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Characterization of facet breaking for nonsmooth mean curvature flow in the convex case

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We investigate the breaking and bending phenomena of a facet of a three-dimensional crystal which evolves under crystalline mean curvature flow. We give necessary and sufficient conditions for a facet to be calibrable, i.e. not to break or bend under the evolution process. We also give a criterion which allows us to predict exactly where a subdivision of a non-calibrable facet takes place in the evolution process.

Keywords: Crystalline mean curvature; anisotropic evolutions; calibrable facet.

1. Introduction

Motion by crystalline mean curvature in three dimensions is an important example of geometric evolution of solid sets. Besides its geometric interest, it finds applications in material sciences and crystal growth: see, for instance, [6, 7, 16, 23]. Among the geometric flows by anisotropic mean curvature, we say that the evolution is crystalline if the anisotropy ϕ is faceted, which means that ϕ is a piecewise linear convex function or, equivalently, that the Wulff shape $\mathcal{W}_\phi := \{\phi \leqslant 1\}$ is a polytope. It has been recently shown [3, 24] that a facet F of a polyhedron E evolving by crystalline mean curvature can subdivide into two or more regions, or can even bend, creating a curved portion on the surface ∂E (see also [22] for numerical computations). In this paper we investigate these phenomena for a generic nonsmooth anisotropy (including the crystalline ones) and give necessary and sufficient conditions for a facet not to break or bend during the evolution. Moreover, in the case of convex facets, we identify explicitly the velocity (denoted by κ_ϕ^E), and therefore we are able to predict exactly where a subdivision will take place. κ_ϕ^E is obtained as the solution of a global variational problem on the whole of ∂E [4], and is expected to coincide with the actual velocity of

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the crystalline evolution. This conjecture is strongly supported by the expression of the first variation of the surface energy computed in [4].

It is remarkable that the analysis of facet breaking/bending phenomena turns out to be equivalent to the study of a variational problem on a given facet F of ∂E: more precisely, the sublevel sets of κ_ϕ^E in F are solutions of a prescribed anisotropic curvature problem with respect to an anisotropy $\widetilde{\phi}$, which is a sort of two-dimensional restriction of the original anisotropy ϕ. Prescribed mean curvature problems in the Euclidean case have been widely studied (see for instance [13, 15, 17]) also because of their connections with capillarity theory [8–10]. For the anisotropic case we refer to [18–20]. As a consequence of these results and the results in [21, 24], it turns out that the connected components of the level sets of κ_ϕ^E lying inside F are portions of the boundary of the corresponding two-dimensional Wulff shape $\{\widetilde{\phi} \leqslant 1\}$. This fact is crucial in the present paper.

Let us describe more precisely the content of this paper. In Section 2 we introduce some notation. In Section 3 we collect some definitions and results from [4] and [5] which are necessary in the sequel. In particular, we recall the notion of Lipschitz ϕ-regular set (Definition 3.1): a Lipschitz set $E \subset \mathbb{R}^3$ is said to be Lipschitz ϕ-regular if ∂E admits a Lipschitz intrinsic normal vector field n_ϕ. The ϕ-mean curvature κ_ϕ^E is defined in (16), through a minimizer $N_{\min}$ of the variational problem (15) on vector fields on ∂E. This variational problem is meaningful only for nonsmooth ϕ. Indeed, when ϕ is smooth and strictly convex, κ_ϕ^E simply reduces to $\mathrm{div} n_\phi$; for a nonsmooth ϕ, this is in general not the case, and the variational problem (15) is necessary in order to naturally define κ_ϕ^E. By the results of [4] and [5], it follows that κ_ϕ^E is bounded on ∂E and has bounded variation on the facets of ∂E. In particular, the jump set of κ_ϕ^E is well defined (on facets), and it should identify the subdivision regions in the geometric evolution problem. In Definition 3.12 we recall the notion of ϕ-calibrable facet, that is a facet $F \subset \partial E$ such that κ_ϕ^E is constant on the interior of F. Such facets are expected not to break or bend during the evolution process. In Section 4 we localize the variational problem (15) on a facet F, see Propositions 4.5, 4.6 and Corollary 4.7. At the basis of the localization argument there is a trace property of the class of ϕ-normal vector fields having bounded divergence (the class $H_{\nu,\phi}^{\mathrm{div}\infty}(\partial E)$). In order to prove that the normal trace for such a nonsmooth ϕ-normal vector field N on ∂F from 'both sides' of ∂F (with respect to the Lipschitz manifold ∂E) does not actually depend on $N \in H_{\nu,\phi}^{\mathrm{div}\infty}(\partial E)$ and coincides with the function c_F defined in (8), we need some assumptions on the shape of ∂E locally around F: essentially we require that ∂E meets transversally the facet F, see Proposition 4.3. In Section 5 we introduce and study the anisotropic prescribed curvature problem on F, see Theorem 5.2. A first characterization of ϕ-calibrable facets is given in Theorem 6.1 of Section 6; in the case of a crystalline and even ϕ this result has been obtained in [24]. Here Theorem 6.1 is proved also in presence of a bounded forcing term g. In Section 7 we prove that, under the assumption that F is convex and that E is convex at F (which means that, locally around F, E lies on one side of the support plane H_F through F), then the sublevel sets of κ_ϕ^E (restricted to F) are convex. In Section 8 we prove one of the main results of the paper, namely a characterization of convex ϕ-calibrable facets which can be concretely handled. More precisely (see Theorem 8.1) if E is convex at F and F is convex, then F is ϕ-calibrable if and only if the $\widetilde{\phi}$-curvature of ∂F is bounded by the quotient of the anisotropic $\widetilde{\phi}$-perimeter of F with the measure of F (this quotient is the mean value of κ_ϕ^E on F, see (41)). In Section 9, under the assumptions that ϕ is crystalline, F is convex, and E is convex at F, we precisely identify the sublevel sets of κ_ϕ^E as union of all the $\widetilde{\phi}$-Wulff shapes with a given radius contained in F, see Theorem 9.1. As a consequence we localize the subdivision region; moreover (see Corollary 9.5) we obtain that κ_ϕ^E is convex on F. This is an indication that convex sets remain convex under crystalline mean curvature flow. Finally, in Section 10 we apply the above results to an explicit example, partially discussed in [3]. This is an example of convex polyhedral set (very close to the Wulff shape) which has a non ϕ-calibrable facet and does not remain polyhedral under crystalline mean curvature flow.

All results of Sections 5–9 refer to a Lipschitz ϕ-regular set (E, n_ϕ), to a facet F corresponding to a facet of the Wulff shape $\mathcal{W}_\phi$, and under the assumption that any $N \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E)$ has normal trace on ∂F coinciding with the function c_F. The extension of the results of Sections 8 and 9 for nonconvex facets F seems to be nontrivial, and deserves further investigation.

2. Notation

In the following we denote by $\cdot$ the Euclidean scalar product in $\mathbb{R}^3$ and by $|\cdot|$ the Euclidean norm of $\mathbb{R}^3$. Given $v \in \mathbb{R}^3$, we set $v^\perp := \{w \in \mathbb{R}^3 : w \cdot v = 0\}$. If $\rho > 0$ and $x \in \mathbb{R}^k$, $k = 2, 3$, we set $B_\rho(x) := \{y \in \mathbb{R}^k : |y - x| < \rho\}$.

Given two vectors $v, w \in \mathbb{R}^3$ we denote by $[v, w]$ (resp. $]v, w[$) the closed (resp. open) segment joining v and w. With the notation $A \Subset B$ we mean that the set A is compactly contained in B.

The symbol $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^3$, $k \in \{1, 2\}$. We often use the symbol $|B|$ to denote the $\mathcal{H}^2$ measure of B. When integrating on a plane of $\mathbb{R}^3$, we will often use the notation $\mathrm{d}x$ in place of $\mathrm{d}\mathcal{H}^2(x)$ for the integration measure. All sets and functions considered in this paper are Borel measurable.

If $A \subset \mathbb{R}^k$, $k = 2, 3$, we denote by 1_A the characteristic function of A and by ∂A the topological boundary of A.

We say that $A \subset \mathbb{R}^k$, $k = 2, 3$, is Lipschitz (or equivalently that ∂A is Lipschitz) if, for any $x \in \partial A$, there exists $\rho > 0$ such that $B_\rho(x) \cap \partial A$ is the graph of a Lipschitz function f and $B_\rho(x) \cap A$ is the subgraph of f (with respect to a suitable orthogonal coordinate system). By $\mathrm{Lip}(\partial A)$ (resp. $\mathrm{Lip}(\partial A; \mathbb{R}^h)$, $h = 2, 3$) we denote the class of all Lipschitz functions (resp. vector fields with values in $\mathbb{R}^h$) defined on ∂A.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. The space $BV(\Omega)$ is defined as the set of all functions $u \in L^1(\Omega)$ whose distributional gradient Du is a Radon measure with bounded total variation in Ω, i.e. $|Du|(\Omega) = \int_\Omega |Du| < +\infty$, see [14]. Ω will play the role, in most cases, of the interior of a facet F of a Lipschitz set $E \subset \mathbb{R}^3$.

We say that a set $B \subseteq \Omega$ is of finite perimeter in Ω if $1_B \in BV(\Omega)$. If B is of finite perimeter in Ω, $\partial^* B$ denotes the reduced boundary of B; $\partial^* B$ is rectifiable and can be endowed with a generalized exterior Euclidean unit normal $\widetilde{\nu}^B$.

We recall the following result, which is a particular case of a theorem proved in [2].

THEOREM 2.1 Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Let $u \in BV(\Omega)$ and $X \in L^\infty(\Omega; \mathbb{R}^2)$ with $\mathrm{div} X \in L^2(\Omega)$. Then the linear functional

$$(X, Du) : \varphi \to -\int_\Omega u\varphi \, \mathrm{div} X \, \mathrm{d}x - \int_\Omega uX \cdot \nabla\varphi \, \mathrm{d}x, \qquad \varphi \in \mathcal{C}^1_c(\Omega)$$

defines a Radon measure (still denoted by (X, Du)) and satisfies

$$|(X, Du)|(B) \leqslant \|X\|_{L^\infty(\Omega;\mathbb{R}^2)} |Du|(B)$$

for any Borel set $B \subseteq \Omega$. If in addition Ω is Lipschitz, then there is a function $[X \cdot \widetilde{\nu}^\Omega] \in L^\infty(\partial\Omega)$

such that $\|[X \cdot \widetilde{\nu}^{\Omega}]\|_{L^\infty(\partial\Omega)} \leqslant \|X\|_{L^\infty(\Omega;\mathbb{R}^2)}$, and

$$\int_\Omega u \,\mathrm{div} X \,\mathrm{d}x + \int_\Omega \theta(X, Du)\,\mathrm{d}|Du| = \int_{\partial\Omega} [X \cdot \widetilde{\nu}^{\Omega}] u \,\mathrm{d}\mathcal{H}^1 \tag{1}$$

where $\theta(X, Du) \in L^\infty_{|Du|}(\Omega)$ denotes the density of (X, Du) with respect to $|Du|$.

The last part of Theorem 2.1 is still valid when Ω is a bounded open set which is locally Lipschitz continuous up to a finite set of points in $\partial\Omega$.

Finsler metrics and duality mappings. We indicate by $\phi : \mathbb{R}^3 \to [0, +\infty[$ a Finsler metric on $\mathbb{R}^3$, i.e. a convex function satisfying the properties

$$\phi(\xi) \geqslant \Lambda|\xi|, \qquad \phi(a\xi) = a\phi(\xi), \qquad \xi \in \mathbb{R}^3,\ a \geqslant 0, \tag{2}$$

for a suitable constant $\Lambda \in\ ]0, +\infty[$. The function $\phi^o : \mathbb{R}^3 \to [0, +\infty[$ is defined as

$$\phi^o(\xi^*) := \sup\{\xi^* \cdot \xi : \phi(\xi) \leqslant 1\}, \tag{3}$$

and is the dual of ϕ. We set

$$\mathcal{W}^o_\phi := \{\xi^* \in \mathbb{R}^3 : \phi^o(\xi^*) \leqslant 1\}, \qquad \mathcal{W}_\phi := \{\xi \in \mathbb{R}^3 : \phi(\xi) \leqslant 1\}.$$

By a facet of $\partial\mathcal{W}_\phi$ (or of $\partial\mathcal{W}^o_\phi$) we always mean a two-dimensional facet.

We say that ϕ is crystalline if $\mathcal{W}_\phi$ is a (convex) polytope. If ϕ is crystalline, then also $\mathcal{W}^o_\phi$ is a (convex) polytope. $\mathcal{W}^o_\phi$ is sometimes called the Frank diagram and $\mathcal{W}_\phi$ the Wulff shape.

By T and T^o we denote the possibly multivalued duality mappings defined by

$$\begin{aligned} T(\xi) &:= \tfrac{1}{2} D^-(\phi(\xi))^2, \qquad \xi \in \mathbb{R}^3, \\ T^o(\xi^*) &:= \tfrac{1}{2} D^-(\phi^o(\xi))^2, \qquad \xi^* \in \mathbb{R}^3, \end{aligned} \tag{4}$$

where D^- denotes the subdifferential.

ϕ-distance function. Given a nonempty set $E \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$, we set

$$\mathrm{dist}_\phi(x, E) := \inf_{y \in E} \phi(x - y), \qquad \mathrm{dist}_\phi(E, x) := \inf_{y \in E} \phi(y - x),$$
$$d^E_\phi(x) := \mathrm{dist}_\phi(x, E) - \mathrm{dist}_\phi(\mathbb{R}^3 \setminus E, x).$$

If $E \subset \mathbb{R}^3$ is Lipschitz, for $\mathcal{H}^2$ almost every $x \in \partial E$ we denote by $\nu^E(x)$ the outward unit Euclidean normal to ∂E at x. At each point x where d^E_ϕ is differentiable, there holds $\nabla d^E_\phi(x) \in \partial\mathcal{W}^o_\phi$; we set $\nu^E_\phi(x) := \nabla d^E_\phi(x)$ at those points $x \in \partial E$. We have $\nu^E_\phi(x) = \frac{\nu^E(x)}{\phi^o(\nu^E(x))}$.

If $E \subset \mathbb{R}^3$ is Lipschitz we define

$$\mathrm{Nor}_\phi(\partial E) := \{N : \partial E \to \mathbb{R}^3 : N(x) \in T^o(\nu^E_\phi(x)) \text{ for } \mathcal{H}^2\text{a.e. } x \in \partial E\}, \tag{5}$$
$$\mathrm{Lip}_{\nu,\phi}(\partial E) := \mathrm{Lip}(\partial E; \mathbb{R}^3) \cap \mathrm{Nor}_\phi(\partial E).$$

Note that if $N_1, N_2 \in \mathrm{Nor}_\phi(\partial E)$, then $N_1 - N_2$ is tangent, since $N_1 \cdot \nu_\phi = 1 = N_2 \cdot \nu_\phi$.

We also set $\mathrm{d}\mathcal{P}_\phi$ to be the measure supported on ∂E with density $\phi^o(\nu^E)$, i.e.

$$\mathrm{d}\mathcal{P}_\phi(B) := \int_B \phi^o(\nu^E)\,\mathrm{d}\mathcal{H}^2, \quad B \subseteq \partial E.$$

If E is Lipschitz and $\psi \in \mathrm{Lip}(\partial E)$ we denote by $\nabla_\tau \psi$ the Euclidean tangential gradient of ψ on ∂E and, if $v \in \mathrm{Lip}(\partial E; \mathbb{R}^3)$, we denote by $\mathrm{div}_\tau v$ the Euclidean tangential divergence of v. In the following, whenever there is no risk of confusion, we do not indicate the dependence on E of the unit normals ν^E and ν_ϕ^E, i.e. we set $\nu := \nu^E$ and $\nu_\phi := \nu_\phi^E$.

DEFINITION 2.2 We say that F is a facet of ∂E if F is the closure of a connected component of the relative interior of $\partial E \cap T_x \partial E$ for some $x \in \partial E$ such that the tangent plane $T_x\partial E$ to ∂E at x exists.

