NEWSLETTER

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European Mathematical Society

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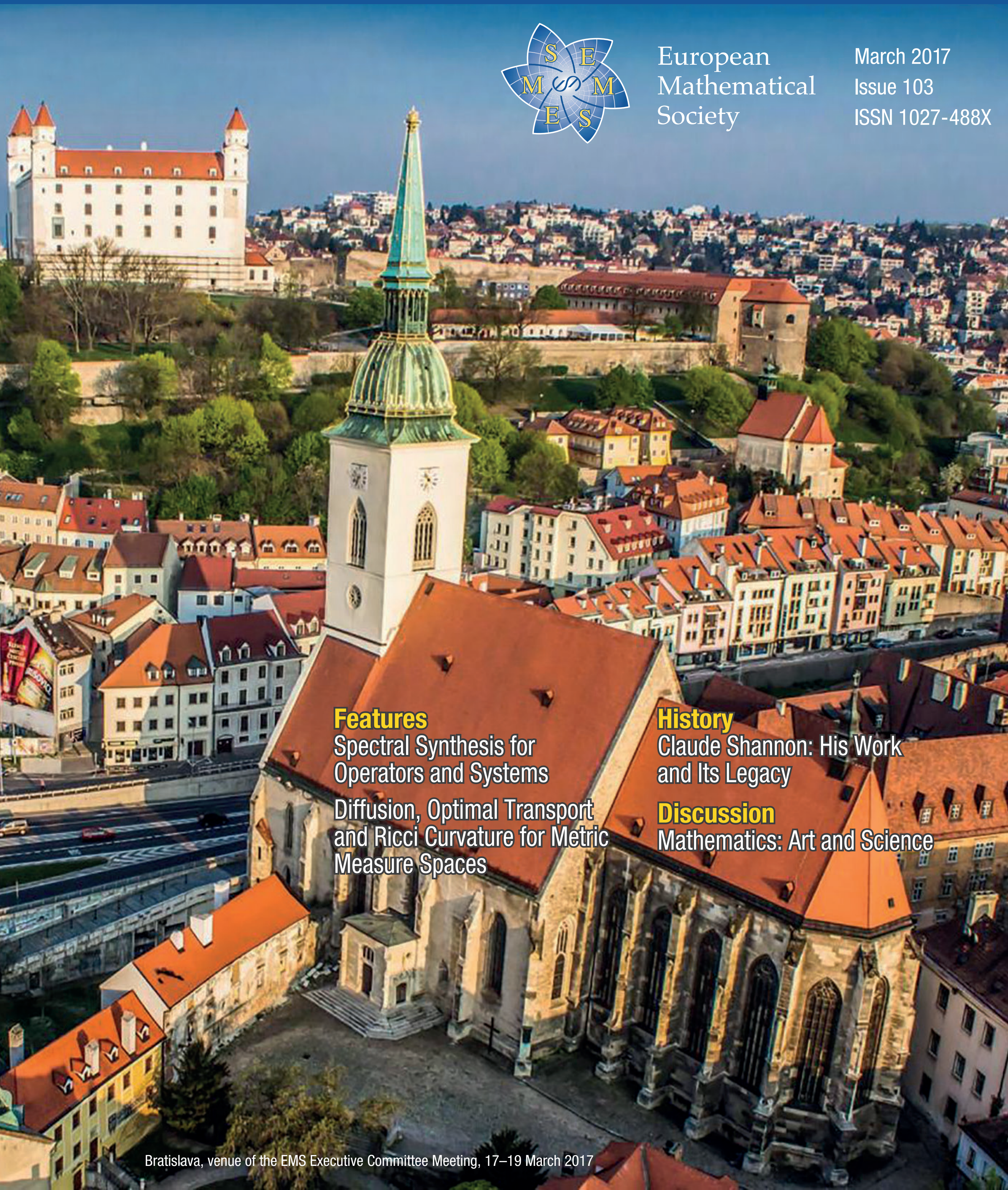

Features
Spectral Synthesis for Operators and Systems
Diffusion, Optimal Transport and Ricci Curvature for Metric Measure Spaces

History
Claude Shannon: His Work and Its Legacy

Discussion
Mathematics: Art and Science

Bratislava, venue of the EMS Executive Committee Meeting, 17–19 March 2017

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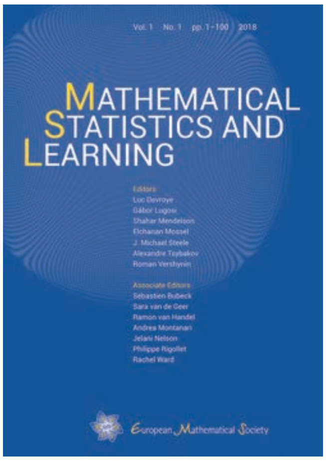

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European Mathematical Society

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EMS Agenda

2017

7–8 April
EMS Committee for Developing Countries (CDC) Meeting, Rome, Italy

28–29 April
European Research Centres on Mathematics (ERCOM) Meeting, Linz, Austria

EMS Scientific Events

2017

29 May – 2 June
Conference "Representation Theory at the Crossroads of Modern Mathematics" in honor of Alexandre Kirillov for his 3^{4}th birthday. Reims, France
http://reims.math.cnrs.fr/pevzner/aak81.html

12–15 June
Meeting of the Catalan, Spanish and Swedish Mathematical Societies (CAT-SP-SW-MATH), Umeå, Sweden
EMS Distinguished Speaker: Kathryn Hess
http://liu.se/mai/catspsw.math?l=en

