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Impulsive Control for Stability of Volterra Functional Differential Equations

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Abstract. Consider the system of Volterra functional differential equations with nonlinear impulsive perturbations of the form

$$\begin{aligned} x'(t) &= F(t, x(\cdot)), \qquad && t > t^*,\ t \neq t_k,\ x \in \mathbb{R}^n \\ \Delta x &= I_k(t, x(t^-)), \qquad && t = t_k,\ k \in \mathbb{N}. \end{aligned}$$

Criteria on asymptotic stability are established for the above system using Lyapunov like functions with Razumikhin techniques or Lyapunov like functionals. It is shown that impulses given in the second equation do contribute to yield stability properties even when the underlying differential equation system does not enjoy any (or same) stability behavior. Some examples are also discussed to illustrate the results.

Keywords: *Impulsive control, stability, functional differential equation*

MSC 2000: 34D20, 34K20

1. Introduction

There are numerous examples of evolutionary systems which at certain instants in time are subjected to rapid changes. In the simulations of such processes it is frequently convenient and valid to neglect the durations of the rapid changes and to assume that the changes can be represented by state jumps. Appropriate mathematical models for processes of the type described above are so-called systems with impulsive effects [5, 19, 21]. Significant progress has been made in the theory of systems of impulsive differential equations in recent years [1, 2, 4, 5–7, 10, 21, 23, 26, 27, 29]. However, the corresponding theory for impulsive

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functional differential equations has yet not been fully developed. There are some difficulties one must face in developing the corresponding theory of impulsive functional differential equations. For example, in the classical theory of functional differential equations, the fact that the continuity of a function $x(t)$ in $\mathbb{R}^n$ implies the continuity of the functional x_t or $x(\cdot)$ in space C plays a key role in establishing the existence of solutions of functional differential equations [3, 13, 15] (for the symbol x_t, see [13], and for $x(\cdot)$, see the Section 2 or [9, 14]). However, if a function $x(t)$ is piecewise continuous, which is typical for solutions of impulsive differential equations, then the functional x_t or $x(\cdot)$ need not be piecewise continuous [2, 7]. In fact, it can be discontinuous everywhere. The same case appears in the Lyapunov functional $V(t, x_t)$ or $V(t, x(\cdot))$ for establishing stability results [5, 18, 26, 27]. Thus, even if $F(t, \phi)$ is continuous in its two variables, one cannot, in general, say anything about the composition function $F(t, x_t)$ or $F(t, x(\cdot))$ when $x(t)$ is piecewise continuous. The same case also appears in $V(t, x_t)$ or $V(t, x(\cdot))$. Therefore, the study of impulsive differential equations is more difficult than that of non-impulsive differential equations. Recently, existence and uniqueness results for impulsive functional differential equations have been presented in [7, 17, 18, 24, 25]. The study on stability theory of impulsive functional differential equations has also tended to focus on special classes of problems such as linear impulsive delay differential equations or delay differential-difference equations such as $x'(t) = f(t, x(t), x(t-\tau))$ together with impulses [1, 4, 10, 20, 29, 30]. As a result, little attention is ever made about stability theory of impulsive functional differential equations in more general form [18, 27]. In particular, the study of the problems of impulsive control for stability, which is an important investigation area for impulsive differential equations, remain neglected even for impulsive ordinary differential equations. Here, the impulsive control for stability or, say, impulsive stabilization is presented in the sense that stability properties are caused by impulsive effects even when the corresponding systems without impulses does not enjoy any (or the same) stability behavior [18, 26]. In this paper, we shall establish some criteria on (uniform) asymptotic stability for impulsive Volterra functional differential equations using Lyapunov like functions with Razumikhin techniques or Lyapunov like functionals. The results obtained for such equations show that impulses do contribute to yield stability properties even when the underlying continuous system does not enjoy any (or the same) stability behavior.

Recall that during the past decades the stability theory of finite and infinite delay functional differential equations based on Lyapunov's direct method has received much attention. The earliest results on Lyapunov's direct method for such equations tended to be patterned on those for ordinary differential equations with the norm in $\mathbb{R}^n$ replaced by the supremum norm in the continuous functions space C (cf. Krasovskii [16]). Stimulated by the applications of Krasovskii's results, two different directions have taken shape. One is to improve the conditions of Krasovskii's theorems, which is mainly directed toward finding a good formulation for a replacement of the boundedness of vector fields. The other one is to consider Lyapunov functions on $\mathbb{R} \times \mathbb{R}^n$ taking the place of Lyapunov functionals on $\mathbb{R} \times C$. Such a method was due to Razumikhin [15], which does not need the boundedness of vector fields and is somewhat more convenient in applications. Razumikhin techniques including its various variation has also been widely used in the treatment of stability for various functional differential equations (cf. [8, 9, 11, 12, 14, 22, 28]). It should be noted that when applying Lyapunov functional $V(t, x_t)$ or $V(t, x(\cdot))$ in its general form to the stability analysis of impulse functional differential equations, one must face some new difficulties as described above. In addition, the question arises as to one how to fix the properties of $V(t, x_t)$ or $V(t, x(\cdot))$ along the solutions of such systems at certain instants in time when state jumps occur. To overcome the difficulties mentioned before which are created actually by the special features possessed in impulsive functional differential equations, when using a Lyapunov like function $V(> 0)$ we will allow V' to be positive along solutions of the equations but we also impose a bounded on the growth rate of V along solutions. In such case, V probably increases along solutions between moments of impulses. However, these allowable increases are counter-balanced by sufficient decreases in V at each subsequent moment of impulses. When using a Lyapunov like function, we will introduce a class of functionals $\nu_0^*(\cdot)$ so that one may use the functions in $\nu_0^*(\cdot)$ to describe the impulsive perturbations.

