

Cauchy Transform and Rectifiability in Clifford Analysis

Juan Bory Reyes and Ricardo Abreu Blaya

Abstract. Let Γ be an n -dimensional rectifiable Ahlfors-David regular surface in \mathbb{R}^{n+1} . Let u be a continuous $\mathbb{R}_{0,n}$ -valued function on Γ , where $\mathbb{R}_{0,n}$ is the Clifford algebra associated with \mathbb{R}^n . Then we prove that the Cliffordian Cauchy transform

$$(\mathcal{C}_\Gamma u)(x) := \int_\Gamma \frac{\overline{y-x}}{A_{n+1}|y-x|^{n+1}} n(y)u(y) d\mathcal{H}^n(y), \quad x \notin \Gamma,$$

has continuous limit values on Γ if and only if the truncated integrals

$$\mathcal{S}_{\Gamma, \epsilon} u(z) := \int_{\Gamma \setminus \{|y-z| \leq \epsilon\}} \frac{\overline{y-z}}{A_{n+1}|y-z|^{n+1}} n(y)(u(y) - u(z)) d\mathcal{H}^n(y)$$

converge uniformly on Γ as $\epsilon \rightarrow 0$.

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1. Introduction

Given a closed Jordan curve γ in \mathbb{C} which bounds a bounded domain Δ_+ and its complement $\Delta_- = \mathbb{C} \setminus (\Delta_+ \cup \gamma)$, the Cauchy transform of a complex valued function f on γ is defined for $z \in \Delta_+ \cup \Delta_-$ by

$$(\mathcal{C}_\gamma f)(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)}{\tau - z} d\tau$$

and represents a function analytic in $\overline{\mathbb{C}} \setminus \gamma$.

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There are often problems in complex analysis which are reduced to the study of the boundary behaviour of $\mathcal{C}_\gamma f$, mainly in applications to solving boundary value problems and singular integral equations.

This paper deals with boundary values properties of a higher dimensional analogue of the Cauchy transform defined within the Clifford analysis setting. More precisely, we find a necessary and sufficient condition for the Cauchy transform of a continuous function on a rectifiable surface in Euclidean space to have continuous boundary values. In such a way, a generalization of a fundamental result in classical complex function theory is obtained.

Properties of the boundary values of this higher-dimensional Cauchy transform (Cliffordian Cauchy transform) were first examined for compact Liapunov surfaces by Iftimie [9]. He proved in 1965 that it has Hölder continuous limit values for any Hölder continuous function, and he established a Plemelj-Sokhotski type formula. More recently, these results have been extended to L^p -spaces over the boundaries of Lipschitz domains; see for instances [11, 12].

Our main interest lies in the study of the existence of the boundary values of the Cliffordian Cauchy transform on n -rectifiable surfaces in \mathbb{R}^{n+1} which at same time satisfy the so-called Ahlfors-David regularity condition. Links to the Ahlfors-David regularity condition and the L^2 -boundedness of the higher dimensional singular Cauchy transform may be found in, e.g., [5, 19].

The question on the existence of the continuous extension of the Cliffordian Cauchy transform is optimally answered by means of Theorem 6 below. The basic idea for our proof goes back to the one introduced by Salaev and Tokov in [21, Theorem 4] for the complex analysis case.

Recently, in solving various hard boundary value problems in Euclidean space the Clifford analysis tools have shown to play an important role. For instance, in [22] Clifford analysis is used to solve a water wave problem in three dimensions.

2. Clifford algebras and monogenic functions

The real Clifford algebra associated with \mathbb{R}^n endowed with the Euclidean metric is the minimal enlargement of \mathbb{R}^n to a real linear associative algebra $\mathbb{R}_{0,n}$ with identity such that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^n$.