26–30 June
EMS-ESMTB Summer School 2017: Mathematical Modeling in Neuroscience, Copenhagen, Denmark
http://dsin.ku.dk/calendar/ems-esmtb-summer-school/

26–30 June
EMS summer school: Interactions between Dynamical Systems and Partial Differential Equations, Barcelona, Spain

10–19 July
Foundations of Computational Mathematics (FoCM'17), Barcelona, Spain
EMS Distinguished Speaker: Mireille Bousquet-Mélou
http://www.ub.edu/focm2017/

17–21 July
Summer School: Between Geometry and Relativity, ESI Vienna, Austria
http://www.univie.ac.at/AGESI_2017/school/

24–28 July
31st European Meeting of Statisticians, Helsinki, Finland
EMS-Bernoulli Society Joint Lecture: Alexander Holevo
http://ems2017.helsinki.fi

3–9 September
EMS Summer School "Rationality, stable rationality and birationally rigidity of complex algebraic varieties", Udine, Italy
http://rational.dimi.uniud.it

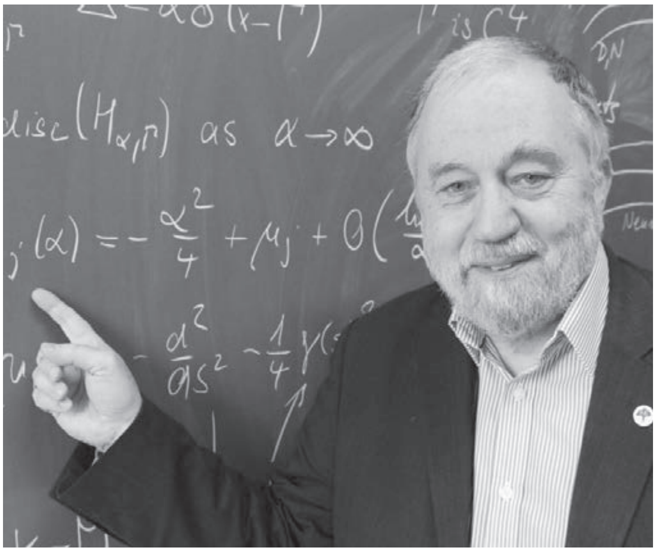
24–29 September
5th Heidelberg Laureate Forum
http://www.heidelberg-laureate-forum.org/

15–19 November
ASTUCON – The 2nd Academic University Student Conference (Science, Technology Engineering, Mathematics), Larnaca, Cyprus

Editorial – Message from the President

Pavel Exner, President of the EMS

Dear EMS Members, Dear Friends,



Any society's life contains both climactic and anticlimactic periods. The year just concluded was full of major events but while the New Year promises many things we look forward to, it is day-to-day work that will provide the leitmotiv of the coming months. This includes the implementation of the decisions taken at our council meeting in Berlin last Summer. To begin with, the society's leadership has been substantially renewed. With deep gratitude for their devoted work, we part with Franco Brezzi and Martin Raussen and we wish success to Volker Mehrmann and Armen Sergeev who replace them as the society's vice-presidents. We also thank the other departing members of the Executive Committee: Alice Fialowski, Gert-Martin Greuel and Laurence Halpern. We welcome the new members: Nicola Fusco, Stefan Jackowski, Vicente Muñoz, Beatrice Pelloni and Betül Tanbay.

Equally important is the renewal of our standing committees, which form the backbone of the EMS. Around half the chairs and a number of the members have reached the end of their tenure. Their replacements were decided at the recent meeting of the Executive Committee in Tbilisi. Let me express our gratitude to all of them, with personal thanks to follow separately. In some committees, the changes run particularly deep. This is especially true for the Education Committee, which will see a majority of new members. We wish them success and hope that the committee's scope will broaden and include some hands-on activities.

Another big change concerns the Publication and Electronic Publication Committees. As technology advances, this separation has become gradually less justifiable and we have invited their members to discuss the formation of a unified committee within the next few months. Before leaving the topic of committees, let me also mention the generous support the EMS has received from the Simons Foundation, targeted at mathematics in Africa. Our Committee for Developing Countries has worked hard to create appropriate grant schemes and its five year programme is now underway.

While none of the largest mathematical meetings will occur this year, some are already looming on the horizon. On a global scale, we look forward to ICM 2018 in Rio and are delighted that the following meeting in 2022 will return, after 16 years, to Europe: either Paris or Saint Petersburg. The EMS, as a society representing the entire European mathematical community, will express no preference between the two bids but we are confident that both offer the prospect of a wonderful meeting. At the European scale, the Berlin Council has decided to hold the 8th European Congress in Portorož in 2020. Our Slovenian colleagues have started working intensively and we are certain of an attractive meeting, which will do much for the standing of mathematics in this part of Europe.

Even if it is still a long time ahead, I encourage you to contemplate possible candidates for the EMS prizes in 2020. Our main award is highly renowned – recent confirmation can be seen from two of its latest laureates, Hugo Duminil-Copin and Geordie Williamson, winning the 2017 New Horizons in Mathematics Prize just a few days ago. It is in all our interests to keep the flag flying high.

A New Year message generally strikes an optimistic tone. However, I hope it won't do any harm to add a few words about our worries. Some of them, frankly, are of our own making; if I were to characterise their common root, I would suggest a lack of loyalty to the mathematical community. To give a few examples, numerous colleagues registered for the Berlin congress but did not then pay, causing a financial headache for the organisers (and we know that at least some such individuals did indeed attend). On the other hand, far from every member of the organising committee of the congress opted to attend the meeting whose programme they had designed!

