Solved and Unsolved Problems

Michael T. Rassias (University of Zürich, Switzerland)

God created the natural numbers. The rest is the work of man.

Leopold Kronecker (1823–1891)

The column *Solved and Unsolved Problems* will continue presenting six proposed problems and two open problems, as has been done over recent years. The set of proposed and open problems in each issue will be devoted to a specific field of mathematics. In every issue featuring this column, solutions will be presented to the proposed problems from the previous issue along with the names of solvers. Possible progress toward the solution of any of the open problems proposed in this column will also be featured. The goal of the *Solved and Unsolved Problems* column is to provide a series of intriguing proposed problems and open problems ranging over several areas of mathematics. Effort will also be made to present problems of an interdisciplinary flavour.

The column in this issue is devoted to number theory. As is well known, number theory is one of the oldest and most vibrant areas of pure mathematics. Over the last few decades, it has also found important applicability in various scientific domains such as cryptography, coding theory, theoretical computer science and even nuclear physics and quantum information theory.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

171. Prove that every integer can be written in infinitely many ways in the form

$$\pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2$$

for some choices of signs + and −.

(Dorin Andrica, Babeşş Bolyai University, Cluj-Napoca, Romania)

172. Show that, for every integer $n \geq 1$ and every real number $a \geq 1$, one has

$$\frac{1}{2n} \leq \frac{1}{n^{a+1}} \sum_{k=1}^{n} k^a - \frac{1}{a+1} < \frac{1}{2n}\left(1 + \frac{1}{2n}\right)^a.$$

(László Tóth, University of Pécs, Hungary)

173. Let $c_n(k)$ denote the Ramanujan sum, defined as the sum of kth powers of the primitive nth roots of unity. Show that, for any integers n, k, a with $n \geq 1$,

$$\sum_{d|n} c_d(k) a^{n/d} \equiv 0 \pmod{n}.$$

(László Tóth, University of Pécs, Hungary)

174. Prove, disprove or conjecture:

1. There are infinitely many primes with at least one 7 in their decimal expansion.
2. There are infinitely many primes where 7 occurs at least 2017 times in their decimal expansion.
3. There are infinitely many primes where at most one-quarter of the digits in their decimal expansion are 7s.
4. There are infinitely many primes where at most half the digits in their decimal expansion are 7s.
5. There are infinitely many primes where 7 does not occur in their decimal expansion.

Note. Let p be a prime. Then, the decimal expansion of $1/p$ is often called the "decimal expansion of p".

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

175. Show that there is an infinite sequence of primes $p_1 < p_2 < p_3 < \cdots$ such that p_2 is formed by appending a number in front of p_1, p_3 is formed by appending a number in front of p_2 and so on. For example, we could have $p_1 = 3$, $p_2 = 13$, $p_3 = 313$, $p_4 = 3313$, $p_5 = 13313, \ldots$. Of course, you might have to add more than one digit at a time. Find a bound on how many digits you need to add to ensure it can be done.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

176. Consider all pairs of integers x, y with the property that $xy - 1$ is divisible by the prime number 2017. If three such integral pairs lie on a straight line on the xy−plane, show that both the vertical distance and the horizontal distance of at least two of such three integral pairs are divisible by 2017.

(W. S. Cheung, Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong)

II Two new open problems (on ζ-functions) by Preda Mihăilescu, Mathematisches Institut, Göttingen, Germany

Let K be a number field, let $I(K)$ denote the set of integral ideals of K, including the trivial ideal $1 = O(K)$, let $P(K) \subset I(K)$ denote the principal ideals and let $C(K)$ be the ideal class group of K. Denote by $N_K = N$ the absolute norm $\mathbf{N}_{K/\mathbb{Q}}$ and let $d = [K : \mathbb{Q}]$. The Dedekind ζ-function of K is

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I} \frac{1}{|N_K \mathfrak{a}|^s}. \tag{1}$$

If $K = \mathbb{Q}$ then

$$\zeta_K(s) = \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

is the Riemann ζ-function. More precisely, the Dirichlet series above define the respective ζ-functions on the half plane $\mathbb{H}_1 = \{s \in \mathbb{C} : \Re(s) > 1\}$, on which the series are absolutely convergent. They have a pole at $s = 1$ and it is proved by means of Mellin transforms that they have an analytic continuation to $\mathbb{C}$, with no other singularity except for $s = 1$. The properties of the $\zeta_K(s)$ have been investigated by a series of classical mathematicians, including Dirichlet, Weierstraß and Hecke. We refer to Lang's Algebraic Number Theory [La], Chapters V–VIII, for a review of the classical results on $\zeta_K(s)$. Let $\mathfrak{K} \in C(K)$ be a class.

