

Identification of Cavities in a Three Dimensional Elastic Body

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Abstract. In this paper the authors prove a uniqueness theorem for identifying cavities in a three dimensional elastic body from displacements and stresses measured on a portion of the outer surface. The cavities, finite in number, are assumed to be stress free. The surfaces of the cavities are assumed to be smooth on the complement of a set that is negligible in the sense that its fractal dimension is less than 2.

Keywords: *Elastic body, cavity identification, fractal dimension, stress singularity, irregular set of the boundary of cavities*

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1. Introduction

In the present paper, we consider the problem of identifying cavities in a three dimensional elastic body from stresses and displacements measured on part of the outer boundary. As is known, the problem is ill-posed. We note at once that the problem of existence of a solution is not considered here. In fact, data given by experimental measurements are usually subject to error. Hence, a solution corresponding to the data does not always exist. As a consequence, the question of existence for given data is less important than that of uniqueness and one has to resort to a regularization, but this is another story. In the two dimensional case, identification problems for cavities and cracks (considered as the limit of a cavity as one of the dimensions goes to zero) has been considered earlier in a number of papers. We refer to Sneddon and Lowengrub [22], for a general reference in which a linear crack is defined (by Griffith) as the limit of elliptic cavities as the minor axis tends to zero. For determination of cracks using the electric method, we refer to [1, 2, 18, 20]. As it is shown in [17], two measurements are needed for uniqueness in the case of cracks. This differs from

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the case of cavities for which only one measurement is sufficient for uniqueness (see. e.g. [10, 24, 25, 26]). For the determination of irregular cavities in a two dimensional elastic body, we refer to [12, 25]. We also refer to [6, 8, 27] (and the references therein) for the problem related to cavities in elastic bodies and for the unique continuation for the Lamé system. Unlike the two dimensional case, the literature on the problem of determination of three dimensional cavities is rather scarce. In a number of papers (see e.g. [2, 13]), the surfaces of cavities are assumed to be smooth. In [13], the author considered the problem of detecting a star-shaped cavity (with a C^1 -boundary) in a solid using electric potentials. In [2], cavities are assumed to be of class $C^{1,\alpha}$. Both of above papers deal with elliptic boundary value problems. In the present paper, we study cavities having a non-smooth boundary, and the problem is associated to the three dimensional Lamé system. Our paper can be seen as a direct extension of [25]. In fact, from the point of view of fracture mechanics, the surfaces of cavities often have cones, sharp edges etc. where stress singularities are produced. Accordingly, it seems natural to consider cavities with surfaces having some irregularities. In fact, we shall assume that the surfaces of cavities are smooth except at an irregular set that is, in a sense to be defined later, negligible.

The remainder of the paper is divided into two sections. In Sections 2, we shall give notations and assumptions. In Section 3, we state (and prove) the main result of our paper.

2. Notations and Assumptions

Let Ω be a connected domain in \mathbb{R}^3 limited by an known outer surface Γ and containing finitely many unknown holes (cavities) represented by simply connected domains $\omega_1, \omega_2, \dots, \omega_n$ such that

$$\bar{\omega}_i \cap \bar{\omega}_j = \emptyset; \quad \partial\omega_i \cap \Gamma = \emptyset \quad (i \neq j; 1 \leq i, j \leq n).$$

It is noted that the set $S = \partial\omega_1 \cup \partial\omega_2 \cup \dots \cup \partial\omega_n$ is the unknown inner boundary of Ω . Assuming Ω to be an elastic body, we have the following system of equations in Ω (see e.g. Timosenko and Goodier [23]) for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$:

$$\frac{\partial\sigma_i}{\partial x_i} + \frac{\partial\tau_{ij}}{\partial x_j} + \frac{\partial\tau_{ik}}{\partial x_k} = -X_i \quad (1)$$

subject to the boundary conditions

$$(u_1, u_2, u_3)|_{\Gamma_0} = (u_{01}, u_{02}, u_{03}) \quad (2)$$

and

$$n_i\sigma_i + n_j\tau_{ij} + n_k\tau_{ik} = \bar{X}_i \quad \text{on } \Gamma_0 \quad (3)$$

Γ_0 is a relatively open subset of the outer surface Γ and $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal to Γ .

In the system (1)-(3), (x_1, x_2, x_3) is a point in $\bar{\Omega}$, $\sigma_i, \tau_{ij}, \tau_{ik}$ are normal and shearing stresses respectively, $(u_1, u_2, u_3) = (u_1, u_2, u_3)(x_1, x_2, x_3)$ is the displacement, $(X_1, X_2, X_3) = (X_1, X_2, X_3)(x_1, x_2, x_3)$ is the body force per unit volume at the point (x_1, x_2, x_3) , $(\bar{X}_1, \bar{X}_2, \bar{X}_3) = (\bar{X}_1, \bar{X}_2, \bar{X}_3)(x_1, x_2, x_3)$ is the surface stress per unit area at $(x_1, x_2, x_3) \in \Gamma_0$. In the present paper, we shall assume that $X_1 = X_2 = X_3 = 0$.

From the theory of elasticity, we have the strain-displacement relations

$$\epsilon_i = \frac{\partial u_i}{\partial x_i}; \quad e = \epsilon_1 + \epsilon_2 + \epsilon_3 \quad \text{in } \Omega, \quad i = 1, 2, 3, \tag{4}$$

and the stress-displacement relations

$$\left. \begin{aligned} \sigma_i &= \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{1+\nu} \epsilon_i, \quad i = 1, 2, 3, \\ \tau_{ij} &= \tau_{ji} = G \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad i \neq j, \quad i, j = 1, 2, 3. \end{aligned} \right\} \tag{5}$$