2. Preliminaries

Let $\mathbb{R} = (-\infty, \infty), \mathbb{R}^+ = [0, \infty)$. For $x \in \mathbb{R}^n, |\cdot|$ denotes the Euclidean norm of x, the ball $S(H)$ of $\mathbb{R}^n$ is denoted by $S(H) = \{x \in \mathbb{R}^n : |x| < H \leq \infty\}$. For $t \geq t^* > \alpha \geq -\infty$, $F(t, x(s)) : \alpha \leq s \leq t$ or $F(t, x(\cdot))$ is a Volterra type functional (cf. [9, 14]), its values are in $\mathbb{R}^n$ and determined by $t \geq t^*$ and the values of $x(s)$ on $[\alpha, t]$. In the case when $\alpha = -\infty$, the interval $[\alpha, t]$ is understood to be replaced by $(-\infty, t]$. Then a system of impulsive Volterra functional differential equations considered has the form

$$x'(t) = F(t, x(\cdot)), \qquad t \neq t_k, \quad t > t^*, \quad x \in \mathbb{R}^n \tag{2.1}$$

$$\Delta x = I_k(t, x(t^-)), \qquad t = t_k, \quad k \in \mathbb{N}, \tag{2.2}$$

where $x'(t)$ denotes the right-hand derivative of x at t; $\mathbb{N} := \{1, 2, \ldots,\}, \Delta x := x(t) - x(t^-)$, where $x(t^-) = \lim_{s \to t-0} x(s)$. It is assumed that $t^* < t_k < t_{k+1}$ with $t_k \to \infty$ as $k \to \infty$, and $I_k(t, x) : [t^*, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are some given functions.

Let $I \subset \mathbb{R}$ be any interval. Define $PC(I, \mathbb{R}^n) = \{x : I \to \mathbb{R}^n, x$ is continuous everywhere except at the points $t = t_k \in I$ and $x(t_k^-), x(t_k^+) = \lim_{t \to t_k+0} x(t)$

exist with $x(t_k^+) = x(t_k)\}$. For any $t \geq t^*, PC([\alpha, t], \mathbb{R}^n)$ will be written as $PC(t)$. Define $PCB(t) = \{x \in PC(t) : x \text{ is bounded } \}$. For any $\phi \in PCB(t)$, the norm of ϕ is defined as

$$||\phi|| = ||\phi||^{[\alpha,t]} = \sup_{\alpha \leq s \leq t} |\phi(s)|.$$

For given $\sigma \geq t^*$ and $\phi \in PCB(t)$, with eqs. (2.1) and (2.2), one associates an initial condition of the form

$$x(t) = \phi(t), \quad \alpha \leq t \leq \sigma. \tag{2.3}$$

Definition 2.1. A function $x(t)$ is called a *solution* corresponding to σ of the initial value problem (2.1) – (2.3) if $x : [\alpha, \beta) \to \mathbb{R}^n$ (for some $t^* < \beta \leq \infty$) is continuous for $t \in [\alpha, \beta)\backslash\{t_k, k = 1, 2, \ldots\}$, $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k)$, and if it satisfies (2.1) – (2.3). We denote by $x(t, \sigma, \phi)$ the solution of the initial value problem (2.1) – (2.3).

We suppose that the following conditions (C_1) – (C_4) hold, so that the initial value problem (2.1) – (2.3) has a unique solution (cf. [7, 25]). We also assume that $F(t, 0) \equiv 0, I_k(t, 0) \equiv 0$ so that $x(t) = 0$ is a solution of (2.1) and (2.2), which is called the zero solution.

(**C₁**) F is continuous on $[t_{k-1}, t_k) \times PC(t), k = 1, 2, \ldots$, where $t_0 = t^*$. For all $\varphi \in PC(t)$ and $k \in \mathbb{N}$, the limit $\lim_{(t,\phi)\to(t_k^-,\varphi)} F(t, \phi) = F(t_k^-, \varphi)$ exists.

(**C₂**) F is locally Lipschitz in ϕ on each compact set in $PCB(t)$. More precisely, for every $a \in [t^*, \beta)$ and every compact set $G \subset PCB(t)$ there exists a constant $L = L(a, G)$ such that $|F(t, \varphi(\cdot)) - F(t, \psi(\cdot))| \leq L||\varphi - \psi||^{[\alpha,t]}$, whenever $t \in [t^*, a]$ and $\varphi, \psi \in G$.

(**C₃**) $I_k(t, x) \in C([t^*, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$ for each $k \in \mathbb{N}$, and there exists some $0 < H_1 \leq H$ such that $x \in S(H_1)$ implies that $x + I_k(t_k, x) \in S(H)$ for all $k \in \mathbb{N}$.

(**C₄**) $F(t, x(\cdot)) \in PC([t^*, \infty), \mathbb{R}^n)$ for $x \in PC([\alpha, \infty), \mathbb{R}^n)$.

For any $t \geq t^*$ and $\rho > 0$, let $PCB_\rho(t) = \{\phi \in PCB(t) : ||\phi|| < \rho\}$.

