The Diversity of Mathematical Cultures: One Past and Some Possible Futures

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Introduction

For a long time now, as a historian as well as an observer of contemporary mathematical practices, I have been struck by the diversity of ways of doing mathematics. I am not speaking here of the variety of individual styles, which has already been the subject of many works, but rather the diversity of collectively shared ways of practising mathematics. I feel this phenomenon deserves more attention than it has received to date. In this article, I would like to explain some of the reasons that convince me of the value that could be represented by the description of these social and cultural realities – realities that it seems to me appropriate to understand as different "mathematical cultures". I have had the opportunity to clarify, with the help of an example developed at length, what I mean by this last expression [1] and I will return to this question below. However, before I go any further, I must rule out one possible source of misunderstanding. While I propose speaking about 'mathematical cultures', this is totally unrelated to another, all-too-common interpretation of the same expression that seems to me meaningless and, what is more, dangerous.

Indeed, since the 19th century, a certain way of thinking about the diversity of mathematical practices has become dominant: it is the antithesis of the thesis for which I argue here. To remain brief, I will forego nuance and illustrate this alternative concept with a statement made by the physicist Jean-Baptiste Biot, which has the merit of, in only a few lines, revealing many facets of the representation that I reject. In an 1841 review of Jean-Jacques Sédillot's translation of a work in Arabic entitled *Traité des instruments astronomiques des Arabes*, Biot published the following verdict (the italics are mine, except for the final sentence in Latin) [2]:

> …one finds [in this book] renewed evidence for this peculiar *habit of mind*, whereby the Arabs, like the Chinese and Hindus, *limited* their scientific writings to the statement of a *series of rules*, that, once given, were only to be *verified by their very application, without requiring any logical demonstration or connection* between them, which gives these *Oriental nations a remarkable character of dissimilarity*, I would even add of *intellectual inferiority, compared to the Greeks*, with whom *all propositions are established by reasoning, and generate logically-deduced consequences.* This *fixed* writing of scientific methods, *in the form of precepts*, must have represented a significant *hindrance for the development of new ideas* for the peoples for which it was in use, and it is in sharp *contrast* with our *European* maxim: *nullius in verba*.[2]

When Biot concludes by quoting the motto of the Royal Society: "take no one's word for it", which enjoined its members to reject all forms of authority, it is to draw a contrast. The maxim calls for a form of "freedom" in thinking, which, for Biot, characterises Europe – elsewhere he says the Occident – and which has been extolled regularly ever since as the specific intellectual attitude that allowed the emergence of "modern science". According to Biot, the "Orientals", however, contented themselves with stating sequences of "rules" (in modern terms: algorithms) and then proceeded with prescriptions ("precepts"), which, in his view, supposed, by contrast, obedience from their users, meaning that it was therefore impossible for them to bring about a scientific revolution. This is one of the key elements of a broader opposition between mathematical practices of different peoples that Biot shapes along these lines. Thus, on another level, the "Orientals" would not feel the need to demonstrate, making do with simple "verifications". It is, in Biot's eyes, the function of the mathematical problems contained in their texts and that he designates as "applications". However, he insists, by contrast, that "the Greeks" demonstrate everything. As a consequence, in "Oriental" mathematical practice, the rules presented no interrelationships, while the "Occidentals", conversely, created deductive edifices.

All in all, as the above statement shows, Biot believed in a fundamental difference in nature between peoples, the presentation of which required only two categories: in one camp, the "Oriental nations" and, in the other, "The Greeks" and the "Europeans", among whom he

[1] I present here some of the results of research carried out in the context of the SAW ("Mathematical Sciences in the Ancient Worlds") project that has been financed by the European Research Council, in the context of the 7th programme framework (FP7/2007–2013, ERC Grant agreement n. 269804). This article is a translation, by Richard Kennedy, of "La diversité des cultures mathématiques: un passé et quelques futurs possibles", *Gazette des mathématiciens*, 150, 2016, p. 16–30 (online at http://www.smf.emath.fr/files/150-bd.pdf). It derives from the plenary lecture that I gave at the European Congress of Mathematics (Berlin, July 2016). A more complete version of this text is to be published in the proceedings of this conference; I will also make it available on HAL-SHS. I am grateful to Bruno Belhoste and Nad Fachard for their invaluable help throughout the preparation of this article.

[2] This document was first published by F. Charette [3].

positioned himself. For him, this difference was reflected in the contrast between their mathematical practices – a contrast to which he assigns long-term consequences (one camp experiences progress while the other advances with great difficulty). If we examine more closely how Biot articulates the difference between peoples and the contrast between mathematics, we notice that, to his eyes, the way in which "Orientals" carry out mathematics derives from a "particular habit of mind" common to these peoples: mathematics here only illustrates a more general fact. It is as much his belief in the truth of the general fact as the confirmation that he believes to have found in the description of their mathematical activities that leads Biot to express a hierarchy between the peoples. However, conversely, the declaration gives mathematics and modern science as the proof of the superiority of the Greeks and of Europe. History of science served for a long time, in fact, as a laboratory for developing conceptions with which some have believed it possible to consider the "characteristics" of peoples and establish the theory of an irreducible disparity between them. In the context of the SAW Project, we started an historical study of these forms of history of science and their uses but pursuit of this here would lead us too far. Rather, let us return to our subject.

Biot wrote these lines in 1841. I can testify that many elements of the representation of the diversity of mathematical practice to which he subscribed still persist today, in various forms, and are even very widespread, if not within the mathematical community, at least more widely in our society. In the context of today's world, the effects are potentially as destructive as they have been in the past. It is interesting to examine the documentary base from which Biot established his verdict. This is quite straightforward for China, as Biot's son, Edouard (1803–1850), was the first specialist of China to publish in Europe on the history of mathematics, and the four articles he wrote on the subject between 1835 and 1841 were all discussed with his father. Like a good number of sinologists of the time, Edouard never travelled to China and his investigations had to be limited to documents available in Europe. The Bibliothèque Royale's collections in Paris gave him access to a book on mathematics, written in Chinese and published in China in 1593, to which he devoted his first two articles. In 1839, he published a study on a second work, which he was able to consult thanks to the fact that his mentor in sinology, Stanislas Julien, loaned it to him. Isolated in his work on the mathematics in Ancient China, Edouard incorrectly dated this book, completed in 1259, to the 8th century and appears not to have understood the algebraic symbolism that was central to the author's project. Finally, once again in the Bibliothèque Royale, he found a work dating from the start of the Common Era and addressing mathematical knowledge necessary for astronomy and cosmography, a translation of which he published in June 1841. It is essentially from these data that, in the same year, Jean-Baptist Biot would formulate his definitive opinion of the mathematics of the Chinese "people" from antiquity up to his time.