If F is a facet of ∂E, we denote by ∂F (resp. $\mathrm{int}(F)$) the relative boundary (resp. the relative interior) of F. Let F be a facet of ∂E; we define $\nu(F)$ to be the outer unit normal to $\mathrm{int}(F)$ (i.e. $\nu(F) := \nu^E(x)$ for any $x \in \mathrm{int}(F) \subset \partial E$), we set $\nu_\phi(F) := \frac{\nu(F)}{\phi^o(\nu_\phi(F))}$, and

$$\widetilde{W}_\phi^F := T^o(\nu_\phi(F)).$$

We denote by H_F the affine plane spanned by the facet F. Whenever necessary, we identify H_F with the plane parallel to H_F and passing through the origin, and F with its orthogonal projection on this latter plane.

Fix $y \in \mathrm{int}(\widetilde{W}_\phi^F)$ and let $\tau_y \widetilde{W}_\phi^F := \widetilde{W}_\phi^F - y$. Let $\widetilde{\phi}_y : H_F \to [0, +\infty[$ be the Finsler metric on H_F such that $\{\widetilde{\phi}_y \leqslant 1\} = \tau_y \widetilde{W}_\phi^F$. Define also $\mathrm{sym}(\widetilde{\phi}_y)$ as the Finsler metric on H_F such that $\{\mathrm{sym}(\widetilde{\phi}_y) \leqslant 1\} = -\tau_y \widetilde{W}_\phi^F$. The classes of Lipschitz $\widetilde{\phi}_y$-regular sets and Lipschitz $\mathrm{sym}(\widetilde{\phi}_y)$-regular sets do not depend on the choice of y. We accordingly often omit specifying the point y (thus addressing, for instance, $\widetilde{\phi}_y$-regularity as $\widetilde{\phi}$-regularity).

We denote by $\widetilde{\phi}^o$ the dual of $\widetilde{\phi}$. The maps $\widetilde{T}, \widetilde{T}^o$ are defined as in (4) with $\widetilde{\phi}$ in place of ϕ and H_F in place of $\mathbb{R}^3$.

If $\psi : H_F \to [0, +\infty[$ is a Finsler metric on H_F and B is a finite perimeter subset of H_F, we denote by $\widetilde{\nu}_\psi^B$ the normalized outward unit normal $\frac{\widetilde{\nu}^B}{\psi^o(\widetilde{\nu}^B)}$ to $\partial^* B$. We use the symbol $\widetilde{\nu}_\phi^B$ in place of $\widetilde{\nu}_{\widetilde{\phi}}^B$. If there is no risk of confusion, we do not indicate the dependence on B of $\widetilde{\nu}^B$ and $\widetilde{\nu}_\phi^B$.

If $\psi : H_F \to [0, +\infty[$ is a Finsler metric on H_F and $B \subset H_F$ is Lipschitz, we set

$$\mathrm{Nor}_\psi(\partial B) := \{\widetilde{N} : \partial B \to H_F, \widetilde{N}(x) \in \widetilde{T}^o(\widetilde{\nu}_\psi(x)) \text{ for } \mathcal{H}^1 \text{ a.e. } x \in \partial B\}, \tag{6}$$

$$\mathrm{Lip}_{\widetilde{\nu},\psi}(\partial B) := \mathrm{Lip}(\partial B; H_F) \cap \mathrm{Nor}_\psi(\partial B). \tag{7}$$

3. Preliminaries

In this section we collect some definitions and results taken from [4] and [5] which will be useful in the sequel.

3.1 *Lipschitz ϕ-regular sets*

DEFINITION 3.1 Let $E \subseteq \mathbb{R}^3$. We say that E is Lipschitz ϕ-regular if ∂E is compact and Lipschitz continuous and there exists a vector field $n_\phi : \partial E \to \mathbb{R}^3$ with $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E)$.

n_ϕ is usually called a Cahn–Hoffman vector field; several different choices of n_ϕ are usually allowed for the same set E, due to the nonsmoothness of ϕ (notice for instance that if ϕ is crystalline then T and T^o are necessarily multivalued).

The standard example of Lipschitz ϕ-regular set is $(\mathcal{W}_\phi, x)$.

Notation. Throughout the paper, the symbols E or (E, n_ϕ) always denote a Lipschitz ϕ-regular set; n_ϕ will be a given selection in $\mathrm{Lip}_{\nu,\phi}(\partial E)$ as in Definition 3.1. The symbol F will always denote a facet of ∂E such that $\widetilde{W}^F_\phi$ is a facet of $\mathcal{W}_\phi$.

DEFINITION 3.2 We say that E is convex (resp. concave) at F if there exists an open set $U \subset \mathbb{R}^3$ such that $F \subset U$ and $F = \overline{E} \cap H_F \cap U$ (resp. $F = \overline{\mathbb{R}^3 \setminus E} \cap H_F \cap U$).

THEOREM 3.3 F is locally Lipschitz, out of a finite set of points in $\partial F \setminus \partial^* F$. Moreover, if E is convex or concave at F, then F is Lipschitz.

DEFINITION 3.4 We define the trace function $c_F \in L^\infty(\partial F)$ as

$$c_F(x) := n_\phi(x) \cdot \widetilde{\nu}^F(x) \qquad \forall x \in \partial^* F. \tag{8}$$

The next result shows that c_F is independent of the choice of $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E)$, but depends only on F, on ∂E locally around F, and on the geometry of $\mathcal{W}_\phi$. We say that ∂E is weakly convex (resp. weakly concave) at $x \in \partial^* F$ if $\widetilde{\nu}^F(x)$ points outside (resp. inside) E.

LEMMA 3.5 Let $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E)$. Then, for any $x \in \partial^* F$ we have

$$\eta(x) \cdot \widetilde{\nu}^F(x) = c_F(x) = \begin{cases} \max\ \{p \cdot \widetilde{\nu}^F(x) : p \in \widetilde{W}^F_\phi\} & \text{if} \quad \partial E \text{ is weakly convex at } x, \\ \min\ \{p \cdot \widetilde{\nu}^F(x) : p \in \widetilde{W}^F_\phi\} & \text{if} \quad \partial E \text{ is weakly concave at } x. \end{cases} \tag{9}$$

DEFINITION 3.6 Let $\psi : H_F \to [0, +\infty[$ be a Finsler metric on H_F. Let $B \subset H_F$. We say that B is Lipschitz ψ-regular if ∂B is compact and Lipschitz continuous and there exists a vector field in $\mathrm{Lip}_{\widetilde{\nu},\psi}(\partial B)$.

In the following proposition, y is any point in the interior of $\widetilde{W}^F_\phi$, see the discussion after Definition 2.2.

PROPOSITION 3.7 If E is convex at F then $(F, n_\phi - y)$ is Lipschitz $\widetilde{\phi}$-regular. If E is concave at F, then $(F, y - n_\phi)$ is Lipschitz $\mathrm{sym}(\widetilde{\phi}_y)$-regular.

In the next definition we prefer to keep the notation $\widetilde{P_\phi}$ instead of $P_{\widetilde{\phi}}$.

DEFINITION 3.8 Let A be an open subset of H_F. For any $B \subseteq F$, we set

$$\widetilde{P_\phi}(B, A) := \sup \left\{ \int_B \mathrm{div}_\tau \eta \, \mathrm{d}x : \eta \in \mathcal{C}^1_c(A; \tau_y \widetilde{W}^F_\phi) \right\}, \tag{10}$$

$$\widetilde{P_\phi}(B) := \widetilde{P_\phi}(B, H_F). \tag{11}$$

Notice that $\widetilde{P_\phi}(F) < +\infty$ by Theorem 3.3.

3.2 *ϕ-tangential divergence*

Let us introduce the ϕ-tangential divergence for vector fields $v \in L^2(\partial E; \mathbb{R}^3)$ as bounded linear operator on $\mathrm{Lip}(\partial E)$. Recall that (E, n_ϕ) is Lipschitz ϕ-regular.

DEFINITION 3.9 Let $v \in L^2(\partial E; \mathbb{R}^3)$. We define $\mathrm{div}_{\phi,n_\phi,\tau}\, v : \mathrm{Lip}(\partial E) \to \mathbb{R}$ as follows: for any $\psi \in \mathrm{Lip}(\partial E)$ we set

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}\, v, \psi\rangle := \int_{\partial E} \psi\; v \cdot \nu_\phi\, \mathrm{div}_\tau n_\phi \,\mathrm{d}\mathcal{P}_\phi - \int_{\partial E} [\nabla_\tau \psi - \nabla_\tau \psi \cdot n_\phi\, \nu_\phi] \cdot v \,\mathrm{d}\mathcal{P}_\phi. \qquad (12)$$

Notice that, if $X \in L^2(\partial E; \mathbb{R}^3)$ is a tangent vector field, then

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} X, \psi\rangle = -\int_{\partial E} \nabla_\tau \psi \cdot X \,\mathrm{d}\mathcal{P}_\phi \qquad \forall \psi \in \mathrm{Lip}(\partial E). \qquad (13)$$

We say that $\mathrm{div}_{\phi,n_\phi,\tau} v$ is independent of the choice of n_ϕ if, given $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E)$ then $\langle \mathrm{div}_{\phi,n_\phi,\tau} v, \psi\rangle = \langle \mathrm{div}_{\phi,\eta,\tau} v, \psi\rangle$ for any $\psi \in \mathrm{Lip}(\partial E)$. When $\mathrm{div}_{\phi,n_\phi,\tau} v$ is independent of the choice of n_ϕ, we simply set $\mathrm{div}_{\phi,\tau} v := \mathrm{div}_{\phi,n_\phi,\tau} v$. It turns out that if $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E)$ then $\langle \mathrm{div}_{\phi,n_\phi,\tau} \eta, \psi\rangle = \int_{\partial E} \psi\, \mathrm{div}_\tau \eta \,\mathrm{d}\mathcal{P}_\phi$ for any $\psi \in \mathrm{Lip}(\partial E)$. Moreover, if $N \in \mathrm{Nor}_\phi(\partial E)$, then $\mathrm{div}_{\phi,n_\phi,\tau} N$ is independent of the choice of n_ϕ and, on $\mathrm{int}(F)$, $\mathrm{div}_{\phi,\tau} N$ coincides with $\mathrm{div}_\tau N$ (we will accordingly use the notation $\mathrm{div}_\tau N$ in place of $\mathrm{div}_{\phi,\tau} N$ on $\mathrm{int}(F)$).

3.3 *The minimum problem on ∂E*

We define

$$H^{\mathrm{div}}_{\nu,\phi}(\partial E) := \{N \in \mathrm{Nor}_\phi(\partial E) : \mathrm{div}_{\phi,\tau} N \in L^2(\partial E)\},$$
$$H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E) := \{N \in \mathrm{Nor}_\phi(\partial E) : \mathrm{div}_{\phi,\tau} N \in L^\infty(\partial E)\}.$$

Let $\mathcal{F} : H^{\mathrm{div}}_{\nu,\phi}(\partial E) \to [0, +\infty[$ be the functional defined as

$$\mathcal{F}(N) := \int_{\partial E} (\mathrm{div}_{\phi,\tau}\, N)^2 \,\mathrm{d}\mathcal{P}_\phi. \qquad (14)$$

The minimum problem

$$\inf\{\mathcal{F}(N) : N \in H^{\mathrm{div}}_{\nu,\phi}(\partial E)\} \qquad (15)$$

admits a solution and, if N_1 and N_2 are two minimizers, then $\mathrm{div}_{\phi,\tau}\, N_1(x) = \mathrm{div}_{\phi,\tau}\, N_2(x)$ for $\mathcal{H}^2$ almost every $x \in \partial E$.

Except for Section 6, in the following we denote by $N_{\min}$ a solution of (15), and we set

$$\kappa^E_\phi := \mathrm{div}_{\phi,\tau} N_{\min} \in L^2(\partial E). \qquad (16)$$

κ^E_ϕ is the natural definition of ϕ-mean curvature of ∂E. The following regularity results hold.

THEOREM 3.10 $\kappa^E_\phi \in L^\infty(\partial E)$. Moreover $\kappa^E_\phi \in BV(\mathrm{int}(F))$.

We set

$$\kappa_{\min}(F) := \operatorname{ess\,inf}_F \kappa_\phi^E, \qquad \kappa_{\max}(F) := \operatorname{ess\,sup}_F \kappa_\phi^E,$$

and for any $\lambda \in \mathbb{R}$ we define

$$\Omega_\lambda^F := \{x \in \operatorname{int}(F) : \kappa_\phi^E(x) < \lambda\}, \qquad \Theta_\lambda^F := \{x \in \operatorname{int}(F) : \kappa_\phi^E(x) \leqslant \lambda\}.$$

THEOREM 3.11 For every $\lambda \in \mathbb{R}$ the set Ω_λ^F is a solution of the following variational problem:

$$\inf\{\widetilde{P_\phi}(B, \operatorname{int}(F)) - \lambda|B| : (B \setminus \Omega_\lambda^F) \cup (\Omega_\lambda^F \setminus B) \Subset \operatorname{int}(F)\}. \tag{17}$$

Moreover, if $\lambda \neq 0$, every connected component of $\operatorname{int}(F) \cap \partial\Omega_\lambda^F$ is contained in a translated of $\frac{1}{\lambda}\partial\widetilde{W}_\phi^F$, and has extrema on ∂F. Same assertions hold for the sets Θ_λ^F.

DEFINITION 3.12 We say that F is ϕ-calibrable if κ_ϕ^E is constant on $\operatorname{int}(F)$.

The following technical result will be very useful in the sequel.

THEOREM 3.13 For any $\lambda \in \mathbb{R}$ we have

$$-\theta(N_{\min}, D1_{\Omega_\lambda^F})(x) = \max\{\, p \cdot \widetilde{\nu}^{\Omega_\lambda^F}(x) : p \in \widetilde{W}_\phi^F\} \qquad \mathcal{H}^1 \text{ a.e. } x \in \operatorname{int}(F) \cap \partial^*\Omega_\lambda^F,$$

$$-\theta(N_{\min}, D1_{\Theta_\lambda^F})(x) = \max\{\, p \cdot \widetilde{\nu}^{\Theta_\lambda^F}(x) : p \in \widetilde{W}_\phi^F\} \qquad \mathcal{H}^1 \text{ a.e. } x \in \operatorname{int}(F) \cap \partial^*\Theta_\lambda^F,$$

where $\theta(N_{\min}, \cdot)$ is given by Theorem 2.1.

We conclude this section with the following definition.

DEFINITION 3.14 If $P \subseteq H_F$ is Lipschitz $\widetilde{\phi}$-regular, we denote by $\widetilde{\kappa}_\phi^P$ the $\widetilde{\phi}$-curvature of ∂P, obtained by taking the divergence of a minimizer of a functional as in (14) with P in place of E and $\widetilde{\phi}$ in place of ϕ.

4. Normal traces on ∂F. Localized minimum problem on facets

The aim of this section is to extend the validity of the first equality in (9) under weaker regularity assumptions on η. In doing this, however, we strengthen the regularity assumptions of ∂E locally around F. We miss the proof of the first equality of (9) for a facet F of a generic (Lipschitz ϕ-regular) set and a generic $N \in H_{\nu,\phi}^{\operatorname{div}_\infty}(\partial E)$. We recall that, thanks to Theorems 2.1 and 3.3, any $N \in H_{\nu,\phi}^{\operatorname{div}_\infty}(\partial E)$ admits a normal trace $[N \cdot \widetilde{\nu}^F] \in L^\infty(\partial F)$.

We begin with the simplest case, where we assume that ∂F is locally the intersection of two half-planes. This situation covers the case when E is polyhedron.

PROPOSITION 4.1 Let $N \in H_{\nu,\phi}^{\operatorname{div}_\infty}(\partial E)$. Assume that there exist $\overline{x} \in \partial F$ and $\rho > 0$ such that $B_\rho(\overline{x}) \cap \partial E$ is the union of $B_\rho(\overline{x}) \cap F$ and $B_\rho(\overline{x}) \cap F_1$, where $F_1 \subseteq \mathbb{R}^3$ is a half-plane nonparallel to H_F. Then

$$[N \cdot \widetilde{\nu}^F] = c_F \qquad \mathcal{H}^1 \text{a.e. on } B_\rho(\overline{x}) \cap \partial F. \tag{18}$$

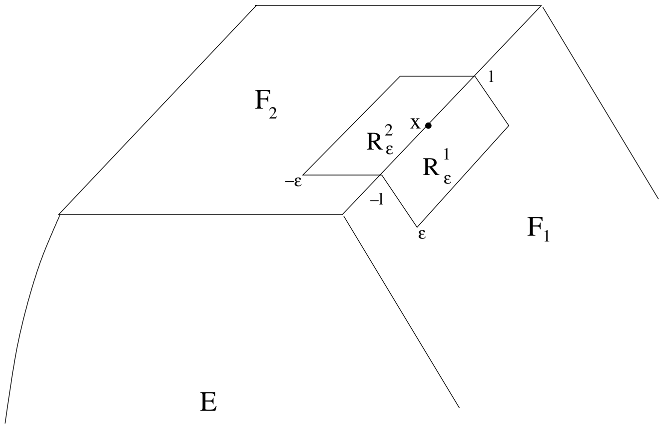

FIG. 1. Case (i) of Proposition 4.1 ($F_2 := F$).

