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Double Positive Solutions of Three-Point Boundary Value Problems for p-Laplacian Difference Equations

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Abstract. In this paper, by means of double fixed-point theorem in a cone, the existence of double positive solutions of three-point boundary value problems for $p-$Laplacian difference equations is considered.

Keywords: *Difference equation, p−Laplacian, boundary value problem, positive solution, double fixed-point theorem, cone*

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1. Introduction

For notation, given $a < b$ in Z, we employ intervals to denote discrete sets such as $[a, b] = \{a, a+1, ..., b\}$, $[a, b) = \{a, a+1, ..., b-1\}$, $[a, \infty) = \{a, a+1, ...\}$, etc. Let $N \geq 1$ be fixed. In this paper, we are concerned with the existence of positive solutions of the following p-Laplacian difference equation

$$\triangle[\phi_p(\triangle u(t-1))] + a(t)f(u(t)) = 0, \quad t \in [1, N+1], \tag{1}$$

satisfying the boundary conditions

$$u(0) - B_0(\triangle u(\eta)) = 0, \quad \triangle u(N+1) = 0, \tag{2}$$

or

$$\triangle u(0) = 0, \quad u(N+2) + B_1(\triangle u(\eta)) = 0, \tag{3}$$

where $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s, p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \eta < N+1$ and

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($\mathbf{H_1}$) $f : \mathbb{R} \to \mathbb{R}^+$ is continuous ($\mathbb{R}^+$ denotes the nonnegative reals);
($\mathbf{H_2}$) $a(t)$ is a positive valued function defined on $[0, N+2]$;
($\mathbf{H_3}$) $B_0(v)$ and $B_1(v)$ are both continuous odd functions defined on $\mathbb{R}$ and satisfies that there exist $A, B > 0$ such that $Bv \le B_j(v) \le Av$ for all $v \ge 0,\ j = 0, 1$.

We remark that by a solution u of (1), (2) (respectively (1), (3)), we mean $u : [0, N+2] \to \mathbb{R}$, u satisfies (1) on $[1, N+1]$, and u satisfies the boundary conditions (2) (respectively (3)). If $\triangle^2 u(t-1) \le 0$ for $t \in [1, N+1]$, then we say $u(t)$ is concave on $[0, N+2]$.

p-Laplacian problems with two-point, three-point and multi-point boundary conditions for ordinary differential equations and finite difference equations have been studied extensively, see [1, 4 − 6, 9 − 16] and references therein. In this paper, by using a new double fixed-point theorem due to Avery and Henderson [3] in a cone, we prove that there exist at least double positive solutions of (1), (2) (respectively (1), (3)). To this end, in Section 2 we provide some background material from the theory of cones in Banach spaces, and we then state the double fixed-point theorem. In Section 3 and Section 4, by defining an appropriate Banach space and cones, we impose the growth conditions on f which allow us to apply the double fixed-point theorem in obtaining existence of double positive solutions of (1), (2) (respectively (1), (3)). Our results are discrete analogues of the recent paper by Liu and Ge [11].

2. Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces, and we then state the double fixed-point theorem for a cone preserving operator. The following definitions can be found in the book by Deimling [7] as well as in the book by Guo and Lakshmikantham [8].

Definition 1. Let E be a real Banach space. A nonempty, closed convex set $P \subset E$ is called a *cone*, if it satisfies the following two conditions:

(i) $x \in P, \lambda \ge 0$ implies $\lambda x \in P$;
(ii) $x, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an *ordering* in E given by

$$x \le_P y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2. Given a cone P in a real Banach space E, a functional $\psi : P \to R$ is said to be *increasing* on P, provided $\psi(x) \le \psi(y)$ for all $x, y \in P$ with $x \le_P y$.

Definition 3. Given a nonnegative continuous functional γ on a cone P of a real Banach space E, we define, for each $d > 0$, the *level* set

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\}.$$

The following double fixed-point theorem due to Avery and Henderson [3] will play an important role in the proof of our results. Applications of this fixed point theorem can be found in recent papers [2, 11, 12].