The counting function $J(\mathfrak{K}, t)$ for $t \in \mathbb{R}_{>1}$ plays an essential role in this context. It counts the number of ideals $\mathfrak{a} \in I(K) \cap \mathfrak{K}$ that have norm less than t. This is done by choosing some fixed ideal $\mathfrak{b} \in \mathfrak{K}^{-1}$ and counting the number of ideals $(\alpha) \in P(K) \cap \mathfrak{b}$ that have norm bounded by $|N(\alpha)| < t|N(\mathfrak{b})|$. It is an elementary fact (proved in [La]) that these ideals are in one-to-one correspondence to the ideals of $\mathfrak{K}$ with norm less than t. Let $E = O^\times(K)$ be the global units; certainly, for $\alpha \in O(K)$, the principal ideal $(\alpha) \in P(K)$ is generated by any element of the orbit αE of α under the action of the units by multiplication. We are thus reduced to the problem of counting orbits of numbers $\alpha \in \mathfrak{b}$ under the action of the units. Here enters the geometry of numbers. For details of the classical estimates, we refer the reader to any detailed deduction of the classical results in any book on algebraic number theory that also treats analytical results – the account of Lang is a possible example.

Briefly, the numbers of the field K have two representations in $\mathbb{R}^{r+1}$, with $r = r_1 + r_2 - 1$ the Dirichlet rank of the units. The first representation is $\mu : \mathbb{K}^\times \to \mathbb{R}^{r+1}$ via $x \mapsto (|\sigma_i(x)|^{\delta_i})_{i=1}^{r+1}$, with $(\sigma_i)_{i=1}^{r_1}$ an enumeration of the real embeddings of K and $(\sigma_j)_{j=r_1+1}^{r+1}$ an enumeration of representatives of pairs of complex conjugate embeddings; the exponents are $\delta_i = 1$ for real embeddings and $\delta_j = 2$ for complex embeddings. The map μ is continued by an additive one $\lambda : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$, defined by $\lambda(\mu(x))_k = \log(|\mu(x)_k|)$ for $k = 1, 2, \ldots, r$ and $\lambda(\mu(x))_{r+1} = |N(x)|^{1/d}$. The fundamental classical result deduced by investigating $J(\mathfrak{K}, t)$ under these maps is

$$J(\mathfrak{K}, t) = \rho_K t + O(t^{1-1/d}). \tag{2}$$

The constant ρ_K is completely determined in terms of the data of the field, which are Δ, R, w – the discriminant, the regulator and the number of roots of unity of the field respectively. It is independent of $\mathfrak{K}$ and its value is, with these notations,

$$\rho_K = \frac{2^{r+1}\pi^{r_2} R}{w\sqrt{\Delta}}.$$

The order of magnitude of the error term is determined by a crude argument involving the fact that the fundamental domain $D(1) \subset \mathbb{R}^{r+1}$ used for estimating $J(\mathfrak{K}, t)$ is Lipschitz-parametrisable. One can rephrase the formula above by stating that there certainly exists some constant $\gamma_K(\mathfrak{K})$, depending only on K and possibly also on the class $\mathfrak{K}$, such that

$$|J(\mathfrak{K}, t) - \rho_K t| \leq \gamma_K(\mathfrak{K}) \cdot t^{1-1/d}, \quad \text{for } t > \Delta.$$