It thus follows that if $\{e_j\}_{j=1}^n$ is the standard basis of \mathbb{R}^n , then we must have that $e_i e_j + e_j e_i = -2\delta_{ij}$. Every element $a \in \mathbb{R}_{0,n}$ is of the form $a = \sum_{A \subseteq N} a_A e_A$, $N = \{1, \dots, n\}$, $a_A \in \mathbb{R}$, where $e_\emptyset = e_0 = 1$, $e_{\{j\}} = e_j$, and $e_A = e_{\beta_1} \cdots e_{\beta_k}$ for $A = \{\beta_1, \dots, \beta_k\}$ where $\beta_j \in \{1, \dots, n\}$ and $\beta_1 < \cdots < \beta_k$. The conjugation is defined by $\bar{a} := \sum_A a_A \bar{e}_A$, where

$$\bar{e}_A := (-1)^k e_{i_k} \cdots e_{i_2} e_{i_1}, \quad \text{if } e_A = e_{i_1} e_{i_2} \cdots e_{i_k}.$$

Put $\mathbb{R}_{0,n}^{(k)} = \text{span}_{\mathbb{R}}(e_A : |A| = k)$. Then clearly $\mathbb{R}_{0,n}^{(k)}$ is a subspace of $\mathbb{R}_{0,n}$ – the space of so-called k -vectors – and

$$\mathbb{R}_{0,n} = \sum_{k=0}^n \oplus \mathbb{R}_{0,n}^{(k)}.$$

The projection operator of $\mathbb{R}_{0,n}$ on $\mathbb{R}_{0,n}^{(k)}$ is denoted by $[\cdot]_k$, and \mathbb{R} and \mathbb{R}^n will be identified with $\mathbb{R}_{0,n}^{(0)}$ and $\mathbb{R}_{0,n}^{(1)}$, respectively.

In what follows, an element $x = (x_0, x_1, \dots, x_n)$ will be identified with

$$x = x_0 + \sum_{j=1}^n x_j e_j \in \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}.$$

Elements of $\mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$ are often called paravectors. Notice that for $x \in \mathbb{R}^{n+1}$, we thus have that

$$x \bar{x} = \bar{x} x = |x|^2.$$

By means of the conjugation, $\mathbb{R}_{0,n}$ may be endowed with the natural Euclidean norm $|a|^2 = [a\bar{a}]_0$. An algebra norm is defined by taking $|a|_0^2 = 2^n |a|^2$.

We consider functions u defined in some subset Ω of \mathbb{R}^{n+1} and taking values in $\mathbb{R}_{0,n}$:

$$u(x) = \sum_A u_A(x) e_A,$$

where u_A are \mathbb{R} -valued functions. We say that u belongs to some classical class of function on Ω if each of its components u_A belongs to that class.

In [4] (see also [7]) a theory of monogenic functions with values in Clifford algebras is considered which generalizes in a natural way the theory of analytic functions of one complex variable to the $(n + 1)$ -dimensional Euclidean space. Monogenic functions are null solutions of the *generalized Cauchy Riemann operator* in \mathbb{R}^{n+1} :

$$\partial_x := \sum_{i=0}^n e_i \frac{\partial}{\partial x_i}.$$

It is a first order elliptic operator whose fundamental solution is given by

$$e(x) = \frac{1}{A_{n+1}} \frac{\bar{x}}{|x|^{n+1}},$$

where A_{n+1} is the area of the unit sphere in \mathbb{R}^{n+1} . If Ω is open in \mathbb{R}^{n+1} and $u \in C^1(\Omega)$, then u is said to be left (resp. right) monogenic in Ω if $\partial_x u = 0$ (resp. $u \partial_x = 0$) in Ω . Notice that the fundamental solution e is both left and right monogenic in $\mathbb{R}^{n+1} \setminus \{0\}$.

