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Hypoellipticity of Hankel Convolution Equations in $\mathcal{D}_{L^1}$-Type Spaces

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Abstract. In this paper we analyze the hypoellipticity of Hankel convolution equations in distribution spaces of L^p-growth. The spaces that we consider are $\mathcal{D}_{L^p}$-type spaces in the Hankel setting.

Keywords: *Hypoelliptic, Hankel transform,convolution equations*

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1. Introduction

The space $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, were studied by L. Schwartz ([14]). Assume that $n \in \mathbb{N} \setminus \{0\}$. If $1 \leq p < \infty$, a smooth function ϕ on $\mathbb{R}^n$ is in $\mathcal{D}_{L^p}$ provided that

$$\|\phi\|_{p,k} = \left(\int_{\mathbb{R}^n} |D^k \phi(x)|^p \, dx \right)^{\frac{1}{p}} < \infty,$$

for every $k \in \mathbb{N}^n$. A smooth function ϕ on $\mathbb{R}^n$ is in $\mathcal{D}_{L^\infty}$ when

$$\|\phi\|_{\infty,k} = \sup_{x \in \mathbb{R}^n} |D^k \phi(x)| < \infty,$$

and $\lim_{|x| \to \infty} D^k \phi(x) = 0$, for every $k \in \mathbb{N}^n$. Here, for each $k = (k_1, k_2, ..., k_n) \in \mathbb{N}^n$ we understand as usual

$$D^k \phi = \frac{\partial^{k_1 + k_2 + ... + k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_n^{k_n}} \, \phi.$$

The space $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, is endowed with the topology associated with the family $\{\| \cdot \|_{p,k}\}_{k \in \mathbb{N}^n}$ of seminorms. Thus, $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$ is a Fréchet space.

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Moreover the space $\mathcal{D}$ of the smooth and compact support functions in $\mathbb{R}^n$ is dense in $\mathcal{D}_{L^p}$, $1 \le p \le \infty$. The dual space $\mathcal{D}'_{L^p}$ of $\mathcal{D}_{L^p}$ is hence a normal space of distributions for each $1 \le p \le \infty$.

In [3] J. J. Betancor and B. González introduced the space $\mathcal{D}_{L^p}$-type in the setting of Hankel transforms. There, the rule of the derivatives was played by the Bessel operator $\Delta_\mu = x^{-2\mu-1}Dx^{2\mu+1}D$. The spaces $H_{\mu,p}$ and $\mathcal{H}_{\mu,p}$, $\mu > -\frac{1}{2}$ and $1 \le p \le \infty$, were defined in [3] as follows. Let $1 \le p \le \infty$ and $\mu > -\frac{1}{2}$. A Lebesgue measurable function f defined on $(0,\infty)$ is in $H_{\mu,p}$ if, for every $k \in \mathbb{N}$, $\Delta_\mu^k f \in L^p(x^{2\mu+1}\,dx)$, that is, there exists $h_k \in L^p(x^{2\mu+1}\,dx)$ such that

$$\int_0^\infty f(x)\Delta_\mu^k(\phi)(x)x^{2\mu+1}\,dx = \int_0^\infty \phi(x)h_k(x)x^{2\mu+1}\,dx, \quad \phi \in S_e.$$

Here by S_e we understand the space that consists of all the even functions in the Schwartz space $S(\mathbb{R})$. The space $H_{\mu,p}$ is equipped with the topology generated by the family $\{\gamma_k^{p,\mu}\}_{k\in\mathbb{N}}$ of seminorms, where

$$\gamma_k^{p,\mu}(\phi) = \|\Delta_\mu^k \phi\|_{\mu,p}, \quad \phi \in H_{\mu,p},\ k \in \mathbb{N},$$

and $\|\cdot\|_{\mu,p}$ being the usual norm in the Lebesgue space $L^p(x^{2\mu+1}\,dx)$. Note that $L^\infty(x^{2\mu+1}\,dx) = L^\infty(\,dx)$. Thus $H_{\mu,p}$ is a Fréchet space ([3, Proposition 2.1]). It is not hard to see that S_e is properly contained in $H_{\mu,p}$. The space $\mathcal{H}_{\mu,p}$ is defined as the closure of S_e into $H_{\mu,p}$.

In [3] the author and B. González investigated the Hankel convolution on the spaces $H_{\mu,p}$, $\mathcal{H}_{\mu,p}$ and their duals. Motivated by the paper of D. H. Pahk [13] in [3, open question 3.3] J. J. Betancor and B. González propose the study of the hypoellipticity of Hankel convolution equations on the spaces $H'_{\mu,1}$, the dual space of $H_{\mu,1}$. This is our objective in this paper.

We now recall some definitions and properties concerning to Hankel transforms and Hankel convolution that will be useful in the sequel.

The *Hankel transform* $h_\mu(f)$ of $f \in L^1(x^{2\mu+1}\,dx)$ is defined by (see [10] and [11], for instance)

$$h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu}J_\mu(xy)f(x)x^{2\mu+1}\,dx, \quad y \in (0,\infty),$$

where J_μ denotes the Bessel function of the first kind and order μ ([16]). Here we assume that $\mu > -\frac{1}{2}$. The Hankel transform, that is also called *Hankel-Schwartz transform* (see [7]), is an automorphism in S_e ([1, Satz 5]). h_μ is defined on S'_e, the dual space of S_e, by transposition. Each function $f \in L^p(x^{2\mu+1}\,dx)$, $1 \le p \le \infty$, defines an element of S'_e, that will be continue denoting by f, as follows:

$$\langle f,\phi\rangle = \int_0^\infty f(x)\phi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dx, \quad \phi \in S_e.$$

Throughout this paper, when we say that a function defines a functional on a suitable function space it must understood as above.

The convolution equation associated with the Hankel transform h_μ was investigated by D. T. Haimo [9] and I. I. Hirschman [11] on the Lebesgue spaces $L^p(x^{2\mu+1}\,dx)$. If $f,g \in L^1(x^{2\mu+1}\,dx)$, the *Hankel convolution* $f\#_\mu g$ of f and g is given through

$$(f\#_\mu g)(x) = \int_0^\infty ({}_\mu\tau_x g)(y)f(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy, \quad x \in (0,\infty),$$

where the *Hankel translation operator* ${}_\mu\tau_x$, $x \in [0,\infty)$, is defined by

$$({}_\mu\tau_x g)(y) = \int_0^\infty D_\mu(x,y,z)g(z)\frac{z^{2\mu+1}}{2^\mu\Gamma(\mu+1)}dz, \quad x,y \in (0,\infty),$$

and ${}_\mu\tau_0 g = g$, and being, for each $x,y,z \in (0,\infty)$,

$$D_\mu(x,y,z) = (2^\mu\Gamma(\mu+1))^2\int_0^\infty (xt)^{-\mu}J_\mu(xt)(yt)^{-\mu}J_\mu(yt)(zt)^{-\mu}J_\mu(zt)t^{2\mu+1}\,dt.$$

The Hankel convolution and the Hankel translations are related to the *Hankel transformation* h_μ by the following formulas (see [11]):

$$h_\mu(f\#_\mu g) = h_\mu(f)h_\mu(g)$$
$$h_\mu({}_\mu\tau_x g)(y) = 2^\mu\Gamma(\mu+1)(xy)^{-\mu}J_\mu(xy)h_\mu(g)(y),$$

where $x,y \in [0,\infty)$ and $f,g \in L^1(x^{2\mu+1}\,dx)$. The Hankel convolution was studied in distribution spaces in [5], [6] and [12].

