

Topological groups of Kac–Moody type, right-angled twinings and their lattices

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Abstract. We construct new groups satisfying the combinatorial axioms of twin root data, by amalgamating spherical parabolic subgroups. The corresponding buildings have right-angled Coxeter groups as Weyl groups. The possibility of mixing several ground fields, not available for Kac–Moody groups, leads to strong non-linearity properties for the groups and some of their subgroups. We also discuss a completion procedure for groups with twin root data which is the starting point for defining a large family of topologically simple groups.

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Introduction

In this paper we construct groups acting on right-angled buildings. Our groups are generated by root groups in a way that places a Moufang structure on the building. This yields a twin building which in turn leads to lattices acting on the original building. As a special case we obtain non-uniform lattices acting on right-angled hyperbolic buildings.

A building is called *right-angled* if its Coxeter system (W, S) is right-angled, where W denotes the Coxeter group, and S a canonical set of generators. This means that the order of the product of any two generators s_i and s_j in S is either 2 or infinity. For such buildings each rank 2 residue is either a complete bipartite graph or a tree. There is no loss of generality in assuming that the Coxeter system is irreducible, and the rank is at least 2, which implies that the building contains tree residues.

The property of being right-angled implies, in a suitable geometric realization, that the panels of each chamber are either at right angles or are disjoint. For example the hyperbolic plane tiled by right-angled n -gons is an apartment of such a type; the chambers are the n -gons, the panels are their edges, and buildings of this type have rank n and are 2-dimensional hyperbolic.

In order to state a theorem, let the *valency* of a panel denote the cardinality of the set of chambers on that panel. If all panels of type s have the same valency $v(s)$ then we say the building has *parameters* $\{v(s)\}_{s \in S}$. Our construction in Section 3.E, based on our construction of Moufang trees in Section 2, yields the following theorem.

Theorem. *Any right-angled building having a parameter system whose parameters are the cardinalities of projective lines admits a Moufang structure.*

A Moufang structure is defined in terms of root group actions (see [Ro1] and [T5]). For the purposes of the present paper we lay out the axioms in terms of a “twin root datum” in Section 1.A. Such data immediately yield twin buildings having a Moufang structure. A twin building is a pair of buildings having the same Coxeter system (W, S) , along with a “codistance”, taking values in W , between chambers of one building and chambers of the other. In a twin building the stabilizer of a face in one building acts on the other building. If Δ is a building that is Moufang and locally finite, a stabilizer in the building twinned with it acts as a lattice on Δ (if the parameters are sufficiently large). We should mention here that the geometric realization of our buildings uses only the spherical residues. When the parameters are finite this implies that the link of each vertex is finite.

Our construction uses an action of the group $\mathrm{SL}_2(\mathbb{K}_s)$ on the set of chambers having a common panel of type s . The action is the same as the action on the projective line over \mathbb{K}_s (see the theorem above). We allow the possibility of mixing fields of different characteristics (for different s in S), and this gives groups containing infinite subgroups of finite exponent, prime to one another. This provides our second result, given more precisely in Section 4.A.

Theorem. *There exist groups having right-angled twin root data such that any Borel subgroup for one building is a non-uniform lattice for the other building of the twin. Moreover any group homomorphism from such a lattice to a product of algebraic groups has infinite kernel.*

These exotic groups share some combinatorial properties with algebraic and arithmetic groups, but provide non-classical applications of recent results generalizing properties of lattices in Lie groups [BS]. We can construct interesting non-linear topological groups for which well-known properties of harmonic analysis or unitary representations are relevant (and sometimes false, see the examples of the existence of a Gelfand pair or the Howe–Moore property considered in [BS, pp. 31–32]). This is done via a geometric completion procedure described in Section 1.B. This procedure is applied to groups having twin root data. For example, Kac–Moody groups admit twin root data (see [T2]), and when the group is over a finite field of characteristic p our completion procedure leads to a topological Kac–Moody group having a refined Tits system in the sense of [KP] (see Section 1.A). In such cases the topological group

is closer to an algebraic group over a local field than the non-linear topological groups mentioned above. More precisely we have the following pro- p -ness result (1.C).

Theorem. *A topological Kac–Moody group admits a refined Tits system. The Tits system gives rise to a building in which any facet-fixator is, up to finite index, a pro- p group.*

This paper is organized as follows. In the first section, we introduce twin root data (1.A), define topological groups of root datum type (1.B), and show that they generalize algebraic groups over local fields (1.C). In Section 2 we sketch a construction of twin root data and apply it to trees; it is a special case of the construction of Moufang twin trees due to J. Tits, but given in a very down-to-earth way. The generators and relations that we provide in Section 2 are then used in Section 3 for our construction of right-angled buildings admitting root groups. Finally, in Section 4 we discuss linearity properties for lattices of topological groups of root datum type.

Let us state a convention we use for group actions. If a group G acts on a set X , the pointwise stabilizer of a subset $Y \subset X$ is called its *fixator* and is denoted by $\text{Fix}_G(Y)$. The global stabilizer is denoted by $\text{Stab}_G(Y)$. When Y is a facet of a building X and G is type-preserving, the two notions coincide. Finally the notation $G|_X$ means the group obtained from G by factoring out the kernel of the action on X .

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1. Closures of groups with twin root data

We introduce a family of totally disconnected topological groups, called *topological groups of root datum type* (1.B). This requires some basic properties of group combinatorics introduced by J. Tits and by V. Kac and D. Peterson (1.A). In the case of topological Kac–Moody groups, some precise structure results are available, leading to analogies with algebraic groups over local fields of characteristic p (1.C).

1.A. Group combinatorics ([A], [Cho], [Hée], [KP], [Ro1], [Ro2], [T2], [T3], [T4], [T5]). Let (W, S) be the Coxeter system $W = \langle s \in S \mid (st)^{M_{st}} = 1 \text{ whenever } M_{st} < \infty \rangle$, where M is a Coxeter matrix. The *Coxeter complex* $\Sigma = \Sigma(W, S)$ is

the union of the translates $wW_J = w\langle J \rangle$ for $J \subset S$ and $w \in W$, partially ordered by reverse inclusion [Ro1, §2]. It is an abstract simplicial complex acted upon by W on which are modelled apartments in buildings of type (W, S) . The elements $w = w\langle \emptyset \rangle$ have maximum dimension and are called *chambers*. The *root system* of (W, S) is defined using the length function $\ell: W \rightarrow \mathbb{N}$ with respect to S [T2, §5]: the *simple root* of index $s \in S$ is the subset $a_s = \{w \in W \mid \ell(sw) > \ell(w)\}$, and a *root* of W is a translate wa_s , $w \in W$, $s \in S$. The set of all roots is denoted by Φ ; it admits an obvious W -action by left multiplication. A root is *positive* if it contains 1; otherwise, it is *negative*. We denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots. The complement of a root a , denoted $-a$, is also a root, called the *opposite* of a . The wall ∂a of a root a is the union of the closed panels having exactly one chamber in a ; it is also the intersection of the closures of a and $-a$. A pair of roots $\{a; b\}$ is called *prenilpotent* if both intersections $a \cap b$ and $(-a) \cap (-b)$ are nonempty. Given a prenilpotent pair of roots $\{a; b\}$, the *interval* $[a; b]$ is by definition the set of roots c with $c \supset a \cap b$ and $(-c) \supset (-a) \cap (-b)$.

Definition. Let Λ be an abstract group containing a subgroup H . Suppose Λ is endowed with a family $\{U_a\}_{a \in \Phi}$ of subgroups, called *root groups*, indexed by the set of roots Φ , and define the subgroups $U_+ = \langle U_a \mid a \in \Phi_+ \rangle$ and $U_- = \langle U_a \mid a \in \Phi_- \rangle$. We say that the triple $(\Lambda, \{U_a\}_{a \in \Phi}, H)$ is a *twin root datum* for Λ (or satisfies the (TRD) axioms) if the following conditions are fulfilled.

- (TRD0) Each U_a is nontrivial and normalized by H .
- (TRD1) For each prenilpotent pair of roots $\{a; b\}$, the commutator subgroup $[U_a, U_b]$ is contained in the subgroup $\langle U_c : c \in [a; b] \setminus \{a; b\} \rangle$.
- (TRD2) For each $s \in S$ and $u \in U_{a_s} \setminus \{1\}$, there exist uniquely defined elements $u', u'' \in U_{-a_s} \setminus \{1\}$ such that $m(u) = u'uu''$ conjugates U_b onto $U_{s.b}$ for every root b . Moreover, $m(u)H = m(v)H$ for all $u, v \in U_{a_s} \setminus \{1\}$.
- (TRD3) For each $s \in S$, we have $U_{a_s} \not\subset U_-$ and $U_{-a_s} \not\subset U_+$.
- (TRD4) The group Λ is generated by H and the U_a 's.

Examples. 1) The simplest example of a group with a twin root datum is SL_2 over a field k . In the computational parts of Sections 2 and 3, we are led to handle several copies of it, so we use an index i in describing certain elements and subgroups. The corresponding Coxeter group is $\mathbb{Z}/2\mathbb{Z} = \langle s_i \rangle$, so $\Phi = \{\pm a_i\}$ and (TRD1) is an empty condition. We set $u_i(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, $u_{-i}(r) = \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}$, $m_i(\lambda) = u_{-i}(\lambda^{-1})u_i(\lambda)u_{-i}(\lambda^{-1})$, $m_i = m_i(1)$ and $t_i(\lambda) = m_i(\lambda)m_i^{-1}$. The twin root datum is given by $U_{\pm a_i} = u_{\pm i}(k)$ and $H = t_i(k^\times)$. In matrix notation: $m_i(\lambda) = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$ and $t_i(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Axiom (TRD3) is obvious and (TRD4) is well-known. Axiom (TRD0) follows from $t_i(\lambda)u_{\pm i}(r)t_i(\lambda)^{-1} = u_{\pm i}(\lambda^{\pm 2}r)$, and finally (TRD2) follows from $m_i(\lambda)u_i(r)m_i(\lambda)^{-1} = u_{-i}(\lambda^{-2}r)$, $m_i(\lambda)u_{-i}(r)m_i(\lambda)^{-1} = u_i(\lambda^2r)$ and

$m_i(\lambda)m_i(\mu) = t_i(-\lambda\mu^{-1})$. Note that we also have $m_i(\lambda)t_i(\mu)m_i(\lambda)^{-1} = t_i(\mu^{-1})$, $m_i(\lambda)^{-1} = m_i(-\lambda)$ and $m_i^2 = -\text{id}$. Note that the uniqueness in (TRD2) follows from the rest of the axioms [T5, 3.3].

2) The special linear groups $\text{SL}_n(k[t, t^{-1}])$ over Laurent polynomials satisfy these axioms for W affine of type A_{n-1} . More generally, the groups $\mathbb{G}(k[t, t^{-1}])$ where \mathbb{G} is a simply connected Chevalley group, satisfy these axioms for a suitable affine Coxeter group W ([A, §1]). In these cases, the root groups are naturally defined as subgroups of the spherical root groups by using polynomials of the form at^r for fixed r as the off-diagonal entries. The commutator condition (TRD1) is carefully examined in [A, Examples 2 and 3].

