

Solved and Unsolved Problems

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The world is continuous, but the mind is discrete.

David Mumford

The problem column in this issue is devoted to discrete mathematics. This beautiful and highly applicable area of mathematics deals with the study of discrete structures and phenomena. The structures studied in discrete mathematics consist of sequences of individual steps. This is in contrast to other areas of mathematics such as differential calculus, where the concept of a continuous process plays an integral role.

Discrete mathematics covers several subjects, among them the theory of sets and relations, mathematical logic, combinatorics and graph theory, as well as some aspects of number theory. Combinatorics and graph theory have a prominent place in the world of discrete mathematics.

It is worth mentioning that in our modern society, discrete models – and thus the techniques to study them – have a wide range of applicability. Apart from the fact that the notion of enumeration appears so naturally in our everyday life, another essential reason for the increased applicability of such models is their intimate connection to computers, which have become so deeply integrated into our culture.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

179. Let $p = p_1 p_2 \cdots p_n$ and $q = q_1 q_2 \cdots q_n$ be two permutations. We say that they are *colliding* if there exists at least one index i so that $|p_i - q_i| = 1$. For instance, 3241 and 1432 are colliding (choose $i = 3$ or $i = 4$), while 3421 and 1423 are not colliding. Let S be a set of *pairwise colliding* permutations of length n . Is it true that $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

180. Let us say that a word w over the alphabet $\{1, 2, \dots, n\}$ is *n-universal* if w contains all $n!$ permutations of the symbols $1, 2, \dots, n$ as a subword, not necessarily in consecutive positions. For instance, the word 121 is 2-universal as it contains both 12 and 21, while the word 1232123 is 3-universal. Let $n \geq 3$. Does an *n-universal* word of length $n^2 - 2n + 4$ exist?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

181. Given natural numbers m and n , let $[m]^n$ be the collection of all n -letter words, where each letter is taken from the alphabet $[m] = \{1, 2, \dots, m\}$. Given a word $w \in [m]^n$, a set $S \subseteq [n]$ and $i \in [m]$, let $w(S, i)$ be the word obtained from w by replacing the j^{th} letter with i for all $j \in S$. The Hales–Jewett theorem then says that for any natural numbers m and r , there exists a natural number n such that every r -colouring of $[m]^n$ contains a monochromatic combinatorial line, that is, a monochromatic set of the form $\{w(S, 1), w(S, 2), \dots, w(S, m)\}$ for some $S \subseteq [n]$. Show that for $m = 2$, it is always possible to take S to be an interval in this theorem, while for $m = 3$, this is not the case.

(David Conlon, Mathematical Institute, University of Oxford, Oxford, UK)

182. (A) Let A_1, A_2, \dots be finite sets, no two of which are disjoint. Must there exist a finite set F such that no two of $A_1 \cap F, A_2 \cap F, \dots$ are disjoint?
(B) What happens if all of the A_i are the same size?

(Imre Leader, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK)

183. The following is from the 2012 Green Chicken maths contest between Middlebury and Williams Colleges. A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \dots, 2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus, 30 and 1593 are connected by an edge as 3 divides each but 30 and 49 are not. The colouring number of a graph is the smallest number of colours needed so that each vertex is coloured *and* if two vertices are connected by an edge then those two vertices are not coloured the same. The Green Chicken says the colouring number of this graph is at most 9. Prove he is wrong and find the correct colouring number.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

184. There are n people at a party. They notice that for every two of them, the number of people at the party that they both know is *odd*. Prove that n is an odd number.¹

(Benny Sudakov, Department of Mathematics, ETH Zürich, Zürich, Switzerland)

II Open Problems: Two combinatorial problems by Endre Szemerédi

(Renyi Alfred Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary. This work was supported by the ERC-AdG. 321104 and OTKA NK 104183 grants.)

