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Hölder-Zygmund Regularity in Algebras of Generalized Functions

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Abstract. We introduce an intrinsic notion of Hölder-Zygmund regularity for Colombeau generalized functions. In case of embedded distributions belonging to some Zygmund-Hölder space this is shown to be consistent. The definition is motivated by the well-known use of Littlewood-Paley decompositions in characterizing Hölder-Zygmund regularity for distributions. It is based on a simple interplay of differentiated convolution-mollification with wavelet transforms, which directly translates wavelet estimates into properties of the regularizations. Thus we obtain a scale of new subspaces of the Colombeau algebra. We investigate their basic properties and indicate first applications to differential equations whose coefficients are non-smooth but belong to some Hölder-Zygmund class (distributional or generalized). In applications problems of this kind occur, for example, in seismology when Earth's geological properties of fractal nature have to be taken into account while the initial data typically involve strong singularities.

Keywords: *Zygmund classes, Hölder continuity, algebras of generalized functions, generalized solutions to differential equations*

AMS subject classification: 46F30, 35D10

1. Introduction

When studying models of wave propagation in highly irregular media, e.g., in seismology, (hyperbolic) partial differential equations have to be considered with coefficients and initial data being generalized functions. The coefficients represent the medium properties, which may be irregular, e.g., due to folds, fault zones, or junctions of different geological units as well as caused by long term physical processes within geological layers. Once the location of layer boundaries through jump discontinuities is completed, refined geological information is reflected in a specific type of regularity patterns of the material properties

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within a certain unit. Often self-similar or multi-fractal behavior can be observed and Hölder continuity, and more generally, Hölder-Zygmund spaces, were found to be a useful tool for a systematic qualitative analysis (cf. [6, Chapter 4] and [22, Chapter IV] for a mathematical justification, and [4, 5, 13, 18, 23] for seismological applications).

In general, differential equations of the type mentioned above need not make sense or may fail to have solutions within the theory of distributions. However, embedding the singular coefficients first into an algebra of generalized functions, here Colombeau algebras, enables one to carry out a detailed analysis and yields unique solvability under mild conditions (cf. [10, 12, 16]).

A preliminary study of this procedure in Colombeau theory was undertaken in [9], where the focus was on microlocal properties and the regularization aspects of wavelet transforms. The feasibility of recovering Zygmund-Hölder spaces of positive regularity in one space dimension after the embedding into Colombeau algebras was proven. In the present paper we extend this result to arbitrary dimension and regularity scale, although by slightly changing the definition proposed earlier. We also give first applications to simple differential equations. In particular, we study a (1+1)-dimensional hyperbolic Cauchy problem with typical geophysical conditions on the coefficients. We show how the regularity of the measured wave depends on the regularity properties of the medium as well as of the initial value.

The outline of this paper is as follows. After a brief introduction to the basics of Colombeau theory in Subsection 1.1 we devote Subsection 1.2 to a review of distributional Hölder-Zygmund spaces and their characterization in terms of Littlewood-Paley decompositions and wavelet transforms. Section 2 introduces the corresponding Colombeau-theoretic notion and discusses basic properties and illustrative examples. Section 3 presents simple case studies in applications to differential equations.

1.1. Colombeau algebras of generalized functions. We recall the basic facts about the so-called special Colombeau algebras on $\mathbb{R}^n$. They can be defined on arbitrary open subsets, or even on smooth manifolds, contain the space of Schwartz distributions, and provide far reaching consistency with respect to analysis in distribution spaces. For further details and applications we refer to [1, 2, 17].

The key ingredient of Colombeau algebras is the regularization by nets of smooth functions and the use of asymptotic estimates with respect to the regularization parameter ε. More precisely, it is based on a quotient construction as follows: we set (with $I = (0, 1]$)

$$\mathcal{E} := C^\infty(\mathbb{R}^n)^I$$

and

$$\mathcal{E}_{\mathrm{M}} := \left\{ (u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E} \;\middle|\; \begin{array}{l} \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n \; \exists N \in \mathbb{N} : \\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \end{array} \right\}$$

$$\mathcal{N} := \left\{ (u_\varepsilon)_{\varepsilon\in I} \in \mathcal{E}_{\mathrm{M}} \;\middle|\; \begin{array}{l} \forall K \subset\subset \mathbb{R}^n, \forall m \in \mathbb{N} : \\ \sup_{x\in K} |u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0 \end{array} \right\}.$$

$\mathcal{E}_{\mathrm{M}}$ is a differential algebra with component-wise operations, $\mathcal{N}$ is an ideal in $\mathcal{E}_{\mathrm{M}}$, and the *special Colombeau algebra* is defined as the quotient space

$$\mathcal{G} := \mathcal{E}_{\mathrm{M}} \,/\, \mathcal{N}\,.$$

Since we consider only this type of algebras here we will omit the term 'special' henceforth. A representative of an element u of $\mathcal{G}$ will be denoted by $(u_\varepsilon)_\varepsilon$, and we will write $u = [(u_\varepsilon)_\varepsilon]$ in this case. Smooth functions are embedded as a differential subalgebra simply by $\sigma(f) = [(f)_\varepsilon]$.

To embed non-smooth distributions we first have to fix a mollifier $\rho \in \mathscr{S}(\mathbb{R}^n)$ with unit integral satisfying the moment conditions $\int \rho(x)\, x^\alpha\, dx = 0$ when $|\alpha| \geq 1$. Setting $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$, compactly supported distributions are embedded by $\iota_0(w) = (w * \rho_\varepsilon)_\varepsilon + \mathcal{N}$. Using partitions of unity and suitable cut-off functions one may explicitly construct an embedding $\iota_\rho \colon \mathcal{D}' \hookrightarrow \mathcal{G}$ extending ι_0, commuting with partial derivatives and its restriction to C^∞ agreeing with σ. Note that although ι_ρ depends on the choice of the mollifier ρ this rather reflects a fundamental property of nonlinear modeling where the interaction of singular objects depends on the regularization. Additional specifications of the regularization from a physical model may and should enter the mathematical theory at this point.

The ring of generalized complex numbers $\widetilde{\mathbb{C}}$ is defined as the set of moderate nets of numbers ($(r_\varepsilon)_\varepsilon \in \mathbb{C}^I$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some N) modulo negligible nets ($|r_\varepsilon| = O(\varepsilon^m)$ for each m).

1.2. Review: Hölder-Zygmund regularity of temperate distributions. This subsection is a synthesis of related parts from the following sources: in the basic notation and setup of Zygmund spaces we stay close to [8]; all wavelet aspects are taken from [15]; for further properties of Zygmund classes and related spaces we refer to [20, 21].

The result reviewed here is not new and neither are the techniques of its proof, given in the Appendix. However, we felt the need to unify various aspects which are crucial to our application later on. The concise summary of our efforts is the formulation of Theorem 1.1.

Continuous Littlewood-Paley decomposition: Following [8, Sect.8.5] we introduce a continuous analog of the Littlewood-Paley decomposition.

Choose $\varphi \in \mathcal{D}(\mathbb{R}^n)$ real valued and symmetric such that $|\xi| \leq 1$ in $\operatorname{supp}(\varphi)$ and $\varphi(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$. Put $\psi = \frac{d}{dt}\varphi(\frac{\xi}{t}) \mid_{t=1} = -\xi \cdot \operatorname{grad}\varphi(\xi)$ so that the support of $\psi(\frac{\cdot}{t})$ is contained in the annulus $\frac{t}{2} \leq |\xi| \leq t$. Observe that we obtain a continuous partition of unity

$$1 = \varphi(\xi) + \int_1^\infty \psi(\tfrac{\xi}{t})\, \tfrac{dt}{t}. \tag{1}$$

If $f \in \mathscr{S}(\mathbb{R}^n)$ is used as a Fourier multiplier for $u \in \mathscr{S}'(\mathbb{R}^n)$ we will sometimes write this in pseudodifferential operator notation, i.e., $f(D)u = \mathcal{F}^{-1}(f\widehat{u}) = (\mathcal{F}^{-1}f) * u$ where $\mathcal{F}$ and $\widehat{\ }$ denote Fourier transform.

Note that for any $u \in \mathscr{S}'$ and $T \geq 1$ arbitrary we have

$$\varphi(D)u + \int_1^T \psi(\tfrac{D}{t})u\, \tfrac{dt}{t} = \varphi(\tfrac{D}{T})u = T^n(\mathcal{F}^{-1}\varphi)(T.) * u$$

which converges to u in $\mathscr{S}'$ when $T \to \infty$. This specifies the meaning of the following decomposition formula, which is valid in $\mathscr{S}'$,

$$u = \varphi(D)u + \int_1^\infty \psi(\tfrac{D}{t})u\, \tfrac{dt}{t}. \tag{2}$$

Hölder-Zygmund spaces: The classical Hölder spaces $\mathcal{C}^s(\mathbb{R}^n)$, for $s > 0$ not integer, as well as their natural extension to $s \in \mathbb{N}$, the so-called Zygmund classes, appear in [8, Section 8.6] in an equivalent realization given by the spaces $C^s_*(\mathbb{R}^n)$. These are defined, for any real s, in terms of a continuous Littlewood-Paley decomposition by

$$C^s_*(\mathbb{R}^n) := \left\{ u \in \mathscr{S}' \;\middle|\; |u|^*_s := \|\varphi(D)u\|_{L^\infty} + \sup_{t>1} \left(t^s \|\psi(\tfrac{D}{t})u\|_{L^\infty}\right) < \infty \right\}. \tag{3}$$

Let $m \in \mathbb{N}$. In the context of this paper we call a function $g \in \mathscr{S}(\mathbb{R}^n)$ a *wavelet of (oscillation) order* m if its first m moments vanish, that is

$$\int x^\alpha g(x)\, dx = 0 \qquad \forall \alpha : \ 0 \leq |\alpha| \leq m-1 \tag{4}$$

and it is *weakly radial* [15, Chapter 1, Equ. (5.6)], i.e.,

$$\int_0^\infty |\widehat{g}(t\xi)|^2\, \tfrac{dt}{t} = 1 \qquad \forall \xi \neq 0. \tag{5}$$

In particular, radial functions can always be normalized so that they satisfy equation (5).