In the sequel, to simplify we will write $\#$, τ_x, $x \in [0,\infty)$, and D, instead $\#_\mu$, ${}_\mu\tau_x$, $x \in [0,\infty)$, and D_μ, respectively. Throughout this paper by C we always denote a suitable positive constant that can be changed from the one to the other line.

2. Hankel convolution operators in the space $H'_{\mu,1}$

We introduce the space $\mathbf{H}_{\mu,\infty}$ as the closure of $H_{\mu,1}$ in $H_{\mu,\infty}$. In this Section, we characterize $\mathbf{H}'_{\mu,\infty}$ as the space of Hankel convolution operators on $H'_{\mu,1}$.

Let $f \in H_{\mu,1}$. As it was proved in [3, Remark 1], $y^k h_\mu(f) \in L^1(x^{2\mu+1}\,dx)$ for every $k \in \mathbb{N}$. Hence, according to [11, Corollary 2.e],

$$f(x) = \int_0^\infty (xy)^{-\mu}J_\mu(xy)h_\mu(f)(y)y^{2\mu+1}\,dy \quad \text{for a.e. } x \in (0,\infty). \tag{2.1}$$

Here, a.e. refers to the Lebesgue measure on $(0,\infty)$. Moreover, the right hand side of (2.1) defines a smooth function on $(0,\infty)$ and by [17, (7), Chapter 5], for every $k \in \mathbb{N}$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^k \int_0^\infty (xy)^{-\mu} J_\mu(xy) h_\mu(f)(y) y^{2\mu+1}\, dy = (-1)^k \int_0^\infty (xy)^{-\mu-k} J_{\mu+k}(xy) h_\mu(f)(y) y^{2\mu+1+2k}\, dy,$$

for each $x \in (0,\infty)$. Thus we can consider $H_{\mu,1}$ as a subspace of $C^\infty(0,\infty)$. Moreover the distributional and classical operator Δ_μ coincide on $H_{\mu,1}$.

Moreover, from (2.1) and [17, (6) and (7), Chapter 5] we can deduce that, for every $k \in \mathbb{N}$,

$$\Delta_\mu^k f(x) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)(-y^2)^k h_\mu(f)(y) y^{2\mu+1}\, dy, \quad f \in H_{\mu,1}. \tag{2.2}$$

Hence, $H_{\mu,1}$ is continuously contained in $H_{\mu,\infty}$. From (2.2), according to the Riemann-Lebesgue Lemma for the Hankel transform, we infer that, for every $f \in H_{\mu,1}$ and $k \in \mathbb{N}$,

$$\lim_{x\to\infty} \Delta_\mu^k f(x) = 0.$$

Also, (2.2) implies that, for every $f \in H_{\mu,1}$ and $k \in \mathbb{N}$,

$$\lim_{x\to 0} \Delta_\mu^k f(x) = \frac{(-1)^k}{2^\mu \Gamma(\mu+1)} \int_0^\infty f(y) y^{2k+2\mu+1}\, dy.$$

Then, it is not hard to see that, for every $f \in \mathbf{H}_{\mu,\infty}$ and $k \in \mathbb{N}$,

$$\lim_{x\to\infty} \Delta_\mu^k f(x) = 0,$$

and, there exists $\lim_{x\to 0} \Delta_\mu^k f(x)$.

By $H'_{\mu,1}$ and $\mathbf{H}'_{\mu,\infty}$ we denote, as usual, the dual spaces of $H_{\mu,1}$ and $\mathbf{H}_{\mu,\infty}$, respectively. Since $H_{\mu,1}$ is a dense subspace of $\mathbf{H}_{\mu,\infty}$, $\mathbf{H}'_{\mu,\infty}$ is a subspace of $H'_{\mu,1}$. We now present characterizations of the elements of $H'_{\mu,1}$ and $\mathbf{H}'_{\mu,\infty}$. They can be proved by employing standard procedures (see [3, Proposition 2.2]).

Proposition 1.

(i) *Let T be a functional on $H_{\mu,1}$. Then $T \in H'_{\mu,1}$ if, and only if, there exist $n \in \mathbb{N}$ and $f_k \in L^\infty(\,dx)$, $k = 0, 1, ..., n$, such that*

$$\langle T, \phi\rangle = \sum_{k=0}^n \int_0^\infty f_k(x) \Delta_\mu^k \phi(x) x^{2\mu+1}\, dx, \quad \phi \in H_{\mu,1}.$$

(ii) *Let T be a functional on $\mathbf{H}_{\mu,\infty}$. Then $T \in \mathbf{H}'_{\mu,\infty}$ if, and only if, there exist $n \in \mathbb{N}$ and regular complex Borel measures γ_k on $[0,\infty)$, $k = 0,1,...,n$, such that*

$$\langle T, \phi \rangle = \sum_{k=0}^{n} \int_0^\infty \Delta_\mu^k \phi(x) d\gamma_k(x), \quad \phi \in \mathbf{H}_{\mu,\infty}.$$

To study the Hankel convolution on the space $H'_{\mu,1}$ we need to analyze the behaviour of the Hankel translation τ_x, $x \in [0,\infty)$, on the space $H_{\mu,1}$.

Proposition 2. *Let $x \in (0,\infty)$. The Hankel translation τ_x defines a continuous linear mapping from $H_{\mu,1}$ into itself.*

Proof. Let $f \in H_{\mu,1}$. Since $f \in L^1(x^{2\mu+1}\,dx)$, by [4, (3.1)], we can write

$$h_\mu(\tau_x f)(y) = 2^\mu \Gamma(\mu+1)(xy)^{-\mu} J_\mu(xy) h_\mu(f)(y), \quad y \in (0,\infty).$$

Moreover, since $h_\mu(f) \in L^1(x^{2\mu+1}\,dx)$ and $z^{-\mu}J_\mu(z)$ is a bounded function on $(0,\infty)$, we get

$$(\tau_x f)(y) = 2^\mu \Gamma(\mu+1) h_\mu((xt)^{-\mu} J_\mu(xt) h_\mu(f)(t))(y), \quad y \in (0,\infty).$$

Then, for every $k \in \mathbb{N}$,

$$\begin{aligned}\Delta_\mu^k(\tau_x f)(y) &= (-1)^k 2^\mu \Gamma(\mu+1) h_\mu((xt)^{-\mu} J_\mu(xt) t^{2k} h_\mu(f)(t))(y) \\ &= \tau_x(\Delta_\mu^k f)(y), \qquad y \in (0,\infty).\end{aligned}$$

Note that the differentiation under the integral sign is justified because, for every $k \in \mathbb{N}$, $y^{2k} h_\mu(f) \in L^1(x^{2\mu+1}\,dx)$.

By invoking now [15, p. 17], for every $k \in \mathbb{N}$, we get

$$\|\Delta_\mu^k(\tau_x f)\|_{\mu,1} \leq C \|\Delta_\mu^k f\|_{\mu,1}. \tag{2.3}$$

Hence $\tau_x f \in H_{\mu,1}$. That the mapping $f \to \tau_x f$ is continuous from $H_{\mu,1}$ into itself follows also from the inequality (2.3). ∎

We now study the behaviour of the Hankel convolution on the space $H_{\mu,1}$.