The procedure to attach a building to a BN -pair (G, B, N, S) is standard, see e.g. [Ro1, §5]: the simplices are in equivariant bijection with the translates gP_J of the standard parabolic subgroups $P_J = B\langle J \rangle B$, $J \subset S$. The (TRD) axioms imply the existence of two BN -pairs in G ([A, Proposition 1]), hence of two buildings. Moreover there is a natural opposition relation on chambers of opposite signs such that two chambers are opposite if the intersection of their stabilizers is minimal. This is the generic position for two chambers, and the set of opposites for a given chamber is sometimes called a *big cell*, referring to a well-known terminology for algebraic groups. The root groups U_a act as root groups on each building of the twinning. The buildings are therefore *Moufang* in the sense of Tits [T5, §4.3]. Geometrically this property implies that given a codimension one simplex Π and a chamber c containing it, there is a group U_a of building automorphisms fixing c and acting simple-transitively on the chambers containing Π and different from c ([Ro1, 6.4]).

Finally, we will also need group combinatorics defined by V. Kac and D. Peterson [KP, §3].

Definition. A *refined Tits system* for a group G is a sextuple (G, N, U_+, U_-, H, S) , where N, U_+, U_-, H are subgroups satisfying the following:

- (RT1) We have $G = \langle N, U_+ \rangle$, $H \triangleleft N$, $H < N_G(U_+) \cap N_G(U_-)$, $W = N/H$ and (W, S) is a Coxeter system.
- (RT2) For each $s \in S$, we set $U_s = U_+ \cap s^{-1}U_-s$; and for any $w \in W$ and $s \in S$ we require:
 - (RT2a) $U_s \neq \{1\}$ and $s^{-1}(U_s \setminus \{1\})s \subset U_s s H U_s$,
 - (RT2b) either $w^{-1}U_s w \subset U_+$ or $w^{-1}U_s w \subset U_-$,
 - (RT2c) $U_+ = U_s(U_+ \cap s^{-1}U_+s)$.
- (RT3) If $u_- \in U_-$, $u_+ \in U_+$ and $n \in N$ are such that $u_- n u_+ = 1$, then $u_- = u_+ = n = 1$.

1.B. Topological groups. Let Λ be a group with twin root datum $(\Lambda, \{U_a\}_{a \in \Phi}, H)$. Until the end of Section 1, we assume that the Weyl group W (hence Φ) is infinite and that all root groups U_a are finite. We denote by Δ the simply connected non-positively curved metric realization, due to M. Davis [D] and G. Moussong [Mou], of the positive building of Λ [A, Proposition 1 and Example 6]. We set $q_{\min} = \min_{a \in \Phi} |U_a| (= \min_{s \in S} |U_{a_s}|)$ and $q_{\max} = \max_{a \in \Phi} |U_a| (= \max_{s \in S} |U_{a_s}|)$. Since only spherical facets appear in the Moussong–Davis realization, Δ is locally finite. Our main motivation in defining topological groups of root datum type is geometric group theory, in which groups are studied via their actions on suitable spaces. The following lemma shows that taking the quotient by the kernel of the Λ -action on Δ is harmless; it follows easily from [Ré1, 1.5.4] and refined Tits system arguments.

Lemma 1. *The kernel K of the Λ -action on Δ consists of the elements in H centralizing all root groups. The groups U_{\pm} embed in Λ/K , hence so do the root groups, and $(\Lambda/K, \{U_a\}_{a \in \Phi}, H/K)$ is again a twin root datum with the same associated Coxeter system and twin buildings. \square*

When Λ is a Kac–Moody group, the kernel K is the center $Z(\Lambda)$ [Ré1, 9.6.2], which is finite when the ground field is. For $\mathrm{SL}_n(\mathbb{K}[t, t^{-1}])$, the buildings of Λ are Euclidean \tilde{A}_{n-1} -buildings, and the kernel K is the group $\mu_n(\mathbb{F}_q)$ of n -th roots of unity in \mathbb{F}_q .

Convention. Until the end of 1.B, Λ denotes a group with twin root datum as above factored out by the kernel of its action on the building Δ . This group has a twin root datum with finite root groups and infinite Weyl group.

Given an automorphism $g \in \mathrm{Aut}(\Delta)$ and a finite union C of facets, $O_C(g) = \{h \in \mathrm{Aut}(\Delta) : h|_C = g|_C\}$ is by definition an open neighborhood of g in $\mathrm{Aut}(\Delta)$. The group Λ is not discrete on the single building Δ because each facet fixator is transitive on opposite facets in Δ_- , of which there are infinitely many.

Definition. (i) We call the closure $G = \overline{\Lambda}^{\mathrm{Aut}(\Delta)}$ a *topological group of root datum type*. When Λ is a Kac–Moody group over a finite field, we call G a *topological Kac–Moody group*.

(ii) We call the fixator $\mathrm{Fix}_G(F)$ of a facet F the *parahoric subgroup* of G associated to F , and we denote it by G_F . We call a chamber fixator an *Iwahori subgroup*.

Remark. An automorphism of Δ lies in G if and only if it coincides with an element of Λ on any finite set of facets.

Example. Consider the image of $\mathrm{SL}_n(\mathbb{F}_q[t, t^{-1}])$ acting on the Euclidean building corresponding to the discrete valuation ring with uniformizer t . The kernel of the

action is $\mu_n(\mathbb{F}_q)$ and the “completion” G is $SL_n(\mathbb{F}_q((t)))/\mu_n(\mathbb{F}_q)$. When $n > 2$, G is close to $\text{Aut}(\Delta)$, but when $n = 2$, Δ is a tree and G is far from being all of $\text{Aut}(\Delta)$.

We now consider discrete subgroups of G , which by the above example are analogous to some arithmetic groups over function fields. Fix a chamber c_- in Δ_- , and let Γ denote its stabilizer. Recall that for $q_{\min} \gg 1$, the parabolic subgroups of a given sign ε are lattices of the locally finite CAT(0)-building $\Delta_{-\varepsilon}$ (see [CG], [Ré2]). Since the group Λ acts by type-preserving isometries on Δ , the stabilizer of a facet F in Λ is also its fixator. We denote it by $\Lambda(F)$. Recall also that two subgroups Γ and Γ' of a given group G are *commensurable* if they share a finite index subgroup. The *commensurator* of Γ in G is the group $\text{Comm}_G(\Gamma) = \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}$.

Lemma 2. (i) *For any two negative facets F and F' , $\Lambda(F)$ and $\Lambda(F')$ are commensurable.*

(ii) *For any negative facet F , Λ lies in the commensurator $\text{Comm}_{\text{Aut}(\Delta)}(\Lambda(F))$.*

Proof. Let F and F' be two negative facets. Since Δ_- is locally finite, the $\Lambda(F)$ -transforms of F' are in finite number; we denote them by $\gamma_1.F', \dots, \gamma_m.F'$ for some $\gamma_i \in \Lambda(F)$. Therefore, given any $\gamma \in \Lambda(F)$ one has $\gamma.F' = \gamma_j.F'$ for some j , and hence $\Lambda(F) = \bigsqcup_{j=1}^m \gamma_j(\Lambda(F) \cap \Lambda(F'))$. Switching F and F' proves (i), and (ii) follows from (i) since $g\Lambda(F)g^{-1} = \Lambda(gF)$. □

Example. The negative vertex fixators in $SL_n(\mathbb{F}_q((t^{-1}))/\mu_n(\mathbb{F}_q)$ are conjugates of the subgroup $SL_n(\mathbb{F}_q[[t^{-1}]])/\mu_n(\mathbb{F}_q)$. In Λ they are $\{0\}$ -arithmetic groups commensurable with $SL_n(\mathbb{F}_q[t^{-1}])/\mu_n(\mathbb{F}_q)$, and the commensurator contains $SL_n(\mathbb{F}_q(t))/\mu_n(\mathbb{F}_q)$.

1.C. Kac–Moody groups. Let Λ be the image in $\text{Aut}(\Delta)$ of the rational points of an almost split Kac–Moody group over \mathbb{F}_q [Ré1, 12.6.3] whose Weyl group is infinite. We denote its twin root datum by $(\Lambda, \{U_a\}_{a \in \Phi}, T)$, and let $G = \bar{\Lambda}^{\text{Aut}(\Delta)}$ denote the corresponding topological Kac–Moody group (1.B). By [Ré1, 12.5.4], for each root a the group U_a is isomorphic to the \mathbb{F}_q -points of the unipotent radical of a Borel subgroup in a rank one finite group of Lie type, so the finiteness of the root groups follows from that of \mathbb{F}_q , and $|U_a| \geq q$ for every $a \in \Phi$.

Example. The group $SU_3(\mathbb{F}_q[t, t^{-1}])$ is an almost split Kac–Moody group over \mathbb{F}_q with infinite dihedral Weyl group. The associated buildings are semihomogeneous trees of valencies $1 + q$ and $1 + q^3$ [Ré4, 3.5].

The group G admits a BN -pair, since its subgroup Λ is strongly transitive (in the sense of W -distance) on Δ [Ro1, §5]. We now prove a stronger result stressing the

analogy between G and semisimple groups over local fields of positive characteristic. Let F be a facet in Δ and let \mathbb{A} be a twin apartment containing it [A, Lemma 2]. We have a Levi decomposition

$$\Lambda(F) = M(\mathbb{A}, F) \ltimes U(F),$$

where $M(\mathbb{A}, F)$ is abstractly isomorphic to the Kac–Moody group associated to a submatrix of the generalized Cartan matrix defining Λ [Ré1, 6.2.2]. Since F is spherical, $M(\mathbb{A}, F)$ is a finite reductive group of Lie type over \mathbb{F}_q . We denote by U_F the closure $\overline{U(F)}^{\text{Aut}(\Delta)}$ of the unipotent radical $U(F)$.

Theorem. (i) *The sextuple (G, N, U_c, U_-, T, S) defines a structure of refined Tits system for the topological Kac–Moody group G .*

(ii) *The group U_F is pro- p and the parahoric subgroup G_F is virtually pro- p since we have $G_F = M(\mathbb{A}, F) \ltimes U_F$.*

Remark. The theorem implies that $\text{SL}_n(\mathbb{F}_q((t)))$ satisfies the axioms of a refined Tits system, and that its parahoric subgroups admit semidirect product decompositions. This splitting of facet fixators is not true for algebraic groups over local fields of characteristic 0. Other arguments, involving torsion, explain why the analogy with the characteristic 0 case is not relevant [Ré3].

To prove the theorem we need two lemmas which will be proved using a filtration $\{E_n\}_{n \geq 1}$ of Δ . The first term E_1 is (the closure of) $\text{st}(F)$, the star of F in Δ and further terms are defined inductively. Write $\mathbb{A}_n = E_n \cap \mathbb{A}$ and choose a chamber c_n of \mathbb{A} sharing a panel with the boundary of \mathbb{A}_n and at minimal distance from \mathbb{A}_n . We define \mathbb{A}_{n+1} to be the convex hull of c_n and \mathbb{A}_n , and E_{n+1} to be $\Lambda(F) \cdot \mathbb{A}_{n+1}$, that is the set of all $\Lambda(F)$ -transforms of \mathbb{A}_{n+1} . By the Bruhat decomposition, for any chamber $c \in \text{st}(F)$, \mathbb{A} is a complete set of representatives for the action of $U(c) < \Lambda(F)$ on Δ . Hence $\{E_n\}_{n \geq 1}$ exhausts Δ . For each $n \geq 1$, the set E_n is $\Lambda(F)$ -stable. We can write the groups G_F and U_F as projective limits: $G_F = \varprojlim_{n \geq 1} \Lambda(F)|_{E_n}$ and $U_F = \varprojlim_{n \geq 1} U(F)|_{E_n}$.