It is important to choose a lower bound for t in order to obtain an accurate order of magnitude but the bound Δ chosen in our definition is not stringent. One may expect, for reasons discussed in the Remarks below, that these constants are quite small. However, the present methods of estimates, which have only recently been worked out by van Order and Murty [MO] to the effect of obtaining explicit bounds on $\gamma_K(\mathfrak{K})$, yield excessively large values for the bound. We shall make the definition of our constant uniform to make it independent of the class and then state our first problem, which is a conjecture. We define the constant γ_K by

$$\gamma_K := \inf_{t > \sqrt{\Delta}, \mathfrak{K} \in C(K)} \left\{ \gamma \in \mathbb{R}_{>0} : |J(\mathfrak{K}, t) - \rho_K t| \leq \gamma \cdot t^{1-1/d} \right\}. \tag{3}$$

We define the surface of the units as follows: for a fundamental system of units $u_i \in E(K)$, we let $S(\vec{u})$ be the surface of the fundamental parallelepiped of the lattice spanned by the vectors $w_i := \lambda(\mu(u_i))$. The surface $S(E(K)) = \inf_{\vec{u}} S(\vec{u})$, the infimum over all the fundamental systems of units of K.

177*. We keep the notation introduced above, in particular the notation in (2) and (3).

(i) Prove that $\gamma_K = c_1 R^{a_1} \Delta^{a_2} + c_2 S(E)^{b_1} \cdot \Delta^{b_2}$, with constants $c_1, c_2 > 0$ and powers $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, which do not depend on the extension degree d.

(ii) Prove that there is an additional constant $0 < C < 1$ such that

$$|J(\mathfrak{K}, t) - \rho_K t| > C\gamma_K t^{1-1/d}$$

for all $t > \Delta$.

We continue our investigation of the counting function J for arbitrary number fields K with a problem on the geometry of numbers. For a given class $\mathfrak{K}$, one can consider the lattices $L_\mathfrak{a}$ spanned by some ideal $\mathfrak{a} \in \mathfrak{K}$ as a $\mathbb{Z}$-module in Minkowski space. We are interested in determining how close such a lattice can come to orthonormal lattices, if we allow $\mathfrak{a}$ to take all the ideals in $\mathfrak{K}$ as its value. The following definition will introduce quantitative measures for the "distance" of a lattice to an orthonormal one. Let $\Lambda \subset \mathbb{R}^n$ be a full lattice, let $(v_i)_{i=1}^n \subset \mathbb{R}^n$ be a spanning set of generators and let A_v be the matrix with these vectors as columns. Let the Euclidean norm of a matrix $B = (b_{i,j})_{i,j=1}^n$ be the norm $\|B\| = \sqrt{\sum_{i,j} b_{i,j}^2}$ and let B^T denote the transpose. Then we define the *orthonormality defect* of this base by

$$\omega_v(\Lambda) = \inf_{\lambda \in \mathbb{R}_+} \|A \cdot A^T - \lambda I\|.$$

The orthonormality defect of the lattice is defined by $\omega(\Lambda) := \inf_v \omega_v(\Lambda)$, the infimum being over all bases of Λ.

Now, let a class $\mathfrak{K} \subset O(K)$ be fixed and $\mathfrak{b} \in \mathfrak{K}^{-1} \cap I(K)$ be any integral ideal. The image of $\mathfrak{b}$ under the map μ is a lattice $L_\mathfrak{b} \subset \mathbb{R}^{r+1}$. Let $s_\mathfrak{b} = |N(\mathfrak{b})|^{1/d}$ and normalise the lattice to $L'_\mathfrak{b} = L_\mathfrak{b}/s_\mathfrak{b}$, a lattice of volume one. The orthonormality defect of $\mathfrak{b}$ is naturally given by $\omega(\mathfrak{b}) = \omega(L'_\mathfrak{b})$. For our counting function, the choice of $\mathfrak{b}$ is arbitrary. We may multiply $\mathfrak{b}$ by field elements (not necessarily integral) and obtain ideals of the same class. This leads to defining the orthonormality defect of the class $\mathfrak{K}$ by

$$\omega(\mathfrak{K}) = \inf_{\mathfrak{b} \in \mathfrak{K}^{-1} \cap I(K)} \omega(\mathfrak{b}). \tag{4}$$

The defect of the class $\mathfrak{K}$ is thus defined by means of ideals in $\mathfrak{K}^{-1}$. The second problem concerns orthonormality defects of classes.