Other basic examples of monogenic functions are obtained by means of the Cliffordian Cauchy transform. Assume that Ω_+ is a bounded domain in \mathbb{R}^{n+1} with a sufficiently smooth boundary $\Gamma := \partial\Omega_+$. Then for each continuous function u in Γ , its Cliffordian Cauchy transform $\mathcal{C}_\Gamma u$ is formally defined by

$$\mathcal{C}_\Gamma u(x) := \int_\Gamma e(y-x)n(y)u(y) d\mathcal{H}^n(y), \quad x \in \mathbb{R}^{n+1} \setminus \Gamma,$$

where \mathcal{H}^n denotes the n -dimensional Hausdorff measure (the definition of the Hausdorff measure can be found in, e.g., [6, p. 3], [8, p. 171] and [10, p. 60]) and $n(y)$ is the outward pointing unit normal vector at $y \in \Gamma$. Clearly $\mathcal{C}_\Gamma u$ is monogenic in $\mathbb{R}^{n+1} \setminus \Gamma$.

Besides the Cauchy transform we also consider its singular version, the principal value singular integral operator $\mathcal{S}_\Gamma u(z) := \lim_{\epsilon \rightarrow 0} \mathcal{S}_{\Gamma, \epsilon} u(z)$, where $\mathcal{S}_{\Gamma, \epsilon}$ denotes the *truncated integral* defined by

$$\mathcal{S}_{\Gamma, \epsilon} u(z) := \int_{\Gamma \setminus \{|y-z| \leq \epsilon\}} e(y-z)n(y)(u(y) - u(z)) d\mathcal{H}^n(y), \quad z \in \Gamma.$$

Actually, the *a priori* smoothness assumption for Γ is not necessary. For instance, there is a very general notion of the unit normal $n(y)$ introduced by Federer [8, Chapter three] such that the Stokes’s Theorem still holds for boundaries with $\mathcal{H}^n(\Gamma) < +\infty$. It is exactly this version of Stokes’s Theorem we need to establish basic formulas in Clifford analysis such as Cauchy’s Theorem, Cauchy’s integral, etc (see [1, 2]).

3. Rectifiable and regular sets in \mathbb{R}^{n+1}

In what follows m will be a fixed integer such that $1 \leq m \leq n$. We denote by $\alpha(m)$ the volume of the m -dimensional unit ball.

If $\mathbf{E} \subset \mathbb{R}^{n+1}$ is an \mathcal{H}^m -measurable set and $0 < \mathcal{H}^m(\mathbf{E}) < +\infty$ we say that \mathbf{E} is an m -set. The geometric condition $0 < \mathcal{H}^m(\mathbf{E}) < +\infty$ is a natural condition without any quantitative estimates on the size of the set \mathbf{E} . In particular, if \mathbf{E} is a compact connected set with $\mathcal{H}^1(\mathbf{E}) < +\infty$, then \mathbf{E} is contained in a rectifiable curve of length at most $2\mathcal{H}^1(\mathbf{E})$ (see [20, p. 875]). In that case, \mathbf{E} can be parametrized nicely by a Lipschitz function. Notice however that for m -dimensional subsets of \mathbb{R}^{n+1} ($m > 1$) one cannot, in general, find such a nice parametrization.

The m -dimensional Hausdorff upper and lower density of $\mathbf{E} \subset \mathbb{R}^{n+1}$ at the point $a \in \mathbb{R}^{n+1}$ are defined by

$$\begin{aligned} \overline{\Theta}^m(\mathbf{E}, a) &= \limsup_{r \searrow 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(\mathbf{E} \cap \mathbf{B}(a, r)) \\ \underline{\Theta}^m(\mathbf{E}, a) &= \liminf_{r \searrow 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(\mathbf{E} \cap \mathbf{B}(a, r)), \end{aligned}$$

where $\mathbf{B}(a, r) := \{x \in \mathbb{R}^n : |x - a| \leq r\}$. When the upper and lower densities are equal, their common value is the m -dimensional density at a :

$$\Theta^m(\mathbf{E}, a) := \lim_{r \searrow 0} \alpha(m)^{-1} (r)^{-m} \mathcal{H}^m(\mathbf{E} \cap \mathbf{B}(a, r)).$$

Furthermore, if the density is equal to one, then a is called an m -regular point. Otherwise, a is called an m -irregular point. An m -set is said to be m -regular if \mathcal{H}^m -almost all its points are m -regular (see [14, p. 264])

