Spectral Synthesis for Operators and Systems

Anton Baranov (St. Petersburg State University, St. Petersburg, Russia) and Yurii Belov (St. Petersburg State University, St. Petersburg, Russia)

Spectral synthesis is the reconstruction of the whole lattice of invariant subspaces of a linear operator from generalised eigenvectors. A closely related problem is the reconstruction of a vector in a Hilbert space from its Fourier series with respect to some complete and minimal system. This article discusses the spectral synthesis problem in the context of operator and function theory and presents several recent advances in this area. Among them is the solution of the spectral synthesis problem for systems of exponentials in $L^2(-\pi, \pi)$.

A more detailed account of these problems can be found in the survey [1]*, to appear in the proceedings of 7ECM.*

1 Introduction

Eigenfunction expansions play a central role in analysis and its applications. We discuss several questions concerning such expansions for special systems of functions, e.g. exponentials in L^2 on an interval and in weighted spaces, phase-space shifts of the Gaussian function in $L^2(\mathbb{R})$ and reproducing kernels in de Branges spaces (which include certain families of Bessel or Airy type functions). We present solutions of some problems in the area (including the longstanding spectral synthesis problem for systems of exponentials in $L^2(-\pi, \pi)$) and mention several open questions, such as the Newman–Shapiro problem about synthesis in Bargmann–Fock space.

Spectral synthesis for operators

One of the basic ideas of operator theory is to consider a linear operator as a "sum" of its simple parts, e.g. its restrictions onto invariant subspaces. In the finite-dimensional case, the possibility of such decomposition is guaranteed by the Jordan normal form. Moreover, any invariant subspace coincides with the span of the generalised eigenvectors it contains (recall that x is said to be a *generalised eigenvector* or a *root vector* of a linear operator A if $x \in \mathrm{Ker}\,(A - \lambda I)^n$ for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$).

However, the situation in the infinite-dimensional case is much more complex. Assume that A is a bounded linear operator in a separable Hilbert space H that has a complete set of generalised eigenvectors (in this case we say that A is *complete*). Is it true that any A-invariant subspace is *spectral*, that is, it coincides with the closed linear span of the generalised eigenvectors it contains? In general, the answer is negative. Therefore, it is natural to introduce the following notion.

Definition 1. A continuous linear operator A in a separable Hilbert (or Banach, or Frechét) space H is said to admit spectral synthesis or to be *synthesable* (we write $A \in \mathrm{Synt}$) if, for

any invariant subspace E of A, we have

$$E = \overline{\mathrm{Span}}\{x \in E : x \in \cup_{\lambda, n}\,\mathrm{Ker}\,(A - \lambda I)^n\}.$$

Equivalently, this means that the restriction $A|_E$ has a complete set of generalised eigenvectors.

The notion of spectral synthesis for a general operator goes back to J. Wermer (1952). In the special context of translation invariant subspaces in spaces of continuous or smooth functions, similar problems were studied by J. Delsarte (1935), L. Schwartz (1947) and J.-P. Kahane (1953). Note that, in this case, the generalised eigenvectors are exponentials and exponential monomials.

Wermer showed, in particular, that any compact normal operator in a Hilbert space admits spectral synthesis. However, both of these conditions are essential: there exist non-synthesable compact operators and there exist non-synthesable normal operators. For a normal operator A with simple eigenvalues λ_n, Wermer showed that $A \notin \mathrm{Synt}$ if and only if the set $\{\lambda_n\}$ carries a complex measure orthogonal to polynomials, i.e. there exists a nontrivial sequence $\{\mu_n\} \in \ell^1$ such that $\sum_n \mu_n \lambda_n^k = 0$, $k \in \mathbb{N}_0$. Existence of such measures follows from Wolff's classical example of a Cauchy transform vanishing outside of the disc: there exist $\lambda_n \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\{\mu_n\} \in \ell^1$ such that

$$\sum_n \frac{\mu_n}{z - \lambda_n} \equiv 0, \qquad |z| > 1.$$

At the same time, there exist compact operators that do not admit spectral synthesis. Curiously, the first example of such a situation was implicitly given by H. Hamburger in 1951 (even before Wermer's paper). Further results were obtained in the 1970s by N. Nikolski and A. Markus. For example, Nikolski [20] proved that any Volterra operator can be a part of a complete compact operator (recall that a *Volterra operator* is a compact operator whose spectrum is $\{0\}$).

Theorem 2 (Nikolski)**.** For any Volterra operator V in a Hilbert space H, there exists a complete compact operator A on a larger Hilbert space $H \oplus H'$ such that H is A-invariant and $A|_H = V$. In particular, $A \notin \mathrm{Synt}$.

A. Markus [16] found a relation between spectral synthesis for a compact operator and the geometric properties of the eigenvectors. We now introduce the required "strong completeness" property.

Hereditarily complete systems

Let $\{x_n\}_{n \in N}$ be a system of vectors in a separable Hilbert space H that is both *complete* (i.e. its linear span is dense in H) and

minimal (meaning that it fails to be complete when we remove any of its vectors). Let $\{y_n\}_{n \in N}$ be its (unique) *biorthogonal* system, i.e. the system such that $(x_m, y_n) = \delta_{mn}$, where δ_{mn} is the Kronecker delta. Note that a system coincides with its biorthogonal system if and only if it is an orthonormal basis.