Lemma 1. *The group U_F is pro- p .*

Remark. This lemma is true for an arbitrary twin root datum as in 1.B, provided its root groups are p -groups.

Proof. For each $n \geq 1$, we have an exact sequence

$$1 \longrightarrow K_n \longrightarrow U(F)|_{E_{n+1}} \longrightarrow U(F)|_{E_n} \longrightarrow 1,$$

where K_n is the fixator $\text{Fix}_{U(F)|_{E_{n+1}}}(E_n)$ of E_n in the restricted group $U(F)|_{E_{n+1}}$.

Since $U(F)$ fixes $\text{st}(F)$, we have $U(F)|_{E_1} = \{1\}$. By definition $U_F = \varprojlim_{n \geq 1} U(F)|_{E_n}$, so in order to show that U_F is pro- p , it is enough by induction to show that K_n is a p -group for each $n \geq 1$.

Let us fix $n \geq 1$ and $u \in K_n$. Let Π_n be the panel of c_n in the boundary of \mathbb{A}_n , and let d_n be the other chamber of \mathbb{A} having Π_n as a panel. Then $d_n \in \mathbb{A}_n$, so u fixes d_n , and by the Moufang property [Ro1, 6.4] there exists $v \in U_a$ such that $v^{-1}u.c_n = c_n$. The element $v^{-1}u$ is in the “unipotent radical” of the Borel subgroup $\text{Fix}_\Lambda(d_n)$ and fixes $c_n \cup d_n$; hence it must fix the whole star of Π_n . Consequently, u and v coincide on $\text{st}(\Pi_n)$ and thus $u^{q_{\max}}$ acts trivially on it. In particular, $u^{q_{\max}}$ fixes $\mathbb{A}_n \cup \{c_n\}$ hence \mathbb{A}_{n+1} by convexity. Every chamber of $E_{n+1} \setminus E_n$ is of the form $v.d$ for some d in the convex hull of \mathbb{A}_n and c_n , and $v \in \Lambda(F)$. Then $u^{q_{\max}}(vd) = v(v^{-1}u^{q_{\max}}v.d) = v((v^{-1}uv)^{q_{\max}}.d)$. Recall that $U(F)$ is normalized by $\Lambda(F)$, so that applying the result of the above paragraph to $v^{-1}uv$ instead of u , we get $(v^{-1}uv)^{q_{\max}}.d = d$, and thus $u^{q_{\max}}(vd) = vd$. This shows that the order of each element $u \in K_n$ divides q_{\max} ; hence K_n is a p -group, as required. \square

Lemma 2. (i) We have $\Lambda \cap U_F = U(F)$, hence $M(\mathbb{A}, F) \cap U_F = \{1\}$ for any facet F .
 (ii) We have $G_F = M(\mathbb{A}, F) \rtimes U_F$.

Remark. In the proof below, the assumption that Λ is Kac–Moody is used in asserting that the center of $M(\mathbb{A}, F)$ is an \mathbb{F}_q -torus, hence must intersect U_F trivially.

Proof. (i) Let $g \in \Lambda \cap U_F < \text{Fix}_\Lambda(F) = \Lambda(F)$. The Levi decomposition of $\Lambda(F)$ gives $g = mu$, with $m \in M(\mathbb{A}, F)$ and $u \in U(F)$. Since U_F fixes $\text{st}(F)$, the set of chambers containing F , we see that both g and u , hence also m , fix $\text{st}(F)$. Therefore m is central in the finite reductive group $M(\mathbb{A}, F)$ of Lie type; this implies that m lies in a torus, so its order is not divisible by p . But m is also a torsion element in the pro- p group U_F , and is hence trivial.

(ii) For each $n \geq 1$, E_n is $\Lambda(F)$ -stable by definition, hence $M(\mathbb{A}, F)$ -stable. Since $M(\mathbb{A}, F)$ normalizes $U(F)$, it normalizes $U(F)|_{E_n}$ in $\Lambda(F)|_{E_n}$ for each $n \geq 1$. Hence U_F is normalized by $M(\mathbb{A}, F)$. Let $g \in \Lambda(F)$. It can be written as $g = \varprojlim_{n \geq 1} g_n$, with $g_n \in \Lambda(F)|_{E_n}$. According to the Levi decomposition of $\Lambda(F)$, we have $g_n = m_n u_n$ with $m_n \in M(\mathbb{A}, F)|_{E_n}$ and $u_n \in U(F)|_{E_n}$. The group $M(\mathbb{A}, F)$ is finite, so up to extracting a subsequence, we shall assume that m_n is a constant element m in $M(\mathbb{A}, F)$. This enables us to write $g = m \cdot (\varprojlim_{n \geq 1} u_n)$, and proves $G_F = M(\mathbb{A}, F) \cdot U_F$. Finally (i) implies the trivial intersection $M(\mathbb{A}, F) \cap U_F = \{1\}$, hence the semidirect product decomposition $G_F = M(\mathbb{A}, F) \rtimes U_F$. \square

Proof of the theorem. By [R 1, 1.5.4], $(\Lambda, N, U_+, U_-, T, S)$ is a refined Tits system with Weyl group W , and $U_+ \cap s^{-1}U_-s = U_{a_s}$ for any $s \in S$. Axiom (RT1): the group T normalizes $U(c)$, and passing to the projective limit it normalizes U_c . The other

statements in axiom (RT1) are either clear or true from the refined Tits system axioms for Λ . Axiom (RT3): suppose we have $\gamma nu_c = 1$, with $\gamma \in \Gamma$, $n \in N$ and $u_c \in U_c$. Then $\gamma n = u_c^{-1}$ belongs to $U_c \cap \Lambda$, which is $U(c)$ by Lemma 2: all the factors are in Λ and we apply axiom (RT3) from the refined Tits system structure of Λ to get (RT3) for G . Axioms (RT2): define $U(s) = U_c \cap s^{-1}\Gamma s$. The first part of Lemma 2 implies $U(s) < U(c) \cap s^{-1}\Gamma s$, hence $U(s)$ is the root group U_{a_s} according to the relation of a refined Tits system and the twin root datum of a Kac–Moody group [Ré1, 1.6]. Axioms (RT2a) and (RT2b) are then clear because they just involve subgroups of Λ . For axiom (RT2c), we need $U_c = U_{a_s} \cdot (U_c \cap s^{-1}U_c s)$. An element $u \in U_c$ fixes c , hence permutes the chambers sharing their panel of type s with c . Since U_{a_s} fixes c and is (simply) transitive on the set of chambers $\neq c$ of the latter type, there exists an element $v \in U_{a_s}$ such that $v^{-1}u$ fixes both c and sc . This proves (RT2c). \square

The theorem gives an analogy between topological Kac–Moody groups and semi-simple algebraic groups over local fields of positive characteristic. In the next two sections, we construct groups exotic enough to show that the analogy is strictly weaker for arbitrary twin root data.

2. Twinning trees

The construction of this section applies to trees as well as to right-angled buildings. Here are its main steps.

1. Define the “Borel subgroup” by means of a presentation described by an apartment.
2. To each type of spherical facets associate a “unipotent radical” and a “Levi factor”.
3. Show that the “Levi factor” admits an action on the “unipotent radical”.
4. Amalgamate these groups along the inclusions given by the closure of a chamber.
5. Check the twin root datum axioms for the amalgam Λ .

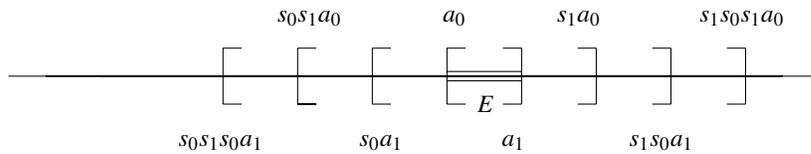
Step 4 amounts to taking the fundamental group of a suitable complex of groups. The computations for Step 3 are done in detail in the present section. This enables us to concentrate on more conceptual arguments in the case of right-angled buildings (Section 3).

2.A. Root system. The Coxeter complex here is a tiling of \mathbb{R} by segments with vertices the integers. We denote by E the segment $[0, 1]$, and by s_0 (resp. s_1) the reflection with respect to 0 (resp. 1). This is the Coxeter complex of the infinite dihedral group $D_\infty = \langle s_0, s_1 \rangle$. The roots of Φ are the half-lines defined by the

integers, the positive roots (in Φ_+) are those containing E . We denote by a_0 (resp. a_1) the positive root having boundary 0 (resp. 1). Each vertex has type 0 or 1, with the obvious notation, and the boundary of a root is called its *vertex*. The *type* of a root a is the type of its vertex, and is denoted by $\iota(a)$. A positive root a is prenilpotent with a_i if and only if $a \supset a_i$. Each positive root contains either a_0 or a_1 , but not both, so the positive roots fall into four disjoint subsets $\mathcal{P}(a_i, j)$ for $i, j \in \{0; 1\}$, where $\mathcal{P}(a_i, j)$ is the set of positive roots of type j which contain a_i . Note that these subsets of positive roots can also be defined as follows:

$$\begin{aligned} \mathcal{P}(a_0, 0) &= \{(s_0s_1)^m a_0\}_{m \geq 0}, & \mathcal{P}(a_0, 1) &= \{(s_0s_1)^m s_0a_1\}_{m \geq 0}, \\ \mathcal{P}(a_1, 0) &= \{(s_1s_0)^m s_1a_0\}_{m \geq 0}, & \mathcal{P}(a_1, 1) &= \{(s_1s_0)^m a_1\}_{m \geq 0}. \end{aligned}$$

We shall use $\mathcal{P}(a_i)$ to mean $\mathcal{P}(a_i, 0) \cup \mathcal{P}(a_i, 1)$, the set of positive roots prenilpotent with a_i .



2.B. Borel subgroup. Unipotent subgroups. We pick two fields \mathbb{K}_0 and \mathbb{K}_1 . To each positive root a of type i is attached a copy of the additive group $(\mathbb{K}_i, +)$. We denote it by $U_a = \{u_a(k) : k \in \mathbb{K}_i\}$ (u_a is the chosen isomorphism between the root group and its field).

Definition. (i) Let the abelian group A_i be the direct sum of the root groups associated to the roots in $\mathcal{P}(a_i)$.

(ii) Let the group U_+ be the free product of A_0 and A_1 , that is $U_+ = A_0 * A_1$.

(iii) Let the group T be the product $\mathbb{K}_0^\times \times \mathbb{K}_1^\times$ of the multiplicative groups of the chosen fields.

For a (positive) root a of type i , we define $\varepsilon_a \in \{\pm 1\}$ to be $(-1)^m$, where m is the exponent appearing in 2.A. It is the parity of the number of vertices of type i in the interior of the segment joining the middle of E and the vertex ∂a . Using the notation of Example 1 in 1.A, T is the maximal torus

$$t_0(\mathbb{K}_0^\times) \times t_1(\mathbb{K}_1^\times) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} : \lambda \in \mathbb{K}_0^\times, \mu \in \mathbb{K}_1^\times \right\}$$

of $SL_2(\mathbb{K}_0) \times SL_2(\mathbb{K}_1)$, and we define the action of T on U_+ by

$$(2B1) \quad t_j(\lambda)u_a(k)t_j(\lambda)^{-1} = u_a(\lambda^{2\varepsilon_a\delta_{j\iota(a)}}k) \text{ for any } j \in \{0; 1\}, \lambda \in \mathbb{K}_j^\times, a \in \Phi_+ \text{ and } k \in \mathbb{K}_{\iota(a)},$$

and where the Kronecker symbol $\delta_{j\iota(a)}$ means that the element $t_j(\lambda)$ induces multiplication of k by $\lambda^{2\varepsilon_a}$ only if the type of the root a is j . In particular, $t_j(\lambda)$ centralizes each root group of type $\neq j$. The action of a single element $t_j(\lambda)$ obviously preserves the relations in U_+ and the actions of any two such elements commute. We therefore obtain an action by T on U_+ .