Theorem 1. *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative, continuous functionals on P, and let θ be a nonnegative, continuous functional on P with $\theta(0) = 0$ such that for some $c > 0$ and $M > 0$,*

$$\gamma(x) \le \theta(x) \le \alpha(x) \quad \text{and} \quad \|x\| \le M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose there exist positive numbers a and b with $a < b < c$ such that

$$\theta(\lambda x) \le \lambda\theta(x) \quad \text{for} \quad 0 \le \lambda \le 1 \quad \text{and} \quad x \in \partial P(\theta, b)\,,$$

and

$$T : \overline{P(\gamma, c)} \to P$$

is a completely continuous operator such that

- **(i)** $\gamma(Tx) > c$ *for all* $x \in \partial P(\gamma, c)$
- **(ii)** $\theta(Tx) < b$ *for all* $x \in \partial P(\theta, b)$
- **(iii)** $P(\alpha, a) \ne \emptyset$ *and* $\alpha(Tx) > a$ *for all* $x \in \partial P(\alpha, a)$.

Then T has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ such that

$$a < \alpha(x_1) \quad \text{with} \quad \theta(x_1) < b$$
$$b < \theta(x_2) \quad \text{with} \quad \gamma(x_2) < c.$$

3. Solutions of (1) and (2) in a cone

In this section, by defining an appropriate Banach space and cones, we impose the growth conditions on f which allow us to apply the double fixed-point theorem in establishing the existence of double positive solutions of (1), (2). We note that, from the nonegativity of a and f, a solution of (1), (2) is nonegative and concave on $[0, N+2]$.

Let

$$E = \{u \mid u : [0, N+2] \to \mathbb{R}\},$$

with norm $\|u\| = \max_{t\in[0,N+2]} |u(t)|$, then $(E, \|\cdot\|)$ is a Banach space. Define a cone $P \subset E$ by

$$P = \left\{ u \in E \;\middle|\; \begin{array}{l} u \text{ is concave and nonnegative valued} \\ \text{on } [0, N+2]\,, \text{ and } \triangle u(N+1) = 0 \end{array} \right\}.$$

Lemma 1. *If* $u \in P$, *then*

$$u(t) \geq \frac{t}{N+2}\|u\|, \quad t \in [0, N+2], \tag{4}$$

where $\|u\| = \max_{t\in[0,N+2]} |u(t)|$.

Proof. From the fact that u is concave on $[0, N+2]$, we see that $\triangle u(t)$ is decreasing. Thus $\triangle u(t) \geq \triangle u(N+1) = 0$ for $t \in [0, N+1]$ and $u(t)$ is increasing on $[0, N+2]$, that is, $u(N+2) \geq u(t) \geq u(0) \geq 0$ for $t \in [0, N+2]$. So, $\|u\| = \max_{t\in[0,N+2]} |u(t)| = u(N+2)$.

Let

$$x(t) = u(t) - \frac{t}{N+2}\|u\|, \quad t \in [0, N+2]. \tag{5}$$

Then

$$\triangle^2 x(t-1) \leq 0 \quad \text{for } t \in [1, N+1], \tag{6}$$

and

$$x(0) \geq 0, \quad x(N+2) = 0. \tag{7}$$

From (6), (7) we get for $t \in [0, N+2]$

$$x(t) = \frac{N+2-t}{N+2}x(0) + \frac{t}{N+2}x(N+2) - \sum_{s=1}^{N+1} G(t,s)\triangle^2 x(s-1) \geq 0, \tag{8}$$

where

$$G(t,s) = \frac{1}{N+2}\begin{cases} s(N+2-t), & 1 \leq s \leq t \leq N+2, \\ t(N+2-s), & 0 \leq t \leq s \leq N+1. \end{cases}$$