With any $x \in H$, we associate its (formal) Fourier series with respect to the biorthogonal pair $\{x_n\}, \{y_n\}$:

$$x \in H \sim \sum_{n \in N} (x, y_n) x_n. \tag{1}$$

It is one of the basic problems of analysis to find conditions on the system $\{x_n\}$ that ensure the convergence of the Fourier series to x in some sense. In applications, the system $\{x_n\}$ is often given as the system of eigenvectors of some operator.

The choice of the coefficients in the expansion (1) is natural: note that if the series $\sum_n c_n x_n$ converges to x in H then, necessarily, $c_n = (x, y_n)$. There are many ways to understand convergence/reconstruction:

- The simplest case (with the exception of orthonormal bases, of course) is a *Riesz basis*: a system $\{x_n\}$ is a Riesz basis if, for any x, we have $x = \sum_n c_n x_n$ (the series converges in the norm) and $A\|\{c_n\}\|_{\ell^2} \leq \|x\| \leq B\|\{c_n\}\|_{\ell^2}$ for some positive constants A, B. Equivalently, $x_n = Te_n$ for an orthonormal basis $\{e_n\}$ and some bounded invertible operator T.
- *Bases with brackets*: there exists a sequence n_k such that $\sum_{n=1}^{n_k} (x, y_n) x_n$ converges to x as $k \to \infty$.
- Existence of a *linear (matrix) summation method* (e.g. Cesàro, Abel–Poisson, etc.): this means that there exists a doubly infinite matrix $(A_{m,n})$ such that $x = \lim_{m \to \infty} \sum_n a_{m,n} S_n(x)$, i.e. some means of the partial sums $S_n(x)$ of the series (1) converge to x.

The following property, known as *hereditary completeness*, may be understood as the weakest form of the reconstruction of a vector x from its Fourier series $\sum_{n \in N} (x, y_n) x_n$.

Definition 3. A complete and minimal system $\{x_n\}_{n \in N}$ in a Hilbert space H is said to be hereditarily complete if, for any $x \in H$, we have

$$x \in \overline{\text{Span}} \{(x, y_n) x_n\}.$$

It is easy to see that hereditary completeness is equivalent to the following property: for any partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, of the index set N, the *mixed system*

$$\{x_n\}_{n \in N_1} \cup \{y_n\}_{n \in N_2} \tag{2}$$

is complete in H. Clearly, hereditary completeness is necessary for the existence of a linear summation method for the series (1) (otherwise, there exists a vector x orthogonal to all partial sums of (1)).

Hereditarily complete systems are also known as *strong Markushevich bases*. We will also say, in this case, that the system admits *spectral synthesis* motivated by the following theorem of Markus [16].

Theorem 4 (Markus). *Let A be a complete compact operator with generalised eigenvectors $\{x_n\}$ and trivial kernel. Then, $A \in \text{Synt}$ if and only if the system $\{x_n\}$ is hereditarily complete.*

Indeed, assume that the system of eigenvectors $\{x_n\}$ is not hereditarily complete and, for some partition $N = N_1 \cup N_2$, the mixed system $\{x_n\}_{n \in N_1} \cup \{y_n\}_{n \in N_2}$ is not complete. The biorthogonal system y_n consists of eigenvectors of the adjoint operator A^*. Then, $E = \overline{\text{Span}}\{y_n : n \in N_2\}^\perp$ is A-invariant and $\{x_n : n \in N_1\} \subset E$ but $E \neq \overline{\text{Span}}\{x_n : n \in N_1\}$.

Note that hereditary completeness includes the requirement that the biorthogonal system $\{y_n\}$ is complete in H, which is by no means automatic. In fact, it is very easy to construct a complete and minimal system whose biorthogonal is not complete.

Example 5. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis in H. Set $x_n = e_1 + e_n$, $n \geq 2$. Then, it is easy to see that $\{x_n\}$ is complete and minimal, while its biorthogonal is clearly given by $y_n = e_n$, $n \geq 2$. Taking direct sums of such examples, one obtains biorthogonal systems with arbitrary finite or infinite codimension.

It is not so trivial to construct a complete and minimal system $\{x_n\}$ with a complete biorthogonal system $\{y_n\}$ that is not hereditarily complete (i.e. the mixed system (2) fails to be complete for some partition of the index set). A first explicit construction was given by Markus (1970). Further results about the structure of nonhereditarily complete systems were obtained by N. Nikolski, L. Dovbysh and V. Sudakov (1977). For an extensive survey of spectral synthesis and hereditary completeness, the reader is referred to [13].

Interesting examples of hereditarily (in)complete systems also appear in papers by D. Larson and W. Wogen (1990), E. Azoff and H. Shehada (1993) and A. Katavolos, M. Lambrou and M. Papadakis (1993) in connection with reflexive operator algebras.

2 Spectral synthesis for exponential systems

Let $e_\lambda(t) = e^{i\lambda t}$ be a complex exponential. For $\Lambda = \{\lambda_n\} \subset \mathbb{C}$, we consider

$$\mathcal{E}(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}$$

as a system in $L^2(-a, a)$. The series $\sum_n c_n e_{\lambda_n}$ are often referred to as *nonharmonic Fourier series*, in contrast to "harmonic" orthogonal series. A good introduction to the subject can be found in [26].