You may also have noticed the council amending the society's By-Laws (Rule 23) to state that committee members must be individual EMS members "in good standing". While this requirement should be self-evident, we have spotted committee members (and even chairs) ignoring it. (I add that this has happened despite our membership dues being far lower than those of mathematical societies on other continents.) A few of our corporate members are also perpetually in arrears. We are, of course, conscious of difficult economic situations in parts of our continent and will never introduce the spirit of "juste retour" to the EMS. Nevertheless, we need at least to see sincere efforts to deal with this matter.

These are problems we can resolve ourselves, with the will to do so. That is less true of difficulties in our relations to the "outside world", including European funding schemes. In last year's message, I spoke about the ERC, a very valuable instrument that covers, however, only a limited segment of mathematical activities. During 2016, the European Commission led an open consultation on the role of mathematics in Horizon 2020. This was a useful exercise in which many of us participated but it would be overly optimistic to expect an enduring effect. It is a task for each of us to seek out opportunities within the funding system and I would like to praise members of the EU-MATHS-IN initiative and other colleagues who have devoted their energies to such activities. This matter regularly features at the annual Meeting of the Presidents of the EMS Member Societies and no doubt will be raised again in April in Lisbon.

Then, we come to a still wider political scene, in which our ability to influence things is close to zero. We received a harsh reminder of this on the eve of the Berlin congress, when an attempted coup (or whatever we should call it) prevented our colleagues in Turkey from attending. An immediate consequence was that the second Caucasian Conference (planned with EMS support) had to be postponed; subsequent events in Turkey have thrown its new date into further doubt. In other countries, we see processes unfolding that may not be as violent but that signify deep instabilities in the political climate. In such a situation, it is useful to keep in mind a double inclusion: geographical Europe is wider than political Europe and mathematical Europe is wider than geographical Europe. We can and must hold together, even as we sail through rough waters.

Let me end on an optimistic note after all: these political tumults are temporary but – as we all know – mathematics is eternal. Happy New Year!

Farewells within the Editorial Board of the EMS Newsletter

With the December 2016 issue, **Lucia Di Vizio, Jorge Buescu** and **Jaap Top** ended their editorship of the Newsletter. We express our deep gratitude for all the work they have carried out with great enthusiasm and competence, and thank them for contributing to a friendly and productive atmosphere.
Two new members have rejoined the Editorial Board in January 2017. It is a pleasure to welcome Fernando P. da Costa and Michael Th. Rassias, introduced below.

New Editors Appointed

Fernando Pestana da Costa is an associate professor at the Department of Sciences and Technology of Universidade Aberta, Lisbon, and is a researcher at the Centre for Mathematical Analysis, Geometry and Dynamical Systems, Técnico, University of Lisbon. His research areas are analysis and differential equations, with an emphasis on aspects of qualitative theory. He graduated from Técnico in chemical engineering and then turned to mathematics, completing his PhD in mathematics at Heriot-Watt University, Edinburgh, under the supervision of Jack Carr. He was Vice-President (2012–4) and then President (2014–6) of the Portuguese Mathematical Society and he is now (2016–8) President of its General Assembly.

Michael Th. Rassias is a postdoctoral researcher in Mathematics at the University of Zürich and a visiting researcher at the Program in Interdisciplinary Studies of the Institute for Advanced Study, Princeton, since September 2015. He holds a Diploma from the School of Electrical and Computer Engineering of the National Technical University of Athens (2010), a Master of Advanced Study in Mathematics from the University of Cambridge (2011), and a PhD in Mathematics from ETH-Zürich (2014). His doctoral thesis was on the "Analytic investigation of cotangent sums related to the Riemann zeta function", written under the supervision of Professor E. Kowalski. During the academic year 2014-2015, he was a Postdoctoral researcher at the Department of Mathematics of Princeton University and the Department of Mathematics of ETH-Zürich, conducting research at Princeton. While at Princeton, he collaborated with Professor John F. Nash, Jr. for the preparation of the volume entitled "Open Problems in Mathematics", Springer, 2016. He has been awarded with two Gold medals in National Mathematical Olympiads in Greece, a Silver medal in the International Mathematical Olympiad of 2003 in Tokyo, as well as with the Notara Prize of the Academy of Athens, 2014. He has authored two problem-solving books in Number Theory and Euclidean Geometry, respectively, as well as a book on Goldbach's Ternary Conjecture and has edited four books, by Springer. He has published several research papers in Mathematical Analysis and Analytic Number Theory. His homepage is http://www.mthrassias.com/ .

New Members of the EC of the EMS

Nicola Fusco It is a great honour for me to be elected to the Executive Committee (EC) of the EMS.

In my professional career as a mathematician, I have had a lot of scientific and personal contact with colleagues from several European countries. On all these occasions, I have experienced, beyond any obvious national differences, that we all face the same problems and the same challenges. One of them is the funding of both pure and applied mathematical research, which is often severely cut in favour of other areas with more immediate impact. Another common and critical issue in many countries is that life is not easy for talented young mathematicians: there is little money for mobility, research funds are scarce and there are inadequate job opportunities. Therefore, I would like to make a contribution to all the actions that the new EC will promote to support and defend (in all countries and European institutions) the fundamental role played by mathematics in the scientific and technological enhancement of our society. For these reasons, I will also be happy to develop, together with my distinguished colleagues of the EC, actions aimed at spreading, throughout the European public, the knowledge of mathematics and the awareness of its great achievements and its important role in so many aspects of everybody's life. In the hard times we are nowadays facing, I would also stress how mathematics can promote peaceful cooperation among people of different cultures and countries.