Proposition 3. *The Hankel convolution defines a continuous bilinear mapping from $H_{\mu,1} \times H_{\mu,1}$ into $H_{\mu,1}$.*

Proof. Let $f, g \in H_{\mu,1}$, $\phi \in S_e$ and $k \in \mathbb{N}$. Then, $f\#\phi \in C^\infty(0,\infty)$ and we can write

$$\Delta_\mu^k(f\#\phi) = (\Delta_\mu^k f)\#\phi = f\#(\Delta_\mu^k \phi).$$

Moreover,

$$\begin{aligned}\langle \Delta_\mu^k (f\#g)(x), \phi\rangle &= \langle f\#g, \Delta_\mu^k \phi\rangle \\ &= \int_0^\infty (f\#g)(y)\Delta_\mu^k\phi(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy \\ &= \int_0^\infty f(y)(g\#\Delta_\mu^k\phi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy \\ &= \int_0^\infty f(y)((\Delta_\mu^k g)\#\phi)(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy \\ &= \int_0^\infty (f\#\Delta_\mu^k g)(y)\phi(y)\frac{y^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dy.\end{aligned}$$

Hence $\Delta_\mu^k(f\#g) = f\#\Delta_\mu^k g$. Then, according to [11, Theorem 2.b], we get

$$\|\Delta_\mu^k(f\#g)\|_{\mu,1} \le \|f\|_{\mu,1}\|\Delta_\mu^k g\|_{\mu,1}.$$

Thus we conclude that the #-convolution defines a bilinear and continuous mapping from $H_{\mu,1}\times H_{\mu,1}$ into $H_{\mu,1}$. ∎

By virtue of Proposition 2 we can define the Hankel convolution $T\#f$ of $T\in H'_{\mu,1}$ and $f\in H_{\mu,1}$ as the function

$$(T\#f)(x) = \langle T, \tau_x f\rangle, \quad x\in[0,\infty).$$

Proposition 4. *Let $T\in H'_{\mu,1}$ and $f\in H_{\mu,1}$. Then $T\#f$ is a continuous and bounded function on $(0,\infty)$. Hence $T\#f\in H'_{\mu,1}$ and, for every $k\in\mathbb{N}$, we have that*

$$\Delta_\mu^k(T\#f) = (\Delta_\mu^k T)\#f = T\#(\Delta_\mu^k f).$$

Proof. According to Proposition 1, (i) we can find $n\in\mathbb{N}$ and $g_k\in L^\infty(\,dx)$, $k=0,1,...,n$, such that

$$\langle T,\phi\rangle = \sum_{k=0}^n \int_0^\infty g_k(x)\Delta_\mu^k\phi(x)x^{2\mu+1}\,dx, \quad \phi\in H_{\mu,1}.$$

Then,

$$\begin{aligned}(T\#f)(x) &= \sum_{k=0}^n \int_0^\infty g_k(y)\Delta_{\mu,y}^k\tau_x(f)(y)y^{2\mu+1}\,dy \\ &= \sum_{k=0}^n \int_0^\infty g_k(y)\tau_x(\Delta_\mu^k f)(y)y^{2\mu+1}\,dy, \quad x\in(0,\infty).\end{aligned}$$

Hence, by taking into account [15, p. 17], we can conclude that $T\#f$ is a continuous and bounded function on $(0,\infty)$. Then $T\#f$ defines an element of $H'_{\mu,1}$ by

$$\langle T\#f,\phi\rangle = \int_0^\infty (T\#f)(x)\phi(x)\frac{x^{2\mu+1}}{2^\mu\Gamma(\mu+1)}\,dx, \quad \phi\in H_{\mu,1}.$$

Moreover, we can write

$$\begin{aligned}\langle T\#f,\phi\rangle &= \sum_{k=0}^n\int_0^\infty \phi(x)\int_0^\infty g_k(y)\tau_x(\Delta_\mu^k f)(y)y^{2\mu+1}\,dy\,2^M\Gamma(\mu+1)\\ &= \sum_{k=0}^n\int_0^\infty g_k(y)\int_0^\infty \phi(x)\tau_y(\Delta_\mu^k f)(x)\,2^M\Gamma(\mu+1)\,y^{2\mu+1}\,dy\\ &= \sum_{k=0}^n\int_0^\infty g_k(y)(\phi\#\Delta_\mu^k f)(y)y^{2\mu+1}\,dy\\ &= \sum_{k=0}^n\int_0^\infty g_k(y)\Delta_\mu^k(\phi\#f)(y)y^{2\mu+1}\,dy\\ &= \langle T,f\#\phi\rangle, \quad \phi\in H_{\mu,1}.\end{aligned}$$

Then, a straightforward manipulation leads, for every $k\in\mathbb{N}$ and $\phi\in H_{\mu,1}$, to

$$\begin{aligned}\langle\Delta_\mu^k(T\#f),\phi\rangle = \langle T\#f,\Delta_\mu^k\phi\rangle \quad &= \langle T,f\#\Delta_\mu^k\phi\rangle \quad = \langle T,\Delta_\mu^k(f\#\phi)\rangle\\ = \langle(\Delta_\mu^k T)\#f,\phi\rangle &= \langle T,(\Delta_\mu^k f)\#\phi\rangle = \langle T\#\Delta_\mu^k f,\phi\rangle\,. \qquad \blacksquare\end{aligned}$$

Our next objective is to characterize to $\mathbf{H}'_{\mu,\infty}$ as the subspace of $H'_{\mu,1}$ that defines convolution operators on $H_{\mu,1}$.

Proposition 5. *Let $T\in H'_{\mu,1}$. If $T\in\mathbf{H}'_{\mu,\infty}$, then $T\#\phi\in H_{\mu,1}$ for every $\phi\in H_{\mu,1}$.*

Proof. Assume that $T\in\mathbf{H}'_{\mu,\infty}$. Then there exist $n\in\mathbb{N}$ and regular complex Borel measures γ_k, $k=0,1,...,n$, on $[0,\infty)$ such that

$$\langle T,f\rangle = \sum_{k=0}^n\int_0^\infty \Delta_\mu^k f(x)d\gamma_k(x), \quad f\in\mathbf{H}_{\mu,\infty}.$$

Let $\phi\in H_{\mu,1}$. We have that

$$\begin{aligned}(T\#\phi)(x) &= \sum_{k=0}^n\int_0^\infty \tau_x(\Delta_\mu^k\phi)(y)d\gamma_k(y)\\ &= \sum_{k=0}^n 2^\mu\Gamma(\mu+1)\int_0^\infty h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\Delta_\mu^k\phi)(t))(y)d\gamma_k(y).\end{aligned}$$

Hence, since $y^{2k}h_\mu(\phi) \in L_1(x^{2\mu+1}\,dx)$, for every $k \in \mathbb{N}$, [17, (7), Chapter 5] allows us to prove that $T\#\phi$ is an smooth function on $(0,\infty)$. Moreover, for every $l \in \mathbb{N}$, we get

$$\Delta_\mu^l(T\#\phi)(x) = \sum_{k=0}^{n} \int_0^\infty \tau_x(\Delta_\mu^{k+l}\phi)(y)d\gamma_k(y), \quad x \in (0,\infty).$$

By interchanging the order of integration and according to [15, p. 17] we can obtain

$$\begin{aligned}\|\Delta_\mu^l(T\#\phi)\|_{\mu,1} &\leq \sum_{k=0}^{n} \int_0^\infty \int_0^\infty |\tau_x(\Delta_\mu^{k+l}\phi)(y)|d|\mu_k|(y)x^{2\mu+1}\,dx \\ &\leq C\sum_{k=0}^{n} \|\Delta_\mu^{k+l}\phi\|_{\mu,1}, \quad l \in \mathbb{N}.\end{aligned}$$

Thus we have proved that $T\#\phi \in H_{\mu,1}$. ∎

We now define the Hankel convolution $T\#S$ of $T \in H'_{\mu,1}$ and $S \in \mathbf{H}'_{\mu,\infty}$ as follows:

$$\langle T\#S,\phi\rangle = \langle T, S\#\phi\rangle, \quad \phi \in H_{\mu,1}.$$

Thus $T\#S \in H'_{\mu,1}$. Next we prove some properties of the #-convolution on the spaces under consideration that will be useful in the sequel.