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(i) Find an optimal estimate for the orthonormality defect $\omega(\mathfrak{K})$ of a class $\mathfrak{K} \in C(K)$.

(ii) Prove or disprove that the radii verify an ultrametric inequality

$$\omega(\mathfrak{K} \cdot \mathfrak{K}') \leq \max(\omega(\mathfrak{K}), \omega(\mathfrak{K}')) .$$

Remarks: The Riemann[1] zeta function $\zeta(z)$ has a Laurent expansion in a neighbourhood of its simple pole at $z = 1$:

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (z-1)^n, \tag{5}$$

where γ_n are the Stieltjes constants

$$\gamma_n = \lim_{m \to \infty} \left(\sum_{k=1}^{m} \frac{\ln^n k}{k} - \frac{\ln^{n+1} m}{m+1} \right), n = 0, 1, \ldots \tag{6}$$

Clearly, γ_0 is the Euler-Mascheroni constant and note that all the terms of the sequence $(\gamma_n)_{n \geq 0}$ are Euler-Mascheroni type constants. Here are the first decimals of γ_n for $n = 0, 1, 2, 3, 4, 5$.

$$\gamma_0 = 0.5772156649\dots, \gamma_1 = -0.0728158454\dots,$$
$$\gamma_2 = 0.0096903631\dots, \gamma_3 = 0.0020538344\dots,$$
$$\gamma_4 = 0.0023253700\dots, \gamma_5 = 0.0007933238\dots.$$

An elementary proof of the expansion (1) can be obtained by the Euler–Maclaurin summation formula. In the paper [AT], formula (1) and some asymptotic evaluations were obtained by using the Laplace transform. The behaviour of these constants suggests that the error term in (2) might be small, despite our present incapacity of finding appropriate estimates – hence the relevance of these two research problems.

References

[AT] D. Andrica and L. Tóth, *Some remarks on Stieltjes constants of the zeta function*, Stud. Cerc. Mat. 43 (1991) 3–9; MR 93c:11066.

[Br] W. E. Briggs, *Some constants associated with the Riemann zeta-function*, Michigan Math. J. 3 (1955–56) 117–121; MR 17,955c.

[La] S. Lang: *Algebraic Number Theory*, Spinger GTM 110, (1986).

[MO] M. Ram Murty and J. Van Order: *Counting integral ideals in a number field*.

III Solutions

163. Find all positive integers m and n such that the integer

$$a_{m,n} = \underbrace{2\dots2}_{m \text{ time}}\underbrace{5\dots5}_{n \text{ time}}$$

is a perfect square.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. We have $a_{1,1} = 25 = 5^2$ and $a_{2,1} = 225 = 15^2$. In the first step, we will show that if $a_{m,n}$ is a perfect square then $n = 1$. We can write

$$\begin{aligned} a_{m,n} &= 2(10^{m+n-1} + \cdots + 10^n) + 5(10^{n-1} + \cdots + 1) \\ &= 2 \cdot 10^n \cdot \frac{10^m - 1}{9} + 5 \cdot \frac{10^n - 1}{9}. \end{aligned}$$

Therefore, the relation $a_{m,n} = x^2$ is equivalent to

$$2 \cdot 10^{m+n} + 3 \cdot 10^n - 5 = (3x)^2. \tag{7}$$

If $n \geq 2$, it follows that $3x$ is divisible by 5, hence $x = 5x_1$ for some positive integer x_1. Replacing in equation (7), we get the equation

$$2 \cdot 2^{m+n} \cdot 5^{m+n-1} + 3 \cdot 2^n \cdot 5^{n-1} - 1 = 5(3x_1)^2,$$

which is not possible.

Now, we will prove that for $m \geq 3$ the integer $a_{m,1} = \underbrace{2\dots2}_{m \text{ time}}5$ is not a perfect square. For $n = 1$, equation (7) is equivalent to

$$2 \cdot 10^{m+1} + 25 = (3x)^2,$$

that is,

$$2 \cdot 10^{m+1} = (3x-5)(3x+5).$$

It follows that $3x - 5 = 2^a \cdot 5^b$ and $3x + 5 = 2^{m+2-a} \cdot 5^{m+1-b}$, where a and b are non-negative integers, hence

$$2^{m+2-a} \cdot 5^{m+1-b} - 2^a \cdot 5^b = 10, \tag{8}$$

that is,

$$2^{m+1-a} \cdot 5^{m-b} - 2^{a-1} \cdot 5^{b-1} = 1. \tag{9}$$

We consider the following cases for equation (9).