Definition. (i) The *Borel subgroup* B is the semi-direct product $T \ltimes U_+$.

(ii) We denote by V^i the subgroup of U_+ generated by all the positive root groups except U_i .

(iii) We denote by U^i the normal closure of V^i in U_+ , i.e. the subgroup generated by $u_a(k)$ for $a \in \mathcal{P}(a_i) \setminus \{a_i\}$ and $u_{a_i}(r)u_a(k)u_{a_i}(r)^{-1}$ for $a \in \mathcal{P}(a_{1-i})$, with $r \in \mathbb{K}_i$.

(iv) The (*standard*) *Levi factor of type i* is the direct product $L_i = \mathrm{SL}_2(\mathbb{K}_i) \times \mathbb{K}_{1-i}^\times$.

2.C. Actions of Levi factors on unipotent radicals. In order to define a parabolic subgroup as a semi-direct product $L_i \ltimes U^i$, we must define an action of L_i on the group U^i . We define actions of $\mathrm{SL}_2(\mathbb{K}_i)$ and \mathbb{K}_{1-i}^\times which are easily seen to commute with one another, so we can deal with each factor separately. The action of the torus \mathbb{K}_{1-i}^\times on U^i is that obtained as a subgroup of T . This makes sense because T obviously stabilizes U^i . We turn now to the action of the factor $\mathrm{SL}_2(\mathbb{K}_i)$. We define the actions of the generators $u_i(r)$ and $m_i(\lambda)$ given in 1.A.

For the root groups, we set for k in \mathbb{K}_0 or \mathbb{K}_1 , and r, s in \mathbb{K}_i :

$$(2C1) \quad u_i(r)u_a(k)u_i(r)^{-1} = u_a(k) \text{ for } a \in \mathcal{P}(a_i) \setminus \{a_i\},$$

$$(2C2) \quad u_i(s)(u_{a_i}(r)u_a(k)u_{a_i}(r)^{-1})u_i(s)^{-1} = u_{a_i}(r+s)u_a(k)u_{a_i}(r+s)^{-1} \text{ for } a \in \mathcal{P}(a_{1-i}).$$

The elements $m_i(\lambda)$ have a conjugation action on the positive root group elements $u_a(k)$:

$$(2C3) \quad m_i(\lambda)u_a(k)m_i(\lambda)^{-1} = u_{s_i a}(\lambda^{-2\varepsilon_a \delta_{i\iota(a)}} k).$$

Note that $s_i a$ is a non-positive root different from a_i since $\Phi_+ \cap s_i \Phi_- = \{a_i\}$ [T2, 5.6, Proposition 3]. We must also define a conjugation action of $m_i(\lambda)$ on elements $u_{a_i}(r)u_a(k)u_{a_i}(r)^{-1}$ in the free product $U_a * U_{a_i}$, whenever a is a positive root containing a_{1-i} , and $u_{a_i}(r) \neq 1$. We set

$$(2C4) \quad m_i(\lambda)(u_{a_i}(r)u_a(k)u_{a_i}(r)^{-1})m_i(\lambda)^{-1} \\ = u_{a_i}\left(\frac{-\lambda^2}{r}\right)u_a\left(\left(\frac{-\lambda}{r}\right)^{2\varepsilon_a \delta_{i\iota(a)}} k\right)u_{a_i}\left(\frac{-\lambda^2}{r}\right)^{-1}.$$

Remark. The Kronecker symbol $\delta_{i\iota(a)}$ in the exponents $2\varepsilon_a\delta_{i\iota(a)}$ involves the types of roots i and $\iota(a)$. It simply means that the element $m_i(\lambda)$ induces a multiplication of the additive parameter k in $u_a(k)$ by a factor $\lambda^{-2\varepsilon_a}$ or $(\frac{-\lambda}{r})^{2\varepsilon_a}$ only if the type of the root a is i .

2.D. Checking the product relation. In this subsection, we make sure that the individual actions above define an action of L_i on U^i . We must show that given any two elements g, h of $SL_2(\mathbb{K}_i) \times \mathbb{K}_{1-i}$ and a generator $v \in U^i$, we always have $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$. This equality involves the product $g.h$, and we will have to describe the product of g and h in SL_2 in terms of the Bruhat decomposition. When g is in the big cell $SL_2 \begin{pmatrix} * & * \\ \neq 0 & * \end{pmatrix}$, we write it $g = u_i(r)m_i(\lambda)u_i(r')$; when it is in the Borel subgroup $SL_2 \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, we write it $g = u_i(r)t_i(\lambda)$. Similarly, when h is in the big cell, we write it $h = u_i(s)m_i(\mu)u_i(s')$; when it is in the Borel subgroup, we write it $h = u_i(s)t_i(\mu)$.

The form of the generator v of U^i will also play a role. A generator $v = u_{a_i}(t)u_a(k)u_{a_i}(t)^{-1}$ with a a root containing a_{1-i} (hence not prenilpotent with a_i), will be referred to as a generator of the first type. A generator $v = u_a(k)$, with a a root containing a_i (hence prenilpotent with a_i), will be referred to as a generator of the second type.

The computation is divided into five cases, according to whether each of the elements g, h or gh belongs to the big cell or its complement the Borel subgroup. Each case is divided into two subcases, according to whether the generator is of the first or of the second type.

We introduce the notation $R = r' + s$ and $S = s' + t$ for the remainder of Section 2.D.

2.D.1. “big cell \cdot big cell \in big cell”, with $g = u_i(r)m_i(\lambda)u_i(r')$ and $h = u_i(s)m_i(\mu)u_i(s')$, in which case $R \neq 0$. A calculation in SL_2 shows that $gh = u_i(r - \frac{\lambda^2}{R})m_i(\frac{-\lambda\mu}{R})u_i(s' - \frac{\mu^2}{R})$.

We deal with a generator v of the first type. By (2C2) we have $hvh^{-1} = [u_i(s)m_i(\mu)]u_{a_i}(S)u_a(k)u_{a_i}(S)^{-1}[\dots]^{-1}$, and assuming that $S \neq 0$, we have by (2C4) and (2C2) that

$$\begin{aligned} hvh^{-1} &= u_{a_i}(s - \mu^2 S^{-1})u_a((-\mu S^{-1})^{2\varepsilon_a\delta_{i\iota(a)}}k)u_{a_i}(\dots)^{-1}, \\ g(hvh^{-1})g^{-1} &= [u_i(r)m_i(\lambda)]u_{a_i}(R - \mu^2 S^{-1}) \\ &\quad \cdot u_a((\mu^2 S^{-1})^{2\varepsilon_a\delta_{i\iota(a)}}k)u_{a_i}(R - \mu^2 S^{-1})^{-1}[\dots]^{-1}. \end{aligned}$$

Under the further assumption that $R - \mu^2 S^{-1} \neq 0$, (2C4) and (2C2) give

$$\begin{aligned} g(hvh^{-1})g^{-1} &= u_{a_i} \left(r - \frac{\lambda^2}{R - \mu^2 S^{-1}} \right) \\ &\quad \cdot u_a \left(\left(\frac{-\lambda}{R - \mu^2 S^{-1}} \right)^{2\varepsilon_a \delta_{ii}(a)} (-\mu S^{-1})^{2\varepsilon_a \delta_{ii}(a)} k \right) \cdot u_{a_i}(\dots)^{-1} \\ &= u_{a_i} \left(r - \frac{\lambda^2 S}{RS - \mu^2} \right) \cdot u_a \left(\left(\frac{\lambda \mu}{RS - \mu^2} \right)^{2\varepsilon_a \delta_{ii}(a)} k \right) \cdot u_{a_i}(\dots)^{-1}. \end{aligned}$$

Now assuming $(R - \mu^2 S^{-1}) \cdot S \neq 0$, we use (2C4) and (2C2) to conjugate v by gh :

$$\begin{aligned} (gh)v(gh)^{-1} &= [u_i(r - \lambda^2 R^{-1})m_i(-\lambda \mu R^{-1})] \\ &\quad \cdot u_{a_i}(S - \mu^2 R^{-1})u_a(k)u_{a_i}(S - \mu^2 R^{-1})^{-1}[\dots]^{-1} \\ &= u_{a_i} \left(r - \frac{\lambda^2}{R} - \frac{\lambda^2 \mu^2}{R^2(S - \mu^2 R^{-1})} \right) \\ &\quad \cdot u_a \left(\left(-\frac{-\lambda \mu}{R(S - \mu^2 R^{-1})} \right)^{2\varepsilon_a \delta_{ii}(a)} k \right) \cdot u_{a_i}(\dots)^{-1} \\ &= u_{a_i} \left(r - \frac{\lambda^2 S}{RS - \mu^2} \right) \cdot u_a \left(\left(\frac{\lambda \mu}{RS - \mu^2} \right)^{2\varepsilon_a \delta_{ii}(a)} k \right) \cdot u_{a_i}(\dots)^{-1}, \end{aligned}$$

which proves $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$ when $S \neq 0$ and $R - \mu^2 S^{-1} \neq 0$.

Now, suppose $S \neq 0$ but $R - \mu^2 S^{-1} = 0$. Then the first equation above for $g(hvh^{-1})g^{-1}$ simplifies, and using (2C3) and (2C1) (which implies that u_{a_i} commutes with $u_{s_i a}$) we obtain

$$\begin{aligned} g(hvh^{-1})g^{-1} &= u_{a_i}(r)u_{s_i a} \left(\left(\frac{\mu}{\lambda S} \right)^{2\varepsilon_a \delta_{ii}(a)} k \right) u_{a_i}(r)^{-1} \\ &= u_{s_i a} \left(\left(\frac{\mu}{\lambda S} \right)^{2\varepsilon_a \delta_{ii}(a)} k \right), \end{aligned}$$

and by (2C3) and (2C1) again

$$\begin{aligned} (gh)v(gh)^{-1} &= [u_i(r - \lambda^2 R^{-1})m_i(-\lambda \mu R^{-1})]u_a(k)[\dots]^{-1} \\ &= u_{s_i a} \left(\left(\frac{R}{\lambda \mu} \right)^{2\varepsilon_a \delta_{ii}(a)} k \right). \end{aligned}$$

Using $R = \frac{\mu^2}{S}$, we obtain the desired equality $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$ in this case.