185* (Erdős' unit distance problem). In 1946, Erdős [7] published a short paper in the *American Mathematical Monthly*, in which he suggested a very natural modification of the Hopf-Pannwitz question. Let P be a set of n points in the plane. What happens if we want to determine or estimate $u(n)$, the largest number of unordered pairs $\{p, q\} \subset P$ such that p and q are at a fixed distance, which is not necessarily the largest distance between two elements of P ? Without loss of generality, we can assume that this distance is the *unit distance*. This explains why Erdős' question is usually referred to as the *unit distance problem*. That is,

$$u(n) = \max_{P \subset \mathbb{R}^2, |P|=n} |\{ \{p, q\} \subset P : |p - q| = 1 \}|.$$

(a) Using classical results of Fermat and Lagrange, Erdős showed that one can choose an integer $x \leq n/10$ that can be written as the sum of two squares in at least $n^{c/\log \log n}$ different ways, for a suitable constant $c > 0$. Thus, among the points of the $\sqrt{n} \times \sqrt{n}$ integer lattice, there are at least $(1/2)n^{1+c/\log \log n}$ pairs whose distance is \sqrt{x} . Scaling this point set by a factor of $1/\sqrt{x}$, we obtain a set of n points with at least $(1/2)n^{1+c/\log \log n}$, i.e. with a superlinear number of unit distance pairs.

Erdős proved that

$$n^{1+c_1/\log \log n} \leq u(n) \leq c_2 n^{3/2},$$

for some $c_1, c_2 > 0$, and he conjectured that the order of magnitude of $u(n)$ is roughly $n^{1+c/\log \log n}$. In spite of many efforts to improve on the upper bound, 70 years after the publication of the paper in *Monthly*, the best known upper bound is still only slightly better than the above estimate. Erdős' upper bound was first improved to $o(n^{3/2})$ by Józsa and Szemerédi [13], and ten years later to $O(n^{13/9})$ by Beck and Spencer [2]. In a joint paper with Spencer and Trotter [17], I proved $u(n) = O(n^{4/3})$, which is currently the best known result.

(b) We say that n points in the plane are in *convex position* if they form the vertex set of a convex polygon.

Erdős and Moser [9] conjectured that the number of unit distances, $u_{\text{conv}}(n)$, among n points in convex position in the plane satisfies $u_{\text{conv}}(n) = \frac{2}{3}n + O(1)$. They were wrong: Edelsbrunner and P. Hajnal [6] exhibited an example with $2n - 7$ unit distance pairs, for every $n \geq 7$. It is widely believed that $u_{\text{conv}}(n) = O(n)$ and perhaps even $u_{\text{conv}}(n) = 2n + O(1)$. The best known upper bound is due to Füredi [11], who proved by a forbidden submatrix argument that $u_{\text{conv}}(n) = O(n \log n)$. A very short and elegant inductive argument for the same bound can be found in [4].

Erdős suggested a beautiful approach to prove that $u_{\text{conv}}(n)$ grows at most linearly with n . He conjectured that every convex n -gon in the plane has a vertex from which there are no $k + 1$ other vertices at the same distance. Originally, he believed that this is also true with $k = 2$ but Danzer constructed a series of counterexamples. Later, Fishburn and Reeds [10] even found convex polygons whose unit distance graphs are 3-regular, that is, for each vertex there are precisely three others at unit distance. If Erdős' latter conjecture is true for some integer k then this immediately implies by induction that $u_{\text{conv}}(n) < kn$.

186* Some problems on Sidon sets. $A \subset [1, n]$ is called a *Sidon sequence* if all sums $a + a', a, a' \in A$ are different.

(a) Prove or disprove that

$$|A| < n^{1/2} + O(1).$$

The best result is due to B. Lindström [15], who proved that $|A| < n^{1/2} + n^{1/4}$.

(b) Prove or disprove that

$$|A| < n^{1/2} + o(n^{1/4}).$$

$A = \{a_1 < a_2 < a_3 < \dots\}$ is an *infinite Sidon sequence* if all sums $a + a', a, a' \in A$ are distinct. $A(n)$ denotes the number of elements of A in $[1, n]$.

(c) For every $\epsilon > 0$, construct (the construction can be a random construction) an infinite Sidon sequence with $A(n) > n^{1/2-\epsilon}$.

The best bound is due to I. Ruzsa [16] and J. Cilleruelo [5]. They constructed an infinite Sidon sequence with $A(n) > n^{\sqrt{2}-1+o(1)}$.