From (5), (8) we obtain

$$u(t) \geq \frac{t}{N+2}\|u\| \quad \text{for } t \in [0, N+2].$$

The proof of Lemma 1 is complete. ∎

Fix an integer l such that $0 < \eta < l < N+2$, and define the increasing, nonnegative continuous functionals γ, θ, and α on P by

$$\gamma(u) = \min_{\eta \leq t \leq l} u(t) = u(\eta)$$
$$\theta(u) = \max_{0 \leq t \leq \eta} u(t) = u(\eta)$$
$$\alpha(u) = \min_{l \leq t \leq N+2} u(t) = u(l).$$

We see that $\gamma(u) = \theta(u) \leq \alpha(u)$ for each $u \in P$. In addition, for each $u \in P$, Lemma 1 implies $\gamma(u) = u(\eta) \geq \frac{\eta}{N+2}\|u\|$. Thus,

$$\|u\| \leq \frac{N+2}{\eta}\gamma(u) \quad \text{for all } u \in P.$$

We also see that $\theta(\lambda u) = \lambda\theta(u)$ for $\lambda \in [0,1]$ and $u \in \partial P(\theta, b)$. For notational convenience, we denote μ, ξ and δ, by

$$\mu = (B+l)\phi_q\Big(\sum_{i=l}^{N+1} a(i)\Big)$$
$$\xi = A\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)\Big) + \sum_{s=0}^{\eta-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)\Big)$$
$$\delta = (B+\eta)\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)\Big).$$

We note that $u(t)$ is a solution of (1) and (2), if and only if for $t \in [0, N+2]$

$$u(t) = B_0\left(\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big)\right) + \sum_{s=0}^{t-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big).$$

Theorem 2. *Assume that conditions* (H_1), (H_2) *and* (H_3) *are satisfied. Let*

$$0 < a < \frac{\mu}{\xi}b < \frac{\eta\mu}{(N+2)\xi}c,$$

and suppose that f satisfies the following conditions:

- $(\mathbf{C_1})$ $f(w) > \phi_p(\frac{c}{\delta})$ *for* $c \le w \le \frac{N+2}{\eta}c$
- $(\mathbf{C_2})$ $f(w) < \phi_p(\frac{b}{\xi})$ *for* $0 \le w \le \frac{N+2}{\eta}b$
- $(\mathbf{C_3})$ $f(w) > \phi_p(\frac{a}{\mu})$ *for* $a \le w \le \frac{N+2}{l}a.$

Then, there exists at least two solutions u_1 and u_2 of (1) *and* (2) *such that*

$$a < \alpha(u_1) \quad \textit{with} \quad \theta(u_1) < b$$
$$b < \theta(u_2) \quad \textit{with} \quad \gamma(u_2) < c.$$

Proof. Define a completely continuous summation operator $T : P \to E$ by

$$(Tu)(t) = B_0\left(\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big)\right) + \sum_{s=0}^{t-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big) \qquad (9)$$

for $u \in P,\ t \in [0, N+2]$. We will seek fixed points of T in the cone P. For $t \in [0, N+2]$, it is easy to see that $(Tu)(t)$ satisfies (1), (2). So each fixed point of T in the cone P is a positive solution of (1), (2).

We now prove that the conditions of Theorem 1 hold with respect to T. Let $u \in \partial P(\gamma, c)$, then $(Tu)(t) \ge 0$ for $t \in [0, N+2]$. In addition, $\triangle^2(Tu)(t) \le 0$ for $t \in [0, N]$, and $\triangle(Tu)(N+1) = 0$. This implies $Tu \in P$, and so $T : P(\gamma, c) \to P$.

To verify that (i) of Theorem 1 holds, we choose $u \in \partial P(\gamma, c)$. Then $\gamma(u) = \min_{\eta \le t \le l} u(t) = u(\eta) = c$. This implies $u(t) \ge c$, $\eta \le t \le N+2$. Recalling that $\|u\| \le \frac{N+2}{\eta}\gamma(u) = \frac{N+2}{\eta}c$, we have

$$c \le u(t) \le \frac{N+2}{\eta}c \quad \text{for } \eta \le t \le N+2.$$

As a consequence of (C_1), $f(u(s)) > \phi_p\left(\frac{c}{\delta}\right)$ for $\eta \le s \le N+2$. Since $Tu \in P$, we have

$$\begin{aligned}
\gamma(Tu) &= (Tu)(\eta) \\
&= B_0\left(\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big)\right) + \sum_{s=0}^{\eta-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big) \\
&> (B+\eta)\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)\Big)\cdot\frac{c}{\delta} \;=\; c.
\end{aligned}$$

Thus, (i) of Theorem 1 is satisfied.