Stefan Jackowski received all his degrees from the University of Warsaw, where he is currently a professor. He has spent several research stays abroad, including at the ETH Zürich, Aarhus University, the University of Oxford, the University of Chicago, IHES, the Fields Institute and Max-Planck Institute Bonn. His research interests are in algebraic topology, including transformation groups, homotopy theory, group theory, homological algebra and their interactions. He is an editor of "Fundamenta mathematicae" and "Journal of Homotopy and Related Structures" and is a former editor of "Algebraic and Geometric Topology".

In the years 2005-2013, he was President of the Polish Mathematical Society and he chaired the Local Organisation Committee of 6ECM in Kraków (2012). He is currently a member of the Committee for Evaluation of [Polish] Scientific Research Institutions.

The goals he would like to work for while on the Executive Committee include promotion of collaboration and exchange of ideas between national societies, facilitation of national and joint initiatives under the auspices of the EMS and, last but not least, strengthening European identity and promoting understanding between the countries represented in the EMS.

Vicente Muñoz received his PhD in 1996 at the University of Oxford (UK) under the supervision of Simon Donaldson. After this, he had positions in Universidad de Málaga, Universidad Autónoma de Madrid and CSIC (Spain). Since 2009, he has been a full professor at the Univesidad Complutense de Madrid. He has had visiting fellowships at IAS Princeton (USA) in 2007 and Université Paris 13 (France) in 2015 and he has been a member of ICMAT (Spain), 2013-2016. His research interests lie in differential geometry, algebraic geometry and algebraic topology and, more specifically, gauge theory, moduli spaces, symplectic geometry, complex geometry and rational homotopy theory. His main results are a proof of the Atiyah-Floer conjecture, the finite type condition for Donaldson invariants of 4-manifolds, the concept of s-formality on rational homotopy theory and the construction of a symplectic simply-connected non-formal 8-manifold (with Marisa Fernández), the realisation of real homology classes via embedded laminations (with Ricardo Pérez-Marco), a Bogomolov inequality associated to Spin(7)-instantons on 8-dimensional tori and a topological field theory method for computing Deligne-Hodge polynomials of character varieties of surfaces (with Peter Newstead, Marina Logares and Javier Martínez). He has supervised four doctoral students and is currently supervising two students. He has published more than 90 research papers and the popular book "Distorting Shapes" (editorial RBA), which has been translated into six languages. He was an editor of the EMS Newsletter from 2004 to 2008 and then became Editor-in-Chief from 2008 to 2012. He is an editor of Revista Matemática Complutense (Springer) and Revista Matemática Iberoamericana (EMS) and he is a member of the Governing Board of the Real Sociedad Matemática Española.

Beatrice Pelloni I am honoured to start serving on the Executive Committee of the EMS this year, after serving for four years on the Council of the London Mathematical Society (LMS). I have also served a 5-year term as a member of the Women in Mathematics Committee of the LMS and one year as Chair of the Women and Mathematics Committee of the EMS. After many years at the University of Reading, in April 2016 I moved to Heriot-Watt University in Edinburgh, as Head of the School of Mathematical and Computer Sciences. I am strongly committed to promoting fairness, equality and transparency in academic life, while maintaining the highest standards of scientific integrity. This is the reason I agreed to take on the Head of School role, giving me the opportunity of supporting these values in the everyday reality within a large school with a high volume of research activity. It is also the reason I am attracted by the opportunity to participate and assist the strategic aims of the EMS. My research interests are at the boundary between pure and applied mathematics. My contributions relate to the study of analytical techniques for nonlinear partial differential equations arising in mathematical physics. I am active in two different parts of this wide area of research: boundary value problems for integrable partial differential equations and the rigorous analysis of nonlinear partial differential equations modelling atmospheric flows. For my work in the former area, I was awarded the Olga Taussky-Todd prize lecture at ICIAM 2011.

Betül Tanbay completed her undergraduate degree at the University of Strasbourg (ULP) and her graduate degrees at the University of California, Berkeley (UCB). She has been a faculty member at the University of Boğaziçi, Istanbul, since 1989. Her research area is operator algebras, mainly focusing on the Kadison-Singer problem. She has held visiting positions at the University of California, Berkeley and Santa Barbara, the Université de Bordeaux, the Institut de Mathématiques de Jussieu, Paris VI, the University of Kansas and Pennsylvania State University. Betül Tanbay is the founder of the Istanbul Center for Mathematical Sciences, which she directed from 2006 to 2012. She was the first woman president of the Turkish Mathematical Society, from 2010 to 2016, where she also became an Honorary Member. She has represented the society at the General Assemblies of the IMU in Germany, Spain and South Korea and at the Councils of the EMS in Holland and Poland, and she was elected as a member to the Executive Committee at the Council in Berlin in 2016. Previously, she worked as a member of the EMS Raising Public Awareness and Ethics Committees and is currently EC-liaison of the latter committee and also a member of the Committee for Women in Mathematics of the IMU. Betül Tanbay is fluent in Turkish, English, French, German and Spanish. She has a daughter and a son both studying mathematics in Turkish public universities.

Report from the Executive Committee Meeting in Tbilisi, 4–6 November 2016

Richard Elwes, EMS Publicity Officer and Sjoerd Verduyn Lunel, EMS Secretary

The Executive Committee of the EMS met in Tbilisi, 4–6 November 2016, at Ivane Javakhishvili Tbilisi State University, on the kind invitation of the Georgian Mathematical Society. As well as the committee's current incumbents, several incoming members for 2017 also attended by invitation, making this event something of a handover. On Friday evening, the gathered assembly enjoyed a warm welcome from Roland Duduchava, Otar Chkadua and Tinatin Davitashvili (President and Vice-Presidents of the Georgian Mathematical Society), as well as Mikhail Chkhenkeli, Vice-Rector of Tbilisi State University. The oldest university in the Caucasus, it was founded in 1918 by the historian Ivane Javakhishvili, with, amongst others, the mathematician Andrea Razmadze, after whom the mathematics institute is named.