Proposition 6. *Let $T \in H'_{\mu,1}$ and $R,S \in \mathbf{H}'_{\mu,\infty}$. Then:*

(i) *$R\#S \in \mathbf{H}'_{\mu,\infty}$ and $R\#S = S\#R$.*

(ii) *$\Delta_\mu(T\#R) = (\Delta_\mu T)\#R = T\#(\Delta_\mu R)$.*

(iii) *$T\#(R\#S) = (T\#R)\#S$.*

Proof. (i): According to Proposition 1, (ii), we can write

$$\begin{aligned}\langle R,\phi\rangle &= \sum_{k=0}^{r} \int_0^\infty \Delta_\mu^k\phi(x)d\gamma_k(x), \quad \phi \in \mathbf{H}_{\mu,\infty} \\ \langle S,\phi\rangle &= \sum_{k=0}^{\alpha} \int_0^\infty \Delta_\mu^k\phi(x)d\nu_k(x), \quad \phi \in \mathbf{H}_{\mu,\infty},\end{aligned} \tag{2.4}$$

where $r,\alpha \in \mathbb{N}$ and $\gamma_0,\ldots,\gamma_k$ and $\nu_0,\ldots,\nu_\alpha$ are complex regular Borel measures on $[0,\infty)$. Let $\phi \in H_{\mu,1}$. We can write

$$\begin{aligned}\langle R\#S,\phi\rangle &= \langle R, S\#\phi\rangle \\ &= \sum_{k=0}^{r} \int_0^\infty \Delta_\mu^k(S\#\phi)(x)d\gamma_k(x) \\ &= \sum_{k=0}^{r} \int_0^\infty \Delta_{\mu,x}^k \sum_{l=0}^{\alpha} \int_0^\infty \tau_x(\Delta_\mu^l\phi)(y)d\nu_l(y)d\gamma_k(x)\end{aligned}$$

and hence, by using [4, (3.1)] and Fubini's theorem,

$$\langle R\#S,\phi\rangle = \sum_{k=0}^{r}\sum_{l=0}^{\alpha}\int_0^\infty\int_0^\infty \tau_x(\Delta_\mu^{k+l}\phi)(y)d\nu_l(y)d\gamma_k(x) \tag{2.5}$$

$$= 2^\mu\Gamma(\mu+1)\sum_{k=0}^{r}\sum_{l=0}^{\alpha}\int_0^\infty\int_0^\infty h_\mu((xt)^{-\mu}J_\mu(xt) \times h_\mu(\Delta_\mu^{k+l}\phi)(t))(y)d\nu_l(y)d\gamma_k(x)$$

$$= 2^\mu\Gamma(\mu+1)\sum_{k=0}^{r}\sum_{l=0}^{\alpha}\int_0^\infty\int_0^\infty (xt)^{-\mu}J_\mu(xt)h_\mu(\Delta_\mu^{k+l}\phi)(t) \times t^{2\mu+1}h_\mu(\nu_l)(t)\,dtd\gamma_k(x)$$

$$= 2^\mu\Gamma(\mu+1)\sum_{k=0}^{r}\sum_{l=0}^{\alpha}\int_0^\infty h_\mu(\Delta_\mu^{k+l}\phi)(t)h_\mu(\nu_l)(t)h_\mu(\gamma_k)(t) \times t^{2\mu+1}\,dt. \tag{2.6}$$

Here, if γ is a complex regular Borel measure on $[0,\infty)$, $h_\mu(\gamma)$ is defined by

$$h_\mu(\gamma)(x) = \int_0^\infty (xt)^{-\mu}J_\mu(xt)d\gamma(t), \quad x\in(0,\infty).$$

From (2.5) we deduce that,

$$|\langle R\#S,\phi\rangle| \le C\sum_{k=0}^{r}\sum_{l=0}^{\alpha}\|\Delta_\mu^{k+l}\phi\|_{\mu,\infty}.$$

Thus, we conclude that $R\#S$ defines a linear and continuous mapping on $H_{\mu,1}$ when we consider on $H_{\mu,1}$ the topology induced by $H_{\mu,\infty}$. Hence, since $H_{\mu,1}$ is dense in $\mathbf{H}_{\mu,\infty}$, $R\#S$ can be extended in a unique way to $\mathbf{H}_{\mu,\infty}$ as an element of $\mathbf{H}'_{\mu,\infty}$. The equality $R\#S = S\#R$ follows immediately from (2.6).

(ii): Suppose that R admits the representation (2.4) and that

$$\langle T,\phi\rangle = \sum_{l=0}^{\alpha}\int_0^\infty f_l(x)\Delta_\mu^l\phi(x)\,dx, \quad \phi\in H_{\mu,1},$$

where $\alpha\in\mathbb{N}$ and $f_l\in L^\infty(\,dx)$, $l=0,...,\alpha$. We can write, for every $\phi\in H_{\mu,1}$,

$$\langle \Delta_\mu(T\#R),\phi\rangle = \langle T\#R,\Delta_\mu\phi\rangle$$

$$= \langle T, R\#(\Delta_\mu\phi)\rangle$$

$$= \sum_{l=0}^{\alpha}\int_0^\infty f_l(t)\Delta_{\mu,t}^l\sum_{k=0}^{r}\int_0^\infty \tau_t(\Delta_\mu^{k+1}\phi)(x)d\gamma_k(x)\,dt$$

$$= \sum_{l=0}^{\alpha}\int_0^\infty f_l(t)\sum_{k=0}^{r}\int_0^\infty \tau_t(\Delta_\mu^{k+l+1}\phi)(x)d\gamma_k(x)\,dt\,,$$

which proves (ii).

(iii): By using the procedure developed in the proof of (i) and (ii) and according to Proposition 1, it is sufficient to prove the property when

$$\langle T,\phi\rangle = \int_0^\infty f(t)\Delta_\mu^k\phi(t)t^{2\mu+1}\,dt, \quad \phi\in H_{\mu,1}$$
$$\langle R,\phi\rangle = \int_0^\infty \Delta_\mu^l\phi(t)d\gamma(t), \qquad \phi\in \mathbf{H}_{\mu,\infty}$$
$$\langle S,\phi\rangle = \int_0^\infty \Delta_\mu^m\phi(t)d\nu(t), \qquad \phi\in \mathbf{H}_{\mu,\infty},$$

where $f\in L^\infty(\,dx)$, γ and ν are complex regular Borel measures on $[0,\infty)$ and $k,l,m\in\mathbb{N}$.