Let $u \in \partial P(\theta, b)$. Then $\theta(u) = \max_{0 \le t \le \eta} u(t) = u(\eta) = b$. This implies $0 \le u(t) \le b$, $0 \le t \le \eta$, and since $u \in P$, we have $b \le u(t) \le \|u\| = u(N+2)$ for $\eta \le t \le N+2$. Note that $\|u\| \le \frac{N+2}{\eta}\gamma(u) = \frac{N+2}{\eta}\theta(u) = \frac{N+2}{\eta}b$. So,

$$0 \le u(t) \le \frac{N+2}{\eta}b \quad \text{for } 0 \le t \le N+2.$$

From (C_2) we have $f(u(s)) < \phi_p\big(\frac{b}{\xi}\big)$ for $0 \le s \le N+2$, and so

$$\begin{aligned}
\theta(Tu) &= (Tu)(\eta) \\
&= B_0\left(\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big)\right) + \sum_{s=0}^{\eta-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big) \\
&\le A\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big) + \sum_{s=0}^{\eta-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big) \\
&< \left(A\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)\Big) + \sum_{s=0}^{\eta-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)\Big)\right)\cdot\frac{b}{\xi} \;=\; b.
\end{aligned}$$

Thus, (ii) of Theorem 1 is satisfied.

We now prove that (iii) of Theorem 1 is also satisfied. We note that $u(t) = \frac{a}{2}$, $t \in [0, N+2]$, is a member of $P(\alpha, a)$ and $\alpha(u) = \frac{a}{2} < a$. So $P(\alpha, a) \ne \emptyset$.

Now, let $u \in \partial P(\alpha, a)$. Then $\alpha(u) = \min_{l \le t \le N+2} = u(l) = a$. Recalling that $\|u\| \le \frac{N+2}{l}\gamma(u) \le \frac{N+2}{l}\alpha(u) = \frac{N+2}{l}a$, we have

$$a \le u(t) \le \frac{N+2}{l}a \quad \text{for } l \le t \le N+2.$$

From assumption (C_3), we get $f(u(s)) > \phi_p\left(\frac{a}{\mu}\right)$ for $l \le s \le N+2$, and so

$$\begin{aligned}
\alpha(Tu) &= (Tu)(l) \\
&= B_0\left(\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big)\right) + \sum_{s=0}^{l-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big) \\
&\ge B\phi_q\Big(\sum_{i=\eta+1}^{N+1} a(i)f(u(i))\Big) + \sum_{s=0}^{l-1}\phi_q\Big(\sum_{i=s+1}^{N+1} a(i)f(u(i))\Big) \\
&> (B+l)\phi_q\Big(\sum_{i=l}^{N+1} a(i)\Big)\cdot\frac{a}{\mu} = a.
\end{aligned}$$

Therefore, Theorem 1 implies that T has at least two fixed points u_1 and u_2, belonging to $\overline{P(\gamma, c)}$, which are positive solutions of (1) and (2) such that

$$\begin{aligned}
a &< \alpha(u_1) \quad \text{with } \theta(u_1) < b \\
b &< \theta(u_2) \quad \text{with } \gamma(u_2) < c\,.
\end{aligned}$$

The proof of Theorem 2 is complete. ∎

4. Solutions of (1) and (3) in a cone

In this section, we use the double fixed-point theorem to establish the existence of double positive solutions of (1), (3).

