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Non-Analyticity in Time of Solutions to the KdV Equation

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Abstract. It is proved that formal power series solutions to the initial value problem $\partial_t u = \partial_x^3 u + \partial_x(u^2), u(0,x) = \varphi(x)$ with analytic data φ belong to the Gevrey class G^2 in time. However, if $\varphi(x) = \frac{1}{1+x^2}$, the formal solution does not belong to the Gevrey class G^s in time for $0 \le s < 2$, so it is not analytic in time. The proof is based on the estimation of a double sum of products of binomial coefficients.

Keywords: *KdV equation, non-analyticity, Gevrey spaces, binomial coefficients*
AMS subject classification: Primary 35A10, 35A20, 35K55, 35Q53, secondary 05A10, A0519, 11B65

1. Introduction

We consider the characteristic Cauchy problem for the Korteweg-de Vries equation

$$\left.\begin{aligned} \partial_t u &= \partial_x^3 u + \partial_x(u^2) \\ u(0,x) &= \varphi(x) \end{aligned}\right\}. \tag{1}$$

The equation appears in the study of a number of different physical systems, e.g. it describes the long time evolution of small amplitude dispersive waves. Since its first derivation in a paper by D. J. Korteweg and G. de Vries in 1895 [11], it was extensively studied and numerous results have been obtained. The reader interested in different aspects of its theory is referred to the papers by A. Jeffrey and T. Kakutani [7], D. M. Kruskal [12], P. D. Lax [13], R. M. Miura [15], J. Bourgain [3], C. E. Kenig, G. Ponce and L. Vega [10], N. Hayashi [6], P. E. Zhidkov [18] and the references given there.

Here we are interested in the analyticity properties of solutions to problem (1). The first result in this direction was obtained by E. Trubowitz who showed that solutions with periodic real analytic data remain spatially real analytic

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for all time (see [17: Section 3, Amplification 2] and [3: Remark (iv)]. Next, T. Kato and K. Masuda proved that if the initial data φ is analytic and L^2 in a strip along $\mathbb{R}$, then the solution $u(t,\cdot)$ has the same property for all time [8: Remark 2.1]. An analytic smoothing effect for Gevrey data was established by A. De Bouard, N. Hayashi and K. Kato. If φ belongs to the Gevrey class of order 3, then there exists $T > 0$ such that for $0 < t < T$ the solution $u(t,\cdot)$ has an analytic continuation to the complex domain $\{z = x + iy \in \mathbb{C} : |x| < R, |y| < At^{\frac{1}{3}}\}$ with some $A = A(R) > 0$ [5: Theorem 1.1, Remark 1.1 (III)]. The result was obtained by using operators which commute or almost commute with the linear part of the KdV equation. A remarkable result was obtained by K. Kato and T. Ogawa in [9]. Namely, under the assumption that $\varphi \in H^s(\mathbb{R})$ $(s > -\frac{3}{4})$ satisfies with some positive A

$$\sum_{k=0}^{\infty} \frac{A^k}{(k!)^3} \|(x\partial_x)^k \varphi\|_{H^s} < \infty$$

they proved analyticity of $u(t,\cdot)$ for any $0 < t < T$ and Gevrey regularity of order 3 of $u(\cdot,x)$ for any $x \in \mathbb{R}$. Moreover, under a stronger condition

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} \|(x\partial_x)^k \varphi\|_{H^s} < \infty$$

(which implies analyticity except at the origin) the solution is analytic in both variables at any point of $(0,T) \times \mathbb{R}$ [9: Theorem 1.1 and Corollary 1.2].

However, the above results do not guarantee analyticity of solutions in time at $t = 0$ even if the initial data is analytic. Indeed, if φ is analytic, then problem (1) has a unique formal power series solution

$$u(t,x) = \sum_{n=0}^{\infty} \varphi_n(x) t^n \tag{2}$$

where φ_n are given by the recurrence relations

$$\left.\begin{aligned} &\varphi_0 = \varphi \\ &\varphi_{n+1} = \frac{1}{n+1}\big(\partial_x^3 \varphi_n + \partial_x \psi_n\big),\ \psi_n = \sum_{n_1+n_2=n} \varphi_{n_1}\varphi_{n_2} \quad (n \in \mathbb{N}_0). \end{aligned}\right\}. \tag{3}$$

We shall prove that this formal solution belongs to the Gevrey class G^2 in time, see Definition 1. (Our definition of Gevrey order differs by one from that used in [5] and [9], but it is consistent with the one used in the summability theory, see [1].) Next we show that the formal solution (2) is divergent if φ does not extend to an entire function of exponential order $\frac{3}{2}$ and has non-negative Taylor coefficients.

Note that in the case of the linear counterpart of problem (1), $\partial_t u = \partial_x^3 u$, $u(0,x) = \varphi(x)$ the formal solution is given by (2) with $\varphi_n(x) = \frac{1}{n!}\partial_x^{3n}\varphi(x)$. Now the condition that φ is entire of exponential order at most $\frac{3}{2}$ is equivalent to (see [2: Section 2.2])

$$\sup_{x\in K} |\partial_x^{3n}\varphi(x)| \le C^{3n+1}(3n)!^{\frac{1}{3}}$$

with some constant $C < \infty$. Hence the formal solution converges if and only if φ is an entire function of exponential order at most $\frac{3}{2}$.

One could expect that the same characterization of convergence of formal solutions holds for problem (1). It appears however that the soliton solution of (1), $u(t,x) = 6a^2\cosh^{-2}(ax + 4a^2t))$ $(a \neq 0)$ is analytic on $\mathbb{R}_t \times \mathbb{R}_x$ but $u(0,x) = 6a^2\cosh^{-2}(ax)$ is not entire.

The main aim of our paper is to show the divergence of the formal solution in the case of $\varphi(x) = \frac{1}{1+x^2}$. This function is analytic and bounded in a strip along $\mathbb{R}$ and it satisfies the conditions of Kato and Ogawa with $s = 0$.

Our main result reads as follows:

Theorem 1. *Let* $\varphi(x) = \frac{c}{1+x^2}$ *with* $c < 0$ *or* $0 < c < 5\frac{305}{359}$. *Then the formal solution* (2) *to the initial value problem* (1) *does not belong to the Gevrey class* G^s *in time for* $0 \le s < 2$. *Thus, the solution of problem* (1) *is not analytic in time at* $t = 0$.

In order to prove the theorem we write φ_n as a power series in x variable and estimate its coefficients. For n even we have

$$\varphi_n(x) = \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{k+\frac{n}{2}}A(n,2k)x^{2k}$$

where $A(n,2k)$ satisfy the recurrence relations (14) (see below). The second term in (14) is due to the influence of the nonlinear part of the equation and it is always non-negative. So we easily get $|A(n,2k)| \ge \frac{(3n+2k)!}{(2k)!}|c|$ for n even and $c < 0$, which implies $|\varphi_n(0)| \ge \frac{(3n)!}{n!}|c|$. Hence the formal solution (2) can not belong to G^s for $0 \le s < 2$. In the case of positive c the proof is much more involved since it requires a subtle estimation from above of the second term in (14).

The required estimation follows by the following lemma which combinatorial proof seems to be of independent interest.

Main Lemma. *For $k, n \in \mathbb{N}_0$ put*

$$C(k,n) = \begin{cases} \sum_{i=0}^{n} \sum_{l=0}^{k+1-[i]_2} \binom{n}{i} \binom{2k+2}{2l+[i]_2} \Big/ \binom{2k+3n+2}{2l+3i+[i]_2} & \text{if } n \text{ is even} \\ \sum_{i=0}^{n} \sum_{l=0}^{k} \binom{n}{i} \binom{2k+1}{2l+[i]_2} \Big/ \binom{2k+3n+1}{2l+3i+[i]_2} & \text{if } n \text{ is odd} \end{cases} \tag{4}$$

where $[i]_2 = i \bmod 2$. Then

$$C(k,n) \leq \begin{cases} 2\frac{9}{70} & \text{for } n \in \mathbb{N}_0 \text{ if } k = 0 \\ k+2 & \text{for } n \in \mathbb{N}_0 \text{ if } k \geq 1. \end{cases} \tag{5}$$

The proof of the Main Lemma follows the method of the proof of a similar result presented in [14], but it is much more involved. Namely, we represent $C(k,n)$ as a linear combination with positive coefficients of sequences of the form

$${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n) = \sum_{l=0}^{n} \binom{2n+\alpha}{2l+\beta} \Big/ \binom{6n+\gamma}{6l+\delta} \tag{6}$$

with some $\alpha, \beta, \gamma, \delta \in \mathbb{N}_0$. The actual form of the representation depends on $k \bmod 3$ and $n \bmod 2$ and is given in Lemma 1. Next we prove that, for $k \in \mathbb{N}$, sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ appearing in the representation are decreasing. So $C(k,n)$ is bounded by $C(k,0) = k+2$.

Recently, P. Byers and A. Himonas have given another examples of non-analytic solutions to the KdV equation with analytic initial data in both the periodic and the non-periodic case [4].

2. Gevrey estimates

In this section we study Gevrey-type estimates for formal solutions to problem (1).