The fact that today we can read several dozen mathematical books written in China between the last centuries before the Common Era and the 19th century does not mean that it makes any more sense to talk about "Chinese mathematics". In any case, it is not "mathematical cultures" conceived in terms of this type that I am thinking of when I propose to argue in favour of the interest there would be in considering the diversity of collectively shared ways of doing mathematics. Entities such as "nations" or "peoples" seem far too vast for what I have in mind. Wanting, at all costs, to say something about mathematics in a context of this magnitude, we would find ourselves condemned, like Biot, to generalising unduly. Or else the search for a common denominator for the mathematics of a "nation" or a "people" would lead us to stand much too far from those whom we are observing (and whom I will, as anthropologists do, call "actors"). At such a distance we would only grasp some commonalities of little significance, frequently minimising everything that contradicts the overall conclusion, and it would be by decree that we give these common points as characteristics of the entity observed. In both cases, it is by postulate that "nation" or "people" are posited as relevant frameworks and therefore we shouldn't be surprised to find the postulate in the conclusions.

Another approach to mathematical cultures

Like the majority of historians, I prefer to work from documents. And what has struck me, in considering the writings produced in a variety of contexts, is that these documents form clusters, which attest shared but different ways of doing mathematics. What types of human collectives do these clusters of writings bear witness to? We cannot give a general answer to this question and it would be necessary to examine them case by case. Below, I will outline some ways of addressing it. My main objective here will be, however, to illustrate, with examples, the phenomena which interest me and that I propose to approach in terms of different "cultures". Along the way, these examples will allow me to explain why I am convinced of the importance of taking these phenomena into account to interpret our documents in a more thorough and rigorous way and, through this exploration, I will also bring out some new general questions that they seem to raise.

The first illustration of what I mean by a "mathematical culture" comes from a field with which I am familiar. This is not by chance: an approach of this type requires an intimate knowledge of the sources. I chose this example from ancient history, as the problems of the interpretation of documents are often more acute when the writings were produced in the distant past. I hope, therefore, that the help in interpretation that can be afforded by an approach in terms of culture will be all the more obvious. I will consider, then, a cluster of Ancient Chinese mathematical works presented to the throne in 656 by Li Chunfeng and the scholars working under his direction: *The Ten Canons of Mathematics*.

By order of the Emperor, Li and his colleagues set about the preparation of this anthology, selecting classics from the past as well as commentaries that had been written about them and then preparing a critical edition and their own commentaries for all these documents. From 656, with the work having just been completed, these books were used in the School of Mathematics that had been established within the Imperial University, where students could follow a specialised curriculum in mathematics in order to gain access to a career in the bureaucracy. These ten canons, in a different chronological order to their composition, formed the content of two curricula [4]. I am interested here in only the most elementary of these curricula and I will, in fact, mention only two of the books studied in this context: the first canon worked on, *The Mathematical Classic of Master Sun*, which was devised around the year 400 (even if the text that we can read shows marks of modifications dating from the 8th century), and the canon that formed the centrepiece of this course, *The Nine Chapters on Mathematical Procedures*, the completion of which I date to the 1st century CE. This book and the commentaries written about it by Liu Hui in the 3rd century and by the team working with Li Chunfeng in the 7th century required, in fact, a far greater number of years of study in comparison to the other books.

These historical elements allow us to formulate two important points of method. If these canons and their commentaries were taught in the same curricula, it means that 7th century actors considered them de facto as associated with the same mathematical culture. In addition, the first six canons of the first curriculum are essentially composed of mathematical problems and algorithms allowing them to be solved; they are therefore difficult to interpret. By contrast, however, the commentaries chosen or written by Li Chunfeng's team comprise discussions on mathematics and explicit references to the practice of mathematics. These commentators are, in fact, the earliest readers of the canons that we are able to observe and they provide us with essential clues to describe the mathematical culture that makes up my first example. I stress this point: the description of a mathematical culture must not derive from impressions or intuition but instead rely on historical demonstrations based on documents. The assertions that I will formulate below are, as far as possible, based on long arguments but I will not elaborate on them here, instead referring the reader to previous publications.

The key question at present is to understand how the mathematical activity testified by these documents was practised. A typical page from canons like *The Nine Chapters* (this is how I will abbreviate the title henceforth) is composed of problems and algorithms, while the commentaries, which appear in smaller characters and often between the sentences making up the algorithms, systematically establish the correctness of these algorithms, interspersing these developments with all sorts of remarks and discussions.

As the oldest editions show, all these writings only contain characters, without any graphical representations of any sort. However, the canons, like the commentaries, make reference to rods, with which the numbers were represented on a surface on which the calculations were carried out. Without representations in the texts of the use made of the rods or the calculating surface, everything that took place on that surface has had to be reconstructed from clues gleaned from the writings and on the basis of historical arguments. Our situation is probably comparable with that of future historians who will concern themselves with understanding the part of the activity of mathematics that takes place today on our blackboards.

The rods constitute the first physical object mentioned in the texts and we will see that they played a key role in the mathematical culture testified by the canons. Furthermore, canons like *The Mathematical Classic of Master Sun* and *The Nine Chapters* neither contain nor mention any figures, nor even any visual aids. However, in the context of certain demonstrations, the commentaries do evoke figures and blocks, opting for one or the other according to whether they are dealing with plane geometry or space geometry. With the blocks, which evoke the plaster and string models used by certain mathematical milieus in the second half of the 19th and the beginning of the 20th centuries, we thus encounter a second type of physical object that mathematics activity had recourse to. The early editions of these classics do not contain the figures that the commentators refer to and the examination of clues that we could gather about them has led me to conclude that they, too, were physical objects at the time. I will refer to them with the term "diagram", to remind us that they are visual aids different from those we usually associate with the term "figure".