Officers' Reports and Membership

President Pavel Exner welcomed all members and guests and opened the meeting by relating his recent activities. The treasurer, Mats Gyllenberg, then presented his report on the society's 2016 income and expenditure, recording healthy results. Thus, the EMS can afford to maintain its recently elevated expenditure on scientific projects. He then presented the society's budget for 2017.

The committee was pleased to approve a list of 54 new individual members. The office has received no new applications for corporate membership, although there have been enquiries that may develop. Too many individual members remain in arrears, although several have belatedly paid their dues after a reminder. The office will continue to chase outstanding payments. The conversation then turned to the thornier problem of member societies that are perpetually in arrears. The society is mindful of the financial difficulties that national mathematical societies can experience. Nevertheless, the committee previously agreed the principle that membership will lapse for societies that neither pay their dues nor respond to letters from the president. In early 2016, the president wrote to all corporate members badly in arrears. The committee thus resolved that he will now write final reminders to such members, warning them of the committee's intention to propose the termination of their membership at the next council meeting.

ECMs 7&8

This was the first meeting of the Executive Committee since the 7th European Congress of Mathematics (ECM) in Berlin in July. The committee considered a report from local organiser Volker Mehrmann. It was agreed that, on most metrics, the meeting was a great success and the committee reiterated its profound thanks to all involved. However, there were also problems, notably a surprising disparity between the number of people registering and those paying the registration fee. Consequently, the meeting ended up in financial deficit. The organisers of ECM8 must be aware of this danger.

The committee opted to amend the profile for members of future ECM Scientific Committees, who should not only be top mathematicians but also people with broad mathematical interests willing to attend the event in person.

This was also the first Executive Committee meeting since the venue of the 8th European Congress was settled at July's council meeting: it will be held in Portorož in Slovenia, 5–11 July 2020. Local organiser Tomaz Pisanski delivered a presentation on preparations so far, including some thoughts on the philosophy of the congress. In the ensuing discussion, the committee contemplated the mission of the ECM in our changing mathematical world. How can we use such events to strengthen our community? How can we help young mathematicians feel welcome? Some initial ideas were aired and it was agreed that this was an important topic that will be revisited.

Other Scientific Meetings

The committee discussed reports from the 2016 EMS Summer Schools and agreed that these were scientifically successful and a worthwhile use of EMS resources. Considering recommendations from the Meetings Committee, support for four 2017 Summer Schools was approved. It was decided to require applicants for Summer School funding to provide more details on topics such as geographic, age and gender distributions of participants. The remit for the Meetings Committee was then adapted to encourage a more proactive approach to attracting proposals and thus hopefully increasing competition. It may also be useful to publicise recent successful Summer Schools.

Committee member Alice Fialowski reported from the EMS Distinguished Speaker talk, delivered by Ernest Vinberg at the 50th Seminar Sophus Lie. It was agreed to increase the level of ceremony at EMS Distinguished Speaker events to reflect the honour of the award. EMS Distinguished Speakers for 2017 were decided: Mireille

Bousquet-Mélou at the Conference on Foundations of Computational Mathematics (Barcelona, 10–19 July 2017) and Kathryn Hess at the Meeting of the Catalan, Spanish and Swedish Math Societies (Umeå, 12–15 June, 2017).

The committee agreed that the EMS will endorse the following events: the 9th European Student Conference in Mathematics, EUROMATH-2017 (Bucharest, 29 March – 2 April 2017); the 2nd Academic University Student Conference 2017 (15–19 November 2017, Larnaca, Cyprus); and the 11th European Conference on Mathematical and Theoretical Biology (Lisbon, 23–27 July 2018).

Meanwhile, the EMS-Bernoulli Society Joint Lecture for 2017 will be given by Alexander Holevo at the 31st European Meeting of Statisticians (Helsinki, 24–28 July 2017).

The committee approved separate funding for the Applied Mathematics Committee for scientific activities, to the tune of 5,000 euros in 2017 and 10,000 euros in 2018 (bearing in mind that 2018 is the Year of Mathematical Biology).

Society Meetings

The Executive Committee discussed the president's report from the council meeting from July and contemplated ways to make it a more interactive event. This conversation will be continued at future meetings. The committee was delighted to accept an invitation from the Czech Mathematical Society to hold the 2018 Council in Prague.

The next Executive Committee meeting will be held on 17–19 March in Bratislava and the next meeting of the Presidents of Member Societies will be held on 1–2 April 2017 in Lisbon.

Standing Committees & Projects

With numerous upcoming vacancies across the society's 11 standing committees (excluding the Executive Committee), the committee expressed its sincere appreciation to all outgoing members for their efforts in carrying out the society's work. It was then pleased to fill these places, including appointing Jürg Kramer as Chair and Tine Kjedsen as Vice-Chair of the Education Committee, Sandra di Rocco as Chair and Stanislaw Janeczko as Vice-Chair of the European Solidarity Committee, Michael Drmota as Chair and Ciro Ciliberto as Vice-Chair of the Meetings Committee, and Alessandra Celletti as Chair of the Women in Mathematics Committee.