Definition 1. We say that the formal power series (2) is in the *Gevrey class $G^s(\Omega)$ in time*, $s \geq 0$ and $\Omega \subset \mathbb{R}$, if for any compact set $K \subset\subset \Omega$ one can find $L < \infty$ such that

$$\sup_{n \in \mathbb{N}_0} \sup_{x \in K} \frac{|\varphi_n(x)|}{L^n (n!)^s} < \infty. \tag{7}$$

Note that since $\varphi_n(x) = \frac{1}{n!}\partial_t^n u(0,x)$, our Gevrey index s is less by one from the Gevrey order used in [5, 9]. However, it agrees with the definition of the Gevrey order commonly used in summability theory (see [1]).

In the proof of Theorem 2 we shall need the formula

$$m! \sum_{j=0}^{m} \frac{(j+\nu)!}{j!} \frac{(m-j+\mu)!}{(m-j)!} = \frac{\nu!\mu!(m+\nu+\mu+1)!}{(\nu+\mu+1)!} \tag{8}$$

for $\nu, \mu, m \in \mathbb{N}_0$ which in the equivalent form

$$\sum_{j=0}^{m} \binom{j+\nu}{\nu} \binom{m-j+\mu}{\mu} = \binom{m+\nu+\mu+1}{\nu+\mu+1}$$

can be found in [16: Formula 4.2.5.36]

Theorem 2. *Let φ be analytic in $\Omega \subset \mathbb{R}$. Then the formal solution* (2) *to the initial value problem* (1) *belongs to $G^2(\Omega)$ in time.*

Proof. Let K be compact in Ω. Since $\varphi_0 = \varphi$ is analytic in Ω, we can find $1 \leq D < \infty$ such that

$$\sup_{x \in K} |\partial^m \varphi_0(x)| \leq D^{m+1} m! \qquad (m \in \mathbb{N}_0). \tag{9}$$

We shall prove inductively the estimation

$$\sup_{x \in K} |\partial^m \varphi_n(x)| \leq 2^n D^{m+3n+1} \frac{(m+3n)!}{n!} \tag{10}$$

which implies (7) with $s = 2$ and $L = 6D^3$. To this end observe that the recurrence relations (3) imply

$$\varphi_n = \frac{1}{n!} \left(\partial^{3n} \varphi_0 + \sum_{k=0}^{n-1} k! \partial^{3n-3k-2} \psi_k \right) \qquad (n \in \mathbb{N}). \tag{11}$$

Next, by the Leibniz rule, the inductive assumption and (8) we estimate

$$\begin{aligned}
\sup_{x \in K} |\partial^m \psi_k(x)| &\leq \sum_{k_1+k_2=k} \sum_{j=0}^{m} \binom{m}{j} \sup_{x \in K} |\partial^j \psi_{k_1}(x)| \sup_{x \in K} |\partial^{m-j} \psi_{k_2}(x)| \\
&\leq 2^k D^{m+3k+2} \sum_{k_1+k_2=k} \frac{m!}{k_1! k_2!} \sum_{j=0}^{m} \frac{(j+3k_1)!(m-j+3k_2)!}{j!(m-j)!} \\
&= 2^k D^{m+3k+2} \frac{(m+3k+1)!}{(3k+1)!} \sum_{k_1+k_2=k} \frac{(3k_1)!(3k_2)!}{k_1! k_2!} \\
&\leq 2^k D^{m+3k+2} \frac{(m+3k+1)!}{k!}
\end{aligned}$$

since

$$\frac{k!}{(3k+1)!}\sum_{k_1+k_2=k}\frac{(3k_1)!(3k_2)!}{k_1!k_2!}=\frac{1}{3k+1}\sum_{k_1=0}^{k}\binom{k}{k_1}\Big/\binom{3k}{3k_1}\le 1.$$

Hence, by (9) and (11) we get for $n \ge 1$

$$\begin{aligned}\sup_{x\in K}&|\partial^m\varphi_n(x)|\\ &\le \frac{1}{n!}\left[D^{m+3n+1}(m+3n)!+\sum_{k=0}^{n-1}k!\cdot 2^k D^{m+3n}\frac{(m+3n-1)!}{k!}\right]\\ &\le 2^n D^{m+3n+1}\frac{(m+3n)!}{n!}\end{aligned}$$

since $D \ge 1$, which concludes the proof ∎

Before formulation of the next result recall that an entire function φ is of exponential order $\rho > 0$ if $|\varphi(x)| \le C\exp\{C|x|^\rho\}$ for $x \in \mathbb{C}$. The condition can be expressed in terms of the growth of derivatives of φ at a point $x_0 \in \mathbb{C}$. Namely, it is equivalent to (see [2: Subsection 2.2])

$$|\partial^n\varphi(x_0)| \le D^{n+1}(n!)^{1-\frac{1}{\rho}} \qquad (n \in \mathbb{N}_0).$$

Theorem 3. *Fix $\rho \ge \frac{3}{2}$. Let φ be analytic in $\Omega \subset \mathbb{R}$ and assume that at a point $x_0 \in \Omega$ the Taylor coefficients of φ are non-negative. If φ does not extend to an entire function of exponential order ρ, then the formal solution* (2) *of problem* (1) *does not belong to $G^s(\Omega)$ in time for any $0 \le s \le 2-\frac{3}{\rho}$. In particular, it is divergent.*

Proof. Since φ_n are given by (11), the assumption about non-negativity of Taylor coefficients of φ implies

$$\varphi_n(x_0) \ge \frac{1}{n!}\partial^{3n}\varphi(x_0). \tag{12}$$

Next, the condition that φ is not an entire function of exponential order ρ is equivalent to

$$\overline{\lim}\sqrt[3n]{\partial^{3n}\varphi(x_0)\big((3n)!\big)^{\frac{1}{\rho}-1}} = \infty$$

which together with (12) contradicts (7) for $s \le 2-\frac{3}{\rho}$ ∎

3. Proof of Theorem 1

Assuming that (2) is a formal power series solution of problem (1) we easily get the recurrence relations (3) for φ_n. Next note that φ_n can be written in the form

$$\varphi_n(x) = \begin{cases} \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{k+\frac{n}{2}}A(n,2k)x^{2k} & \text{if } n \text{ is even} \\ \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^{k+\frac{n-1}{2}}A(n,2k+1)x^{2k+1} & \text{if } n \text{ is odd} \end{cases} \tag{13}$$

where the coefficients $A(n,2k)$ and $A(n,2k+1)$ satisfy recurrence relations

$$\begin{aligned} A(0,2k) &= c \\ A(n+1,2k+1) &= (2k+2)(2k+3)(2k+4)A(n,2k+4) \\ &\quad - (2k+2)\sum_{i=0}^{n}\binom{n}{i}B(i,n-i,2k+2) \quad \text{for } n \text{ even} \\ A(n+1,2k) &= (2k+1)(2k+2)(2k+3)A(n,2k+3) \\ &\quad - (2k+1)\sum_{i=0}^{n}\binom{n}{i}B(i,n-i,2k+1) \quad \text{for } n \text{ odd} \end{aligned} \tag{14}$$

and, for n even or odd, respectively

$$\begin{aligned} &B(i,n-i,2k+2) \\ &\quad = \begin{cases} \sum_{l=0}^{k+1} A(i,2l)A(n-i,2k+2-2l) & \text{if } i \text{ is even} \\ \sum_{l=0}^{k} A(i,2l+1)A(n-i,2k+1-2l) & \text{if } i \text{ is odd} \end{cases} \\ &B(i,n-i,2k+1) \\ &\quad = \begin{cases} \sum_{l=0}^{k} A(i,2l)A(n-i,2k+1-2l) & \text{if } i \text{ is even} \\ \sum_{l=0}^{k} A(i,2l+1)A(n-i,2k-2l) & \text{if } i \text{ is odd.} \end{cases} \end{aligned} \tag{15}$$

Indeed, $\varphi_0(x) = \sum_{k=0}^{\infty}(-1)^k c x^{2k}$. Next, assuming inductively (13), by (3) we get for n even

$$\varphi_{n+1}(x) = \frac{1}{n+1}\Bigg[\frac{1}{n!}\sum_{k=2}^{\infty}(-1)^{k+\frac{n}{2}}2k(2k-1)(2k-2)A(n,2k)x^{2k-3}$$

$$+\sum_{\substack{i=0\\ i-\text{even}}}^{n}\frac{1}{i!(n-i)!}\sum_{k=1}^{\infty}(-1)^{k+\frac{n}{2}}2k$$

$$\times\sum_{l=0}^{k}A(i,2l)A(n-i,2k-2l)x^{2k-1}$$

$$+\sum_{\substack{i=0\\ i-\text{odd}}}^{n}\frac{1}{i!(n-i)!}\sum_{k=0}^{\infty}(-1)^{k+\frac{n-1}{2}}(2k+2)$$

$$\times\sum_{l=0}^{k}A(i,2l+1)A(n-i,2k+1-2l)x^{2k+1}\Bigg]$$

$$=\frac{1}{(n+1)!}\sum_{k=0}^{\infty}(-1)^{k+\frac{n}{2}}A(n+1,2k+1)x^{2k+1}$$

where $A(n+1,2k+1)$ is given by (14). Similarly we get (14) for n odd.