In summary, and in contrast to what later documents attest, the mathematical activity evidenced by our first cluster of writings is based on books containing only text and also on three types of object: rods, blocks and diagrams [5].

In the course of a series of articles, I have shown how the description of what the actors did with the elements contained in the writings, as well as with the objects we have just identified (which only partly covers what I mean by the expression "way of doing mathematics"), is essential to interpreting the writings and obtaining a more complete grasp of the mathematical knowledge they had. Here, I will illustrate this thesis with the aid of only one of these aspects, concentrating on the way in which the actors worked with the calculating surface, according to what can be reconstructed, and showing how this approach allows us to understand the knowledge that they had developed around arithmetical operations.

My reasoning starts with the first pages of *The Mathematical Classic of Master Sun*, that is to say, the start of the elementary curriculum. Here, the work describes, among other things, the use of rods to represent numbers on the calculating surface: without entering into detail, in this description, we can recognise a decimal place-value system, in the sense that writing the symbols 123 in these positions implies that 1 means a hundred, 2 twenty and 3 three.

Then, based on this system of numeration (which, therefore, was purely physical and did not appear at the time in the writings), the book offers two algorithms, one for multiplying and the other for dividing – the division here is called *chu*. The text of this second algorithm does not open with a prescription but with an assertion: "this algorithm is exactly opposed to that for multiplication". The meaning of this statement is not obvious solely on the basis of the text of these two algorithms. However, the calculations for the execution of these two operations that we can reconstruct on the basis of the texts, and for which I give an example in Figure 1, suggest an interpretation. They will be essential to my argument and hence I enter into detail here.

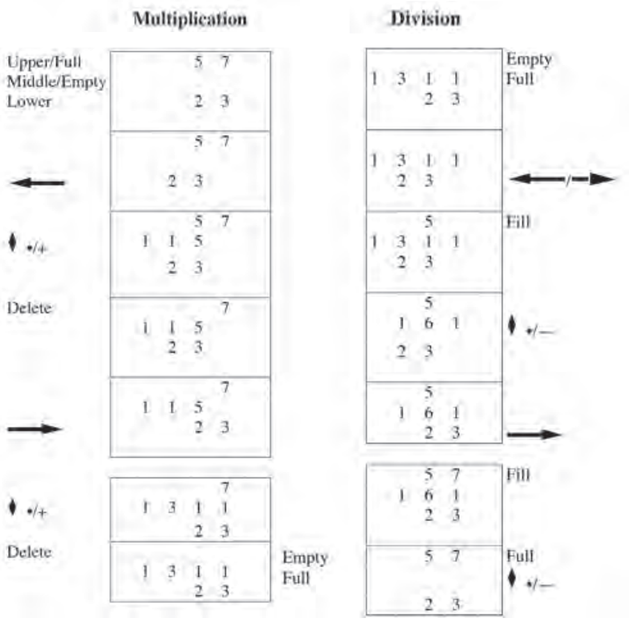

Figure 1.

These algorithms are based on two types of "positions", both designated by the same Chinese term (*wei*). Firstly, the numbers are written horizontally, as a series of decimal positions. This place-value notation echoes the property of the algorithms to iterate the same series of elementary operations along this sequence of digits. Moreover, both algorithms make use of three vertical positions, one above the other (upper, middle and lower). The multiplication starts by placing the multiplier (23 in my example) and the multiplicand (57) in the upper and lower positions respectively, leaving the middle position empty. The initial layout for the division of 1311 by 23 is completely opposed with respect to the middle and upper positions: contrary to the multiplication, it is the upper position that is empty at the start of the calculation, while the middle position is full, as it contains the dividend. For both operations, the calculation proceeds in the same way, by filling the line of these two that is empty while emptying the line that is full. Operating on initial configurations that are opposed, the processes that follow are themselves opposed to each other. The "result" of the multiplication is produced in the middle, while that of the division is produced at the top. Here, we thus see that a relationship of opposition between the two operations is shaped, through the precise fashioning of the processes of execution on the calculating surface. It is the first property of interest for us in these flows of operations that execute multiplication and division. We will return to it.

In contrast to the middle and upper positions, which are opposed to each other between multiplication and division, the lower position similarly receives operators that are the multiplier and the divisor. Both act in the same way during the execution of their respective operations: their significant digits are not modified, but their decimal positions are, being displaced at each iteration. The layout of the two operations and the algorithms have the effect that the execution of the multiplication ends at the starting point of the division and vice versa: if you run multiplication and division one after the other, the operations cancel each other out. This is a second property of interest to us in these flows of operations.

These arrangements, partly opposed and partly identical, of the algorithms on the calculating surface correspond to flows of calculations that allow practitioners to see the relationship of opposition between multiplication and division. Thus, once the multiplication operands are positioned, the multiplier 23 is moved to the left until its units digit is vertically under the digit with the highest magnitude in the multiplicand (5). The multiplier is thus multiplied by the power of 10 corresponding to this latter digit. The products of the digits of 23 by 5 can then be added progressively to the middle position, immediately above the corresponding digits in the multiplier. Once this sub-procedure is completed, 5 is deleted from the upper line, 23 is shifted one position to the right and the same sub-procedure is repeated with 7, the following digit, which in turn will be deleted at the end of the execution. Thus, it is in this way that "that which the multiplication produces" finds itself "in the middle" while the multiplicand is, for its part, deleted. The execution of a division will "produce", in an opposed way, the result "in the upper position", while the number in the middle position will be progressively deleted. By contrast, the digits in the quotient are, in effect, progressively added to the upper position (5 then 7), while, in the appropriate corresponding position, the products of the digits in the divisor and first 5, and then 7, are progressively subtracted from (and not added to) the dividend. Incidentally, if we had divided not 1311 but 1312 by 23, the quotient would be given as 57 + 1/23. The fact that the results of divisions are always exact plays a critical role, but that necessitates another development that I am not able to give here.