The society's standing committees will soon reduce to 10, following a decision to merge the Publishing and Electronic Publishing Committees. The details, and further appointments to the combined committee, will await the outcome of a discussion between the two. The Executive Committee then approved an updated remit for the Ethics Committee.

The Chair of the Committee for Developing Countries (CDC) Giulia Di Nunno, in attendance by invitation, then delivered a presentation on the CDC's work, including on the Emerging Regional Centres of Excellence programme and plans for the grant scheme for researchers from Africa (made possible by generous funding from the Simons Foundation for Africa). The Executive Committee thanked her for the CDC's excellent work and discussed ways to increase its visibility. EMS members should be reminded of the option of donating to the CDC when paying their EMS membership fees (the amount raised by this route has decreased in recent years).

Reports from the chairs of other standing committees were also discussed, along with reports from the Publicity Officer and Editor-in-Chief of the Newsletter, and reports on other EMS-affiliated projects including the European Digital Mathematical Library, Zentralblatt MATH and EU-MATHS-IN (the European Service Network of Mathematics for Industry and Innovation). The latter has had a proposal ("Mathematical Modelling, Simulation and Optimization for Societal Challenges with Scientific Computing") funded under the Horizon 2020 programme on user-driven e-infrastructure innovation. EU-MATHS-IN is also involved in making mathematics more visible in H2020 generally.

Funding, Political and Scientific Organisations

The president reported on recent developments in Horizon2020 and the European Research Council. He also discussed the new legal status of the Initiative for Science in Europe (ISE), reminding the committee that the fee for ISE membership for the EMS is set to double to 3,000 euros. Thus, after two years, the benefits of ISE membership will be re-evaluated. The president gave a short report about ESOF (EuroScience Open Forum) in Manchester in July 2016, noting that the Committee for Raising Public Awareness had run a successful panel session on "The Myth of Turing".

There are live bids from both Paris and St. Petersberg for the ICM in 2022. The EMS is supporting both. The committee also discussed EMS ongoing business with other mathematical organisations, including ICIAM, the Abel Prize, CIMPA, the Fermat Prize and various research facilities around Europe.

Conclusion

The meeting was closed with enthusiastic thanks to our hosts, the Georgian Mathematical Society (particularly its President Roland Duduchava) and Tbilisi State University for excellent organisation and generous hospitality.

Call for Nominations for the Fermat Prize 2017

The new edition of the Fermat Prize for Mathematics Research opened in October 2016.

The call for nominations of candidates will be open until 30 June 2017 and the results will be announced in December 2017.

The Fermat Prize rewards the research of one or more mathematicians in fields where the contributions of Pierre de Fermat have been decisive: statements of variational principles or, more generally, partial differential equations; foundations of probability and analytical geometry; and number theory.

More information about the Fermat Prize, in particular concerning the application process, are available at

http://www.math.univ-toulouse.fr/FermatPrize.

Call for Nominations for the Ostrowski Prize 2017

The aim of the Ostrowski Foundation is to promote the mathematical sciences. Every second year it provides a prize for recent outstanding achievements in pure mathematics and in the foundations of numerical mathematics. The value of the prize for 2017 is 100,000 Swiss francs.

The prize has been awarded every two years since 1989. The most recent winners are Ben Green and Terence Tao in 2005, Oded Schramm in 2007, Sorin Popa in 2009, Ib Madsen, David Preiss and Kannan Soundararajan in 2011, Yitang Zhang in 2013, and Peter Scholze in 2015. See

https://www.ostrowski.ch/index_e.php

for the complete list and further details.

The jury invites nominations for candidates for the 2017 Ostrowski Prize. Nominations should include a CV of the candidate, a letter of nomination and 2–3 letters of reference.

The Chair of the jury for 2017 is Gil Kalai of the Hebrew University of Jerusalem, Israel. Nominations should be sent to kalai@math.huji.ac.il by May 15, 2017.

Call for Nominations for the 2019 ICIAM Prizes

The ICIAM Prize Committee for 2019 is calling for nominations for the five ICIAM Prizes to be awarded in 2019 (the Collatz Prize, the Lagrange Prize, the Maxwell Prize, the Pioneer Prize and the Su Buchin Prize). Each ICIAM Prize has its own special character but each one is truly international in character. Nominations are therefore welcome from every part of the world. A nomination should take into account the specifications of each particular prize (see below and at http://www.iciam.org/iciam-prizes) and should contain the following information:

- Full name and address of the person nominated.
- Web homepage if any.
- Name of particular ICIAM Prize.
- Justification for nomination (cite nominator's reason for considering the candidate to be deserving of the prize, including explanations of the scientific and practical influence of the candidate's work and publications).
- Proposed citation (concise statement about the outstanding contribution in fewer than 250 words).
- CV of the nominee.
- 2–3 letters of support from experts in the field and/or 2–3 names of experts to be consulted by the Prize Committee.
- Name and contact details of the proposer.

Nominations should be made electronically through the website https://iciamprizes.org/. The deadline for nominations is 15 July 2017.

Please contact president@iciam.org if you have any questions regarding the nomination procedure.

ICIAM Prize Committee for 2019:
Committee Chair: Maria J. Esteban
Zdenek Strakos (Chair of Collatz Prize Subcommittee)
Alexandre Chorin (Chair of Lagrange Prize Subcommittee)
Alexander Mielke (Chair of Maxwell Prize Subcommittee)
Denis Talay (Chair of Pioneer Prize Subcommittee)
Zuowei Shen (Chair of Su Buchin Prize Subcommittee)
Margaret H. Wright

The International Council for Industrial and Applied Mathematics (ICIAM) is the world organisation for applied and industrial mathematics.