Now, if $c<0$, we easily get (since in (14) we subtract a positive term)

$$A(n,2k+[n]_2)\le\frac{(2k+3n+[n]_2)!}{(2k+[n]_2)!}\,c \tag{16}$$

where $[n]_2=n \bmod 2$. So, for n even,

$$|\varphi_n(0)|=\frac{|A(n,0)|}{n!}\ge\frac{(3n)!}{n!}\,|c|$$

and taking $K=\{0\}$ in Definition 1 we see that the formal solution (2) does not belong to $G^s(\mathbb{R})$ in time for $0\le s<2$.

For $c>0$ estimation (16) does not prove the theorem. Instead, by the Main Lemma we show the following

Claim. *Let* $0<c<5\frac{305}{359}$. *Then*

$$\begin{aligned}c\,\frac{(2k+3n+[n]_2)!}{(2k+[n]_2)!}\bigg(1-\sum_{i=1}^{n}\varepsilon\big(i,2k+3n+[n]_2-3i\big)\bigg)&\\ \le A(n,2k+[n]_2)&\\ \le c\frac{(2k+3n+[n]_2)!}{(2k+[n]_2)!}&\end{aligned} \tag{17}$$

with $\varepsilon(i,l)$ $(i\in\mathbb{N},l\in\mathbb{N}_0)$ *defined by*

$$(l+3i-1)(l+3i)\,\varepsilon(i,l)=\begin{cases}2\frac{9}{70}\,c & \text{if } l=0,i\ge1\\ \big(\frac{l-[l]_2}{2}+2\big)\,c & \text{if } l\ge1,i\ge1.\end{cases} \tag{18}$$

Furthermore,

$$\sum_{i=1}^{n} \varepsilon\big(i, 2k+3n+[n]_2-3i\big) \le \frac{359}{2100}\,c. \tag{19}$$

Proof. First of all we show (19). To this end we derive: for n even, $n \ge 2$ and $k=0$

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon(i, 2k+3n-3i) &= \frac{\sum_{j=1}^{n/2}(3n-6j+5)+\frac{9}{70}}{3n(3n-1)}\,c\\ &= \frac{3n^2+4n+\frac{18}{35}}{12n(3n-1)}\,c\\ &\le \frac{359}{2100}\,c,\end{aligned}$$

for n even, $n \ge 2$ and $k \ge 1$

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon(i, 2k+3n-3i) &= \frac{\sum_{j=1}^{n/2}(3n-6j+5)}{(3n+2k)(3n+2k-1)}\,c\\ &= \frac{3n^2+4n}{4(3n+2k)(3n+2k-1)}\,c\\ &\le \frac{c}{6}\end{aligned}$$

and finally for n odd and $k \in \mathbb{N}_0$

$$\begin{aligned}\sum_{i=1}^{n} \varepsilon(i, 2k+3n+1-3i) &= \frac{k+2+\sum_{j=1}^{(n-1)/2}(2k+3n-6j+6)}{(3n+2k)(3n+2k+1)}\,c\\ &= \frac{3n^2+(4k+6)n-1}{4(3n+2k)(3n+2k+1)}\,c\\ &\le \frac{c}{6}.\end{aligned}$$

To prove (17) observe that it trivially holds for $n=0$ since $A(0,2k)=c$. Next, if $n=1$, we get by (14)

$$\begin{aligned}A(1,2k+1) &= (2k+2)(2k+3)(2k+4)c-(2k+2)(k+2)\,c^2\\ &\le c\frac{(2k+4)!}{(2k+1)!}\big(1-\varepsilon(1,2k+1)\big)\end{aligned}$$

with $\varepsilon(1,2k+1)=c\frac{k+2}{(2k+3)(2k+4)}$. Now fix $m \in \mathbb{N}$ and assume that (17) holds for $n \le m$ and $k \in \mathbb{N}_0$. Since $A(m+1, 2k+1-[m]_2)$ is given by (14) (with

$n = m$) and by (19) in (14) we subtract a positive term (since $0 < c < 5\frac{305}{359}$), we easily get the estimation from above

$$A(m+1, 2k+1-[m]_2) = A(m, 2k+4-[m]_2)\prod_{l=2}^{4}(2k+l-[m]_2)$$
$$\le c\frac{(2k+3(m+1)+1-[m]_2)!}{(2k+1-[m]_2)!}.$$

To estimate $A(m+1, 2k+1-[m]_2)$ from below we need to estimate the second term of (14) from above. By the inductive assumption, (15) and (4) we derive for m even

$$\sum_{i=0}^{m}\binom{m}{i}B(i, m-i, 2k+2)$$
$$\le \sum_{\substack{i=0\\ i-\text{even}}}^{m}\binom{m}{i}\sum_{l=0}^{k+1}c^2\frac{(2l+3i)!}{(2l)!}\,\frac{(2k+2-2l+3m-3i)!}{(2k+2-2l)!}$$
$$+ \sum_{\substack{i=0\\ i-\text{odd}}}^{m}\binom{m}{i}\sum_{l=0}^{k}c^2\frac{(2l+3i+1)!}{(2l+1)!}\,\frac{(2k-2l+3m-3i+1)!}{(2k-2l+1)!}$$
$$= c^2\frac{(2k+3m+2)!}{(2k+2)!}\cdot C(k,m)$$

and for m odd

$$\sum_{i=0}^{m}\binom{m}{i}B(i, m-i, 2k+1)$$
$$\le \sum_{\substack{i=0\\ i-\text{even}}}^{m}\binom{m}{i}\sum_{l=0}^{k}c^2\frac{(2l+3i)!}{(2l)!}\,\frac{(2k-2l+3m-3i+1)!}{(2k-2l+1)!}$$
$$+ \sum_{\substack{i=0\\ i-\text{odd}}}^{m}\binom{m}{i}\sum_{l=0}^{k}c^2\frac{(2l+3i+1)!}{(2l+1)!}\,\frac{(2k-2l+3m-3i)!}{(2k-2l)!}$$
$$= c^2\frac{(2k+3m+1)!}{(2k+1)!}\,C(k,m).$$

So, by the Main Lemma and (18) we get

$$A(m+1, 2k+1-[m]_2)$$
$$\ge c\,\frac{(2k+3m+4-[m]_2)!}{(2k+1-[m]_2)!}\left(1-\sum_{i=1}^{m}\varepsilon(i, 2k+4-[m]_2+3m-3i)\right)$$
$$- c\,\frac{(2k+3m+4-[m]_2)!}{(2k+1-[m]_2)!}\,\varepsilon(m+1, 2k+1-[m]_2).$$

Hence (17) holds for $n = m + 1$ ∎

Returning to the proof of Theorem 1 take $K = \{0\}$ in Definition 1. Since for n even the Claim implies

$$|\varphi_n(0)| = \frac{A(n,0)}{n!} \geq c\frac{(3n)!}{n!}\Big(1 - \sum_{i=1}^{n} \epsilon(i, 3n-3i)\Big) \geq c\big(1 - \frac{359}{2100}c\big)\frac{(3n)!}{n!}$$

the formal solution (2) does not belong to $G^s(\mathbb{R})$ in time for $0 \leq s < 2$ if $0 < c < 5\frac{305}{359}$ ∎

4. The representation of $C(k, n)$

The rest of the paper is devoted to the proof of the Main Lemma. It is based on the inequality $C(k,n) \geq C(k,n+2)$. In order to prove it we represent $C(k,n)$ as a linear combination of sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n)$ given by (6) with some $\alpha, \beta, \gamma, \delta \in \mathbb{N}_0$. The actual form of the representation depends on $k \bmod 3$ and $n \bmod 2$ and is given in Lemma 1. We shall give the proof of Lemma 1 only in the case when n is even and k is divisible by 3. The other cases can be treated analogously. In the next Section we prove that the sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ appearing in the representation are decreasing.

Lemma 1. *The $C(k,n)$ given by (4) can be represented as follows:*

`Case A.` *Let $k = 3\overline{k}$ and $n = 2\overline{n}$ with some $\overline{k}, \overline{n} \in \mathbb{N}_0$. Then*

$$\begin{aligned}
C(k,n) = 2 &\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(k,6i)\; {}^{0}_{0}D^{12i+2}_{6i}(\overline{n} + \overline{k} - 2i) \\
&+ 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+2)\; {}^{1}_{0}D^{12i+8}_{6i+2}(\overline{n} + \overline{k} - 2i - 1) \\
&+ 2 \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+3)\; {}^{2}_{1}D^{12i+14}_{6i+6}(\overline{n} + \overline{k} - 2i - 2) \\
&+ 2 \sum_{i=0}^{\lfloor \overline{k}/2 - 1 \rfloor} a(k,6i+5)\; {}^{1}_{0}D^{12i+14}_{6i+6}(\overline{n} + \overline{k} - 2i - 2) \\
&+ \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} b(k,6i+1)\; {}^{2}_{1}D^{12i+8}_{6i+4}(\overline{n} + \overline{k} - 2i - 1) \\
&+ \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} b(k,6i+4)\; {}^{0}_{0}D^{12i+8}_{6i+4}(\overline{n} + \overline{k} - 2i - 1)
\end{aligned} \qquad (20A)$$

where $\lfloor m \rfloor$ *denotes the integer part of* m *and*

$$\begin{aligned} a(k,3l) &= 3^{3l}\binom{2\overline{k}+l}{3l} && \mathit{for}\quad 0\le l\le \overline{k} \\ a(k,3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+1}{3l+2} && \mathit{for}\quad 0\le l\le \overline{k}-1 \\ b(k,3l+1) &= \frac{3^{3l}(6\overline{k}+2)}{3l+1}\binom{2\overline{k}+l}{3l} && \mathit{for}\quad 0\le l\le \overline{k}. \end{aligned} \tag{21A}$$