Consider the Banach space

$$E = \{u \mid u : [0, N+2] \to R\},$$

with norm $\|u\| = \max_{t\in[0,N+2]}|u(t)|$, and define a cone $P_1 \subset E$ by

$$P_1 = \left\{ u \in E \;\middle|\; \begin{array}{l} u \text{ is concave and nonnegative valued} \\ \text{on } [0, N+2]\,,\ \text{and } \triangle u(0) = 0 \end{array} \right\}.$$

Lemma 2. *If* $u \in P_1$, *then*

$$u(t) \ge \frac{N+2-t}{N+2}\|u\|\,, \quad t \in [0, N+2], \tag{10}$$

where $\|u\| = \max_{t\in[0,N+2]}|u(t)|$.

Proof. From the fact that u is concave on $[0, N+2]$, we see that $\triangle u$ is decreasing. Thus $\triangle u(t) \le \triangle u(0) = 0$ for $t \in [0, N+1]$ and $u(t)$ is decreasing on $[0, N+2]$, that is, $u(0) \ge u(t) \ge u(N+2) \ge 0$ for $t \in [0, N+2]$. So, $\|u\| = \max_{t\in[0,N+2]} |u(t)| = u(0)$.

Let

$$y(t) = u(t) - \frac{N+2-t}{N+2}\|u\|\,, \quad t \in [0, N+2]. \tag{11}$$

Then

$$\triangle^2 y(t-1) \le 0\,, \quad t \in [1, N+1], \tag{12}$$

and

$$y(0) = 0, \quad y(N+2) \ge 0. \tag{13}$$

From (12), (13) we get

$$y(t) = \frac{N+2-t}{N+2} y(0) + \frac{t}{N+2} y(N+2) - \sum_{s=1}^{N+1} G(t,s)\triangle^2 y(s-1) \ge 0 \tag{14}$$

for $t \in [0, N+2]$, where

$$G(t,s) = \frac{1}{N+2}\begin{cases} s(N+2-t), & 1 \le s \le t \le N+2 \\ t(N+2-s), & 0 \le t \le s \le N+1. \end{cases}$$

From (11), (14) we obtain

$$u(t) \ge \frac{N+2-t}{N+2}\|u\|\,, \quad t \in [0, N+2].$$

The proof of Lemma 2 is complete. ∎

Fix an integer r such that $0 < r < \eta$, and define the increasing, nonnegative, continuous functionals γ, θ and α on P_1 by

$$\begin{aligned} \gamma(u) &= \min_{r\le t\le \eta} u(t) = u(\eta) \\ \theta(u) &= \max_{\eta\le t\le N+2} u(t) = u(\eta) \\ \alpha(u) &= \min_{0\le t\le r} u(t) = u(r)\,. \end{aligned}$$

We see that, for each $u \in P_1$, $\gamma(u) = \theta(u) \le \alpha(u)$. In addition, for each $u \in P_1$, $\gamma(u) = u(\eta) \ge \frac{N+2-\eta}{N+2}\|u\|$. Thus,

$$\|u\| \le \frac{N+2}{N+2-\eta}\gamma(u)\,, \quad u \in P_1.$$

We also see that $\theta(\lambda u) = \lambda\theta(u)$ for $\lambda \in [0,1]$ and $u \in \partial P_1(\theta, b)$. Set

$$\mu_1 = (B + N + 2 - r)\phi_q\Big(\sum_{i=1}^{r} a(i)\Big)$$
$$\xi_1 = A\phi_q\Big(\sum_{i=1}^{\eta} a(i)\Big) + \sum_{s=\eta}^{N+1}\phi_q\Big(\sum_{i=1}^{s} a(i)\Big)$$
$$\delta_1 = (B + N + 2 - \eta)\phi_q\Big(\sum_{i=1}^{\eta} a(i)\Big).$$

We note that $u(t)$ is a solution of (1) and (3), if and only if for $t \in [0, N+2]$

$$u(t) = B_1\left(\phi_q\Big(\sum_{i=1}^{\eta} a(i)f(u(i))\Big)\right) + \sum_{s=t}^{N+1}\phi_q\Big(\sum_{i=1}^{s} a(i)f(u(i))\Big).$$

In analogy to the existence results of the previous section, we have the following theorem for positive solutions of (1) and (3).