Case 1: $a = 1$. We obtain $2^m \cdot 5^{m+1-b} - 5^{b-1} = 1$. If $b = 1$ then it follows that $5^m = 2$, which is not possible for $m \geq 1$.

If $b = m$, we obtain $2^m - 5^{m-1} = 1$, which is not possible because $5^{m-1} > 2^m$ for $m \geq 1$.

Case 2: $a = m+1$. It follows that $5^{m-b} - 2^m \cdot 5^{b-1} = 1$. If $b = 1$, we get $5^{m-1} - 2^m = 1$, which is not possible because $5^{m-1} > 2^{m+1} > 2^m + 1$ when $m \geq 1$.

If $b = m$, we obtain $2^{m+2} \cdot 5^m = 0$, which is not possible.

In conclusion, the only solutions are $m = 1, n = 1$ and $m = 2, n = 1$. □

Also solved by Panagiotis T. Krasopoulos (Athens, Greece), Hans J. Munkholm, Ellen S. Munkholm (University of Southern Denmark, Odense, Denmark), F. Plastria (BUTO-Vrije Universiteit Brussel), José Hernández Santiago (Morelia, Michoacan, Mexico)

164. Prove that every power of 2015 can be written in the form $\frac{x^2+y^2}{x-y}$, with x and y positive integers.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. We have $2015 = 5 \cdot 13 \cdot 31$. Because 31 is congruent to 3 modulo 4, it follows that 31 divides both x and y, etc. We get $x = 31^n x_1, y = 31^n y_1$ and replace in the equation to obtain $x_1^2 + y_1^2 = 5^n \cdot 13^n(x_1 - y_1)$. But $5 \cdot 13 = 65 = 8^2 + 1^2$, hence $5^n \cdot 13^n = (8^2 + 1^2)^n = a^2 + b^2$, where we can assume that $a > b$. The equation is equivalent to $(x_1 + y_1)^2 + (x_1 - y_1)^2 - 2 \cdot 5^n \cdot 13^n(x_1 - y_1) + (5^n \cdot 13^n)^2 = (5^n \cdot 13^n)^2$, that is,

$$(x_1 + y_1)^2 + (5^n \cdot 13^n - x_1 + y_1)^2 = (5^n \cdot 13^n)^2.$$

The last equation is Pythagorean and we select solutions as

$$5^n \cdot 13^n = a^2 + b^2,\ x_1 + y_1 = a^2 - b^2,\ 5^n \cdot 13^n - x_1 + y_1 = 2ab\,,$$

where a and b are positive integers such that

$$5^n \cdot 13^n = (8^2 + 1^2)^n = a^2 + b^2 \text{ and } a > b\,.$$

It follows that

$$x_1 = a^2 - ab = a(a-b), y_1 = ab - b^2 = b(a-b)\,.$$

Finally, it follows that the equation is solvable and has solution

$$(x, y) = (31^n a(a-b), 31^n b(a-b))\,.$$

For example, for $n = 1$, we have $a = 8, b = 1$, hence we get the solution to the reduced equation modulo 31, $(x_1, y_1) = (8(8-1), 1(8-1)) = (56, 7)$. Finally, it follows that the equation is solvable and has solution

$$(x, y) = (31 \cdot 56, 31 \cdot 7) = (1736, 217)\,.$$

□

Also solved by Mihály Bencze (Brasov, Romania), Panagiotis T. Krasopoulos (Athens, Greece), Hans J. Munkholm, Ellen S. Munkholm (University of Southern Denmark, Odense, Denmark), F. Plastria (BUTO-Vrije Universiteit Brussel)