`Case B`. *Let* $k=3\overline{k}+1$ *and* $n=2\overline{n}$ *with some* $\overline{k},\overline{n}\in\mathbb{N}_0$. *Then*

$$\begin{aligned} C(k,n) = {} & 2\sum_{i=0}^{\lfloor \overline{k}/2\rfloor} a(k,6i)\ {}_0^1D_{6i}^{12i+4}(\overline{n}+\overline{k}-2i) \\ & + 2\sum_{i=0}^{\lfloor \overline{k}/2\rfloor} a(k,6i+1)\ {}_1^2D_{6i+4}^{12i+10}(\overline{n}+\overline{k}-2i-1) \\ & + 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} a(k,6i+3)\ {}_0^1D_{6i+4}^{12i+10}(\overline{n}+\overline{k}-2i-1) \\ & + 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} a(k,6i+4)\ {}_0^0D_{6i+4}^{12i+10}(\overline{n}+\overline{k}-2i-1) \\ & + \sum_{i=0}^{\lfloor \overline{k}/2\rfloor} b(k,6i+2)\ {}_0^0D_{6i+2}^{12i+4}(\overline{n}+\overline{k}-2i) \\ & + \sum_{i=0}^{\lfloor (\overline{k}-1)/2\rfloor} b(k,6i+5)\ {}_1^2D_{6i+8}^{12i+16}(\overline{n}+\overline{k}-2i-2) \end{aligned} \tag{20B}$$

where

$$\begin{aligned} a(k,3l) &= 3^{3l}\binom{2\overline{k}+l+1}{3l} && \mathit{for}\quad 0\le l\le \overline{k} \\ a(k,3l+1) &= 3^{3l+1}\binom{2\overline{k}+l+1}{3l+1} && \mathit{for}\quad 0\le l\le \overline{k} \\ b(k,3l+2) &= \frac{3^{3l+1}(6\overline{k}+4)}{3l+2}\binom{2\overline{k}+l+1}{3l+1} && \mathit{for}\quad 0\le l\le \overline{k}. \end{aligned} \tag{21B}$$

`Case C`. *Let* $k=3\overline{k}+2$ *and* $n=2\overline{n}$ *with some* $\overline{k},\overline{n}\in\mathbb{N}_0$. *Then*

$$C(k,n) = \sum_{i=0}^{\lfloor (\overline{k}+1)/2\rfloor} b(k,6i)\ {}_0^0D_{6i}^{12i}(\overline{n}+\overline{k}-2i+1)$$

$$
\begin{aligned}
&+ \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} b(k,6i+3)\ {}^{2}_{1}D^{12i+12}_{6i+6}(\overline{n}+\overline{k}-2i-1) \\
&+ 2\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(k,6i+1)\ {}^{1}_{0}D^{12i+6}_{6i+2}(\overline{n}+\overline{k}-2i) \\
&+ 2\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(k,6i+2)\ {}^{0}_{0}D^{12i+6}_{6i+2}(\overline{n}+\overline{k}-2i) \\
&+ 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+4)\ {}^{1}_{0}D^{12i+12}_{6i+4}(\overline{n}+\overline{k}-2i-1) \\
&+ 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+5)\ {}^{2}_{1}D^{12i+18}_{6i+8}(\overline{n}+\overline{k}-2i-2)
\end{aligned}
\tag{20C}
$$

where

$$
\begin{aligned}
a(k,3l+1) &= 3^{3l+1}\binom{2\overline{k}+l+2}{3l+1} && \text{for} \quad 0 \le l \le \overline{k} \\
a(k,3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+2}{3l+2} && \text{for} \quad 0 \le l \le \overline{k} \\
b(k,0) &= 1 \\
b(k,3l) &= \frac{3^{3l-1}(6\overline{k}+6)}{3l}\binom{2\overline{k}+l+1}{3l-1} && \text{for} \quad 1 \le l \le \overline{k}+1.
\end{aligned}
\tag{21C}
$$

`Case D.` *Let* $k = 3\overline{k}$ *and* $n = 2\overline{n}+1$ *with some* $\overline{k}, \overline{n} \in \mathbb{N}_0$. *Then*

$$
\begin{aligned}
C(k,n) = {} & 2\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(k,6i)\ {}^{1}_{0}D^{12i+4}_{6i}(\overline{n}+\overline{k}-2i) \\
&+ 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+1)\ {}^{2}_{1}D^{12i+10}_{6i+4}(\overline{n}+\overline{k}-2i-1) \\
&+ 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+3)\ {}^{1}_{0}D^{12i+10}_{6i+4}(\overline{n}+\overline{k}-2i-1) \\
&+ 2\sum_{i=0}^{\lfloor \overline{k}/2-1 \rfloor} a(k,6i+4)\ {}^{0}_{0}D^{12i+10}_{6i+4}(\overline{n}+\overline{k}-2i-1)
\end{aligned}
$$

$$+\sum_{i=0}^{\lfloor(\overline{k}-1)/2\rfloor} b(k,6i+2)\ {}_0^0D_{6i+2}^{12i+4}(\overline{n}+\overline{k}-2i)$$
$$+\sum_{i=0}^{\lfloor\overline{k}/2-1\rfloor} b(k,6i+5)\ {}_1^2D_{6i+8}^{12i+16}(\overline{n}+\overline{k}-2i-2) \tag{20D}$$

where

$$\begin{aligned}
a(k,3l) &= 3^{3l}\binom{2\overline{k}+l}{3l} && \textit{for} \quad 0\le l\le \overline{k}\\
a(k,3l+1) &= 3^{3l+1}\binom{2\overline{k}+l}{3l+1} && \textit{for} \quad 0\le l\le \overline{k}-1\\
b(k,3l+2) &= \frac{3^{3l+1}(6\overline{k}+1)}{3l+2}\binom{2\overline{k}+l}{3l+1} && \textit{for} \quad 0\le l\le \overline{k}-1.
\end{aligned} \tag{21D}$$

`Case E.` *Let* $k=3\overline{k}+1$ *and* $n=2\overline{n}+1$ *with some* $\overline{k},\overline{n}\in\mathbb{N}_0$. *Then*

$$\begin{aligned}
C(k,n) = &\sum_{i=0}^{\lfloor\overline{k}/2\rfloor} b(k,6i)\ {}_0^1D_{6i}^{12i}(\overline{n}+\overline{k}-2i+1)\\
&+\sum_{i=0}^{\lfloor(\overline{k}-1)/2\rfloor} b(k,6i+3)\ {}_1^2D_{6i+6}^{12i+12}(\overline{n}+\overline{k}-2i-1)\\
&+2\sum_{i=0}^{\lfloor\overline{k}/2\rfloor} a(k,6i+1)\ {}_0^1D_{6i+2}^{12i+6}(\overline{n}+\overline{k}-2i)\\
&+2\sum_{i=0}^{\lfloor(\overline{k}-1)/2\rfloor} a(k,6i+2)\ {}_0^0D_{6i+2}^{12i+6}(\overline{n}+\overline{k}-2i)\\
&+2\sum_{i=0}^{\lfloor(\overline{k}-1)/2\rfloor} a(k,6i+4)\ {}_0^1D_{6i+4}^{12i+12}(\overline{n}+\overline{k}-2i-1)\\
&+2\sum_{i=0}^{\lfloor\overline{k}/2-1\rfloor} a(k,6i+5)\ {}_1^2D_{6i+8}^{12i+18}(\overline{n}+\overline{k}-2i-2)
\end{aligned} \tag{20E}$$

where

$$\begin{aligned}
a(k,3l+1) &= 3^{3l+1}\binom{2\overline{k}+l+1}{3l+1} && \textit{for} \quad 0\le l\le \overline{k}\\
a(k,3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+1}{3l+2} && \textit{for} \quad 0\le l\le \overline{k}-1\\
b(k,0) &= 1
\end{aligned} \tag{21E}$$

$$b(k,3l) = \frac{3^{3l-1}(6\overline{k}+3)}{3l}\binom{2\overline{k}+l}{3l-1} \quad for \quad 1 \le l \le \overline{k}.$$