Here, the positive constants ν, E, G are Poisson's ratio, the modulus of elasticity in tension and the modulus of elasticity in shear, respectively. We have the following relation:

$$E = 2G(1 + \nu); \quad 0 < \nu < \frac{1}{2}.$$

Now, we specify conditions on the surfaces of the cavities. We recall that the surfaces are, in general, not smooth, in fact, they can have cones or sharp edges at which points the stresses can be infinite. We are interested in the problem of stress singularities in the vicinity of crack tips. Before going further, we point out that a cavity is a crack having an interior whereas a Griffith crack, also called a mathematical crack, has no interior. Sneddon [21], Payne [19] and others considered the problem of two elastic half-spaces $z \geq 0$ and $z \leq 0$ having a penny shaped crack of radius 1, centered at $x = 0, y = 0, z = 0$, the whole system being subjected to a uniform tension along the axis $x = 0, y = 0, \infty < z < \infty$ found a stress intensity factor κ given by the following formula

$$\kappa = \lim_{\rho \rightarrow 1^+} \sqrt{2(\rho - 1)} \sigma_{zz}(\rho, 0).$$

If the crack does have an interior, our consideration of external force exerted on the edge of the crack led to a singularity of the order $\rho^{-\alpha}, 0 \leq \alpha < 1$, at the edge of the crack.

Although the problem of stress singularity in the vicinity of cone points, is of interest, we have not been able to locate any significant reference. The problem of producing an explicit example of a cone shape cavity with stress free surface, and a stress singularity of the type conjectured above is of great interest and

certainly deserves investigation, which attempt should prove rewarding. The authors would like to thank one of the referees for raising the problem.

As announced earlier, we shall assume that the surfaces of the cavities are smooth on the complement of an irregular set $P \subset \cup_{i=1}^n \partial\omega_i$ and that the surface stresses vanish on the smooth part. More precisely, on $S \setminus P$ ($S = \partial\omega_1 \cup \dots \cup \partial\omega_n$), we assume

$$n_i\sigma_i + n_j\tau_{ij} + n_k\tau_{ik} = 0 \tag{6}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

Now, some remarks related to the properties of the irregular set P are necessary. This set is negligible in the sense that it has the fractal dimension $dim P < 2$. Here, we define the fractal dimension of a set $A \subset \mathbb{R}^3$ as the quantity

$$dim A = \lim_{\delta \downarrow 0} \left\{ \sup_{0 < \varepsilon < \delta} \frac{\ln \mathcal{N}(A, \varepsilon)}{\ln \frac{1}{\varepsilon}} \right\}$$

where $\mathcal{N}(A, \varepsilon)$ denotes the minimum number of balls of radius ε needed to cover A (see e.g. [15]).

As discussed, the set P can be splitted into many subsets as sharp edges, cones. It is worth pointing out that the fractal dimension of a smooth surface in \mathbb{R}^3 is 2. If the boundary of cavities has a cone then the vertex of the cone has fractal dimension 0. If the sharp edges of the boundary of cavities are a union of Lipschitzian closed arcs then its fractal dimension is 1. We shall point out that the singularities of the stresses are different in the two cases.

From the point of view of fracture mechanics, some assumptions related the behavior of stresses in the neighborhood of irregular points are needed. Near every sharp edge of the boundary of cavities, no net force is exerted on the edge, hence (in local cylindrical coordinates) we have

$$\left. \begin{aligned} r \int_a^b \int_c^d \sigma_i(r, \theta, z) d\theta dz &\longrightarrow 0 \quad \text{as } r \rightarrow 0, \quad i = 1, 2, 3, \\ r \int_a^b \int_c^d \tau_{ij}(r, \theta, z) d\theta dz &\longrightarrow 0 \quad \text{as } r \rightarrow 0, \quad i, j = 1, 2, 3, \quad i \neq j \end{aligned} \right\} \tag{7}$$

where r is the distance from the edge of the surface, $d - c > 0$ is the notch angle and $b - a > 0$ is the length of an interval of the sharp edge that can be seen as a Lipschitzian arc. Using the same argument, we can assume that, in a neighborhood of a cone, we have for $i, j = 1, 2, 3$

$$\left. \begin{aligned} r^2 \int_{\varphi_0}^{\varphi_1} \int_0^{2\pi} \sigma_i(r, \theta, \varphi) \sin \varphi d\theta d\varphi &\longrightarrow 0 \quad \text{as } r \rightarrow 0, \\ r^2 \int_{\varphi_0}^{\varphi_1} \int_0^{2\pi} \tau_{ij}(r, \theta, \varphi) \sin \varphi d\theta d\varphi &\longrightarrow 0 \quad \text{as } r \rightarrow 0, \quad i \neq j \end{aligned} \right\} \tag{8}$$

where r, θ, φ are local spherical coordinates, r is the distance from the vertex of the cone and $\varphi_1 - \varphi_0 > 0$ is the notch angle of the cone.

Now let P_a be a Lipschitzian arc on the edge of the boundary of cavities then $\dim P_a = 1$. The conditions (7) are satisfied if there exists an α_a satisfying $0 < \alpha_a < 1 = 2 - \dim P_a$ such that

$$r^{\alpha_a} \sigma_i, r^{\alpha_a} \tau_{ij} \longrightarrow 0 \quad \text{as } r \rightarrow 0.$$

Similarly, letting P_c be a vertex of a cone, we have $\dim P_c = 0$. Conditions (8) are satisfied if there exists an α_c satisfying $0 < \alpha_c < 2 = 2 - \dim P_c$ such that

$$r^{\alpha_c} \sigma_i, r^{\alpha_c} \tau_{ij} \longrightarrow 0 \quad \text{as } r \rightarrow 0.$$

Therefore, we can (and shall) assume generally that the singular set P is a finite union of sets P_ℓ , $\ell = 1, 2, \dots, L$ having fractal dimensions $\dim P_\ell < 2$, i.e.

$$P = \bigcup_{\ell=1}^L P_\ell; \quad \dim P_\ell < 2, \quad \ell = 1, 2, \dots, L. \quad (9)$$

Further, we assume that for each $\ell = 1, \dots, L$, there is an $\alpha_\ell \in (0, 2 - \dim P_\ell)$ such that

$$\lim_{\delta \downarrow 0} \sup_{d(\xi, P_\ell) < \delta} |d(\xi, P_\ell)|^{\alpha_\ell} \left(\sum_{i,j=1}^3 (|\sigma_i(\xi)|^2 + |\tau_{ij}(\xi)|^2) \right)^{\frac{1}{2}} = 0 \quad (10)$$

where $\xi = (x_1, x_2, x_3)$.