165. Find the smallest positive integer k such that, for any $n \geq k$, every degree n polynomial $f(x)$ over $\mathbb{Z}$ with leading coefficient 1 must be irreducible over $\mathbb{Z}$ if $|f(x)| = 1$ has not less than $[\frac{n}{2}] + 1$ distinct integral roots.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer. Suppose $f(x)$ is a degree n polynomial with leading coefficient 1 such that $|f(x)| = 1$ has at least $[\frac{n}{2}] + 1$ distinct integral roots. Assume that $f(x)$ is reducible, say, $f(x) = g(x)h(x)$, with $\deg g \leq \deg h$. Clearly we have $\deg g \leq [\frac{n}{2}]$.
Suppose $|f(x_i)| = 1$ for $i = 1, \ldots, m$ with $m \geq [\frac{n}{2}] + 1$, where $x_i \in \mathbb{Z}$ are distinct. We have $g(x_i) = \pm 1$ for all $i = 1, \ldots, m$. Without loss of generality, assume that $g(x_i) = 1$ for $1 \leq i \leq \ell$, $g(x_j) = -1$ for $\ell + 1 \leq j \leq m$, and $\ell \geq \frac{m}{2}$.
Then,

$$g(x) - 1 = (x - x_1)(x - x_2) \cdots (x - x_\ell)P(x)$$

for some polynomial $P(x)$. Observe that

$$\ell \leq \deg g \leq \left[\frac{n}{2}\right] < \left[\frac{n}{2}\right] + 1 \leq m\,.$$

Since

$$g(x_j) = -1 \qquad \forall\, \ell + 1 \leq j \leq m\,,$$

we have

$$(x_j - x_1)(x_j - x_2) \cdots (x_j - x_\ell)P(x_j) = -2 \qquad \forall\, \ell + 1 \leq j \leq m\,,$$

and so

$$(x_j - x_1)(x_j - x_2) \cdots (x_j - x_\ell)|2 \qquad \forall\, \ell + 1 \leq j \leq m\,. \qquad (*)$$

If $\ell \geq 4$, $(x_j - x_1)(x_j - x_2) \cdots (x_j - x_\ell)$ is a product of 4 or more distinct non-zero integers and so its absolute value is ≥ 4 and cannot divide 2. Hence $\ell \leq 3$.

If $\ell = 3$, (*) reduces to

$$(x_j - x_1)(x_j - x_2)(x_j - x_3)|2\,.$$

Observe that there can be at most one $a \in \mathbb{Z}$ satisfying

$$(a - x_1)(a - x_2)(a - x_3)|2\,.$$

Thus, we must have $m = 3$ or 4.
If $\ell \leq 2$, since $\ell \geq \frac{m}{2}$, we also have $m \leq 4$.

Since $m \geq [\frac{n}{2}] + 1$, we have $n \leq 7$.

This shows that, for any $n > 7$, if $|f(x)| = 1$ has not less than $[\frac{n}{2}] + 1$ distinct integral roots then $f(x)$ is irreducible.

Finally, observe that $k = 7$. In fact, for $n = 7$, the function $f(x)$ defined by

$$h(x) = 1 + x(x-3)(x-2)(x-1)$$
$$g(x) = 1 + x(x-3)(x-1)$$
$$f(x) = g(x)h(x)$$

is reducible, whereas $|f(x)| = 1$ when $x = 0, 1, 2, 3$. So k cannot be made smaller. □

Also solved by Mihály Bencze (Brasov, Romania), F. Plastria (BUTO-Vrije Universiteit Brussel)

166. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing (f not necessarily continuous). If $f(0) > 0$ and $f(100) < 100$, show that there exists $x \in \mathbb{R}$ such that $f(x) = x$.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer. Define $A := \{x \in [0, 100] : f(x) \geq x\}$. Since $0 \in A$, $A \neq \phi$, let $a := \sup A$. Clearly, $a < 100$. For any $\varepsilon > 0$, there exists $x \in A$ such that $a - \varepsilon < x \leq a$. Hence,

$$a - f(a) \leq a - f(x) < x + \varepsilon - f(x) < \varepsilon\,.$$

As $\varepsilon > 0$ is arbitrary, we have $a \leq f(a)$.