The following basic density estimates for \mathcal{H}^m hold (see [8, Section 2.10.19, p. 181] and [10, Section 2.3, pp. 71 – 75]):

Theorem 1. *Let $\mathbf{E} \subset \mathbb{R}^{n+1}$ be \mathcal{H}^m -measurable with $\mathcal{H}^m(\mathbf{E}) < +\infty$. Then*

$$\begin{aligned} \overline{\Theta}^m(\mathbf{E}, x) &\leq 1 && \text{for } \mathcal{H}^m \text{ almost all } x \in \mathbf{E} \\ \Theta^m(\mathbf{E}, x) &= 0 && \text{for } \mathcal{H}^m \text{ almost all } x \in \mathbb{R}^n \setminus \mathbf{E}. \end{aligned}$$

Among m -sets, rectifiable sets in the sense of Federer [8, p. 251] form essentially the largest class wherein many basic geometric properties of smooth surfaces have reasonable analogues. Such properties are, for example, the existence of tangent planes (defined in a measure-theoretic way), parametrization by Lipschitz maps (or rather covering with Lipschitz surfaces) and the analogue of Lebesgue’s density point Theorem.

Definition 1. An m -set $\mathbf{E} \subset \mathbb{R}^{n+1}$ is said to be m -rectifiable if it is the Lipschitz image of some bounded subset of \mathbb{R}^m .

The above definition is a special case of the more general Federer’s definition of (\mathcal{H}^m, m) -rectifiability (see [8, p. 251]). An m -set \mathbf{E} is said to be (\mathcal{H}^m, m) -rectifiable if it is the union of an \mathcal{H}^m -zero set and countably many Lipschitz images of subsets of \mathbb{R}^m . The notion of (\mathcal{H}^m, m) -rectifiability is surprisingly stable in the sense that many other conditions give an equivalent definition. For instance, instead of Lipschitz images one can chose C^1 -submanifolds or Lipschitz graphs. Notice that rectifiability expresses a qualitative condition about the set (see also [6]).

Obviously, rectifiability generalizes the notion of smooth submanifolds. It is well known that each (\mathcal{H}^m, m) -rectifiable set is m -regular (see [8, Theorem 3.2.19, pp. 256]). However, it is not obvious that each m -regular set is in fact (\mathcal{H}^m, m) -rectifiable. In 1975 P. Mattila [14, Theorem 2.1, p. 264] proved the following theorem.

Theorem 2. *If $\mathbf{E} \subset \mathbb{R}^{n+1}$ and $\Theta^m(\mathbf{E}, x) = 1$ for \mathcal{H}^m almost every $x \in \mathbf{E}$ (i.e. \mathbf{E} is m -regular), then \mathbf{E} is (\mathcal{H}^m, m) -rectifiable.*

Theorem 2 was proved by Marstrand [15, Theorem 3, p. 94] in the special case $n = 2$, $m = 2$. Moreover David Preiss (see [18] and [6, p. 5]) proved, introducing the concept of *tangent measures*, a much stronger conjecture which had been open for a long time.

Theorem 3. *Whenever the m -dimensional density of \mathbf{E} exists as a positive finite number for \mathcal{H}^m almost every $x \in \mathbf{E}$, then \mathbf{E} is (\mathcal{H}^m, m) -rectifiable.*

Another kind of regular sets which recently has been intensively studied, is the class of so-called *Ahlfors-David regular sets of dimension m* (AD_m -regular sets). To be more precise, an m -dimensional closed subset \mathbf{E} of \mathbb{R}^{n+1} is said to be *AD_m -regular* if there exists a constant $C > 0$ such that for all $x \in \mathbf{E}$ and $0 < r < \text{diam}(\mathbf{E})$

$$C^{-1}r^m \leq \mathcal{H}^m(\mathbf{E} \cap \mathbf{B}(x, r)) \leq Cr^m. \quad (1)$$

Such sets are frequently also called regular (see [6, Definition 1.13, p. 9]), which may cause some confusion with the previously defined notion of regularity. If the set \mathbf{E} is m -regular in this paper's sense, then the m -dimensional Hausdorff density exists and it is equal to one for almost all points in \mathbf{E} . However, the AD_m -regularity condition does not imply the existence of the density at any point of the set, although it implies a uniform positive and finite bound on \mathbf{E} for the upper and lower density. Many curves as well as countable unions of curves and surfaces are AD_m -regular.