3. The main result and its proof

Before stating (and proving) our main result, we shall assume some restrictions on the smooth part of the surfaces of the cavities. Letting $\xi \in S \setminus P$, we say that $S \setminus P$ has a local representation at ξ if we can find a cube $N(\xi) = I_1(\xi) \times I_2(\xi) \times I_3(\xi)$ such that $\xi \in N(\xi)$ and that

$$\begin{aligned} N(\xi) \cap S &= \{(x_1, x_2, x_3) \mid x_3 = h_\xi(x_1, x_2), (x_1, x_2) \in \overline{I_1}(\xi) \times \overline{I_2}(\xi)\} \\ N(\xi) \cap \Omega &= \{(x_1, x_2, x_3) \mid x_3 < h_\xi(x_1, x_2), (x_1, x_2) \in \overline{I_1}(\xi) \times \overline{I_2}(\xi)\} \end{aligned}$$

where the axes of \mathbb{R}^3 are chosen with the unit normal vector $n(p)$ to S at ξ pointing in the x_3 -direction and h is a C^1 -function on $\overline{I_1}(\xi) \times \overline{I_2}(\xi)$.

Let Ω_0 be a fixed simply connected domain and suppose that, at each point of $\Gamma \equiv \partial\Omega_0$, we have a local representation. We shall consider Ω in a class \mathcal{F} of three dimensional domains satisfying the following conditions:

- (A) Each domain Ω in \mathcal{F} satisfies $\Omega = \overset{\circ}{\overline{\Omega}}$ and has the form $\Omega = \Omega_0 \setminus \bigcup_{i=1}^n \overline{\omega}_i$ where $\omega_1, \dots, \omega_n \subset \Omega_0$ are simply connected domains such that $\overline{\omega}_i \cap \overline{\omega}_j = \emptyset$ ($i, j = 1, 2, \dots, n$) and that $\Gamma \cap \overline{\omega}_i = \emptyset$.

(B) The set $S = \cup_{i=1}^n \partial\omega_i$ has a local representation at each $\xi \in S \setminus P$ where $\dim P < 2$ holds for $P \subset S$.

For every Ω in \mathcal{F} , we have the following proposition pointing out that the cavities can be "approximated" by polyhedrons (the proof of which will be given in the final part of our paper).

Proposition. *Let Ω be in the class \mathcal{F} and let $\mathbf{F} = (f_1, f_2, f_3)$ satisfy the conditions*

- a) $f_i \in C(\bar{\Omega}) \cap C^1(\Omega \cup (S \setminus P)) \cap H^1(\Omega)$, $i = 1, 2, 3$
- b) *there are subsets of P_1, \dots, P_L of P and $\alpha_\ell \in (0, 2 - \dim P_\ell)$, $\ell = 1, \dots, L$ such that $P = \bigcup_{\ell=1}^L P_\ell$ and that*

$$\lim_{\delta \downarrow 0} \sum_{\ell=1}^L \sup_{\xi \in \Omega, d(\xi, P_\ell) < \delta} |d(\xi, P_\ell)|^{\alpha_\ell} \left(\sum_{i=1}^3 |f_i(\xi)|^2 \right)^{\frac{1}{2}} = 0$$

- c) $\mathbf{F}(\xi) \cdot \mathbf{n}(\xi) = 0$ for $\xi \in S \setminus P$.

Then there exists, for each $\delta > 0$, an open set ω_δ such that

- (i) $\omega_\delta \supset \bar{\omega} = \bigcup_{i=1}^n \bar{\omega}_i$,
- (ii) $\partial\omega_\delta$ is a union of planar sets,
- (iii) $\lim_{\delta \downarrow 0} d(\partial\omega_\delta, \partial\omega) = 0$
- (iv) $\lim_{\delta \downarrow 0} \int_{\partial\omega_\delta} |\mathbf{F}(\xi) \cdot \mathbf{n}_\delta(\xi)| d\sigma = 0$,

where $\mathbf{n}_\delta(\xi) = (n_{1\delta}(\xi), n_{2\delta}(\xi), n_{3\delta}(\xi))$ is the unit outer normal vector to $\Omega \setminus \bar{\omega}_\delta$ at $\xi \in \partial\omega_\delta$ and $d(A, B)$ is the semidistance of two sets A and B in \mathbb{R}^3 , i.e.

$$d(A, B) = \sup_{\xi \in A} \{ \inf_{\eta \in B} d(\xi, \eta) \}.$$

Now, we state the main result. For convenient, we shall rewrite our system. We recall that the problem is to identify a pair (Ω, u) ($u = u_1, u_2, u_3$) satisfying

$$\frac{\partial \sigma_i}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial \tau_{ik}}{\partial x_k} = 0$$

$((i,j,k)=(1,2,3), (2,3,1), (3,1,2))$ subject to the boundary conditions

$$(u_1, u_2, u_3)|_{\Gamma_0} = (u_{01}, u_{02}, u_{03})$$

and

$$\begin{aligned} n_i \sigma_i + n_j \tau_{ij} + n_k \tau_{ik} &= \bar{X}_i && \text{on } \Gamma_0, \\ n_i \sigma_i + n_j \tau_{ij} + n_k \tau_{ik} &= 0 && \text{on } S \setminus P, \end{aligned}$$

((i,j,k)=(1,2,3), (2,3,1), (3,1,2)) where Γ_0 is a relatively open subset of the outer surface Γ and $\mathbf{n} = (n_1, n_2, n_3)$ is the unit normal to $\partial\Omega$.