In the context of this way of doing mathematics, inculcated from the beginning of the first curriculum in the School of Mathematics, the algorithms for multiplication and division have therefore been shaped to allow a global vision, position by position, of a network of oppositions and similarities in the very dynamic of the executions on the calculating surface. It is, I think, to this and not the fact that multiplication and division cancel the effect of each other, that the declaration in *The Mathematical Classic of Master Sun* (placed at the beginning of the text on the algorithm for division) refers when it states "this algorithm is exactly opposed to that for multiplication". This conclusion deserves further examination.

Firstly, it implies that the physical practices that mathematical activity brought into play in this context must be reconstituted so that we can fully interpret the writings. This assertion has a de facto general validity. Moreover, the interpretations I suggest for both the declaration in the text and the procedures for calculation imply that the processes for carrying out the operations on the calculating surface do not only have the aim of producing results but also of expressing properties – here a form of relation between multiplication and division.

This basic example suffices to illustrate what I mean by different "mathematical cultures" and it also provides a glimpse of the interest their description takes on. The ideas brought into play in the algorithms represented in Figure 1 are identical to those that inspire the way we ourselves have learned to carry out multiplication and division. Yet, in the eyes of the actors who employed one or the other, the *meaning* of the two sets of algorithms differs in part and we will see that this difference has important consequences. By contrast to this other way of working, our practices for calculation do not invite us to interpret as meaningful the relations between flows of operations executing multiplication and division, or to work with these flows. This is one of the features that confers its uniqueness to the practice of calculation provided in the first curriculum of the 7th century in China, and the declaration in *The Mathematical Classic of Master Sun* allows us to grasp what is at stake. Let us now analyse what is brought to us by the knowledge of this specific element of such a "way of doing mathematics".

Work on the relations between the operations

The statement in *The Mathematical Classic of Master Sun*, combined with the flows of calculations we can reconstitute on the basis of the texts of algorithms, allows us to establish the existence of a practice of calculation unique to a certain context: the use of "position" to explore and express an interpretation of the relation between operations. In doing so, it reveals the existence of mathematical interest in such relations. Understanding this practice will, more generally, allow us to grasp mathematical knowledge on the relation between operations as it was produced in this context. This is all the more important because historians had not really perceived this knowledge before. Only by reading what the texts and the physical inscriptions express in a specific way do we uncover part of the actors' mathematical knowledge and also a fundamental question that inspired their research.

Moreover, the fact of having uncovered such a practice also provides us with tools for interpreting other texts in the same corpus and for going further into the reconstitution of the actors' practices on the calculating surface. Thus, we can better understand the theoretical work that the actors carried out on the operations and also comprehend the history of this work. The operations of multiplication, and especially of division, as well as their execution on the calculating surface described above, will then prove to have played a key role in this history.

To establish this point, we will return firstly to *The Nine Chapters*, whose text attests the same practice of calculation on the calculating surface, as well as the same interest for the relations between operations. Let us examine, for example – without, for the moment, trying to interpret them – the texts of the algorithms provided for the extraction of square and cube roots (I only quote the beginnings here, which are sufficient to bring out the phenomena that interest me):[3]

> "Procedure for the extraction of the square root: One places the number-product as **dividend**. Borrowing one rod, one **moves it forward**, jumping *one* column. Once the **quotient** is obtained, one multiplies *once* the borrowed rod by it, which makes the **divisor**, then one **divides by this**. After having **divided**, one *doubles* the **divisor**, which makes the determined **divisor**. If one **divides again**, one reduces the **divisor by moving it backward**. One again places a borrowed rod, and moves it forward like at the beginning. One **multiplies this once** by **the new quotient**. (…)"
>
> "Procedure for the extraction of the cube root: One places the number-product as **dividend**. Borrowing one rod, one **moves it forward**, jumping *two* columns. Once the **quotient** is obtained, one multiplies *twice* the borrowed rod by it, which makes the **divisor**, then **one divides by this**. After having **divided**, one *triples* this, which makes the determined **divisor**. **If one divides again**, one **reduces (the divisor) by moving it backward**. One multiplies the quantity obtained by three, and one places this in the middle row. Once more borrowing a rod, one places it in the row underneath. One moves them forward, that which is in the middle jumping one column, that which is underneath jumping *two* columns. **One again places a quotient** and multiplies by it that which is in the middle once, and that which is underneath, *twice*. (…)"

If we consider these algorithm texts independently of any context, they are difficult to interpret with certainty. In particular, the layout of the calculations to which they refer seem unfathomable. However, two key points are evident.

I have marked in bold type the terms that these texts take from the algorithm for division. They clearly show that the formulations of the algorithms for extraction – tacitly, i.e. without any other form of commentary – shape these calculation procedures as types of divisions. Based on what we have seen above, we can advance the hypothesis that the texts, like the executions, state a form of relation between extraction and division. We thus again find the interest that the actors manifested for this very question and its exploration with the help of the same working tools, as well as, now, also the algorithm texts.

Furthermore, in the translations of the two texts, I have underlined the terms and expressions that indicate how the root extractions are not real divisions. They show the modifications to the division algorithm

[3] [6] contains a complete, annotated translation of these texts.

through which the extractions have been cast in the divisions mould. These terms and expressions do, however, also demonstrate an interest for the relations between operations since they match each other from one text to another. This correspondence reveals how correlated modifications of the division algorithm lead to the extraction of square and cube roots respectively. The words in italics highlight how even the differences between these modifications are correlated from one algorithm to the other. The use, in the square root text, of an expression like "multiply once" instead of simply "multiply", which accentuates the parallel with the expression "multiply twice" in the corresponding statement of the cube root text, brings out the authors' wish to write the texts in relation to each other.

All these properties confirm what I have advanced above: alongside the work on the flows of calculation executing the operations on the calculating surface, we see emerging, through the formulations of the algorithm texts, a second facet in the modalities of exploring the relations between operations. Above, we encountered a specific practice using physical objects (rods and positions on the calculating surface). We discover now a specific way of working – and of expressing mathematical meanings – with certain elements that make up the texts themselves. These remarks provide us with tools to rigorously reconstitute the flows of calculation to which the extraction procedures for square and cube roots refer. The key hypothesis that the previous argument allows us to advance, a hypothesis that plays a key role in this reconstitution, states that the processes of execution highlight, or "write", the similarity between extractions and division on the calculating surface in the same way that they allowed the reading above of the opposition between multiplication and division. Therefore, we know that the first digit of the root (or "quotient") $a.10^n$ and then those that followed were placed successively in the upper position, while the number A whose root was being sought was positioned as "dividend". In the lower position, a number acting as "divisor" distinguished itself from the homonymous position of the division by the fact that its value had to be adjusted. The interpretation gives the flow of calculations reconstituted in Figure 2.