Its members are mathematical societies based in more than 30 countries.

For more information, see the council's webpage at http://www.iciam.org/.

Simon Donaldson Awarded Doctor Honoris Causa

Sir Simon Kirwan Donaldson FRS, professor in pure mathematics at Imperial College London (UK) and permanent member of the Simons Center for Geometry and Physics at Stony Brook University (US), received the honorary degree Doctor Honoris Causa by Universidad Complutense de Madrid (Spain) on 20 January 2017.

The ceremony was presided over by the Rector, Carlos Andradas, who is also a professor in algebra at UCM. Vicente Muñoz, a professor in geometry and topology at UCM and a member of the Executive Committee of the EMS, was in charge of reading the Laudatio. There were 150 people in attendance, including the President of the Spanish Mathematical Society and the President of the Spanish Academy of Sciences. The ceremony took place in the Faculty of Mathematics of UCM. This is the first time that such an event has been hosted in this relatively new building. This allowed for an affluence of undergraduate and postgraduate students and served to commemorate the 25th anniversary of the faculty building.

Professor Simon Donaldson is a renowned geometer who has received numerous prizes, including the Fields Medal, the Shaw Prize and the Breakthrough Prize. The ceremony can be viewed at www.youtube.com/watch?v=xoC53Smg-WI. A report (in Spanish) can be found at tribuna.ucm.es/43/art2605.php.

Simon Donaldson (left) and Carlos Andradas (right) during the ceremony (photo: Jesús de Miguel/UCM)

Spectral Synthesis for Operators and Systems

Anton Baranov (St. Petersburg State University, St. Petersburg, Russia) and Yurii Belov (St. Petersburg State University, St. Petersburg, Russia)

Spectral synthesis is the reconstruction of the whole lattice of invariant subspaces of a linear operator from generalised eigenvectors. A closely related problem is the reconstruction of a vector in a Hilbert space from its Fourier series with respect to some complete and minimal system. This article discusses the spectral synthesis problem in the context of operator and function theory and presents several recent advances in this area. Among them is the solution of the spectral synthesis problem for systems of exponentials in $L^2(-\pi,\pi)$.

A more detailed account of these problems can be found in the survey [1]*, to appear in the proceedings of 7ECM.*

1 Introduction

Eigenfunction expansions play a central role in analysis and its applications. We discuss several questions concerning such expansions for special systems of functions, e.g. exponentials in L^2 on an interval and in weighted spaces, phase-space shifts of the Gaussian function in $L^2(\mathbb{R})$ and reproducing kernels in de Branges spaces (which include certain families of Bessel or Airy type functions). We present solutions of some problems in the area (including the longstanding spectral synthesis problem for systems of exponentials in $L^2(-\pi,\pi)$) and mention several open questions, such as the Newman–Shapiro problem about synthesis in Bargmann–Fock space.

Spectral synthesis for operators

One of the basic ideas of operator theory is to consider a linear operator as a "sum" of its simple parts, e.g. its restrictions onto invariant subspaces. In the finite-dimensional case, the possibility of such decomposition is guaranteed by the Jordan normal form. Moreover, any invariant subspace coincides with the span of the generalised eigenvectors it contains (recall that x is said to be a *generalised eigenvector* or a *root vector* of a linear operator A if $x \in \operatorname{Ker}(A-\lambda I)^n$ for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$).

However, the situation in the infinite-dimensional case is much more complex. Assume that A is a bounded linear operator in a separable Hilbert space H that has a complete set of generalised eigenvectors (in this case we say that *A is complete*). Is it true that any A-invariant subspace is *spectral*, that is, it coincides with the closed linear span of the generalised eigenvectors it contains? In general, the answer is negative. Therefore, it is natural to introduce the following notion.

Definition 1. A continuous linear operator A in a separable Hilbert (or Banach, or Frechét) space H is said to admit spectral synthesis or to be *synthesable* (we write $A \in \mathrm{Synt}$) if, for any invariant subspace E of A, we have

$$E = \overline{\operatorname{Span}}\{x \in E : x \in \cup_{\lambda,n} \operatorname{Ker}(A-\lambda I)^n\}.$$

Equivalently, this means that the restriction $A|_E$ has a complete set of generalised eigenvectors.

The notion of spectral synthesis for a general operator goes back to J. Wermer (1952). In the special context of translation invariant subspaces in spaces of continuous or smooth functions, similar problems were studied by J. Delsarte (1935), L. Schwartz (1947) and J.-P. Kahane (1953). Note that, in this case, the generalised eigenvectors are exponentials and exponential monomials.

Wermer showed, in particular, that any compact normal operator in a Hilbert space admits spectral synthesis. However, both of these conditions are essential: there exist non-synthesable compact operators and there exist non-synthesable normal operators. For a normal operator A with simple eigenvalues λ_n, Wermer showed that $A \notin \mathrm{Synt}$ if and only if the set $\{\lambda_n\}$ carries a complex measure orthogonal to polynomials, i.e. there exists a nontrivial sequence $\{\mu_n\} \in \ell^1$ such that $\sum_n \mu_n \lambda_n^k = 0$, $k \in \mathbb{N}_0$. Existence of such measures follows from Wolff's classical example of a Cauchy transform vanishing outside of the disc: there exist $\lambda_n \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\{\mu_n\} \in \ell^1$ such that

$$\sum_n \frac{\mu_n}{z-\lambda_n} \equiv 0, \qquad |z| > 1.$$

At the same time, there exist compact operators that do not admit spectral synthesis. Curiously, the first example of such a situation was implicitly given by H. Hamburger in 1951 (even before Wermer's paper). Further results were obtained in the 1970s by N. Nikolski and A. Markus. For example, Nikolski [20] proved that any Volterra operator can be a part of a complete compact operator (recall that a *Volterra operator* is a compact operator whose spectrum is $\{0\}$).