I. Ruzsa's construction was a random one and J. Cilleruelo's construction was a deterministic one.

$A_h = \{a_1 < a_2 < a_3 < \dots\}$ is a sequence such that all sums $a_1 + a_2 + \dots + a_h, a_1, a_2, \dots, a_h \in A$ are distinct. $A_h(n)$ is the number of elements of A_h in $[1, n]$.

(d) Prove or disprove that for $h = 3$,

$$A_3(n) = o(n^{1/3}).$$

Here, there are no results. $A_3(n) = O(n^{1/3})$ is trivial.

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III Solutions

171. Prove that every integer can be written in infinitely many ways in the form

$$\pm 1^2 \pm 3^2 \pm 5^2 \pm \dots \pm (2k+1)^2$$

for some choices of signs + and –.

(Dorin Andrica, Babeş Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. The proof uses induction by step 16. In this respect, we note that we have, for any positive integer m , the identity

$$16 = (2m - 1)^2 - (2m + 1)^2 - (2m + 3)^2 + (2m + 5)^2 \quad (1)$$

and the representations

$$\begin{aligned} 0 &= -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2, \\ 1 &= 1^2, \\ 2 &= 1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2, \\ 3 &= 1^2 - 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 - 15^2 - 17^2 - 19^2 + 21^2, \\ 4 &= -1^2 - 3^2 - 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 - 17^2 + 19^2, \\ 5 &= 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 - 17^2 - 19^2 - 21^2 \\ &\quad - 23^2 - 25^2 + 27^2 + 29^2, \\ 6 &= -1^2 - 3^2 + 5^2 - 7^2 - 9^2 + 11^2, \\ 7 &= 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 - 19^2 + 21^2 \\ &\quad - 23^2 - 25^2 - 27^2 + 29^2, \\ 8 &= -1^2 + 3^2, \\ 9 &= -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 - 17^2 + 19^2 - \\ &\quad - 21^2 - 23^2 - 25^2 - 27^2 + 29^2 + 31^2 + 33^2, \\ 10 &= 1^2 + 3^2, \\ 11 &= -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 + 17^2 - 19^2 - \\ &\quad - 21^2 + 23^2 + 25^2, \\ 12 &= -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 + 15^2 + 17^2, \\ 13 &= -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 - 15^2 + 17^2, \\ 14 &= -1^2 - 3^2 - 5^2 + 7^2, \\ 15 &= -1^2 - 3^2 + 5^2. \end{aligned}$$

For example, to write 16 in this form, we use the representation of 0 and we consider $m = 9$ in identity (1) to get $16 = 17^2 - 19^2 - 21^2 + 23^2$. We obtain

$$\begin{aligned} 16 &= 0 + 16 \\ &= -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 + 17^2 \\ &\quad - 19^2 - 21^2 + 23^2. \end{aligned}$$

To show that there are infinitely many such representations, we observe that, from (1), we have $16 = (2m + 7)^2 - (2m + 9)^2 - (2m + 11)^2 + (2m + 13)^2$. Hence, for any positive integer m , the following identity holds:

$$\begin{aligned} 0 &= (2m - 1)^2 - (2m + 1)^2 - (2m + 3)^2 + (2m + 5)^2 - (2m + 7)^2 \\ &\quad + (2m + 9)^2 + (2m + 11)^2 - (2m + 13)^2. \end{aligned}$$

In this way, we can add 0 to a representation for a suitable value of m to get a new representation and then continue. \square

Also solved by José Hernández Santiago (Morelia, Michoacán, Mexico) and Alexander Vauth (Lübbecke, Germany)

172. Show that, for every integer $n \geq 1$ and every real number $a \geq 1$, one has

$$\frac{1}{2n} \leq \frac{1}{n^{a+1}} \sum_{k=1}^n k^a - \frac{1}{a+1} < \frac{1}{2n} \left(1 + \frac{1}{2n}\right)^a.$$

(László Tóth, University of Pécs, Hungary)

Solution by the proposer. We prove by induction on n . For $n = 1$, we have

$$\frac{1}{2} \leq 1 - \frac{1}{a+1} < \frac{1}{2} \left(\frac{3}{2}\right)^a.$$

Here, the first inequality is equivalent to $a \geq 1$, which holds true by the condition. For the second one, if $1 \leq a < 3$ then $1 - \frac{1}{a+1} < \frac{3}{4} \leq \frac{1}{2} \left(\frac{3}{2}\right)^a$; if $a \geq 3$ then $1 - \frac{1}{a+1} < 1 < \frac{1}{2} \left(\frac{3}{2}\right)^a$.