Theorem 3. *Assume that conditions* (H_1), (H_2) *and* (H_3) *are satisfied. Let*

$$0 < a < \frac{\mu_1}{\xi_1}b < \frac{(N+2-\eta)\mu_1}{(N+2)\xi_1}c,$$

and suppose that f *satisfies the following conditions*

- $(\mathbf{D_1})$ $f(w) > \phi_p(\frac{c}{\delta_1})$ *for* $c \le w \le \frac{N+2}{N+2-\eta}c$
- $(\mathbf{D_2})$ $f(w) < \phi_p(\frac{b}{\xi_1})$ *for* $0 \le w \le \frac{N+2}{N+2-\eta}b$
- $(\mathbf{D_3})$ $f(w) > \phi_p(\frac{a}{\mu_1})$ *for* $a \le w \le \frac{N+2}{N+2-r}a.$

Then, there exists at least two solutions of (1) *and* (3) *such that*

$$a < \alpha(u_1) \quad \text{with} \quad \theta(u_1) < b$$
$$b < \theta(u_2) \quad \text{with} \quad \gamma(u_2) < c.$$

5. Example

In this section, we present an example to explain our result. Consider the $p-$Laplacian difference equation

$$\triangle[\phi_p(\triangle u(t-1))] + f(u(t)) = 0\,, \quad t \in [1, 99], \tag{15}$$

satisfying the boundary conditions

$$u(0) - 2\triangle u(45) = 0\,, \quad \triangle u(99) = 0, \tag{16}$$

where $p = \frac{3}{2}, q = 3, a(t) \equiv 1, A = B = 2, \eta = 45, N = 98$, and

$$f(u) = \begin{cases} 0.4, & 0 \le u \le \frac{1}{9} \cdot 10^6 \\ 0.4 + \frac{9u - 10^6}{8 \cdot 10^5}, & \frac{1}{9} \cdot 10^6 \le u \le 2 \cdot 10^5 \\ 1.4, & u \ge 2 \cdot 10^5. \end{cases}$$

Then, the system (15), (16) has at least two positive solutions.

Proof. Choose $a = 10^4, b = 5 \cdot 10^4, c = 2 \cdot 10^5$ and $l = 50$. Then

$$\mu = 52\phi_3\Big(\sum_{i=50}^{99} a(i)\Big) = 130000$$

$$\xi = 2\phi_3\Big(\sum_{i=46}^{99} a(i)\Big) + \sum_{s=0}^{44} \phi_3\Big(\sum_{i=s+1}^{99} a(i)\Big) = 280227$$

$$\delta = 47\phi_3\Big(\sum_{i=46}^{99} a(i)\Big) = 137052.$$

It is easy to see that $0 < a < \frac{\mu}{\xi} b < \frac{\eta\mu}{(N+2)\xi} c$, and f satisfies

$$f(w) > \phi_p\left(\frac{c}{\delta}\right) = \sqrt{\frac{2 \cdot 10^5}{137052}} \approx 1.208 \quad \text{for } 2 \cdot 10^5 \le w \le \frac{4}{9} \cdot 10^6$$

$$f(w) < \phi_p\left(\frac{b}{\xi}\right) = \sqrt{\frac{5 \cdot 10^5}{280227}} \approx 0.422 \quad \text{for } 0 \le w \le \frac{1}{9} \cdot 10^6$$

$$f(w) > \phi_p\left(\frac{a}{\mu}\right) = \sqrt{\frac{10^4}{130000}} \approx 0.277 \quad \text{for } 10^4 \le w \le 2 \cdot 10^4.$$

Therefore by Theorem 2, the problem (15), (16) has at least two positive solutions u_1, u_2 satisfying

$$10^4 < \min_{t \in [50,100]} u_1(t) \qquad \text{with} \quad \max_{t \in [0,45]} u_1(t) < 5 \cdot 10^4$$

$$5 \cdot 10^4 < \max_{t \in [0,45]} u_2(t) \qquad \text{with} \quad \min_{t \in [45,50]} u_2(t) < 2 \cdot 10^5. \qquad \blacksquare$$

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