 \quad \text{for } i = 1, 4, 7, \tag{3.23}$$

$$\begin{aligned}\langle \partial_\varepsilon G(0; \omega_\theta), \xi_2(\theta)\rangle &= -2\pi \int_0^1 \partial_\varepsilon T(0, R_\phi z + p)z \cdot e_1 \\ &= -2\pi M_3(\theta),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\langle \partial_\varepsilon G(0; \omega_\theta), \xi_3(\theta)\rangle &= -2\pi \int_0^1 \partial_\varepsilon T(0, R_\phi z + p)z \cdot e_2 \\ &= -2\pi M_4(\theta),\end{aligned} \tag{3.25}$$

$$\begin{aligned}\langle \partial_\varepsilon G(0; \omega_\theta), \xi_5(\theta)\rangle &= -\frac{4\pi^2}{\sqrt{1+2\pi^2}} \int_0^1 \partial_\varepsilon K(0, R_\phi z + p)z \cdot e_1 \\ &= -\frac{4\pi^2}{\sqrt{1+2\pi^2}} M_1(\theta),\end{aligned} \tag{3.26}$$

$$\begin{aligned}\langle \partial_\varepsilon G(0; \omega_\theta), \xi_6(\theta)\rangle &= -\frac{4\pi^2}{\sqrt{1+2\pi^2}} \int_0^1 \partial_\varepsilon K(0, R_\phi z + p)z \cdot e_2 \\ &= -\frac{4\pi^2}{\sqrt{1+2\pi^2}} M_2(\theta).\end{aligned} \tag{3.27}$$

Set

$$\begin{aligned}(x_1(\theta), x_2(\theta)) &:= F_0'(\omega_\theta)^{-1}\Big(\sum_{i=1}^{7} \langle \partial_\varepsilon G(0; \omega_\theta), \xi_i(\theta)\rangle \xi_i(\theta) - \partial_\varepsilon G(0; \omega_\theta)\Big), \\ (y_1(\theta), y_2(\theta)) &:= F_0'(\omega_\theta)[x_1(\theta), x_2(\theta)],\end{aligned}$$

and note that, by (3.4),

$$(x_1(\theta), x_2(\theta)) \in (\ker F_0'(\omega_\theta))^\perp.$$

Moreover, using (3.23)–(3.27), one can compute

$$\begin{aligned}y_1(\theta) &= -\frac{4\pi^2}{1+2\pi^2} M_1(\theta)R_\phi e_1 - \frac{4\pi^2}{1+2\pi^2} M_2(\theta)R_\phi e_2 \\ &\qquad + 4\pi^2 \partial_\varepsilon K(0, R_\phi z + p)R_\phi z, \\ y_2(\theta) &= -2\pi M_3(\theta)R_\phi e_1 - 2\pi M_4(\theta)R_\phi e_2 - \frac{8\pi^3}{1+2\pi^2} M_1(\theta)R_\phi(e_1 \wedge z) \\ &\qquad - \frac{8\pi^3}{1+2\pi^2} M_2(\theta)R_\phi(e_2 \wedge z) + 2\pi\partial_\varepsilon T(0, R_\phi z + p)R_\phi z,\end{aligned}$$

so that, by (3.20) and the formula in Remark 4.4 of [7],

$$
\begin{aligned}
J'(\omega_\theta)[x_1(\theta), x_2(\theta)] &= (R_\phi z + p)'(0) \cdot x_2(\theta)(0) + R_\phi e_3 \cdot x_1(\theta)'(0) \\
&= 2\pi R_\phi e_2 \cdot x_2(\theta)(0) + R_\phi e_3 \cdot x_1(\theta)'(0) \\
&= \int_0^1 (1-t) e_3 \cdot R_\phi^{-1} y_1(\theta)(t)\, dt \\
&\quad + \int_0^1 ((1-t)z'(t) + z(t) - e_1) \cdot R_\phi^{-1} y_2(\theta)(t)\, dt \\
&= 4\pi M_3(\theta) + 2\pi \int_0^1 (1 - z(t) \cdot e_1) \partial_\varepsilon T(0, R_\phi z(t) + p)\, dt \\
&= 2\pi (M_5(\theta) + M_3(\theta)).
\end{aligned}
$$

Hence, setting $\Phi q := (q_5, q_6, q_2, q_3, q_8, q_1, q_4, q_7)$ for all $q = (q_1, \dots, q_8) \in \mathbb{R}^8$, we have

$$
\begin{aligned}
\Phi \bar{f}_\varepsilon(\theta) = -2\pi\varepsilon \Big(&\frac{2\pi}{\sqrt{1+2\pi^2}} M_1(\theta), \frac{2\pi}{\sqrt{1+2\pi^2}} M_2(\theta), M_3(\theta), M_4(\theta), \\
&-M_5(\theta) - M_3(\theta), 0, 0, 0 \Big) + o(\varepsilon)
\end{aligned}
$$

as $\varepsilon \to 0$ and we conclude by obvious definition of Ψ, f_ε and $\tilde{f}_\varepsilon$.