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	(Step)
			2	2	2	2	2	Upper
55225	55225	55225	55225	55225	15225	15225	15225	Middle
1	1	1	1	2	2	4	4	Lower

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8
			$a.10^n$	$a.10^n$	$a.10^n$	$a.10^n$	$a.10^n$
A	A	A	A	A	$A - a.a.10^{2n}$	$A-(a.10^n)^2$	$A-(a.10^n)^2$
1	10^2	10^{2n}	10^{2n}	$a.10^{2n}$	$a.10^{2n}$	$2a.10^{2n}$	$2a.10^{2n-1}$

Figure 2.

If we had read the texts with the sole intention of knowing how roots were extracted – as most historians have actually read them – we would have missed the *work* carried out to shape a set of relations between these operations as well as the *ways of working* developed in order to carry out this research (use of positions and the dynamic of the calculations, and formulation of the algorithm texts). Certainly, we would have convinced ourselves, once again, that the ideas applied are essentially identical to those used in the algorithms that some of us learned in our youth for extracting roots. But we would have missed out on what makes the difference between this latter algorithm and the one in *The Nine Chapters*. The reconstitution of the practice of writing algorithm texts, like the practice of calculation on the calculating surface (two facets of this specific way of doing mathematics that illustrate what I mean by "mathematical culture"), invites a different reading of the texts as flows of calculation and consequently allows us to grasp another facet of the actors' mathematical work that no other discourses express. I think this point clearly illustrates the link I stated between, on one hand, the description of the actors' "mathematical culture" and, on the other, a better understanding of their mathematical knowledge, as well as the questions they were pursuing.

Another clue confirms the conclusions that one can draw from this form of interpretation, which derives from attention being paid to the practices: it comes from the way in which these operations were prescribed in the algorithm texts. Indeed, the texts refer to the operand of a root extraction by the term 'dividend" and prescribe the operation, as appropriate, by the expressions: "one divides this by extraction of the square root" or "one divides this by extraction of the cube root". In other words, the prescription states, again without further ado, the same structure for all the operations, signalling that *chu* division was their foundation.

This is not all and, for us to go further, it will be useful to evoke the demonstrations that the commentator Liu Hui developed to establish the correctness of root extraction algorithms and, in particular, the diagram on which the proof is based in the case of the square root extraction. These demonstrations are the opportunity for Liu Hui to correlate the elementary steps of the extractions with those of *chu* division. Moreover, in order to develop the meaning of the steps in the extraction, the commentator introduces a diagram for the square root and blocks for the cube root. While the text of the commentary refers to these, there are no illustrations in the text and, here again, it is down to historians to reconstitute them. Figure 3 illustrates the reconstitution that

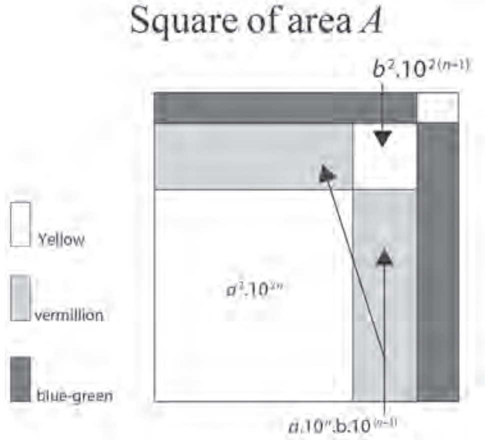

Figure 3.

historians all agree to propose for the visual aid relating to the square root upon which Liu Hui based the explanation of the meaning of the calculations. I reproduce the colours that the text of the commentary indicates – this is a common feature of the diagrams in the context of this culture. Furthermore, I add marks allowing the execution of the extraction to be linked to the figure. It is possible that the diagram used by the commentator contained characters performing the same function but we have no evidence of this. We will return later to this diagram, insofar as we will see that it plays a role in the structuring of an altogether broader set of operations.

Chu division as the foundation for a set of operations

To summarise the conclusions we have obtained thus far: we have encountered several characteristic features of a mathematical culture by concentrating on the practice of computation. Among these features, we have identified the use of positions on the calculating surface to establish links between the operations through the flows of calculation. The decimal positions of the place-value notation for numbers are a part of this landscape, insomuch as they constitute one of the types of position that the practice of computation brings into play. Their utilisation meshes with the use of algorithms operating uniformly on sequences of digits of the operands and producing, with regard to the operations of the division family, the results digit by digit. Furthermore, *chu* division has been shown to play a central role in this context. The combination of all of these features is found in two other subjects dealt with in *The Nine Chapters*. We will analyse them one after the other.

The first concerns the resolution of systems of linear equations, which are the subject of Chapter 8 in the book. The central algorithm describes, firstly, an initial layout of the data (i.e. the coefficients of the equation) on the calculating surface and, thus, easily allows the reconstitution as follows (in modern terms, the system given on the left corresponds to the inscription on the surface reconstituted on the right):

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \\ \end{array} \qquad \begin{array}{cccc} a_{n1} & \cdots & a_{21} & a_{11} \\ a_{n2} & \cdots & a_{22} & a_{12} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{nn} & \cdots & a_{2n} & a_{1n} \\ b_n & \cdots & b_2 & b_1 \end{array}$$

Thus, in the layout described in *The Nine Chapters*, each linear equation corresponds to a column and the coefficients attached to the same unknown are all placed in the same line. Here, again, the actors have developed a place-value notation for the system of equations. The algorithm itself corresponds to the Gauss elimination method. It operates as follows. Assuming that the upper terms in the two right-most columns are non-zero, the upper term in the right column multiplies the column immediately to its left, whereupon the upper term in this second column is eliminated by operating on these two columns. This sub-procedure is repeated until the following triangular system is obtained:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ 0 \quad + c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\ \cdot \\ \cdot \\ \cdot \\ 0 + 0 + \ldots + 0 + c_{nn}x_n = d_n \\ \end{array} \qquad \begin{array}{cccc} 0 & \ldots & 0 & a_{11} \\ 0 & \ldots & c_{22} & a_{12} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ c_{nn} & \ldots & c_{2n} & a_{1n} \\ d_n & \ldots & d_2 & b_1 \end{array}$$

The algorithm is concluded by determining x_n by a simple division then successively calculating the other unknowns in a similar manner. Note in passing that the division that produces x_n presents the dividend under the divisor. Here, too, a uniform algorithm meshes with a place-value notation of the system, since it determines the sequence of the unknowns by means of an iteration of the same sub-procedures, which deal with the positions in a uniform way. Positive and negative marks are introduced during the chapter to allow the operations to be completed in all cases and then to extend the set of systems that the algorithm can handle. Finally, let us note that the way the data are structured on the calculating surface is central to the operations the algorithm uses.