Exponential systems play a most prominent role in analysis and its applications. Geometric properties of exponential systems in $L^2(-a, a)$ were one of the major themes of 20th century harmonic analysis. Let us briefly mention some of the milestones of the theory.

(i) *Completeness of exponential systems.*
This basic problem was studied in the 1930–1940s by N. Levinson and B. Ya. Levin. One of the most important contributions is the famous result of A. Beurling and P. Malliavin (1967), who gave an explicit formula for the radius of completeness of a system $\mathcal{E}(\Lambda)$ in terms of a certain density of Λ. By the radius of completeness of $\mathcal{E}(\Lambda)$, we mean

$$r(\Lambda) = \sup \{a > 0 : \mathcal{E}(\Lambda) \text{ is complete in } L^2(-a, a)\}.$$

A new approach to these (and related) problems and their far-reaching extensions can be found in [9, 10, 17, 18].

(ii) *Riesz bases of exponentials.*
The first results about Riesz bases of exponentials, which go back to R. Paley and N. Wiener (1930s), were of perturbative nature. Assume that the frequencies λ_n are small perturbations of integers,

$$\sup_{n \in \mathbb{Z}} |\lambda_n - n| < \delta.$$

Paley and Wiener showed that $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(-\pi, \pi)$ if $\delta = \pi^{-2}$. It was a longstanding problem to find the best possible δ; finally, M. Kadets (1965) showed that the sharp bound is $1/4$. However, it was clear that one cannot describe all Riesz bases in terms of "individual" perturbations. A complete description of exponential bases in terms of the Muckenhoupt (or Helson–Szegö) condition was given by B. S. Pavlov (1979) and was further extended by S. V. Hruschev, N. K. Nikolski and B. S. Pavlov (see [11] for a detailed account of the problem). Yu. Lyubarskii and K. Seip (1997) extended this description to the L^p-setting. These results revealed the connection of the problem to the theory of singular integrals.

(iii) *Exponential frames* (*sampling sequences*).
A system $\{x_n\}$ in a Hilbert space H is said to be a frame if there exist positive constants A, B such that

$$A \sum_n |(x, x_n)|^2 \leq \|x\|^2 \leq B \sum_n |(x, x_n)|^2,$$

i.e. a generalised Parseval identity holds. Here, we omit the requirement of minimality to gain in "stability" of the reconstruction; there is a canonical choice of coefficients so that the series $\sum_n c_n x_n$ converge to x. If a frame $\{x_n\}$ is minimal then it is a Riesz basis.

Exponential frames were introduced by R. Duffin and A. C. Schaeffer (1952), while their complete description was obtained relatively recently by J. Ortega–Cerdà and K. Seip [22]; this solution involves the theory of de Branges spaces of entire functions, which is to be discussed below. For an extensive review on exponential frames on disconnected sets, see a recent monograph by A. Olevskii and A. Ulanovskii [21].

Synthesis up to codimension 1

The *spectral synthesis* (or *hereditary completeness*) problem for exponential systems was also a longstanding problem in nonharmonic Fourier analysis. Let $\mathcal{E}(\Lambda)$ be a complete and minimal system of exponentials in $L^2(-a, a)$ and let $\{\tilde{e}_\lambda\}$ be the biorthogonal system. It was shown by R. Young (1981) that the biorthogonal system $\{\tilde{e}_\lambda\}$ is always complete.

Problem 6. Is it true that any complete and minimal system of exponentials $\{e_\lambda\}_{\lambda \in \Lambda}$ in $L^2(-a, a)$ is hereditarily complete, i.e. any function $f \in L^2(-a, a)$ belongs to the closed linear span of its "harmonics" $(f, \tilde{e}_\lambda) e_\lambda$?

This question was answered in the negative by the authors jointly with Alexander Borichev [2]. Surprisingly, it turned out, at the same time, that spectral synthesis for exponential systems always holds up to at most one-dimensional defect.

Theorem 7. There exists a complete and minimal system of exponentials $\{e_\lambda\}_{\lambda \in \Lambda}$, $\Lambda \subset \mathbb{R}$, in $L^2(-\pi, \pi)$ that is not hereditarily complete.

Thus, in general, there exists no linear summation method for nonharmonic Fourier series $\sum_{\lambda \in \Lambda}(f, \tilde{e}_\lambda) e_\lambda$ associated to a complete and minimal exponential system.

It is worth mentioning that "bad" sequences Λ can be regularly distributed, e.g. be a bounded perturbation of integers: in Theorem 7 one can choose a uniformly separated sequence Λ so that $|\lambda_n - n| < 1$, $n \in \mathbb{Z}$.