Let $S_a(n) = 1^a + 2^a + \dots + n^a$ and assume that the inequalities hold true for n , that is,

$$\frac{1}{2}n^a + \frac{1}{a+1}n^{a+1} \leq S_a(n) < \frac{1}{2} \left(n + \frac{1}{2}\right)^a + \frac{1}{a+1}n^{a+1}, \quad (2)$$

and prove (2) for $n + 1$.

Adding $(n + 1)^a$ to (2), we get

$$\begin{aligned} \frac{1}{2}n^a + \frac{1}{a+1}n^{a+1} + (n + 1)^a &\leq S_a(n + 1) \\ &< \frac{1}{2} \left(n + \frac{1}{2}\right)^a + \frac{1}{a+1}n^{a+1} + (n + 1)^a. \end{aligned} \quad (3)$$

Applying the inequalities

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(t) dt \leq \frac{f(x) + f(y)}{2}$$

for the convex function $f(t) = t^a$ with $a \geq 1$ and for $x = n, y = n + 1$, we deduce

$$\left(n + \frac{1}{2}\right)^a \leq \frac{1}{a+1} \left((n + 1)^{a+1} - n^{a+1}\right) \leq \frac{n^a + (n + 1)^a}{2}. \quad (4)$$

Now, by the second inequality of (4), we obtain

$$\frac{1}{a+1}(n + 1)^{a+1} - \frac{1}{2}(n + 1)^a \leq \frac{1}{2}n^a + \frac{1}{a+1}n^{a+1},$$

that is,

$$\frac{1}{a+1}(n + 1)^{a+1} + \frac{1}{2}(n + 1)^a \leq \frac{1}{2}n^a + \frac{1}{a+1}n^{a+1} + (n + 1)^a. \quad (5)$$

By (5) and (3), we deduce

$$\frac{1}{2}(n+1)^a + \frac{1}{a+1}(n+1)^{a+1} \leq S_a(n+1). \quad (6)$$

On the other hand, the first inequality of (4) gives

$$\frac{1}{a+1}n^{a+1} \leq \frac{1}{a+1}(n+1)^{a+1} - \left(n + \frac{1}{2}\right)^a \quad (7)$$

and, by using the well known inequality

$$\left(\frac{x+y}{2}\right)^a < \frac{x^a + y^a}{2}, \quad (a > 1, x \neq y)$$

for $x = n + \frac{1}{2}$, $y = n + \frac{3}{2}$, we have

$$(n+1)^a < \frac{1}{2}\left(n + \frac{1}{2}\right)^a + \frac{1}{2}\left(n + \frac{3}{2}\right)^a. \quad (8)$$

Now, by summing the inequalities (7) and (8),

$$\frac{1}{2}\left(n + \frac{1}{2}\right)^a + \frac{1}{a+1}n^{a+1} + (n+1)^a < \frac{1}{2}\left(n + \frac{3}{2}\right)^a + \frac{1}{a+1}(n+1)^{a+1} \quad (9)$$

and finally, by (9) and (3),

$$S_a(n+1) < \frac{1}{2}\left(n + \frac{3}{2}\right)^a + \frac{1}{a+1}(n+1)^{a+1}. \quad (10)$$

Taking into account inequalities (6) and (10), we conclude that (2) holds true for $n+1$ and the proof is complete. \square

Remarks. 1. The equality holds if and only if $a = 1$ and $n \geq 1$ is arbitrary.

2. We deduce by these inequalities the following well known results:

$$\lim_{n \rightarrow \infty} \frac{1^a + 2^a + \dots + n^a}{n^{a+1}} = \frac{1}{a+1},$$

$$\lim_{n \rightarrow \infty} n \left(\frac{1^a + 2^a + \dots + n^a}{n^{a+1}} - \frac{1}{a+1} \right) = \frac{1}{2},$$

valid for every fixed real $a \geq 1$.