`Case F.` *Let* $k = 3\overline{k}+2$ *and* $n = 2\overline{n}+1$ *with some* $\overline{k}, \overline{n} \in \mathbb{N}_0$. *Then*

$$\begin{aligned} C(k,n) = 2 &\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(k,6i)\ {}^{0}_{0}D^{12i+2}_{6i}(\overline{n}+\overline{k}-2i+1) \\ &+ 2\sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} a(k,6i+2)\ {}^{1}_{0}D^{12i+8}_{6i+2}(\overline{n}+\overline{k}-2i) \\ &+ 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+3)\ {}^{2}_{1}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-1) \\ &+ 2\sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} a(k,6i+5)\ {}^{1}_{0}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-1) \\ &+ \sum_{i=0}^{\lfloor \overline{k}/2 \rfloor} b(k,6i+1)\ {}^{2}_{1}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i) \\ &+ \sum_{i=0}^{\lfloor (\overline{k}-1)/2 \rfloor} b(k,6i+4)\ {}^{0}_{0}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i) \end{aligned} \tag{20F}$$

where

$$\begin{aligned} a(k,3l) &= 3^{3l}\binom{2\overline{k}+l+1}{3l} && for \quad 0 \le l \le \overline{k}, \\ a(k,3l+2) &= 3^{3l+2}\binom{2\overline{k}+l+2}{3l+2} && for \quad 0 \le l \le \overline{k}, \\ b(k,3l+1) &= \frac{3^{3l}(6\overline{k}+5)}{3l+1}\binom{2\overline{k}+l+1}{3l} && for \quad 0 \le l \le \overline{k}. \end{aligned} \tag{21F}$$

Proof. We shall prove Lemma 1 only in Case A, since the proofs of the other cases are analogous. So let $n = 2\overline{n}$ be even. Assuming that $\binom{m}{i} = 0$ if $|m-2i| > m$ with $m \in \mathbb{N}_0$ and $i \in \mathbb{Z}$, we get by (4)

$$\begin{aligned} C(k,n) &= \sum_{i=0}^{\overline{n}}\sum_{l=0}^{k+1} \frac{\binom{2\overline{n}}{2i}\binom{2k+2}{2l}}{\binom{2k+6\overline{n}+2}{2l+6i}} + \sum_{i=0}^{\overline{n}-1}\sum_{l=0}^{k} \frac{\binom{2\overline{n}}{2i+1}\binom{2k+2}{2l+1}}{\binom{2k+6\overline{n}+2}{2l+6i+4}} \\ &= \sum_{j=0}^{\overline{n}+\lfloor (k+1)/3 \rfloor} \frac{\sum_{i=0}^{\overline{n}}\binom{2\overline{n}}{2i}\binom{2k+2}{6j-6i} + \sum_{i=0}^{\overline{n}-1}\binom{2\overline{n}}{2i+1}\binom{2k+2}{6j-6i-3}}{\binom{2k+6\overline{n}+2}{6j}} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{\overline{n}+\lfloor k/3\rfloor} \frac{\sum\limits_{i=0}^{\overline{n}} \binom{2\overline{n}}{2i}\binom{2k+2}{6j-6i+2} + \sum_{i=0}^{\overline{n}-1}\binom{2\overline{n}}{2i+1}\binom{2k+2}{6j-6i-1}}{\binom{2k+6\overline{n}+2}{6j+2}} \\
&+\sum_{j=0}^{\overline{n}+\lfloor (k-1)/3\rfloor} \frac{\sum\limits_{i=0}^{\overline{n}} \binom{2\overline{n}}{2i}\binom{2k+2}{6j-6i+4} + \sum_{i=0}^{\overline{n}-1}\binom{2\overline{n}}{2i+1}\binom{2k+2}{6j-6i+1}}{\binom{2k+6\overline{n}+2}{6j+4}} \\
&= \sum_{j=0}^{\overline{n}+\lfloor (k+1)/3\rfloor} \frac{1}{\binom{2k+6\overline{n}+2}{6j}} \sum_{l=0}^{\lfloor (2k+2)/3\rfloor} \binom{2k+2}{3l}\binom{2\overline{n}}{2j-l} \\
&+\sum_{j=0}^{\overline{n}+\lfloor k/3\rfloor} \frac{1}{\binom{2k+6\overline{n}+2}{6j+2}} \sum_{l=0}^{\lfloor 2k/3\rfloor} \binom{2k+2}{3l+2}\binom{2\overline{n}}{2j-l} \\
&+\sum_{j=0}^{\overline{n}+\lfloor (k-1)/3\rfloor} \frac{1}{\binom{2k+6\overline{n}+2}{6j+4}} \sum_{l=0}^{\lfloor (2k+1)/3\rfloor} \binom{2k+2}{3l+1}\binom{2\overline{n}}{2j+1-l} \\
&=: C_1(k,n) + C_2(k,n) + C_3(k,n).
\end{aligned}
$$

Let us first consider $C_3(k,n)$. In Case A we have $k = 3\overline{k}$ and $n = 2\overline{n}$. So

$$
C_3(k,n) = \sum_{j=0}^{\overline{n}+\overline{k}-1} \binom{6\overline{k}+6\overline{n}+2}{6j+4}^{-1} \sum_{l=0}^{2\overline{k}} \binom{6\overline{k}+2}{3l+1}\binom{2\overline{n}}{2j+1-l}.
$$

Now we apply

$$
\sum_{l=l_1}^{2\overline{k}-l_2} \binom{2\overline{k}-l_1-l_2}{l-l_1}\binom{2\overline{n}}{2j-l} = \binom{2\overline{n}+2\overline{k}-l_1-l_2}{2j-l_1} \tag{22}
$$

with $l_1 = l_2 = 0$ and (6) with $\alpha = 2, \beta = 1, \gamma = 8$ and $\delta = 4$ to get

$$
\begin{aligned}
C_3(k,n) &= b(k,1)\, {}^{2}_{1}D^{8}_{4}(\overline{n}+\overline{k}-1) \\
&\quad + \sum_{j=0}^{\overline{n}+\overline{k}-1} \binom{6\overline{k}+6\overline{n}+2}{6j+4}^{-1} \\
&\quad \times \sum_{l=1}^{2\overline{k}-1} \left\{ \binom{6\overline{k}+2}{3l+1} - b(k,1)\binom{2\overline{k}}{l} \right\} \binom{2\overline{n}}{2j+1-l}
\end{aligned}
$$

where $b(k,1) = \binom{6\overline{k}+2}{1}$.

In the next step we apply (22) with $l_1 = l_2 = 1$ and (6) with $\alpha = \beta = 0, \gamma = 8$ and $\delta = 4$ to get

$$\begin{aligned} C_3(k,n) &= b(k,1)\ {}^{2}_{1}D^{8}_{4}(\overline{n}+\overline{k}-1) + b(k,4)\ {}^{0}_{0}D^{8}_{4}(\overline{n}+\overline{k}-1) \\ &\quad + \sum_{j=1}^{\overline{n}+\overline{k}-2} \binom{6\overline{k}+6\overline{n}+2}{6j+4}^{-1} \\ &\quad \times \sum_{l=2}^{2\overline{k}-2} \left\{ \binom{6\overline{k}+2}{3l+1} - b(k,1)\binom{2\overline{k}}{l} - b(k,4)\binom{2\overline{k}-2}{l-1} \right\} \\ &\quad \times \binom{2\overline{n}}{2j+1-l} \end{aligned}$$

where $b(k,4) = \binom{6\overline{k}+2}{4} - b(k,1)\binom{2\overline{k}}{1}$. Continuing the above procedure one can prove inductively that, for $m \in \{0, 1, ..., \lfloor \overline{k}/2 \rfloor + 1\}$,

$$C_3(k,n) = \sum_{i=0}^{m-1} A(i) + \sum_{j=m}^{\overline{n}+\overline{k}-1-m} B(j) \tag{23}$$

where

$$\begin{aligned} A(i) &= b(k,6i+1)\ {}^{2}_{1}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i-1) \\ &\quad + b(k,6i+4)\ {}^{0}_{0}D^{12i+8}_{6i+4}(\overline{n}+\overline{k}-2i-1) \\ B(j) &= \binom{6\overline{n}+6\overline{k}+2}{6j+4}^{-1} \sum_{l=2m}^{2\overline{k}-2m} \binom{2\overline{n}}{2j+1-l} \left\{ \binom{6\overline{k}+2}{3l+1} \right. \\ &\quad \left. - \sum_{i=0}^{m-1} \left[b(k,6i+1)\binom{2\overline{k}-4i}{l-2i} + b(k,6i+4)\binom{2\overline{k}-4i-2}{l-2i-1} \right] \right\} \end{aligned}$$

and the coefficients $b(k, 3l+1)$ $(l \in \mathbb{N}_0)$ satisfy a recurrence relation

$$b(k,3l+1) = \binom{6\overline{k}+2}{3l+1} - \sum_{j=0}^{l-1} b(k,3j+1)\binom{2\overline{k}-2j}{l-j}. \tag{24}$$