We introduce the notation $\check{f}(y) = f(-y)$ and $f_\varepsilon(y) = \varepsilon^{-n} f(\frac{y}{\varepsilon})$ for a function f on $\mathbb{R}^n$ (and the bar denoting complex conjugation). If g is a wavelet we consider the *wavelet transform* $W_g \colon \mathscr{S}'(\mathbb{R}^n) \to \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, mapping $u \in \mathscr{S}'$ into

$$W_g u(x, \varepsilon) = u * \overline{\check{g}_\varepsilon}(x) \qquad \forall (x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{6}$$

(Note that $\check{}$ denotes reflection, not inverse Fourier transform.) It is immediate that the image $W_g(\mathscr{S}')$ is contained in the subspace $\mathcal{O}_\mathrm{M}(\mathbb{R}^n \times \mathbb{R}_+)$ of smooth functions all of whose derivatives have polynomial bounds in x, ε and $\frac{1}{\varepsilon}$ (our notation deviates from [6] where this space is denoted by $\mathscr{S}'(\mathbb{R}^n \times \mathbb{R}_+)$). On the space $\mathcal{O}_\mathrm{M}(\mathbb{R}^n \times \mathbb{R}_+)$ we can define the *wavelet synthesis operator* M_g, mapping $H \in \mathcal{O}_\mathrm{M}$ into an element $M_g H \in \mathscr{S}'(\mathbb{R}^n)$, defined by

$$M_g H = \lim_{r \to 0, R \to \infty} \int_r^R H(\cdot, \varepsilon) * g_\varepsilon \tfrac{d\varepsilon}{\varepsilon} \tag{7}$$

with convergence being understood weakly in $\mathscr{S}'(\mathbb{R}^n)$ (cf. [6, Chapter 1, Sections 24, 25, and 30]). With the aid of M_g distributions in $\mathscr{S}'$ can be reconstructed from their wavelet transforms modulo polynomials, i.e. for each $u \in \mathscr{S}'$ there is a polynomial p on $\mathbb{R}^n$ such that

$$u = M_g(W_g u) + p. \tag{8}$$

The crucial observation that motivates the definition of Zygmund regularity within Colombeau generalized functions is a characterization which is valid for temperate distributions. As mentioned above, this can be found in [15, Chapter 3] in the framework of Bony's two-microlocal spaces. However, we repeat the arguments given there in a 'stripped down' version appropriate for the current context. A detailed proof can be found in the Appendix.

Theorem 1.1. *Let s be a real number and $g \in \mathscr{S}(\mathbb{R}^n)$ have m vanishing moments. Let u be a temperate distribution on $\mathbb{R}^n$.*

(i) *Let $m > s$. If $u \in C^s_*(\mathbb{R}^n)$, then its wavelet transform satisfies*

$$\sup_{\varepsilon \in (0,1]} \varepsilon^{-s} \left\| W_g u(\cdot, \varepsilon) \right\|_{L^\infty} < \infty. \tag{9}$$

(ii) *Let $m > -s$ and g be weakly radial. Then* (9) *implies that there is $u_0 \in \mathrm{C}^\infty(\mathbb{R}^n)$ such that $u - u_0 \in C^s_*(\mathbb{R}^n)$.*

Remark 1. Once more we want to emphasize that the statement of Theorem 1.1 is included in the corresponding, and more general, results presented in Meyer's book [15, Chapter 3]. The characterization of Hölder-Lipschitz-Zygmund regularity $0 < s \leq 1$ via the asymptotic behavior of a wavelet-type transform at small scales has a forerunner in terms of Poisson integrals, e.g. in [24, Section VII.5] for the one-dimensional case and in [19, Section V.4.2] on $\mathbb{R}^n$.

2. Intrinsic Hölder-Zygmund regularity of Colombeau functions

2.1. Basic notions and coherence properties. We recall that a mollifier is a function $\rho \in \mathscr{S}(\mathbb{R}^n)$ with $\int \rho = 1$. In addition, we will henceforth assume ρ to be radial.

Mollifiers and wavelets: We restate the following facts from [9, Section 3.3]

(i) Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$. Then the function $\rho^\alpha := \overline{(\partial^\alpha \rho)\check{}}$ has $m = |\alpha|$ vanishing moments and, for any $u \in \mathscr{S}'(\mathbb{R}^n)$,

$$\partial^\alpha (u * \rho_\varepsilon)(x) = \varepsilon^{-|\alpha|} W_{\rho^\alpha} u(x,\varepsilon) \qquad \forall (x,\varepsilon) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{10}$$

In particular, $\overline{(\Delta^k \rho)\check{}}$ is a wavelet of oscillation order $2k$.

(ii) If $\int x^\alpha \rho(x)\, dx = 0$ when $1 \leq |\alpha| \leq m-1$, then $\overline{\check{\mu}} := -\frac{d}{d\varepsilon}(\rho_\varepsilon)\mid_{\varepsilon=1}$ defines a wavelet of oscillation order m and, for any $u \in \mathscr{S}'(\mathbb{R}^n)$,

$$(u * \rho_\varepsilon)(x) = (u * \rho)(x) + \int_\varepsilon^1 W_\mu u(x,r)\, \tfrac{dr}{r} \qquad \forall (x,\varepsilon) \in \mathbb{R}^n \times \mathbb{R}_+. \tag{11}$$

In view of Theorem 1.1 equation (10) suggests to test for Zygmund regularity after embedding by looking at the asymptotic properties of high-order derivatives. The following definition is based on this idea and refines it in order to ensure mapping properties with respect to differentiations. Note that it differs from the definition proposed earlier in [9].

Definition 2.1. Let $s \in \mathbb{R}$ and $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$. We say that u is of *(generalized) Zygmund regularity* s, denoted $u \in \mathcal{G}_*^s(\mathbb{R}^n)$, if for $\alpha \in \mathbb{N}_0^n$

$$\|\partial^\alpha u_\varepsilon\|_{L^\infty} = \begin{cases} O(1) & \text{if} \quad 0 \leq |\alpha| < s \\ O(\log(\frac{1}{\varepsilon})) & \text{if} \quad |\alpha| = s \in \mathbb{N}_0 \\ O(\varepsilon^{s-|\alpha|}) & \text{if} \quad |\alpha| > s \end{cases} \qquad (\varepsilon \to 0). \tag{12}$$

Remark 2. As a matter of fact, equation (10) and Theorem 1.1 directly suggest to include the third line in (12) of the above definition. This would be already suitable to characterize the embedded Zygmund classes (modulo smooth functions) among all embedded temperate distributions as can be seen from the proof of Theorem 2.1 below. However, if we want the family of spaces $\mathcal{G}_*^s$ ($s \in \mathbb{R}$) to be a scale, in the sense that $s' \geq s$ implies $\mathcal{G}_*^{s'} \subseteq \mathcal{G}_*^s$, then the testing of decrease properties must not start at a derivative order which depends on the (prospective) regularity number. In particular, the case that s is an integer has to be taken into account, which is done here by the minimum possible, i.e., logarithmic, growth rate compatible with embeddings.

Proposition 2.1. *Let s, s', and r be real numbers. Then:*

(i) *The spaces $\mathcal{G}_*^s$ are nested, i.e., $s' \geq s$ implies $\mathcal{G}_*^{s'} \subseteq \mathcal{G}_*^s$.*

(ii) *For each $\beta \in \mathbb{N}_0^n$ we have a linear map $\partial^\beta \colon \mathcal{G}_*^s \to \mathcal{G}_*^{s-|\beta|}$.*

(iii) *Regularity of products: $\mathcal{G}_*^r \cdot \mathcal{G}_*^s \subseteq \mathcal{G}_*^p$ where $p = r + s$ if $r, s < 0$, $p = \min(r, s)$ if $\max(r, s) > 0$, and $p = \min(r, s)_-$ if $\max(r, s) = 0$. (Here, c_- denotes any number $c - \sigma$ for $\sigma > 0$ arbitrary.)*

Proof. *Part* (i): If $0 \leq |\alpha| < s \leq s'$ the assertion is trivial. If $|\alpha| = s \leq s'$, then $\|\partial^\alpha u\|_{L^\infty}$ is $O(1)$ ($s = s'$) or $O(\log(\frac{1}{\varepsilon}))$ ($s < s'$), that is $O(\log(\frac{1}{\varepsilon})$ in any case.

The case $|\alpha| > s$ leaves us with three subcases for the asymptotic bounds of $\|\partial^\alpha u\|_{L^\infty}$: $s < |\alpha| < s'$ yields $O(1)$ which is $O(\varepsilon^{s-|\alpha|})$; $s < |\alpha| = s'$ gives $O(\log(\frac{1}{\varepsilon})$ and hence also $O(\varepsilon^{s-|\alpha|}$; finally, in case $|\alpha| > s' \geq s$ we obtain $O(\varepsilon^{s'-|\alpha|})$ being again $O(\varepsilon^{s-|\alpha|})$.

Part (ii): We use (12) with α replaced by $\alpha + \beta$ and note that $|\alpha + \beta| = |\alpha| + |\beta|$. This gives asymptotic bounds $O(1)$ if $0 \leq |\alpha| < s - |\beta|$, $O(\log(\frac{1}{\varepsilon}))$ if $|\alpha| = s - |\beta|$, and $O(\varepsilon^{s-|\beta|-|\alpha|})$ if $|\alpha| > s - |\beta|$.

Part (iii): We may assume that $r \leq s$, the opposite case being completely analogous. Let $u \in \mathcal{G}_*^r$, $v \in \mathcal{G}_*^s$, and $\alpha \in \mathbb{N}_0^n$. In estimating $\partial^\alpha(uv)$ we use the Leibniz rule and thus have to find asymptotic upper bounds for the typical term of the form $\partial^\beta u_\varepsilon \cdot \partial^{\alpha-\beta} v_\varepsilon$ with $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. This is done by combination of the asymptotic growth information about each factor separately.

If $s < 0$, then the largest growth is due to combinations of the form $O(\varepsilon^{r-|\beta|}) \cdot O(\varepsilon^{s-|\alpha|+|\beta|}) = O(\varepsilon^{r+s-|\alpha|})$. This proves the thirst case for the regularity p.

If $s = 0$, we only have to check the case $|\alpha| = 0$ separately. To see this, note that adding $-\sigma$ in the exponents does not decrease the bounds established above and also captures any occurring logarithmic factors stemming from $\|v_\varepsilon\|_{L^\infty}$. In order 0 the dominating terms are $O(\varepsilon^r) \cdot O(\log(\frac{1}{\varepsilon})) = O(\varepsilon^{r-\sigma})$ which proves the second case for p.

Finally, let $s > 0$. Assuming $|\alpha| < s$ implies $|\beta| < r$ as well as $|\alpha - \beta| < r \leq s$ and hence produces only $O(1)$ factors. If $|\alpha| = s$, then $|\alpha - \beta| = s$ if and only if $|\beta| = 0$ in which case the zero order bound for $\|u_\varepsilon\|_{L^\infty}$ is to be multiplied by $\log(\frac{1}{\varepsilon})$. Otherwise, i.e., if $|\beta| > 0$, then the factor corresponding to v gives only $O(1)$. It follows that we obtain the upper bound $O(\log(\frac{1}{\varepsilon})$ if $r = s > 0$ and $O(\varepsilon^{r-s}) = O(\varepsilon^{r-|\alpha|})$ if $r < s$. If $|\alpha| > s$, all possible nine combinations of upper bounds may have to be employed but $O(\varepsilon r - |\alpha|)$ is dominating all of them (since $s > 0$) ∎

Remark 3.

(i) Compare part (iii) of the Proposition with the distribution-theoretic result on products in Zygmund spaces (cf. [8, Proposition 8.6.8]): If $u \in C_*^r$, $v \in C_*^s$, then $u \cdot v$ can be defined (as a weakly sequentially continuous bilinear map $\mathcal{D}' \times \mathcal{D}' \to \mathcal{D}'$) if $r + s > 0$ and gives an element of Zygmund regularity $\min(r, s)$.