Theorem 8. If the system of exponentials $\{e_\lambda\}_{\lambda \in \Lambda}$ is complete and minimal in $L^2(-a, a)$ then, for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, the corresponding mixed system has defect at most 1, that is,

$$\dim \left(\{e_\lambda\}_{\lambda \in \Lambda_1} \cup \{\tilde{e}_\lambda\}_{\lambda \in \Lambda_2} \right)^\perp \leq 1.$$

It turns out that incomplete mixed systems are always highly asymmetric. Given a complete and minimal system $\mathcal{E}(\Lambda)$ in $L^2(-\pi, \pi)$, it is natural to expect that "in the main" Λ is similar to $\mathbb{Z}$. This is indeed the case. As a very rough consequence of more delicate results (such as the Cartwright–Levinson theorem), let us mention that Λ always has density 1:

$$\lim_{r \to \infty} \frac{n_r(\Lambda)}{2r} = 1,$$

where $n_r(\Lambda)$ is the usual counting function, $n_r(\Lambda) = \#\{\lambda \in \Lambda, |\lambda| \leq r\}$. Analogously, one can define the upper density $D_+(\Lambda)$:

$$D_+(\Lambda) = \limsup_{r \to \infty} \frac{n_r(\Lambda)}{2r}.$$

Theorem 9. Let $\Lambda \subset \mathbb{C}$, let the system $\mathcal{E}(\Lambda)$ be complete and minimal in $L^2(-a, a)$ and let the partition $\Lambda = \Lambda_1 \cup \Lambda_2$ satisfy $D_+(\Lambda_2) > 0$. Then, the mixed system $\{e_\lambda\}_{\lambda \in \Lambda_1} \cup \{\tilde{e}_\lambda\}_{\lambda \in \Lambda_2}$ is complete in $L^2(-a, a)$.

Theorem 9 shows that there is a strong asymmetry between the systems of reproducing kernels and their biorthogonal systems. The completeness of a mixed system may fail only when we take a sparse (but infinite!) subsequence Λ_1.

We conclude this subsection with one open problem.

Problem 10. Given a hereditarily complete system of exponentials, does there exist a linear summation method for the corresponding nonharmonic Fourier series?

Translation to the entire functions setting

A classical approach to a completeness problem is to translate it (via a certain integral transform) to a uniqueness problem in some space of analytic functions. In the case of exponentials on an interval, the role of such a transform is played by the standard Fourier transform $\mathcal{F}$,

$$(\mathcal{F}f)(z) = \frac{1}{2\pi} \int_{-a}^{a} f(t) e^{-izt} dt.$$

By the classical Paley–Wiener theorem, $\mathcal{F}$ maps $L^2(-\pi, \pi)$ unitarily onto the space

$$\mathcal{PW}_a = \{F - \text{entire}, \ F \in L^2(\mathbb{R}), \ |F(z)| \leq Ce^{a|z|}\}.$$

Paley–Wiener space $\mathcal{P}W_a$ (also known as the space of bandlimited functions of bandwidth $2a$) plays a remarkably important role in signal processing.

The Fourier transform maps exponentials in $L^2(-a, a)$ to the cardinal sine functions:

$$(\mathcal{F}e_{\bar\lambda})(z) = k_\lambda(z) = \frac{\sin a(z - \bar\lambda)}{\pi(z - \bar\lambda)}.$$

Note that the functions k_λ are the *reproducing kernels* in Paley–Wiener space $\mathcal{P}W_a$, i.e. the function k_λ generates the evaluation functional at the point λ:

$$F(\lambda) = (F, k_\lambda), \qquad F \in \mathcal{P}W_a.$$

In particular, the orthogonal expansion $f = \sum_{n \in \mathbb{Z}} c_n e^{int}$ in $L^2(-\pi, \pi)$ becomes

$$F(z) = \sum_{n \in \mathbb{Z}} c_n \frac{\sin \pi(z - n)}{\pi(z - n)}, \quad c_n = F(n), \qquad (3)$$

the classical Shannon–Kotelnikov–Whittacker sampling formula.

Moreover, this translation makes it possible to find an explicit form of the biorthogonal system, which is not possible when staying inside $L^2(-a, a)$. Let $\{k_\lambda\}_{\lambda \in \Lambda}$ be a complete and minimal system in $\mathcal{P}W_a$. Its biorthogonal system may then be obtained from one function G_Λ known as the generating function of Λ. This is an entire function with the zero set Λ, which can be defined by the formula

$$G_\Lambda(z) = \lim_{R \to \infty} \prod_{\lambda \in \Lambda, |\lambda| < R} \left(1 - \frac{z}{\lambda}\right),$$

with the properties that $G_\Lambda \notin \mathcal{P}W_a$ (otherwise this would be a contradiction to completeness) but $\frac{G_\Lambda}{z - \lambda} \in \mathcal{P}W_a$ for any $\lambda \in \Lambda$ by the minimality of the system $\{k_\lambda\}_{\lambda \in \Lambda}$. The biorthogonal system is then given by

$$g_\lambda(z) = \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z - \lambda)}. \qquad (4)$$

Thus, with any function $F \in \mathcal{P}W_a$, we can associate two (formal) Fourier series expansions:

$$F \in \mathcal{P}W_a \sim \sum_{\lambda \in \Lambda} c_\lambda \frac{\sin a(z - \bar\lambda)}{\pi(z - \bar\lambda)}, \qquad c_\lambda = (F, g_\lambda),$$

$$F \in \mathcal{P}W_a \sim \sum_{\lambda \in \Lambda} F(\lambda) \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z - \lambda)}.$$

The first series is an expansion with respect to cardinal sine functions while the second one is a Lagrange-type interpolation series.

Our results on exponential systems can be reformulated for reproducing kernels of Paley–Wiener space: *for any complete and minimal system $\{k_\lambda\}_{\lambda \in \Lambda}$ and any partition $\Lambda = \Lambda_1 \cup \Lambda_2$,*

$$\dim\left(\{k_\lambda\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2}\right)^\perp \leq 1 \qquad (5)$$

but the defect 1 is possible.