Finally if $S = 0$, by (2C3) and (2C1) we have

$$\begin{aligned} hvh^{-1} &= [u_i(s)m_i(\mu)]u_a(k)[\dots]^{-1} \\ &= u_{a_i}(s)u_{s_i a}\left(\frac{k}{\mu^{2\varepsilon_a\delta_{i(a)}}}\right)u_{a_i}(s)^{-1} = u_{s_i a}\left(\frac{k}{\mu^{2\varepsilon_a\delta_{i(a)}}}\right). \end{aligned}$$

By (2C3), $m_i(\lambda)u_{s_i a}(k)m_i(\lambda)^{-1} = u_a(\lambda^{2\varepsilon_a\delta_{i(a)}})$. Using this and (2C2),

$$\begin{aligned} g(hvh^{-1})g^{-1} &= [u_i(r)m_i(\lambda)]u_{s_i a}(\mu^{-2\varepsilon_a\delta_{i(a)}}k)[\dots]^{-1} \\ &= u_{a_i}(r) \cdot u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\varepsilon_a\delta_{i(a)}}k\right) \cdot u_{a_i}(r)^{-1}. \end{aligned}$$

Again with $S = 0$, conjugation of v by gh gives by (2C4) and (2C2) that

$$\begin{aligned} (gh)v(gh)^{-1} &= [u_i(r - \lambda^2 R^{-1})m_i(-\lambda\mu R^{-1})u_i(-\mu^2 R^{-1})]u_a(k)[\dots]^{-1} \\ &= u_i\left(r - \frac{\lambda^2}{R}\right)u_{a_i}\left(\frac{-\lambda^2\mu^2 R^{-2}}{-\mu^2 R^{-1}}\right)u_a\left(\left(\frac{\lambda\mu R^{-1}}{-\mu^2 R^{-1}}\right)^{2\varepsilon_a\delta_{i(a)}}k\right) \\ &\quad \cdot u_{a_i}(\dots)^{-1}u_i(\dots)^{-1} \\ &= u_{a_i}(r) \cdot u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\varepsilon_a\delta_{i(a)}}k\right) \cdot u_{a_i}(r)^{-1}, \end{aligned}$$

hence $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$.

The case of a generator $v = u_a(k)$ of the second type is simpler, because $a \in \mathcal{P}_i \setminus \{a_i\}$, so u_{a_i} commutes with u_a . Using (2C2), (2C3) and (2C4) we obtain

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_{a_i}\left(r - \frac{\lambda^2}{R}\right) \cdot u_{s_i a}\left(\left(\frac{R}{\lambda\mu}\right)^{2\varepsilon_a\delta_{i(a)}}\right) \cdot u_{a_i}(\dots)^{-1}.$$

2.D.2. “big cell · big cell ∈ Borel”, with g and h as above. We now have $R = 0$ and a calculation in SL_2 shows that $gh = u_i\left(r + \frac{\lambda^2}{\mu^2}s'\right)t_i\left(\frac{-\lambda}{\mu}\right)$. Let v be a generator of the first type. For $g(hvh^{-1})g^{-1}$, we compute as in 2.D.1 (with $R = 0$), the case $S \neq 0$ but $R - \mu^2 S^{-1} = 0$ being excluded. We obtain in any case $g(hvh^{-1})g^{-1} = u_{a_i}\left(r + \frac{S\lambda^2}{\mu^2}\right) \cdot u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\varepsilon_a\delta_{i(a)}}k\right) \cdot u_{a_i}(\dots)^{-1}$, which equals $(gh)v(gh)^{-1}$ by (2C2) and (2B1). For a generator of the second type, we have $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_a\left(\left(\frac{\lambda}{\mu}\right)^{2\varepsilon_a\delta_{i(a)}}k\right)$.

2.D.3. “Borel · big cell ∈ big cell”, with $g = u_i(r)t_i(\lambda)$, $h = u_i(s)m_i(\mu)u_i(s')$. A calculation in SL_2 shows that $gh = u_i\left(r + \lambda^2 s\right)m_i(\lambda\mu)u_i(s')$.

For a generator of the first type, we have for $S \neq 0$

$$\begin{aligned} g(hvh^{-1})g^{-1} &= (gh)v(gh)^{-1} \\ &= u_{a_i} \left(r + \lambda^2 s - \frac{\lambda^2 \mu^2}{S} \right) u_a \left(\left(\frac{\lambda \mu}{S} \right)^{2\varepsilon_a \delta_{ii(a)}} k \right) u_{a_i}(\dots)^{-1}, \end{aligned}$$

and for $S = 0$

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_{s_i a} \left(\left(\frac{1}{\mu \lambda} \right)^{2\varepsilon_a \delta_{ii(a)}} k \right).$$

For a generator of the second type, we have

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_{a_i}(r + \lambda^2 s) u_{s_i a} \left(\left(\frac{1}{\mu \lambda} \right)^{2\varepsilon_a \delta_{ii(a)}} k \right) u_{a_i}(\dots)^{-1}.$$

2.D.4. “big cell \cdot Borel \in big cell”, with $g = u_i(r)m_i(\lambda)u_i(r')$, $h = u_i(s)t_i(\mu)$. A calculation in SL_2 shows that $gh = u_i(r)m_i(\frac{\lambda}{\mu})u_i(\frac{R}{\mu^2})$. Write $T = R + \mu^2 t$, and note that $R \neq 0$ in this case. Let us deal with a generator of the first type. Using (2C) and (2B1) we have $hvh^{-1} = u_{a_i}(s + \mu^2 t) \cdot u_a(\mu^{2\varepsilon_a \delta_{ii(a)}} k) \cdot u_{a_i}(\dots)^{-1}$. Hence under the assumption that $T \neq 0$, by (2C4) and (2C2) we get $g(hvh^{-1})g^{-1} = u_{a_i}(r - \frac{\lambda^2}{T}) \cdot u_a(\left(\frac{\lambda \mu}{T}\right)^{2\varepsilon_a \delta_{ii(a)}} k) \cdot u_{a_i}(\dots)^{-1}$, which equals $(gh)v(gh)^{-1}$, since by (2C4) and (2C2) we obtain

$$\begin{aligned} (gh)v(gh)^{-1} &= \left[u_i(r)m_i(\lambda \mu^{-1})u_i\left(\frac{T}{\mu^2}\right) \right] u_a(k)[\dots]^{-1} \\ &= \left[u_i(r)u_i\left(\frac{-\lambda^2}{T}\right) \right] u_a\left(\left(\frac{\lambda \mu}{T}\right)^{2\varepsilon_a \delta_{ii(a)}} k\right)[\dots]^{-1}. \end{aligned}$$

When $T = 0$, we simply have

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_{s_i a} \left(\left(\frac{\mu}{\lambda} \right)^{2\varepsilon_a \delta_{ii(a)}} k \right).$$

For a generator of the second type, we have in any case

$$g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1} = u_i(r) \cdot u_{s_i a} \left(\left(\frac{\mu}{\lambda} \right)^{2\varepsilon_a \delta_{ii(a)}} k \right) \cdot u_i(\dots)^{-1}.$$

2.D.5. “Borel \cdot Borel \in Borel”, with $g = u_i(r)t_i(\lambda)$, $h = u_i(s)t_i(\mu)$ and $gh = u_i(r + \lambda^2 s)t_i(\lambda \mu)$. We have $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$, equal to $u_{a_i}(r + \lambda^2 s + (\lambda \mu)^2 t) u_a((\lambda \mu)^{2\varepsilon_a \delta_{ii(a)}} k) u_{a_i}(\dots)^{-1}$ for a generator of the first type and to $u_a((\lambda \mu)^{2\varepsilon_a \delta_{ii(a)}} k)$ for a generator of the second type.

2.E. Twin root datum. Let $i \in \{0; 1\}$. By definition of U^i as a normal closure, the product map provides a surjective group homomorphism $U_{a_i} \times U^i \twoheadrightarrow U_+$. By the very presentation of U_+ , we have another surjective group homomorphism $U_+ \twoheadrightarrow U_{a_i} \times U^i$ whose composition with the previous one gives the identity automorphism of U_+ . Therefore we have $U_+ \cong U_{a_i} \times U^i$.

Definition. (i) We call $P_i = L_i \times U^i$ the (standard) parabolic subgroup of type i .
 (ii) Associated to the fields \mathbb{K}_0 and \mathbb{K}_1 , we define the amalgam $\Lambda = P_0 *_B P_1$.

Note that (i) makes sense precisely thanks to the computation made in 2.D. By the remark before the definition we have $B \cong (T \times U_{a_i}) \times U^i$, and $T \times U_{a_i}$ is isomorphic to the upper triangular Borel subgroup $T \times U_i$ of the Levi factor $L_i = \text{SL}_2(\mathbb{K}_i) \times \mathbb{K}_{1-i}^\times$. In view of the actions of the subgroups $T = \mathbb{K}_i^\times \times \mathbb{K}_{1-i}^\times$ and U_i of L_i on U^i , the subgroup $TU_i \times U^i$ of P_i is isomorphic to B , and the amalgam in (ii) above makes sense. Moreover by the Bruhat decomposition of SL_2 , each parabolic subgroup admits the decomposition

$$(\star) \quad P_i = B \sqcup U_i m_i B$$

for each $i \in \{0; 1\}$. We call N the subgroup of Λ generated by the elements $m_i(\lambda)$ for $i \in \{0; 1\}$ and λ in \mathbb{K}_i^\times ; it is an extension of the Weyl group $D_\infty = \langle s_0, s_1 \rangle$ by T , which provides a natural isomorphism $\varphi: W \cong N/T$ under which s_i is sent to the class $m_i(\lambda)T$. Let now $a = w.a_i$ be a root in Φ . The conjugation of a simple root group U_{a_i} by $n \in N$ only depends on the class nT (since T normalizes each root group), and in fact $\varphi(w)U_{a_i}\varphi(w)^{-1}$ only depends on the root a (by uniqueness of the writing $a = w.a_i$). Moreover the action of the elements $m_i(\lambda)$ on U^i shows that in the case of a positive root a , the group $\varphi(w)U_{a_i}\varphi(w)^{-1}$ corresponds to the group U_a attached to a in the presentation defining U_+ (2.B).

Theorem. *The group Λ satisfies the axioms of a twin root datum for the family of root groups $\{U_a = u_a(\mathbb{K}_{\iota(a)})\}_{a \in \Phi}$ above. As a consequence, there exists a semihomogeneous Moufang twin tree of valencies $1 + |\mathbb{K}_0|$ and $1 + |\mathbb{K}_1|$ for any choice of two fields \mathbb{K}_0 and \mathbb{K}_1 .*

Proof. The group Λ is defined as an amalgam, so by [S, I.4, Theorem 7] it acts edge-transitively on the semihomogeneous tree Δ of valencies $[P_0 : B]$ and $[P_1 : B]$. By (\star) above, these valencies are $1 + |\mathbb{K}_0|$ and $1 + |\mathbb{K}_1|$. The stabilizer of an edge (or a vertex of type 0, resp. of type 1) is conjugate to B (or to P_0 , resp. P_1). We have the following identification of Λ -sets:

$$\Delta \simeq \Lambda/B \sqcup \Lambda/P_0 \sqcup \Lambda/P_1.$$

We recover the simplicial structure of the tree Δ by the reversed inclusion relation on stabilizers. Let us set $t = m_1 m_0$. The set of edges $\{t^n B\}_{n \in \mathbb{Z}} \sqcup \{t^n m_1 B\}_{n \in \mathbb{Z}}$ defines

a geodesic in the tree: this is our standard apartment \mathbb{A} , containing the standard edge B . Let us check the (TRD) axioms. The positive root groups are $t^{-n}U_0t^n$, $t^n m_1 U_0 m_1^{-1} t^{-n}$, $t^n U_1 t^{-n}$ or $t^{-n} m_0 U_1 m_0^{-1} t^n$ for $n \geq 0$, and the negative root groups are defined similarly. A pair of roots $\{a; b\}$ is prenilpotent if $a \supset b$ or $b \supset a$. Conjugating by a suitable power of t allows us to see prenilpotent pairs of roots as pairs of positive roots. Since the group U_+ is defined in such a way that the commutation relations are trivial for all prenilpotent pairs, we get axiom (TRD1). By definition, Λ is generated by P_0 and P_1 , hence by the positive root groups, by T and by the two root groups indexed by the opposite of the simple roots: this is (TRD4). Axiom (TRD2) follows from the relations in SL_2 defining the elements $m_i(\lambda)$, and from the definition of the root groups. By the definition of B , T normalizes the root groups, which is the second half of axiom (TRD0). By the definition of Δ , U_i is simply transitive on the edges whose closure contains the vertex P_i and different from the standard edge: this proves the first assertion of (TRD0) and the second one in (TRD3). The first assertion in (TRD3) follows from what has already been proved [A, §1, Remark 2]. \square

3. Twinning right-angled buildings

The previous section provided Moufang twin trees defined by generators and relations. A more general construction is stated in [T4], but for our purposes we need the specific generators and relations of Section 2. In this section we fix a right-angled Coxeter system (W, S) , meaning that for all s_i and s_j in S , either s_i and s_j commute or generate an infinite dihedral group (i.e. $m_{ij} = 2$ or ∞ , so this is a system in which all edges of the diagram are labelled by ∞). We assume a connected diagram. For the definition of the root system Φ we refer again to 1.A.