(ii) We note that the subalgebra $\mathcal{G}^\infty$, defined in [17, Section 25], reflects a somewhat different concept of regularity. First of all, the $\mathcal{G}^\infty$-property is tested on compact sets only with ε-asymptotic constant with respect to derivative orders but dependent on the compact set. Furthermore, it is easy to give examples of Colombeau functions being very regular in one sense but not in the other: if p is a polynomial and χ a smooth cutoff function, then the class of $\chi(x)p(\frac{x}{\varepsilon^r})$ is in $\mathcal{G}^\infty$ but it has poor Zygmund regularity if $r > 0$; on the other hand, for any $s \in \mathbb{R}$, $\varepsilon^s \sin(\frac{x}{\varepsilon})$ defines a $\mathcal{G}^s_*$-class which is not in $\mathcal{G}^\infty$.

Let ρ be a radial mollifier with all higher moments vanishing. (Hence ρ can be used to construct wavelets of any oscillation order.) Then we have the embedding $\iota_\rho \colon \mathscr{S}' \hookrightarrow \mathcal{G}$, $v \mapsto [(v * \rho_\varepsilon)_\varepsilon]$. We show that under these embeddings the above definition of the subspaces $\mathcal{G}^s_* \subseteq \mathcal{G}$ is compatible with the distributional Zygmund classes C^s_*.

Theorem 2.1. *For any* $s \in \mathbb{R}$*:*

(i) $\iota_\rho(C^s_*(\mathbb{R}^n)) \subseteq \mathcal{G}^s_*(\mathbb{R}^n)$.

(ii) *If* $v \in \mathscr{S}'(\mathbb{R}^n)$ *and* $\iota_\rho(v) \in \mathcal{G}^s_*(\mathbb{R}^n)$*, then there is* $v_0 \in \mathrm{C}^\infty(\mathbb{R}^n)$ *such that* $v - v_0 \in C^s_*(\mathbb{R}^n)$.

Proof. *Part* (i): Let $v \in C^s_*$ and $\alpha \in \mathbb{N}_0$. We work through all cases to be distinguished about the relation of $|\alpha|$ and s.

Case $|\alpha| > s$ and $|\alpha| > 1$: Application of (10) and Theorem 1.1, (i) (with $m = |\alpha| > s$) yields

$$|\partial^\alpha (v * \rho_\varepsilon)(x)| = \varepsilon^{-|\alpha|} |W_{\rho_\alpha} v(x, \varepsilon)| = \varepsilon^{-|\alpha|} O(\varepsilon^s) \qquad (\varepsilon \to 0)$$

uniformly in $x \in \mathbb{R}^n$.

Case $0 \le |\alpha| < s$: In this case $v \in \mathrm{C}^{\lfloor s \rfloor}_b$ and we have

$$\|\partial^\alpha (v * \rho_\varepsilon)\|_{L^\infty} = \|(\partial^\alpha v) * \rho_\varepsilon\|_{L^\infty} \le \|\partial^\alpha v\|_{L^\infty} \|\rho\|_{L^1} = O(1),$$

where we have used that $C^t_* \subset \mathrm{L}^\infty$ if $t > 0$ ([21, Section 2.3.2/Remark 3]).

Case $|\alpha| = s \in \mathbb{N}_0$: Since $\partial^\alpha v \in C^0_*$ we obtain $\|W_\mu \partial^\alpha v(\cdot, r)\|_{L^\infty} = O(1)$ in formula (11) and hence

$$\|\partial^\alpha (v * \rho_\varepsilon)\|_{L^\infty} \le \|(\partial^\alpha v) * \rho\|_{L^\infty} + C \int_\varepsilon^1 \frac{dr}{r} = O(\log(\tfrac{1}{\varepsilon})).$$

Case $|\alpha| = 0 > s$: Again, by (11) and Theorem 1.1 (i), noting that $s < 0$, we conclude that

$$|v * \rho_\varepsilon| \le |v * \rho| + C \int_\varepsilon^1 r^{s-1}\, dr = O(\varepsilon^s).$$

Part (ii): Choose $2k > |s|$. Then by (10) with $\rho^{(2k)} := \overline{(\Delta^k \rho)}\check{}$ and applying (12) to $v_\varepsilon = v * \rho_\varepsilon$ we have

$$\|W_{\rho^{(2k)}} v(\cdot, \varepsilon)\|_{L^\infty} = \varepsilon^{2k} \|\Delta^k(v_\varepsilon)\|_{L^\infty} = \varepsilon^{2k} O(\varepsilon^{s-2k}) = O(\varepsilon^s) \quad (\varepsilon \to 0).$$

The assertion follows from Theorem 1.1, (ii) (with $m = 2k > |s|$) ∎

The global L^∞-bounds used in Definition 2.1 may be somewhat too restrictive in certain applications and instead of using a formulation like 'is in $\mathcal{G}_*^s$ modulo a very regular function' we may prefer to use the following localized version of Zygmund regularity.

Definition 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be open and $s \in \mathbb{R}$. The Colombeau function $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ is said to be *locally of generalized Zygmund regularity s in* Ω, denoted $u \in \mathcal{G}_{*,\mathrm{loc}}^s(\Omega)$, if for all $K \subset\subset \Omega$ and $\alpha \in \mathbb{N}_0^n$

$$\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)} = \begin{cases} O(1) & \text{if} \quad 0 \le |\alpha| < s \\ O(\log(\frac{1}{\varepsilon})) & \text{if} \quad |\alpha| = s \in \mathbb{N}_0 \\ O(\varepsilon^{s-|\alpha|}) & \text{if} \quad |\alpha| > s \end{cases} \qquad (\varepsilon \to 0). \tag{13}$$

2.2. Examples of regularity under composition. We denote by $\mathcal{G}_{*,\mathrm{loc}}^\infty(\Omega) := \cap_{s\in\mathbb{R}} \mathcal{G}_{*,\mathrm{loc}}^s(\Omega)$ the set of functions of arbitrarily high generalized local Zygmund regularity. In contrast to it we say that u has no Zygmund regularity, or regularity $-\infty$, if it is not contained in $\cup_{s\in\mathbb{R}} \mathcal{G}_{*,\mathrm{loc}}^s(\Omega)$.

In the following we will consider the set $\mathcal{O}_{\mathrm{C}}(\mathbb{R}^n)$ of smooth functions all of whose derivatives are of the same polynomial growth, i.e., $u \in \mathrm{C}^\infty$ and there is $M \in \mathbb{R}$ such that for all $\alpha \in \mathbb{N}_0^n$ we have $|u(x)| = O(|x|^M)$ as $|x| \to \infty$; in this case, we will say that u is of growth order M.

We determine the Zygmund regularity of a simple class of Colombeau functions obtained by scaling the arguments of smooth functions. These are not obtained by embedding of distributions and it is a special case of composing a smooth function with a generalized function. However, non-trivial regularity assertions about more general cases remain open at this stage. (A very useful result in C_*^s spaces concerning composition with smooth functions is [8, Proposition 8.6.12].)

Proposition 2.2. *Let r be a real number. Then:*

(i) *Let $f \in \mathcal{O}_{\mathrm{C}}(\mathbb{R})$ of growth order $M \in \mathbb{R}$. Then $u_\varepsilon(x) = f(\frac{x}{\varepsilon^r})$ defines a Colombeau function $u \in \mathcal{G}(\mathbb{R})$ which is (at least) of local Zygmund regularity s if $r < 1$. We have $s = -rM$ if $0 < r \le 1$ and $M > 0$, $s = 1 - r$ in case $0 < r \le 1$ and $M \le 0$, and may put $s = \infty$ when $r \le 0$. In general, we have no Zygmund regularity if $r > 1$.*

(ii) *Let p be a polynomial of degree $m \neq 0$. Then $u_\varepsilon(x) = p(\frac{x}{\varepsilon^r})$ defines a Colombeau function of local Zygmund regularity ∞ if $r \le 0$. If $r > 0$, we have: $u \in \mathcal{G}_{*,loc}^s(\mathbb{R})$ if and only if $s \le -rm$.*

Proof. *Part* (i): If $r > 1$ we consider the (one dimensional) example $u_\varepsilon(x) = \sin(\frac{x}{\varepsilon^r})$. The derivative of order $2k$, evaluated at $x = \frac{\pi\varepsilon^r}{2}$, gives $\pm\varepsilon^{-2kr}$. But this can never be dominated by ε^{s-2k} for all $k \in \mathbb{N}$ and s fixed. Thus u has no Zygmund regularity.

The other extreme case is $r \leq 0$ which always leads to ε-independent bounds over compact sets in each derivative. Thus we have regularity of arbitrary order.

We are left with the case $0 < r \leq 1$. Let $\alpha \in \mathbb{N}_0^n$, then

$$|\partial^\alpha u_\varepsilon(x)| = \varepsilon^{-r|\alpha|} \left|(\partial^\alpha f)\left(\tfrac{x}{\varepsilon^r}\right)\right| \leq C_\alpha\, \varepsilon^{-r|\alpha|}\left(1 + \tfrac{|x|}{\varepsilon^r}\right)^M.$$

Let x stay in a fixed compact set. If $M > 0$, the right-hand side is bounded by some constant times $\varepsilon^{-r(|\alpha|+M)} = O(\varepsilon^{-rM-|\alpha|})$. Finally, if $M \leq 0$, all we can say (in general) is that the right-hand side is $O(\varepsilon^{-r|\alpha|}) = O(\varepsilon^{(1-r)|\alpha|-|\alpha|})$ which is $O(1)$ if $|\alpha| = 0$ and $O(\varepsilon^{1-r-|\alpha|})$ otherwise.

Part (ii): The case $r \leq 0$ is obvious since all derivatives have upper bounds independent of ε then. So we assume $r > 0$ and note that p is not the zero polynomial since it has degree $m \geq 1$.

Let $\alpha \in \mathbb{N}_0^n$ and assume $0 \leq |\alpha| \leq m$. Then all higher derivatives vanish. We have $|\partial^\alpha u_\varepsilon(x)| = \varepsilon^{-r|\alpha|}|(\partial^\alpha p)(\frac{x}{\varepsilon^r})|$ which is $\varepsilon^{-r|\alpha|}O(\varepsilon^{-r(m-|\alpha|)}) = O(\varepsilon^{-rm})$ if x varies in a compact set. Furthermore, since $\partial^\alpha p$ is a polynomial (non-zero for some α of each occurring order) the estimates cannot be improved.

Assume that $u \in \mathcal{G}^s_{*,\mathrm{loc}}$. Since $-rm$ is strictly negative, ε^{-rm} is never dominated by a constant or logarithmic growth. Hence we have the conditions $s - k \leq -rm$ when $0 \leq k \leq m$. Setting $k = 0$ yields $s \leq -rm$.

On the other hand, $s \leq -rm$ is sufficient to establish the corresponding Zygmund regularity by the above estimates ∎

We end this section with two examples falling into the range of the above proposition and that further illustrate the different behavior of the notions of Zygmund- and $\mathcal{G}^\infty$-regularity, in particular, with respect to stability under smooth compositions.

Example 2.1.

(i) We have $v = [(\frac{x}{\varepsilon})_\varepsilon] \in \mathcal{G}^\infty$, and $v \in \mathcal{G}^s_*$ if and only if $s \leq -1$ (put $m = r = 1$ in the proposition above). Consider the composition $u = \sin \circ v$. Then $u \notin \mathcal{G}^\infty$, but $u \in \mathcal{G}^0_*$ ($M = 0$, $r = 1$).