Let $\phi\in H_{\mu,1}$. By (2.6) we have

$$\langle R\#S,\phi\rangle = 2^\mu\Gamma(\mu+1)\int_0^\infty h_\mu(\Delta_\mu^{l+m}\phi)(t)h_\mu(\gamma)(t)h_\mu(\nu)(t)t^{2\mu+1}\,dt.$$

Then $\langle T\#(R\#S),\phi\rangle = \langle T,(R\#S)\#\phi\rangle$, and hence

$$\begin{aligned}\langle T\#(R\#S),\phi\rangle &= \int_0^\infty f(t)\Delta_\mu^k((R\#S)\#\phi)(t)t^{2\mu+1}\,dt\\ &= \int_0^\infty f(x)\Delta_{\mu,x}^k\Big(2^\mu\Gamma(\mu+1)\int_0^\infty h_\mu(\Delta_\mu^{l+m}\tau_x\phi)(t)\\ &\quad\times h_\mu(\gamma)(t)h_\mu(\nu)(t)t^{2\mu+1}\,dt\Big)x^{2\mu+1}\,dx\\ &= 2^\mu\Gamma(\mu+1)\int_0^\infty f(x)\int_0^\infty h_\mu(\tau_x(\Delta_\mu^{l+m+k}\phi))(t)\\ &\quad\times h_\mu(\gamma)(t)h_\mu(\nu)(t)t^{2\mu+1}\,dtx^{2\mu+1}\,dx\\ &= (2^\mu\Gamma(\mu+1))^2\int_0^\infty f(x)\int_0^\infty (xt)^{-\mu}J_\mu(xt)h_\mu(\Delta_\mu^{k+l+m}\phi)(t)\\ &\quad\times h_\mu(\gamma)(t)h_\mu(\nu)(t)t^{2\mu+1}\,dt\;x^{2\mu+1}\,dx.\end{aligned}$$

Also we can write

$$\begin{aligned}\langle T\#R,\phi\rangle &= \langle T,R\#\phi\rangle\\ &= \int_0^\infty f(x)\Delta_{\mu,x}^k\int_0^\infty \Delta_{\mu,t}^l(\tau_x\phi)(t)d\gamma(t)x^{2\mu+1}\,dx\\ &= 2^\mu\Gamma(\mu+1)\int_0^\infty f(x)\int_0^\infty h_\mu((xz)^{-\mu}J_\mu(xz)\\ &\quad\times h_\mu(\Delta_\mu^{l+k}\phi)(z))(t)d\gamma(t)x^{2\mu+1}\,dx\\ &= 2^\mu\Gamma(\mu+1)\int_0^\infty f(x)\int_0^\infty (xz)^{-\mu}J_\mu(xz)\\ &\quad\times h_\mu(\Delta_\mu^{l+k}\phi)(z)h_\mu(\gamma)(z)z^{2\mu+1}dzx^{2\mu+1}\,dx.\end{aligned}$$

Hence, by using (ii),

$$\begin{aligned}
\langle (T\#R)\#S,\phi\rangle &= \langle T\#R, S\#\phi\rangle \\
&= 2^{\mu}\Gamma(\mu+1)\int_0^{\infty} f(x)\int_0^{\infty}(xz)^{-\mu}J_{\mu}(xz)h_{\mu}(\Delta_{\mu}^{l+k}(S\#\phi))(z) \\
&\quad\times h_{\mu}(\gamma)(z)(zx)^{2\mu+1}\,dz\,dx \\
&= 2^{\mu}\Gamma(\mu+1)\int_0^{\infty} f(x)\int_0^{\infty}(xz)^{-\mu}J_{\mu}(xz)h_{\mu}(\gamma)(z) \\
&\quad\int_0^{\infty}(tz)^{-\mu}J_{\mu}(tz)\int_0^{\infty}\tau_t(\Delta_{\mu}^{k+l+m}\phi)(y)d\nu(y)\,(tzx)^{2\mu+1}\,dt\,dz\,dx \\
&= (2^{\mu}\Gamma(\mu+1))^2\int_0^{\infty} f(x)\int_0^{\infty}(xz)^{-\mu}J_{\mu}(xz)h_{\mu}(\gamma)(z) \\
&\quad\int_0^{\infty}(yz)^{-\mu}J_{\mu}(yz)d\nu(y)h_{\mu}(\Delta_{\mu}^{k+l+m}\phi)(y)(zx)^{2\mu+1}\,dz\,dx \\
&= (2^{\mu}\Gamma(\mu+1))^2\int_0^{\infty} f(x)\int_0^{\infty}(xz)^{-\mu}J_{\mu}(xz)h_{\mu}(\gamma)(z)h_{\mu}(\nu)(z) \\
&\quad\times h_{\mu}(\Delta_{\mu}^{k+l+m}\phi)(y)(zx)^{2\mu+1}\,dz\,dx.
\end{aligned}$$

Thus we conclude that $T\#(R\#S) = (T\#R)\#S$. ∎

Proposition 7. *Let $T \in H'_{\mu,1}$. Then, $T \in \mathbf{H}'_{\mu,\infty}$ provided $T\#\phi \in H_{\mu,1}$, for every $\phi \in H_{\mu,1}$.*

Proof. Suppose that $T\#\phi \in H_{\mu,1}$, for every $\phi \in H_{\mu,1}$. By [3, Proposition 1.1], for every $m \in \mathbb{N}$ there exists an $r \in \mathbb{N}$ such that

$$\delta = (1-\Delta_{\mu})^r\varphi + \psi, \tag{2.7}$$

where δ is the Dirac functional, $\psi \in S_e$ and $\varphi \in C^{2m}(0,\infty)$, for which $\psi(x) = \varphi(x) = 0$, $x \geq 1$, and

$$\lim_{x\to 0^+}\left(\frac{1}{x}\frac{d}{dx}\right)^k\varphi(x) = 0,$$

for every $k = 0, 1, ..., 2m$. The equality in (2.7) is understood in [3] in S'_e. It is not hard to see that (2.7) also holds in $\mathbf{H}'_{\mu,\infty}$.

We now choose $m \in \mathbb{N}$ large enough. According to Proposition 6 we have

$$T = T\#\delta = (1-\Delta_{\mu})^r(T\#\varphi) + T\#\psi.$$

By the assumption, $T\#\psi \in H_{\mu,1}$. Then $T\#\psi \in \mathbf{H}'_{\mu,\infty}$.

Assume now $(k_n)_{n\in\mathbb{N}}$ is a sequence in S_e such that, for every $n \in \mathbb{N}$, k_n satisfies the following properties:

(a) $k_n \geq 0$

(b) $k_n(x) = 0$, $x \notin (\frac{1}{n+1}, \frac{1}{n})$

(c) $\int_0^\infty k_n(x)x^{2\mu+1}dx = 2^\mu\Gamma(\mu+1)$.

From [4, Proposition 3.5] we deduce that $k_n\#\varphi \in S_e$ and $(k_n\#\varphi)(x) = 0$, $x \geq 2$, for every $n \in \mathbb{N}$, and

$$\sup_{x\in(0,\infty)} |\Delta_\mu^k(k_n\#\varphi - \varphi)(x)| \to 0, \quad as\ n \to \infty,$$

for $k = 0, 1, ..., m$. According to Proposition 1 (i), there exist $l \in \mathbb{N}$ and $f_k \in L^\infty(dx)$, $k = 0, ..., l$, such that, for every $\phi \in H_{\mu,1}$,

$$(T\#\phi)(x) = \sum_{k=0}^{l} \int_0^\infty f_k(y)\tau_x(\Delta_\mu^k\phi)(y)y^{2\mu+1}\, dy, \quad x \in (0,\infty).$$

Then, by [15, p. 17] we get

$$\begin{aligned} |(T\#(k_n\#\varphi) - T\#\varphi)(x)| &\leq C\sum_{k=0}^{l} \|\Delta_\mu^k(\varphi\#k_n - \varphi)\|_{\mu,1} \\ &\leq C\sum_{k=0}^{l} \sup_{x\in(0,\infty)} |\Delta_\mu^k(\varphi\#k_n - \varphi)(x)| \to 0 \end{aligned}$$

as $n \to \infty$, uniformly in $(0, \infty)$.