As follows directly from Rademacher's theorem (see [10, p. 81]) for an n -rectifiable surface $\Gamma \subset \mathbb{R}^{n+1}$ there are conventional tangent plane to Γ at z for almost every $z \in \Gamma$. This fact allows us to study the behavior of the Cauchy transform near the boundary in almost all of its points, what is carried out in Section 4. If we desire to study the limit values of the Cauchy transform in every $z \in \Gamma$, then it is necessary to assume also a global condition that guarantees an uniform bound for some truncated integrals; as will be seen in the Section 5, this is achieved precisely under the Ahlfors-David regularity condition.

We notice that the combination of these two conditions (n -rectifiability and Ahlfors-David regularity) produces a very wide class of surfaces that contains the classes of surfaces classically considered in the literature: Liapunov surfaces, smooth surfaces and Lipschitz surfaces. Finally we also note that Ahlfors-David regular sets are not always n -rectifiable (see [13, Example 2 on p. 798]), but if γ is a closed Jordan curve in the complex plane which is AD_1 -regular, then it is automatically 1-rectifiable.

4. The Cauchy transform on n -rectifiable surfaces

Throughout this section, Ω_+ will be a bounded oriented connected open subset of \mathbb{R}^{n+1} whose boundary is a compact topological surface. In this section we study the Cliffordian Cauchy transform on such domains in \mathbb{R}^{n+1} .

Relying upon the theory of the Calderon-Zygmund operator, pioneering work about the existence of the non-tangential limit values of the Cauchy transform for Lipschitz surfaces has been done in [16, 17].

Theorem 4. *Let $\Omega_+ \subset \mathbb{R}^{n+1}$ have Lipschitz boundary Σ . Then for any $u \in L^p(\Sigma)$, $1 < p < +\infty$, the non-tangential limit values $(\mathcal{C}_\Sigma u)^+$ and $(\mathcal{C}_\Sigma u)^-$ of the Cauchy transform $\mathcal{C}_\Sigma u$ on Σ exist at almost every $z \in \Sigma$, and*

$$\begin{aligned}
 (\mathcal{C}_\Sigma u)^+(z) &= p.v. \int_\Sigma e(y-z)n(y)u(y) d\mathcal{H}^n(y) + \frac{1}{2}u(z) \\
 (\mathcal{C}_\Sigma u)^-(z) &= p.v. \int_\Sigma e(y-z)n(y)u(y) d\mathcal{H}^n(y) - \frac{1}{2}u(z).
 \end{aligned}$$

Taking into account that any n -rectifiable surface Γ can be covered by a countable family of Lipschitz surfaces, except for a set of n -dimensional Hausdorff measure zero, it is clear that Theorem 4 still holds when Ω_+ is bounded by an n -rectifiable surface Γ .

We note that in the case of continuous functions u on Γ , a simpler proof of the above Theorem can be derived using different techniques. The aim of this section is to give a brief description of it. To this end, let us first introduce some notations and lemmas.

Let $z \in \Gamma$ and $0 < r \leq d$, where $d := \text{diam}(\Gamma)$. Then the function $\vartheta_z^n(r)$ is defined by

$$\vartheta_z^n(r) := \mathcal{H}^n(\Gamma \cap \{x \in \mathbb{R}^{n+1} : |x - z| \leq r\}).$$

Notice that, as $\mathcal{H}^n(\Gamma) < +\infty$, ϑ_z^n is a bounded and non-decreasing function on $(0, d]$.