Now assume that $f_\varepsilon(\theta) = 0$ for some fixed $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$ and $\theta = (\phi, p) \in \mathbb{T}^2 \times \overline{B}_r$. In particular we have

$$\mu_i^\varepsilon(\theta) = 0 \quad \text{for } i = 2, 3, 5, 6.$$

Setting $(u_1, u_2) = \omega_\theta + \eta^\varepsilon(\theta)$, from (3.3) we deduce that

$$F_\varepsilon(u_1, u_2) = \frac{1}{2\pi} \mu_1^\varepsilon(\theta)(R_\phi z', 0) + \mu_4^\varepsilon(\theta)(0, R_\phi e_3) + \frac{1}{\sqrt{1+4\pi^2}} \mu_7^\varepsilon(\theta)(R_\phi e_3, R_\phi z'),$$

that is,

$$-u_1'' + \frac{N(u_1')}{N(u_2)}(1 + K(\varepsilon, u_1)) u_2 \wedge u_1' = \bar{\mu}_1 R_\phi z' + \bar{\mu}_7 R_\phi e_3, \tag{3.28}$$

$$-u_2' + T(\varepsilon, u_1) u_2 \wedge u_1' = \bar{\mu}_4 R_\phi e_3 + \bar{\mu}_7 R_\phi z', \tag{3.29}$$

where

$$
\begin{aligned}
\bar{\mu}_1 &:= \frac{1}{2\pi} \mu_1^\varepsilon(\theta), \\
\bar{\mu}_4 &:= \mu_4^\varepsilon(\theta), \\
\bar{\mu}_7 &:= \frac{1}{\sqrt{1+4\pi^2}} \mu_7^\varepsilon(\theta).
\end{aligned}
$$

Letting now $\eta^\varepsilon(\theta) =: (\eta_1, \eta_2)$, we multiply both (3.28) and (3.29) by $u_1' = R_\phi z' + \eta_1'$ and $u_2 = R_\phi e_3 + \eta_2$ to get

$$-u_1'' \cdot u_1' = 4\pi^2 \bar{\mu}_1 + \bar{\mu}_1 R_\phi z' \cdot \eta_1' + \bar{\mu}_7 R_\phi e_3 \cdot \eta_1', \tag{3.30}$$

$$-u_1'' \cdot u_2 = \bar{\mu}_1 R_\phi z' \cdot \eta_2 + \bar{\mu}_7 + \bar{\mu}_7 R_\phi e_3 \cdot \eta_2, \tag{3.31}$$

$$-u_2' \cdot u_1' = \bar{\mu}_4 R_\phi e_3 \cdot \eta_1' + 4\pi^2 \bar{\mu}_7 + \bar{\mu}_7 R_\phi z' \cdot \eta_1', \tag{3.32}$$

$$-u_2' \cdot u_2 = \bar{\mu}_4 + \bar{\mu}_4 R_\phi e_3 \cdot \eta_2 + \bar{\mu}_7 R_\phi z' \cdot \eta_2. \tag{3.33}$$

By the periodicity of η_1, u_1' and u_2 and since $\langle \eta_2, R_\phi e_3 \rangle = \langle \eta^\varepsilon(\theta), \xi_4(\theta) \rangle = 0$, by (3.4), upon integrating (3.30), (3.33) and the sum of (3.31) and (3.32), we respectively obtain

$$\bar{\mu}_1 \left(4\pi^2 + \int_0^1 R_\phi z' \cdot \eta_1' \right) = 0, \tag{3.34}$$

$$\bar{\mu}_4 + \bar{\mu}_7 \int_0^1 R_\phi z' \cdot \eta_2 = 0, \tag{3.35}$$

$$\bar{\mu}_1 \int_0^1 R_\phi z' \cdot \eta_2 + \bar{\mu}_7 \left(1 + 4\pi^2 + \int_0^1 R_\phi z' \cdot \eta_1' \right) = 0. \tag{3.36}$$