3 Spectral synthesis in de Branges spaces and applications

Preliminaries on de Branges spaces

The theory of Hilbert spaces of entire functions was created by L. de Branges at the end of the 1950s and the beginning of the 1960s. It was the main tool in his famous solution of the direct and inverse spectral problems for two-dimensional canonical systems. These are second order ODEs that include, as particular cases, the Schrödinger equation on an interval, the Dirac equation and Krein's string equation. For the general theory of de Branges spaces, we refer to the original monograph by de Branges [8]; for information relating to inverse problems, see [23, 24].

De Branges spaces proved to be highly nontrivial and are interesting objects from the point of view of function theory. Surprisingly, they appear to be unavoidable in substantially different branches of analysis, for example:

- Polynomial approximations on the real line.
- Orthogonal polynomials and random matrix theory (see, for example, [15]).
- Model (backward shift invariant) subspaces of Hardy space: $K_\Theta = H^2 \ominus \Theta H^2$, where H^2 is Hardy space and Θ is an inner function (for a discussion of this relation, see, for example, [9]).
- Functional models for different classes of linear operators [5, 12].

and even

- Analytic number theory, the Riemann Hypothesis and Dirichlet L-functions [14].

There are equivalent ways to introduce de Branges spaces: an axiomatic approach or a definition in terms of the generating Hermite–Biehler entire function. Here, we will not go into details, instead confining ourselves to an equivalent representation of de Branges spaces via *spectral data*. This representation of a de Branges space in terms of a weighted Cauchy transform (which can already be found in the work of de Branges) turns out to be a very useful tool; it relates the study of de Branges spaces with singular integral operators (see, for example, [6]).

Let $T = \{t_n\}_{n \in N} \subset \mathbb{R}$ be an increasing sequence (one-sided or two sided, the index set being a subset of $\mathbb{Z}$) such that $|t_n| \to \infty$, $|n| \to \infty$. Let $\mu = \sum_n \mu_n \delta_{t_n}$ be a measure supported by T such that $\sum_n (t_n^2 + 1)^{-1} \mu_n < \infty$. Consider the class of entire functions

$$\mathcal{H} = \left\{F : \; F(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n}\right\}, \qquad (6)$$

where A is some (fixed) entire function that is real on $\mathbb{R}$ and vanishes exactly on T, and $\{c_n\} \in \ell^2$.

Set $\|F\|_{\mathcal{H}} = \|\{c_n\}\|_{\ell^2}$. With this norm, $\mathcal{H}$ is a reproducing kernel Hilbert space. It is an axiomatic de Branges space and any de Branges space can be represented in this way.

We call the pair (T, μ) the *spectral data* for space $\mathcal{H}$. Of course, formally, the space also depends on the choice of the entire function A but spaces with the same spectral data and different functions A are canonically isomorphic.

Example 11. If $T = \mathbb{Z}$, $\mu_n = 1$ and $A(z) = \sin \pi z$ then $\mathcal{H} = \mathcal{P}W_\pi$. The corresponding representation of the elements

of $\mathcal{H}$ coincides with the Schannon–Kotelnikov–Whittaker sampling formula (3).

An important feature of de Branges spaces is that they have the so-called *division property*: if a function f in a de Branges space $\mathcal{H}$ vanishes at some point $w \in \mathbb{C}$ then the function $f(z)/(z - w)$ also belongs to $\mathcal{H}$. Another characteristic property of a de Branges space is existence of *orthogonal bases of reproducing kernels*. It is clear from the definition that the functions $A(z)/(z - t_n)$ form an orthogonal basis in $\mathcal{H}$ and are reproducing kernels of $\mathcal{H}$ up to normalisation.

Spectral synthesis problem and its solution

Let $\{k_\lambda\}$ be a complete and minimal system of reproducing kernels in a de Branges space $\mathcal{H}$. As in the Paley–Wiener space, its biorthogonal system is given by formula (4) for some appropriate generating function G_Λ. However, in contrast to the Paley–Wiener case, the biorthogonal system in general need not be complete (Baranov, Belov, 2011).

We will say that a de Branges space has the *spectral synthesis property* if any complete and minimal system of reproducing kernels with the complete biorthogonal system (this assumption is included) is also hereditarily complete, i.e. all mixed systems are complete. In [3], the following problem is addressed.

Problem 12. Which de Branges spaces have the spectral synthesis property? If a space does not have the spectral synthesis property, what is the possible size of the defect for a mixed system?

Why is hereditary completeness of reproducing kernels in de Branges spaces an interesting and significant topic? There are several motivations for that:

- Relation to exponential systems and nonharmonic Fourier series as discussed above.
- N. Nikolski's question: whether there exist nonhereditarily complete systems of reproducing kernels in model spaces $K_\Theta = H^2 \ominus \Theta H^2$? Note that de Branges spaces form an important special subclass of model spaces.
- Spectral synthesis for rank one perturbations of self-adjoint operators (see Subsection 3).

In [3], a complete description of de Branges spaces with the spectral synthesis property was obtained. To state it, we need one more definition. An increasing sequence $T = \{t_n\}$ is said to be *lacunary* (or *Hadamard lacunary*) if

$$\liminf_{t_n \to \infty} \frac{t_{n+1}}{t_n} > 1, \qquad \liminf_{t_n \to -\infty} \frac{|t_n|}{|t_{n+1}|} > 1,$$

i.e. the moduli of $|t_n|$ tend to infinity at least exponentially.