For an $\mathbb{R}_{0,n}$ -valued continuous function u on Γ ($u \in C(\Gamma)$) the *modulus of continuity* ω_u of u is defined by

$$\omega_u(\tau) := \tau \sup_{t \geq \tau} t^{-1} \sup_{x,y \in \Gamma, |x-y| \leq t} |u(x) - u(y)|.$$

This function is much appropriate to describe smoothness properties of $\mathcal{C}_\Gamma u$.

Lemma 1. *Let $u \in C(\Gamma)$, $\mathcal{H}^n(\Gamma) < +\infty$ and $z \in \Gamma$.*

(1) *For $x \in \Omega_+$ with $|x - z| = \frac{\epsilon}{2}$ we have*

$$|\mathcal{C}_\Gamma u(x) - \mathcal{S}_{\Gamma, \epsilon} u(z) - u(z)| \leq C \left(\frac{\vartheta_z^n(\epsilon)}{\text{dist}(x, \Gamma)^n} \omega_u(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_u(\tau)}{\tau^{n+1}} d\vartheta_z^n(\tau) \right).$$

(2) *For $x \in \Omega_-$ with $|x - z| = \frac{\epsilon}{2}$ we have*

$$|\mathcal{C}_\Gamma u(x) - \mathcal{S}_{\Gamma, \epsilon} u(z)| \leq C \left(\frac{\vartheta_z^n(\epsilon)}{\text{dist}(x, \Gamma)^n} \omega_u(\epsilon) + \epsilon \int_\epsilon^d \frac{\omega_u(\tau)}{\tau^{n+1}} d\vartheta_z^n(\tau) \right).$$

Proof. The proof follows from a repetition of the arguments corresponding for the complex analysis case (compare with [21, Lemma 1]; see also [3] for the quaternionic version). The idea is to use the decomposition

$$\begin{aligned} \mathcal{C}_\Gamma u(x) - \mathcal{S}_{\Gamma, \epsilon} u(z) - u(z) &= \int_{\Gamma \cap \{|y-z| \leq \epsilon\}} e(y-x)n(y)(u(y) - u(z)) d\mathcal{H}^n(y) \\ &\quad + \int_{\Gamma \setminus \{|y-z| \leq \epsilon\}} (e(y-x) - e(y-z))n(y)(u(y) - u(z)) d\mathcal{H}^n(y), \end{aligned}$$

and then, to estimate each one of the integrals appearing in the right hand side of the above equality. For the first estimation the procedure is obvious, for the second one it is necessary to use Lemma 2.1 in [2]. ■

From now on, we assume that Ω_+ is a bounded open domain in \mathbb{R}^{n+1} with n -rectifiable boundary Γ . Therefore, as at almost all points $z \in \Gamma$ the conventional tangent plane exists, we can consider a right circular cone $V_\phi(z)$ with vertex at z such that its axis coincides with the normal vector to Γ at z and its angle ϕ between the axis and the generator of the cone being less than $\frac{\pi}{2}$.

For any $0 < \phi < \frac{\pi}{2}$ there is a sufficiently small positive number ϵ_ϕ such that for $\epsilon < \epsilon_\phi$ we have

$$V_\phi(z) \cap \{|x-z| < \epsilon\} \cap \Gamma = \{z\}.$$

When approaching x to z non-tangentially inside $V_\phi(z)$ it is possible to find a constant $C(z)$ independent of x , such that $|x-z| \leq C(z) \text{dist}(x, \Gamma)$. Furthermore, according to Theorem 1, for \mathcal{H}^n -almost every $z \in \Gamma$, there is a constant $C'(z)$ such that $\vartheta_z^n(r) \leq C'(z) r^n$ for every $r \in (0, d]$.

Combining the above remarks, it follows easily that the right hand side of the estimate (1) in Lemma 1 tend to zero for \mathcal{H}^n -almost every $z \in \Gamma$ after replacing $\vartheta_z^n(r)$ by r^n and letting x tend to z non-tangentially.

By virtue of Lemma 1, the following formal version of our main result (see Theorem 6) may be obtained.