Since (3.1) yields

$$\left| \int_0^1 R_\phi z' \cdot \eta_1' \right| \le 2\pi \max_{t \in [0,1]} |\eta_1'(t)| \le 2\pi \|\eta^\varepsilon(\theta)\|_X < 4\pi^2,$$

(3.34) implies $\bar{\mu}_1 = 0$. Then, in turn, (3.36) implies $\bar{\mu}_7 = 0$ and finally (3.35) gives $\bar{\mu}_4 = 0$. Therefore $\mu_i^\varepsilon(\theta) = 0$ also for $i = 1, 4, 7$ and hence $\bar{f}_\varepsilon(\theta) = 0$. □

REMARK 3.4. Notice that the implication $f_\varepsilon(\theta) = 0 \Rightarrow \bar{f}_\varepsilon(\theta) = 0$ in the previous lemma has been proved without using the vanishing of $J(\omega_\theta + \eta^\varepsilon(\theta))$; only the fact that $\mu_i^\varepsilon(\theta) = 0$ for $i = 2, 3, 5, 6$ was needed.

4. PROOF OF THEOREMS 1.1 AND 1.2 AND EXAMPLES

For the proof of Theorem 1.1 the following lemma will be useful.

LEMMA 4.1. *If there exists a nonempty, bounded open set $\mathcal{O}$ in $\mathbb{R}^5$ such that $\deg(M, \mathcal{O}, 0) \neq 0$ then for $|\varepsilon|$ small enough there is $\theta_\varepsilon \in \mathcal{O}$ such that $\bar{f}_\varepsilon(\theta_\varepsilon) = 0$.*

PROOF. Let $\Psi \in GL(5, \mathbb{R})$ be given by Lemma 3.3. Since $\det \Psi \neq 0$, one has $0 \notin \Psi M(\partial\mathcal{O})$ and $|\deg(\Psi M, \mathcal{O}, 0)| = |\deg(M, \mathcal{O}, 0)|$. As $\mathcal{O}$ is bounded, there exists $r > 0$ such that $\overline{\mathcal{O}} \subset \mathbb{R}^2 \times \overline{B}_r$ and $d := \inf_{\theta \in \partial\mathcal{O}} |\Psi M(\theta)| > 0$. Let ε_r be given by Lemma 3.1. For $\varepsilon \in (-\varepsilon_r, \varepsilon_r)$ define the homotopy $H_\varepsilon : \overline{\mathcal{O}} \times [0,1] \to \mathbb{R}^5$ by setting

$$H_\varepsilon(\theta, s) := s f_\varepsilon(\theta) + (1-s)\varepsilon \Psi M(\theta) \quad \text{for } (\theta, s) \in \overline{\mathcal{O}} \times [0,1]$$

where f_ε is defined in Lemma 3.3. We claim that the homotopy H_ε is admissible for $|\varepsilon|$ small enough. Indeed, since $\partial\mathcal{O} \subset \mathbb{R}^2 \times \overline{B}_r$, using Lemma 3.3, we find that $f_\varepsilon(\theta) - \varepsilon \Psi M(\theta) = o(\varepsilon)$ as $\varepsilon \to 0$, uniformly in $\theta \in \partial\mathcal{O}$. Hence, for every $(\theta, s) \in \partial\mathcal{O} \times [0,1]$ and $\varepsilon \neq 0$,

$$\begin{aligned} \frac{|H_\varepsilon(\theta, s)|}{|\varepsilon|} &= \frac{1}{|\varepsilon|} |\varepsilon \Psi M(\theta) + s(f_\varepsilon(\theta) - \varepsilon \Psi M(\theta))| \ge \frac{1}{|\varepsilon|} \big| |\varepsilon| \, |\Psi M(\theta)| - s|o(\varepsilon)| \big| \\ &\ge d - \left| \frac{o(\varepsilon)}{\varepsilon} \right| > 0 \end{aligned}$$

provided that $|\varepsilon|$ is small enough. Thus the claim is proved. Finally, the homotopy invariance property of Brouwer's degree gives

$$\deg(f_\varepsilon, \mathscr{O}, 0) = \deg(\varepsilon \Psi M, \mathscr{O}, 0) = \deg(\Psi M, \mathscr{O}, 0) \neq 0$$