Theorem 13. Let $\mathcal{H}$ be a de Branges space with spectral data (T, μ). Then, $\mathcal{H}$ has the spectral synthesis property if and only if one of the following conditions holds:

(i) $\sum_n \mu_n < \infty$.

(ii) The sequence $\{t_n\}$ is lacunary and, for some $C > 0$ and any n,

$$\sum_{|t_k| \leq |t_n|} \mu_k + t_n^2 \sum_{|t_k| > |t_n|} \frac{\mu_k}{t_k^2} \leq C\mu_n. \qquad (7)$$

Note that condition (3) implies that the sequence of masses μ_n also grows at least exponentially.

Now we turn to the problem of the size of the defect (i.e. the dimension of the complement to a mixed system). It turns out that one can construct examples of systems of reproducing kernels with large or even infinite defect.

Theorem 14. For any increasing sequence $T = \{t_n\}$ with $|t_n| \to \infty$, $|n| \to \infty$, and for any $N \in \mathbb{N} \cup \{\infty\}$, there exists a measure μ such that, in the de Branges space with spectral data (T, μ), there exists a complete and minimal system of reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ whose biorthogonal system $\{g_\lambda\}_{\lambda \in \Lambda}$ is also complete but, for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$,

$$\dim (\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{k_\lambda\}_{\lambda \in \Lambda_2})^\perp = N.$$

The key role in this construction plays the balance between the "spectrum" $\{t_n\}$ and the masses $\{\mu_n\}$. If $\sum_n \mu_n = \infty$, but there exists a subsequence t_{n_k} of T such that $\sum_k t_{n_k}^{2N-2} \mu_{n_k} < \infty$, then, in the corresponding de Branges space, one has mixed systems with any defect up to N. Conversely, the estimate $\mu_n \geq |t_n|^{-M}$ for some $M > 0$ and all n implies an estimate from the above in terms of M for the defect.

Spectral theory of rank one perturbations of compact self-adjoint operators

Let A be a compact self-adjoint operator in a separable Hilbert space H with simple point spectrum $\{s_n\}$ and trivial kernel. In other words, A is the simplest infinite-dimensional operator one can imagine, diagonalisable by the classical Hilbert–Schmidt theorem. Surprisingly, the spectral theory of rank one perturbations of such operators is already highly nontrivial.

For $a, b \in H$, consider the rank one perturbation L of A,

$$L = A + a \otimes b, \qquad Lf = Af + (f, b)a, \qquad f \in H.$$

For example, one may obtain examples of rank one perturbations changing one boundary condition in a second order differential equation.

In [5], the spectral properties of rank one perturbations are studied via a functional model. Several similar functional models for rank one perturbations (or close classes of operators) have been developed, e.g. by V. Kapustin (1996) and G. Gubreev and A. Tarasenko (2010). Let us present the idea of this model without going into technicalities.

For $t_n = s_n^{-1}$, consider a de Branges space $\mathcal{H}$ with spectral data (T, μ), where μ is some measure supported by T. Let G be an entire function such that $G \notin \mathcal{H}$ but $G(z)/(z - w) \in \mathcal{H}$ whenever $G(w) = 0$. This means that the function G has growth just slightly larger than is possible for the elements of $\mathcal{H}$. Assume also that $G(0) = 1$. Consider the linear operator

$$(MF)(z) = \frac{F(z) - F(0)G(z)}{z}, \qquad F \in \mathcal{H}. \qquad (8)$$

It is easily seen that M is a rank one perturbation of a compact self-adjoint operator with spectrum $t_n^{-1} = s_n$. The functional model theorem from [5] proves that any rank one perturbation of A is unitary equivalent to a model operator M of the form (8) for some de Branges space $\mathcal{H}$ and function G. Therefore, while the spectrum $T = \{t_n\}$ is fixed, the masses μ_n and the function G are free parameters of the model.

It is easy to see that the eigenfunctions of M are of the form $G(z)/(z - \lambda)$ for $\lambda \in Z_G$, where $Z(G)$ denotes the zero set of G. The point spectrum of M is thus given by $\{\lambda^{-1} : \lambda \in Z_G\}$. Multiple zeros of G correspond to generalised eigenvectors but we assume, for simplicity, that G has simple zeros only. Thus, the system of eigenfunctions of the rank one perturbation L is unitary equivalent to a system of the form $\{g_\lambda\}_{\lambda \in \Lambda}$, as in (4), while eigenfunctions of L^* (which is also a rank one perturbation of A) correspond to a system $\{k_\lambda\}_{\lambda \in \Lambda}$ of reproducing kernels in $\mathcal{H}$.

Thus, we relate the spectral properties of rank one perturbations to geometric properties of systems of reproducing kernels (in view of the symmetry, we interchange the roles of L and L^*):

- Completeness of $L \iff$ completeness of a system of reproducing kernels $\{k_\lambda\}$ in $\mathcal{H}(E)$.
- Completeness of $L^* \iff$ completeness of the system biorthogonal to the system of reproducing kernels.
- Spectral synthesis for $L \iff$ hereditary completeness of $\{k_\lambda\}_{\lambda \in \Lambda}$, i.e. for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the system $\{k_\lambda\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2}$ is complete in $\mathcal{H}$.