Definition 2.2. The zero solution of (2.1) and (2.2) is said to be

(**S₁**) *stable*, if for any $\sigma \geq t^*$ and $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\varphi \in PCB_\delta(\sigma)$ implies that $|x(t, \sigma, \varphi)| \leq \varepsilon$ for $t \geq \sigma$.

(**S₂**) *uniformly stable*, if the δ in (S_1) is independent of σ.

(**S₃**) *asymptotically stable*, if it is stable and there exists a $\delta = \delta(\sigma) > 0$ such that $\varphi \in PCB_\delta(\sigma)$ implies that $|x(t, \sigma, \varphi)| \to 0$ as $t \to \infty$.

(**S₄**) *uniformly asymptotically stable*, if it is uniformly stable and there exists a $\delta > 0$ such that for any $\varepsilon > 0$ there is a $T = T(\varepsilon) > 0$ such that $\sigma \geq t^*$ and $\phi \in PCB_\delta(\sigma)$ imply that $|x(t, \sigma, \varphi)| \leq \varepsilon$ for $t \geq \sigma + T$.

We define the following Lyapunov like function and functional.

Definition 2.3. A function $V(t,x) : [t^*,\infty) \times S(H) \to \mathbb{R}^+$ belongs to the class ν_0 if

1. V is continuous on each of the sets $[t_{k-1},t_k) \times S(H)$ and for all $x \in S(H), k \in \mathbb{N}$, the limit $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-,x)$ exists.
2. V is locally Lipschitz in x and $V(t,0) \equiv 0$.

Definition 2.4. A functional $V(t,\phi) : [t^*,\infty) \times PCB(t) \to \mathbb{R}^+$ belongs to the class $\nu_0(\cdot)$ if

1. V is continuous on each of the sets $[t_{k-1},t_k) \times PCB(t)$ and for all $\varphi \in PCB(t), k \in \mathbb{N}$, the limit $\lim_{(t,\phi)\to(t_k^-,\varphi)} V(t,\phi) = V(t_k^-,\varphi)$ exists.
2. V is locally Lipschitz in ϕ and $V(t,0) \equiv 0$.

Definition 2.5. A functional $V(t,\phi)$ belongs to the class $\nu_0^*(\cdot)$ if $V \in \nu_0(\cdot)$ and for any $x \in PC([\alpha,\infty),\mathbb{R}^n)$, $V(t,x(\cdot))$ is continuous for $t \geq t^*$.

Remark 2.6. The class ν_0 is an analogue of Lyapunov functions as introduced in [5, 6]. The class $\nu_0(\cdot)$ is an analogue of Lyapunov functionals. We will use respectively these Lyapunov functions with Razumikhin technique and Lyapunov functionals to establish impulsive control stability results. It is to be noted that the class $\nu_0^*(\cdot)$ will play an important role in the application of the Lyapunov functional method to impulsive functional differential equations. Since it is difficult to describe the impulsive perturbations in eq. (2.2) by using the general functionals in the class $\nu_0(\cdot)$, one has to introduce the class $\nu_0^*(\cdot)$ so that it is possible to use the functions in the class ν_0 to describe the impulsive perturbations. A function class which is similar to $\nu_0^*(\cdot)$ was introduced in [26]. It should be pointed out that such a class $\nu_0^*(\cdot)$ is common in applications.

Let $V \in \nu_0$, for any $(t,x) \in [t_{k-1},t_k) \times S(H)$, the right hand derivative of V along the solution $x(t)$ of (2.1) and (2.2) is defined by

$$D^+V(t,x(t)) = \limsup_{h\to 0^+} \frac{1}{h} \left\{V(t+h,x(t+h)) - V(t,x(t))\right\}.$$

Let $V \in \nu_0(\cdot)$, for any $(t,\phi) \in [t_{k-1},t_k) \times PCB(t)$, the right hand derivative of V along the solution $x(t)$ of (2.1) and (2.2) is defined by

$$D^+V(t,x(\cdot)) = \limsup_{h\to 0^+} \frac{1}{h} \left\{V(t+h,x(\cdot)) - V(t,x(\cdot))\right\}.$$

Let us define the following class of functions for later use:

$$K_1 = \left\{u \in C(\mathbb{R}^+,\mathbb{R}^+) \mid u(0)=0, u(s) \text{ is strictly increasing in } s\right\}$$

$$K_2 = \left\{u \in C(\mathbb{R}^+,\mathbb{R}^+) \;\middle|\; \begin{array}{l} u(0)=0, u(s)>0 \text{ for } s>0, \\ u(s) \text{ is nondecreasing in } s \end{array}\right\}$$

$$K_3 = \left\{u \in C(\mathbb{R}^+,\mathbb{R}^+) | u(0)=0, u(s)>0 \text{ for } s>0\right\}.$$

3. Main Results

We shall establish, in this section, two theorems which provide sufficient conditions for uniform asymptotic stability and asymptotic stability of the zero solution of (2.1) and (2.2) by using Lyapunov like functions and functionals, respectively. It should be pointed out that, in general, it is very difficult to obtain a stability result similar to the first theorem by employing Lyapunov functionals. In what follows, we assume that hypotheses (C_1) – (C_4) are satisfied.