(ii) Similarly, $v = [(\frac{1+x^2}{\varepsilon})_\varepsilon] \in \mathcal{G}^\infty$, and $v \in \mathcal{G}^s_*$ if and only if $s \leq -1$ (use the proposition with $m = 2$, $r = \frac{1}{2}$). Since $v_\varepsilon \geq 1$ for all $\varepsilon > 0$ we may form $u = \frac{1}{v} \in \mathcal{G}$. We observe that $u \notin \mathcal{G}^\infty$: at $x = 0$, the values of the derivatives can be read off the coefficients in the power series expansion $\sum_k x^{2k}(-1)^k \frac{1}{\varepsilon^k}$, valid in the interval $(-\sqrt{\varepsilon}, \sqrt{\varepsilon})$. From the proposition, with $M = -2$, $r = \frac{1}{2}$, we deduce that $u \in \mathcal{G}^{1/2}_*$.

3. Application to linear differential equations with nonsmooth coefficients

3.1. Solutions with classical Hölder continuity. We start with the simplest possible case of a differential equation and mention the well-known elliptic case only briefly. Finally, we sketch how a gain of regularity can be observed in the hyperbolic case too.

Primitive functions in one dimension: Let s be any real number and $u \in C^s_*(\mathbb{R})$. If $v \in \mathcal{D}'(\mathbb{R})$ is a primitive distribution of u, i.e., $v' = u$, then there is $f \in \mathrm{C}^\infty(\mathbb{R})$ such that

$$v - f \in C^{s+1}_*(\mathbb{R}). \tag{14}$$

To see this we can employ an explicit parametrix of $\frac{d}{dx}$, given as pseudodifferential operator with symbol $h(\xi) = \frac{\chi(\xi)}{i\xi}$ where $\chi \in \mathrm{C}^\infty(\mathbb{R})$ vanishes near $\xi = 0$ and $\chi(\xi) = 1$ when $|\xi| \geq 1$. (Note that $u \in \mathscr{S}'(\mathbb{R})$ and $\mathcal{F}((h(D)u)' - u) = (\chi - 1)\widehat{u}$ has compact support; hence $(h(D)u)' - u$ is smooth.) It follows that $(v - h(D)u)' = u - (h(D)u)'$ is smooth and so $v - h(D)u$ must be. But $h(D)$ being of order -1 maps C^s_* into C^{s+1}_* (see [8, Theorem 8.6.14]) which proves (14). Alternatively, we could state that v is locally in C^{s+1}_* in the sense that φv belongs to this space for any test function $\varphi \in \mathcal{D}$.

Elliptic partial differential operators: Consider $P(x, D)u = f$ where P is an elliptic partial differential operator of order m with coefficients and right-hand side f in C^s_*, $s > 0$. Then $u \in C^{s+m}_*$, i.e., we observe a gain in regularity by the order of the operator. More precise statements and related results can be found in [11, Chapter 3], a concise summary is [7, Theorem 17.1.1'].

The embryonic hyperbolic case: As a resemblance of more realistic models from geophysics we consider the Cauchy problem

$$\begin{aligned} \partial_t u + a(x)\partial_x u &= 0 \\ u\,|_{t=0} &= b \end{aligned} \tag{15}$$

where $a \in C^s_*(\mathbb{R})$, $0 < s < 1$, and $b \in C^{s+1}_*(\mathbb{R})$. In addition, we make the following strong positivity and boundedness assumption on the coefficient: there exist constants c_1, c_2 such that

$$0 < c_1 \leq a(x) \leq c_2 \qquad \forall x \in \mathbb{R}. \tag{16}$$

This condition is justified, e.g., if a is of the nature of sound speed in a certain medium or fluid.

The Cauchy problem (15) is easily solved by the method of characteristics. We point out that, by continuity and positivity of the coefficient a, the characteristic Ordinary Differential Equation has indeed a unique C^1 solution. To make this more explicit we define $A(x) = \int_0^x \frac{dr}{a(r)}$. Note that A is C^1, strictly monotone, and that $|A(x)| \leq \frac{|x|}{c_1}$. Then we set

$$u(x,t) = b\left(A^{-1}(A(x) - t)\right) \tag{17}$$

which is directly checked to be the C^1 solution of (15). As an introduction to the subject of the following two sections we investigate its Hölder-Zygmund regularity in some detail.

Proposition 3.1. *Let u be the solution of* (15) *given by* (17). *Then the first order derivatives of u are Hölder continuous of order s.*

Proof. Note that $\frac{1}{a}$ is in C^s_* which can be seen directly or, alternatively, be deduced from [8, Proposition 8.6.12] since a is bounded away from zero. We proceed straightforward in two steps.

The function $h(x,t) = A^{-1}(A(x) - t)$ clearly is C^1. We first show that its first order derivatives are Hölder continuous with exponent s. We have $\operatorname{grad} h(x,t) = a(h(x,t)) \cdot (\frac{1}{a(x)}, -1)$, which is bounded, and

$$\begin{aligned}
&|\operatorname{grad} h(x,t) - \operatorname{grad} h(y,r)| \\
&\quad \leq |a(h(x,t))| \left| \Big(\frac{1}{a(x)} - \frac{1}{a(y)}, 0\Big) \right| + |a(h(x,t)) - a(h(y,r))| \left| \Big(\frac{1}{a(y)}, -1\Big) \right| \\
&\quad \leq C\big(|x-y|^s + |h(x,t) - h(y,r)|^s\big) \\
&\quad \leq C'\big(|x-y|^s + |(x-y, t-r)|^s\big)
\end{aligned}$$

with generic constants C, C' depending on a only. Hence $\operatorname{grad} h$ is Hölder continuous of order s.

The second step is the composition with b. We have $\operatorname{grad} u = b'(h) \cdot \operatorname{grad} h$ and therefore obtain

$$\begin{aligned}
|\operatorname{grad} u(x,t) - \operatorname{grad} u(y,r)|& \\
&\leq |b'(h(x,t))||\operatorname{grad} h(x,t) - \operatorname{grad} h(y,r)| \\
&\quad + |b'(h(x,t)) - b'(h(y,r))||\operatorname{grad} h(y,r)| \\
&\leq C\big(|x-y|^s + |h(x,t) - h(y,r)|^s\big) \\
&\leq C'\big(|x-y|^s + |(x-y, t-r)|^s\big)
\end{aligned}$$

where we have used the Hölder continuity, as well as the boundedness, of $\operatorname{grad} h$ and b' ∎

3.2. Primitive functions and a linear first order Ordinary Differential Equation. The simplest inhomogeneous case is that of primitive functions in one dimension. Unlike primitive distributions, a Colombeau primitive function need not gain regularity, as the following examples illustrate.

Example 3.1. The generalized constants $[(\frac{1}{\varepsilon^r})_\varepsilon]$, $r > 0$, do not have Zygmund regularity higher than $-r$ but nevertheless are primitive functions of 0. As a consequence, any Colombeau function allows for primitive functions with Zygmund regularity arbitrarily low. Furthermore, all primitive functions of $[(\frac{1}{\varepsilon^r})_\varepsilon]$ are of the form $[(\frac{x}{\varepsilon^r})_\varepsilon] + c$ where c is any generalized constant. The latter can never be of Zygmund regularity higher than $-r$ thereby showing the existence of Colombeau functions possessing no primitive function of any higher regularity.

However, saving a minimum of the classical intuition, we can still control the regularity of primitive functions obtained from embedded distributions via integration.

Proposition 3.2. *Let $u \in \iota_\rho(C^s_*(\mathbb{R}))$, $x_0 \in \mathbb{R}$ arbitrary, and define a primitive function v by the representative $v_\varepsilon(x) = \int_{x_0}^x u_\varepsilon(y)\,dy$. Then v belongs to $\mathcal{G}^{s+1}_{*,loc}$.*

Proof. There is $u_0 \in C^s_*$ such that $u = \iota_\rho(u_0)$. By (14) we can find $w \in C^{s+1}_*$ of u_0 such that we have $\iota_\rho(u_0) = \iota_\rho(w') + \sigma(g)$ for some smooth function g. Hence there is $(n_\varepsilon)_\varepsilon \in \mathcal{N}$ such that

$$v_\varepsilon(x) = \int_{x_0}^x (w' * \rho_\varepsilon)(y)\,dy + \int_{x_0}^x f(y)\,dy + n_\varepsilon(x).$$

We observe that, in general, any derivative of order $l \geq 1$ has the asserted asymptotic estimates since $v_\varepsilon^{(l)}(x) = u_\varepsilon^{(l-1)}$, so only the zero order estimate has to be investigated separately.

Using $\int_{x_0}^x (w' * \rho_\varepsilon)(y)\,dy = (w * \rho_\varepsilon)(x) - (w * \rho_\varepsilon)(x_0)$ we obtain, for any compact interval I containing x, x_0, and of length $|I|$,

$$|v_\varepsilon(x)| \leq 2 \sup_{y \in I} |(w * \rho_\varepsilon)(y)| + \sup_{y \in I} \big(|I|\ |f(y)| + |n_\varepsilon(y)|\big).$$

The second term on the right-hand side is $O(1)$ on compact subsets with respect to x. Finally, since $w \in C^{s+1}_*$ we deduce from Theorem 2.1 the required growth properties, according to regularity $s+1$, of the complete expression ∎

In the proposition to follow we give a lower bound for the regularity of the solution to a linear homogeneous ODE with coefficient from a generalized Zygmund class. We will impose an additional boundedness condition on this coefficient and recall: $v \in \mathcal{G}$ is said to be *locally bounded* if for all $K \subset\subset \mathbb{R}^n$ there are C, $\varepsilon_0 > 0$ such that $\sup_{x \in K} |v_\varepsilon(x)| \leq C$ for all $0 < \varepsilon < \varepsilon_0$.

Proposition 3.3. *Assume $s \geq -1$ and let $a \in \mathcal{G}^s_{*,loc}(\mathbb{R})$ such that $Re(a)$ is locally bounded. Let b be a generalized constant which, considered as a generalized function, is of generalized Zygmund regularity t ($t \in \mathbb{R}$). Then the unique solution $u \in \mathcal{G}(\mathbb{R})$ to the initial value problem*

$$\begin{aligned} \frac{d}{dx}u(x) &= a(x)u(x) \\ u(0) &= b \end{aligned} \tag{18}$$

belongs to $\mathcal{G}^r_{*,loc}(\mathbb{R})$ with $r = s+1$ if $t > 0$ and $r = t$ if $t < 0$. When $t = 0$, we have $r = 0_-$ if $s = -1$ and $r = 0$ if $s > -1$. Here, 0_- stands for any negative number, arbitrarily close to 0.