Also solved by Mihály Bencze (Brasov, Romania) and Panagiotis T. Krasopoulos (Athens, Greece)

173. Let $c_n(k)$ denote the Ramanujan sum, defined as the sum of k th powers of the primitive n th roots of unity. Show that, for any integers n, k, a with $n \geq 1$,

$$\sum_{d|n} c_d(k) a^{n/d} \equiv 0 \pmod{n}.$$

(László Tóth, University of Pécs, Hungary)

Solution by the proposer. The proof is based on the congruence

$$M_n(a) := \sum_{d|n} \mu(d) a^{n/d} \equiv 0 \pmod{n}, \quad (11)$$

represented several times in the literature (see, for example, [1] and [2]), and on Hölder's relation,

$$c_n(k) = \sum_{\delta|(n,k)} \delta \mu(n/\delta).$$

We obtain

$$R_n(k, a) := \sum_{d|n} c_d(k) a^{n/d} = \sum_{d|n} \left(\sum_{\delta|(d,k)} \delta \mu(d/\delta) \right) a^{n/d},$$

where, by denoting $k = \delta a$, $d = \delta b$, $n = dj$ and regrouping the terms,

$$R_n(k, a) = \sum_{\substack{\delta b=jn \\ \delta a=k}} \delta \mu(b) a^j = \sum_{\substack{\delta m=n \\ \delta a=k}} \delta \sum_{b=jm} \mu(b) a^j = \sum_{\delta|(n,k)} \delta M_{n/\delta}(a).$$

We have from (11) that, for any δ , $M_{n/\delta}(a)$ is a multiple of n/δ , hence $\delta M_{n/\delta}(a)$ is a multiple of n . This shows that $R_n(k, a)$ is a multiple of n . \square

Remarks 1. If $k = 0$ then $c_n(0) = \varphi(n)$ is Euler's totient function and we have, as a consequence,

$$\sum_{d|n} \varphi(d) a^{n/d} \equiv 0 \pmod{n},$$

which is also known in the literature.

2. For $k = 1$, one has $c_n(1) = \mu(n)$, the Möbius function, and the given congruence reduces to (11).

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Also solved by Mihály Bencze (Brasov, Romania) and Sotirios E. Louridas (Athens, Greece)

174. Prove, disprove or conjecture:

- (1) There are infinitely many primes with at least one 7 in their decimal expansion.
- (2) There are infinitely many primes where 7 occurs at least 2017 times in their decimal expansion.
- (3) There are infinitely many primes where at most one-quarter of the digits in their decimal expansion are 7s.
- (4) There are infinitely many primes where at most half the digits in their decimal expansion are 7s.
- (5) There are infinitely many primes where 7 does not occur in their decimal expansion.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

Solution by the proposer. (1) This follows from Dirichlet's Theorem of Primes (if a and b are relatively prime then there are infinitely many primes congruent to a modulo b), as 7 and 10 are relatively prime. The same is true for part (2).

We tackle part (4) first as it is easier than (3). Assume it is not true. Let us count how many numbers in $[10^k, 10^{k+1})$ have at least half their digits as 7s and, if k is large, there will be no primes in this interval with at most half their digits as 7s (by assumption). How many numbers are there? For each $\ell \in [k/2, k]$, we have $\binom{k}{\ell}$ ways to choose which ℓ of k digits are 7s and thus the number of such numbers is

$$\sum_{\ell=k/2}^k \binom{k}{\ell} 1^\ell 9^{k-\ell} \leq \frac{k}{2} \binom{k}{k/2} 9^{k/2}$$

because the largest binomial coefficient is in the middle. While we could use Stirling, note $\binom{k}{k/2} < (1+1)^k = 2^k$ by the binomial theorem. Thus, the number of numbers in $[10^k, 10^{k+1})$ with at least half their digits as 7s is at most $k2^k 9^{k/2} = k \cdot 6^k$; as there are more than 10^k such numbers, we see the percentage of numbers that have at least half their digits as 7s is at most $k6^k/10^k = k(6/10)^k$, which tends to zero VERY rapidly. By Chebyshev's Theorem we have that