Theorem 3.1. *Assume that there exist functions $a, b \in K_1, c \in K_2, V \in \nu_0, q \in K_3$ and $P \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that $P(s)$ is strictly increasing with $P(0) = 0$, $P(s) > s$ for $s > 0$, and the following conditions are satisfied:*

- **(i)** $a(|x|) \le V(t,x) \le b(|x|)$ *for all* $(t,x) \in [\alpha, \infty) \times S(H)$;
- **(ii)** *for any solution $x(t)$ of* (2.1) *and* (2.2), $V(s, x(s)) \le P(V(t, x(t)))$ *for* $\max\{\alpha, t - q(V(t,x(t)))\} \le s \le t$, *implies that*

$$D^+V(t,x(t)) \le g(t)c(V(t,x(t))), \quad t \ne t_k,$$

  where $g : [t^*, \infty) \to \mathbb{R}^+$, locally integrable.
- **(iii)** *for all $k \in \mathbb{N}$ and $x \in S(H_1)$,*

$$V(t_k, x + I_k(t_k, x)) \le h_k(V(t_k^-, x)),$$

  where $h_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $h_k(s) \le P^{-1}(s)$ for $s \ge 0$ and $k \in \mathbb{N}$, where P^{-1} is the inverse of the function P.
- **(iv)** $\sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\} < \infty$, *and there exists a constant $A \ge 0$ such that for all $\mu \in (0, \infty)$ and $k \in \mathbb{N}$,*

$$\int_{P^{-1}(\mu)}^{\mu} \frac{du}{c(u)} - \int_{t_{k-1}}^{t_k} g(s)ds > A.$$

Then the zero solution of (2.1) *and* (2.2) *is uniformly stable; moreover, it is uniformly asymptotically stable if $A > 0$.*

Proof. We first show uniform stability. Let $\varepsilon > 0$ and assume without loss of generality that $\varepsilon \le H_1$. Choose $\delta = \delta(\varepsilon) > 0$ so that $\delta < b^{-1}(P^{-1}(a(\varepsilon)))$ and note that $\delta < \varepsilon$. For $\sigma \ge t^*, \phi \in PCB_\delta(\sigma)$, let $x(t) = x(t, \sigma, \varphi)$ be the solution of (2.1) and (2.2), where $\sigma \in [t_{m-1}, t_m)$ for some $m \in \mathbb{N}, t_0 = t^*$. Then we have for $\alpha \le t \le \sigma$,

$$a(|x(t)|) \le V(t,x(t)) \le b(|x(t)|) \le b(\delta) < P(b(\delta)) < a(\varepsilon). \tag{3.1}$$

Let $[\alpha, \sigma + \beta)$ be the maximal interval of existence of $x(t)$. If $\beta < \infty$, then by (3.1) there exists some $t \in (\sigma, \sigma + \beta)$ satisfying $|x(t)| > \varepsilon$. We will prove that $|x(t)| \le \varepsilon$ for $t \in [\sigma, \sigma+\beta)$, which in turn will imply that $\beta = \infty$ and that the zero solution of (2.1) and (2.2) is thereby uniformly stable.

Suppose for the sake of contradiction that $|x(t)| > \varepsilon$ for some $t \in (\sigma, \sigma+\beta)$. Let $\tau_1 = \inf\{t \in (\sigma, \sigma+\beta) | \, |x(t)| > \varepsilon\}$. Then $|x(t)| \le \varepsilon \le H_1$ for $t \in [\alpha, \tau_1)$, and either $|x(\tau_1)| = \varepsilon$ or $|x(\tau_1)| > \varepsilon$ and $\tau_1 = t_k$ for some k. In the latter case, $|x(\tau_1)| \le H$ since condition (C_3) implies that $|x(\tau_1)| = |x(t_k)| = |x(t_k^-) + I_k(t_k, x(t_k^-))| = |x(\tau_1^-) + I_k(t_k, x(\tau_1^-))| \le H$. Thus, in either case $V(t, x(t))$ is defined for $t \in [\alpha, \tau_1]$. For $t \in [\alpha, \tau_1]$, define

$$V(t) = V(t, x(t)). \tag{3.2}$$

Then, for $t \in [\alpha, \tau_1]$ by condition (i) we have $a(|x(t)|) \le V(t) \le b(|x(t)|)$, specially, $V(t) \le b(\delta) < P^{-1}(a(\varepsilon))$ for $t \in [\alpha, \sigma]$. Since V is locally Lipschitz in x, we have, by condition (ii),

$$D^+V(t) = \limsup_{h \to 0^+} \frac{1}{h} [v(t+h) - V(t)] \le g(t)c(V(t)) \tag{3.3}$$

for all $t \ne t_k$ in $(\sigma, \tau_1]$, whenever $V(s) \le P(V(t))$ for $\max\{\alpha, t - q(V(t))\} \le s \le t$. Also, by condition (iii), we have

$$V(t_k) \le h_k(V(t_k^-)) \le P^{-1}(V(t_k^-)), \tag{3.4}$$

for all $t_k \in (\sigma, \tau_1]$.

Let $\tau_2 = \inf\{t \in [\sigma, \tau_1] | V(t) \ge a(\varepsilon)\}$. Since $V(\sigma) < P^{-1}(a(\varepsilon)) < a(\varepsilon)$ and $V(\tau_1) \ge a(\varepsilon)$, it follows that $\tau_2 \in (\sigma, \tau_1]$, and $V(t) < a(\varepsilon)$ for $t \in [\alpha, \tau_2)$. We claim that $V(\tau_2) = a(\varepsilon)$ and that $\tau_2 \ne t_k$ for any k. Clearly, we must have $V(\tau_2) \ge a(\varepsilon) > 0$. If $\tau_2 = t_k$ for some k, then $0 < a(\varepsilon) \le V(\tau_2) \le h_k(V(\tau_2^-)) \le P^{-1}(V(\tau_2^-)) < V(\tau_2^-) \le a(\varepsilon)$ by (3.4), which is impossible. Thus, $\tau_2 \ne t_k$ for any k and that in turn implies $V(\tau_2) = a(\varepsilon)$ since $V(t)$ is continuous at τ_2.