Proof. Let $N \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E)$ and let χ be the tangent vector field defined by $\chi := N - n_\phi$. Let $x \in B_\rho(\overline{x}) \cap \partial F$ be a Lebesgue point of $[\chi \cdot \widetilde{\nu}^F]$. Set $F_2 := F$. Let l be a fixed positive number small enough, and let $0 < \epsilon \ll l$. Let $R_\epsilon := R^1_\epsilon \cup R^2_\epsilon \subset B_\rho(\overline{x})$ be the set 'centred' at x as in Fig. 1, where we identify the rectangle R^2_ϵ (resp. the rectangle R^1_ϵ) with $[-\epsilon, 0] \times [-l, l]$ (resp. $[0, \epsilon] \times [-l, l]$). We also sometimes identify the edges of the rectangles with their lengths.

To prove the assertion, it is enough to show that

$$\int_{\{0\}\times[-l,l]} [\chi \cdot \widetilde{\nu}^F]\, \mathrm{d}\mathcal{H}^1 = 0. \tag{19}$$

Indeed, since (19) holds for any l small enough we deduce $[\chi \cdot \widetilde{\nu}^F](x) = 0$, and (18) follows recalling (8).

Let δ be a positive number with $\delta \ll \epsilon$. For any $y \in \partial E$ define $\psi(y) := \frac{1}{\delta}\mathrm{dist}(y, \partial E \setminus R_\epsilon) \wedge 1$. Then $\psi \in \mathrm{Lip}(\partial E)$ and $\mathrm{spt}(\psi) \subseteq R_\epsilon$.

Recalling that $\mathrm{div}_{\phi,\tau}\chi$ is a bounded function on ∂E, it is immediate to check that

$$\left|\int_{R_\epsilon} \psi\, \mathrm{div}_{\phi,\tau}\chi\, \mathrm{d}\mathcal{P}_\phi\right| = l\mathrm{O}(\epsilon), \qquad \left|\int_{R^i_\epsilon} \psi\, \mathrm{div}_\tau \chi\, \mathrm{d}\mathcal{P}_\phi\right| = l\mathrm{O}(\epsilon), \quad i = 1, 2. \tag{20}$$

We also claim that

$$\int_{R^i_\epsilon} \nabla_\tau \psi \cdot \chi\, \mathrm{d}\mathcal{P}_\phi = l\mathrm{O}(\epsilon) + \mathrm{O}(\epsilon), \qquad i = 1, 2. \tag{21}$$

Indeed, from (20) we get

$$-\int_{R^2_\epsilon} \nabla_\tau \psi \cdot \chi\, \mathrm{d}\mathcal{P}_\phi = l\mathrm{O}(\epsilon) + \int_{R^1_\epsilon} \nabla_\tau \psi \cdot \chi\, \mathrm{d}\mathcal{P}_\phi. \tag{22}$$

By general properties of Lipschitz ϕ-regular sets (see [5: Lemma 4.1 and Theorem 4.4]) it follows that, if $z \in B_\rho(\overline{x}) \cap F_1 \cap F_2$, then $n_\phi(z) \in \widetilde{W}^{F_1}_\phi \cap \widetilde{W}^{F_2}_\phi$, and $\widetilde{\nu}^{F_i}(z)$ belongs to the outward normal cone to $\partial \widetilde{W}_\phi^{F_i}$ at $n_\phi(z)$. Therefore

$$\widetilde{\nu}^{F_i}(z) \cdot (p - n_\phi(z)) \leqslant 0 \qquad \text{for any } p \in \widetilde{W}_\phi^{F_i}, \quad i = 1, 2. \tag{23}$$

Given $y \in R_\epsilon^i$, we denote by $\pi_i(y) \in [-l, l]$ the point of minimal distance of y from $[-l, l]$. Clearly $|y - \pi_i(y)| = \mathrm{O}(\epsilon)$. Since n_ϕ is Lipschitz continuous on ∂E and $N(y) \in \widetilde{W}_\phi^{F_2}$ (resp. $N(y) \in \widetilde{W}_\phi^{F_1}$) for $\mathcal{H}^2$ almost every $y \in F_2$ (resp. for $\mathcal{H}^2$ almost every $y \in F_1 \cap \partial E$), using (23) we have, for $i = 1, 2$ and $y \in F_i$,

$$\begin{aligned} \widetilde{\nu}^{F_i}(\overline{x}) \cdot \chi(y) &= \widetilde{\nu}^{F_i}(\overline{x}) \cdot (N(y) - n_\phi(\pi_i(y))) + \widetilde{\nu}^{F_i}(\overline{x}) \cdot (n_\phi(\pi_i(y)) - n_\phi(y)) \\ &= \widetilde{\nu}^{F_i}(\pi_i(y)) \cdot (N(y) - n_\phi(\pi_i(y))) + \widetilde{\nu}^{F_i}(\overline{x}) \cdot (n_\phi(\pi_i(y)) - n_\phi(y))) \\ &\leqslant \widetilde{\nu}^{F_i}(\overline{x}) \cdot (n_\phi(\pi_i(y)) - n_\phi(y)) = \mathrm{O}(\epsilon). \end{aligned} \tag{24}$$

Recalling the definition of ψ and the properties of the distance function, we have

$$-\int_{R_\epsilon^2} \nabla_\tau \psi \cdot \chi \, \mathrm{d}\mathcal{P}_\phi = \frac{1}{\delta} \int_{A_\delta} \widetilde{\nu}^{F_2}(\overline{x}) \cdot \chi \, \mathrm{d}\mathcal{P}_\phi + \frac{1}{\delta} \int_{B_\delta} \widetilde{\nu}^{p} \cdot \chi \, \mathrm{d}\mathcal{P}_\phi, \tag{25}$$

where $A_\delta := [-\epsilon, -\epsilon + \delta] \times [-l, l]$, $B_\delta := \{y \in R_\epsilon^2 \setminus A_\delta : \mathrm{dist}(y, \partial E \setminus R_\epsilon) \leqslant \delta\}$, and $\widetilde{\nu}^p$ denotes the outward unit normal to the level sets of ψ. A similar formula holds when R_ϵ^2 is replaced by R_ϵ^1. Therefore, using (24) and (25), we get

$$-\int_{R_\epsilon^i} \nabla_\tau \psi \cdot \chi \, \mathrm{d}\mathcal{P}_\phi \leqslant l\mathrm{O}(\epsilon) + \mathrm{O}(\epsilon), \qquad i = 1, 2. \tag{26}$$

From (26) and (22) we deduce

$$l\mathrm{O}(\epsilon) + \mathrm{O}(\epsilon) \geqslant -\int_{R_\epsilon^2} \nabla_\tau \psi \cdot \chi \, \mathrm{d}\mathcal{P}_\phi = l\mathrm{O}(\epsilon) + \int_{R_\epsilon^1} \nabla_\tau \psi \cdot \chi \, \mathrm{d}\mathcal{P}_\phi \geqslant l\mathrm{O}(\epsilon) + \mathrm{O}(\epsilon),$$

which proves claim (21).

Using (20) and (1) we have

$$l\mathrm{O}(\epsilon) = \int_{R_\epsilon^1} \psi \, \mathrm{div}_\tau \chi \, \mathrm{d}\mathcal{P}_\phi = -\int_{R_\epsilon^1} \nabla_\tau \psi \cdot \chi \, \mathrm{d}\mathcal{P}_\phi + \int_{\partial R_\epsilon^1} \psi \, [\chi \cdot \widetilde{\nu}^{R_\epsilon^1}] \, \mathrm{d}\mathcal{P}_\phi. \tag{27}$$

Observe that ψ vanishes on ∂R_ϵ and, when restricted to ∂R_ϵ^1, is nonzero only on the segment $[-l, l]$, and is equal to one on $[-l + \delta, l - \delta]$. Hence

$$\int_{\partial R_\epsilon^1} \psi \, [\chi \cdot \widetilde{\nu}^{R_\epsilon^1}] \, \mathrm{d}\mathcal{P}_\phi = \int_{[-l+\delta, l-\delta]} [\chi \cdot \widetilde{\nu}^{R_\epsilon^1}] \, \mathrm{d}\mathcal{P}_\phi + \mathrm{O}(\delta). \tag{28}$$

Inserting (28) into (27) and using (21) we have

$$\int_{[-l+\delta, l-\delta]} [\chi \cdot \widetilde{\nu}^{R_\epsilon^1}] \, \mathrm{d}\mathcal{P}_\phi = l\mathrm{O}(\epsilon) + \mathrm{O}(\epsilon) + \mathrm{O}(\delta).$$

Letting first $\delta \to 0^+$ and then $\epsilon \to 0^+$, we get (19), and the proposition is proved. □

We now extend the class of sets E for which Proposition 4.1 is valid. For any $x \in \partial E$ and $\rho > 0$ we let $E_\rho(x) := \frac{E-x}{\rho}$. Recall that (E, n_ϕ) is a Lipschitz ϕ-regular set, and that $\nu_\phi = \nu_\phi^E$. We begin with the following lemma on the structure of the blow-up of ∂E.

LEMMA 4.2 Let $x \in \partial E$. There exist a set $E_0 = E_0(x) \subset \mathbb{R}^3$ and a sequence $(\rho_n)_n$ of positive numbers converging to 0 such that

(a) $1_{E_{\rho_n}(x)} \rightharpoonup 1_{E_0}$ weakly in $BV_{\rm loc}(\mathbb{R}^3)$,

(b) ∂E_0 is an entire Lipschitz graph and $n_\phi(x) \in T^o(\nu_\phi^{E_0}(y))$ for $\mathcal{H}^2$ almost every $y \in \partial E_0$,

(c) E_0 minimizes P_ϕ between all subsets of $\mathbb{R}^3$ of finite perimeter which coincide with E_0 out of some ball.

In contrast with the Euclidean case, in general E_0 is not a cone over x.

Proof. Point (a) is standard in the theory of finite-perimeter sets. Let us prove (b). Let $x = 0$ for simplicity. Let $\Pi \subset \mathbb{R}^3$ be a plane and $f : \Pi \to \mathbb{R}$ be a Lipschitz function such that ∂E coincide with the graph of f in a neighbourhood of 0. Then ∂E_ρ can be written (locally around 0) as the graph of the Lipschitz function $f_\rho(y) := \frac{f(\rho y)}{\rho}$. Since f_ρ are equi-Lipschitz on any bounded set, using the Ascoli-Arzelà theorem, f_ρ converges uniformly on compact subsets of Π (possibly passing to a subsequence) to a Lipschitz function f_0 whose subgraph is E_0. We can also assume that f_ρ converges to f_0 weakly in $H^1_{\rm loc}(\Pi)$. By [5], Lemma 4.2, we have that for any $R > 0$

$$\lim_{\rho \to 0^+} \sup_{y \in B_R(0) \cap \partial^* E_\rho} \operatorname{dist}(\nu_\phi^{E_\rho}(y), T(n_\phi(0))) = 0. \tag{29}$$

Since $T(n_\phi(0))$ is a convex set and $\nu_\phi^{E_\rho}(\cdot + f_\rho(\cdot)\nu^\Pi)$ converges to $\nu_\phi^{E_0}(\cdot + f_0(\cdot)\nu^\Pi)$ weakly in $L^2_{\rm loc}(\Pi)$, from (29) it follows that

$$\nu_\phi^{E_0}(y) \in T(n_\phi(0)) \qquad \text{for } \mathcal{H}^2 \text{ a.e. } y \in \partial E_0.$$

It follows $T^o(\nu_\phi^{E_0}(y)) \supseteq T^o(\operatorname{int}(T(n_\phi(0)))) \ni n_\phi(0)$, and (b) is proved (note therefore that ∂E_0 admits a constant ϕ-normal vector field $n_\phi(0)$).

Let us prove (c). Let $A \subset \mathbb{R}^3$ be a set of finite perimeter such that $(E_0 \setminus A) \cup (A \setminus E_0) \Subset B_R := B_R(0)$ for some $R > 0$. From the Gauss–Green theorem we get

$$\begin{aligned} 0 = \int_{B_R} \operatorname{div} n_\phi(0)\, (1_{E_0} - 1_A)\, \mathrm{d}x &= (D1_{E_0}(B_R) - D1_A(B_R)) \cdot n_\phi(0) \\ &\geqslant (D1_{E_0}(B_R)) \cdot n_\phi(0) - P_\phi(A, B_R), \end{aligned}$$

where the last inequality follows from the inequality $\nu^A \cdot n_\phi(0) \leqslant \phi^o(\nu^A)$. Since $\nu_\phi^{E_0} \cdot n_\phi(0) = 1$ on $\partial^* E_0$, we obtain $P_\phi(A, B_R) \geqslant P_\phi(E_0, B_R)$, and (c) is proved. □

PROPOSITION 4.3 Assume that for $\mathcal{H}^1$ almost any $x \in \partial^* F$ the boundary $\partial E_0(x)$ of the blow-up set $E_0(x)$ defined in Lemma 4.2 is the union of two closed nonparallel half-planes P_1, P_2, with P_2 parallel to F. Assume also that the Lipschitz functions f_ρ in the proof of Lemma 4.2, converge to f_0 strongly in $H^1_{\rm loc}(\Pi)$, and that $|D1_{\frac{F-x}{\rho}}|(K) \to |D1_{P_2}|(K)$ for any compact set K contained in the plane spanned by P_2. Then, for any $N \in H^{\operatorname{div}_\infty}_{\nu,\phi}(\partial E)$ we have

$$[N \cdot \widetilde{\nu}^F] = c_F \qquad \mathcal{H}^1 \text{ a.e. on } \partial F. \tag{30}$$

Proof. Fix $\overline{x} \in \partial^* F$ and assume for simplicity $\overline{x} = 0$. In a neighbourhood V of $\overline{x} = 0$, the set E coincides with the subgraph of a Lipschitz function $f : \Pi \to \mathbb{R}$. Up to a translation, we can assume that $0 \in \Pi$ and $f(0) = 0$. Let also $U := V \cap \Pi$ and $\pi : \mathbb{R}^3 \to \Pi$ be the orthogonal projection such that $\pi(y, f(y)) = y$ for $y \in \Pi$. For $\rho > 0$ we let $U_\rho := U/\rho$, and we define $N_\rho \in L^\infty(U_\rho; \mathbb{R}^3)$, $n_\rho \in \mathrm{Lip}(U_\rho; \mathbb{R}^3)$ and $\xi_\rho \in L^\infty(U_\rho; \mathbb{R}^3)$ as

$$\begin{aligned} N_\rho(y) &:= N(\rho(y, f(y))), \quad n_\rho(y) := n_\phi(\rho(y, f(y))), \\ \xi_\rho(y) &:= \phi^o(-\nabla f_\rho(y), 1)(N_\rho(y) - n_\rho(y)), \end{aligned} \tag{31}$$

where $y \in U_\rho$. We divide the proof into four steps.

Step 1. We have $\operatorname{div} \xi_\rho \in L^\infty(U_\rho)$.