there are at least $.9\pi(10^{k+1}) - 1.1\pi(10^k)$ primes in this interval, or at least $10^{k-2}/k$ primes. Thus, even if every number with at least half its digits as 7s were prime, there wouldn't be enough such numbers to account for all the primes in $[10^k, 10^{k+1})$. Thus, there are infinitely many primes with at most half their digits as 7s. It is not unreasonable to expect that a typical large prime has about 10% of its digits as 7s. *Do you expect there to be infinitely many primes where there are at most $c\%$ of the digits as 7s, where c is any number strictly less than $1/7$?*

One can argue similarly for (3) but it is a little more involved. I found it easiest to break the counting into primes with between $k/4$ and $k/3$ of their digits as 7s and then $k/3$ and $k/2$ of their digits as 7s (by (4), we don't need to worry about more than $k/2$ of their digits as 7s or we could just look directly at $k/3$ to k of their digits as 7s). The proof follows from estimating the sums – Stirling was useful for $k/4$ to $k/3$. Let's analyse from $k/3$ to $k/2$. Arguing as above, we have

$$\sum_{\ell=k/3}^{k/2} \binom{k}{\ell} 1^\ell 9^{k-\ell} \leq \frac{k}{6} \binom{k}{k/2} 9^{2k/3} \leq \frac{k}{6} 2^k (9^{2/3})^k \leq k(2 \cdot 9^{2/3})^k.$$

As $2 \cdot 9^{2/3} \approx 8.65 < 10$, the number of such numbers tends to zero so rapidly that, arguing as in part (4), there just aren't enough of these numbers to matter. We are thus left with $\ell \in [k/4, k/3]$:

$$\sum_{\ell=k/4}^{k/3} \binom{k}{\ell} 1^\ell 9^{k-\ell} \leq \frac{k}{12} \binom{k}{k/3} 9^{3k/4}.$$

If we used $\binom{k}{k/3} \leq 2^k$, we would find the above is at most $k(2 \cdot 9^{3/4})^k$ but $2 \cdot 9^{3/4} \approx 10.39 > 10$; this is why we must be more careful and why we have to split up into different ranges. By Stirling,

$$\begin{aligned} \binom{k}{k/3} &\sim \frac{k^k e^{-k} \sqrt{2\pi k}}{(k/3)^{k/3} e^{-k/3} \sqrt{2\pi k/3} \cdot (2k/3)^{2k/3} e^{-2k/3} \sqrt{2\pi 2k/3}} \\ &\ll \frac{1}{(1/3)^{k/3} \cdot (2/3)^{2k/3} \sqrt{k}} \\ &\leq \left(\frac{3}{2^{2/3}}\right)^k. \end{aligned}$$

Substituting this in above gives

$$\sum_{\ell=k/4}^{k/3} \binom{k}{\ell} 1^\ell 9^{k-\ell} \ll k \cdot \left(\frac{3}{2^{2/3}}\right)^k 9^{3k/4} \leq k \cdot \left(\frac{3 \cdot 9^{3/4}}{2^{2/3}}\right)^k.$$

As $3 \cdot 9^{3/4} / 2^{2/3} \approx 9.82 < 10$, arguing as before, we see that there are negligibly many numbers of this form. I really like this problem, as it highlights how careful we must be. We just need to get a number less than 10, so we keep splitting things up into different regions and using different estimates in each. We always replace the binomial coefficients with their largest value in the interval (which is at the right end point for $\ell \leq k/2$) and the $9^{k-\ell}$ term with its largest value (which is at the left end point for $\ell \leq k/2$). It would be interesting to do a more careful analysis and not bound things so crudely but this is what we number theorists do whenever possible: arguing as crudely as possible to get the required result.