Now let us first consider the case where $t_{m-1} \le \sigma < \tau_2 < t_m$. Let $\tau_3 = \sup\{t \in [\sigma, \tau_2] | V(t) \le P^{-1}(a(\varepsilon))\}$. Since $V(\sigma) < P^{-1}(a(\varepsilon))$, $V(\tau_2) = a(\varepsilon) > P^{-1}(a(\varepsilon))$, and $V(t)$ is continuous on $[\sigma, \tau_2]$, then $\tau_3 \in (\sigma, \tau_2)$, $V(\tau_3) = P^{-1}(a(\varepsilon))$, and $V(t) \ge P^{-1}(a(\varepsilon))$ for $t \in [\tau_3, \tau_2]$. Therefore, for $t \in [\tau_3, \tau_2]$ and $\max\{\alpha, t - q(V(t))\} \le s \le t$, we have $P^{-1}(V(s)) \le P^{-1}(a(\varepsilon)) \le V(t)$. Then, by condition (ii), inequality (3.3) holds for all $t \in (\tau_3, \tau_2]$. Integrating this differential inequality yields

$$\int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)} \le \int_{\tau_3}^{\tau_2} g(s)ds \le \int_{t_{m-1}}^{t_m} g(s)ds.$$

On the other hand, by condition (iv), we have

$$\int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)} = \int_{P^{-1}(a(\varepsilon))}^{a(\varepsilon)} \frac{ds}{c(s)} > \int_{t_{m-1}}^{t_m} g(s)ds + A \ge \int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)},$$

which is our desired contradiction.

Alternatively, suppose that $t_k < \tau_2 < t_{k+1}$ for some $k \geq m$. Then $V(t_k) \leq h_k(V(t_k^-)) \leq P^{-1}(V(t_k^-)) \leq P^{-1}(a(\varepsilon))$ by (3.4). Similar to before, define $\tau_3 = \sup\{t \in [t_k, \tau_2] | V(t) \leq P^{-1}(a(\varepsilon))\}$. Then $\tau_3 \in [t_k, \tau_2), V(\tau_3) = P^{-1}(a(\varepsilon))$, and $V(t) \geq P^{-1}(a(\varepsilon))$ for $t \in [\tau_3, \tau_2]$. Applying exactly the same argument as before yields a contradiction.

So in either case, we get a contradiction, which proves that the zero solution of (2.1) and (2.2) is uniformly stable. Now we shall show that it is uniformly asymptotically stable provided $A > 0$.

For $\varepsilon = H_1$ find a δ of uniform stability such that if $\phi \in PCB_\delta(\sigma)$, then $|x(t, \sigma, \phi)| \leq H_1$ for all $t \geq \alpha$, where $x(t) = x(t, \sigma, \varphi)$ is any solution of (2.1) and (2.2) and $\sigma \geq t^*$. Moreover, $V(t, x(t)) \leq b(|x(t)|) \leq b(H_1)$ for $t \geq \alpha$.

Let $r > 0$ be given and assume without loss of generality that $r < H_1$. Set $M = M(r) = \sup\{c^{-1}(s) \mid P^{-1}(a(r)) \leq s \leq b(H_1)\}$, and note that $0 < M < \infty$. For $a(r) \leq l \leq b(H_1)$, we have $P^{-1}(a(r)) \leq P^{-1}(l) < l \leq b(H_1)$ and so

$$A < \int_{P^{-1}(l)}^{l} \frac{ds}{c(s)} \leq M[l - P^{-1}(l)],$$

for which we get $P^{-1}(l) < l - d$, where $d = \frac{A}{M}$. Let $N = N(r)$ be the positive integer such that $a(r) + (N-1)d < b(H_1) \leq a(r) + Nd$. Let $\sigma \in [t_{m-1}, t_m)$ for some $m \in \mathbb{N}$ and set

$$m_i = \min\{k \in \mathbb{N} \mid t_k - t_{m_{i-1}} \geq q(a(r))\}, \quad i = 1, 2, \ldots, N,$$

where we let $m_0 = m$. Set $\tau = \sup_{k\in\mathbb{N}}\{t_k - t_{k-1}\}$, then $0 < \tau < \infty$. Let $n_i = n_i(r), i = 1, 2, \ldots, N$, be the numbers of impulsive points t_k in the intervals $(t_{m_{i-1}}, t_{m_i})$. Since $t_k - t_{k-1} \leq \tau$, then it is easy to see that $t_{m_i} - t_{m_{i-1}} \leq (n_i + 1)\tau, i = 1, 2, \ldots, N$. Thus,

$$t_{m_N} - t_{m_0} \leq N\tau + (n_1 + \ldots + n_N)\tau. \tag{3.5}$$

We now define $T = T(r) = (N + n_1 + \ldots + n_N)\tau$ and will prove that $\phi \in PCB_\delta(\sigma)$ implies that $|x(t)| \leq r$ for all $t \geq \sigma + T$. Set $V(t)$ as in (3.2) for $t \geq \alpha$. Then $V(t) \leq b(H_1)$ for $t \geq \alpha$. Given $0 < B \leq a(r)$ and $m_j \geq m$ we will prove that if $V(t) \leq B$ for $t \in [t_{m_{j-1}}, t_{m_j})$, then $V(t) \leq B$ for $t \geq t_{m_j}$ and if in addition $a(r) \leq B \leq b(H_1)$, then $V(t) \leq B - d$ for $t \geq t_{m_j}$.