Indeed, for any function $\psi \in \mathcal{C}^1_c(U_\rho)$ we have, setting $\hat{\psi} := \psi \circ \pi$,

$$\begin{aligned} \int_{U_\rho} \xi_\rho(y) \cdot \nabla\psi(y)\, \mathrm{d}y &= \int_{U_\rho} (N_\rho(y) - n_\rho(y)) \cdot \nabla\psi(y) \phi^o(-\nabla f_\rho(y), 1)\, \mathrm{d}y \\ &= \int_{\partial E_\rho \cap (V/\rho)} (N_\rho - n_\rho) \cdot \nabla\hat{\psi}\, \mathrm{d}\mathcal{P}_\phi \\ &= \frac{1}{\rho^2} \int_{\partial E \cap V} (N(x) - n_\phi(x)) \cdot (\nabla\hat{\psi})(x/\rho)\, \mathrm{d}\mathcal{P}_\phi \\ &= \frac{1}{\rho} \int_{\partial E \cap V} (N(x) - n_\phi(x)) \cdot \nabla(\hat{\psi}(x/\rho))\, \mathrm{d}\mathcal{P}_\phi. \end{aligned}$$

Since $N - n_\phi \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E)$ is a tangent vector field, from the previous equality we deduce

$$\int_{U_\rho} \xi_\rho(y) \cdot \nabla\psi(y)\, \mathrm{d}y \leqslant \frac{C}{\rho} \int_{\partial E} |\hat{\psi}(x/\rho)|\, \mathrm{d}\mathcal{P}_\phi = C\rho \|\hat{\psi}\|_{L^1(\partial E_\rho)} \leqslant \widetilde{C}\rho \|\psi\|_{L^1(U_\rho)},$$

for some positive constant C, $\widetilde{C}$ independent of ρ. This proves Step 1.

Step 2. Definition of ξ_0.

Letting $\rho \to 0$, up to a subsequence, we can assume that, for all $n \in \mathbb{N}$, ξ_ρ weakly* converges, in $H^{\mathrm{div}_\infty}_{\nu,\phi}(B_n(0) \cap \Pi)$ to a divergence free vector field $\xi_0 \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\Pi)$, that f_ρ converge to $f_0 \in \mathrm{Lip}(\Pi)$ uniformly on compact subsets of Π, strongly in $H^1_{\mathrm{loc}}(\Pi)$ (by assumption) and $\nabla f_\rho \to \nabla f_0$ almost everywhere in Π.

Step 3. We have

$$\xi_0(y) \in C_0(y) := [T^o(\nu_\phi^{E_0}(y)) - n_\phi(0)]\phi^o(-\nabla f_0(y), 1) \qquad \text{for a.e. } y \in \Pi.$$

Indeed

$$\xi_\rho(y) \in C_\rho(y) := [T^o(\nu_\phi(\rho y, \rho f(y))) - n_\phi(\rho y, \rho f(y))]\phi^o(-\nabla f_\rho(y), 1) \qquad \text{for a.e. } y \in U_\rho.$$

From the upper semicontinuity of T^o it follows that for almost every $y \in \Pi$

$$\bigcap_{\epsilon > 0} \overline{\bigcup_{\rho < \epsilon} C_\rho(y)} \subseteq C_0(y).$$

Since $C_0(y)$ is a convex set and $\xi_\rho \rightharpoonup \xi_0$ weakly in $L^2_{\mathrm{loc}}(\Pi)$, it follows $\xi_0(y) \in C_0(y)$ for almost every $y \in \Pi$.

Step 4. Definition of N_0.

For $\mathcal{H}^2$ almost every $x \in \partial E_0$ let us define

$$N_0(x) := n_\phi(0) + \frac{\xi_0(\pi(x))}{\phi^o(-\nabla f_0(\pi(x)), 1)}.$$

Clearly, $N_0 \in T^o(\nu_\phi^{E_0})$; we now prove that $N_0 \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E_0)$. Indeed, since $\xi_0 \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\Pi)$ and $\mathrm{div}\xi_0 = 0$, for any $\psi \in \mathrm{Lip}(\partial E_0)$ with compact support, we have

$$\int_{\partial E_0} (N_0 - n_\phi(0)) \cdot \nabla\psi \, \mathrm{d}\mathcal{P}_\phi = \int_\Pi \xi_0 \cdot \nabla(\psi \circ \pi^{-1}) \, \mathrm{d}y = 0,$$

which implies $N_0 \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E_0)$ and $\mathrm{div}_{\phi,\tau} N_0 = 0$.

We now conclude the proof of the proposition. Assume that $\overline{x} \in \partial^* F$ is a Lebesgue point for $[N \cdot \widetilde{\nu}^F]$ on ∂F. For simplicity we let $\overline{x} = 0$. Recalling that $\widetilde{\nu}^{P_2} = \widetilde{\nu}^F(0)$, by Proposition 4.1 we have

$$[N_0 \cdot \widetilde{\nu}^{P_2}] = c_F(0), \qquad \mathcal{H}^1\text{a.e. on } P_1 \cap P_2.$$

To conclude it is enough to show

$$[N_0 \cdot \widetilde{\nu}^{P_2}] = [N \cdot \widetilde{\nu}^F](0), \qquad \mathcal{H}^1\text{a.e. on } P_1 \cap P_2. \tag{32}$$

Let $\psi \in \mathcal{C}^1_c(\mathbb{R}^3)$, $0 \leqslant \psi \leqslant 1$ be a radially symmetric function such that $\psi \equiv 1$ in $B_1(0)$ and $\mathrm{spt}(\psi) \subset B_2(0)$. We have

$$\begin{aligned}
[N \cdot \widetilde{\nu}^F](0) &= \lim_{\rho\to 0} \frac{1}{\int_{\partial F} \psi(x/\rho) \, \mathrm{d}\mathcal{H}^1} \int_{\partial F} [N \cdot \widetilde{\nu}^F] \psi(x/\rho) \, \mathrm{d}\mathcal{H}^1 \\
&= \lim_{\rho\to 0} \frac{1}{\rho \int_{\partial F/\rho} \psi \, \mathrm{d}\mathcal{H}^1} \int_{\partial F} [N \cdot \widetilde{\nu}^F] \psi(x/\rho) \, \mathrm{d}\mathcal{H}^1 \\
&= \lim_{\rho\to 0} \left(\frac{1}{\rho \int_{\partial F/\rho} \psi \, \mathrm{d}\mathcal{H}^1} \int_F \mathrm{div}_\tau N \psi(x/\rho) \, \mathrm{d}x \right. \\
&\quad \left. + \frac{1}{\rho^2 \int_{\partial F/\rho} \psi \, \mathrm{d}\mathcal{H}^1} \int_F N \cdot \nabla_\tau \psi(x/\rho) \, \mathrm{d}x \right) \\
&= \lim_{\rho\to 0} \frac{1}{\int_{\partial F/\rho} \psi \, \mathrm{d}\mathcal{H}^1} \int_{F/\rho} N_\rho \cdot \nabla_\tau \psi \, \mathrm{d}x \\
&= \frac{1}{\int_{\partial P_2} \psi \, \mathrm{d}\mathcal{H}^1} \int_{P_2} N_0 \cdot \nabla_\tau \psi \, \mathrm{d}x = [N_0 \cdot \widetilde{\nu}^{P_2}],
\end{aligned}$$

where, in the first equality of the last line, we used the convergence assumption on $\partial F/\rho$. The proof of (32) is complete. □

REMARK 4.4 Notice that any convex set E such that $\partial E \setminus F$ intersects F transversally verifies the assumptions of Proposition 4.3.

ASSUMPTION In what follows, we will always assume that E and F are such that any vector field $N \in H^{\mathrm{div}_\infty}_{\nu,\phi}(\partial E)$ verifies $[N \cdot \widetilde{\nu}^F] = c_F$ on ∂F (see the hypotheses in Propositions 4.1 and 4.3).

We let

$$H^{\mathrm{div}}_{\nu,\phi}(F) := \{N \in \mathrm{Nor}_\phi(F) : \mathrm{div}_\tau N \in L^2(F),\ [N \cdot \widetilde{\nu}^F] = c_F\},$$
$$H^{\mathrm{div}_\infty}_{\nu,\phi}(F) := \{N \in \mathrm{Nor}_\phi(F) : \mathrm{div}_\tau N \in L^\infty(F),\ [N \cdot \widetilde{\nu}^F] = c_F\},$$

where $\mathrm{Nor}_\phi(F)$ is as in (5) with ∂E replaced by F, and we define the functional $\mathcal{F}(\cdot, F) : H^{\mathrm{div}}_{\nu,\phi}(F) \to [0, +\infty[$ as

$$\mathcal{F}(N, F) := \int_F (\mathrm{div}_\tau\ N)^2 \,\mathrm{d}\mathcal{P}_\phi = \phi^o(\nu(F)) \int_F (\mathrm{div}_\tau\ N)^2 \,\mathrm{d}x. \tag{33}$$

PROPOSITION 4.5 The minimum problem

$$\inf\{\mathcal{F}(N, F) : N \in H^{\mathrm{div}}_{\nu,\phi}(F)\} \tag{34}$$

admits a solution. Moreover, if N_1 and N_2 are two minimizers, then $\mathrm{div}_\tau\ N_1(x) = \mathrm{div}_\tau\ N_2(x)$ for $\mathcal{H}^2$ almost every $x \in \mathrm{int}(F)$.

Proof. Let $C := \{\mathrm{div}_\tau N : N \in H^{\mathrm{div}}_{\nu,\phi}(F),\ [N \cdot \widetilde{\nu}^F] = c_F\}$. Then C is a convex subset of $L^2(F)$. Let us prove that C is closed in $L^2(F)$. Let $f_k := \mathrm{div}_\tau N_k \in C$ be such that $f_k \to f$ in $L^2(F)$ as $k \to \infty$. We have to prove that $f \in C$. Localizing the arguments of Proposition 6.1 in [4] to the facet F, one can prove that $f = \mathrm{div}_\tau N$, for some $N \in L^2(F; \mathbb{R}^3)$. It remains to check that $[N \cdot \widetilde{\nu}^F] = c_F$. Let $u \in \mathcal{C}^1(F)$; since $[N_k \cdot \widetilde{\nu}^F] = c_F$ for any k, we have

$$\int_F u \mathrm{div}_\tau N_k \,\mathrm{d}x + \int_F N_k \cdot \nabla u \,\mathrm{d}x = \int_{\partial F} c_F u \,\mathrm{d}\mathcal{H}^1, \qquad k \in \mathbb{N}.$$

Noticing that $\sup_k \|N_k\|_{L^\infty(F)} < +\infty$, we may, possibly extracting a subsequence, pass to the limit as $k \to \infty$, and we get

$$\int_F u \mathrm{div}_\tau N \,\mathrm{d}x + \int_F N \cdot \nabla u \,\mathrm{d}x = \int_{\partial F} c_F u \,\mathrm{d}\mathcal{H}^1.$$

As $u \in \mathcal{C}^1(F)$ is arbitrary, we obtain that $[N \cdot \widetilde{\nu}^F] = c_F$. The existence of a (unique in the divergence) minimizer of (34) is a standard consequence of minimization on convex sets of convex functionals on Hilbert spaces. □

The following proposition, based on the trace property discussed in Propositions 4.1 and 4.3, shows that the divergence of a solution to (34) is the divergence of $N_{\min}$ restricted to F.

PROPOSITION 4.6 $N_{\min|F}$ is a solution of (34).

Proof. By our assumptions on E and F we have that $[N_{\min} \cdot \widetilde{\nu}^F] = c_F$ on ∂F. Assume by contradiction that $N_{\min|F}$ is not a solution of (34). Let $\eta \in H^{\mathrm{div}_\infty}_{\nu,\phi}(F)$ be a solution of (34), and define

$$\overline{\eta} := \begin{cases} \eta & \text{on } \mathrm{int}(F), \\ N_{\min} & \text{on } \partial E \setminus F. \end{cases}$$

To reach a contradiction, it is enough to show that

$$\mathrm{div}_{\phi,\tau}\overline{\eta} = \begin{cases} \mathrm{div}_\tau \eta & \text{on } \mathrm{int}(F), \\ \mathrm{div}_{\phi,\tau} N_{\min} & \text{on } \partial E \setminus F, \end{cases} \tag{35}$$

since this implies that $\mathcal{F}(\overline{\eta}) < \mathcal{F}(N_{\min})$, thus violating the minimality of $N_{\min}$. Relation (35) is equivalent to showing that

$$\langle \mathrm{div}_{\phi,\tau}\overline{\eta}, \psi \rangle = \int_F \psi \, \mathrm{div}_\tau \eta \, \mathrm{d}\mathcal{P}_\phi + \int_{\partial E \setminus F} \psi \, \mathrm{div}_{\phi,\tau} N_{\min} \, \mathrm{d}\mathcal{P}_\phi \qquad \forall \psi \in \mathrm{Lip}(\partial E). \tag{36}$$

We first observe that $[\eta \cdot \widetilde{\nu}^F] = c_F$ on ∂F, hence

$$\int_{\partial F} \psi \, [(\eta - n_\phi) \cdot \widetilde{\nu}^F] \, \mathrm{d}\mathcal{H}^1 = 0. \tag{37}$$

As $\eta - n_\phi$ is a tangent vector field, (37) implies that

$$\int_F \psi \, \mathrm{div}_\tau (\eta - n_\phi) \, \mathrm{d}\mathcal{P}_\phi = - \int_F \nabla_\tau \psi \cdot (\eta - n_\phi) \, \mathrm{d}\mathcal{P}_\phi. \tag{38}$$

Equality (38) holds also with $N_{\min}$ in place of η; since, moreover, by (13)

$$\int_{\partial E} \psi \, \mathrm{div}_{\phi,\tau} (N_{\min} - n_\phi) \, \mathrm{d}\mathcal{P}_\phi = - \int_{\partial E} \nabla_\tau \psi \cdot (N_{\min} - n_\phi) \, \mathrm{d}\mathcal{P}_\phi,$$

we deduce

$$\int_{\partial E \setminus F} \psi \, \mathrm{div}_{\phi,\tau} (N_{\min} - n_\phi) \, \mathrm{d}\mathcal{P}_\phi = - \int_{\partial E \setminus F} \nabla_\tau \psi \cdot (N_{\min} - n_\phi) \, \mathrm{d}\mathcal{P}_\phi. \tag{39}$$

To conclude the proof, it is now enough to observe that (36) is equivalent to the sum of (38) and (39) (recall that $N_{\min} \cdot \nu_\phi = \eta \cdot \nu_\phi = 1$). □

The following result is a consequence of Propositions 4.5, 4.6 and Theorem 3.10.

COROLLARY 4.7 If N is a solution of (34) then $\mathrm{div}_\tau N$ coincides with κ^E_ϕ restricted to F, hence belongs to $L^\infty(F) \cap BV(\mathrm{int}(F))$.

5. Prescribed anisotropic curvature problem on convex facets

The following result will be useful in the sequel.

PROPOSITION 5.1 Assume that E is convex at F. Then for any $\lambda \in [\kappa_{\min}(F), \kappa_{\max}(F)]$ we have

$$\int_{\Omega_\lambda^F} \kappa_\phi^E \, dx = \widetilde{P_\phi}(\Omega_\lambda^F), \qquad \int_{\Theta_\lambda^F} \kappa_\phi^E \, dx = \widetilde{P_\phi}(\Theta_\lambda^F). \tag{40}$$

In particular

$$\int_F \kappa_\phi^E \, dx = \widetilde{P_\phi}(F). \tag{41}$$

Proof. Let $\lambda \in [\kappa_{\min}(F), \kappa_{\max}(F)]$. We apply (1) with the choice $\Omega := \mathrm{int}(F)$ (recall Theorem 3.3), $X := N_{\min}$, $u = 1_{\Omega_\lambda^F}$, so that, being $[N_{\min} \cdot \widetilde{\nu}^F] = c_F$ on ∂F,

$$\int_{\Omega_\lambda^F} \kappa_\phi^E \, dx = -\int_{\mathrm{int}(F)\cap\partial^*\Omega_\lambda^F} \theta(N_{\min}, D1_{\Omega_\lambda^F}) \, d\mathcal{H}^1 + \int_{\partial F} [N_{\min} \cdot \widetilde{\nu}^F] 1_{\Omega_\lambda^F} \, d\mathcal{H}^1.$$

Then the first equality in (40) follows, using a localization argument, from the definition of $\widetilde{P_\phi}$, from Theorem 3.13 and from the expression of c_F given by the second equality in (9) in the weakly convex case (recall that, if E is convex at F, then ∂E is weakly convex at any $x \in \partial F$). The proof of the second equality in (40) follows in a similar way. □

The following result is crucial to characterize ϕ-calibrable facets and extends the first assertion of Theorem 3.11; it shows that the sets Ω_λ^F solve a minimum problem which is the anisotropic version of the so-called prescribed curvature problem: see for instance [9] and references therein, [18–20].