Finally, part (5) is interesting. It's natural to conjecture that there are infinitely many. The following is a very common heuristic. Assume the two events are independent, namely having no 7s and being a prime. Let us label all such numbers a_1, a_2, a_3, \dots . The probability a number x is prime is essentially $1/\log x$, thus the expected number of numbers at most x that are prime and 7-free is $\sum_{a_i \leq x} 1/\log a_i$. We break this into sums of $a_i \in [10^k, 10^{k+1})$. There are 9^k numbers in this interval that are 7-free. We obtain an upper bound for the sum

by replacing each a_i with 10^k and a lower bound by replacing with 10^{k+1} . This yields

$$\sum_{k=1}^K \frac{9^k}{(k+1)\log 10} \leq \sum_{a_i \leq 10^{K+1}} \frac{1}{\log a_i} \leq \sum_{k=1}^K \frac{9^k}{k \log 10}.$$

Both the upper and lower bounds clearly tend to infinity with K , though much more slowly than $\pi(10^{K+1}) \approx 10^{K+1}/(K+1)\log 10$. As an aside, there are some sequences that are so sparse that we do not expect infinitely many primes. The standard example is the Fermat numbers: $F_n = 2^{2^n} + 1$. It is conjectured that only the first four are prime; see, for example, tinyurl.com/yarhbtu3. Using $a_n = 2^{2^n} + 1 \approx 2^{2^n}$, we find that the expected number of prime Fermat numbers is about

$$\sum_{n=0}^{\infty} \frac{1}{\log 2^{2^n}} \approx \sum_{n=0}^{\infty} \frac{1}{2^n \log 2} \approx \frac{2}{\log 2} \approx 3.$$

Returning to the problem at hand, it was recently successfully resolved by James Maynard; see his arXiv post "Primes with restricted digits", available at <https://arxiv.org/pdf/1604.01041v1>, where he shows there are infinitely many primes base 10 omitting any given digit. \square

Also solved by Mihály Bencze (Brasov, Romania), Cristinel Mortici (Targoviste, Romania) and Socratis Varelogiannis (National Technical University of Athens, Greece)

175. Show that there is an infinite sequence of primes $p_1 < p_2 < p_3 < \dots$ such that p_2 is formed by appending a number in front of p_1 , p_3 is formed by appending a number in front of p_2 and so on. For example, we could have $p_1 = 3, p_2 = 13, p_3 = 313, p_4 = 3313, p_5 = 13313, \dots$. Of course, you might have to add more than one digit at a time. Find a bound on how many digits you need to add to ensure it can be done.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

Solution by the proposer. One way to solve this problem is to use Dirichlet's Theorem for Primes in Arithmetic Progression, which states that if a and m are relatively prime then there are infinitely many primes congruent to a modulo m . Start with any prime number, and call that p_1 , and define the function $g(n)$ to be the number of digits of n . By Dirichlet's theorem, since p_1 and $10^{g(p_1)}$ are relatively prime, there are infinitely many primes congruent to p_1 modulo $10^{g(p_1)}$; note that all of these primes will have their final $g(p_1)$ digits as p_1 , and thus are constructed by appending digits in front of p_1 . For definiteness, take the smallest such prime and call that p_2 . We continue by induction. If we have formed p_m then p_{m+1} is obtained by applying Dirichlet's result to the pair $p_m, 10^{g(p_m)}$.

Unfortunately, as usually stated, Dirichlet's theorem is not constructive; it just states that there are infinitely many primes but says nothing about how far we must go before we find the first such prime. Fortunately, with a bit more work, one can find upper bounds on how far we must search. Interestingly, however, what we need is the *second* smallest prime in arithmetic progressions, and thus many of the results in the literature are not directly applicable. If we wish to use them, however, we can easily modify our work. Start off with a prime p_1 , append a 1 to the front of it and then construct p_2 by choosing the first prime congruent to $10^{g(p_1)} + p_1$ modulo $10^{g(p_1)+1}$, and so on. It is conjectured that the first prime congruent to a modulo m can be found by going up to $C_\epsilon m^{1+\epsilon}$ (where, for each $\epsilon > 0$, there is some C_ϵ) but this is far from known. Linnik proved in 1944 that there are c and L such that the first prime is found before cm^L , though he didn't

provide a value for L . The best current value is $L = 5$, which is due to Xylouris.