To prove the first part, suppose for the sake of contradiction that there exists some $t \geq t_{m_j}$ for which $V(t) > B$. Then let $\tau_2 = \inf\{t \geq t_{m_j} | V(t) > B\}$. Thus, $\tau_2 \in [t_k, t_{k+1})$ for some $k \geq m_j$, and $V(t) \leq B$ for $t \in [t_{m_{j-1}}, \tau_2)$. Since $V(t_k) \leq h_k(V(t_k^-)) \leq P^{-1}(V(t_k^-)) \leq P^{-1}(B) < B$, then $\tau_2 \in (t_k, t_{k+1})$. Moreover, $V(\tau_2) = B$ and $V(t) \leq B$ for $t \in [t_{m_{j-1}}, \tau_2]$.

Let $\tau_3 = \sup\{t \in [t_k, \tau_2] | V(t) \le P^{-1}(B)\}$. Since $V(\tau_2) = B > P^{-1}(B) \ge V(t_k)$, then $\tau_3 \in [t_k, \tau_2), V(\tau_3) = P^{-1}(B)$, and $V(t) \ge P^{-1}(B)$ for $t \in [\tau_3, \tau_2]$. Thus, for $t \in [\tau_3, \tau_2]$ and $\max\{\alpha, t - q(V(t))\} \le s \le t$, we have $P^{-1}(V(s)) \le P^{-1}(B) \le V(t)$ which implies that inequality (3.3) holds for all $t \in (\tau_3, \tau_2]$, and so

$$\int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)} \le \int_{\tau_3}^{\tau_2} g(s)ds \le \int_{t_k}^{t_{k+1}} g(s)ds. \tag{3.6}$$

On the other hand, by condition (iv), we have

$$\int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)} = \int_{P^{-1}(B)}^{B} \frac{ds}{c(s)} > \int_{t_k}^{t_{k+1}} g(s)ds + A > \int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)}.$$

Then we get a contradiction proving the first.

Below we give the proof of the second part. Assume for the sake of contradiction that there exists some $t \ge t_{m_j}$ for which $V(t) > B - d$. Then define $\tau_2 = \inf\{t \ge t_{m_j} | V(t) > B - d\}$ and let $k \ge m_j$ be chosen so that $\tau_2 \in [t_k, t_{k+1})$. Since $a(r) \le B \le b(H_1)$, then $P^{-1}(B) < B - d$ and so $V(t_k) \le P^{-1}(V(t_k^-)) \le P^{-1}(B) < B - d$. Thus, $\tau_2 \in (t_k, t_{k+1})$. Moreover, $V(\tau_2) = B - d$ and $V(t) \le B$ for $t \in [t_k, \tau_2]$. Define τ_3 as before. Since $V(\tau_2) = B - d > P^{-1}(B) \ge V(t_k)$, then $\tau_3 \in [t_k, \tau_2), V(\tau_3) = P^{-1}(B)$ and $V(t) \ge P^{-1}(B)$ for $t \in [\tau_3, \tau_2]$. Thus, we obtain inequality (3.6) as before. However,

$$\int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)} = \int_{P^{-1}(B)}^{B-d} \frac{ds}{c(s)} = \int_{P^{-1}(B)}^{B} \frac{ds}{c(s)} - \int_{B-d}^{B} \frac{ds}{c(s)}. \tag{3.7}$$

Since $a(r) \le B \le b(H_1)$, then $P^{-1}(a(r)) \le P^{-1}(B) < B - d \le b(H_1)$ and so $\frac{1}{c(s)} \le M$ for $B - d \le s \le B$. Thus, from (3.7), we have

$$\int_{V(\tau_3)}^{V(\tau_2)} \ge \int_{P^{-1}(B)}^{B} \frac{ds}{c(s)} - Md > \int_{t_k}^{t_{k+1}} g(s)ds + A - Md \ge \int_{V(\tau_3)}^{V(\tau_2)} \frac{ds}{c(s)}.$$

This is a contradiction, establishing the second part.

We now claim that for each $i = 0, 1, 2, \ldots, N, V(t) \le b(H_1) - id$ for $t \ge t_{m_i}$. Since $V(t) \le b(H_1)$ for $t \in [\alpha, t_{m_0})$, then by setting $B = b(H_1)$ in our earlier argument, we get $V(t) \le b(H_1) - d$ for $t \ge t_{m_0}$, which establishes the base case. We now proceed by induction and assume $V(t) \le b(H_1) - jd$ for $t \ge t_{m_j}$ for some $1 \le j \le N - 1$. Let $B = b(H_1) - jd$, then $B \ge b(H_1) - (N-1)d \ge a(r)$. Since $t_{m_j} \le t_{m_{j+1}} - q(a(r))$, then $V(t) \le B$ for $t \in [t_{m_{j+1}} - q(a(r)), t_{m_{j+1}})$ and so $V(t) \le B - d = b(H_1) - (j+1)d$ for $t \ge t_{m_{j+1}}$. Thus, we have proved our claim by induction.