Theorem. *Let \mathcal{F} be a class of domains in \mathbb{R}^3 satisfying the conditions (A) and (B), let Γ_0 be a smooth relatively open subset of the outer boundary Γ , and let E, ν be in $C^2(\mathbb{R}^3)$. If $(\bar{X}_1, \bar{X}_2, \bar{X}_3) \neq (0, 0, 0)$ then the above system has at most one solution $(\Omega, (u_1, u_2, u_3))$ such that $\Omega \in \mathcal{F}$,*

$$u_1, u_2, u_3 \in C^2(\overline{\Omega \cup \Gamma_0 \cup (S \setminus P)}) \cap C(\overline{\Omega}) \cap H^1(\Omega),$$

and that (9) and (10) hold.

To prove the main theorem, we shall use the following lemma (which will be proved later).

Lemma. *Under the assumptions of the Theorem, one has*

$$\int_{\Omega^1 \setminus \bar{W}} \left(\sigma_i^1 \frac{\partial u_i^1}{\partial x_i} + \tau_{ij}^1 \frac{\partial u_i^1}{\partial x_j} + \tau_{ik}^1 \frac{\partial u_i^1}{\partial x_k} \right) d\xi = 0. \quad (11)$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$

Proof of the Theorem. Suppose that $(\Omega^\kappa, (u_1^\kappa, u_2^\kappa, u_3^\kappa))$, $\kappa = 1, 2$, satisfy the assumptions of the Theorem. We claim that $\Omega^1 = \Omega^2$. Denote by W the connected component of $\Omega^1 \cap \Omega^2$ such that $\Gamma \subset \bar{W}$. By uniqueness of continuation for the system (1)-(6) (see [8, 11, 27]) one has

$$(u_1^1, u_2^1, u_3^1) = (u_1^2, u_2^2, u_3^2) \quad \text{on } \bar{W}. \quad (12)$$

Suppose by contradiction that $\Omega^1 \neq \Omega^2$. If $(\Omega^1 \setminus \bar{\Omega}^2) \cup (\Omega^2 \setminus \bar{\Omega}^1) = \emptyset$ then $\bar{\Omega}^1 = \bar{\Omega}^2$. From the properties of Ω^1, Ω^2 , one has

$$\Omega^1 = \overset{\circ}{\bar{\Omega}^1} = \overset{\circ}{\bar{\Omega}^2} = \Omega^2$$

which contradicts the assumption $\Omega^1 \neq \Omega^2$. Hence $(\Omega^1 \setminus \bar{\Omega}^2) \cup (\Omega^2 \setminus \bar{\Omega}^1) \neq \emptyset$. Without loss of generality, we can assume that $\Omega^1 \setminus \bar{\Omega}^2 \neq \emptyset$.

Let S_1, S_2 be the inner boundaries of Ω^1, Ω^2 respectively. By assumption, S^κ ($\kappa = 1, 2$) is C^1 -smooth except at a finite union of sets $P_1^\kappa, \dots, P_{L_\kappa}^\kappa$ which have fractal dimensions $\dim P_1^\kappa, \dots, \dim P_{L_\kappa}^\kappa$ satisfying $0 \leq \dim P_\ell^\kappa < 2$, $1 \leq \ell \leq L_\kappa$. Put

$$P^\kappa = \bigcup_{\ell=1}^{L_\kappa} P_\ell^\kappa, \quad \kappa = 1, 2.$$

One has

$$\partial(\Omega^1 \setminus \bar{W}) \subset (\partial W \setminus \Gamma) \cup S^1.$$

Noting that $\partial W \subset \partial\Omega^1 \cup \partial\Omega^2 = \Gamma \cup S^1 \cup S^2$ for $\xi \in \partial(\Omega^1 \setminus \bar{\Omega}^2) \setminus (P^1 \cup P^2)$, one has to consider the following two cases:

- (i) $\xi \in (S^1 \setminus \partial W) \setminus (P^1 \cup P^2)$
- (ii) $\xi \in \partial W \cap S^2 \setminus (P^1 \cup P^2)$.

Case (i): If (i) holds, then relation (6) holds for σ_i, τ_{ij} ($i, j = 1, 2, 3, i \neq j$) replaced by σ_i^1, τ_{ij}^1 where $\sigma_i^\kappa, \tau_{ij}^\kappa$ ($\kappa = 1, 2$) are calculated from $(u_1^\kappa, u_2^\kappa, u_3^\kappa)$ by (4), (5).

Case (ii): We get in view of (12)

$$\sigma_i^1(\xi) = \sigma_i^2(\xi), \quad \tau_{ij}^1(\xi) = \tau_{ij}^2(\xi), \quad i, j = 1, 2, 3, i \neq j.$$

Since $\xi \in \partial W \cap S^2 \setminus (P^1 \cup P^2) \subset S^2 \setminus P^2$, the relations in (6) hold for σ_i, τ_{ij} replaced by σ_i^1, τ_{ij}^1 . This gives for $\xi \in \partial(\Omega^1 \setminus \overline{\Omega^2}) \setminus (P^1 \cup P^2)$ that

$$n_i(\xi)\sigma_i^1(\xi) + n_j(\xi)\tau_{ij}^1(\xi) + n_k(\xi)\tau_{ik}^1(\xi) = 0, \tag{13}$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

Substituting $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ into (11) of the Lemma, respectively, and adding the results thus obtained, we get after some rearrangements

$$\int_{\Omega^1 \setminus \overline{W}} \left(\frac{\nu E}{(1 + \nu)(1 - 2\nu)} (e^1)^2 + \frac{E}{1 + \nu} \sum_{i=1}^3 \left(\frac{\partial u_i^1}{\partial x_i} \right)^2 + G \sum_{1 \leq i < j \leq 3} \left(\frac{\partial u_i^1}{\partial x_j} + \frac{\partial u_j^1}{\partial x_i} \right)^2 \right) d\xi = 0$$

where $e^1 = \frac{\partial u_1^1}{\partial x_1} + \frac{\partial u_2^1}{\partial x_2} + \frac{\partial u_3^1}{\partial x_3}$.