In the same way as before, the interpretation of the algorithm as identical to the Gauss elimination method is relevant but it only partially captures the mathematical knowledge developed. Indeed, the observation of the same elements as above (the terms employed to designate the operands, the algorithm texts and the calculation flows) highlights something else quite unexpected here. It appears that the constant terms in the equations are given the name "dividends", while the coefficients of the unknowns are described as forming "divisors in square" – this is, in my view, the meaning of the name of the algorithm (in Chinese *fang cheng*, "measures in square"). Finally, the central operation of eliminating the upper non-zero terms from the columns is prescribed as a "vertical *chu* division". It appears, once again, that the actors' work was not limited to determining an algorithm to produce the results. In addition, they further carried out a conceptual reflection on the relations between the operations, which led to conceiving the resolution of systems of linear operations as a generalised division, opposing a sequence of dividends to a square of divisors (here the dividends are also under the divisors), and articulating the forms of horizontal and vertical division [7]. Again, we find, on one hand, an interest for the structuring of a set of operations and, on the other, *chu* division as the foundation of this enlarged set. In this context, the positions seem, once more, to have served as the work tool for carrying out the exploration.

We have brought out, by means of the observation of aspects of the mathematical culture, a reflection by the actors on the operations and the relations between them. We notice that bringing this work programme to light, which no text appears to formulate explicitly, allows us to give meaning to a growing set of clues contained in the texts and to grasp a facet of the mathematical knowledge specific to the actors that have, up to this point, remained invisible. That the linear equation is conceived in this context as the opposition between a dividend and a set of divisors is actually only one aspect of a more general fact, as we will now see by turning to the second subject dealt with in *The Nine Chapters*, where positions and *chu* division also play a key role.

I will introduce this subject by showing how the description of facets of the mathematical culture in the context in which the text was written, with the restitution of the flows of calculation on the surface and the diagrams that can be deduced from clues in the text, provide essential tools for interpretation. The algorithm to understand is formulated following the problem that I represent in Figure 4 (by respecting the representation of the cardinal directions usual at that time).

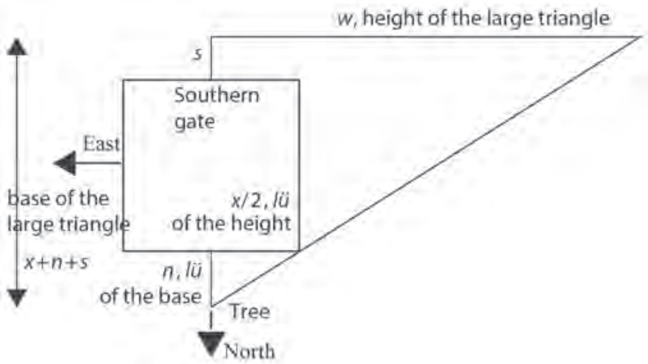

Figure 4.

It is a question of determining the (length of the) side of a square town, knowing that a person walking *s bu* (unit of distance) outside the southern gate then westwards *w bu* sees a tree situated *n bu* from the northern gate. The algorithm *The Nine Chapters* proposes is formulated as follows (I emphasise certain words in bold text):

> "One multiplies the quantity of *bu* walked to the west by the quantity of *bu* outside the northern gate, and one doubles this which gives the **dividend**. Adding together the quantities of *bu* outside the southern gate and the northern gate makes the **joined divisor**. And one **divides this by the extraction of the square root**, which gives the side of the square town."

The algorithm thus calculates two operands ("dividend" and "joined divisor") and prescribes the operation as a "square root extraction". What is the meaning of this operation? Actually, it can neither be a square root extraction, as this operation should only have one operand, nor can it be a division. Here, Liu Hui's commentary provides valuable clues for dealing with the conundrum. The commentator describes a graphical process that does not correspond to any illustration in the text and that I translate as a sequence of figures. In Figure 4, I have marked the height and the base of a large triangle. The term *lü* attached to the height and the base of a second triangle in the figure indicates the similarity of these two triangles. From this observation, Liu Hui draws the equality of the areas of the horizontal rectangle, with sides w and n, and the vertical rectangle, with sides $x/2$ and $n + x + s$ (see Figure 5).

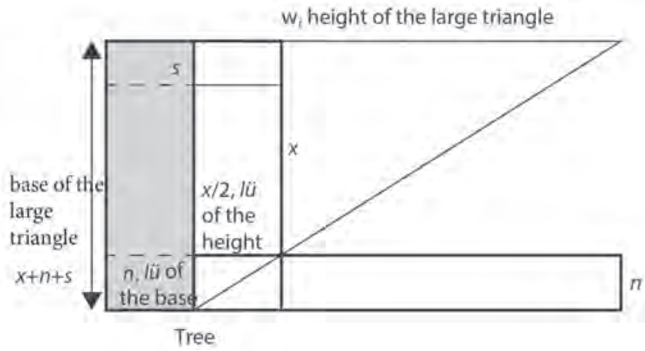

Figure 5.

Twice this area corresponds to what the algorithm calls the "dividend": Liu Hui interprets this as the area of the vertical rectangle to which one adds the grey rectangle. It corresponds to the rectangle in Figure 6a. Liu Hui finally interprets the calculation of the "joined divisor" by the prescription of joining the upper and lower rectangles, which produces the shaded rectangle in Figure 6b. Establishing this figure concludes his commentary.