In particular, we have $V(t) \le b(H_1) - Nd \le a(r)$ for $t \ge t_{m_N}$. Finally, by (3.5) we see that $V(t) \le a(r)$ for all $t \ge \sigma + T$. The proof thereby is complete. ∎

In Theorem 3.1, condition (iii) ensures that along solutions of (2.1) and (2.2), the Lyapunov function must decrease at each impulse time. Condition (ii) effectively imposes a bound on the growth rate of V along solutions through a Razumikhin-type argument. Condition (iv) ensures that any possible growth in V between impulses is more than offset by a reduction in V at impulses. Stability results along this line for impulsive differential equations without delay and impulsive functional differential equations with finite delay can be found in [6, 18, 27].

The importance of Theorem 3.1 is mainly in its applicability to Volterra functional differential equations that are not already stable but that can be stabilized through the incorporation of impulses. The following is an illustrative example.

Example. Consider the impulsive functional differential equation with infinite delay

$$x'(t) = a(t)x(t) + b(t)x(t-r) + \int_{-\infty}^{t} f(t, u-t, x(u))du, \quad t \geq 0 \tag{3.8}$$

$$\Delta x = I_k(t, x(t^-)), \quad t = t_k, \quad k \in \mathbb{N}, \tag{3.9}$$

where $r > 0, a(t), b(t) \in C(\mathbb{R}^+)$ such that $a(t) \leq a, |b(t)| \leq b, f(t,u,v)$ is continuous on $\mathbb{R}^+ \times (-\infty, 0] \times \mathbb{R}, I_k(t,x) \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, and $|x + I_k(t_k, x)| \leq \gamma|x|$ for all $k \in \mathbb{N}$, where a, b, γ are some constants. Let the following conditions hold:

(i) $|f(t,u,v)| \leq m(u)|v|, t \geq 0$, and $\int_{-\infty}^{0} m(u)du \leq M$.

(ii) $0 < \gamma < 1$, and there exist constant $\tau > 0$ such that

$$t_{k+1} - t_k \leq \tau < -\frac{\ln \gamma}{a + b\gamma^{-1} + M\gamma^{-1}}, \quad k \in \mathbb{N}.$$

Then the zero solution of (3.8) and (3.9) is uniformly asymptotically stable.

In fact, we let $V(t,x) = V(x) = \frac{x^2}{2}, h_k(s) = h(s) = \gamma^2 s, c(s) = s$. Then

$$V(x + I_k(t_k, x)) = \frac{1}{2}[x + I_k(t_k, x)]^2 \leq \frac{1}{2}\gamma^2 x^2 = h(V(x)).$$

From condition (ii) we may choose a constant $A > 0$ such that

$$t_{k+1} - t_k \leq -\frac{2\ln\gamma + A}{2(a + b\gamma^{-1} + M\gamma^{-1} + A)}, \quad k \in \mathbb{N}. \tag{3.10}$$

From (i) we see that there exists a continuous function $q : (0,\infty) \to (0,\infty)$, $q(s) \geq r$ for $s > 0, q$ is non-increasing, such that $\int_{-\infty}^{-q(s)} m(u)du \leq A\sqrt{2s}$. By Theorem 3.1, one can easily see that the zero solution is uniformly stable.

Thus, without loss of generality, we may assume that $||x(t)||^{(-\infty,t]} \le 1$. Let $P(s) = h^{-1}(s)$. Then $P(s) > s$ for $s > 0$. If $V(s, x(s)) \le P(V(t, x(t)))$ and $\max\{-\infty, t - q(V(t, x(t)))\} \le s \le t$, then we have $|x(s)| \le \gamma^{-1}|x(t)|$, and so

$$\begin{aligned}
V'(t, x(t)) &\le ax^2(t) + |b(t)||x(t)||x(t-r)| + |x(t)| \int_{-\infty}^{t} m(v-t)|x(v)|\,dv \\
&\le ax^2(t) + b\gamma^{-1}x^2(t) + |x(t)| \int_{-\infty}^{t-q(V(t,x(t)))} m(v-t)|x(v)|\,dv \\
&\quad + |x(t)| \int_{t-q(V(t,x(t)))}^{t} m(v-t)|x(v)|\,dv \\
&\le (a + b\gamma^{-1})x^2(t) + |x(t)| \int_{-\infty}^{-q(V(t,x(t)))} m(u)\,du \\
&\quad + \gamma^{-1}x^2(t) \int_{-\infty}^{0} m(u)\,du \\
&\le (a + b\gamma^{-1} + M\gamma^{-1})x^2(t) + A|x(t)|\sqrt{2V(t, x(t))} \\
&= (a + b\gamma^{-1} + M\gamma^{-1} + A)x^2(t) \\
&= g(t)c(V(t, x(t))),
\end{aligned}$$

where $g(t) = 2(a + b\gamma^{-1} + M\gamma^{-1} + A)$. From (3.10) we see that for all $\mu > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\int_{P^{-1}(\mu)}^{\mu} \frac{du}{c(u)} - \int_{t_k}^{t_{k+1}} g(s)\,ds &= \int_{\gamma^2\mu}^{\mu} \frac{du}{u} - 2\int_{t_k}^{t_{k+1}} (a + b\gamma^{-1} + M\gamma^{-1} + A)\,ds \\
&= -2\ln\gamma - 2\big(a + b\gamma^{-1} + Mc\gamma^{-1} + A\big)(t_{k+1} - t_k) \\
&\ge A.
\end{aligned}$$

Therefore, we may conclude from Theorem 3.1 that the zero solution of (3.8) and (3.9) is uniformly asymptotically stable.