Theorem 5. *Let Γ be n -rectifiable and let $u \in C(\Gamma)$. Then:*

- (1) *If the Cauchy transform $\mathcal{C}_\Gamma u(x)$ has non-tangential limit values from Ω_+ or Ω_- almost everywhere on Γ , then the principal value integral $\mathcal{S}_\Gamma u(z)$ exists for \mathcal{H}^n -almost every $z \in \Gamma$ and the following Plemelj-Sokhotski formulae hold:*

$$(\mathcal{C}_\Gamma u)^+(z) = \mathcal{S}_\Gamma u(z) + u(z), \quad (\mathcal{C}_\Gamma u)^-(z) = \mathcal{S}_\Gamma u(z) \tag{2}$$

- (2) *Conversely, if $\mathcal{S}_\Gamma u(z) = \lim_{\epsilon \rightarrow 0} \mathcal{S}_{\Gamma, \epsilon} u(z)$ exists for \mathcal{H}^n -almost every $z \in \Gamma$, then the Cauchy transform $\mathcal{C}_\Gamma u(x)$ has non-tangential limit values $(\mathcal{C}_\Gamma u)^+$ and $(\mathcal{C}_\Gamma u)^-$ \mathcal{H}^n -almost everywhere in Γ and the Plemelj-Sokhotski formulae (2) hold.*

5. The Cauchy transform on rectifiable AD_n -regular surfaces

In this section (see Theorem 6) it will be proved that for $u \in C(\Gamma)$, $\mathcal{C}_\Gamma u$ has continuous boundary values if and only if $\mathcal{S}_{\Gamma,\epsilon} u$ converges uniformly to $\mathcal{S}_\Gamma u$ as $\epsilon \rightarrow 0$.

Let us start with two lemmas.

Lemma 2. *Let $\mathcal{H}^n(\Gamma) < +\infty$, $z \in \Gamma$ and $r \in (0, d]$.*

(1) *If $u^+ \in C(\Omega_+ \cup \Gamma)$ is monogenic in Ω_+ , then*

$$|\mathcal{S}_{\Gamma,r} u^+(z)| \leq C \max_{x \in \Omega_+ \cup \Gamma, |x-z|=r} |u^+(x) - u^+(z)|.$$

(2) *If $u^- \in C(\Omega_- \cup \Gamma)$ is monogenic in Ω_- with $u^-(\infty) = 0$, then*

$$|\mathcal{S}_{\Gamma,r} u^-(z) + u^-(z)| \leq C \max_{x \in \Omega_- \cup \Gamma, |x-z|=r} |u^-(x) - u^-(z)|.$$

Proof. The proof proceeds along similar lines as in [3], where the case $n = 2$ was dealt with, and we omit the details. ■

Lemma 3. *Let Γ be n -rectifiable and let $u \in C(\Gamma)$. If the Cliffordian Cauchy transform $\mathcal{C}_\Gamma u$ has continuous limit values $(\mathcal{C}_\Gamma u)^\pm$ on Γ , then the truncated integrals $\mathcal{S}_{\Gamma,\epsilon} u$ converge uniformly on Γ as $\epsilon \rightarrow 0$.*

Proof. First note that if the Cauchy transform $\mathcal{C}_\Gamma u$ has continuous limit values $(\mathcal{C}_\Gamma u)^\pm$ on Γ , then in view of Theorem 5 we have

$$(\mathcal{C}_\Gamma u)^+(z) - (\mathcal{C}_\Gamma u)^-(z) = u(z) \text{ for } \mathcal{H}^n\text{-almost every } z \in \Gamma.$$

The continuity property of the functions $(\mathcal{C}_\Gamma u)^\pm$ and u implies that the above equality holds everywhere on Γ . For the sake of brevity we put $u^+ = (\mathcal{C}_\Gamma u)^+$ and $u^- = (\mathcal{C}_\Gamma u)^-$.