The results of Subsection 3 lead to a number of striking examples for rank one perturbations of compact self-adjoint operators. These examples show that the spectral theory of such perturbations is a rich and complicated subject that is far from being completely understood.

Theorem 15 (Baranov, Yakubovich)**.** For any compact self-adjoint operator A with simple spectrum, there exists its rank one perturbation $L = A + a \otimes b$ such that:

(i) Ker $L =$ Ker $L^* = 0$ and L is complete but the eigenvectors of L^* span a subspace with infinite codimension.

(ii) L and L^* are complete but L admits no spectral synthesis with infinite defect (i.e. there exists an L-invariant subspace E such that the generalised vectors of L that belong to E have infinite codimension in E).

For more results about completeness and spectral synthesis of rank one perturbations, see [5]. One may also ask for which compact self-adjoint operators A there exists a rank one perturbation that is a Volterra operator (i.e. the spectrum can be destroyed by a rank one perturbation). This problem was solved in [4], where it was shown that the spectrum $\{s_n\}$ is "destructible" if and only if $t_n = s_n^{-1}$ form the zero set of an entire function of some special class introduced by M. G. Krein (1947).

4 Spectral synthesis in Fock-type spaces

Classical Bargmann–Fock space

Fock-type spaces form another important class of Hilbert spaces of entire functions. In contrast to de Branges spaces, where the norm is defined as an integral over the real axis (with respect to some continuous or discrete measure), in Fock-type spaces the norm is defined as an area integral. Classical Fock space $\mathcal{F}$ (also known as Bargmann, Segal–Bargmann or Bargmann–Fock space) is defined as the set of all entire functions F for which

$$\|F\|_{\mathcal{F}}^2 := \frac{1}{\pi} \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < \infty,$$

where m stands for the area Lebesgue measure. This space (as well as its multi-dimensional analogues) plays a most prominent role in theoretical physics, serving as a model of the phase space of a particle in quantum mechanics. It also appears naturally in time-frequency analysis and Gabor frame theory. There is a canonical unitary map from $L^2(\mathbb{R})$ to $\mathcal{F}$ (the Bargmann transform), which plays a role similar to that of the Fourier transform in the Paley–Wiener space setting. Note that, clearly, all functions in $\mathcal{F}$ are of order at most 2 and satisfy the estimate $|F(z)| \le C \exp(\pi|z|^2/2)$ (which can be slightly refined).

The reproducing kernels of $\mathcal{F}$ are the usual complex exponentials, $k_\lambda(z) = e^{\pi \bar\lambda z}$. Moreover, the Bargmann transform of the phase-space shift of the Gaussian, i.e. of the function $e^{2\pi i \eta t} e^{-\pi(t-\xi)^2}$, coincides up to normalisation with $e^{\pi \lambda z}$, $\lambda = \xi + i\eta$. Thus, geometric properties (e.g. spectral synthesis) of the phase-space shifts of the Gaussian are equivalent to the corresponding properties of the exponentials in Fock space.

Radial Fock-type spaces

Considering radial weights differing from the Gaussian weight, one obtains a wide class of Hilbert spaces of entire functions. Namely, for a continuous function $\varphi : [0, \infty) \to (0, \infty)$, we define the radial Fock-type space as

$$\mathcal{F}_\varphi = \left\{ F \text{ entire } : \|F\|_{\mathcal{F}_\varphi}^2 := \frac{1}{\pi} \int_{\mathbb{C}} |F(z)|^2 e^{-\varphi(|z|)} dm(z) < \infty \right\}.$$

We always assume that $\log r = o(\varphi(r))$, $r \to \infty$, to exclude finite-dimensional spaces.

Any Fock-type space is a reproducing kernel Hilbert space. It was shown by K. Seip [25] that in classical Fock space there are no Riesz bases of reproducing kernels. Recently, A. Borichev and Yu. Lyubarskii [7] showed that Fock-type spaces with slowly growing weights $\varphi(r) = (\log r)^\gamma$, $\gamma \in (1, 2]$, have Riesz bases of reproducing kernels corresponding to real points and, thus, can be realised as de Branges spaces with equivalence of norms (this is clear from the representation of de Branges spaces via their spectral data). Moreover, $\varphi(r) = (\log r)^2$ is in a sense the sharp bound for this phenomenon. Namely, it is shown in [7] that if $(\log r)^2 = o(\varphi(r))$, $r \to \infty$, and φ has a certain regularity then $\mathcal{F}_\varphi$ has no Riesz bases of reproducing kernels.

In view of the examples above, one may ask which de Branges spaces can be realised as radial Fock-type spaces, that is, there is an area integral norm that is equivalent to the initial de Branges space norm. Surprisingly, it turns out that this class of de Branges spaces exactly coincides with the class of de Branges spaces (ii) with the spectral synthesis property in Theorem 13.

Theorem 16 (Baranov, Belov, Borichev)**.** Let $\mathcal{H}$ be a de Branges space with spectral data (T, μ). Then, the following statements are equivalent:

(i) There exists a Fock-type space $\mathcal{F}_\varphi$ such that $\mathcal{H} = \mathcal{F}_\varphi$.

(ii) $\mathcal{H}$ is rotation invariant, that is, the operator $R_\theta : f(z) \mapsto f(e^{i\theta} z)$ is a bounded invertible operator in $\mathcal{H}$ for some (all) $\theta \in (0, \pi)$.