3.A. Root system. For a right-angled Coxeter system (W, S) , the Moussong–Davis realization ([D] [Mou]) of the Coxeter complex is a cubical complex. The chamber, which we denote by C , is constructed topologically as follows. Let E_S be the standard Euclidean vector space of finite dimension $|S|$ with orthonormal basis $\{e_s\}_{s \in S}$. Let $I \subset S$ be a spherical subset of S (in our case a subset such that $W_I = \langle s : s \in I \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{|I|}$). Let us consider the subspace E_I generated by the vectors e_s for $s \in I$. We denote by C_I the cube generated by the $2^{|I|}$ vectors of E_I with coordinates equal to 0 or 1. Then the (closed) Moussong–Davis chamber C is the union in E_S of all the cubes C_I when I runs over the spherical subsets of S . Its panel Π_s of type $\{s\}$ is the intersection of C with the hyperplane of vectors of coordinate 1 along e_s .

Vertices in C correspond to maximal spherical subsets of S and the inclusion of a canonical generator s in a maximal spherical subset I (with at least two elements) corresponds to the inclusion of a vertex v_I in the panel Π_s . The sub-

group $\langle t : t \in I \setminus \{s\} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{|I|-1}$ acts simple transitively on the panels of type s around v_I in the Coxeter complex and preserves ∂a_s . This implies that all panels in the boundary wall ∂a_s have the same type, and that the Coxeter subgroup $W^{\{s\}} = \langle j : j \in S \setminus \{s\} \text{ and } [j, s] = 1 \rangle$ acts transitively on the panels in ∂a_s . The latter fact also implies that $\text{Stab}_W(\partial a_s) = W^{\{s\}} \times \langle s \rangle$ and $\text{Stab}_W(a_s) = W^{\{s\}}$ (since if $w \in \text{Stab}_W(\partial a_s)$ then there is $w' \in W^{\{s\}}$ such that $(w')^{-1}w \cdot \Pi_s = \Pi_s$).

We have seen that given a root a the panels on its boundary wall all have the same type. We shall refer to this as the *type* of the root a , and denote it by $\iota(a) \in S$. Finally for a root a , we define $\varepsilon_a \in \{\pm 1\}$ to be $(-1)^m$, where m is the number of walls of type $\iota(a)$ meeting the interior of any geodesic segment from a point $x \in C \setminus \bigcup_{s \in S} \Pi_s$ to the wall ∂a .

Example. Let us consider a regular right-angled r -gon R of the hyperbolic plane \mathbb{H}^2 . We number cyclically by $\mathbb{Z}/r\mathbb{Z}$ the edges $\{E_i\}_{i \in \mathbb{Z}/r\mathbb{Z}}$ of R and for each $i \in \mathbb{Z}/r\mathbb{Z}$ we let s_i denote the reflection in the geodesic supporting E_i . Then the subgroup W of $\text{PSL}_2(\mathbb{R})$ generated by the reflections s_i is a Coxeter group, and $(W, \{s_i\}_{i \in \mathbb{Z}/r\mathbb{Z}})$ is a right-angled Coxeter system. In the above construction, the chamber C is the union in \mathbb{R}^r of r squares $C_{\{i; i+1\}}$, and each panel Π_i is the union of two edges, one in $C_{\{i-1; i\}}$ and one in $C_{\{i; i+1\}}$. To recover the original tiling of \mathbb{H}^2 by R , we can so to speak “hyperbolize” each Euclidean square $C_{\{i; i+1\}}$ into a hyperbolic square, by replacing the right angle of $C_{\{i; i+1\}}$ at the origin of \mathbb{R}^r by an angle $\frac{2\pi}{r}$.

3.B. Borel subgroup. Unipotent radicals. For each $s \in S$, we pick a field \mathbb{K}_s . To each positive root a of type s is attached a copy of the additive group $(\mathbb{K}_s, +)$. We denote it by $U_a = \{u_a(k) : k \in \mathbb{K}_s\}$, where u_a is a chosen isomorphism from $(\mathbb{K}_s, +)$ to U_a .

Definition. (i) We define the *standard torus* to be the direct product $T = \prod_{s \in S} \mathbb{K}_s^\times$ of the multiplicative groups of the chosen fields.

(ii) We define the group U_+ by generators and relations: it is the quotient of the free product $*_{a \in \Phi_+} U_a$ by the relations $[U_a, U_b] = \{1\}$ whenever $\{a; b\}$ is prenilpotent.

We view T as the maximal torus $\prod_{s \in S} t_s(\mathbb{K}_s^\times)$ of $\prod_{s \in S} \text{SL}_2(\mathbb{K}_s)$ (the notation t_s refers to the same parametrization of the diagonal matrices of SL_2 as in Example 1 of 1.A). As in (2B1), we make T act on U_+ by

$$t_s(\lambda)u_a(k)t_s(\lambda)^{-1} = u_a(\lambda^{2\varepsilon_a \delta_{s\iota(a)}} k)$$

for any $s \in S$, any $a \in \Phi_+$ and any $\lambda \in \mathbb{K}_s^\times$ and $k \in \mathbb{K}_{\iota(a)}$. In particular, $t_s(\lambda)$ centralizes any root group of type $\neq s$.

Definition. We define the (*standard*) *Borel subgroup* B to be the semi-direct product $T \ltimes U_+$.

We can now introduce unipotent radicals. Let I be a spherical subset of S , and let $R(I)$ denote the corresponding set of simple roots $\{a_i\}_{i \in I}$. Thus for any two $i, j \in I$, we have $[U_{a_i}, U_{a_j}] = \{1\}$.

Definition. We denote by V^I the subgroup of U_+ generated by all the positive root groups except the groups U_{a_i} for $i \in I$, and we denote by U^I the normal closure of V^I in U_+ .

For any positive root $a \notin R(I)$, we set $I(a) = \{j \in I : a_j \text{ is not prenilpotent with } a\}$. Then for any choice of parameters $k_a \in \mathbb{K}_{I(a)}$ and $k_j \in \mathbb{K}_j$, $j \in I(a)$, the element

$$\left(\prod_{j \in I(a)} u_{a_j}(k_j) \right) \cdot u_a(k_a) \cdot \left(\prod_{j \in I(a)} u_{a_j}(k_j) \right)^{-1}$$

lies in U^I , and the family of all such elements is a generating system for U^I .

3.C. Actions of Levi factors on unipotent radicals. We first define the Levi factors.

Definition. Let I be a spherical subset of S . We define the (*standard*) *Levi factor of type I* to be the group $L_I = \prod_{s \in I} \mathrm{SL}_2(\mathbb{K}_s) \times \prod_{s \notin I} t_s(\mathbb{K}_s^\times)$.

Let us define now the action of individual elements in the group L_I on U^I . First, the T -action was defined in 3.B (its action on U_+ restricts to U^I). Let $v = \left(\prod_{j \in I(a)} u_{a_j}(k_j) \right) \cdot u_a(k_a) \cdot \left(\prod_{j \in I(a)} u_{a_j}(k_j) \right)^{-1}$ be an element of U^I in the form given in 3.B. Let us pick $s \in I$ and $k \in \mathbb{K}_s$. If $s \notin I(a)$ we assume that $u_s(k)$ centralizes v . Otherwise, we define $u_s(k)v u_s(k)^{-1}$ to be

$$\left(\prod_{j \in I(a) \setminus \{s\}} u_{a_j}(k_j) \right) u_{a_s}(k_s + k) \cdot u_a(k_a) \cdot u_{a_s}(k_s + k)^{-1} \left(\prod_{j \in I(a) \setminus \{s\}} u_{a_j}(k_j) \right)^{-1}.$$

Let us now pick $s \in I$ and $\lambda \in \mathbb{K}_s^\times$. If $s \notin I(a)$ or if $s \in I(a)$ but $k_s = 0$, we set

$$m_s(\lambda) v m_s(\lambda)^{-1} = \left(\prod_{j \in I(a)} u_{a_j}(k_j) \right) \cdot u_{s(a)}(\lambda^{-2\varepsilon_a \delta_{s(a)}} k_a) \cdot \left(\prod_{j \in I(a)} u_{a_j}(k_j) \right)^{-1}.$$

Otherwise, i.e. if $s \in I(a)$ and $k_s \neq 0$, we define $m_s(\lambda) v m_s(\lambda)^{-1}$ to be

$$\begin{aligned} & \left(\prod_{j \in I(a) \setminus \{s\}} u_{a_j}(k_j) \right) u_{a_s} \left(\frac{-\lambda^2}{k_s} \right) \\ & \cdot u_a \left(\left(\frac{-\lambda}{k_s} \right)^{2\varepsilon_a \delta_{s(a)}} k_a \right) \cdot u_{a_s} \left(\frac{-\lambda^2}{k_s} \right)^{-1} \left(\prod_{j \in I(a) \setminus \{s\}} u_{a_j}(k_j) \right)^{-1}. \end{aligned}$$

In other words, up to the conjugating element $\left(\prod_{j \in I(a) \setminus \{s\}} u_{a_j}(k_j) \right)$, the formulas are copied from the tree case (2.C).

3.D. Checking the product relation. We now check that these individual actions provide an action by L_I ; this is done by checking that $g(hvh^{-1})g^{-1} = (gh)v(gh)^{-1}$ for any generator $v \in U^I$ and any $g, h \in L_I$. The Levi factor of type I is a direct product and the actions of different factors in 3.C are easily seen to commute with one another, so it is enough to check each factor separately. We are reduced to showing that for a single factor $SL_2(\mathbb{K}_s)$, the actions of $u_s(k)$, $t_s(\lambda)$ and $m_s(\lambda)$ as above provide an $SL_2(\mathbb{K}_s)$ -action. We noted in 3.C that the formulas defining the individual actions are the same as those of the tree case up to conjugation by elements of the form $(\prod_{j \in I(a) \setminus \{s\}} u_{a_j}(k_j))$, which are all centralized by $SL_2(\mathbb{K}_s)$. Therefore the latter verification follows from the computation made in 2.D. This finally shows that the Levi factor L_I acts on the unipotent radical U^I .