As about $C_1(k,n)$ and $C_2(k,n)$ we follow the procedure described for $C_3(k,n)$. In fact, noting the relation ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n) = {}^{\alpha}_{\alpha-\beta}D^{\gamma}_{\gamma-\delta}(n)$ and the symmetry of binomial coefficients one can prove inductively that, for $m \in \{0, 1, ..., \lfloor \overline{k}/2 \rfloor + 1\}$,

$$C_1(k,n) = C_2(k,n) = \sum_{i=0}^{m-1} A(i) + \sum_{j=m}^{\overline{n}+\overline{k}-m} B(j) \tag{25}$$

where

$$\begin{aligned} A(i) = {} & a(k,6i)\, {}^{0}_{0}D^{12i+2}_{6i}(\overline{n}+\overline{k}-2i) \\ & + a(k,6i+2)\, {}^{1}_{0}D^{12i+8}_{6i+2}(\overline{n}+\overline{k}-2i-1) \\ & + a(k,6i+3)\, {}^{2}_{1}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-2) \\ & + a(k,6i+5)\, {}^{1}_{0}D^{12i+14}_{6i+6}(\overline{n}+\overline{k}-2i-2) \end{aligned}$$

$$\begin{aligned} B(j) = {} & \binom{6\overline{n}+6\overline{k}+2}{6j}^{-1} \sum_{l=2m}^{2\overline{k}-2m} \binom{2\overline{n}}{2j-l} \Bigg\{ \binom{6\overline{k}+2}{3l} \\ & - \sum_{i=0}^{m-1} \Bigg[a(k,6i)\binom{2\overline{k}-4i}{l-2i} + a(k,6i+2)\binom{2\overline{k}-4i-1}{l-2i-1} \\ & + a(k,6i+3)\binom{2\overline{k}-4i-2}{l-2i-1} + a(k,6i+5)\binom{2\overline{k}-4i-3}{l-2i-2} \Bigg] \Bigg\} \end{aligned}$$

and the coefficients $a(k,3l)$ and $a(k,3l{+}2)$ $(l \in \mathbb{N}_0)$ satisfy recurrence relations

$$\begin{aligned} a(k,3l) = {} & \binom{6\overline{k}+2}{3l} \\ & - \sum_{j=0}^{l-1} \left[a(k,3j)\binom{2\overline{k}-2j}{l-j} + a(k,3j+2)\binom{2\overline{k}-2j-1}{l-j-1} \right] \\ a(k,3l+2) = {} & \binom{6\overline{k}+2}{3l+2} - a(k,3l) \\ & - \sum_{j=0}^{l-1} \left[a(k,3j)\binom{2\overline{k}-2j}{l-j} + a(k,3j+2)\binom{2\overline{k}-2j-1}{l-j} \right]. \end{aligned} \tag{26}$$

Finally, noting that for $m = \lfloor \overline{k}/2 \rfloor + 1$ the second summands in (23) and in (25) vanish, by Lemma 2 (stated below) we get (20A) and (21A), proving Lemma 1 in Case A. The proofs of Cases B - E can be done in the same way ∎

Lemma 2. *Let* $a(k,3l)$, $a(k,3l+2)$ *and* $b(k,3l+1)$ $(\overline{k}, l \in \mathbb{N}_0)$ *satisfy recurrence relations* (24) *and* (26)*. Then* (21A) *holds.*

Proof. For $l = 0$ we clearly have

$$\begin{aligned} a(k,0) &= 1 \\ a(k,2) &= \binom{6\overline{k}+2}{2} - 1 = 3^2\binom{2\overline{k}+1}{2} \\ b(k,1) &= 6\overline{k}+2. \end{aligned}$$

To prove (21A) for $l \geq 1$ put $2\overline{k} = \tilde{k}$ and note that it is sufficient to show that

$$\binom{3\tilde{k}+2}{3l} = \sum_{j=0}^{l} 3^{3j}\left[\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j} + 9\binom{\tilde{k}+j+1}{3j+2}\binom{\tilde{k}-2j-1}{l-j-1}\right] \tag{27}$$

$$\binom{3\tilde{k}+2}{3l+2} = \sum_{j=0}^{l} 3^{3j}\left[\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j} + 9\binom{\tilde{k}+j+1}{3j+2}\binom{\tilde{k}-2j-1}{l-j}\right] \tag{28}$$

$$\binom{3\tilde{k}+2}{3l+1} = \sum_{j=0}^{l} \frac{3^{3j}(3\tilde{k}+2)}{3j+1}\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j}. \tag{29}$$

To show (27) for a fixed $l \in \mathbb{N}$ we observe that the left-hand side is a polynomial on $\tilde{k}$ of degree $3l$, with the leading coefficient $\frac{3^{3l}}{(3l)!}$, vanishing at $\tilde{k} = 0, ..., l-1$ and at $\tilde{k} = -\frac{1}{3} + m$ and $\tilde{k} = -\frac{2}{3} + m$ with $m = 0, ..., l-1$. Clearly, the first two statements also hold for the right-hand side of (27). So we only need to prove the third one. To this end we compute for $j = 0, ..., l$

$$\begin{aligned} &3^{3j}\left[\binom{\tilde{k}+j}{3j}\binom{\tilde{k}-2j}{l-j} + 9\binom{\tilde{k}+j+1}{3j+2}\binom{\tilde{k}-2j-1}{l-j-1}\right] \\ &\quad = (\tilde{k}-l+1)\cdots\tilde{k} \times \left(1 + \frac{9(l-j)(\tilde{k}+j+1)}{(3j+1)(3j+2)}\right) \\ &\quad\quad \times \frac{3^{3j}(\tilde{k}-l-j+1)\cdots(\tilde{k}-l)\cdot(\tilde{k}+1)\cdot\ldots\cdot(\tilde{k}+j)}{(l-j)!(3j)!}. \end{aligned}$$

So it is sufficient to show that the polynomial

$$\begin{aligned} W_l(\tilde{k}) = 1 + \frac{9l(\tilde{k}+1)}{2} + \sum_{j=1}^{l} \frac{l!}{(l-j)!}\left(1 + \frac{9(l-j)(\tilde{k}+j+1)}{(3j+1)(3j+2)}\right) \\ \times \frac{3^{3j}(\tilde{k}-l-j+1)\cdots(\tilde{k}-l)(\tilde{k}+1)\cdot\ldots\cdot(\tilde{k}+j)}{(3j)!} \end{aligned}$$

vanishes for $\tilde{k} = -\frac{1}{3} + m$ and $\tilde{k} = -\frac{2}{3} + m$ with $m = 0, ..., l-1$. But for

$m = 0, ..., l-1$ we derive

$$W_l(-\frac{1}{3}+m) = 1 + \frac{3l(3m+2)}{2}$$
$$+ \sum_{j=1}^{l} (-1)^j \frac{l!}{(l-j)!j!} \left(1 + \frac{3(l-j)(3m+3j+2)}{(3j+1)(3j+2)}\right)$$
$$\times \prod_{i=0}^{j-1} \frac{(3l-3m+3i+1)(3m+3i+2)}{(3i+1)(3i+2)}$$
$$= \sum_{j=0}^{l} (-1)^j \binom{l}{j} P_l(j)$$
$$= 0$$

since for $m = 0, 1, ..., l-1$

$$P_l(j) = \frac{(9l-9m+3)j + 9lm + 6l + 2}{(3j+1)(3j+2)}$$
$$\times \prod_{i=0}^{l-m-1} \frac{3j+3i+1}{3i+1} \prod_{i=0}^{m-1} \frac{3j+3i+2}{3i+2}$$

is a polynomial of degree $l-1$ and, for such a polynomial P, $\sum_{j=0}^{l} (-1)^j \binom{l}{j} P(j) = 0$.

Analogously, for $m = 0, ..., l-1$,

$$W_l(-\frac{2}{3}+m) = 1 + \frac{3l(3m+1)}{2}$$
$$+ \sum_{j=1}^{l} (-1)^j \frac{l!}{(l-j)!j!} \left(1 + \frac{3(l-j)(3m+3j+2)}{(3j+1)(3j+2)}\right)$$
$$\times \prod_{i=0}^{j-1} \frac{(3m+3i+1)(3l-3m+3i+2)}{(3i+1)(3i+2)}$$
$$= \sum_{j=0}^{l} (-1)^j \binom{l}{j} P_l(j)$$
$$= 0$$

where

$$P_l(j) = \frac{(9l-9m+6)j + 9lm + 3l + 2}{(3j+1)(3j+2)}$$
$$\times \prod_{i=0}^{m-1} \frac{3j+3i+1}{3i+1} \prod_{i=0}^{l-m-1} \frac{3j+3i+2}{3i+2}$$

is a polynomial of degree $l-1$. The proofs of (28) and (29) go along the same lines ∎

5. An auxiliary lemma

In the proof of the Main Lemma we shall also need

Lemma 3. *Let* ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ *be given by* (6). *Assume one of the following cases:*
Case 1° : $\alpha = 0, \beta = 0, \delta \geq 0, \gamma = 2\delta + \eta$ *with* $\eta = 0, 2$.
Case 2° : $\alpha = 1, \beta = 0, \delta \geq 0, \gamma = 2\delta + \eta$ *with* $\eta = 2, 4$.
Case 3° : $\alpha = 2, \beta = 1, \delta \geq 2, \gamma = 2\delta + \eta$ *with* $\eta = 0, 2$.
Then, for $n \geq 2$,

$$ {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1). \tag{30}$$

Furthermore, (30) *holds for* $n = 0$ *if* $\delta \geq 1$, *and for* $n = 1$ *except Case* 1° *with* $\delta = \gamma = 0$.