Suppose $f(a) - a = \delta > 0$. Then, for any $x \in (a, a+\delta) \cap [0, 100]$, x does not belong to A and, by the monotonicity of f, we have

$$f(x) \geq f(a) = a + \delta > f(a + \delta) \geq f(x)\,,$$

which is absurd. Thus $f(a) = a$. □

Also solved by A. M. Encinas (Universitat Politècnica de Catalunya, Spain), Laurent Moret-Bailly (IRMAR, Université de Rennes 1, France), F. Plastria (BUTO-Vrije Universiteit Brussel, Belgium).

167. Show that, for any $a, b > 0$, we have

$$\frac{1}{2}\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2. \qquad (10)$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

Solution by the proposer. Integrating by parts, we have

$$\int_a^b \frac{b-t}{t^2}dt = \frac{b-a}{a} - \ln b + \ln a \qquad (11)$$

for any $a, b > 0$.

If $b > a$ then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \geq \int_a^b \frac{b-t}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{b^2}. \qquad (12)$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \geq \int_b^a \frac{t-b}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{a^2}. \qquad (13)$$

Therefore, by (12) and (13), we have for any $a, b > 0$ that

$$\int_a^b \frac{b-t}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \leq \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2.$$

By the representation (11), we then get the desired result (10). □

Also solved by Panagiotis T. Krasopoulos (Athens, Greece), John N. Lillington (Wareham, UK), F. Plastria (BUTO-Vrije Universiteit Brussel)

168. Let $f : I \to \mathbb{C}$ be an n-time differentiable function on the interior $\mathring{I}$ of the interval I, and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on $\mathring{I}$. Show that, for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in \mathbb{R} \backslash \{0, 1\}$, we have the representation

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a)(x-a)^k + (-1)^k \lambda f^{(k)}(b)(b-x)^k \right] + S_{n,\lambda}(x, a, b), \quad (14)$$

where the remainder $S_{n,\lambda}(x, a, b)$ is given by

$$S_{n,\lambda}(x, a, b) := \frac{1}{n!} \left[(1-\lambda)(x-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n \, ds + (-1)^{n+1} \lambda (b-x)^{n+1} \int_0^1 f^{(n+1)}((1-s)x + sb) s^n ds \right]. \quad (15)$$

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Solution by the proposer. Using Taylor's representation with the integral remainder, we can write the following two identities:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt \quad (16)$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k + \frac{(-1)^{n+1}}{n!} \int_x^b f^{(n+1)}(t)(t-x)^n \, dt \quad (17)$$

for any $x, a, b \in \mathring{I}$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0, 1]$, that

$$\int_c^d h(t) \, dt = (d-c) \int_0^1 h((1-s)c + sd) \, ds.$$

Therefore,

$$\int_a^x f^{(n+1)}(t)(x-t)^n \, dt = (x-a) \int_0^1 f^{(n+1)}((1-s)a + sx)(x - (1-s)a - sx)^n \, ds = (x-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n \, ds$$

and

$$\int_x^b f^{(n+1)}(t)(t-x)^n \, dt = (b-x) \int_0^1 f^{(n+1)}((1-s)x + sb)((1-s)x + sb - x)^n \, ds = (b-x)^{n+1} \int_0^1 f^{(n+1)}((1-s)x + sb) s^n ds.$$

The identities (16) and (17) can then be written as

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{n!} (x-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n \, ds \quad (18)$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k + (-1)^{n+1} \frac{(b-x)^{n+1}}{n!} \int_0^1 f^{(n+1)}((1-s)x + sb) s^n ds. \quad (19)$$

Now, if we multiply (18) by $(1-\lambda)$ and (19) by λ and add the resulting equalities, a simple calculation yields the desired identity (14) with the reminder from (15). □

Also solved by Mihály Bencze (Brasov, Romania), Panagiotis T. Krasopoulos (Athens, Greece), John N. Lillington (Wareham, UK)

Remark 1. Note that Problems 155 and 159 were also solved by John N. Lillington (Poundbury, Dorchester, UK)

Remark 2. K. P. Hart noted that the answer to problem 157 can be found in the article by Freudenthal and Hurewicz from 1936, https://eudml.org/doc/212824.

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, and by email to michail.rassias@math.uzh.ch.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Discrete Mathematics*.