and the conclusion follows, by using again Lemma 3.3. $\square$

Proof of Theorem 1.1. As noted in Section 2, our first goal is to find solutions $(u_1, u_2) \in \Omega$ of problem (3.19) for small $|\varepsilon|$. By Lemma 3.1 and by the definition (3.21), $(u_1^\varepsilon, u_2^\varepsilon) = \omega_\theta + \eta^\varepsilon(\theta)$ solves (3.19) if $\bar{f}_\varepsilon(\theta) = 0$. Lemma 4.1 ensures that there exists a mapping $\varepsilon \mapsto \theta_\varepsilon \in \mathscr{O}$, defined for $|\varepsilon|$ small, such that $\bar{f}_\varepsilon(\theta_\varepsilon) = 0$. Hence, for $|\varepsilon|$ small, the function $u_\varepsilon := R_{\phi_\varepsilon} z + p_\varepsilon + \eta_1^\varepsilon(\theta_\varepsilon)$ is a parametrization of a $(\kappa_\varepsilon, \tau_\varepsilon)$-loop, where $(\phi_\varepsilon, p_\varepsilon) = \theta_\varepsilon$ and $\eta_1^\varepsilon(\theta) \in C^2_{\text{per}}$ is the first component of $\eta^\varepsilon(\theta)$. Moreover, since $\theta_\varepsilon \in \mathscr{O}$ and $\overline{\mathscr{O}}$ is compact, every sequence $\varepsilon_n \to 0$ has a subsequence, again denoted (ε_n), such that $\theta_{\varepsilon_n} \to \bar{\theta}$ for some $\bar{\theta} = (\bar{\phi}, \bar{p}) \in \overline{\mathscr{O}}$. By Lemma 3.1, $\|\eta^{\varepsilon_n}(\theta_{\varepsilon_n})\|_X \to 0$ and then $u_{\varepsilon_n} \to R_{\bar{\phi}} z + \bar{p}$ in C^2_{per}. In particular, since the parametrization $R_{\bar{\phi}} z + \bar{p}$ is injective on $\mathbb{R}/\mathbb{Z}$, for $|\varepsilon|$ small, Γ_ε is a simple curve. Finally, by Theorem 1.1 in [7], $M(\bar{\phi}, \bar{p}) = 0$. $\square$

Before the proof of Theorem 1.2, we state the following lemma.

LEMMA 4.2. *Assuming K and T are of class C^2, if M admits a nondegenerate zero at some $\bar{\theta} = (\bar{\phi}, \bar{p}) \in \mathbb{T}^2 \times \mathbb{R}^3$, then there exists a C^1 mapping $\varepsilon \mapsto \theta_\varepsilon \in \mathbb{T}^2 \times \mathbb{R}^3$, defined on some interval $(-\bar{\varepsilon}, \bar{\varepsilon})$, such that $\bar{f}_\varepsilon(\theta_\varepsilon) = 0$ for $|\varepsilon| < \bar{\varepsilon}$ and $\theta_0 = \bar{\theta}$.*

PROOF. Fix $r > 0$ such that $\bar{\theta} \in U_r := \mathbb{R}^2 \times B_r$, let ε_r be given by Lemma 3.1, and, for $|\varepsilon| < \varepsilon_r$, let f_ε be given by Lemma 3.3. Let us introduce the mapping $\hat{f} : (-\varepsilon_r, \varepsilon_r) \times U_r \to \mathbb{R}^5$ defined by

$$\hat{f}(\varepsilon, \theta) := \begin{cases} \dfrac{1}{\varepsilon} f_\varepsilon(\theta) & \text{if } \varepsilon \neq 0, \\ \Psi M(\theta) & \text{if } \varepsilon = 0, \end{cases}$$

where $\Psi \in GL(5, \mathbb{R})$ is given by Lemma 3.3. Our goal is to find the mapping $\varepsilon \mapsto \theta_\varepsilon$ satisfying the statement of the lemma, by applying the Implicit Function Theorem with respect to the equation $\hat{f}(\varepsilon, \theta) = 0$ in a neighbourhood of $(0, \bar{\theta})$. By hypothesis, $\hat{f}(0, \bar{\theta}) = 0$ and $\partial_\theta \hat{f}(0, \bar{\theta}) = \Psi DM(\bar{\theta})$ is invertible. We claim that $\hat{f}$ is of class C^1 on its domain. Clearly $\partial_\varepsilon \hat{f}$ and $\partial_\theta \hat{f}$ are well defined and continuous in $((-\varepsilon_r, \varepsilon_r) \setminus \{0\}) \times U_r$, since they are continuous with respect to θ and with respect to ε uniformly in $\theta \in U_r$. Since K and T are of class C^2, according to Lemma 3.1 the mapping $\varepsilon \mapsto f_\varepsilon$ is of class C^2 from $(-\varepsilon_r, \varepsilon_r)$ into $C^0(\overline{U}_r, \mathbb{R}^5)$ and thus, using also Lemma 3.3, we can write