It follows that, for $i, j = 1, 2, 3, i \neq j$,

$$\frac{\partial u_i^1}{\partial x_i} = \frac{\partial u_i^1}{\partial x_j} + \frac{\partial u_j^1}{\partial x_i} = 0 \quad \text{on } \Omega^1 \setminus \overline{W}.$$

Let U_0 be an open ball in $\Omega^1 \setminus \overline{W}$, then we can show that there are constants a, b, c, m_1, m_2, m_3 such that

$$\left. \begin{aligned} u_1^1(x_1, x_2, x_3) &= ax_2 + bx_3 + m_1, \\ u_2^1(x_1, x_2, x_3) &= -ax_1 + cx_3 + m_2, \\ u_3^1(x_1, x_2, x_3) &= -bx_1 - cx_2 + m_3. \end{aligned} \right\}$$

for $(x_1, x_2, x_3) \in U_0$. Putting

$$\left. \begin{aligned} \bar{u}_1(x_1, x_2, x_3) &= ax_2 + bx_3 + m_1, \\ \bar{u}_2(x_1, x_2, x_3) &= -ax_1 + cx_3 + m_2, \\ \bar{u}_3(x_1, x_2, x_3) &= -bx_1 - cx_2 + m_3. \end{aligned} \right\}$$

and

$$\bar{\sigma}_i = \bar{\tau}_{ij} = 0, \quad i, j = 1, 2, 3, \quad i < j,$$

we can verify directly that $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ satisfies (1) with $X = Y = Z = 0$. Moreover, one has $(u_1^1, u_2^1, u_3^1) = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ on U_0 . Hence, using the unique continuation properties for solutions of (1) (see e.g. [8, 27]) and the connectedness of Ω^1 one has $(u_1^1, u_2^1, u_3^1) = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ on Ω^1 . From (3) and the latter inequality, we get $\bar{X} = \bar{Y} = \bar{Z} \equiv 0$ on Γ_0 , which contradicts the assumptions of Theorem. The proof of this theorem will be completed once the Lemma is proved. \blacksquare

Proof of the Lemma. Let ω_δ^κ satisfy the Proposition corresponding to Ω^κ, u^κ ($\kappa = 1, 2$). Put $\Omega_\delta^\kappa = \Omega^\kappa \setminus \bar{\omega}_\delta^\kappa$ and let W_δ be the (connected) component of $\Omega_\delta^1 \cap \Omega_\delta^2$ satisfying $\Gamma \subset W_\delta$. From the Proposition part (iii) and the assumption $\Omega^1 \setminus \bar{\Omega}^2 \neq \emptyset$, we get for $\delta > 0$ sufficient small that $\Omega_\delta^1 \setminus \bar{W}_\delta \supset \Omega_\delta^1 \setminus \bar{\Omega}_\delta^2 \neq \emptyset$ and that

$$\partial(\Omega_\delta^1 \setminus \bar{W}_\delta) = \partial(\Omega_\delta^1 \cap (\mathbb{R}^3 \setminus \bar{W}_\delta)) \subset \Gamma \cup \partial\omega_\delta^1 \cup (\omega_\delta^2 \cap W)$$

where, we recall that W is the connected component of $\Omega^1 \cap \Omega^2$ satisfying $\Gamma \subset W$. From the definition of W_δ , one has $\partial(\Omega_\delta^1 \setminus \bar{W}_\delta) \cap \Gamma = \emptyset$. It follows that

$$\partial(\Omega_\delta^1 \setminus \bar{W}_\delta) \subset \partial\omega_\delta^1 \cup (\partial\omega_\delta^2 \cap \bar{W}). \quad (14)$$

Multiplying relation (1) by u_i^1 and integrating this on $\Omega_\delta^1 \setminus \bar{W}_\delta$ gives by the divergence theorem that

$$\int_{\Omega_\delta^1 \setminus \bar{W}_\delta} \left(\sigma_i^1 \frac{\partial u_i^1}{\partial x_i} + \tau_{ij}^1 \frac{\partial u_i^1}{\partial x_j} + \tau_{ik}^1 \frac{\partial u_i^1}{\partial x_k} \right) d\xi = - \int_{\partial(\Omega_\delta^1 \setminus \bar{W}_\delta)} \mathbf{F}_i^1(\xi) \cdot \mathbf{n}_{i\delta}(\xi) d\sigma(\xi) \quad (15)$$

where $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, $\mathbf{n}_{i\delta} = (n_{i\delta}, n_{j\delta}, n_{k\delta})$ and

$$\mathbf{F}_i^\kappa = u_i^\kappa(\xi) (\sigma_i^\kappa(\xi), \tau_{ij}^\kappa(\xi), \tau_{ik}^\kappa(\xi)), \quad \kappa = 1, 2.$$

From (14) one has

$$\begin{aligned} \left| - \int_{\partial(\Omega_\delta^1 \setminus \bar{W}_\delta)} \mathbf{F}_i^1(\xi) \cdot \mathbf{n}_{i\delta}(\xi) d\sigma(\xi) \right| &\leq \int_{\partial\omega_\delta^1} |\mathbf{F}_i^1(\xi) \cdot \mathbf{n}_{i\delta}(\xi)| d\sigma(\xi) \\ &\quad + \int_{\partial\omega_\delta^2 \cap \bar{W}} |\mathbf{F}_i^1(\xi) \cdot \mathbf{n}_{i\delta}(\xi)| d\sigma(\xi). \end{aligned}$$

We get from (12) that $\mathbf{F}_i^1(\xi) = \mathbf{F}_i^2(\xi)$ on $\partial\omega_\delta^2 \cap \bar{W}$. Hence,

$$\left| \int_{\partial(\Omega_\delta^1 \setminus \bar{W}_\delta)} \mathbf{F}_i^1(\xi) \cdot \mathbf{n}_{i\delta}(\xi) d\sigma(\xi) \right| \leq \sum_{\kappa=1}^2 \int_{\partial\omega_\delta^\kappa} |\mathbf{F}_i^\kappa(\xi) \cdot \mathbf{n}_{i\delta}(\xi)| d\sigma(\xi).$$