Define

$$\mathcal{G}_\lambda(B) := \widetilde{P_\phi}(B) - \lambda|B|, \qquad B \subseteq F.$$

THEOREM 5.2 Assume that E is convex at F. Then for every $\lambda \in [\kappa_{\min}(F), \kappa_{\max}(F)]$ the sets Ω_λ^F and Θ_λ^F are solutions of the following variational problem:

$$\inf\{\mathcal{G}_\lambda(B) : B \subseteq F\}. \tag{42}$$

In addition, if $\widetilde{\Omega}$ is a solution of (42) then

$$\Omega_\lambda^F \subseteq \widetilde{\Omega} \subseteq \Theta_\lambda^F. \tag{43}$$

Proof. For any $B \subseteq F$ it holds

$$\mathcal{G}_\lambda(B) \geqslant \int_B (\kappa_\phi^E - \lambda) \, dx. \tag{44}$$

Since $\Omega_\lambda^F = \mathrm{int}(F) \cap \{\kappa_\phi^E - \lambda < 0\}$, if follows that

$$\int_B (\kappa_\phi^E - \lambda) \, dx \geqslant \int_{\Omega_\lambda^F} (\kappa_\phi^E - \lambda) \, dx. \tag{45}$$

As E is convex at F, using Proposition 5.1, we get

$$\int_{\Omega_\lambda^F} (\kappa_\phi^E - \lambda) \, dx = \widetilde{P_\phi}(\Omega_\lambda^F) - \lambda|\Omega_\lambda^F|. \tag{46}$$

From (44)–(46) it follows that Ω_λ^F is a solution of (42). In a similar way one proves that Θ_λ^F is also a solution of (42).

Finally, let $\widetilde{\Omega}$ be another solution of (42). Then the equality must hold in (45) with B replaced by $\widetilde{\Omega}$. Similarly, the equality in (45) must hold with B replaced by $\widetilde{\Omega}$ and Ω_λ^F replaced by Θ_λ^F. These observations imply (43). □

REMARK 5.3 Assume that E is convex at F. Then

$$\kappa_{\min}(F) \geqslant 2\sqrt{\frac{\pi}{|F|}}. \tag{47}$$

Indeed, if λ is such that $\Omega_\lambda^F \neq \emptyset$, then by the isoperimetric inequality (see for instance [11]) it follows $\widetilde{P_\phi}(\Omega_\lambda^F) \geqslant 2\sqrt{\pi|\Omega_\lambda^F|}$. Therefore, by Theorem 5.2 we have

$$0 = \mathcal{G}_\lambda(\emptyset) \geqslant \mathcal{G}_\lambda(\Omega_\lambda^F) \geqslant 2\sqrt{\pi|\Omega_\lambda^F|} - \lambda|\Omega_\lambda^F|.$$

Hence

$$|F| \geqslant |\Omega_\lambda^F| \geqslant \frac{4\pi}{\lambda^2}, \tag{48}$$

which implies (47). Notice that from (48) it follows that $\Theta_{\kappa_{\min}(F)}^F \neq \emptyset$, since $\Theta_{\kappa_{\min}(F)}^F = \bigcap_{\lambda>\kappa_{\min}(F)} \Omega_\lambda^F$.

6. Characterization of general ϕ-calibrable facets

This is the only section of the paper where we consider also the presence of a forcing term g. We also do not assume here any convexity-type assumption on E and F.

Let $g \in L^\infty(\partial E)$; all results of Section 3.3 still hold [4], [5] when the functional $\mathcal{F}$ in (14) is replaced by

$$\int_{\partial E} (\mathrm{div}_{\phi,\tau} N - g)^2 \,\mathrm{d}\mathcal{P}_\phi, \qquad N \in H_{\nu,\phi}^{\mathrm{div}}(\partial E), \tag{49}$$

provided we replace κ_ϕ^E with $d_{\min}^E - g$, where $d_{\min}^E := \mathrm{div}_{\phi,\tau}\mathcal{N}_{\min}$, $\mathcal{N}_{\min}$ a minimizer of (49). Accordingly, the functional $\mathcal{F}(\cdot, F)$ in (33) must be modified into

$$\int_F (\mathrm{div}_\tau N - g)^2 \,\mathrm{d}\mathcal{P}_\phi, \qquad N \in H_{\nu,\phi}^{\mathrm{div}}(F). \tag{50}$$

Again (see Corollary 4.7) if N is a minimizer of the functional in (50), then $\mathrm{div}_\tau N - g$ coincides with $d_{\min}^E - g$ restricted to F.

For any $B \subseteq F$ we set

$$\overline{g}_B := \frac{1}{|B|}\int_B g \,\mathrm{d}x.$$

We also define the constant V_F as follows:

$$V_F := \frac{1}{|F|}\int_{\partial F} c_F \,\mathrm{d}\mathcal{H}^1 - \overline{g}_F.$$

Notice that by the results of Sections 4 and by (1) (we recall that by Theorem 3.3 F is Lipschitz up to a finite set of points) we have

$$V_F = \frac{1}{|F|}\int_{\partial F} [\mathcal{N}_{\min} \cdot \widetilde{\nu}^F]\, d\mathcal{H}^1 - \overline{g}_F = \frac{1}{|F|}\int_F (d^E_{\min} - g)\, dx. \tag{51}$$

If B has finite perimeter in H_F, for $x \in \partial^* B$ we define

$$c_B(x) := \begin{cases} \max\left\{p \cdot \widetilde{\nu}^B(x) : p \in \widetilde{W}^F_\phi\right\} & \text{if } x \in \partial^* B \setminus \partial F \\ c_F(x), & \text{otherwise.} \end{cases} \tag{52}$$

A weaker form of the implication (i) ⇒ (ii) of the following result was proved in [3].

THEOREM 6.1 The following two conditions are equivalent.

(i) F is ϕ-calibrable (i.e. $d^E_{\min} - g$ is constant on $\mathrm{int}(F)$);
(ii) for any $B \subseteq F$ of finite perimeter in H_F there holds

$$\frac{1}{|B|}\int_{\partial^* B} c_B\, d\mathcal{H}^1 - \overline{g}_B \geqslant \frac{1}{|F|}\int_{\partial F} c_F\, d\mathcal{H}^1 - \overline{g}_F. \tag{53}$$

Proof. (ii) ⇒ (i). Suppose by contradiction that F is not ϕ-calibrable, i.e. $d^E_{\min} - g$ is not constantly equal to V_F on $\mathrm{int}(F)$. It follows that $\Omega^F_{V_F} = \{d^E_{\min} - g < V_F\} \cap \mathrm{int}(F)$ is nonempty. By Corollary 4.7, we can find $\overline{\lambda} < V_F$ such that $\Omega^F_{\overline{\lambda}}$ is a nonempty set of finite perimeter. Set for simplicity $Q := \Omega^F_{\overline{\lambda}}$. From (1) we have

$$\begin{aligned}\int_Q d^E_{\min}\, dx &= -\int_{\mathrm{int}(F)\cap\partial^* Q} \theta(\mathcal{N}_{\min}, D1_Q)\, d\mathcal{H}^1 + \int_{\partial F} [\mathcal{N}_{\min} \cdot \widetilde{\nu}^F] 1_Q\, d\mathcal{H}^1 \\ &= -\int_{\mathrm{int}(F)\cap\partial^* Q} \theta(\mathcal{N}_{\min}, D1_Q)\, d\mathcal{H}^1 + \int_{\partial F\cap\partial^* Q} [\mathcal{N}_{\min} \cdot \widetilde{\nu}^F]\, d\mathcal{H}^1.\end{aligned}$$

Recalling Theorem 3.13 (which is still valid for $\mathcal{N}_{\min}$ [5]) and definition (52) of c_Q, we have $-\theta(\mathcal{N}_{\min}, D1_Q) = c_Q$ on $\partial^* Q \cap \mathrm{int}(F)$; moreover $[\mathcal{N}_{\min} \cdot \widetilde{\nu}^F] = c_F = c_Q$ on $\partial F \cap \partial^* Q$. Therefore $\int_Q d^E_{\min}\, dx = \int_{\partial^* Q} c_Q\, d\mathcal{H}^1$. It follows, using (ii),

$$V_F > \overline{\lambda} > \frac{1}{|Q|}\int_Q d^E_{\min}\, dx - \overline{g}_Q = \frac{1}{|Q|}\int_{\partial^* Q} c_Q\, d\mathcal{H}^1 - \overline{g}_Q \geqslant V_F, \tag{54}$$

which is a contradiction.

(i) ⇒ (ii). Let $B \subseteq F$ be a set of finite perimeter in H_F. If we integrate $d^E_{\min} - g$ over B, using (1) and (52), we get

$$\begin{aligned}V_F = \frac{1}{|B|}\int_B V_F\, dx &= -\frac{1}{|B|}\int_{\mathrm{int}(F)\cap\partial^* B} \theta(\mathcal{N}_{\min}, D1_B)\, d\mathcal{H}^1 \\ &+ \frac{1}{|B|}\int_{\partial F\cap\partial^* B} c_F\, d\mathcal{H}^1 - \overline{g}_B \leqslant \frac{1}{|B|}\int_{\partial^* B} c_B\, d\mathcal{H}^1 - \overline{g}_B,\end{aligned}$$

which is (ii). □

7. Convexity of the sets Ω_λ^F and Θ_λ^F

Our aim is to prove the following result.

THEOREM 7.1 Assume that E is convex at F and that F is convex. Then Ω_λ^F is convex for any $\lambda > \kappa_{\min}(F)$, and Θ_λ^F is convex for any $\lambda \geqslant \kappa_{\min}(F)$.

In Corollary 9.5 we will prove a stronger result, namely that κ_ϕ^E is (continuous and) convex on F. We will prove Theorem 7.1 only for the sets Ω_λ^F since the assertion on Θ_λ^F follows from the convexity of Ω_λ^F and the equality

$$\Theta_\lambda^F = \bigcap_{\mu>\lambda} \Omega_\mu^F, \qquad \forall\lambda \geqslant \kappa_{\min}(F). \tag{55}$$

To prove Theorem 7.1 we need some preliminary lemmas.

LEMMA 7.2 Assume that E is convex at F and that F is convex. Let $\lambda > \kappa_{\min}(F)$. Then $\mathrm{int}(\Omega_\lambda^F)$ consists of a finite union of convex open sets whose closures are pairwise disjoint.

Proof. Since Ω_λ^F has finite perimeter, by [1] it follows that

$$\mathrm{int}(\Omega_\lambda^F) = \bigcup_{i\in I} C_i, \qquad \widetilde{P_\phi}(\Omega_\lambda^F) = \sum_{i\in I} \widetilde{P_\phi}(C_i), \tag{56}$$

where I is at most countable and C_i are nonempty open connected sets, pairwise disjoint. Observe that each C_i is simply connected by Theorem 5.2, because filling the holes strictly decreases the functional $\mathcal{G}_\lambda$ (we use here the property that, if E is convex at F, then $\lambda > \kappa_{\min}(F) > 0$, see (47)). This fact, together with the property that ∂C_i has finite length, implies that ∂C_i is parametrizable in a Lipschitz way by a closed Jordan curve. Let us show that C_i is convex for any $i \in I$. Let $\mathrm{co}(C_i)$ be the (open) convex envelope of C_i, and assume by contradiction that $\mathrm{co}(C_i) \neq C_i$ for some $i \in I$. It follows that the set $A := \bigcup_{i\in I} \mathrm{co}(C_i)$ properly contains Ω_λ^F, hence $|A| > |\Omega_\lambda^F|$; moreover A is contained in F, since F is convex. Parametrizing ∂C_i, we can use Jensen's inequality to prove that $\widetilde{P_\phi}(C_i) \geqslant \widetilde{P_\phi}(\mathrm{co}(C_i))$. Therefore, by (56)

$$\widetilde{P_\phi}(\Omega_\lambda^F) = \sum_{i\in I} \widetilde{P_\phi}(C_i) \geqslant \sum_{i\in I} \widetilde{P_\phi}(\mathrm{co}(C_i)) \geqslant \widetilde{P_\phi}(A).$$

Hence $\mathcal{G}_\lambda(A) < \mathcal{G}_\lambda(\Omega_\lambda^F)$, which contradicts Theorem 5.2. It follows that each C_i is convex. In view of the different scaling factors of $\widetilde{P_\phi}(\cdot)$ and $|\cdot|$ it is easy to see that I is finite. Indeed, eliminating the connected components with volume sufficiently small decreases the functional $\mathcal{G}_\lambda$. It remains to prove that $\overline{C_i} \cap \overline{C_j} = \emptyset$ for $i \neq j$. Assume by contradiction that $\overline{C_i} \cap \overline{C_j} \neq \emptyset$. By Jensen's inequality it follows again that $\mathcal{G}_\lambda$ strictly decreases by substituting $C_i \cup C_j$ with $\mathrm{co}(C_i \cup C_j)$, thus contradicting Theorem 5.2. □

In the following lemma we prove that the part of ∂F lying 'above' or 'below' a connected component of $\mathrm{int}(F) \cap \Omega_\lambda^F$ can be written as a graph on a segment $[x, y]$, with possibly a 'vertical' part at x or at y, but not at x and at y, see Fig. 2.

LEMMA 7.3 Let F be convex. Let $\lambda > 0$ be such that $\Omega_\lambda^F \notin \{\emptyset, \mathrm{int}(F)\}$. Denote by Σ the closure of a connected component of $\mathrm{int}(F) \cap \partial\Omega_\lambda^F$, and set $\{x, y\} := \Sigma \cap \partial F$. Let $\widetilde{\nu}^\Sigma$ be the outward unit

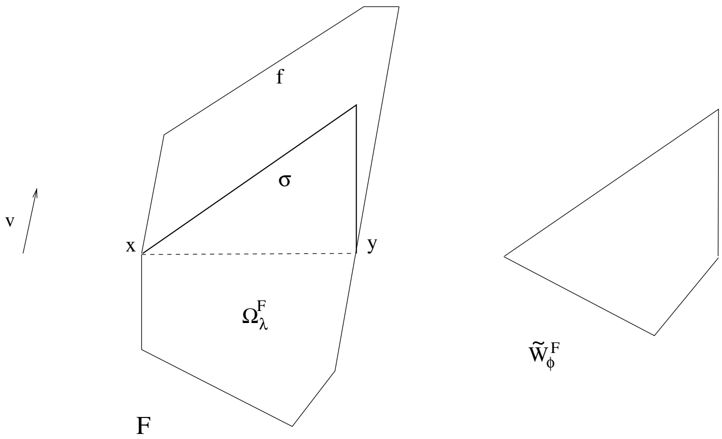

FIG. 2. Lemma 7.3: ∂F is locally graph of a function f, possibly discontinuous at one extremum.

normal on $[x, y]$ to the convex set bounded by Σ and $[x, y]$ (when $\Sigma = [x, y]$ we set $\widetilde{\nu}^{\Sigma} := -\widetilde{\nu}^{\Omega_\lambda^F}$). Then there exist a vector v such that $v \cdot \widetilde{\nu}^{\Sigma} < 0$ and a convex function $f : [x, y] \to \mathbb{R}v$ such that either $f(x) = 0$ or $f(y) = 0$, and $\operatorname{graph}(f) \cup [x, x + f(x)] \cup [y, y + f(y)] \subseteq \partial F$. A similar statement holds for Θ_λ^F.