$$f_\varepsilon(\theta) = \varepsilon \Psi M(\theta) + \frac{\varepsilon^2}{2} g(\theta) + h(\varepsilon, \theta) \quad \text{for every } (\varepsilon, \theta) \in (-\varepsilon_r, \varepsilon_r) \times \overline{U}_r \tag{4.1}$$

where

$$g := \left. \frac{d^2 f_\varepsilon}{d\varepsilon^2} \right|_{\varepsilon = 0} \in C^0(\overline{U}_r, \mathbb{R}^5),$$

and $h(\varepsilon, \cdot) \in C^0(\overline{U}_r, \mathbb{R}^5)$ is such that

$$h(\varepsilon, \theta) = o(\varepsilon^2) \quad \text{as } \varepsilon \to 0 \text{ uniformly in } \theta \in \overline{U}_r. \tag{4.2}$$

On the other hand, by Lemma 3.3, we also have

$$f_\varepsilon(\theta) = \varepsilon \Psi M(\theta) + \tilde{h}(\varepsilon, \theta) \quad \text{for every } (\varepsilon, \theta) \in (-\varepsilon_r, \varepsilon_r) \times \overline{U}_r \tag{4.3}$$

where $\tilde{h}(\varepsilon, \cdot) \in C^1(\overline{U}_r, \mathbb{R}^5)$ is such that

$$\partial_\theta \tilde{h}(\varepsilon, \theta) = o(\varepsilon) \quad \text{as } \varepsilon \to 0 \text{ uniformly in } \theta \in \overline{U}_r. \tag{4.4}$$

Then (4.3) and the definition of $\hat{f}$ yield

$$\partial_\theta \hat{f}(\varepsilon, \theta) = \begin{cases} \Psi DM(\theta) + \frac{1}{\varepsilon} \partial_\theta \tilde{h}(\varepsilon, \theta) & \text{if } \varepsilon \neq 0, \\ \Psi DM(\theta) & \text{if } \varepsilon = 0. \end{cases}$$

Hence, thanks to (4.4), $\partial_\theta \hat{f}$ is also continuous at every point $(0, \theta)$ with $\theta \in U_r$. Moreover, using again the definition of $\hat{f}$ and (4.1), we deduce that

$$\partial_\varepsilon \hat{f}(\varepsilon, \theta) = \frac{1}{2} g(\theta) + \frac{1}{\varepsilon} \partial_\varepsilon h(\varepsilon, \theta) - \frac{1}{\varepsilon^2} h(\varepsilon, \theta) \quad \text{for } \varepsilon \neq 0,$$
$$\partial_\varepsilon \hat{f}(0, \theta) = \lim_{\varepsilon \to 0} \frac{\hat{f}(\varepsilon, \theta) - \hat{f}(0, \theta)}{\varepsilon} = \lim_{\varepsilon \to 0} \left(\frac{1}{2} g(\theta) + \frac{h(\varepsilon, \theta)}{\varepsilon^2} \right) = \frac{1}{2} g(\theta),$$

where the last equality follows from (4.2). In addition, by the definition of g, we have

$$\frac{\partial_\varepsilon h(\varepsilon, \theta)}{\varepsilon} = \frac{1}{\varepsilon} \left(\frac{df_\varepsilon}{d\varepsilon}(\theta) - \Psi M(\theta) - \varepsilon g(\theta) \right) = \frac{1}{\varepsilon} \left(\frac{df_\varepsilon}{d\varepsilon}(\theta) - \left. \frac{df_\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} (\theta) \right) - g(\theta) \to 0$$

as $\varepsilon \to 0$, uniformly with respect to $\theta \in U_r$. Therefore, using (4.2), also $\partial_\varepsilon \hat{f}$ is continuous at every point $(0, \theta)$ with $\theta \in U_r$. In conclusion $\hat{f}$ is of class C^1 and the assertion of the lemma can be obtained as an application of the Implicit Function Theorem. □

Proof of Theorem 1.2. One argues as in the proof of Theorem 1.1, by exploiting Lemma 4.2 instead of Lemma 4.1. □

We conclude this section with a couple of examples, focusing on the case

$$\kappa_\varepsilon(p) = 1 + \varepsilon K(p) \quad \text{and} \quad \tau_\varepsilon(p) = \varepsilon T(p).$$