Proof. Existence and uniqueness of the solution u follows from [3]. A representative is given by

$$u_\varepsilon(x) = b_\varepsilon e^{\int_0^x a_\varepsilon(y)\,dy}$$

where $(b_\varepsilon)_\varepsilon$ is a representative of b. By our assumption on a we have on any compact set K

$$\|u_\varepsilon\|_{L^\infty(K)} = O(|b_\varepsilon|) \quad (\varepsilon \to 0).$$

To find sharp asymptotic bounds for the derivatives we first investigate their algebraic structure. The following assertion is easily proved using the ODE itself and induction on the derivative order k. Let $k \in \mathbb{N}$. Then $u_\varepsilon^{(k)}$ is a linear combination of terms of the following form: with $m \in \mathbb{N}$, $1 \leq m \leq k$, and $\lambda \in \mathbb{N}_0^m$ such that $|\lambda| = k - m$ we have the expression

$$u_\varepsilon \cdot \prod_{j=1}^m a_\varepsilon^{(\lambda_j)}. \tag{19}$$

As noted above, the first factor u_ε is $O(|b_\varepsilon|)$, so we focus on the product of derivatives of a_ε.
Claim: for any $s \geq -1$ we have, with the notation as in (19),

$$\prod_{j=1}^m \|a_\varepsilon^{(\lambda_j)}\|_{L^\infty(K)} = \begin{cases} O(1) & \text{if} \quad k < s+1 \\ O(\log(\frac{1}{\varepsilon})) & \text{if} \quad k = s+1 \\ O(\varepsilon^{s+1-k}) & \text{if} \quad k > s+1 \end{cases} \tag{20}$$

on compact sets with respect to x.

• If $s < 0$, then each $\lambda_j \geq 0 > s$ and hence we have the asymptotic bound $O(\varepsilon^{ms-|\lambda|}) = O(\varepsilon^{m(s+1)-k}) = O(\varepsilon^{s+1-k})$.

• If $s = 0$, let n be the number of j's such that $\lambda_j = 0$. Then we have the asymptotic upper bound involving $(\log(\frac{1}{\varepsilon}))^n \varepsilon^{-|\lambda|} = (\log(\frac{1}{\varepsilon}))^n \varepsilon^{m-k}$. When

$m \geq 2$, the second factor is $O(\varepsilon^{1-k}\varepsilon)$ where ε can compensate for the logarithmic terms. Hence we have a bound $O(\varepsilon^{1-k})$. When $m = 1$, we obtain $O(\log(\frac{1}{\varepsilon}))$ if $k = 1$ and $O(\varepsilon^{1-k})$ otherwise (since $n = 0$ then).

• Finally, we have to consider the case $s > 0$. We have further to distinguish three subcases for the relation between k and $s + 1$.

Subcase $k < s + 1$: Since $|\lambda| = k - m \leq k - 1 < s$, we have that $\lambda_j < s$ for each j, and hence an upper bound $O(1)$.

Subcase $k = s + 1$: Now $|\lambda| \leq s$ and for at most one j we have $\lambda_j = s$, all others are less than s; hence we obtain an estimate of the form $O(\log(\frac{1}{\varepsilon})$.

Subcase $k > s + 1$: Denote by n the number of j's such that $\lambda_j = s$ and define $N' := \{j \mid \lambda_j > s\}$, $n' := |N'|$ (the cardinality of N'). Put $\lambda'_j = 0$ if $j \notin N'$ and $\lambda'_j = \lambda_j$ otherwise. The asymptotic upper bound in question is now expressible as $O((\log(\frac{1}{\varepsilon}))^n \varepsilon^{n's-|\lambda'|})$. If $n' = |\lambda'| = 0$, this clearly is $O(\varepsilon^{s+1-k})$, so we may assume that $n' \geq 1$. Since $k - m = |\lambda| \geq ns + |\lambda'|$ we obtain $\varepsilon^{-|\lambda'|} \leq \varepsilon^{m+ns-k}$. Inserting this into the expression for the asymptotic upper bound we arrive at $O(\log(\frac{1}{\varepsilon})^n \varepsilon^{ns}) O(\varepsilon^{n's+m-k})$. Here, the first factor is $O(1)$ since $s > 0$ and the second factor is $O(\varepsilon^{s+1-k})$, due to $n' \geq 1$ and $m \geq 1$, as claimed.

Now we come back to (19) and use the information from (20). If $t < 0$, the order zero estimate implies $r \leq t$. By combining (19) with (20) we see that regularity $r = t$ can indeed be established for any $s \geq -1$.

If $t = 0$ the order zero estimate is logarithmic, due to $|b_\varepsilon|$, and forces $r \leq 0$. If $s + 1 > 0$, it is seen from (20) that $r = 0$ holds. In case $s = -1$ and $k > 0$ we have to cope with appearing upper bounds of the form $O(\log(\frac{1}{\varepsilon})\varepsilon^{-k})$. This requires subtraction of an arbitrary small, but still positive, number σ in the exponent to incorporate the additional logarithmic factor. (Compare with the situation in the general multiplication result.)

Finally, if $t > 0$, the factor $|b_\varepsilon| = O(1)$ and the regularity $r = s + 1$ is established directly from (20) ∎

Remark 4.

(i) Note that if k is very large in (19), it may happen that each $\lambda_j > s$. In this case, a general upper bound will be of the form $O(\varepsilon^{ms-|\lambda|}) = O(\varepsilon^{m(s+1)-k})$. Since m may also become arbitrarily large, this indicates that the condition $s + 1 \geq 0$ cannot be dropped in general while expecting Zygmund regularity of the solution. This is illustrated by the constant coefficient problem with $a_\varepsilon(x) = \frac{i}{\varepsilon^r}$, $b = 1$ and $r > 0$. The solution (representative) is then $\exp(i\frac{x}{\varepsilon^r})$, a sort of 'standard counter example' in Colombeau regularity theory.

(ii) The boundedness condition on the real part of the coefficient cannot be dropped. Indeed, this can be seen from the constant coefficient problem with $a_\varepsilon(x) = \log(\frac{1}{\varepsilon})$ and $b = 1$. A Colombeau solution representative is given by $\exp(x \log(\frac{1}{\varepsilon}))$ which is not Zygmund-regular: The L^∞-norm taken over a compact set K grows like $\varepsilon^{-m(K)}$ if $m(K)$ denotes the maximum of K.

3.3. A linear hyperbolic Cauchy problem. As we have indicated in the introduction, if we think of modeling seismic wave propagation we may encounter fractal-like variations in sound speed, for example. By the very nature of coefficients representing physical observables like sound speed, density, elasticity tensors, we see that a positivity condition on the coefficient(s) is not artificial. We state a first regularity result for a simple model of this type in one space dimension. It fits nicely with the classical embryonic case discussed in Subsection 4.1.

Theorem 3.1. *Let $a = [(a_\varepsilon)_\varepsilon] \in \mathcal{G}^s_{*,loc}(\mathbb{R})$, $s \geq 0$, and assume there are positive constants c_1, c_2 such that $c_1 \leq a_\varepsilon(x) \leq c_2$ for all $x \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Let t be a real number and $b \in \mathcal{G}^t_{*,loc}(\mathbb{R})$. If u is the (unique) solution of the Cauchy problem*

$$\begin{aligned} \partial_t u + a(x)\partial_x u &= 0 \\ u(0) &= b \end{aligned} \tag{21}$$

then $u \in \mathcal{G}^r_{*,loc}(\mathbb{R}^2)$ with $r = \min(t,1)_-$ if $s = 0$, and $r = \min(t, s+1)$ if $s > 0$. (As above, $\min(t,1)_-$ denotes any number approximating $\min(t,1)$ from below.)

Proof. We have to determine asymptotic upper bounds of all derivatives of $u_\varepsilon(x,t) = b_\varepsilon(A_\varepsilon^{-1}(A_\varepsilon(x) - t))$ on compact sets.

We first note that the assumptions on a imply that $h_\varepsilon(x,t) = A_\varepsilon^{-1}(A_\varepsilon(x)-t)$ maps a compact subset K of $\mathbb{R}^2$ into a fixed compact subset K' of $\mathbb{R}$, independently of ε. Therefore when doing estimates on K we may essentially ignore the argument $h_\varepsilon(x,t)$ whenever appearing as inner function in compositions and write instead the supremum over K'. However, the chain rule will bring out derivatives of h_ε as additional factors. Thus the order zero estimate for u_ε is simply

$$\|u_\varepsilon\|_{\mathrm{L}^\infty(K)} \leq \|b_\varepsilon\|_{\mathrm{L}^\infty(K')}. \tag{22}$$

In the following, let $\alpha \in \mathbb{N}_0^2$ such that $|\alpha| \geq 1$.

As a preparation we have to investigate the structure of the higher order derivatives of u_ε. To simplify notation we drop the subscript ε in doing this algebra.

Claim 1: $\partial^\alpha u$ is a linear combination of terms of the form

$$b^{(l)}(h(x,t)) \cdot \prod_{i=1}^{m} a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j=1}^{n} \frac{a^{(\mu_j)}(x)}{a^k(x)} \tag{23}$$

where $1 \leq l \leq m \leq |\alpha|$, $0 \leq n \leq |\alpha|$, $0 \leq k \leq |\alpha|$, $|\lambda| = m - l$, and $|\mu| = |\alpha| - m$, with the notation $\lambda := (\lambda_i)_{i=1}^m$ and $\mu := (\mu_j)_{j=1}^n$.

We prove (23) by induction on $|\alpha|$. Concerning the inner derivatives when applying the chain rule we note that, by definition of A, we have $\partial_t h(x,t) = -a(h(x,t))$ and $\partial_x h(x,t) = \frac{a(h(x,t))}{a(x)}$.

The base cases correspond to first order derivatives

$$\begin{aligned} \partial_t u(x,t) &= -b'(h(x,t)) \cdot a(h(x,t)) \\ \partial_x u(x,t) &= b'(h(x,t)) \, \frac{a(h(x,t))}{a(x)}, \end{aligned}$$

both complying with the structure of (23).

Assume the claim to be proven already for $|\alpha|$ and let $\beta \in \mathbb{N}_0^2$ with $|\beta| = |\alpha| + 1$. We distinguish the two cases $\beta = \alpha + e_1$ and $\beta = \alpha + e_2$ (e_j denoting the standard unit vector in direction j).

Case $\beta = \alpha + e_1$: By the induction hypothesis, $\partial^\beta u = \partial_x(\partial^\alpha u)$ is a linear combination of terms

$$\partial_x \left(b^{(l)}(h(x,t)) \cdot \prod_{i=1}^m a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j=1}^n \frac{a^{(\mu_j)}(x)}{a^k(x)} \right).$$

Application of the Leibniz and chain rules yields four types of terms.
Type 1 is

$$b^{(l+1)}(h(x,t)) \, a(h(x,t)) \, a(x)^{-1} \cdot \prod_{i=1}^m a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j=1}^n \frac{a^{(\mu_j)}(x)}{a^k(x)}$$

which matches the claim with new quantities $l+1$, $m+1$, $k+1$, and $\lambda_{m+1} := 0$.
Type 2, for any $1 \leq r \leq m$, is

$$b^{(l)}(h(x,t)) \cdot \prod_{i \neq r, i=1}^m a^{(\lambda_i)}(h(x,t)) \cdot a^{(\lambda_r+1)}(h(x,t)) \cdot \frac{a(h(x,t))}{a(x)^{-1}} \cdot \prod_{j=1}^n \frac{a^{(\mu_j)}(x)}{a^k(x)}$$

and satisfies (23) with $k+1$, $m+1$, $\lambda_r + 1$, and $\lambda_{m+1} := 0$ instead.
Type 3, for any $1 \leq r \leq n$, is

$$b^{(l)}(h(x,t)) \cdot \prod_{i=1}^m a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j \neq r, j=1}^n a^{(\mu_j)}(x) \cdot \frac{a^{(\mu_r+1)}(x)}{a^k(x)}$$

where we may use the new component $\mu_r + 1$ in (23).
Type 4 is

$$b^{(l)}(h(x,t)) \cdot \prod_{i=1}^m a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j=1}^n a^{(\mu_j)}(x) \cdot \frac{-ka'(x)}{a^{k+1}(x)}$$

and matches the claim with new quantities $k+1$, $n+1$, and $\mu_{n+1} := 1$.