On the other hand, $T\#(k_n\#\varphi) \in H_{\mu,1}$, for every $n \in \mathbb{N}$. Moreover, we can write, for certain $l \in \mathbb{N}$,

$$\|T\#(k_n\#\varphi)\|_{\mu,1} \leq C\sum_{k=0}^{l} \|\Delta_\mu^k\varphi\|_{\mu,1}\|k_n\|_{\mu,1} \leq C\sum_{k=0}^{l} \|\Delta_\mu^k\varphi\|_{\mu,1}$$

for every $n \in \mathbb{N}$. Hence we conclude that $T\#\varphi \in L^1(x^{2\mu+1}dx)$.

Thus, since the operator Δ_μ defines a continuous linear mapping from $\mathbf{H}_{\mu,\infty}$ into itself, we establish that $T \in \mathbf{H}'_{\mu,\infty}$. ∎

Proposition 8. *Let $T \in \mathbf{H}'_{\mu,1}$. Then, $T\#\phi \in H_{\mu,1}$, for every $\phi \in H_{\mu,1}$, if, and only if, the mapping $\phi \longrightarrow T\#\phi$ is continuous from $H_{\mu,1}$ into itself.*

Proof. Assume that $T\#\phi \in H_{\mu,1}$, for every $\phi \in H_{\mu,1}$. To see that the mapping $\phi \longrightarrow T\#\phi$ is continuous from $H_{\mu,1}$ into itself, we are going to use the closed graph theorem. Suppose that $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $H_{\mu,1}$ such that $\phi_n \to \phi$ and $T\#\phi_n \to \psi$, as $n \to \infty$, in $H_{\mu,1}$, where $\phi, \psi \in H_{\mu,1}$. We can write (Poposition 1, (i))

$$\langle T, \varphi\rangle = \sum_{k=0}^{l} \int_0^\infty f_k(y)\Delta_\mu^k\varphi(y)y^{2\mu+1}\, dy, \quad \varphi \in H_{\mu,1},$$

for certain $l \in \mathbb{N}$ and $f_k \in L^\infty(dx)$, $k = 0, 1, ..., l$.

We assume that, by taking a subsequence if it is necessary, $T\#\phi_n \to \psi$, a.e. on $(0, \infty)$. Moreover, we have that

$$\begin{aligned} |(T\#\phi_n - T\#\phi)(x)| &\leq \sum_{k=0}^{l} \int_0^\infty |f_k(y)| \tau_x(|\Delta_\mu^k(\phi_n - \phi)|)(y) y^{2\mu+1}\, dy \\ &\leq C \sum_{k=0}^{l} \|\Delta_\mu^k(\phi_n - \phi)\|_{\mu,1}, \quad x \in \mathbb{R}. \end{aligned}$$

Hence $T\#\phi_n \to T\#\phi$, as $n \to \infty$, uniformly in $(0, \infty)$. Thus we conclude that $T\#\phi = \psi$, and the proof is finished. ∎

We summarize the above results in the following theorem, where the space $\mathbf{H}'_{\mu,\infty}$ is characterized as the space of convolution operators of $H_{\mu,1}$.

Theorem 1. *Let $T \in H'_{\mu,1}$. Then the following assertions are equivalent:*

(i) $T \in \mathbf{H}'_{\mu,\infty}$.

(ii) $T\#\phi \in H_{\mu,1}$, *for every* $\phi \in H_{\mu,1}$.

(iii) *The mapping $\phi \longrightarrow T\#\phi$ is continuous from $H_{\mu,1}$ into itself.*

3. The Hankel transformation on the space $H'_{\mu,1}$

We now define the Hankel transformation on the space $H'_{\mu,1}$. We denote by $S_{\mu,1}$ the space of Hankel transforms of functions in $H_{\mu,1}$, that is,

$$S_{\mu,1} = \{h_\mu(\phi) : \phi \in H_{\mu,1}\}.$$

Since h_μ is one to one on $H_{\mu,1}$, we endow to $S_{\mu,1}$ with the topology induced on it by $H_{\mu,1}$ via h_μ. Thus, $S_{\mu,1}$ is a Fréchet space. It is not hard to see that if P is a polynomial, then the mapping $\psi \to P(x^2)\psi$ is continuous from $S_{\mu,1}$ into itself.

Let $T \in H'_{\mu,1}$. The Hankel transform $h'_\mu T$ is defined as the element of $S'_{\mu,1}$, the dual space of $S_{\mu,1}$, given by

$$\langle h'_\mu T, h_\mu \phi \rangle = \langle T, \phi \rangle, \quad \phi \in H_{\mu,1}.$$

Then if $T \in \mathbf{H}'_{\mu,\infty}$ is given by

$$\langle T, \phi \rangle = \sum_{k=0}^{l} \int_0^\infty \Delta_\mu^k \phi(x) d\gamma_k(x), \quad \phi \in \mathbf{H}_{\mu,\infty},$$

where $l \in \mathbb{N}$ and γ_k is a regular complex Borel measure on $[0,\infty)$, $k = 0, 1, ..., l$, we can write

$$\begin{aligned}\langle h'_\mu T, \phi\rangle &= \sum_{k=0}^{l} \int_0^\infty \Delta_\mu^k h_\mu(\phi)(y) d\gamma_k(y)\\ &= \sum_{k=0}^{l}(-1)^k \int_0^\infty \int_0^\infty x^{2k}(xy)^{-\mu} J_\mu(xy)\phi(x) x^{2\mu+1} dx d\gamma_k(y)\\ &= \sum_{k=0}^{l}(-1)^k \int_0^\infty \phi(x) x^{2k} \int_0^\infty (xy)^{-\mu} J_\mu(xy) d\gamma_k(y) x^{2\mu+1} dx, \quad \phi \in S_{\mu,1}.\end{aligned}$$

Hence, we obtain that $h_\mu(T)$ is a continuous function on $(0,\infty)$ and

$$h'_\mu(T) = 2^\mu \Gamma(\mu+1) \sum_{k=0}^{l} (-1)^k x^{2k} h_\mu(\gamma_k).$$

Hence $h'_\mu(T)$ is an slow growth function.

We now prove interchange distributional formulas.