Hence the mapping M is

$$M(\phi, p) = \begin{pmatrix} \int_0^1 K(R_\phi z(t) + p) \cos(2\pi t)\, dt \\ \int_0^1 K(R_\phi z(t) + p) \sin(2\pi t)\, dt \\ \int_0^1 T(R_\phi z(t) + p) \cos(2\pi t)\, dt \\ \int_0^1 T(R_\phi z(t) + p) \sin(2\pi t)\, dt \\ \int_0^1 T(R_\phi z(t) + p)\, dt \end{pmatrix}.$$

EXAMPLE 1. Let $K, T \in C^2(\mathbb{R}^3)$ be such that

$$K(p) = p_2 p_3 \quad \text{and} \quad T(p) = p_1 \quad \text{for } p = (p_1, p_2, p_3) \in \mathbb{R}^3 \text{ with } |p| < r,$$

for some $r > 1$. One can check that for $|p| < r - 1$,

$$M(\phi, p) = \frac{1}{2} \begin{pmatrix} p_3 \sin\phi_2 \\ p_3 \cos\phi_2 \cos\phi_1 + p_2 \sin\phi_1 \\ \cos\phi_2 \\ -\sin\phi_2 \cos\phi_1 \\ 2p_1 \end{pmatrix}. \tag{4.5}$$

If we set $\bar{\phi} = (\pi/2, \pi/2)$ and $\bar{p} = (0, 0, 0)$, the point $(\bar{\phi}, \bar{p})$ turns out to be a nondegenerate zero of M. Thus Theorem 1.2 applies.

EXAMPLE 2. Let $K, T \in C^1(\mathbb{R}^3)$ be such that

$$K(p) = p_2 p_3 \quad \text{and} \quad T(p) = p_1 \quad \text{for } p = (p_1, p_2, p_3) \in \mathbb{R}^3 \text{ with } |p| \geq r,$$

for some $r > 0$. Let $\mathscr{O}$ be the set of pairs $(\phi, p) \in \mathbb{R}^2 \times \mathbb{R}^3$ such that:

$$\begin{aligned} &\pi/4 < \phi_i < 3\pi/4 \quad (i = 1, 2), \\ &-\rho < p_i < \rho \quad (i = 1, 3), \\ &-\rho^2 < p_2 < \rho^2, \end{aligned}$$

with $\rho \geq r + 1$ large enough. Then, for $(\phi, p) \in \partial\mathscr{O}$ the vector $M(\phi, p)$ is given by (4.5) and, denoting by M_i the i-th component of M, one can check that, for ρ sufficiently large:

$$M_1(\phi, p) = \begin{cases} \frac{1}{2}\rho \sin\phi_2 > 0 & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } p_3 = \rho, \\ -\frac{1}{2}\rho \sin\phi_2 < 0 & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } p_3 = -\rho, \end{cases}$$

$$M_2(\phi, p) = \begin{cases} \frac{1}{2}(p_3 \cos\phi_2 \cos\phi_1 + \rho^2 \sin\phi_1) > 0 & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } p_2 = \rho^2, \\ \frac{1}{2}(p_3 \cos\phi_2 \cos\phi_1 - \rho^2 \sin\phi_1) < 0 & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } p_2 = -\rho^2, \end{cases}$$

$$M_3(\phi, p) = \begin{cases} \frac{\sqrt{2}}{4} & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } \phi_2 = \frac{\pi}{4}, \\ -\frac{\sqrt{2}}{4} & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } \phi_2 = \frac{3\pi}{4}, \end{cases}$$

$$M_4(\phi, p) = \begin{cases} -\frac{\sqrt{2}}{4} \sin\phi_2 < 0 & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } \phi_1 = \frac{\pi}{4}, \\ \frac{\sqrt{2}}{4} \sin\phi_2 > 0 & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } \phi_1 = \frac{3\pi}{4}, \end{cases}$$

$$M_5(\phi, p) = \begin{cases} \rho & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } p_1 = \rho, \\ -\rho & \text{for } (\phi, p) \in \partial\mathscr{O} \text{ with } p_1 = -\rho. \end{cases}$$

Hence, by the Miranda theorem [17], $\deg(M, \mathscr{O}, 0) \neq 0$ and Theorem 1.1 applies.

The first author is supported by MIUR-PRIN project "Metodi variazionali ed equazioni differenziali nonlineari".

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Received 22 September 2005,
and in revised form 11 October 2005.

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