Proof. For $n, l \in \mathbb{N}_0$ put

$$ {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l) = \binom{2n+\alpha}{2l+\beta} \Big/ \binom{6n+\gamma}{6l+\delta}.$$

Note that it is sufficient to show that for n even the following inequalities hold:

$$\text{(IN1)} \begin{cases} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,l) \\ \qquad\qquad\quad \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,l) \end{cases} \qquad (l = 0, ..., \tfrac{n}{2} - 1, n \geq 2)$$

$$\text{(IN2)} \begin{cases} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,l+1) \\ \qquad\qquad\quad \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,l+2) \end{cases} \qquad (l = \tfrac{n}{2} + 1, ..., n, n \geq 2)$$

$$\text{(IN3)} \begin{cases} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,\tfrac{n}{2}) \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,\tfrac{n}{2}) \\ \qquad\qquad\quad + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,\tfrac{n}{2}+1) \end{cases} \qquad \begin{pmatrix} n \geq 2 \\ \text{and} \\ n = 0 \text{ if } \delta \geq 1 \end{pmatrix}$$

$$\text{(IN4)} \begin{cases} {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,\tfrac{n}{2}) + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+1,\tfrac{n}{2}+1) \\ \geq {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,\tfrac{n}{2}) + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,\tfrac{n}{2}+1) \\ \quad + {}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n+2,\tfrac{n}{2}+2) \end{cases} \qquad \begin{pmatrix} n \geq 2 \\ \text{and} \\ n = 0 \text{ except Case } 1^\circ \\ \text{with } \gamma = \delta = 0 \end{pmatrix}.$$

`Proof of inequality (IN1).` Using the definition of ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}(n,l)$, expanding the binomial coefficients and cancelling similar factors we see that the first inequality in (IN1) is equivalent to

$$\prod_{i=1}^{2} \frac{2n+\alpha+i}{2n-2l+\alpha-\beta+i} \prod_{j=1}^{6} \frac{6n-6l+\gamma-\delta+j}{6n+\gamma+j} \leq 1 \tag{31}$$

which in tern is implied by (since $\delta \geq 0$)

$$\Big(1 - \frac{6l+\delta}{6n+\gamma+5}\Big)^3 \leq 1 - \frac{2l+\beta}{2n+\alpha+1} \tag{31a}$$

$$\Big(1 - \frac{6l+\delta}{6n+\gamma+6}\Big)^3 \leq 1 - \frac{2l+\beta}{2n+\alpha+2}. \tag{31b}$$

To show inequality (30a) we fix $l \in \mathbb{N}_0$ and put $n = 2l + 2m + 2$ with some $m \in \mathbb{N}_0$,

$$x = \frac{6l+\delta}{12l+12m+\gamma+17}, \quad y = \frac{2l+\beta}{4l+4m+\alpha+5}.$$

Next note that since $x \geq 0$ it is sufficient to show that $3x - 3x^2 \geq y$. But $\gamma = 2\delta + \eta$ with $\eta \geq -17$ implies that $x \leq \frac{1}{2}$ and so $3x - 3x^2 \geq \frac{3}{2}x$. Now $\frac{3}{2}x \geq y$ is equivalent to

$$\begin{aligned} &24l^2 + (24m + 12\delta - 4\gamma + 18\alpha - 24\beta + 22)l \\ &\quad + (12\delta - 24\beta)m + 3\delta(\alpha+5) - (2\gamma+34)\beta \geq 0 \end{aligned} \tag{32a}$$

which in Cases 1° - 3° holds for any $l \in \mathbb{N}, m \in \mathbb{N}_0$ and for $l = 0, m \in \mathbb{N}_0$ except Case 3° with $\delta = \eta = 2$. Inequality (31b) we treat in the same way with

$$x = \frac{6l+\delta}{12l+12m+\gamma+18}, \quad y = \frac{2l+\beta}{4l+4m+\alpha+6}$$

and (32a) replaced by

$$\begin{aligned} &24l^2 + (24m + 12\delta - 4\gamma + 18\alpha - 24\beta + 36)l \\ &\quad + (12\delta - 24\beta)m + 3\delta(\alpha+6) - (2\gamma+36)\beta \geq 0. \end{aligned} \tag{32b}$$

Finally, we directly show inequality (31) for $l = 0$ in the exceptional Case 3° with $\delta = 2$ and $\gamma = 6$.

The second inequality in (IN1) is equivalent to

$$\prod_{i=3}^{4} \frac{2n+\alpha+i}{2n-2l+\alpha-\beta+i} \prod_{j=7}^{12} \frac{6n-6l+\gamma-\delta+j}{6n+\gamma+j} \leq 1 \tag{33}$$

which in tern is implied by (since $\delta \geq 0$)

$$\Big(1 - \frac{6l+\delta}{6n+\gamma+11}\Big)^3 \leq 1 - \frac{2l+\beta}{2n+\alpha+3} \tag{33a}$$

$$\Big(1 - \frac{6l+\delta}{6n+\gamma+12}\Big)^3 \leq 1 - \frac{2l+\beta}{2n+\alpha+4}. \tag{33b}$$

To show inequality (33a) we follow the proof of inequality (31a) with

$$x = \frac{6l+\delta}{12l+12m+\gamma+23}, \quad y = \frac{2l+\beta}{4l+4m+\alpha+7}$$

and inequality (32a) replaced by

$$\begin{aligned} 24l^2 &+ (24m + 12\delta - 4\gamma + 18\alpha - 24\beta + 34)l \\ &+ (12\delta - 24\beta)m + 3\delta(\alpha+7) - (2\gamma+46)\beta \geq 0 \end{aligned} \tag{34a}$$

which in Cases 1° - 3° holds for any $l \in \mathbb{N}, m \in \mathbb{N}_0$ and for $l = 0, m \in \mathbb{N}_0$ except Case 3° with $\delta = \eta = 2$. Inequality (33b) can be treated in the same way with

$$x = \frac{6l+\delta}{12l+12m+\gamma+24}, \quad y = \frac{2l+\beta}{4l+4m+\alpha+8}$$

and inequality (34a) replaced by

$$\begin{aligned} 24l^2 &+ (24m + 12\delta - 4\gamma + 18\alpha - 24\beta + 48)l \\ &+ (12\delta - 24\beta)m + 3\delta(\alpha+8) - (2\gamma+48)\beta \geq 0. \end{aligned} \tag{34b}$$

Finally, we directly can show inequality (33) for $l = 0$ in the exceptional Case 3° with $\delta = 2$ and $\gamma = 6$.

`Proof of inequality (IN2).` Note that inequality (IN2) is equivalent to inequalities

$${}_{\alpha-\beta}^{\ \ \alpha}D_{\gamma-\delta}^{\gamma}(n,l) \geq {}_{\alpha-\beta}^{\ \ \alpha}D_{\gamma-\delta}^{\gamma}(n+1,l) \geq {}_{\alpha-\beta}^{\ \ \alpha}D_{\gamma-\delta}^{\gamma}(n+2,l)$$

for $l = 0, ..., \frac{n}{2} - 1$. Hence we have to show inequality (IN1) in the following three cases:

`Case` $1'$: $\alpha = 0, \beta = 0, \delta \geq 0, \gamma = 2\delta + \eta$ with $\eta = 0, -2$

`Case` $2'$: $\alpha = 1, \beta = 1, \delta \geq 2, \gamma = 2\delta + \eta$ with $\eta = -2, -4$

`Case` $3'$: $\alpha = 2, \beta = 1, \delta \geq 2, \gamma = 2\delta + \eta$ with $\eta = 0, -2$.

In these cases inequalities (32a), (32b) and (34a), (34b) hold for any $l, m \in \mathbb{N}_0$ except Case $2'$ with $\delta = 2, \gamma = 2$ and $l = 0$. In this exceptional case we directly check (31) and (33).

`Proof of inequality (IN3).` Expanding the binomial coefficients and cancelling the similar factors we see that inequality (IN3) is equivalent to the inequality

$$\begin{aligned} &\prod_{i=1}^{2} \frac{2n+\alpha+i}{n+\alpha-\beta+i} \prod_{j=1}^{6} \frac{3n+\gamma-\delta+j}{6n+\gamma+j} \\ &\quad + \prod_{i=1}^{2} \frac{2n+\alpha+i}{n+\beta+i} \prod_{j=1}^{6} \frac{3n+\delta+j}{6n+\gamma+j} \leq 1. \end{aligned} \tag{35}$$

`Case 1°.` Then (35) is implied by

$$4 \cdot \frac{2n+1}{n+2} \prod_{j=1}^{6} \frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j} \leq 1$$

which holds for $n \geq 2$, and for $n = 0$ if $\delta \geq 1$.

`Case 2°.` Then (35) takes the form

$$\frac{2n+2}{n+2} \cdot \frac{2n+3}{n+3} \prod_{j=1}^{6} \frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j} + 2 \cdot \frac{2n+3}{n+1} \prod_{j=1}^{6} \frac{3n+\delta+j}{6n+2\delta+\eta+j} \leq 1$$

which holds for $n \geq 2$, and for $n = 0$ if $\delta \geq 1$.

`Case 3°.` Then (35) takes the form

$$2 \cdot \frac{2n+3}{n+3} \left(\prod_{j=1}^{6} \frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j} + \prod_{j=1}^{6} \frac{3n+\delta+j}{6n+2\delta+\eta+j} \right) \leq 1$$

which holds for $n \geq 2$, and for $n = 0$ if $\delta \geq 1$.