Proposition 9. *Let $T \in H'_{\mu,1}$, $S \in \mathbf{H}'_{\mu,\infty}$ and $\phi \in H_{\mu,1}$. Then:*

$$h_\mu(S\#\phi) = h'_\mu(S) h_\mu(\phi) \tag{3.1}$$

$$h_\mu(T\#S) = h'_\mu(T) h'_\mu(S). \tag{3.2}$$

Proof. Assume that

$$\langle S, \psi\rangle = \sum_{k=0}^{l} \int_0^\infty \Delta_\mu^k \psi(y) d\gamma_k(y), \quad \psi \in \mathbf{H}_{\mu,\infty},$$

where $l \in \mathbb{N}$ and γ_k, $k = 0, 1, ..., l$ is a regular complex Borel measure on $[0,\infty)$. Then,

$$(S\#\phi)(x) = \sum_{k=0}^{l} \int_0^\infty \tau_x(\Delta_\mu^k \phi)(y) d\gamma_k(y), \quad x \in (0,\infty).$$

Hence, by interchanging the order of integration, we obtain

$$\begin{aligned}h_\mu(S\#\phi)(x) &= \sum_{k=0}^{l} \int_0^\infty \int_0^\infty \tau_t(\Delta_\mu^k \phi)(y) d\gamma_k(y) (xt)^{-\mu} J_\mu(xt) t^{2\mu+1}\, dt\\ &= 2^\mu \Gamma(\mu+1) \sum_{k=0}^{l} (-1)^k \int_0^\infty h_\mu(\phi)(x)(xy)^{-\mu} J_\mu(xy) x^{2k} d\gamma_k(y)\\ &= h_\mu(\phi)(x) h'_\mu(S)(x), \quad x \in (0,\infty).\end{aligned}$$

Let now $\psi \in S_{\mu,1}$. We can write

$$\begin{aligned}\langle h'_\mu(T\#S), \psi\rangle = \langle T\#S, h_\mu(\psi)\rangle &= \langle T, S\#h_\mu(\psi)\rangle \\ = \langle h'_\mu(T), h'_\mu(S)\psi\rangle &= \langle h'_\mu(T)h'_\mu(S), \psi\rangle.\end{aligned}$$

Thus (3.2) is shown. ∎

4. Hypoellipticity of Hankel convolution equations.

Our next objective is to analyze the hypoellipticity of Hankel convolution equations in $H'_{\mu,1}$. Firstly we prove a useful result.

Proposition 10. *Assume that* $(a_j)_{j\in\mathbb{N}} \subset \mathbf{C}$ *and* $(\xi_j)_{j\in\mathbb{N}} \subset (0,\infty)$ *such that* $2^j \leq 2\xi_{j-1} < \xi_j$, $j \in \mathbb{N}$, *and that* $|a_j| = O(\xi_j^m)$, *as* $j \to \infty$, *for some* $m \in \mathbb{N}$. *We define the functional* $S \in S'_e$ *as follows*

$$S(x) = 2^\mu \Gamma(\mu+1) \sum_{j=1}^{\infty} (x\xi_j)^{-\mu} J_\mu(x\xi_j) a_j.$$

Then $S \in H'_{\mu,1}$. *Moreover,* $S \in H_{\mu,\infty}$ *if, and only if,* $|a_j| = o(\xi_j^{-n})$ *as* $j \to \infty$, *for every* $n \in \mathbb{N}$.

Proof. Let $l_1, l_2 \in \mathbb{N}$, $l_1 < l_2$, and $\phi \in H_{\mu,1}$. We can write

$$\begin{aligned}\left|\sum_{j=l_1}^{l_2} a_j \int_0^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j)\phi(x)x^{2\mu+1}dx\right| &\leq \sum_{j=l_1}^{l_2} |a_j||h_\mu(\phi)(\xi_j)| \\ &\leq C\sum_{j=l_1}^{l_2} \xi_j^m |h_\mu(\phi)(\xi_j)|.\end{aligned}$$

Since $y^k h_\mu(\phi)$ is bounded on $(0,\infty)$, for every $k \in \mathbb{N}$ (because, for each $k \in \mathbb{N}$, $\Delta_\mu^k \phi \in L^1(x^{2\mu+1}dx)$), and since $\xi_j > 2^j$, $j \in \mathbb{N}$, the series $\sum_{j=1}^\infty a_j h_\mu(\phi)(\xi_j)$ converges and $S \in H'_{\mu,1}$.

Suppose now that $|a_j| = o(\xi_j^n)$, as $j \to \infty$, for every $n \in \mathbb{N}$. By [17, (6) and (7), Chapter 5] we can see that $S \in C^\infty(0,\infty)$ and that, for every $l \in \mathbb{N}$,

$$\Delta_\mu^l S(x) = 2^\mu \Gamma(\mu+1) \sum_{j=1}^{\infty} (-\xi_j)^{2l} (x\xi_j)^{-\mu} J_\mu(x\xi_j) a_j, \quad x \in (0,\infty).$$

Then, for every $l \in \mathbb{N}$, $\|\Delta_\mu^l S\|_{\mu,\infty} \leq C\sum_{j=1}^\infty \xi_j^{2l}|a_j| < \infty$. Thus we have proved that $S \in H_{\mu,\infty}$.

Assume now that $S \in H_{\mu,\infty}$. To see that $|a_j| = o(\xi_j^n)$, as $j \to \infty$, for every $n \in \mathbb{N}$, we proceed as in the proof of [6, Proposition 3.2]. Let $k \in \mathbb{N}$ and $\phi \in S_e$. By [4, (3.1)] and integrating by parts we can write

$$\begin{aligned}
&\big\langle (xt)^{-\mu} J_\mu(xt) \Delta_\mu^k S(x), \phi \big\rangle \\
&\qquad = \big\langle \Delta_\mu^k S(x), (xt)^{-\mu} J_\mu(xt)\phi(x) \big\rangle \\
&\qquad = \big\langle S(x), \Delta_{\mu,x}^k ((xt)^{-\mu} J_\mu(xt)\phi(x)) \big\rangle \\
&\qquad = 2^\mu \Gamma(\mu+1) \sum_{j=1}^\infty \big\langle (x\xi_j)^{-\mu} J_\mu(x\xi_j) a_j, \Delta_{\mu,x}^k ((xt)^{-\mu} J_\mu(xt)\phi(x)) \big\rangle \\
&\qquad = \sum_{j=1}^\infty a_j \int_0^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j) \Delta_{\mu,x}^k ((xt)^{-\mu} J_\mu(xt)\phi(x))\, x^{2\mu+1}\, dx \\
&\qquad = \frac{1}{2^\mu \Gamma(\mu+1)} \sum_{j=1}^\infty a_j (-1)^k \xi_j^{2k} \tau_{\xi_j}(h_\mu \phi)(t), \quad t \in (0,\infty). \qquad (4.1)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\big\langle (xt)^{-\mu} J_\mu(xt) \Delta_\mu S(x), \phi(x) \big\rangle \\
&\qquad = \int_0^\infty (xt)^{-\mu} J_\mu(xt) \Delta_\mu^k S(x) \phi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dx, \quad t \in (0,\infty).
\end{aligned}$$

Since $\Delta_\mu^k S(x)\phi \in L^1(x^{2\mu+1}\, dx)$, the Riemann-Lebesgue theorem for Hankel transforms implies that

$$\lim_{t\to\infty} \langle (xt)^{-\mu} J_\mu(xt) \Delta_\mu S(x), \phi(x) \rangle = 0. \qquad (4.2)$$

We now choose a function $\phi \in S_e$ such that $h_\mu(\phi)(x) = 0$, $x > 1$, and $h_\mu(\phi)(x) > \frac{1}{2}$, $x \in (0, \frac{1}{2})$. By (4.1), since $\tau_z(h_\mu \phi)(x) = 0$, provided that $|z - x| > 1$, we have

$$\begin{aligned}
|\langle (x\xi_j)^{-\mu} J_\mu(x\xi_j) \Delta_\mu^k S, \phi \rangle &= |a_j (-1)^k \xi_j^{2k} \tau_{\xi_j}(h_\mu \phi)(\xi_j)| \\
&\geq |a_j| \xi_j^{2k-2\mu-1}, \quad j \in \mathbb{N}.
\end{aligned}$$

To see last inequality we have used ([8, (8.11,31)]. Then, (4.2) allows us to conclude that $|a_j| = o(\xi^{-2k+2\mu+1})$, as $j \to \infty$. Thus the proof is finished. ∎

Let $S \in \mathbf{H}'_{\mu,\infty}$. We say that S is hypoelliptic in $H'_{\mu,1}$ when the folllowing property holds: if $T \in H'_{\mu,1}$ and $T \# S \in H_{\mu,\infty}$ then $T \in H_{\mu,\infty}$.