Case $\beta = \alpha + e_2$: By the induction hypothesis, $\partial^\beta u = \partial_t(\partial^\alpha u)$ is a linear combination of terms

$$\partial_t\left(b^{(l)}(h(x,t)) \cdot \prod_{i=1}^{m} a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j=1}^{n} \frac{a^{(\mu_j)}(x)}{a^k(x)} \right).$$

Application of the Leibniz and chain rules yields two types of terms.
Type 1 is

$$-b^{(l+1)}(h(x,t))\, a(h(x,t)) \cdot \prod_{i=1}^{m} a^{(\lambda_i)}(h(x,t)) \cdot \prod_{j=1}^{n} \frac{a^{(\mu_j)}(x)}{a^k(x)}$$

which matches the claim with new quantities $l+1$, $m+1$, and $\lambda_{m+1} := 0$.
Type 2 is

$$-b^{(l)}(h(x,t)) \cdot \prod_{i\neq r, i=1}^{m} a^{(\lambda_i)}(h(x,t)) \cdot a^{(\lambda_r+1)}(h(x,t))\, a(h(x,t)) \cdot \prod_{j=1}^{n} \frac{a^{(\mu_j)}(x)}{a^k(x)}$$

and satisfies (23) with new values $m+1$, λ_r+1, and $\lambda_{m+1} := 0$.
The claim is proved.

According to claim 1 and the remark at the beginning of this proof we deduce that on any compact set we have

$$\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)} = O\Big(\|b_\varepsilon^{(l)}\|_{\mathrm{L}^\infty(K')} \cdot \prod_{i=1}^{m} \|a_\varepsilon^{(\lambda_i)}\|_{\mathrm{L}^\infty(K')} \cdot \prod_{j=1}^{n} \|a_\varepsilon^{(\mu_j)}\|_{\mathrm{L}^\infty(K)} \Big). \tag{24}$$

To evaluate this carefully we first focus on all the factors having bounds depending on a or its derivatives.

With the notation of (23) define the sets $L_0 = \{i \mid \lambda_i = s\}$, $L_1 = \{i \mid \lambda_i > s\}$, $M_0 = \{j \mid \mu_j = s\}$, and $M_1 = \{j \mid \lambda_j > s\}$. Let $l_0 = |L_0|$ and define similarly l_1, m_0, m_1 as the respective cardinalities.

Claim 2: On compact sets we can give asymptotic upper bounds of the form

$$\prod_{i=1}^{m} \|a_\varepsilon^{(\lambda_i)}\|_{\mathrm{L}^\infty(K')} \cdot \prod_{j=1}^{n} \|a_\varepsilon^{(\mu_j)}\|_{\mathrm{L}^\infty(K)} = \begin{cases} O\big((\log(\frac{1}{\varepsilon}))^{l_0+m_0} \varepsilon^{(l_0+m_0)s} \varepsilon^{(l_1+m_1)s+l-|\alpha|} \big) & \text{if} \quad l_1+m_1 > 0 \\ O\big((\log(\frac{1}{\varepsilon}))^{l_0+m_0} \big) & \text{if} \quad l_1+m_1 = 0. \end{cases} \tag{25}$$

Using the notation introduced above the proof is easy. We observe that each $i \in L_0$ and $j \in M_0$ contributes a factor $\log(\frac{1}{\varepsilon})$, whereas each $i \in L_1$, resp.

$j \in M_1$, gives rise to a factor $\varepsilon^{s-\lambda_i}$, resp. $\varepsilon^{s-\mu_j}$. We define the tuples λ' and μ' by setting all components in λ, resp. μ, which are less than s to 0. Then we obtain a total bound $O((\log(\frac{1}{\varepsilon}))^{l_0+m_0}\varepsilon^{(l_1+m_1)s-|\lambda'|-|\mu'|})$. If $l_1+m_1=0$, then also $|\lambda'|+|\mu'|=0$ which proves the second case in (25). If $l_1+m_1\geq 1$, we note that $m-l=|\lambda|\geq|\lambda'|+l_0 s$ and $|\alpha|-m=|\mu|\geq|\mu'|+m_0 s$. This implies $-|\lambda'|-|\mu'|\geq(l_0+m_0)s+l-|\alpha|$ and hence $\varepsilon^{-|\lambda'|-|\mu'|}\leq\varepsilon^{(l_0+m_0)s+l-|\alpha|}$. Inserting this into the above total bound matches the first case in (25) and proves claim 2.

We are now in a position to estimate the regularity r of u using (22) and (24). From (22) we learn that $r\leq t$; and since $s\geq 0$, this is compatible with the assertion in (3.1). In order to investigate the asymptotic behavior of (24) if $|\alpha|\geq 1$, we consider the cases $s=0$ and $s>0$ separately.

Case $s=0$*:* We recall that $r=\min(r,1)-\sigma<1$ and we have to show that (24) is $O(\varepsilon^{r-|\alpha|})$. Combination of (25) (note that $|\alpha|\geq l\geq 1$) with the three possible cases $O(1)$, $O(\log(\frac{1}{\varepsilon})$, $O(\varepsilon^{t-l})$ of the growth rate of $|b_\varepsilon^{(l)}|$ directly yields an upper bound of the form $O(\varepsilon^{\min(t,1)-|\alpha|}(\log(\frac{1}{\varepsilon}))^k)$ (where $k\leq l_0+m_0+1$). Since the logarithmic factor is dominated by $\varepsilon^{-\sigma}$, for any $\sigma>0$, the assertion is proved.

Case $s>0$*:* Now $r=\min(t,s+1)$ can be any real number and we have to go through all cases relating the possible values of $|\alpha|$ and r.

Case $|\alpha|<r$: Since $1\leq l\leq|\alpha|$ the factor $|b_\varepsilon^{(l)}|$ is $O(1)$, and in (23), (25) we find $|\lambda|+|\mu|=|\alpha|-l<s$, which in turn yields $l_1=m_1=l_0=m_0=0$. Therefore we have an overall bound $O(1)$.

Case $|\alpha|=r$: If $l=|\alpha|$, then $|\lambda|+|\mu|=0$ and hence $l_0+m_0=0$ in (25) which means $O(1)$ for this part. The factor $|b_\varepsilon^{(l)}|$ gives at most $O(\log(\frac{1}{\varepsilon}))$. If $l<|\alpha|$, then $l<t$ and so $|b_\varepsilon^{(l)}|$ is $O(1)$. Since $|\lambda|+|\mu|\leq s$, we deduce $l_0+m_0\leq 1$ and (25) ensures an overall logarithmic bound. Hence we have an upper bound of logarithmic order in both (sub)subcases.

We note that $s>0$ implies $(\log(\frac{1}{\varepsilon}))^{l_0+m_0}\varepsilon^{(l_0+m_0)s}=O(1)$ whatever the value of $l_0+m_0\geq 0$. Therefore, the first case in (25), $l_1+m_1\geq 1$, always yields a bound $O(\varepsilon^{s+l-|\alpha|})$.

Case $l<t$: The b-dependent factor in (24) is $O(1)$ and both cases in (25) are dominated by $O(\varepsilon^{s+1-|\alpha|})$.

Case $l=t$: $|b_\varepsilon^{(t)}|$ contributes a logarithmic factor. The first case in (25) then gives $O(\varepsilon^{s+t-|\alpha|})$ of which the part ε^s can be used to suppress this logarithmic factor; hence a bound is $O(\varepsilon^{t-|\alpha|})$. The second case in (25) yields an overall bound which is some power of $\log(\frac{1}{\varepsilon})$ and therefore clearly dominated by $\varepsilon^{r-|\alpha|}$.

Case $l>t$: Here $|b_\varepsilon^{(l)}|=O(\varepsilon^{t-l})$. Adding the factor according to the first case in (25) then gives a bound $O(\varepsilon^{t-|\alpha|+s})=O(\varepsilon^{t-|\alpha|})$. On the other hand, using the second line in (25) provides an overall bound $O(\varepsilon^{t-l}\cdot(\log(\frac{1}{\varepsilon}))^{l_0+m_0})$. If

$l < |\alpha|$, splitting off $\varepsilon^{t-|\alpha|}$ leaves an additional positive ε-power to compensate for the logarithmic term. If $l = |\alpha|$, we can again reason, like in earlier cases, that $l_0 + m_0 = 0$. So, all branches of this (subsub)subcase lead to a bound $O(\varepsilon^{t-|\alpha|})$.

Collecting the results of all (sub)subcases we have established the asymptotic upper bound $O(\varepsilon^{\min(t,s+1)-|\alpha|})$ of (24) ∎

The previous theorem indicates that we may expect a seismic wave to be about one degree smoother than the irregular medium variation if the source is prepared appropriately. In principle this would enable one to deduce from measurements of the wave an upper bound of the (global) medium regularity: first, estimate a strict upper bound of the wave's Zygmund regularity r via wavelet analysis of the data; then the medium regularity cannot be better than $r - 1$.

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Appendix: Characterization of Zygmund regularity via continuous wavelet transform

The proof to be presented below is a destillation of methods and basic setups drawing from a variety of sources. We briefly sketch the basics of these as a preparation.

(i) Zygmund classes can alternatively be defined by a discrete Littlewood-Paley decomposition (cf. [15]), also called dyadic resolution (e.g. in [21]). Let $\varphi_0 = \varphi$ and for $j \in \mathbb{N}$ put

$$\varphi_j(\xi) = \int_{2^{j-1}}^{2^j} \psi(\tfrac{\xi}{t})\,\tfrac{dt}{t} = \varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi).$$

We have $\varphi_{j+1}(\xi) = \varphi_j(\frac{\xi}{2})$ and the support of φ_j ($j \geq 1$) is contained in the annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$. By construction, the family $(\varphi_j)_{j\geq 0}$ is a dyadic partition of unity: $\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$. Similarly, the equation $\sum_j \varphi_j(D)u = u$ holds with convergence of the series in $\mathscr{S}'$.

(ii) The classical Hölder-Zygmund spaces can also be considered as the special cases $B^s_{\infty,\infty}(\mathbb{R}^n)$ in Triebel's family of Besov-Hardy-Sobolev-type spaces (cf. [21, Chapter 2, in particular Equation 2.6.5/(1)]). These spaces are defined, for any $s \in \mathbb{R}$, by $B^s_{\infty,\infty} := \{u \in \mathscr{S}' \mid \|u\|_{B^s_{\infty,\infty}} := \sup_{j\geq 0} 2^{js}\|\varphi_j(D)u\|_{L^\infty} < \infty\}$. The definition is independent of the particular choice of φ (cf. [21, Section 2.3.2]).