`Proof of inequality (IN4).` Expanding the binomial coefficients and cancelling similar factors we see that inequality (IN4) is equivalent to the inequality

$$\begin{aligned}
&\prod_{i=1}^{4} \frac{2n+\alpha+i}{n+\alpha-\beta+i} \prod_{j=1}^{12} \frac{3n+\gamma-\delta+j}{6n+\gamma+j} \\
&\quad + \prod_{i=1}^{4} \frac{2n+\alpha+i}{n+\beta+i} \prod_{j=1}^{12} \frac{3n+\delta+j}{6n+\gamma+j} \\
&\quad + \prod_{i=1}^{2} \frac{2n+\alpha+i}{n+\alpha-\beta+i} \prod_{i=1}^{2} \frac{2n+\alpha+i+2}{n+\beta+i} \\
&\quad \times \prod_{j=1}^{6} \frac{3n+\gamma-\delta+j}{6n+\gamma+j} \prod_{j=1}^{6} \frac{3n+\delta+j}{6n+\gamma+j+6} \\
&\quad \leq \prod_{i=1}^{2} \frac{2n+\alpha+i}{n+\alpha-\beta+i} \prod_{j=1}^{6} \frac{3n+\gamma-\delta+j}{6n+\gamma+j} \\
&\quad + \prod_{i=1}^{2} \frac{2n+\alpha+i}{n+\beta+i} \prod_{j=1}^{6} \frac{3n+\delta+j}{6n+\gamma+j}.
\end{aligned} \tag{36}$$

Note that, for $\gamma = 2\delta + \eta$ with $0 \le \eta \le 6$,

$$2\prod_{j=1}^{6}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}\prod_{j=1}^{6}\frac{3n+\delta+j}{6n+\gamma+j+6} \le \prod_{j=1}^{12}\frac{3n+\gamma-\delta+j}{6n+\gamma+j}+\prod_{j=1}^{12}\frac{3n+\delta+j}{6n+\gamma+j}.$$

Hence, (36) is implied by

$$\left(\prod_{i=3}^{4}\frac{2n+\alpha+i}{n+\alpha-\beta+i}+\frac{1}{2}\prod_{i=1}^{2}\frac{2n+\alpha+i+2}{n+\beta+i}\right)\prod_{j=7}^{12}\frac{3n+\gamma-\delta+j}{6n+\gamma+j} \le 1 \quad (36a)$$

$$\left(\prod_{i=3}^{4}\frac{2n+\alpha+i}{n+\beta+i}+\frac{1}{2}\prod_{i=1}^{2}\frac{2n+\alpha+i+2}{n+\alpha-\beta+i}\right)\prod_{j=7}^{12}\frac{3n+\delta+j}{6n+\gamma+j} \le 1. \quad (36b)$$

`Case 1°.` Then (36b) is weaker then (36a) which takes the form

$$\left(\frac{2n+3}{n+3}\,\frac{2n+4}{n+4}+\frac{2n+3}{n+1}\right)\prod_{j=7}^{12}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j} \le 1,$$

and it clearly holds for $n \ge 2$, and for $n = 0$ if $\delta \ge 4$. If $n = 0$, we directly check (36) for $\delta = 1, 2, 3$, $\eta = 0$ and for $\delta = 0, 1, 2, 3$, $\eta = 2$.

`Case 2°.` Then (36a) and (36b) take the form

$$\left(\frac{2n+4}{n+4}\cdot\frac{2n+5}{n+5}+\frac{2n+5}{n+1}\right)\prod_{j=7}^{12}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j} \le 1$$

$$\left(\frac{2n+4}{n+3}\cdot\frac{2n+5}{n+4}+\frac{2n+5}{n+3}\right)\prod_{j=7}^{12}\frac{3n+\delta+j}{6n+2\delta+\eta+j} \le 1.$$

Clearly, both inequalities hold for $n \ge 2$, and for $n = 0$ if $\delta \ge 8$. If $n = 0$, we directly check (36) for $\delta = 0, ..., 7$, $\eta = 2, 4$.

`Case 3°.` Then (36b) is weaker then (36a) which takes the form

$$\left(\frac{2n+5}{n+4}\cdot\frac{2n+6}{n+5}+\frac{2n+5}{n+2}\right)\prod_{j=7}^{12}\frac{3n+\delta+\eta+j}{6n+2\delta+\eta+j} \le 1,$$

and it clearly holds for $n \ge 2$, and for $n = 0$ if $\delta \ge 4$. If $n = 0$, we directly check (36) for $\delta = 1, 2, 3$, $\eta = 0, 2$ ∎

6. Proof of the Main Lemma

First of all observe that all sequences ${}^{\alpha}_{\beta}D^{\gamma}_{\delta}$ appearing in the representation of $C(k,n)$ given by (20A) - (20F) fall within one of the cases of Lemma 3. Next, the coefficients $a(k,l), b(k,l)$ in (20A) - (20F) are given by (21A) - (21F), and so they are non-negative. Hence, by Lemma 3 we get $C(k,n) \geq C(k,n+2)$ for $n \geq 2$, for $n \geq 1$ if $k \geq 2$, and for $n \geq 0$ if $k \geq 3$. But

$$C(1,0) = 3 > C(1,2) = 2\frac{37}{105}$$
$$C(1,1) = 2\frac{2}{5} > C(1,3) = 2\frac{103}{770}$$
$$C(2,0) = 4 > C(2,2) = 2\frac{67}{110}.$$

So

$$C(k,n) \geq C(k,n+2) \qquad \text{for } \begin{cases} n \geq 2 & \text{if } k \geq 0 \\ n \geq 0 & \text{if } k \geq 1. \end{cases}$$

Now, if $n = 0$, we easily get $C(k,0) = k+2$ for $k \in \mathbb{N}_0$. Next, if $n = 1$, we derive for $k \in \mathbb{N}_0$

$$C(k,1) = 2\sum_{l=0}^{k} \frac{(2l+2)(2l+3)(2l+4)}{(2k+2)(2k+3)(2k+4)} = \frac{(k+2)(k+3)}{2k+3} \leq k+2.$$

Hence, $C(k,n) \leq C(k,0) = k+2$ for $n \in \mathbb{N}_0$ if $k \geq 1$. To finish the proof we compute $C(0,2) = 2\frac{9}{70} > C(0,3) = 2\frac{2}{70}$.

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References

[1] Balser, W.: *Formal power series and linear systems of meromorphic ordinary differential equations.* New York: Springer-Verlag 2000.

[2] Boas, R. P.: *Entire Functions.* New York: Academic Press 1954.

[3] Bourgain, J.: *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations.* Part II: *The KdV equation.* Geom. & Funct. Anal. 3 (1993), 209 – 262.

[4] Byers, P. and A. Himonas: *Non-analytic solutions of the KdV equation.* Abstr. & Appl. Anal. (to appear).

[5] De Bouard, A., Hayashi, N. and K. Kato: *Regularizing effect for the (generalized) Korteweg-de Vries equation and nonlinear Schrödinger equation.* Ann. Inst. Henri Poincaré, Anal. nonlin. 12 (1995), 673 – 725.

[6] Hayashi, N.: *Analytic function spaces and their applications to nonlinear evolution equations.* In: Analytic Extension Formulas and Their Applications (Fukuola 1999/Kyoto 2000; eds.: S. Saitoh et al.). Dordrecht et al.: Kluwer Acad. Publ. 2001, pp. 59 – 72.

[7] Jeffery, A. and T. Kakutani: *Weak nonlinear dispersive waves: A discussion centered around Korteweg-de Vries equation.* SIAM Review 14 (1972), 582 – 643.

[8] Kato, T. and M. Masuda: *Nonlinear evolutions equations and analyticity*, Part I. Ann. Inst. Henri Poincaré, Anal. nonlin. 3 (1986), 455 – 467.

[9] Kato, K. and T. Ogawa: *Analyticity and smoothing effect for the Korteweg de Vries equation with a single point singularity.* Math. Ann. 316 (2000), 577 – 608.

[10] Kenig, C. E., Ponce, G. and L. Vega: *A bilinear estimate with the application to the KdV equation.* Trans. Amer. Math. Soc. 348 (1996), 573 – 603.

[11] Korteweg, D. J. and G. de Vries: *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves.* Philos. Mag. 39 (1895), 422 – 443.

[12] Kruskal, M. D.: *Nonlinear wave equations.* In: Dynamical Systems, Theory and Applications (ed.: J. Moser). New York: Springer-Verlag 1975, pp. 310 – 354.

[13] Lax, P. D.: *Almost periodic solutions of the KdV equation.* SIAM Review 18 (1976), 351 – 375.

[14] Łysik, G.: *Nonanalyticity of solutions to* $\partial_t u = \partial_x^2 u + u^2$. Coll. Math. 95 (2003), 255 – 266.

[15] Miura, R. M.: *The Korteweg-de Vries equation: a survey of results.* SIAM Review 18 (1976), 412 – 459.

[16] Prudnikov, A. P., Brychkov. Yu. A. and O. I. Marichev: *Integrals and Series. Elementary functions* (in Russian). Moscow: Nauka 1981; English transl.: Gordon and Breach 1986.

[17] Trubowitz, E.: *The inverse problem for periodic potentials.* Comm. Pure Appl. Math. 30 (1977), 321 – 337.

[18] Zhidkov, P. E.: *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory.* Lect. Notes Math. 1756 (2001).

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