The following theorem shows that certain impulses may make a stable system asymptotically stable.

Theorem 3.2. *Let the function $\Phi \in C(\mathbb{R}^+, \mathbb{R}^+)$ be bounded and satisfy $\Phi \in L^1(\mathbb{R}^+)$. Suppose that there exist functions $W_i \in K_1 (i = 1 - 4), V_1(t, x) \in v_0, V_2(t, \phi) \in v_0^*(\cdot)$, and $h \in K_2$ such that*

(i) $W_1(|\phi(t)|) \le V(t, \phi(\cdot)) \le W_2(|\phi(t)|) + W_3\big(\int_\alpha^t \Phi(t-s)W_4(|\phi(s)|)\,ds\big)$, *where* $V(t, \phi(\cdot)) = V_1(t, \phi(t)) + V_2(t, \phi(\cdot)) \in v_0(\cdot)$.

(ii) *For any* $x \in S(H_1)$, $V_1(t_k, x + I_k(t_k, x)) - V_1(t_k^-, x) \le -\mu_k h(V_1(t_k^-, x))$, $k \in \mathbb{N}$, *where* $\mu_k \ge 0$ *with* $\sum_{k=1}^{\infty} \mu_k = \infty$.

(iii) *For any solution $x(t)$ of* (2.1) *and* (2.2)*, the right hand derivative of $V(t, x(\cdot))$ along the solution satisfies $D^+V(t, x(\cdot)) \le 0$ and for any $\sigma \ge t^*$ and $r_1 > 0$, there exists $r_2 > 0$ such that $V(t, x(\cdot)) \ge r_1$ for $t \ge \sigma$ implies that $V_1(t, x(t)) \ge r_2$ for $t \ge \sigma$.*

Then the zero solution of (2.1) *and* (2.2) *is uniformly stable and asymptotically stable.*

Proof. We first prove the uniform stability. For given $\varepsilon > 0\, (\varepsilon \le H_1)$, we may choose a $\delta = \delta(\varepsilon) > 0$ such that $W_2(\delta) < W_1(\varepsilon)/2$ and $W_3(JW_4(\delta)) \le W_1(\varepsilon)/2$, where $J = \int_0^\infty \Phi(u)du$. For any $\sigma \ge t^*$ and $\phi \in PCB_\delta(\sigma)$, let $x(t) = x(t, \sigma, \phi)$ be the solution of (2.1) and (2.2). We will prove that

$$|x(t, \sigma, \phi)| \le \varepsilon, \quad t \ge \sigma.$$

Let $V_1(t) = V_1(t, x(t)), V_2(t) = V_2(t, x(\cdot))$ and $V(t) = V(t, x(\cdot))(= V_1(t)+V_2(t))$. Let $\sigma \in [t_{m-1}, t_m)$ for some $m \in \mathbb{N}$. Then by condition (iii), $D^+V(t) \le 0$, $\sigma \le t < t_m$, $t_{m+k-1} \le t < t_{m+k}$, $k \in \mathbb{N}$. Thus, $V(t)$ is non-increasing on the intervals of the form $[\sigma, t_m), [t_{m+k-1}, t_{m+k})$, $k \in \mathbb{N}$. From condition (ii) we have for all $t_i > \sigma$,

$V(t_i) - V(t_i^-) = V_1(t_i, x(t_i^-) + I_i(t_i, x(t_i^-))) - V_1(t_i^-, x(t_i^-)) \le 0$. Therefore $V(t)$ is non-increasing on $[\sigma, \infty)$. Then we have for $t \ge \sigma$

$$\begin{aligned} W_1(|x(t)|) \le V(t) \le V(\sigma) &\le W_2(\delta) + W_3\Big(W_4(\delta) \int_\alpha^t \Phi(t-s)\, ds\Big) \\ &= W_2(\delta) + W_3\Big(W_4(\delta) \int_0^{t-\alpha} \Phi(u)\, du\Big) \\ &\le W_2(\delta) + W_3\Big(W_4(\delta) \int_0^{\infty} \Phi(u)\, du\Big) \\ &= W_2(\delta) + W_3(JW_4(\delta)) \\ &\le W_1(\varepsilon). \end{aligned}$$

This implies that $|x(t)| \le \varepsilon$ for $t \ge \sigma$, and so the zero solution of (2.1) and (2.2) is uniformly stable.

To prove the asymptotic stability, it suffices to prove that $\lim_{t\to\infty} x(t) = 0$. Let $\lim_{t\to\infty} V(t) = r_1$. If $r_1 > 0$, then condition (iii) implies that there is a number $r_2 > 0$ such that $V_1(t) \ge r_2$ for $t \ge \sigma$. Set $r = \inf_{r_2 \le s \le W_1(\varepsilon)} h(s)$. Then $r > 0$. From condition (ii) we have $V_1(t_i) - V_1(t_i^-) \le -\mu_i h(V_1(t_i^-)) \le -r\mu_i$, $i = m, m+1, \ldots$. Since $V(t)$ is non-increasing, it follows that

$$V(t_i)-V(t_{i-1}) \le V(t_i)-V(t_i^-) = V_1(t_i)-V_1(t_i^-) \le -r\mu_i, \quad i = m+1, m+2, \ldots ,$$

and so

$$V(t_i) \le V(t_m) - r \sum_{j=m+1}^{i} \mu_j \to -\infty, \quad \text{as } i \to \infty.$$

This is a contradiction. Thus, we have $r_1 = 0$ which in turn implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. ∎

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