By (13) and the Proposition, the latter inequality implies

$$\lim_{\delta \downarrow 0} \int_{\partial(\Omega_\delta^1 \setminus \overline{W_\delta})} \mathbf{F}_i^1(\xi) \cdot \mathbf{n}_{i\delta}(\xi) d\sigma(\xi) = 0.$$

Combining (15) with the latter equality gives (11). This completes the proof of the Lemma. ■

Proof of the Proposition. This proof is divided into 3 steps. In Step A we shall cover $P = \bigcup_{\ell=1}^L P_\ell$ by a finite union of cubes. In Step B, we shall approximate (in the sense of the semidistance) $S \setminus P$ locally in the neighborhood of every $\xi \in S \setminus P$ by a surface that is a union of planar sets. Finally, in Step C, combining Step A and Step B, we shall approximate S globally by a surface that is a union of planar sets.

Step A: Let $\delta > 0$, we cover $P = \bigcup_{\ell=1}^L P_\ell$ by a finite union of cubes U_δ such that $d(\xi, P) \geq 2\delta$ for $\xi \in \partial U_\delta$

For $\delta > 0$ and $1 \leq \ell \leq L$, there exist $\mathcal{N}(P_\ell, \delta)$ sets $Q_1^\ell, \dots, Q_{\mathcal{N}(P_\ell, \delta)}^\ell$ such that

$$\text{diam } Q_m^\ell \leq \delta \quad \text{and} \quad P^\ell = \bigcup_{m=1}^{\mathcal{N}(P_\ell, \delta)} Q_m^\ell, \tag{16}$$

for $\ell = 1, \dots, L$; $m = 1, \dots, \mathcal{N}(P_\ell, \delta)$. We define

$$D(\xi, \delta) = (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times (x_3 - \delta, x_3 + \delta)$$

for $\xi = (x_1, x_2, x_3)$. Choosing $a_{\ell m} \in Q_m^\ell$ for $\ell = 1, \dots, L$ and $m = 1, \dots, \mathcal{N}(P_\ell, \delta)$, we get in view of (16) that

$$P \subset \bigcup_{\ell=1}^L \bigcup_{m=1}^{\mathcal{N}(P_\ell, \delta)} D(a_{\ell m}, 2\delta). \tag{17}$$

Put

$$U_\delta = \bigcup_{\ell=1}^L \bigcup_{m=1}^{\mathcal{N}(P_\ell, \delta)} D(a_{\ell m}, 4\delta). \tag{18}$$

One has $P \subset U_\delta$. We claim that

$$d(\xi, P) \geq 2\delta \quad \text{for } \xi \in \partial U_\delta.$$

In fact, let $\eta \in P$. From (17), we can find ℓ, m ($1 \leq \ell \leq L$, $1 \leq m \leq \mathcal{N}(P_\ell, \delta)$) such that $|\eta - a_{\ell m}| < 2\delta$ where $|\eta|$ is the Euclidean norm of $\eta \in \mathbb{R}^3$. Hence, for every $\xi \in \partial U_\delta$ one has $|\xi - \eta| \geq |\xi - a_{\ell m}| - |\eta - a_{\ell m}| \geq 4\delta - 2\delta = 2\delta$. Thus, $d(\xi, P) \geq 2\delta$ for every $\xi \in \partial U_\delta$.

Step B: From the assumptions of the Theorem, for each $\xi \in S \setminus P$, the surface S has a local representation at ξ , i.e., there are a cube $N(\xi) = I_1(\xi) \times I_2(\xi) \times I_3(\xi)$ and a C^1 -function h_ξ defined on $\overline{I_1(\xi)} \times \overline{I_2(\xi)}$ such that

$$\begin{aligned} N(\xi) \cap S &= \{(x, y, z) \mid z = h_\xi(x, y), (x, y) \in I_1(\xi) \times I_2(\xi)\} \\ N(\xi) \cap \Omega &= \{(x, y, z) \mid z < h_\xi(x, y), (x, y) \in I_1(\xi) \times I_2(\xi)\} \end{aligned}$$

where the axes of \mathbb{R}^3 are chosen with normal vector $\mathbf{n}(\xi)$ to $\partial\Omega$ at ξ pointing in the z -direction. We shall approximate $N(\xi) \cap S$ by a surface that is a finite union of planar sets. In fact, we shall prove that for each pair (δ, ξ) ($\delta > 0$, $\xi \in S \setminus P$) there exist a $\theta(\delta, \xi) > 0$ and polygons $W_\theta(\xi)$ ($0 < \theta < \theta(\delta, \xi)$) such that $d(W_\theta(\xi), N(\xi) \cap S) \rightarrow 0$ as $\theta \rightarrow 0$.

Let $I_1(\xi) = (a(\xi), b(\xi))$ and $I_2(\xi) = (c(\xi), d(\xi))$ be two open intervals in \mathbb{R} where $a(\xi), b(\xi), c(\xi), d(\xi) \in \mathbb{R}$. For $n \in \mathbf{N}$, $0 \leq i, j \leq n-1$, we put

$$\begin{aligned} x_i &= a(\xi) + \frac{i(b(\xi) - a(\xi))}{n} \\ y_j &= c(\xi) + \frac{j(d(\xi) - c(\xi))}{n}. \end{aligned}$$

In the plane xOy , let $\Delta_{ij}^1, \Delta_{ij}^2$ be the right isocetes triangles having the vertices $(x_i, y_j), (x_i, y_{j+1}), (x_{i+1}, y_{j+1})$ and $(x_i, y_j), (x_{i+1}, y_j), (x_{i+1}, y_{j+1})$, respectively. Using these notations, one has

$$I_1(\xi) \times I_2(\xi) = \bigcup_{i,j=1}^{n-1} (\Delta_{ij}^1 \cup \Delta_{ij}^2).$$