We now characterize the hypoellipticity of $S \in \mathbf{H}'_{\mu,\infty}$ in terms of the growth of the Hankel transform $h'_\mu(S)$ of S.

Proposition 11. *Let $S \in \mathbf{H}'_{\mu,\infty}$ such that there exists a polynomial p for which*

$$\left|\frac{d^l}{dx^l} h'_\mu(S)(x)\right| \leq p(x), \quad x \in (0,\infty),\ l = 0,1,2,...,2s,$$

where $s = [\mu + 2]$. Then, S is hypoelliptic in $H'_{\mu,1}$ if, and only if, there exist $a, M > 0$ such that

$$|h'_\mu(S)(x)| \geq x^{-a}, \quad x \in (M,\infty). \tag{4.3}$$

Proof. Suppose that firstly (4.3) does not hold for every $a, M > 0$. Then we can find a sequence $(\xi_j)_{j\in\mathbb{N}} \subset (0,\infty)$ such that, for every $j \in \mathbb{N}$, $|h'_\mu(S)(\xi_j)| < \xi_j^{-j}$ and $2^j < 2\xi_{j-1} < \xi_j$. We now define the functional $T \in H'_{\mu,1}$ by

$$T(x) = 2^\mu \Gamma(\mu+1) \sum_{j=1}^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j).$$

According to Proposition 10, $T \in H'_{\mu,1}$. Moreover, for every $\phi \in H_{\mu,1}$, from (3.1) we infer

$$\begin{aligned}
\langle T\#S, \phi\rangle &= \langle T, S\#\phi\rangle \\
&= \sum_{j=1}^\infty \int_0^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j)(S\#\phi)(x) x^{2\mu+1}\, dx \\
&= \sum_{j=1}^\infty h'_\mu(S)(\xi_j) h_\mu(\phi)(\xi_j) \\
&= \sum_{j=1}^\infty h'_\mu(S)(\xi_j) \int_0^\infty (x\xi_j)^{-\mu} J_\mu(x\xi_j)\phi(x) x^{2\mu+1}\, dx \\
&= \int_0^\infty \sum_{j=1}^\infty h_\mu(S)(\xi_j)(x\xi_j)^{-\mu} J_\mu(x\xi_j)\phi(x) x^{2\mu+1}\, dx.
\end{aligned}$$

Hence, we can conclude that

$$T\#S = 2^\mu \Gamma(\mu+1) \sum_{j=1}^\infty h'_\mu(S)(\xi_j)(x\xi_j)^{-\mu} J_\mu(x\xi_j).$$

By Proposition 10 we deduce that $T\#S \in H_{\mu,\infty}$ because the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$. However, T is not in $H_{\mu,\infty}$. Thus we prove that S is not hypoelliptic in $H'_{\mu,1}$.

Assume that $|h'_\mu(S)(x)| > x^{-a}$, $x \in (M,\infty)$, for certain $a, M > 0$. We choose a function $\phi \in S_e$ such that $\phi(x) = 1$, $x \in (0, M)$, and $\phi(x) = 0$,

$x \in (M+1, \infty)$, and we define the function F as follows

$$F(x) = \begin{cases} \frac{1-\phi(x)}{h'_\mu(S)(x)}, & x \in (M, \infty) \\ 0, & x \in (0, M]. \end{cases}$$

By using an iterated Leibniz rule, since there exists a polynomial p such that

$$\left| \frac{d^l}{dx^l} h'_\mu(S)(x) \right| \leq p(x), \quad x \in (0, \infty),\, l = 0, 1, ..., 2s,$$

where $s = [\mu + 2]$, we can find a $k \in \mathbb{N}$ such that by defining $f = h_\mu(\frac{F(x)}{(1+x^2)^k})$, we have

$$f(x) = \frac{1}{(1+x^2)^s} h_\mu\bigg((1-\Delta_\mu)^s \bigg(\frac{F(x)}{(1+x^2)^k} \bigg) \bigg)(x), \quad x \in (0, \infty),$$

where $s = [\mu + 2]$, being $\Delta_\mu^s(\frac{F(x)}{(1+x^2)^k}) \in L^1(x^{2\mu+1}\, dx)$. Hence $f \in L^1(x^{2\mu+1}\, dx)$ and $(1-\Delta_\mu)^k f = h'_\mu(F) \in \mathbf{H}'_{\mu,1}$. Moreover, $F h'_\mu(S) = 1 - \phi$. Then

$$G\#S = \delta - \psi,$$

where $G = (1-\Delta_\mu)^k f$ and $\psi = h_\mu(\phi)$. Indeed, for every $\varphi \in H_{\mu,1}$, (3.2) implies that

$$\begin{aligned} \langle G\#S, \varphi \rangle &= \langle h'_\mu(G) h_\mu(S), h_\mu(\varphi) \rangle \\ &= \langle 1 - \phi, h_\mu(\varphi) \rangle \\ &= \int_0^\infty h_\mu(\varphi) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dx - \int_0^\infty \phi(x) h_\mu(\varphi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dx \\ &= \varphi(0) - \int_0^\infty h_\mu(\phi)(x) \varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)}\, dx \\ &= \langle \delta, \varphi \rangle - \langle \psi, \varphi \rangle. \end{aligned}$$

Suppose now that $T\#S = R \in H_{\mu,\infty}$, where $T \in H'_{\mu,1}$. Since $T\#\delta = T$, by using Proposition 6, we can write

$$T = T\#\delta = T\#(G\#S) + T\#\psi = (T\#S)\#G + T\#\psi = R\#G + T\#\psi.$$

There exist $k \in \mathbb{N}$ and $f_j \in L^\infty(\, dx)$, $j = 0, 1, ..., k$, such that

$$\langle T, \varphi \rangle = \sum_{j=0}^k \int_0^\infty f_j(x) \Delta_\mu^j \varphi(x) x^{2\mu+1}\, dx, \quad \varphi \in H_{\mu,1}.$$

In particular,

$$(T\#\psi)(x) = \sum_{j=0}^{k} \int_0^\infty f_j(y)\tau_x(\Delta_\mu^j \psi)(y) y^{2\mu+1}\, dx.$$

According to [15, p. 17], we get that, for every $m \in \mathbb{N}$,

$$\|\Delta_\mu^m (T\#\psi)\|_{\mu,\infty} \le C \sum_{j=0}^{k+m} \|\Delta_\mu^j \psi\|_{\mu,1}.$$

Hence $T\#\psi \in H_{\mu,\infty}$.

On the other hand, we have

$$R\#G = R\#(1-\Delta_\mu)^k f = (1-\Delta_\mu)^k R\#f.$$

Moreover, for every $m \in \mathbb{N}$,

$$\begin{aligned} \|\Delta_\mu^m (R\#G)\|_{\mu,\infty} &= \|(\Delta_\mu^m (1-\Delta_\mu)^k R)\#f\|_{\mu,\infty} \\ &\le C\|\Delta_\mu^m (1-\Delta_\mu)^k R\|_{\mu,\infty} \|f\|_{\mu,1}. \end{aligned}$$

Hence $T\#G \in H_{\mu,\infty}$. Thus we conclude that $T \in H_{\mu,\infty}$ and the hypoellipticity of S on $H'_{\mu,1}$ is established. ∎

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