Proof. Let $\Pi := \{w : (w - x) \cdot \widetilde{\nu}^{\Sigma} \leqslant 0\}$. Let τ_x, τ_y be the tangent unit vectors to $\partial F \cap \Pi$ at x and y respectively, pointing inside Π (τ_x and τ_y exist because F is convex). Let us prove that τ_x and τ_y are 'weakly convergent', i.e. $(\tau_y - \tau_x) \cdot (y - x) \leqslant 0$. Assume by contradiction that $(\tau_y - \tau_x) \cdot (y - x) > 0$. Choose $\widetilde{v}$, $|\widetilde{v}| = 1$, such that $\tau_y \cdot (y - x) > \widetilde{v} \cdot (y - x) > \tau_x \cdot (y - x)$. Let C be the (convex) connected component of Ω_λ^F such that $\partial C \supset \Sigma$. It is easy to realize that we can slightly translate C in the direction of $\widetilde{v}$ still remaining inside F, and this translated set does not intersect $\Omega_\lambda^F \setminus C$ (recall Lemma 7.2). Precisely, there exists $\epsilon > 0$ such that

$$s\widetilde{v} + C \subset F, \qquad (\Omega_\lambda^F \setminus C) \cap (s\widetilde{v} + C) = \emptyset, \quad \forall s \in \,]0, \epsilon[. \tag{57}$$

Let us fix $0 < s_1 < \epsilon$ and define $\widetilde{\Omega} := (\Omega_\lambda^F \setminus C) \cup (s_1\widetilde{v} + C)$. Then $\widetilde{\Omega}$ is a minimum of $\mathcal{G}_\lambda$ which does not contain Ω_λ^F, contradicting (43).

It follows that $(\tau_y - \tau_x) \cdot (y - x) \leqslant 0$. This and the convexity of F imply that there are a unit vector v and a convex function $f : [x, y] \to \mathbb{R}v$ such that $\partial F \cap \Pi = \operatorname{graph}(f) \cup [x, x + f(x)] \cup [y, y + f(y)]$. It remains to check that either $f(x) = 0$ or $f(y) = 0$. Indeed, if by contradiction $f(x) \cdot v > 0$ and $f(y) \cdot v > 0$, then we can perform a slight translation of C in the direction of v obtaining a contradiction, exactly as in the previous argument.

The assertion on Θ_λ^F follows from similar considerations. □

REMARK 7.4 As $\Sigma \subseteq F$ and Σ is contained in a translated of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ (Theorem 3.11), from Lemma 7.3 it follows that Σ can be written as a graph of a convex function $\sigma : [x, y] \to \mathbb{R}v$ such that $\sigma(x) = \sigma(y) = 0$.

We are now in the position to prove Theorem 7.1.

Proof. By Lemma 7.2, it is enough to show that Ω_λ^F is connected. Assume by contradiction that Ω_λ^F has (at least) two connected components C, C' and let $\Sigma \subset \partial C$, $x, y \in \Sigma$, τ_x, τ_y, Π, v, f be as in Lemma 7.3 and its proof. We can assume, without loss of generality, that $C' \subset (F \setminus C) \cap \Pi$. In the same way, we can find $\Sigma' \subset \partial C'$, x', $y' \in \Sigma'$, $\tau_{x'}$, $\tau_{y'}$, Π' such that $C \subset (F \setminus C') \cap \Pi'$. By Lemma 7.3 we have

$$(\tau_y - \tau_x) \cdot (y - x) \leqslant 0, \qquad (\tau_{y'} - \tau_{x'}) \cdot (y' - x') \leqslant 0. \tag{58}$$

Since F is convex and $C \subset (F \setminus C') \cap \Pi'$, from the first inequality in (58) it follows

$$(\tau_{y'} - \tau_{x'}) \cdot (y' - x') \geqslant 0.$$

Hence $(\tau_{y'} - \tau_{x'}) \cdot (y' - x') = 0$. In the same way we obtain $(\tau_y - \tau_x) \cdot (y - x) = 0$. It follows that $\partial F \cap \Pi \cap \Pi'$ is the union of two parallel segments, which implies $f(x) \cdot v > 0$ and $f(y) \cdot v > 0$, contradicting Lemma 7.3. □

8. Characterization of ϕ-calibrable facets in the convex case

The aim of this section is to prove the following theorem, which is one of the main results of this paper.

THEOREM 8.1 Assume that E is convex at F and that F is convex. Then F is ϕ-calibrable if and only if

$$\operatorname*{ess\,sup}_{\partial F} \widetilde{\kappa}_\phi^F \leqslant \frac{\widetilde{P_\phi}(F)}{|F|}. \tag{59}$$

Proof of the implication:

$$\operatorname*{ess\,sup}_{\partial F} \widetilde{\kappa}_\phi^F \leqslant \frac{\widetilde{P_\phi}(F)}{|F|} \Rightarrow F \text{ is } \phi\text{-calibrable}. \tag{60}$$

We need the following local comparison lemma, whose proof (well known in the crystalline case [12]) is omitted. Recall that, if $\lambda > 0$, the $\widetilde{\phi}$-curvature of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ is constantly equal to λ.

LEMMA 8.2 Let $P \subseteq H_F$ be a closed convex Lipschitz $\widetilde{\phi}$-regular set, let $x \in \partial P$ and $\lambda > 0$. Assume that there exist a neighbourhood $N(x)$ of x and a translated $\mathcal{B}_{\frac{1}{\lambda}}$ of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ such that $x \in \partial\mathcal{B}_{\frac{1}{\lambda}}$, and

$$P \supseteq N(x) \cap \mathcal{B}_{\frac{1}{\lambda}}.$$

Then

$$\operatorname*{ess\,inf}_{\partial P \cap N(x)} \widetilde{\kappa}_\phi^P \leqslant \lambda.$$

Similarly, if

$$P \cap N(x) \subseteq \mathcal{B}_{\frac{1}{\lambda}},$$

then

$$\operatorname*{ess\,sup}_{\partial P \cap N(x)} \widetilde{\kappa}_\phi^P \geqslant \lambda.$$

Assume by contradiction that (60) is false, i.e. F is not ϕ-calibrable. Since E is convex at F, by (41) we have

$$\frac{1}{|F|}\int_F \kappa_\phi^E \,\mathrm{d}x = \frac{\widetilde{P_\phi}(F)}{|F|}.$$

Therefore we can pick $\overline{\lambda} > 0$ with the following properties:

$$\overline{\lambda} > \frac{\widetilde{P_\phi}(F)}{|F|}, \qquad \Omega^F_{\overline{\lambda}} \notin \{\emptyset, \mathrm{int}(F)\}, \qquad \Omega^F_{\overline{\lambda}} \text{ of finite perimeter.} \tag{61}$$

Let $\Sigma \subset \partial\Omega^F_{\overline{\lambda}}$, x, y, v, Π be as in Lemma 7.3 and its proof. From Lemma 7.3 and Remark 7.4 it follows that there exist two convex functions $f, \sigma : [x, y] \to \mathbb{R}v$ such that $f \cdot v \geqslant \sigma \cdot v$, $\Sigma = \mathrm{graph}(\sigma)$ and $\Pi \cap \partial F = \mathrm{graph}(f) \cup [x, x + f(x)] \cup [y, y + f(y)]$. Let

$$M := \{z \in [x, y] : f(z) - \sigma(z) = \max_{[x,y]}(f - \sigma)\}.$$

We divide the proof into two cases.

Case 1. Assume that $M \cap \,]x, y[\, \neq \emptyset$.

Let $z \in M \cap \,]x, y[$. Then F is a convex set which is Lipschitz $\widetilde{\phi}$-regular by Proposition 3.7, and is contained, locally in a neighbourhood of the point $z + f(z)v$, in the set $f(z)v + \Omega^F_{\overline{\lambda}}$. Recall that, by Theorem 3.11, we know that Σ is contained in a translated of $\frac{1}{\overline{\lambda}}\widetilde{W}^F_{\overline{\lambda}}$. Therefore, using Lemma 8.2, it follows

$$\operatorname*{ess\,sup}_{\partial F} \widetilde{\kappa}_\phi^F \geqslant \overline{\lambda}. \tag{62}$$

From (62) and the inequality in (61) it follows $\operatorname{ess\,sup}_{\partial F} \widetilde{\kappa}_\phi^F > \frac{\widetilde{P_\phi}(F)}{|F|}$, which contradicts (59).

Case 2. Assume that $M \cap \,]x, y[\, = \emptyset$.

In this case we can suppose $M = \{x\}$, since by Lemma 7.3 if $x \in M$ then $f(x) \neq \sigma(x) = 0$ and $f(y) = \sigma(y) = 0$, which implies $y \notin M$.

Define $\sigma_\epsilon(\cdot) := \sigma(\cdot + \epsilon(y - x))$ on $I_\epsilon := [x - \epsilon(y - x), y - \epsilon(y - x)]$. If $\epsilon > 0$ is sufficiently small, the set $M_\epsilon := \{z \in I_\epsilon : f(z) - \sigma_\epsilon(z) = \max_{I_\epsilon}(f - \sigma_\epsilon)\}$ cannot intersect ∂I_ϵ. We now reason as in Case 1 considering σ_ϵ in place of σ and taking a point $z' \in M_\epsilon$ in place of z. The proof of (60) is concluded.

Proof of the implication:

$$F \text{ is } \phi\text{-calibrable} \Rightarrow \operatorname*{ess\,sup}_{\partial F} \widetilde{\kappa}_\phi^F \leqslant \frac{\widetilde{P_\phi}(F)}{|F|}. \tag{63}$$

We need some preliminaries. The following lemma is a sort of converse of Lemma 8.2. It concerns the existence of an 'obsculating' Wulff shape. By definition, we set $\inf\emptyset = +\infty$.

LEMMA 8.3 Let $P \subseteq H_F$ be a closed convex Lipschitz $\widetilde{\phi}$-regular set. Let $x \in \partial P$ be a point of differentiability of ∂P and where $\widetilde{\kappa}_\phi^P(x)$ exists. Define $\mathrm{O}(x)$ as the set of all $R > 0$ such that P is locally contained, in a neighbourhood of x, in a translated $\mathcal{B}_R$ of $R\widetilde{W}_\phi^F$ with $x \in \partial\mathcal{B}_R$; define also

$I(x)$ as the set of all $r > 0$ such that a translated $\mathcal{B}_r$ of $r\widetilde{W}_\phi^F$ with $x \in \partial\mathcal{B}_r$ is locally contained, in a neighbourhood of x, in P. Then

$$\widetilde{\kappa}_\phi^P(x) = (\sup I(x))^{-1} = (\inf \mathrm{O}(x))^{-1}.$$

Proof. The assertion is well known when $\widetilde{\phi}$ is smooth and strictly convex. Here, we shall give the proof only in the crystalline case. Since P is Lipschitz $\widetilde{\phi}$-regular, there exists $\widetilde{n}_\phi \in \mathrm{Lip}(\partial P; H_F)$ with $\widetilde{n}_\phi(x) \in \widetilde{T}^o(\widetilde{\nu}_\phi^P(x))$ for $\mathcal{H}^1$ almost every $x \in \partial P$. As P is also convex and ϕ is crystalline, only two possibilities occur: either x is in the interior of an arc or of an edge where $\widetilde{n}_\phi$ is constantly equal to a vertex of $\widetilde{W}_\phi^F$ or x is in the interior of an edge of $L \subset \partial P$ parallel to some edge $l \subset \partial\widetilde{W}_\phi^F$. In the first case we have $\widetilde{\kappa}_\phi^P(x) = 0$, and since $\widetilde{\phi}$ is crystalline and ∂P is differentiable at x, it is immediate to check that $\mathrm{O}(x) = \emptyset$ and $I(x) = \,]0, +\infty[$. In the second case we have $\widetilde{\kappa}_\phi^P(x) = \frac{l}{L}$, and $I(x) = \,]0, L/l[$, $\mathrm{O}(x) = \,]L/l, +\infty[$, which gives the assertion. □

The following lemma concerns minimizers of the functional $\mathcal{G}_\lambda$ computed on graphs of functions u.

LEMMA 8.4 Let $a, b \in \mathbb{R}$, $a < b$, $\lambda > 0$ and $G_\lambda : H_0^1([a,b]) \to \mathbb{R}$ be defined as

$$G_\lambda(u) := \int_{[a,b]} \widetilde{\phi}^o(-u'(s), 1) - \lambda u(s)\, \mathrm{d}\mathcal{H}^1(s). \tag{64}$$

Assume that there exists a function $u_\lambda \in H_0^1([a,b])$ whose graph is contained in a translated of $\frac{1}{\lambda}\partial\widetilde{W}_\phi^F$. Then u_λ minimizes G in $H_0^1([a,b])$.

Proof. Assume first that $\widetilde{W}_\phi^F$ is smooth and strictly convex, and let $\widetilde{\phi}^o = \widetilde{\phi}^o(\xi_1, \xi_2)$, $(\xi_1, \xi_2) \in \mathbb{R}^2 \simeq H_F$. Then the Euler equation associated to G_λ reads as

$$\frac{\partial}{\partial s}\left(\frac{\partial\widetilde{\phi}^o}{\partial\xi_1}(-u'(s), 1)\right) = \lambda,$$

which is equivalent to

$$\frac{\partial\widetilde{\phi}^o}{\partial\xi_1}(-u'(s), 1)) = \lambda s + c, \qquad \text{for some } c \in \mathbb{R}. \tag{65}$$

Since the functional G_λ is strictly convex in $H_0^1([a,b])$, if we prove that u_λ solves (65), then u_λ minimizes G_λ in $H_0^1([a,b])$. By assumption, there exists a point $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^2$ such that $\mathrm{graph}(u_\lambda) \subset \bar{z} + \frac{1}{\lambda}\partial\widetilde{W}_\phi^F$. Letting $\widetilde{\nu}_\phi^\lambda(s) := (-u_\lambda'(s), 1)/\widetilde{\phi}^o(-u_\lambda'(s), 1)$ we have

$$\nabla\widetilde{\phi}^o(-u_\lambda'(s), 1) = \widetilde{T}^o(\widetilde{\nu}_\phi^\lambda(s)) = \lambda(s, u_\lambda(s)) - \bar{z} = (\lambda s - \bar{x}, \lambda u_\lambda(s) - \bar{y})$$

which implies (65) with $c = -\bar{x}$. Then u_λ minimizes G_λ on $H_0^1([a,b])$.

Let us consider now a general Finsler metric ϕ. Choose a sequence of functions $(\widetilde{\phi}_k^o)_k$, with $\widetilde{\phi}_k^o > \widetilde{\phi}^o$, which converges uniformly on compact subsets of $\mathbb{R}^2$ to $\widetilde{\phi}^o$ and such that $\{\widetilde{\phi}_k^o \leqslant 1\}$ are smooth and strictly convex. Let G_k be defined as G_λ with $\widetilde{\phi}^o$ replaced by $\widetilde{\phi}_k^o$. The functionals G_k converge uniformly, as $k \to +\infty$, to G_λ on bounded subsets of $H_0^1([a,b])$. Since $\widetilde{\phi}_k^o > \widetilde{\phi}^o$, we can find functions $u_\lambda^k \in H_0^1([a,b])$ whose graphs are contained in a translated of $\frac{1}{\lambda}\partial\{\widetilde{\phi}_k \leqslant 1\}$. By the previous argument, u_λ^k minimizes G_k on $H_0^1([a,b])$. Since $u_\lambda^k \to u_\lambda$ in $H_0^1([a,b])$ as $k \to +\infty$, we obtain that u_λ minimizes G_λ. □

Let us now prove (63). Assume that F is ϕ-calibrable, so that

$$\mathrm{int}(F) = \Omega_\lambda^F \qquad \forall \lambda > \kappa_{\min}(F), \tag{66}$$

and suppose by contradiction that (59) does not hold. Let $x \in \partial F$ be a point where ∂F is differentiable, where there exists $\widetilde{\kappa}_\phi^F(x)$ and $\widetilde{\kappa}_\phi^F(x) > \frac{\widetilde{P_\phi}(F)}{|F|}$. Choose

$$\lambda \in \left] \frac{\widetilde{P_\phi}(F)}{|F|}, \widetilde{\kappa}_\phi^F(x) \right[. \tag{67}$$

By Lemma 8.3, there exist $\rho > 0$ and a translated $\mathcal{B}_{\frac{1}{\lambda}}$ of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ such that $x \in \partial \mathcal{B}_{\frac{1}{\lambda}}$ and

$$F \cap B_\rho(x) \subseteq \mathcal{B}_{\frac{1}{\lambda}}.$$

We divide the proof into three cases.

Case 1. Assume that $\widetilde{T}^o(\widetilde{\nu}_\phi^F(x))$ is a singleton.