Now, for $\theta > 0$, let $h_{\theta\xi}$ be a continuous function defined on $\overline{I_1(\xi)} \times \overline{I_2(\xi)}$ satisfying

$$h_{\theta\xi}(x_i, y_j) = h_\xi(x_i, y_j) - \theta, \quad 0 \leq i, j \leq n-1$$

and

$$h_{\theta\xi}|_{\Delta_{ij}^\kappa} \text{ is affine for } \kappa = 1, 2; \quad 0 \leq i, j \leq n-1.$$

Put

$$S_{ij}^\kappa = \{(x, y, z) \mid (x, y) \in \Delta_{ij}^\kappa, z = h_{\theta\xi}(x, y)\}$$

and $\mathbf{n}_{\delta\theta}(S_{ij}^\kappa)$ is the unit outer normal vector to $\partial V_\theta(\xi)$ for any point ξ of S_{ij}^κ with

$$V_\theta(\xi) = \{(x, y, z) \in N(\xi) \mid z < h_{\theta\xi}(x, y)\}. \quad (19)$$

Since h_ξ is C^1 on $\overline{I_1(\xi)} \times \overline{I_2(\xi)}$, we can find $n = n(\theta)$ such that

$$h_{\theta\xi}(x, y) < h_\xi(x, y) \quad \forall (x, y) \in \overline{I_1(\xi)} \times \overline{I_2(\xi)}. \quad (20)$$

Moreover, there are an $M > 0$ and a $\theta = \theta(\delta, \xi)$ such that, for $0 < \theta < \theta(\delta, \xi)$, we have the following three properties:

$$0 < h_\xi(x, y) - h_{\theta\xi}(x, y) < M\theta \quad \forall (x, y) \in \overline{I_1}(\xi) \times \overline{I_2}(\xi),$$

$$\sup_{0 \leq i, j \leq n-1} |\mathbf{n}_{\delta\theta}(S_{ij}^\kappa) - \mathbf{n}(x_i, y_j, h_\xi(x_i, y_j))| < M\theta \quad (21)$$

and

$$| |S_\theta(\xi)| - |N(\xi) \cap S| | < M\theta, \quad (22)$$

where $\kappa = 1, 2$, $\mathbf{n}(x_i, y_j, h_\xi(x_i, y_j))$ is the unit outer normal vector to $\partial\Omega$ at $(x_i, y_j, h_\xi(x_i, y_j))$, $|S|$ is the area of the surface S and

$$S_\theta(\xi) = \{(x, y, z) \in N(\xi) \mid z = h_{\theta\xi}(x, y)\}. \quad (23)$$

It is clear that

$$S_\theta(\xi) = \bigcup_{i,j=1}^{n-1} (S_{ij}^1 \cup S_{ij}^2). \quad (24)$$

We shall put

$$W_\theta(\xi) = N(\xi) \setminus \overline{V_\theta(\xi)}.$$

From (20) and (19) one has

$$W_\theta(\xi) \supset N(\xi) \cap S. \quad (25)$$

From (21) we have $\lim_{\theta \rightarrow 0} d(W_\theta(\xi), N(\xi) \cap S) = 0$ as desired.

Step C: We shall approximate S by a surface that is a union of planar sets. One has

$$U_\delta \cup \bigcup_{\xi \in S \setminus P} N(\xi) \supset S,$$

where U_δ is chosen as in (18). From the compactness of S , there exist $\xi_1, \dots, \xi_{k_\delta}$ such that

$$U_\delta \cup \bigcup_{s=1}^{k_\delta} N(\xi_s) \supset S. \quad (26)$$

From (25) and (26) one has

$$U_\delta \cup \bigcup_{s=1}^{k_\delta} W_\theta(\xi_s) \supset S \quad \text{for } 0 < \theta < \theta(\delta) \equiv \min_{1 \leq s \leq k_\delta} \theta(\delta, \xi_s). \quad (27)$$

Put $B = \bigcup_{s=1}^{k_\delta} \partial N(\xi_s) \cap S$ and note that

$$\bigcup_{s=1}^{k_\delta} (W_\theta(\xi_s) \cap S) = \bigcup_{s=1}^{k_\delta} \partial N(\xi_s) \cap S \equiv B$$

is independent of θ . Since $\dim(\partial N(\xi_s) \cap S) = 1$, we get in view of the definition of B that $\dim B = 1$. Hence, for any θ we can find sets $Q'_1, \dots, Q'_{\mathcal{N}(B, \delta)}$ such that

$$\left. \begin{aligned} \text{diam } Q'_m &\leq M\theta \\ B &= \bigcup_{m=1}^{\mathcal{N}(B, \delta)} Q'_m \\ \limsup_{\delta \downarrow 0} \frac{\ln \mathcal{N}(B, \delta)}{\ln \delta^{-1}} &= 1. \end{aligned} \right\} \quad (28)$$

Let $b_m \in Q'_m$ $m = 1, 2, \dots, \mathcal{N}(B, \delta)$. We put

$$B_{1\theta} = \bigcup_{m=1}^{\mathcal{N}(B, \delta)} D_m(b_m; 2M\theta). \quad (29)$$

One has

$$B_{1\theta} \supset B. \quad (30)$$

Put

$$\omega_{\delta\theta} = \omega \cup B_{1\theta} \cup U_\delta \cup \left(\bigcup_{s=1}^{k_\delta} W_\theta(\xi_s) \setminus B_{1\theta} \right). \quad (31)$$

It follows from (27), (30) that $\omega_{\delta\theta} \supset \bar{\omega}$ and that $\partial\omega_{\delta\theta}$ is a union of finite planar sets, i.e. the assertions (i) and (ii) hold. From (21), (25), (28) and (30) it follows

$$d(\partial\omega_{\delta\theta}, S) \leq \max\{d(\partial B_{1\theta}, S), d(\partial W_\theta, S)\} \leq M\theta.$$

If we choose θ as a function of δ such that $\lim_{\delta \downarrow 0} \theta(\delta) = 0$ then, in view of the latter inequality, assertion (iii) holds.