(iii) Both families of spaces, $B^s_{\infty,\infty}$ as well as C^s_*, are realizations of the classical Hölder-Zygmund spaces when $s > 0$. Therefore we clearly have $B^s_{\infty,\infty} = C^s_*$ in this case. In fact, equality holds for all real s: By [20, Section 2.3.8] (resp. [8, Proposition 8.6.6]), for any $r \in \mathbb{R}$ the operators $(1-\Delta)^{r/2}$ (resp. $(1-\Delta)^{-r/2}$) on $\mathscr{S}'$ map $B^s_{\infty,\infty}$ (resp. C^{s-r}_*) isomorphically into $B^{s-r}_{\infty,\infty}$ (resp. C^s_*); therefore we obtain $B^s_{\infty,\infty} = C^s_*$ for all $s \in \mathbb{R}$ with equivalent norms $\|u\|_{B^s_{\infty,\infty}}$ and $|u|^*_s$. We refer to these spaces as *Zygmund spaces of regularity* s. In particular, we deduce that the definition of C^s_* is independent of the choice of φ.

(iv) In Meyer's book (cf. [15, Chapter 3]) the Hölder-Zygmund spaces are treated as special cases of Bony's two-microlocal spaces $C^{s,s'}_{x_0}$ (where $s > 0$, $s' = 0$, x_0 arbitrary). In fact, it is this point of view which is underlying the proof of the characterization via the ('continuous') wavelet transform given in the following.

Proof of Theorem 1.1. Recall that $\mathscr{S}_0(\mathbb{R}^n)$ is the subspace of $\mathscr{S}(\mathbb{R}^n)$ consisting of functions with vanishing moments of all orders. Throughout the proof we will make use of the following fact which will allow us to balance vanishing moment conditions with regularity properties in occurring convolutions.

Lemma 3.1. *If $f \in \mathscr{S}$ with moments up to order $m-1$ vanishing, then one can find functions $f_\alpha \in \mathscr{S}$ ($|\alpha| = m$) such that*

$$f = \sum_{|\alpha|=m} \partial^\alpha f_\alpha.$$

If in addition $f \in \mathscr{S}_0$, the functions f_α can be chosen to be in $\mathscr{S}_0$.

This can be shown by adapting the proof of [14, Section 2.6, Lemma 12].

Concerning the notation of various constants in the estimates to follow we will use the generic letter C, with subscripts if we want to indicate dependence on certain parameters.

Part (i): Applying the above lemma to g we have $g_\varepsilon = \varepsilon^m \sum_{|\alpha|=m} \partial^\alpha((g_\alpha)_\varepsilon)$ and obtain

$$W_g u(\cdot,\varepsilon) = u * \overline{\check{g}_\varepsilon} = (-1)^m \varepsilon^m \sum_{|\alpha|=m} (\partial^\alpha u) * (\overline{\check{g}_\alpha})_\varepsilon.$$

Since $\partial^\alpha u \in C_*^{s-m}$ with $s-m<0$ and $g_\alpha \in \mathscr{S}$, we have reduced the proof of (9) to the task of estimating $\|u * g_\varepsilon\|_{L^\infty}$ where $u \in C_*^s$ with $s<0$ and $g \in \mathscr{S}$.

If $T>1$ arbitrary, then

$$u = \varphi(\tfrac{D}{T})u + \int_T^\infty \psi(\tfrac{D}{T})u \,\tfrac{dt}{t}$$

(with $\mathscr{S}'$-convergence) and therefore we have for any $\varepsilon>0$ fixed

$$u * g_\varepsilon = \varphi(\tfrac{D}{T})u * g_\varepsilon + \int_T^\infty \psi(\tfrac{D}{t})u * g_\varepsilon \,\tfrac{dt}{t}. \tag{26}$$

Let $T \geq \frac{1}{\varepsilon} \geq \frac{T}{2}$ and estimate the two terms in (26) separately.

Since $\varphi(\frac{D}{T})u = \varphi(D)u + \int_1^T \psi(\frac{D}{T})u \,\frac{dt}{t}$, we deduce (recalling that we may assume $s<0$)

$$\begin{aligned}
\|\varphi(\tfrac{D}{T})u\|_{L^\infty} &\leq \|\varphi(D)u\|_{L^\infty} + \int_1^T \|\psi(\tfrac{D}{t})u\|_{L^\infty} \,\tfrac{dt}{t} \\
&\leq C\Big(1 + \int_1^T t^{-s} \,\tfrac{dt}{t}\Big) \\
&\leq CT^{-s} \\
&\leq 2^{-s} C\varepsilon^s.
\end{aligned}$$

Therefore we obtain

$$\|\varphi(\tfrac{D}{t})u * g_\varepsilon\|_{L^\infty} \leq \|\varphi(\tfrac{D}{t})u\|_{L^\infty} \|g_\varepsilon\|_{L^1} \leq C\varepsilon^s. \tag{27}$$

To estimate the integrand in the second term of (26) we assume $t \geq T$ and choose $\tilde\psi \in \mathcal{D}$ with $\tilde\psi = 0$ near 0 and $\tilde\psi = 1$ on $\operatorname{supp}(\psi)$. It follows that $\mathcal{F}^{-1}\tilde\psi \in \mathscr{S}_0$ and $\psi(\frac{D}{t})u * g_\varepsilon = \psi(\frac{D}{t})u * \tilde\psi(\frac{D}{t})g_\varepsilon$.

Choose $r \in \mathbb{N}$ such that $r+s>0$ and apply Lemma 3.1 to obtain functions $\tilde\psi_\alpha$, $|\alpha|=r$, satisfying $\mathcal{F}^{-1}\tilde\psi_\alpha \in \mathscr{S}_0$ and

$$\mathcal{F}^{-1}\tilde\psi = \sum_{|\alpha|=r} D^\alpha \mathcal{F}^{-1}\tilde\psi_\alpha = \sum_{|\alpha|=r} \mathcal{F}^{-1}(\xi^\alpha \tilde\psi_\alpha).$$

Then

$$\tilde\psi(\tfrac{D}{t})g_\varepsilon = t^{-r}\varepsilon^{-r} \textstyle\sum_{|\alpha|=r} \tilde\psi_\alpha(\tfrac{D}{t})(D^\alpha g)_\varepsilon$$

and since $\|\psi(\frac{D}{t})u\|_{L^\infty} \leq Ct^{-s}$ we have the estimate

$$\begin{aligned}
&\|\psi(\tfrac{D}{t})u * g_\varepsilon\|_{L^\infty} \\
&\quad\leq t^{-r}\varepsilon^{-r} \sum_{|\alpha|=r} \|\psi(\tfrac{D}{t})u * \tilde\psi_\alpha(\tfrac{D}{t})(D^\alpha g)_\varepsilon\|_{L^\infty} \\
&\quad\leq t^{-r}\varepsilon^{-r} \sum_{|\alpha|=r} \|\psi(\tfrac{D}{t})u\|_{L^\infty} \|\tilde\psi_\alpha(\tfrac{D}{t})(D^\alpha g)_\varepsilon\|_{L^1} \\
&\quad\leq Ct^{-(r+s)}\varepsilon^{-r} \max_{|\alpha|=r} \|\tilde\psi_\alpha(\tfrac{D}{t})(D^\alpha g)_\varepsilon\|_{L^1}.
\end{aligned}$$

We show that the appearing L^1-norms have bounds independent of t and ε.

Writing $\tilde{\psi}_\alpha(\frac{D}{t})(D^\alpha g)_\varepsilon$ explicitly as a convolution and rescaling by t via substitution of the integration variable we have

$$\tilde{\psi}_\alpha(\tfrac{D}{t})(D^\alpha g)_\varepsilon(x) = \varepsilon^{-n} \int \mathcal{F}^{-1}(\tilde{\psi}_\alpha)(y)(D^\alpha g)\left(\frac{x}{\varepsilon} - \frac{y}{t\varepsilon}\right) dy.$$

For any l, the second factor in the integrand is bounded by

$$\begin{aligned} C_l \left(1 + \left|\frac{x}{\varepsilon} - \frac{y}{t\varepsilon}\right|^2\right)^{-l} &\leq 2^l C_l \left(1 + \left|\frac{x}{\varepsilon}\right|^2\right)^{-l} \left(1 + \left|\frac{y}{t\varepsilon}\right|^2\right)^{l} \\ &\leq 2^l C_l \left(1 + \left|\frac{x}{\varepsilon}\right|^2\right)^{-l} \left(1 + |y|^2\right)^{l} \end{aligned}$$

since $t\varepsilon \geq T\varepsilon \geq 1$. Assuming $l > \frac{n}{2}$ and integrating also over x we finally obtain a bound for $\|\tilde{\psi}_\alpha(\frac{D}{t})(D^\alpha g)_\varepsilon\|_{L^1}$ of the form

$$C_{l,\alpha} \int \varepsilon^{-n} \left(1 + \left|\frac{x}{\varepsilon}\right|^2\right)^{-l} dx = C_{l,\alpha} \int (1 + |z|^2)^{-l} \, dz$$

which is indeed independent of t and ε.

Taking the maximum of all bounds over $|\alpha| = r$ we arrive at the conclusion that for all $t \geq T$

$$\|\psi(\tfrac{D}{t})u * g_\varepsilon\|_{L^\infty} \leq C\varepsilon^{-r} t^{-(r+s)}.$$

If $R > T$ arbitrary, then

$$\begin{aligned} \left\| \int_T^R \psi(\tfrac{D}{t})u * g_\varepsilon \, \tfrac{dt}{t} \right\|_{L^\infty} &\leq \int_T^R \left\|\psi(\tfrac{D}{t})u * g_\varepsilon\right\|_{L^\infty} \tfrac{dt}{t} \\ &\leq C\varepsilon^{-r} \int_T^R t^{-(r+s)-1} \, dt \\ &= \frac{C\varepsilon^{-r}}{r+s} \left(T^{-(r+s)} - R^{-(r+s)}\right). \end{aligned}$$

When $R \to \infty$ this upper bound tends to $\frac{C}{r+s}(\varepsilon T)^{-r} T^{-s} \leq C\varepsilon^s$. This completes the proof of (9).

Part (ii):

Lemma 3.2. *Let $r > 0$ and $k \in \mathbb{N}$ such that $k > r$. Assume that h_j ($j \in \mathbb{N}_0$) is a sequence of functions in $C^k(\mathbb{R}^n)$ with the property that there is $B > 0$ such that for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k$*

$$\|\partial^\beta h_j\|_{L^\infty} \leq B\, 2^{j|\beta|}. \tag{28}$$

Then the infinite series

$$h(x) := \sum_{j=0}^{\infty} 2^{-jr} h_j(x) \tag{29}$$

converges uniformly and defines an element in $C_*^r(\mathbb{R}^n)$.