Definition. Let I be a spherical subset of S . We define the (standard) parabolic subgroup of type I to be $P_I = L_I \ltimes U^I$.

The definition of U^I in 3.B implies that $U_+ = U_I \cdot U^I$, where $U^I \triangleleft U_+$ and $U_I = \prod_{s \in I} U_{a_s}$. This provides a surjective group homomorphism $U_I \ltimes U^I \twoheadrightarrow U_+$. The presentation defining U_+ implies the existence of another surjective group homomorphism $U_+ \twoheadrightarrow U_I \ltimes U^I$, and the composition of these two homomorphisms is the identity map of U_+ . Therefore we have $U_+ \cong U_I \ltimes U^I$ for any spherical subset I of S . This implies that for an inclusion $J \subset I$ of spherical subsets, we have $U^J \cong U_{I \setminus J} \ltimes U^I$ where $U_{I \setminus J} = \prod_{s \in I \setminus J} U_{a_s}$.

3.E. Twin root datum. We consider the group

$$P_I^J = \prod_{s \in J} SL_2(\mathbb{K}_s) \times \prod_{s \in I \setminus J} (t_s(\mathbb{K}_s^\times) \ltimes U_s) \times \prod_{s \notin I} t_s(\mathbb{K}_s^\times),$$

which is a standard parabolic subgroup of the Levi factor L_I , containing the Levi factor L_J . In the middle factor, the group $t_s(\mathbb{K}_s^\times) \ltimes U_s$ is the upper triangular Borel subgroup of $SL_2(\mathbb{K}_s)$. In the language of parabolic subgroups of reductive groups, the group $\prod_{s \in I \setminus J} U_s$ is the unipotent radical of the parabolic subgroup P_I^J of L_I , and we can also write

$$P_I^J = L_J \ltimes \prod_{s \in I \setminus J} U_s.$$

We have

$$P_I = L_I \ltimes U^I \supset P_I^J \ltimes U^I = \left(L_J \ltimes \prod_{s \in I \setminus J} U_s \right) \ltimes U^I \cong L_J \ltimes \left(\prod_{s \in I \setminus J} U_s \ltimes U^I \right).$$

The group $\prod_{s \in I \setminus J} U_s$ lies in the Levi factor L_I whereas the group $U_{I \setminus J} = \prod_{s \in I \setminus J} U_{a_s}$ of the previous subsection lies in U^J . But these two groups are naturally isomorphic and under the isomorphism, the L_I -action on U^I (defined in 3.C)

restricted to $\prod_{s \in I \setminus J} U_s$ is the $U_{I \setminus J}$ -action on U^I by conjugation given by the presentation of U_+ . By $U^J \cong U_{I \setminus J} \times U^I$ (end of 3.D), this shows that for any inclusion $J \subset I$ as above we have an injective group homomorphism $P_J \hookrightarrow P_I$. Finally, varying such inclusions we can see the finite set of standard spherical parabolic subgroups as an inductive system of groups with injective transition maps.

Definition. (i) We denote by Λ the limit of the inductive system indexed by the inclusions of spherical subsets I , where the groups are the corresponding parabolic subgroups P_I .

(ii) We denote by N its subgroup generated by the elements $m_s(\lambda)$ when s varies in S and λ in \mathbb{K}_s^\times .

Remarks. 1) Since any transition map of the inductive system is injective, any parabolic subgroup P_I is naturally a subgroup of Λ .

2) In the terminology of [HP, Section 5], Λ is the fundamental group of the complex of groups indexed by the cone over the first barycentric subdivision of the finite nerve of the Coxeter system (W, S) . This complex of groups is described precisely in the remark following [loc. cit., Proposition 5.1], and to recover our situation it is enough to replace the group G_J therein by the parabolic subgroup P_J (with the convention that $B = P_\emptyset$).

In view of the relation $m_s(\lambda)m_s(-1) = t_s(\lambda)$, we see that N contains T , and in fact $T \triangleleft N$. For each spherical subset I we denote by N_I the subgroup of the Levi factor L_I generated by T and the elements $m_s(1)$, $s \in I$. Then we obtain another inductive system indexed by the spherical subsets, whose limit is N . The Coxeter group W itself is also the limit of the inductive system obtained from the previous one by factoring out by the intersections with T , and there is a natural isomorphism $\varphi: W \cong N/T$ under which s is sent to the class $m_s(\lambda)T$.

Now let $a = w.a_s = w'.a_t$ be a root of Φ . Since we are dealing with right-angled Coxeter groups (3.A) we have $s = t$ and $w^{-1}w' \in W^{\{s\}} = \langle j : j \in S \setminus \{s\} \text{ and } [j, s] = 1 \rangle$. As usual we can identify w with a coset of T in N , and since T normalizes the root groups U_{a_s} , the conjugate $wU_{a_s}w^{-1}$ is well-defined. Therefore we can write

$$w'U_{a_t}w'^{-1} = w(w^{-1}w')U_{a_s}(w^{-1}w')^{-1}w^{-1} = wU_{a_s}w^{-1},$$

which shows that $wU_{a_s}w^{-1}$ only depends on the root a . Moreover the action of the elements $m_s(\lambda)$ on $U^{\{s\}}$ shows that in the case of a positive root a , the group $wU_{a_s}w^{-1}$ corresponds to the group U_a attached to a in the presentation defining U_+ (3.B).

Our main constructive result about twinings of right-angled buildings can now be stated.

Theorem. (i) *The triple $(\Lambda, \{U_a\}_{a \in \Phi}, T)$ is a twin root datum in which for any prenilpotent pair of roots the corresponding root groups commute.*

(ii) *Every right-angled building with a parameter system of cardinalities of projective lines belongs to a Moufang twinning.*

Proof. (i) We have to prove axioms (TRD0) to (TRD4) of 1.A, together with the improvement of (TRD1) on the trivial commutation of root groups. (TRD0) and the second part of (TRD2) are clear by the discussion preceding the statement of the theorem, where the subgroup N and the root groups are defined. The first part of (TRD2) follows from a simple computation in SL_2 : $m_s(\lambda) = u_{-a_s}(\lambda^{-1})u_{a_s}(\lambda)u_{-a_s}(\lambda^{-1})$ (see Example 1 in 1.A) and [T5, comment following 3.3] for the uniqueness assertion). Let now $\{a; b\}$ be a prenilpotent pair of roots. There is $w \in W$ such that $w.\{a; b\} \subset \Phi_+$, so that the conjugates $\varphi(w)U_a\varphi(w)^{-1}$ and $\varphi(w)U_b\varphi(w)^{-1}$ lie in U_+ , in which the commutation relation is trivial by definition. This proves the improvement of (TRD1). The parabolic subgroup $P_{\{s\}}$ injects in Λ and we have $P_{\{s\}} = U_{a_s}m_sB \sqcup B$ with $U_{-a_s} \setminus \{1\} \subset U_{a_s}m_sB$. This proves the second assertion of (TRD3) and the first assertion follows from all the other axioms [A, §1, Remark 2]. It remains to prove (TRD4). This is clear because Λ is generated by the parabolic subgroups P_I and any such group is generated by the root groups it contains.

(ii) The twin root datum structure implies that Λ naturally acts on a twin building Δ_{\pm} with Weyl group $W \cong N/T$ and that Δ_{\pm} enjoys the Moufang property [T5]. Moreover any residue of type $\{s\}$, $s \in S$, is isomorphic to the building $\mathbb{P}^1(\mathbb{K}_s)$. But it follows from [HP, Proposition 5.1] that, up to isomorphism, there is a unique right-angled building of given Coxeter type (W, S) and parameter system. This uniqueness implies (ii) when the parameter system consists of cardinalities of projective lines. \square

Remark. In 4.B, we will need a slight modification of the construction. We will need to replace the group SL_2 by the group GL_2 , in order to obtain local actions on residues of panels via PGL_2 instead of PSL_2 . This can be done as follows. In the definition of T (3.B), each factor $t_s(\mathbb{K}_s^{\times}) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{K}_s^{\times} \right\}$ can be replaced by the group $\left\{ d_s(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} : \lambda, \mu \in \mathbb{K}_s^{\times} \right\}$, and in the definition of the Levi subgroups L_I , each factor $SL_2(\mathbb{K}_s)$ can be replaced by a factor $GL_2(\mathbb{K}_s)$ (3.C). In the actions given in 3.B and 3.C, the only change concerns the diagonal matrices of $GL_2(\mathbb{K}_s)$ acting on root groups, for which we shall set the formula

$$d_s(\lambda, \mu)u_a(k)d_s(\lambda, \mu)^{-1} = u_a\left(\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}^{\varepsilon_a\delta_{st(a)}} k \right)$$

for any $s \in S$, any $a \in \Phi_+$ and any $\lambda, \mu \in \mathbb{K}_s^{\times}$ and $k \in \mathbb{K}_{t(a)}$. This generalizes the formula $t_s(\lambda)u_a(k)t_s(\lambda)^{-1} = u_a(\lambda^{2\varepsilon_a\delta_{st(a)}}k)$ of 3.B since $t_s(\lambda) = d_s(\lambda, \lambda^{-1})$. Then the group Λ generated by T and the root groups has a twin root datum structure $(\Lambda, \{U_a\}_{a \in \Phi}, T)$ with Weyl group the right-angled Coxeter group W .

4. Non-linearities

A Kac–Moody group defined over \mathbb{F}_q has no reason to be linear in characteristic p (where q is a power of p) merely because it is defined by a presentation involving subgroups that are linear over \mathbb{F}_q . Moreover an infinite Kac–Moody group over \mathbb{F}_q cannot be linear over any field of characteristic different from p [R 3]. In Sections 2 and 3 we constructed groups of root datum type with more than one ground field and we now show that if these fields have different characteristics, then the group cannot be linear.

4.A. Non-linearity of Borel subgroups. Our result actually shows that even much smaller subgroups of the alluded to above groups have strong non-linearity properties.

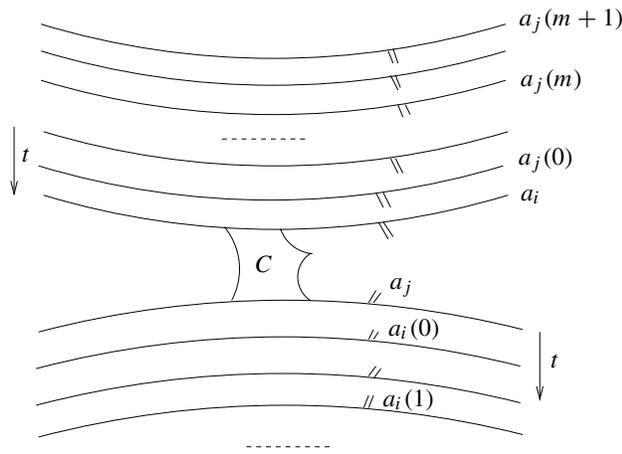
Theorem. *Let (W, S) be a right-angled Coxeter system with Coxeter diagram \mathcal{D} . Let $\{\mathbb{K}_s\}_{s \in S}$ be a choice of fields with two fields \mathbb{K}_s and \mathbb{K}_t of different characteristics for $s \neq t$ in the same connected component of \mathcal{D} . Let Λ be the corresponding group defined in 3.E and let Γ be the fixator of some chamber in the associated twinned buildings. Then, any group homomorphism $\rho: \Gamma \rightarrow \prod_{\alpha \in A} \mathbb{G}_\alpha(\mathbb{F}_\alpha)$ has infinite kernel, whenever the index set A is finite and \mathbb{G}_α is a linear algebraic group defined over the field \mathbb{F}_α for each $\alpha \in A$.*

Proof. Any two chamber-fixators are isomorphic, so we deal with the standard positive chamber C (3.A). Its fixator is $B = T \times U_+$ (3.B), and we shall use the notation B rather than Γ . Let \mathcal{C} be the connected component of the vertices s and t in the Coxeter diagram of (W, S) . We let \mathcal{C}_s denote the subset of vertices in \mathcal{C} to which are attached fields having the same characteristic as \mathbb{K}_s . The complement $\mathcal{C} \setminus \mathcal{C}_s$ is non-empty since it contains t . Taking an edge connecting \mathcal{C}_s to $\mathcal{C} \setminus \mathcal{C}_s$ we obtain an edge with one vertex, say i , with corresponding field \mathbb{K}_i of characteristic p_i and the other vertex, say j , with corresponding field \mathbb{K}_j of characteristic $p_j \neq p_i$.