Finally, we prove assertion (iv). Using the local representation of S at $\xi_1, \dots, \xi_{k_\delta}$ (in Step B) we can prove that

$$\partial\omega_{\delta\theta} \subset \partial U_\delta \cup \partial B_{1\theta} \cup \bigcup_{s=1}^{k_\delta} S_\theta(\xi_s) \quad (32)$$

where $U_\delta, B_{1\theta}, S_\theta$ are defined in (18), (29) and (23), respectively. In fact, at $\xi_s \in S$ as in Step B, one has

$$\partial W_\theta(\xi_s) \setminus \bar{\omega} = S_\theta(\xi_s) \cup S'_{1\theta}(\xi_s) \cup S'_{2\theta}(\xi_s) \quad (33)$$

with

$$\begin{aligned} S_\theta(\xi_s) &= \{(x, y, z) \in N(\xi_s) \mid z = h_{\theta\xi_s}(x, y)\} \\ S'_{1\theta}(\xi_s) &= \{(x, y, z) \in N(\xi_s) \mid x = a(\xi_s) \text{ or } x = b(\xi_s) \\ &\quad \text{and } h_{\theta\xi_s}(x, y) \leq z \leq h_{\xi_s}(x, y)\} \\ S'_{2\theta}(\xi_s) &= \{(x, y, z) \in N(\xi_s) \mid y = c(\xi_s) \text{ or } y = d(\xi_s) \\ &\quad \text{and } h_{\theta\xi_s}(x, y) \leq z \leq h_{\xi_s}(x, y)\}. \end{aligned}$$

For $\xi = (x, y, z) \in S'_{1\theta}(\xi_s)$, $0 < \theta < \theta(\delta)$, one has

$$\zeta \equiv (x, y, h_{\xi_s}(x, y)) \in \partial W_\theta \cap S \tag{34}$$

and

$$|\xi - \zeta| = |z - h_{\xi_s}(x, y)| \leq h_{\xi_s}(x, y) - h_{\theta\xi_s}(x, y) < M\theta.$$

From (34), (28) and (29) we can find a j such that $|z - b_j| \leq M\theta$. It follows that

$$x \in D(b_j; 2M\theta) \subset B_{1\theta}. \tag{35}$$

So $S'_{1\theta}(\xi_s) \subset B_{1\theta}$. From (30),(31),(33), (35) we get (32).

Now, we verify (iv). From (32) we have

$$\int_{\partial\omega_{\delta\theta}} |\mathbf{F}(\xi) \cdot \mathbf{n}_{\delta\theta}(\xi)| d\sigma(\xi) \leq C(K_\delta + K_{1\delta\theta} + K_{\delta\theta})$$

where $\mathbf{n}_{\delta\theta}(\xi) = (n_{1\delta\theta}(\xi), n_{2\delta\theta}(\xi), n_{3\delta\theta}(\xi))$ is the outer unit normal vector to $\partial\omega_{\delta\theta}$ at ξ (except for the set of points of the union of the edges of $\partial\omega_{\delta\theta}$ which has fractal dimension 1), and

$$\begin{aligned} K_\delta &= \int_{U_\delta} (|f_1|^2 + |f_2|^2 + |f_3|^2) d\sigma(\xi), \\ K_{1\delta\theta} &= \int_{\partial B_{1\theta}} |\mathbf{F}(\xi) \cdot \mathbf{n}_{\delta\theta}(\xi)| d\sigma(\xi) \\ K_{\delta\theta} &= \sum_{m=1}^{k_\delta} \int_{S_\theta(\xi_m)} |\mathbf{F}(\xi) \cdot \mathbf{n}_{\delta\theta}(\xi)| d\sigma(\xi). \end{aligned}$$

From (16),(18) and b) in the assumptions of the Proposition we get

$$\begin{aligned} K_\delta &\leq C \sum_{\ell=1}^L \sum_{m=1}^{\mathcal{N}(P_\ell, \delta)} |d(\xi, P_\ell)|^{-\alpha_\ell} \delta^2 \\ &\leq C' \sum_{\ell=1}^L \mathcal{N}(P_\ell, \delta) \delta^{2-\alpha_\ell}. \end{aligned}$$

Hence $\lim_{\delta \downarrow 0} K_\delta = 0$. Similarly, $\lim_{\delta \downarrow 0} K_{1\delta\theta} = 0$. Finally, to estimate $K_{\delta\theta}$ we use the local representation of S at $\xi_m \in S$. One has in view of (24)

$$\int_{S_\theta(\xi_m)} |\mathbf{F}(\xi) \cdot \mathbf{n}_{\delta\theta}(\xi)| d\sigma(\xi) = \sum_{r,s=1}^{n-1} \sum_{\kappa=1}^2 \int_{S_{\kappa rs}} |\mathbf{F}(\xi) \cdot \mathbf{n}_{\delta\theta}(S_{\kappa rs})| d\sigma(\xi).$$

From (21) and (22), the latter equality implies

$$\lim_{\theta \downarrow 0} \int_{S_\theta(\xi_m)} |\mathbf{F}(\xi) \cdot \mathbf{n}_{\delta\theta}(\xi)| d\sigma(\xi) = 0.$$

Hence

$$\lim_{\theta \downarrow 0} K_{\delta\theta} = 0. \quad (36)$$

So we can choose a function $\theta = \theta(\delta)$ such that

$$\lim_{\delta \downarrow 0} K_{\delta} = \lim_{\delta \downarrow 0} K_{1\delta\theta} = \lim_{\delta \downarrow 0} K_{\delta\theta} = 0.$$

For $\omega_{\delta} = \omega_{\delta\theta(\delta)}$, $\mathbf{n}_{\delta} = \mathbf{n}_{\delta\theta(\delta)}$ we shall get the conclusion of the proposition. ■

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