Proof. Since $\|h_j\|_{L^\infty} \leq B$ for all j, the series is absolutely and uniformly convergent and defines a continuous bounded function h. Hence it is immediate that $\|\varphi(D)h\|_{L^\infty} \leq \|\mathcal{F}^{-1}\varphi\|_{L^1} \|h\|_{L^\infty}$. It remains to estimate $\|t^r \psi(\frac{D}{t})h\|_{L^\infty}$ for all $t \geq 1$. We start by picking $q \in \mathbb{N}_0$ such that $2^q \leq t < 2^{q+1}$ and split the necessary summation according to

$$\begin{aligned}
&|t^r \psi(\tfrac{D}{t})h(x)| \\
&\quad \leq \sum_{j=0}^{\infty} 2^{-jr} t^r |\psi(\tfrac{D}{t})h_j(x)| \\
&\quad = \sum_{j=0}^{q-1} 2^{-jr} t^r |\psi(\tfrac{D}{t})h_j(x)| + \textstyle\sum_{j=q}^{\infty} 2^{-jr} t^r |\psi(\tfrac{D}{t})h_j(x)| \\
&\quad =: S_1(x) + S_2(x).
\end{aligned}$$

The terms in S_2 can be estimated

$$\begin{aligned}
2^{-jr} t^r |\psi(\tfrac{D}{t})h_j(x)| &\leq (\tfrac{t}{2^q})^r 2^{-r(j-q)} \|\mathcal{F}^{-1}\psi\|_{L^1} \|h_j\|_{L^\infty} \\
&\leq 2^r C' B 2^{-r(j-q)} \\
&= C 2^{-r(j-q)}
\end{aligned}$$

and hence S_2 is dominated uniformly by a convergent geometric series.

To find a bound for $S_1(x)$ we apply Lemma 3.1 and rewrite $\psi(\frac{D}{t})$, as with $\tilde{\psi}$ in the proof of part (i), in the form $\psi(\frac{D}{t}) = t^{-k} \sum_{|\alpha|=k} \psi_\alpha(\frac{D}{t}) D^\alpha$. Hence

$$\left|\psi(\tfrac{D}{t})h_j(x)\right| \leq t^{-k} \textstyle\sum_{|\alpha|=k} \|\mathcal{F}^{-1}\psi_\alpha\|_{L^1} \|D^\alpha h_j\|_{L^\infty} \leq t^{-k} C_\psi B 2^{jk} = C' t^{-k} 2^{jk}$$

and we obtain

$$\begin{aligned}
S_1(x) &\leq C' \sum_{j=0}^{q-1} t^{r-k} 2^{j(k-r)} \\
&\leq C' 2^{-q(k-r)} \sum_{j=0}^{q-1} (2^{(k-r)})^j \\
&= C' 2^{-q(k-r)} \frac{2^{q(k-r)} - 1}{2^{k-r} - 1} \\
&\leq C.
\end{aligned}$$

Since $t \geq 1$ was arbitrary and the constants in the estimates are independent of q the lemma is proved ∎

Lemma 3.3. *If $W \in \mathcal{O}_\mathrm{M}(\mathbb{R}^n \times \mathbb{R}_+)$ and satisfies (9), with W substituted for $W_g u$, then*

$$\int_0^1 W(\cdot,\varepsilon) * g_\varepsilon \tfrac{d\varepsilon}{\varepsilon} \in C^s_*(\mathbb{R}^n).$$

Proof. We show that the limit of $u^{(N)} := \int_{2^{-N}}^1 W(\cdot,\varepsilon) * g_\varepsilon \frac{d\varepsilon}{\varepsilon}$, as $N \to \infty$, defines an element in C^s_*. As used already in part (i) Lemma 3.1 implies $g_\varepsilon = \varepsilon^m \sum_{|\alpha|=m} \partial^\alpha((g_\alpha)_\varepsilon)$ and hence

$$u^{(N)} = \sum_{|\alpha|=m} \partial^\alpha_x \underbrace{\Big(\int_{2^{-N}}^1 \varepsilon^{m-1} W(\cdot,\varepsilon) * (g_\alpha)_\varepsilon \Big)}_{u_\alpha^{(N)}} d\varepsilon.$$

For any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$ the map $\partial^\alpha \colon C^{s+m}_* \to C^s_*$ is continuous, hence it suffices to prove convergence of $u_\alpha^{(N)}$ in C^{s+m}_* (as $N \to \infty$) for each such α.

By a dyadic subdivision of the interval $[2^{-N}, 1]$ we find a corresponding series representation of $u_\alpha^{(N)}$ in the form

$$\begin{aligned} u_\alpha^{(N)}(x) &= \sum_{j=0}^{N-1} \int_{2^{-j-1}}^{2^{-j}} \varepsilon^{m-1} W(\cdot,\varepsilon) * (g_\alpha)_\varepsilon(x)\, d\varepsilon \\ &= \sum_{j=0}^{N-1} 2^{-j(m+s)} \int_{1/2}^1 2^{js} W(\cdot, 2^{-j}\eta) * (g_\alpha)_{2^{-j}\eta}(x)\, \eta^{m-1}\, d\eta \\ &=: \sum_{j=0}^{N-1} 2^{-j(m+s)} v_{j,\alpha}(x) \end{aligned}$$

where we have changed the variable $\varepsilon = 2^{-j}\eta$. Note that $\|2^{js} W(\cdot, 2^{-j}\eta)\|_{L^\infty} \leq C\eta^s \leq C$ independent of j. Therefore the sequence $v_{j,\alpha}$ satisfies the condition (28) of Lemma 3.2 for any $k \in \mathbb{N}$ with $k > m + s > 0$ since

$$\begin{aligned} |\partial^\gamma v_{j,\alpha}(x)| &\leq \int_{1/2}^1 \big| 2^{js} W(\cdot, 2^{-j}\eta) * (\partial^\gamma g_\alpha)_{2^{-j}\eta}(x) \big|\, 2^{j|\gamma|}\, \eta^{m-|\gamma|-1}\, d\eta \\ &\leq 2^{j|\gamma|} \|2^{js} W(\cdot, 2^{-j}\eta)\|_{L^\infty} \|\partial^\gamma g_\alpha\|_{L^1} \int_{1/2}^1 \eta^{m-|\gamma|-1}\, d\eta \\ &= C_{\gamma,\alpha} 2^{j|\gamma|}. \end{aligned}$$

Application of Lemma 3.2 completes the proof ∎

Lemma 3.4. *Let* $W \in \mathcal{O}_{\mathrm{M}}(\mathbb{R}^n \times \mathbb{R}_+)$. *Then*

$$\int_1^\infty W(\cdot, \varepsilon) * g_\varepsilon \tfrac{d\varepsilon}{\varepsilon} \in \mathrm{C}^\infty(\mathbb{R}^n).$$

Proof. Let $R > 1$ and put $v_R = \int_1^R W(\cdot, \varepsilon) * g_\varepsilon \frac{d\varepsilon}{\varepsilon}$. Then v_R is smooth, temperate, and converges weakly to some $v \in \mathscr{S}'$ as $R \to \infty$ (cf. (7)).

Clearly, any derivative $\partial^\alpha v_R$ converges to $\partial^\alpha v$ then. But letting the derivative fall on the factor g_ε inside the integral defining v_R produces additional factors $\varepsilon^{-|\alpha|}$. When $|\alpha|$ is large enough to compensate for the polynomial growth of $W(y, \varepsilon)$ with respect to ε this ensures absolute convergence of the classical integral. Hence for all $|\alpha|$ sufficiently large $\partial^\alpha v$ is smooth, yielding that v itself is smooth ∎

To finish the proof of part (ii) we apply (7) together with (8) and obtain, with some polynomial p,

$$u = \int_0^1 W_g u(\cdot, \varepsilon) * g_\varepsilon \tfrac{d\varepsilon}{\varepsilon} + \int_1^\infty W_g u(\cdot, \varepsilon) * g_\varepsilon \tfrac{d\varepsilon}{\varepsilon} + p.$$

The second term is smooth by Lemma 3.4 and the first term is of Zygmund regularity s by Lemma 3.3. It follows that u differs from an element in C_*^s only by some smooth function.

This completes the proof of Theorem 1.1.

References

[1] Colombeau, J. F.: *Elementary Introduction to New Generalized Functions.* Amsterdam: North-Holland 1985.

[2] Grosser, M., Kunzinger, M., Oberguggenberger, M. and R. Steinbauer: *Geometric Theory of Generalized Functions.* Dordrecht: Kluwer 2001.

[3] Hermann, R. and M. Oberguggenberger: *Ordinary differential equations and generalized functions.* In: Nonlinear Theories of Generalized Functions (eds.: M. Grosser et al.) Proceedings of the workshop at the Erwin Schrödinger Institute for Mathematical Physics, Vienna 1997. Boca Raton: Chapman & Hall/CRC. 1999, pp. 85–98.

[4] Herrmann, F. J.: *A scaling medium representation. A discussion on well-logs, fractals and waves.* PhD thesis. Technische Universiteit Delft 1997.

[5] Herrmann, F. J.: *A scaling medium representation and its implication for acoustic wave propagation.* Expanded Abstracts Soc. Expl. Geophys. 1997.

[6] Holschneider, M.: *Wavelets: An Analysis Tool.* New York: Oxford University Press 1995.

[7] Hörmander, L.: *The Analysis of Linear Partial Differential Operators.* Volume III. Berlin und Heidelberg: Springer-Verlag 1985. Second printing 1994.

[8] Hörmander, L.: *Lectures on Nonlinear Hyperbolic Differential Equations.* Berlin and Heidelberg: Springer-Verlag 1997.

[9] Hörmann, G. and M. V. de Hoop: *Geophysical modeling with Colombeau functions: Microlocal properties and Zygmund regularity.* Preprint 2001: arXiv.org e-Print math.AP/0104007.

[10] Hörmann, G. and M. V. de Hoop: *Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients.* Acta Appl. Math. 67 (2001), 173–224.

[11] Ladyženskaya, O. A. and N. N. Ural'tseva: *Linear and Quasilinear Elliptic Equations.* New York and London: Academic Press 1968.

[12] Lafon, F. and M. Oberguggenberger: *Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: The multidimensional case.* J. Math. Anal. Appl. 160 (1991), 93–106.

[13] Li, X. and J. Haury: *Characterization of heterogeneities from sonic velocity measurements using the wavelet transform.* Expanded Abstracts Soc. Expl. Geophys. 1995.

[14] Meyer, Y.: *Wavelets and Operators.* (Cambridge studies in advanced mathematics 37.) Camebridge: Cambridge University Press 1992.

[15] Meyer, Y.: *Wavelets, vibrations and scalings.* (CRM Monograph Series 9.) Providence: Amer. Math. Soc. 1998.

[16] Oberguggenberger, M.: *Hyperbolic systems with discontinuous coefficients: Generalized solutions and a transmission problem in acoustics.* J. Math. Anal. Appl. 142 (1989), 452–467.

[17] Oberguggenberger, M.: *Multiplication of Distributions and Applications to Partial Differential Equations.* Longman Scientific & Technical 1992.

[18] Saucier. A. and J. Muller: *Use of multifractal analysis in the characterization of geological information.* Fractals 1 (1993) 3, 617–628.

[19] Stein, E. M.: *Singular integrals and differentiability properties of functions.* Princeton, New Jersey: Princeton University Press 1970.

[20] Triebel, H.: *Theory of Function Spaces.* Leipzig and Basel: Akad. Verlagsges. Geest & Portig and Birkhäuser Verlag 1983.

[21] Triebel, H.: *Theory of Function Spaces II.* Basel: Birkhäuser Verlag 1992.

[22] Triebel, H.: *Fractals and Spectra.* Basel: Birkhäuser Verlag 1997.

[23] Wapenaar, K.: *Seismic reflection and transmission coefficients of a self-similar interface.* Geophys. J. Int. 135 (1998), 585–594.

[24] Zygmund, A.: *Trigonometric Series.* Second edition. Cambridge: Cambridge University Press 1968.

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