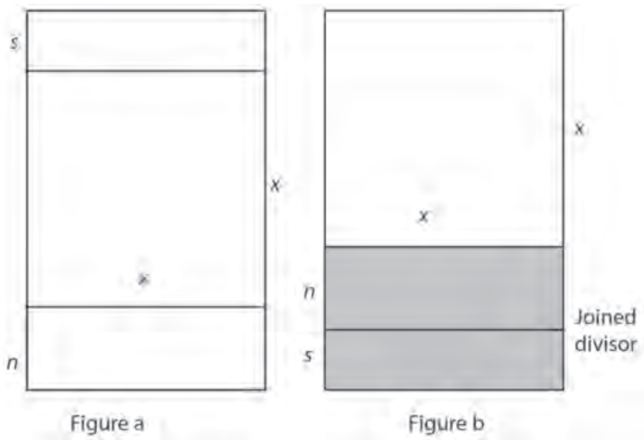

Figure 6.

The commentator thus interprets the operation used in the algorithm as the equation represented in modern terms as:

$$x^2 + (s + n)\,x = 2\,nw.$$

Why is its execution prescribed as a root extraction? The answer to this question is obtained by considering the demonstration Liu Hui formulated for the algorithm executing this last operation and, in particular, the diagram that he introduced to state the meaning of the operations in the algorithm, whose reconstitution is provided in Figure 3. The commentator interprets steps 1 to 6 of the algorithm (see Figure 2) as having the aim of subtracting the area of the square of side $a.10^n$ from A. If this square is removed from the figure, we are left with a gnomon, shown in Figure 7.

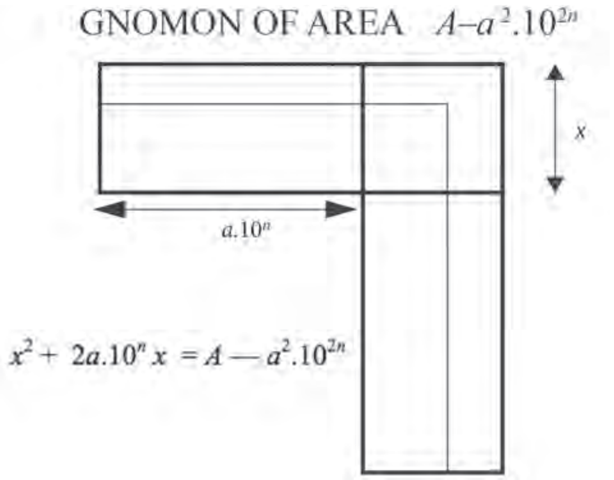

Figure 7.

By unfolding this gnomon, one obtains a figure comparable to that with which Liu Hui established the quadratic equation. Indeed, by omitting the first part of the root extraction algorithm (thus by removing from the result the digit of the root calculated to this point) and by commencing the algorithm at step 8, one solves the quadratic equation that writes this gnomon (or rectangle). Now, if we observe the configuration of the calculating surface at step 8, we notice that there are, at this point in the calculation, two terms that correspond exactly to the operands of the quadratic equation described by the algorithm, the interpretation of which is under consideration. Thus, the quadratic equation is an operation that derives from root extraction, in that the procedure that executes it is a subprocedure of the execution of the extraction: we understand at one and the same time how it is introduced and how it is executed in this context. Several consequences follow from this.

Firstly, we note that a large number of ingredients enter into the development of our interpretation: the demonstrations by which the commentator established the correctness of both the algorithm solving the problem given and the root extraction algorithm; and the reconstitutions of the diagrams and the flows of calculation on the calculating surface, based on knowledge of the practices at work in this context. We see that, in the context of a given way of "doing mathematics", the elementary practices (practices of diagrams, practices of computations, etc.) mesh with one another in a specific way.

Furthermore, we observe here, once again, that the operations for square root extraction and quadratic equations are linked by processes of calculation on the calculating surface, and especially by the way of managing the positions. Both in terms of the processes of calculation and the role played by diagrams, the relation is established in a different way from that which we have described for multiplication and division. Nevertheless, as above, the link between the operations is also expressed by the terminology chosen to designate the operands and the operations. All this explains, at one and the same time, the graphical means used to establish the equation and the fact that only two operands are identified for the quadratic equation (in the context of traditions that developed on this basis, the term in x^2 seems to have been identified only in the 11th century).

This last remark raises a crucial and particularly interesting question. The fact that the quadratic equation is only associated with two operands highlights a *correlation* between the ways of doing mathematics (here, in particular, the practice of computations on the surface linked to the establishment of relations between operations) and the concepts or, more broadly, the mathematical knowledge produced. This fact is, I think, wholly general and only a careful examination of "mathematical cultures" will allow us to explore it further. For me, this issue provides a fundamental reason to justify the interest in the diversity of ways of doing mathematics. What is at stake is understanding how mathematical knowledge is correlated to shared, collective ways of working. This is one of the new questions to which we are led and one which I hope historians of mathematics and mathematicians will consider jointly.

But there is more. If we return to the quadratic equation, we realise that the text of *The Nine Chapters* only contains the names of the operands and the formulation of the prescription. By methodically reconstituting the ways of working, we have been able to reveal a representation of the equation on the calculating surface and a process of execution, as well as a graphical representation essential to its establishment. In fact, all the quadratic equations established in Ancient Chinese sources correspond to the reading of gnomons or of rectangles in geometrical configurations in the same way. In other words, if we had not paid attention to concrete practices with physical objects, we would have missed key aspects of the ways in which the actors worked in this context with this mathematical object and the tools they forged for this purpose.

More important for our purposes, we would also have missed out on the work and the resources that the actors deployed to structure a series of operations. Yet we are now discovering the extent of the knowledge developed on this subject. We see that, in this context, both linear and quadratic equations were conceptualised as forms of division. In fact, many traditions that gained momentum by relying on the canons at the centre of this mathematical culture, be they in China, Korea or Japan, would develop knowledge about algebraic equations in this conceptual framework. And I show, in the complete version of the article to appear in the proceedings of the EMC, that it is again only one aspect of a much more general phenomenon.