Next, we affirm that $\mathcal{S}_\epsilon u \rightarrow u^-$ uniformly on Γ as $\epsilon \rightarrow 0$. In fact, by Lemma 2, it is straightforward to check that

$$\begin{aligned} & |\mathcal{S}_{\Gamma,\epsilon} u(z) - u^-(z)| \\ & \leq |\mathcal{S}_{\Gamma,\epsilon} u^+(z)| + |\mathcal{S}_{\Gamma,\epsilon} u^-(z) + u^-(z)| \\ & \leq C \left(\max_{x \in \Gamma \cup \Omega_+, |x-z|=\epsilon} |u^+(x) - u^+(z)| + \max_{x \in \Gamma \cup \Omega_-, |x-z|=\epsilon} |u^-(x) - u^-(z)| \right) \end{aligned}$$

Using the uniform continuity property of u^+ and u^- we obtain the desired convergences. ■

Now we can state and prove our main result.

Theorem 6. *Let Γ be an n -rectifiable AD_n -regular surface and let $u \in C(\Gamma)$. Then the following two conditions are equivalent:*

- (1) $\mathcal{C}_\Gamma u$ has continuous limit values $(\mathcal{C}_\Gamma u)^\pm$ on Γ
- (2) $\mathcal{S}_{\Gamma,\epsilon} u \rightarrow \mathcal{S}_\Gamma u$ uniformly on Γ as $\epsilon \rightarrow 0$.

Proof. Notice that (1) \Rightarrow (2) is proved in Lemma 3.

In order to prove that (2) implies (1), we shall prove that

$$(\mathcal{C}_\Gamma u)^+(z) = \mathcal{S}_\Gamma u(z) + u(z), \quad (\mathcal{C}_\Gamma u)^-(z) = \mathcal{S}_\Gamma u(z).$$

For the sake of brevity we restrict ourselves to the case $(\mathcal{C}_\Gamma u)^+$.

Fix $z \in \Gamma$ for the moment, and let $x \in \Omega_+$. There is a point $z_x \in \Gamma$ such that $|x - z_x| = \delta := \text{dist}(x, \Gamma)$. Then

$$\begin{aligned} |(\mathcal{C}_\Gamma u)(x) - \mathcal{S}_\Gamma u(z) - u(z)| &\leq |(\mathcal{C}_\Gamma u)(x) - \mathcal{S}_\Gamma u(z_x) - u(z_x)| \\ &\quad + |\mathcal{S}_\Gamma u(z) - \mathcal{S}_\Gamma u(z_x)| + |u(z) - u(z_x)|. \end{aligned}$$

Now, split $\mathcal{S}_\Gamma u$ into two pieces corresponding to the decomposition

$$\Gamma = (\Gamma \setminus \{x \in \mathbb{R}^{n+1} : |x - z_x| \leq \delta\}) \cup (\Gamma \cap \{x \in \mathbb{R}^{n+1} : |x - z_x| \leq \delta\}).$$

Then

$$\begin{aligned} |(\mathcal{C}_\Gamma u)(x) - \mathcal{S}_\Gamma u(z) - u(z)| &\leq |\mathcal{C}_\Gamma u(x) - \mathcal{S}_{\Gamma,\delta} u(z_x) - u(z_x)| + \omega_{\mathcal{S}_\Gamma u}(|z - z_x|) \\ &\quad + \omega_u(|z - z_x|) + \left| \int_{\Gamma \cap \{|x - z_x| \leq \delta\}} e(y - z_x) n(y) (u(y) - u(z_x)) d\mathcal{H}^n(y) \right|. \end{aligned}$$

We estimate separately each term on the right hand side in the above inequality. With Lemma 1 estimate (1) at our disposal and the AD_n -regularity condition for Γ , we have that the first summand tends to zero as $x \rightarrow z$ for every $z \in \Gamma$.

On the other hand, our assumptions on u , $\mathcal{S}_\Gamma u$ and $\mathcal{S}_{\Gamma,\epsilon} u$ imply that for every $z \in \Gamma$ the convergence to zero as $x \rightarrow z$ of the other summands are true and fairly simple, so altogether we have $(\mathcal{C}_\Gamma u)^+(z) = \mathcal{S}_\Gamma u(z) + u(z)$, and the proof is complete. ■

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