As an aside, this problem bears some similarity to searches for Cunningham chains, which are sequences of primes with specific relations between terms. A Cunningham chain of the first kind is a set of primes where $p_n = 2p_{n-1} + 1$ (the second kind is $p_n = 2p_{n-1} - 1$). It is believed that there are Cunningham chains of arbitrarily long length and this follows from standard conjectures (the world record of either is 19, which is due to Wroblewski from 2014). \square

Also solved by Mihály Bencze (Brasov, Romania) and Sotirios E. Louridas (Athens, Greece)

176. Consider all pairs of integers x, y with the property that $xy - 1$ is divisible by the prime number 2017. If three such integral pairs lie on a straight line on the xy -plane, show that both the vertical distance and the horizontal distance of at least two of such three integral pairs are divisible by 2017.

(W. S. Cheung, Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer. By assumption, there are real numbers $a, b, c \in \mathbb{R}$ with $(a, b, c) = 1$ and integers $k_i, i = 1, 2, 3$, such that

$$ax_i + by_i = c \tag{1}$$

and

$$x_i y_i = 1 + k_i \cdot 2017 \tag{2}$$

for all $i = 1, 2, 3$. Without loss of generality, we may assume that $2017 \nmid b$. By (1), we have

$$a(x_1 - x_i) = b(y_i - y_1), \quad i = 1, 2, 3.$$

By (2),

$$x_i y_i = x_1 y_1 + (k_i - k_1) \cdot 2017, \quad i = 1, 2, 3.$$

Hence,

$$\begin{aligned} (x_1 - x_i)by_i &= bx_1y_i - bx_iy_i \\ &= bx_1(y_i - y_1) - b(k_i - k_1) \cdot 2017 \\ &= ax_1(x_1 - x_i) - b(k_i - k_1) \cdot 2017 \end{aligned}$$

and so we have

$$(x_1 - x_i)(ax_1 - by_i) = b(k_i - k_1) \cdot 2017.$$

Hence,

$$2017 \mid (x_1 - x_i)(ax_1 - by_i). \tag{3}$$

Observe that this forces that at least one of

$$2017 \mid (x_1 - x_2), \quad 2017 \mid (x_1 - x_3) \quad \text{and} \quad 2017 \mid (y_2 - y_3)$$

should hold. In fact, if $2017 \nmid (x_1 - x_2)$ and $2017 \nmid (x_1 - x_3)$, by (3), we must have

$$2017 \mid (ax_1 - by_2), \quad 2017 \mid (ax_1 - by_3),$$

and so

$$2017 \mid b(y_2 - y_3).$$

Since $2017 \nmid b$, we have $2017 \mid (y_2 - y_3)$.

Take, for example, $2017 \mid (y_2 - y_3)$. Then, by

$$2017 \mid x_2(y_2 - y_3), \quad 2017 \mid (1 - x_2y_2), \quad 2017 \mid (x_3y_3 - 1), \tag{4}$$

we have

$$2017 \mid (x_3 - x_2)y_3.$$

By (4), $2017 \nmid y_3$, so we have $2017 \mid (x_2 - x_3)$. Hence, both the vertical and horizontal distances of (x_2, y_2) and (x_3, y_3) are divisible by 2017. The remaining cases can be proven analogously. \square

Also solved by Cristinel Mortici (Targoviste, Romania) and Panagiotis T. Krasopoulos (Athens, Greece)

Remark 1. The following much shorter solution to Problem 164 (Newsletter, March 2017, Issue 103) was provided by Panagiotis T. Krasopoulos (Greece), Hans J. Munkholm (Denmark) and Ellen S. Munkholm (Denmark).

Any power of 2015, say $P = 2015^n$, has the form $P = 5k$ with k a positive integer. Therefore

$$P = 5k = \frac{k^2}{k} \cdot \frac{2^2 + 1^2}{2 - 1} = \frac{(2k)^2 + k^2}{2k - k}.$$

Remark 2. Problems 163, 164, 166 and 167 (Newsletter, March 2017, Issue 103) were also solved by Dimitrios Koukakis (Greece).

We would like you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions either by ordinary mail to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, or by email to michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Fundamentals of Mathematical Analysis*.