Notice that s_i and s_j generate an infinite dihedral group; set $t = s_j s_i$. To simplify notation the root a_{s_i} will be denoted a_i . For any integer $m \geq 0$, the root $a_i(m) = t^m \cdot (s_j \cdot a_i)$ is a positive root of Φ . Similarly, for any integer $m \geq 0$ we set $a_j(m) = t^{-m} \cdot (s_i \cdot a_j)$ and we have $a_j(m) \in \Phi_+$. For any $m' \geq m$ we have $a_i(m') \supset a_i(m)$, implying that the pair of roots $\{a_i(m'); a_i(m)\}$ is prenilpotent. In view of the defining relations of Λ , the subgroup of B generated by the root groups $U_{a_i(m)}$, $m \geq 0$, is the direct sum of countably many copies of \mathbb{K}_i ; so it is an infinite group of exponent p_i , which we denote by V_i . Similarly, the group $V_j = \langle U_{a_j(m)} : m \geq 0 \rangle$ is an infinite subgroup of exponent p_j in B .

We set $A_i = \{\alpha \in A : \text{char}(\mathbb{F}_\alpha) = p_i\}$ and $A_j = \{\alpha \in A : \text{char}(\mathbb{F}_\alpha) = p_j\}$. For each $\alpha \in A$, let $\text{pr}_\alpha : \prod_{\alpha \in A} \mathbb{G}_\alpha(\mathbb{F}_\alpha) \rightarrow \mathbb{G}_\alpha(\mathbb{F}_\alpha)$ be the natural projection. By [Mar, VIII.3.7] for each $\alpha \notin A_i$ the group $(\text{pr}_\alpha \circ \rho)(V_i)$ is finite, so there is a finite index normal subgroup $N_i \triangleleft V_i$ such that $\prod_{\alpha \notin A_i} (\text{pr}_\alpha \circ \rho)(N_i)$ is trivial, and similarly

for j replacing i . The subgroup generated by $U_{a_i}, U_{a_j}, U_{-a_i} = m_i U_{a_i} m_i^{-1}$ and $U_{-a_j} = m_j U_{a_j} m_j^{-1}$ satisfies the axioms of a twin root datum with infinite dihedral Weyl group, and by applying [KP, Proposition 3.5 (c)] to the corresponding refined Tits system (1.A), we have a free product decomposition of its subgroup generated by the positive root groups of type i and j . This shows that $V_i * V_j$ injects in B , and if we pick $v \in N_i \setminus \{1\}$ and $v' \in N_j \setminus \{1\}$, then vv' has infinite order. But since $v \in N_i$ and $v' \in N_j$ the images $\rho(v)$ and $\rho(v')$ commute with one another and both have finite order. We have found in B a subgroup isomorphic to \mathbb{Z} with finite image under ρ . \square



If we assume that each field \mathbb{K}_s in the theorem is a finite field \mathbb{F}_{q_s} , then we are in the situation of 1.B and 1.C: we have a locally finite building $\Delta_+ = \Delta$ and a locally compact group G (1.B), which we see as a completion of Λ .

Recall moreover that a discrete subgroup D in a topological group H is called a *lattice* if the homogeneous space H/D carries a finite H -invariant measure [Mar, I.0.40]. A lattice $D < H$ is called *uniform* if the quotient space H/D is compact. The theory of lattices in Lie groups shows that it is extremely useful to see an abstract discrete group D as a lattice in a topological group H (with a much richer structure). The connection with our specific situation is the following. Let $W(t)$ denote the growth series of the Weyl group W with respect to the generating system S , meaning the power series $\sum_{n \geq 0} c_n t^n$ where c_n is the number of words of length n . If $W(\frac{1}{q})$ converges for $q = \min_{s \in S} q_s$, then this implies that any spherical negative parabolic subgroup Γ of Λ is a (non-uniform) lattice of G [R 2]. Combining this with the previous theorem provides the next result.

Corollary. *In the same situation as in the theorem, we assume further that each field \mathbb{K}_s is a finite field \mathbb{F}_{q_s} , that Γ is the fixator of some negative facet and that*

$q = \min_{s \in S} q_s$ is large enough so that Γ is a lattice in G . Then, any group homomorphism $\rho: \Gamma \rightarrow \prod_{\alpha \in A} \mathbb{G}_\alpha(\mathbb{F}_\alpha)$ as in the theorem has infinite kernel. \square

We therefore obtain a wide family of group inclusions $\Gamma < G$ generalizing lattices in Lie groups, the difference being that no group here is linear. Non-linearity follows from the use of ground fields of different characteristics (which is not available in the case of a Kac–Moody group over a finite field), and uses our construction of exotic twin root data in 3.E. Linearity problems for Kac–Moody groups over finite fields are considered in [R 5], but the results in the latter work concern the full Kac–Moody groups Λ rather than their parabolic subgroups Γ .

4.B. Two kinds of lattices. In this last subsection we show that, in some cases, the topological group G of the corollary not only contains non-linear lattices but also lattices which are linear in characteristic 0. For this we assume that, given (W, S) and $\{\mathbb{F}_{q_s}\}_{s \in S}$ as in the corollary, the group Λ is defined by means of the modification described in the concluding remark of 3.E. This is to have the group Λ acting locally on residues of panels of type s via $\text{PGL}_2(\mathbb{F}_{q_s})$ rather than via $\text{PSL}_2(\mathbb{F}_{q_s})$.

Lemma. (i) *The group Λ acts on a right-angled twin building Δ_\pm , where Δ_\pm is the unique building of type (W, S) and parameter system $\{1 + q_s\}_{s \in S}$.*
(ii) *The kernel of the action on each building Δ_\pm is the finite center $Z(\Lambda)$.*

Proof. (i) follows from 3.E and uniqueness of the buildings under consideration [HP, Proposition 5.1]. (ii) follows from Tits system arguments as in [R 1, 8.4.3]. In particular, $Z(\Lambda)$ is the subgroup of T made of the elements centralizing each root group. Note that it is the product $\prod_{s \in S} Z(\text{GL}_2(\mathbb{F}_{q_s})) \cong \prod_{s \in S} \mathbb{F}_{q_s}^\times$, so it intersects trivially the group U_+ , and hence any unipotent radical. \square

By [GP] and the remark before the proof of [HP, Proposition 5.1] about complexes of groups, Δ_\pm is the universal cover of a simple complex of groups indexed by a cone over the barycentric subdivision of the finite nerve of (W, S) . In this theory, the building Δ_\pm is constructed together with a uniform lattice which is the limit of an inductive system indexed by the same directed set as in 3.E (the set of spherical subsets), but in which the groups can be chosen to be much simpler (e.g. finite) than the parabolic subgroups. The simplest choice is when the parabolic subgroup P_I is replaced by the product $\prod_{s \in I} \mathbb{Z}/(q_s + 1)\mathbb{Z}$, with the obvious inclusions as transition maps. We call $\Gamma_{(W,S),1+q}$ the uniform lattice defined by this choice of groups.

Proposition. *The subgroup $\Lambda/Z(\Lambda)$ of $\text{Aut}(\Delta_\pm)$ contains the uniform lattice $\Gamma_{(W,S),1+q}$.*

Proof. The fixator in $\Lambda/Z(\Lambda)$ of the standard facet of spherical type I admits a semi-direct product decomposition $L_I/Z(\Lambda) \rtimes U^I$, with $L_I/Z(\Lambda) = \prod_{s \in I} \text{PGL}_2(\mathbb{F}_{q_s}) \times$

$\prod_{s \notin I} \mathbb{F}_{q_s}^\times$. According to Tits' amalgam theorem [T1, §14], the group $\Lambda/Z(\Lambda)$ is the limit of the inductive system of the groups $L_I/Z(\Lambda) \rtimes U^I = P^I/Z(\Lambda)$ indexed by the finite spherical subsets I .

For any prime power q , $\mathrm{GL}_2(\mathbb{F}_q)$ contains the multiplicative group of the field \mathbb{F}_{q^2} , a cyclic subgroup of order $q^2 - 1$, and hence $\mathrm{PGL}_2(\mathbb{F}_q)$ contains a cyclic subgroup of order $q + 1$. Since the first group is transitive on the projective line, so is the second. Therefore we can see $\mathbb{Z}/(q + 1)\mathbb{Z}$ as a subgroup of $\mathrm{PGL}_2(\mathbb{F}_q)$ which acts simply transitively on the chambers of the corresponding building $\mathbb{P}^1(\mathbb{F}_q)$ [Ch, Proposition 5]. For each spherical subset I , this remark gives the first inclusion in the chain $\prod_{s \in I} \mathbb{Z}/(q_s + 1)\mathbb{Z} < L_I/Z(\Lambda) < P_I/Z(\Lambda)$. Forgetting the middle term we obtain a morphism of inductive systems, from the one defining $\Gamma_{(W,S),1+q}$ to the one defining $\Lambda/Z(\Lambda)$. We obtain the desired inclusion by taking the corresponding direct limits. \square

Finally, we have coexistence of lattices linear in characteristic 0 and non-linear lattices in the same topological group of root datum type whenever the uniform lattice $\Gamma_{(W,S),1+q}$ is known to be linear. This is the case for the lattices considered in [Bou], where the Weyl group is right-angled Fuchsian (see the example in 3.A).

Corollary. *Let (W, S) be the right-angled Coxeter system corresponding to the tiling of the hyperbolic plane \mathbb{H}^2 by regular right-angled r -gons. Let $\{1 + q_s\}_{s \in S}$ be a parameter system consisting of cardinalities of finite projective lines, with q_s prime to q_t for some $s \neq t$. Let Λ be defined as above and let G be the topological group attached to Λ . If $q = \min_{s \in S} q_s$ is large enough, then G contains non-linear non-uniform lattices, whereas all its uniform lattices are linear in characteristic 0.*

Proof. The assertion on non-linear lattices follows from the corollary in 4.A, and it follows from [Hag, Introduction] that any uniform lattice in a right-angled Fuchsian building is linear in characteristic 0. \square

Remark. If Γ is a lattice in a higher-rank non-archimedean simple Lie group G , and if X is a proper $\mathrm{CAT}(-1)$ -space with cocompact isometry group, then any Γ -action on X has a global fixed point [BM, Corollary 0.5]. The coexistence phenomenon of the above corollary is thus excluded in the classical algebraic situation, unless the building of G is a tree (i.e. G is a rank one group).

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