Another mathematical culture in Ancient China and some issues at stake

Let us recapitulate what the observation of certain facets of a mathematical culture (in the main, the use of positions and the processes for calculation) has allowed us to do so far. We have relied on it for a more rigorous interpretation of the texts. We have also reconstituted ways of working with mathematical entities. Finally, we have grasped a body of knowledge that the actors had developed on the subject of the relations that link certain operations and the systematic study, which had been, until now, overlooked by historians. In this context, the operation of *chu* division has emerged as pivotal. I have approached all these aspects from the basis of a cluster of documents originating from Ancient China: the canons published with certain commentaries in the 7th century and used as textbooks in the official mathematics curriculum.

Recently, two other clusters of mathematical documents also originating from Ancient China have resurfaced and a quick observation of the way of doing mathematics they bear witness to allows us to raise some very interesting questions, both specific and general. I will only refer here to the first cluster of documents, directing the reader to the article published in the proceedings for the operation of the second.

I will speak, therefore, only about documents newly provided by archaeology. Since the 1970s, a growing number of tombs sealed in China in the last centuries before the Common Era have been excavated and archaeologists soon discovered that, in some of them, libraries had been interred among the funerary objects supposed to accompany the dead person in the afterlife. These documents provide fresh perspectives on the final centuries before the Common Era and have shaken up our knowledge of this period. During the Winter of 1983–1984, a first mathematical document, the size of a book, came to light among a series of writings of this type. Since then, excavations and the antiquities market have produced several other similar documents and we can expect new finds, all profoundly altering our understanding of the history of mathematics in China at the time. For the time being, only two of the mathematical texts discovered have been published completely (the first in 2001); for the others, we can only consult some extracts pending their full publication. The conclusions I propose are thus fragile and could be contradicted by new discoveries.

Of these documents, those we can study all seem to reflect the same way of practising mathematics: in the terms that I introduced above, they form the same cluster. Moreover, as far as we can see, the writings have several features in common with the canons and their commentaries. We can suppose then that all the documents had close historical ties, without, for the moment, being able to specify their exact nature. What is important for us is that these two clusters of documents also present significant differences to each other, which leads me to advance the hypothesis that these two clusters bear witness to different ways of doing mathematics, even if both present similarities. For my purposes, I will concentrate here on a set of similarities and key differences.

Firstly, like the canons and their commentaries, these documents contain no illustrations, being made up solely of Chinese characters and punctuation marks. They do, however, also refer to counting rods to represent numbers and to the practice of laying out numerical values away from the text. However, no traces have been detected of the use of a decimal place-value system for writing numbers. On the contrary, a certain number of clues gleaned from the operations suggest that the numbers were represented using a different number system.

This is the first of a series of facts that appear to indicate that the surface on which the calculations were carried out was the subject of a different practice. In fact, more generally, no reference is made to the use of a system of positions in the execution of the algorithms and nor do any of these writings use terms like “line”, “column” or “position”.

So much for the physical aspects of the practice of calculation. If we now turn to the operations, an initial fact is immediately striking: division seems to have been seen as a specific operation, different from all the others. A first clue for this is the fact that, while the other operations can all be prescribed by simple verbs, division is always, at least in this context, prescribed by complex expressions. In particular, the term *chu* alone cannot prescribe a division, contrary to what we have seen earlier for the other cluster of texts. And when it is encountered in isolation, it refers in fact to a subtraction. It seems then that one can perceive a change in the meaning of the verb *chu* and a change in the practice of division.

These recently discovered documents contain algorithms for square root extraction. But these procedures do not determine the roots decimal position by decimal position and do not seem to iterate sub-procedures on numbers written in a place-value form. Neither do they appear to present a relation to a process like that of division as we saw in the canons. More generally, no trace appears to reveal an interest for the relations between the operations.

Finally, none of the algorithms in which we have seen the close relation with division and the use of positions, such as the resolution of systems of linear or quadratic equations, appear, for the moment, among the subjects dealt with in these documents.

In conclusion, whether from the perspective of ways of working with the processes of calculation or from the perspective of knowledge or of the projects that actors formed, in these documents we have none of the elements from the constellation of facts described earlier. This suggests another issue of interest which is, in my opinion, wholly general. In fact, these new documents invite the thought that *The Nine Chapters* and the other canons bear witness to the emergence, no later than the 1st century of the Common Era, of two closely linked things: on one hand, a way of doing mathematics (more precisely a way of working with the processes of calculation and an interest in uniform algorithms) and, on the other, new knowledge, among which I include new ways of carrying out known operations, several new operations, a way of understanding the relations between these operations and a decimal place-value numbering system. Thus, at the same time, a way of working and a body of knowledge appear in concert.

When historians of mathematics have become interested in the activity of mathematics as such, they have, in general, studied, with a few exceptions I cannot develop here, the history of mathematical knowledge. Yet, the phenomena that I have mentioned above suggest that a history of ways of doing mathematics is also meaningful. What is more, these two dimensions (mathematical knowledge and mathematical practice) appear to constitute inseparable facets of the same reality. This is what we have seen for Ancient China, and I think it is the same everywhere and at all times.

I pose the conjecture that these two facets transformed themselves jointly. It is, without doubt, one of the fundamental reasons why the description of ways of doing could help in the interpretation of writings and allow a better understanding of the knowledge to which they attest. This close articulation between these two types of facts constitutes another reason why the history of mathematics should be interested in the description of mathematical cultures. After all, ways of doing mathematics do not appear from nowhere. They have been shaped and transformed by the actors during the process of exploring the problems that they sought to solve and the questions they pondered. Ways of doing mathematics represent one of the results of actors' research: mathematical work thus produces both knowledge and practices. This is, in any case, one of the principal motivations of my plea for the history of mathematics to take as a subject of study not only the knowledge but also the practices and ultimately the relations that exist between one and the other.

I have presented arguments on the value of studying the ways of doing mathematics by illustrating my arguments with examples taken from Ancient Chinese documents. Many other clusters of texts produced closer to us, even today, seem to me to call for the same analysis. I conclude this article with the wish that the general issues that I have formulated inspire discussion and research into the mathematical cultures in other periods and in other fields, and even cooperation between mathematicians and historians to address contemporary mathematical practices in this light. I am convinced that such cooperation would be fruitful for the historians and, who knows, could provide some interesting insights to today's mathematics.

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