In this case we have, for $\rho > 0$ sufficiently small,

$$\partial F \cap \partial \mathcal{B}_{\frac{1}{\lambda}} \cap B_\rho(x) = \{x\}.$$

Choose a unit vector v and $\rho > 0$ small enough such that $\partial F \cap B_\rho(x)$ and $\partial \mathcal{B}_{\frac{1}{\lambda}} \cap B_\rho(x)$ are both graphs of two convex functions of class H^1 along v, with $F \cap B_\rho(x)$ and $\mathcal{B}_{\frac{1}{\lambda}} \cap B_\rho(x)$ as corresponding subgraphs. Let $A_\delta := \mathcal{B}_{\frac{1}{\lambda}} - \delta v$, for $\delta > 0$ sufficiently small. Let $\{y_1, y_2\} := \partial F \cap \partial A_\delta$. Denote by Π the half-plane containing v and with y_1, y_2 in its boundary. Then $\partial F \cap \Pi$ and $\partial A_\delta \cap \Pi$ are both graphs of two convex functions on $[y_1, y_2]$ along v. Applying Lemma 8.4 (and a suitable change of coordinates) we have that, letting $H_\lambda := (F \setminus \Pi) \cup (A_\delta \cap \Pi)$, then $\mathcal{G}_\lambda(H_\lambda) \leqslant \mathcal{G}_\lambda(F)$. By (66) we have $\mathcal{G}_\lambda(F) = \mathcal{G}_\lambda(\Omega_\lambda^F)$. We deduce $\mathcal{G}_\lambda(H_\lambda) \leqslant \mathcal{G}_\lambda(\Omega_\lambda^F)$, and this contradicts Theorem 5.2, since H_λ does not contain Ω_λ^F.

Case 2. Assume that $\widetilde{T}^o(\widetilde{\nu}_\phi^F(x))$ is not a singleton and that $\partial \widetilde{W}_\phi^F$ can be written as the graph of a convex function (with respect to some direction) in a neighbourhood of $\widetilde{T}^o(\widetilde{\nu}_\phi^F(x))$.

Note that necessarily $\widetilde{T}^o(\widetilde{\nu}_\phi^F(x))$ is an edge of $\partial \widetilde{W}_\phi^F$. As F is a convex Lipschitz $\widetilde{\phi}$-regular set, we have that x belongs to an edge L of ∂F. Since we may avoid subsets of ∂F with $\mathcal{H}^1$ zero measure in the computation of the essential supremum, we can assume that x belongs to the interior of an edge L of ∂F. Reasoning as in Case 1, we can find a neighbourhood $N(L)$ of L and a translated $\mathcal{B}_{\frac{1}{\lambda}}$ of $\frac{1}{\lambda}\partial \widetilde{W}_\phi^F$ such that $x \in \partial \mathcal{B}_{\frac{1}{\lambda}}$ and

$$F \cap N(L) \subseteq \mathcal{B}_{\frac{1}{\lambda}}.$$

Possibly reducing $N(L)$, we can also assume

$$\partial F \cap \partial \mathcal{B}_{\frac{1}{\lambda}} \cap N(L) = L.$$

Noticing that ∂F can be written as a graph of a convex function in a neighbourhood of L, we conclude as in Case 1, making use of Lemma 8.4.

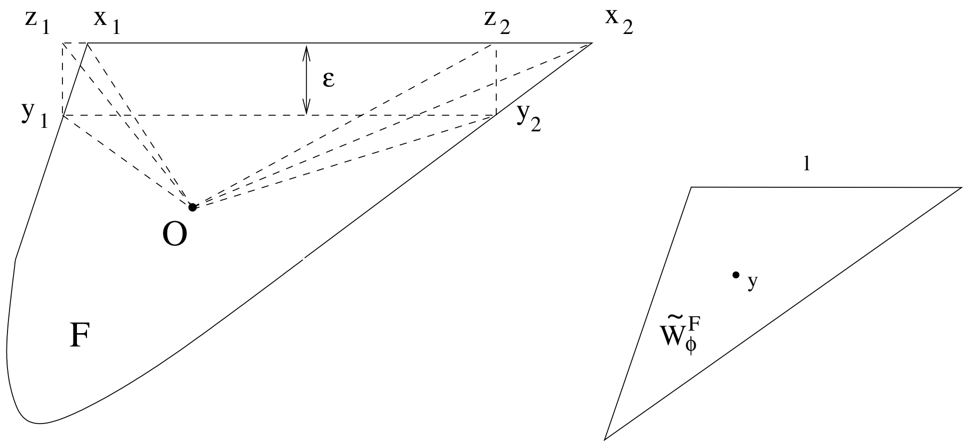

FIG. 3. Case 3 of the proof of (63): $\widetilde{W}^F_\phi$ is not locally graph around l.

Case 3. Assume that $\widetilde{T}^o(\widetilde{\nu}^F_\phi(x))$ is not a singleton and that $\partial\widetilde{W}^F_\phi$ cannot be written as a graph in a neighbourhood of $\widetilde{T}^o(\widetilde{\nu}^F_\phi(x))$, see Fig. 3.

Let L be the edge of ∂F containing x in its interior, and denote by x_1, x_2 its extrema. We often identify L with its length. We need the following lemma. We denote by $y \in \mathrm{int}(\widetilde{W}^F_\phi)$ the point such that $\widetilde{\phi} = \widetilde{\phi}_y$, see the comments after Definition 2.2.

LEMMA 8.5 Let $\mu > 0$ and let $C \subset H_F$ be an open cone centred at μy. Then

$$\widetilde{P_\phi}(\mu\widetilde{W}^F_\phi, C) = \frac{2}{\mu}\,|C \cap \mu\widetilde{W}^F_\phi|.$$

Proof. We take $\mu = 1$, the general case follows by rescaling. For $x \in \partial\widetilde{W}^F_\phi$ we have $\widetilde{\phi}^o(\widetilde{\nu}^{\widetilde{W}^F_\phi}(x)) = \widetilde{\nu}^{\widetilde{W}^F_\phi}(x)\cdot x$, while for $x \in \partial C \setminus \{y\}$ we have $\widetilde{\nu}^C(x)\cdot x = 0$. Therefore

$$\begin{aligned}\widetilde{P_\phi}(\widetilde{W}^F_\phi, C) &= \int_{C\cap\partial\widetilde{W}^F_\phi} \widetilde{\phi}^o(\widetilde{\nu}^{\widetilde{W}^F_\phi}(x))\,\mathrm{d}\mathcal{H}^1 = \int_{\partial(C\cap\widetilde{W}^F_\phi)} \widetilde{\nu}^{\widetilde{W}^F_\phi}(x)\cdot x\,\mathrm{d}\mathcal{H}^1\\ &= \int_{C\cap\widetilde{W}^F_\phi} \mathrm{div}x\,\mathrm{d}x = 2|C\cap\widetilde{W}^F_\phi|.\end{aligned}$$

□

We now prove the assertion in Case 3. Let $\epsilon > 0$; we denote by F_ϵ the set of all points of F whose (Euclidean) distance from the line passing through L is greater than $\epsilon > 0$. We will prove that, if ϵ is small enough, then

$$\mathcal{G}_\lambda(F_\epsilon) < \mathcal{G}_\lambda(F). \tag{68}$$

Denote by l the (length of the) edge of $\widetilde{W}_\phi^F$ corresponding to L. We claim that

$$\widetilde{P_\phi}(F) - \widetilde{P_\phi}(F_\epsilon) = \epsilon l + \mathrm{o}(\epsilon). \tag{69}$$

If ϵ is small enough, we can assume that F, in a neighbourhood of L coincides with a corresponding portion of $w + \frac{L}{l}\widetilde{W}_\phi^F$ for some $w \in H_F$. Indeed, if we modify F locally around L into a new set F' which coincides with a portion of a translated of $\frac{L}{l}\widetilde{W}_\phi^F$, then $\widetilde{P_\phi}(F') = \widetilde{P_\phi}(F)$. Let y_1, y_2 be the extrema of the edge of F_ϵ parallel to L, let z_1, z_2 be the orthogonal projections of y_1, y_2 onto the line passing through L and let δ_i, $i \in \{1, 2\}$, be equal to 0 if the point z_i belongs to L and equal to 1 otherwise (see Fig. 3 where $\delta_1 = 1$ and $\delta_2 = 0$). Let $\mathrm{O} := w + \frac{L}{l}y$, where $\widetilde{\phi} = \widetilde{\phi}_y$. Finally let X_1, X_2 be the intersection of F with the triangles with vertices O, x_1, z_1 and O, x_2, z_2 respectively, let Y_1, Y_2 be the intersection of F with the triangles with vertices O, z_1, y_1 and O, z_2, y_2 respectively, and let Z_1, Z_2 be the quadrilaterals with vertices $\mathrm{O}, x_1, z_1, y_1$ and $\mathrm{O}, x_2, z_2, y_2$.

Notice that $2(|Y_1| + |Y_2|) = L\epsilon + \mathrm{o}(\epsilon)$ since $|Y_1|, |Y_2|$ have a basis with length $|y_i - z_i| = \epsilon$ and the sum of their heights is $|y_1 - y_2| = L + \mathrm{O}(\epsilon)$. Recalling the observation that F coincides with a portion of a translated of $\frac{L}{l}\widetilde{W}_\phi^F$ locally around L, we can apply Lemma 8.5 with $\mu := \frac{L}{l}$ to the cones containing X_i and Y_i, $i = 1, 2$ and we obtain

$$\begin{aligned}\widetilde{P_\phi}(F) - \widetilde{P_\phi}(F_\epsilon) &= \frac{2l}{L}(|Z_1| + |Z_2| - \delta_1|X_1| - \delta_2|X_2|) + \mathrm{o}(\epsilon)\\ &= \frac{2l}{L}(|Y_1| + |Y_2|) + \mathrm{o}(\epsilon) = \epsilon l + \mathrm{o}(\epsilon),\end{aligned}$$

where we have used the fact that the area of the triangles $x_1y_1z_1$, $x_2y_2z_2$ is of order $\mathrm{o}(\epsilon)$. The proof of (69) is complete.

Observe now that

$$|F| - |F_\epsilon| = \epsilon L + \mathrm{o}(\epsilon). \tag{70}$$

Moreover, by (67) we have that the $\widetilde{\phi}$ curvature of L, which is $\frac{l}{L}$, is strictly larger than λ, hence $\lambda L - l < 0$. Using (69) and (70) we have

$$\begin{aligned}\mathcal{G}_\lambda(F_\epsilon) &= \widetilde{P_\phi}(F_\epsilon) - \lambda|F_\epsilon|\\ &= \widetilde{P_\phi}(F) - \epsilon l + \mathrm{o}(\epsilon) - \lambda(|F| - \epsilon L + \mathrm{o}(\epsilon))\\ &= \mathcal{G}_\lambda(F) + \epsilon(\lambda L - l) + \mathrm{o}(\epsilon) < 0\end{aligned}$$

for $\epsilon > 0$ small enough. This gives (68). From (68) we deduce that F is not a minimizer of $\mathcal{G}_\lambda$ and this fact, coupled with (66), contradicts Theorem 5.2. The proof of Case 3, and therefore the proof of the implication (63), is complete.

9. Characterization of the sets Ω_λ^F and Θ_λ^F in the convex case

Given a set $A \subseteq F$ and $r > 0$, we set

$$\begin{aligned}A_r^- &:= \{x \in F : \mathrm{dist}_{\widetilde{\phi}}(\mathbb{R}^2 \setminus A, x) > r\}, \qquad & A_r^+ &:= \{x \in F : \mathrm{dist}_{\widetilde{\phi}}(x, A) < r\},\\ A_-^r &:= \{x \in F : \mathrm{dist}_{\widetilde{\phi}}(\mathbb{R}^2 \setminus A, x) \geqslant r\}, \qquad & A_+^r &:= \{x \in F : \mathrm{dist}_{\widetilde{\phi}}(x, A) \leqslant r\},\\ A_r^\pm &:= (A_r^-)_r^+ \qquad A_\pm^r := (A_-^r)_+^r. & &\end{aligned}$$

Notice that

$$\begin{aligned} A_r^{\pm} &= \bigcup\{\mathcal{B}_r : \mathcal{B}_r \subseteq \mathrm{int}(A) \text{ is a translated of } r\widetilde{W}_\phi^F\}, \\ A_{\pm}^r &= \bigcup\{\mathcal{B}_r : \mathcal{B}_r \subseteq \overline{A} \text{ is a translated of } r\widetilde{W}_\phi^F\}. \end{aligned} \tag{71}$$

Moreover $A_r^{\pm} \subseteq \mathrm{int}(A)$, $A_{\pm}^r \subseteq \overline{A}$, and $r < \rho$ implies $A_r^{\pm} \supseteq A_\rho^{\pm}$ and $A_{\pm}^r \supseteq A_{\pm}^{\rho}$. Note also that $\partial A_r^{\pm} \cap \partial F \neq \emptyset$ and $\partial A_{\pm}^r \cap \partial F \neq \emptyset$.

The aim of this section is to prove the following result, which exactly identifies the sublevels of κ_ϕ^E on $\mathrm{int}(F)$.

THEOREM 9.1 Let ϕ be crystalline. Assume that E is convex at F and that F is convex. Then

$$\mathrm{int}(\Omega_\lambda^F) = F_{\frac{1}{\lambda}}^{\pm} \qquad \forall \lambda > \kappa_{\min}(F), \tag{72}$$

$$\overline{\Theta_\lambda^F} = F_{\pm}^{\frac{1}{\lambda}} \qquad \forall \lambda \geqslant \kappa_{\min}(F). \tag{73}$$

In general, it may happen that, for some $\lambda < \kappa_{\min}(F)$, the sets $F_{\frac{1}{\lambda}}^{\pm}$ are nonempty, whereas the sets Ω_λ^F are obviously empty: see Section 10 for a concrete example of this phenomenon.

To prove Theorem 9.1 we need some preliminary lemmas.

LEMMA 9.2 Let $P \subset H_F$ be a Lipschitz $\widetilde{\phi}$-regular closed convex set and let $\lambda > 0$. Then

$$\operatorname*{ess\,sup}_{\partial P} \widetilde{\kappa}_\phi^P \leqslant \lambda \Rightarrow P = P_{\pm}^{\frac{1}{\lambda}}.$$

Proof. We divide the proof into two steps.

Step 1. Let us prove that $P_{\pm}^{\frac{1}{\lambda}} \neq \emptyset$.

Fix $\mu > \lambda$ and let $x \in \partial P$ be a point where ∂P is differentiable and there exists $\widetilde{\kappa}_\phi^P(x) < \mu$. Since $P_{\pm}^{\rho} = \bigcap_{r<\rho} P_{\pm}^r$, it is enough to show that $\mathcal{B}_{\frac{1}{\mu}}$ is contained in P. Indeed, in this case $P_{\pm}^{\frac{1}{\mu}} \neq \emptyset$, and we conclude by compactness, letting $\mu \to \lambda$, that $P_{\pm}^{\frac{1}{\lambda}} \neq \emptyset$.

By Lemma 8.3, there exist an open neighbourhood $N(x)$ of x and a translated $\mathcal{B}_{\frac{1}{\mu}}$ of $\frac{1}{\mu}\widetilde{W}_\phi^F$ such that $x \in \partial\mathcal{B}_{\frac{1}{\mu}}$ and $N(x) \cap \mathcal{B}_{\frac{1}{\mu}} \subseteq P$.

Assume by contradiction that $\mathcal{B}_{\frac{1}{\mu}}$ is not contained in P. So $\mathcal{B}_{\frac{1}{\mu}}$ is locally (around x) but not globally contained in P. The connected component Γ of $\partial P \setminus \mathrm{int}(\mathcal{B}_{\frac{1}{\mu}})$ containing x is homeomorphic to the interval $[0, 1]$. Then $\Gamma \setminus \{x\} = \Lambda_1 \cup \Lambda_2$, where Λ_i are two arcs, whose interior parts are pairwise disjoint, having x as the common extremum. There are only two possible cases.

Case 1. One of these two arcs, say Λ_1, can be written as the union of a (possibly empty) segment and the graph of a convex function with respect to a suitable orthogonal coordinate system. Reasoning exactly as in the proof of (60) of Theorem 8.1 (with F replaced by P and Ω_λ^F replaced by $\mathcal{B}_{\frac{1}{\mu}}$) we deduce that there exists a point $y \in \Lambda_1$ such that $\widetilde{\kappa}_\phi^P(y) \geqslant \mu > \lambda$, which is a contradiction.