(iii) The sequence T is lacunary and (7) holds.

Thus, in the space $\mathcal{F}_\varphi$ with $\varphi(r) = (\log r)^\gamma$, $\gamma \in (1, 2]$, any complete and minimal system of reproducing kernels is hereditarily complete. We also mention that Riesz bases in some de Branges spaces with lacunary spectral data have been described by Yu. Belov, T. Mengestie and K. Seip [6].

Synthesis in Fock space and the Newman–Shapiro problem

Now we turn to the case of classical Fock space $\mathcal{F}$. Though it has no Riesz bases of reproducing kernels, there exist many complete and minimal systems of reproducing kernels. The two-dimensional lattice $\mathbb{Z} + i\mathbb{Z}$ plays for Fock space a role similar to the role of the lattice $\mathbb{Z}$ for Paley–Wiener space $\mathcal{PW}_\pi$. In particular, if $\Lambda = (\mathbb{Z} + i\mathbb{Z}) \setminus \{0\}$ then $\{k_\lambda\}_{\lambda \in \Lambda} = \{e^{\pi \bar{\lambda} z}\}_{\lambda \in \Lambda}$ is a complete and minimal system, whose generating function is the Weierstrass sigma-function (up to the factor z). The second author proved (2015) the following Young-type theorem for Fock space.

Theorem 17 (Belov). For any complete and minimal system of reproducing kernels (i.e. exponentials) in $\mathcal{F}$, its biorthogonal system is also complete.

On the other hand, the first author recently proved that classical Fock space has no spectral synthesis property. Equivalently, this means that there exist nonhereditarily complete systems of phase-space shifts of the Gaussian in $L^2(\mathbb{R})$. At the same time, there are good reasons to believe that there exists a universal upper bound for the defects of mixed systems. The proofs of these results are to appear elsewhere.

Theorem 18 (Baranov). There exist complete and minimal systems $\{e^{\pi \bar{\lambda} z}\}_{\lambda \in \Lambda}$ of reproducing kernels in $\mathcal{F}$ that are not hereditarily complete, that is, for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the mixed system $\{e^{\pi \bar{\lambda} z}\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2}$ is not complete in $\mathcal{F}$.

Moreover, this example of a nonhereditarily complete system of reproducing kernels in $\mathcal{F}$ admits the following reformulation. Given a function $G \in \mathcal{F}$, let us denote by $\mathcal{R}_G$ the subspace of $\mathcal{F}$ defined as

$$\mathcal{R}_G = \{GF : GF \in \mathcal{F}, F - \text{entire}\}.$$

Thus, $\mathcal{R}_G$ is the (closed) subspace in $\mathcal{F}$ that consists of functions in $\mathcal{F}$ that vanish at the zeros of G with appropriate multiplicities. The example of Theorem 18 shows that there exists $G \in \mathcal{F}$ such that $z^n G \in \mathcal{F}$ for any $n \geq 1$ and $\overline{\text{Span}}\{z^n G : n \in \mathbb{Z}_+\} \neq \mathcal{R}_G$.

We stated this result to compare it with a longstanding problem in Fock space that has a similar form. This problem was posed in the 1960s by D. J. Newman and H. S. Shapiro [19], who were motivated by an old paper of E. Fisher (1917) on differential operators. Assume that a function G from $\mathcal{F}$ is such that $e^{wz} G \in \mathcal{F}$ for any $w \in \mathbb{C}$, that is, its growth is smaller than the critical one. One may define on the linear span of all exponentials the (unbounded) multiplication operator $M_G F = GF$. It is natural to expect that the adjoint of M_G will then be given by an infinite order differential operator $G^*(\frac{\partial}{\partial z})$, where $G^*(z) = \overline{G(\bar{z})}$. Newman and Shapiro showed that the positive answer to this question is equivalent to the positive solution of the following problem.

Problem 19. Let $G \in \mathcal{F}$ be such that $e^{wz} G \in \mathcal{F}$ for any $w \in \mathbb{C}$. Is it true that

$$\overline{\text{Span}}\{e^{wz} G : w \in \mathbb{C}\} = \mathcal{R}_G\ ?$$

Newman and Shapiro showed that the equality holds in the case where G is a linear combination of exponential monomials. In our example in Theorem 18, however, the function G admits multiplication by polynomials in Fock space but not multiplication by the exponents. Thus, the spectral synthesis problem of Newman and Shapiro remains open.

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Anton Baranov [anton.d.baranov@gmail.com] is a professor of mathematics at St. Petersburg State University. He works in harmonic analysis, function theory and operator theory, and is particularly interested in the interaction between these areas, such as the theory of holomorphic function spaces and functional models. He has held invited professor positions at Aix-Marseille University and University Paris-Est. He received the Young Mathematician Prize of the Saint Petersburg Mathematical Society and was an invited speaker at 7ECM in Berlin.

Yurii Belov [j_b_juri_belov@mail.ru] is an associate professor of mathematics at St. Petersburg State University. He works in complex and harmonic analysis and function theory, and is particularly interested in structural properties of Hilbert spaces of entire functions. He has held an invited professor position at University Paris-Est. He received the Young Mathematician Prize of the Saint Petersburg Mathematical Society and the Euler Award from the Government of Saint Petersburg.