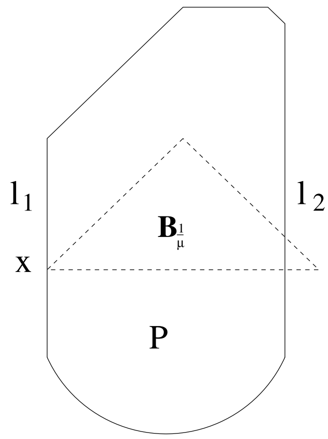

FIG. 4. The set $\mathcal{B}_{\frac{1}{\mu}}$ locally but not globally contained in P.

Case 2. Both Λ_1 and Λ_2 are union of two segments and the graph of a convex function which is not continuous at the extrema.

We are in the situation depicted in Fig. 4, where ∂P contains two parallel segments l_1, l_2, and $\mathcal{B}_{\frac{1}{\mu}}$ is 'tangent' to one of them, say l_1, from inside, and $\mathrm{int}(\mathcal{B}_{\frac{1}{\mu}})$ intersects l_2. We now slightly translate $\mathcal{B}_{\frac{1}{\mu}}$ in the direction of $\widetilde{\nu}^P(x)$ (i.e. toward the left in Fig. 4) in such a way that the interior part of the new translated set intersects both l_1 and l_2. Reasoning as in the proof of (60) of Theorem 8.1, we conclude as in Case 1. The proof of Step 1 is complete.

Step 2. Let us prove that $P = P_{\pm}^{\frac{1}{\lambda}}$.

Assume by contradiction that $P_{\pm}^{\frac{1}{\lambda}}$ is strictly contained in P. This implies that $P_{\pm}^{\frac{1}{\mu}}$ is strictly contained in P for some $\mu > \lambda$. Let A be a connected component of $\mathrm{int}(P) \setminus P_{\pm}^{\frac{1}{\mu}}$ and let $\Sigma := \partial A \cap \partial P_{\pm}^{\frac{1}{\mu}}$. Recalling (71) with $r = 1/\mu$ and using the fact that $P_{\pm}^{\frac{1}{\mu}}$ is convex, it follows that Σ is contained in a translated of $\frac{1}{\mu}\widetilde{W}_{\phi}^F$. Recalling again (71) and the fact that F is convex, with similar arguments as in Lemma 7.3, it follows that both $\overline{\partial A \setminus \Sigma}$ and Σ can be written as graphs (in the same direction) of two convex functions f, σ respectively, such that f can be discontinuous in at most one of the extrema. We can reason again as in the proof of (60) of Theorem 8.1 obtaining a contradiction as in Step 1. □

The following lemma proves that there is a point x in the boundary of a convex not Lipschitz $\widetilde{\phi}$-regular set P with the following property: P is, locally around x, contained in any (translated of the) $\widetilde{\phi}$-Wulff shape with the proper radius and having x in its boundary. Heuristically, the $\widetilde{\phi}$-curvature of ∂P at x is $+\infty$.

LEMMA 9.3 Let $\widetilde{\phi}$ be crystalline. Let $P \subset H_F$ be a compact convex set which is *not* Lipschitz $\widetilde{\phi}$-regular. Then we can find a point $x \in \partial P$ having the following property: for any $\lambda > 0$ there exist $\rho > 0$ and a translated $\mathcal{B}_{\frac{1}{\lambda}}$ of $\frac{1}{\lambda}\widetilde{W}_{\phi}^F$ such that $x \in \partial\mathcal{B}_{\frac{1}{\lambda}}$ and $P \cap B_\rho(x) \subseteq \mathcal{B}_{\frac{1}{\lambda}}$.

Proof. Since P is convex and $\widetilde{\phi}$ is crystalline, P is Lipschitz $\widetilde{\phi}$-regular if and only if any edge of $\partial \widetilde{W}_\phi^F$ has a corresponding parallel edge of ∂P. Therefore, if P is not Lipschitz $\widetilde{\phi}$-regular there exist a point $x \in \partial P$ and a straight line s parallel to some edge of $\partial \widetilde{W}_\phi^F$ such that $s \cap \partial P = \{x\}$. One can verify that x satisfies the thesis. □

LEMMA 9.4 Let $\widetilde{\phi}$ be crystalline. Let $\lambda > \kappa_{\min}(F)$. Then Ω_λ^F is Lipschitz $\widetilde{\phi}$-regular and

$$\operatorname*{ess\,sup}_{\partial \Omega_\lambda^F} \widetilde{\kappa}_\phi^{\Omega_\lambda^F} \leqslant \lambda. \tag{74}$$

Similarly, if $\lambda \geqslant \kappa_{\min}(F)$, then Θ_λ^F is Lipschitz $\widetilde{\phi}$-regular and

$$\operatorname*{ess\,sup}_{\partial \Theta_\lambda^F} \widetilde{\kappa}_\phi^{\Theta_\lambda^F} \leqslant \lambda. \tag{75}$$

Proof. Let us prove that Ω_λ^F verifies the assertions. Let $\lambda > \kappa_{\min}(F)$. By Theorem 7.1 we know that Ω_λ^F is a convex subset of F. We argue by contradiction. If Ω_λ^F is Lipschitz $\widetilde{\phi}$-regular and $\operatorname{ess\,sup}_{\partial \Omega_\lambda^F} \widetilde{\kappa}_\phi^{\Omega_\lambda^F} > \lambda$, then by Lemma 8.3 there exist $x \in \partial \Omega_\lambda^F$, a neighbourhood $N(x)$ of x and a translated $\mathcal{B}_{\frac{1}{\lambda}}$ of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ such that $x \in \partial \mathcal{B}_{\frac{1}{\lambda}}$ and $\mathcal{B}_{\frac{1}{\lambda}} \supseteq N(x) \cap \Omega_\lambda^F$. We then reach a contradiction reasoning as in the proof of (63) of Theorem 8.1.

Assume now that Ω_λ^F is not Lipschitz $\widetilde{\phi}$-regular. We apply Lemma 9.3 and we reach a contradiction as in the previous case.

Finally, the assertions on Θ_λ^F follow from the assertions on Ω_λ^F and (55). □

We are now in the position to prove Theorem 9.1.

We will prove Theorem 9.1 only for the sets Θ_λ^F, since the assertion on Ω_λ^F follows then from the equality $\Omega_\lambda^F = \bigcup_{\mu<\lambda} \Theta_\mu^F$.

Fix $\lambda \geqslant \kappa_{\min}(F)$. From Lemma 9.4 we have that Θ_λ^F is Lipschitz $\widetilde{\phi}$-regular and (75) holds. Therefore, from Lemma 9.2 we have $\overline{\Theta_\lambda^F} = \left(\Theta_\lambda^F\right)_\pm^{\frac{1}{\lambda}}$. Since $\Theta_\lambda^F \subseteq F$ we have $\overline{\Theta}_\lambda^F \subseteq F_\pm^{\frac{1}{\lambda}}$, which proves that $F_\pm^{\frac{1}{\lambda}}$ is not empty.

Assume by contradiction that $\overline{\Theta_\lambda^F}$ is strictly contained in $F_\pm^{\frac{1}{\lambda}}$. Let $\Sigma \subseteq \partial \overline{\Theta_\lambda^F}$, $\{x, y\} := \Sigma \cap \partial F$, Π be as in Lemma 7.3 such that $\Sigma \cap \operatorname{int}(F_\pm^{\frac{1}{\lambda}}) \neq \emptyset$. By Lemma 9.2 and Lemma 9.4, there exists a translated $\mathcal{B}^1_{\frac{1}{\lambda}}$ of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ such that $\mathcal{B}^1_{\frac{1}{\lambda}} \subseteq \overline{\Theta_\lambda^F}$ and $\Sigma \subset \partial \mathcal{B}^1_{\frac{1}{\lambda}}$. Moreover, by definition of $F_\pm^{\frac{1}{\lambda}}$, there exists a translated $\mathcal{B}^2_{\frac{1}{\lambda}} \subseteq F$ of $\frac{1}{\lambda}\widetilde{W}_\phi^F$ such that $\mathcal{B}^2_{\frac{1}{\lambda}} \cap (F \setminus \overline{\Theta_\lambda^F}) \cap \Pi \neq \emptyset$. Since F is convex it must contain the convex combination of $\mathcal{B}^1_{\frac{1}{\lambda}}$ and $\mathcal{B}^2_{\frac{1}{\lambda}}$, which implies that $\partial F \cap \Pi$ cannot be written as the graph of a (convex) function over $[x, y]$, which is continuous at one extreme, and this contradicts Lemma 7.3. The proof of Theorem 9.1 is concluded.

The following result suggests that, at least initially, convex sets remain convex during the evolution by crystalline mean curvature.

COROLLARY 9.5 The function κ_ϕ^E is continuous and convex on F.

Proof. Thanks to Theorem 9.1, we have $\overline{(\mathrm{int}(F) \cap \partial \Omega_\lambda^F)} \bigcap \overline{\mathrm{int}(F) \cap \partial \Omega_\mu^F)} = \emptyset$ for $\lambda \neq \mu$, which implies that κ_ϕ^E is continuous on F.

Let us prove that κ_ϕ^E is convex on F. Let $x, y \in F$, and let $\lambda := \kappa_\phi^E(x)$, $\mu := \kappa_\phi^E(y)$. We have to prove that $\frac{x+y}{2} \in \Theta^F_{\frac{\lambda+\mu}{2}}$. If $\lambda = \mu$ the assertion follows from the convexity of Θ_λ^F (Theorem 7.1), so we can assume $\lambda > \mu$. Since $x \in \Theta_\lambda^F$ and $y \in \Theta_\mu^F$, by Theorem 9.1 there exist $z_x, z_y \in F$ such that

$$x \in z_x + \frac{1}{\lambda}\widetilde{W}_\phi^F \subseteq F, \qquad y \in z_y + \frac{1}{\mu}\widetilde{W}_\phi^F \subseteq F.$$

Using the convexity of F we observe that

$$\frac{x+y}{2} \in \frac{z_x + z_y}{2} + \frac{\lambda+\mu}{2\lambda\mu}\,\widetilde{W}_\phi^F \subseteq F.$$

Therefore $\frac{x+y}{2} \in F_\pm^{\frac{\lambda+\mu}{2\lambda\mu}}$. Since $\frac{2}{\lambda+\mu} \leqslant \frac{\lambda+\mu}{2\lambda\mu}$, we have $\frac{x+y}{2} \in F_\pm^{\frac{2}{\lambda+\mu}} = \Theta^F_{\frac{\lambda+\mu}{2}}$, where the last equality follows again by Theorem 9.1. □

The assumption that ϕ is crystalline in Theorem 9.1 is necessary because we apply Lemma 9.3, where it is required that $\widetilde{\phi}$ is crystalline. We expect that Lemma 9.3 is still valid for a generic $\widetilde{\phi}$, and therefore that Theorem 9.1 is still valid for a generic anisotropy ϕ.

10. An example of a convex set with non ϕ-calibrable facets

We show an example of Lipschitz ϕ-regular set, partially discussed in [3]. We justify the computation of the 'velocity' κ_ϕ^E given in [3] and the subsequent crystalline mean curvature evolution. This flow shows that the frontal facet F_ϵ of E, for ϵ in a suitable range, bends inside E at the initial time [22]. In this example we make use of both Theorems 6.1 and 8.1: we could avoid the use of these two results together, but we find it interesting to apply both of them.

Let $\mathcal{W}_\phi \subset \mathbb{R}^3$ be the prism with hexagonal basis in Fig. 5; the apothem of the hexagon has unit length. Let also E be the convex Lipschitz ϕ-regular set as depicted in Fig. 5. The apothem of the frontal hexagonal facet F_ϵ of E has unit length. Notice that E satisfies the assumptions of Proposition 4.1.

PROPOSITION 10.1 Let $\overline{\epsilon} := 7 - \sqrt{42} \in\ ]0, 1[$. Then F_ϵ is ϕ-calibrable if and only if $\epsilon \in [\overline{\epsilon}, 1]$.

Proof. Let us prove that if F_ϵ is ϕ-calibrable, then $\epsilon \in [\overline{\epsilon}, 1]$. Given $\epsilon \in [0, 1]$ we have $|F_\epsilon| = \frac{1}{\sqrt{3}}(7 - \epsilon^2)$, $\widetilde{P_\phi}(F_\epsilon) = \int_{\partial F_\epsilon} c_{F_\epsilon}\, d\mathcal{H}^1 = \mathcal{H}^1(\partial F_\epsilon) = \frac{2}{\sqrt{3}}(7 - \epsilon)$. Hence

$$V_{F_\epsilon} := \frac{\widetilde{P_\phi}(F_\epsilon)}{|F_\epsilon|} = \frac{2(7-\epsilon)}{7-\epsilon^2} \leqslant 2, \qquad \forall \epsilon \in [0, 1]. \tag{76}$$

The function $\epsilon \to V_{F_\epsilon}$ is strictly convex on $[0, 1]$, with $V_{F_0} = V_{F_1} = 2$, and attains its minimum for $\epsilon = \overline{\epsilon}$, with value $V_{F_{\overline{\epsilon}}} = (7 + \sqrt{42})/7 < 2$. In particular

$$V_{F_{\overline{\epsilon}}} < V_{F_\epsilon} \qquad \text{and} \qquad F_{\overline{\epsilon}} \subset F_\epsilon \qquad \forall \epsilon \in\ ]0, \overline{\epsilon}[.$$

Hence, by Theorem 6.1 (here $g = 0$), the facet F_ϵ is not ϕ-calibrable for any $\epsilon \in\ ]0, \overline{\epsilon}[$.

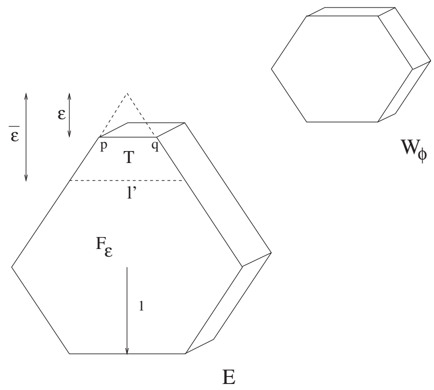

FIG. 5. For $\epsilon \in\ ]0, \overline{\epsilon}[$ the frontal facet $F_\epsilon \subset \partial E$ is not ϕ-calibrable. The dotted line l' separates the region where κ_ϕ^E is constant from the region T where κ_ϕ^E is continuous but not constant.

Let us now prove that if $\epsilon \in [\overline{\epsilon}, 1]$ then F_ϵ is ϕ-calibrable. Thanks to Theorem 8.1 and (76), it is enough to prove that

$$\operatorname*{ess\,sup}_{\partial F_\epsilon} \widetilde{\kappa}_\phi^{F_\epsilon} \leqslant \frac{2(7-\epsilon)}{7-\epsilon^2} \qquad \forall \epsilon \in [\overline{\epsilon}, 1]. \tag{77}$$

Denote by $[p, q]$ the shortest edge of ∂F_ϵ, see Fig. 5. Observe that the supremum of $\widetilde{\kappa}_\phi^{F_\epsilon}$ is attained on l and is equal to $\frac{2}{\sqrt{3}|p-q|}$ (recall that the length of the edges of $\widetilde{W}_\phi^{F_\epsilon}$ is $\frac{2}{\sqrt{3}}$). In addition $\frac{2}{\sqrt{3}|p-q|} = \frac{1}{\epsilon}$. Since $\frac{1}{\epsilon} \leqslant \frac{2(7-\epsilon)}{7-\epsilon^2}$ for any $\epsilon \in [\overline{\epsilon}, 1]$, (77) follows. □

Proposition 10.1 identifies κ_ϕ^E on the frontal facet F_ϵ and on its opposite one. Since, by [3: Lemma 5.1] all remaining facets of E are ϕ-calibrable, we can compute explicitly κ_ϕ^E on the whole of ∂E.

We finally observe that, given $\epsilon \in\ ]0, \overline{\epsilon}[$, we have $\kappa_{\min}(F_\epsilon) = \frac{7+\sqrt{42}}{7}$, hence $\Omega_\lambda^{F_\epsilon} = \emptyset$ for any $\lambda \leqslant \frac{7+\sqrt{42}}{7}$, whereas $F_{\frac{1}{\lambda}}^{\pm} \neq \emptyset$ for any $\lambda \in\ ]1, \frac{7+\sqrt{42}}{7}]$.

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