Theorem 2 (Nikolski)**.** For any Volterra operator V in a Hilbert space H, there exists a complete compact operator A on a larger Hilbert space $H \oplus H'$ such that H is A-invariant and $A|_H = V$. In particular, $A \notin \mathrm{Synt}$.

A. Markus [16] found a relation between spectral synthesis for a compact operator and the geometric properties of the eigenvectors. We now introduce the required "strong completeness" property.

Hereditarily complete systems

Let $\{x_n\}_{n\in N}$ be a system of vectors in a separable Hilbert space H that is both *complete* (i.e. its linear span is dense in H) and *minimal* (meaning that it fails to be complete when we remove any of its vectors). Let $\{y_n\}_{n\in N}$ be its (unique) *biorthogonal* system, i.e. the system such that $(x_m, y_n) = \delta_{mn}$, where δ_{mn} is the Kronecker delta. Note that a system coincides with its biorthogonal system if and only if it is an orthonormal basis.

With any $x \in H$, we associate its (formal) Fourier series with respect to the biorthogonal pair $\{x_n\}$, $\{y_n\}$:

$$x \in H \sim \sum_{n\in N} (x, y_n)x_n. \tag{1}$$

It is one of the basic problems of analysis to find conditions on the system $\{x_n\}$ that ensure the convergence of the Fourier series to x in some sense. In applications, the system $\{x_n\}$ is often given as the system of eigenvectors of some operator.

The choice of the coefficients in the expansion (1) is natural: note that if the series $\sum_n c_n x_n$ converges to x in H then, necessarily, $c_n = (x, y_n)$. There are many ways to understand convergence/reconstruction:

- The simplest case (with the exception of orthonormal bases, of course) is a *Riesz basis*: a system $\{x_n\}$ is a Riesz basis if, for any x, we have $x = \sum_n c_n x_n$ (the series converges in the norm) and $A\|\{c_n\}\|_{\ell^2} \le \|x\| \le B\|\{c_n\}\|_{\ell^2}$ for some positive constants A, B. Equivalently, $x_n = Te_n$ for an orthonormal basis $\{e_n\}$ and some bounded invertible operator T.
- *Bases with brackets*: there exists a sequence n_k such that $\sum_{n=1}^{n_k} (x, y_n)x_n$ converges to x as $k \to \infty$.
- Existence of a *linear* (*matrix*) *summation method* (e.g. Cesàro, Abel–Poisson, etc.): this means that there exists a doubly infinite matrix $(A_{m,n})$ such that $x = \lim_{m\to\infty} \sum_n a_{m,n} S_n(x)$, i.e. some means of the partial sums $S_n(x)$ of the series (1) converge to x.

The following property, known as *hereditary completeness*, may be understood as the weakest form of the reconstruction of a vector x from its Fourier series $\sum_{n\in N}(x, y_n)x_n$.

Definition 3. A complete and minimal system $\{x_n\}_{n\in N}$ in a Hilbert space H is said to be hereditarily complete if, for any $x \in H$, we have

$$x \in \overline{\operatorname{Span}}\,\{(x, y_n)x_n\}.$$

It is easy to see that hereditary completeness is equivalent to the following property: for any partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, of the index set N, the *mixed system*

$$\{x_n\}_{n\in N_1} \cup \{y_n\}_{n\in N_2} \tag{2}$$

is complete in H. Clearly, hereditary completeness is necessary for the existence of a linear summation method for the series (1) (otherwise, there exists a vector x orthogonal to all partial sums of (1)).

Hereditarily complete systems are also known as *strong Markushevich bases*. We will also say, in this case, that the system admits *spectral synthesis* motivated by the following theorem of Markus [16].

Theorem 4 (Markus)**.** Let A be a complete compact operator with generalised eigenvectors $\{x_n\}$ and trivial kernel. Then, $A \in$ Synt if and only if the system $\{x_n\}$ is hereditarily complete.

Indeed, assume that the system of eigenvectors $\{x_n\}$ is not hereditarily complete and, for some partition $N = N_1 \cup N_2$, the mixed system $\{x_n\}_{n\in N_1} \cup \{y_n\}_{n\in N_2}$ is not complete. The biorthogonal system y_n consists of eigenvectors of the adjoint operator A^*. Then, $E = \overline{\operatorname{Span}}\{y_n : n \in N_2\}^\perp$ is A-invariant and $\{x_n : n \in N_1\} \subset E$ but $E \ne \overline{\operatorname{Span}}\{x_n : n \in N_1\}$.

Note that hereditary completeness includes the requirement that the biorthogonal system $\{y_n\}$ is complete in H, which is by no means automatic. In fact, it is very easy to construct a complete and minimal system whose biorthogonal is not complete.

Example 5. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in H. Set $x_n = e_1 + e_n$, $n \ge 2$. Then, it is easy to see that $\{x_n\}$ is complete and minimal, while its biorthogonal is clearly given by $y_n = e_n$, $n \ge 2$. Taking direct sums of such examples, one obtains biorthogonal systems with arbitrary finite or infinite codimension.

It is not so trivial to construct a complete and minimal system $\{x_n\}$ with a complete biorthogonal system $\{y_n\}$ that is not hereditarily complete (i.e. the mixed system (2) fails to be complete for some partition of the index set). A first explicit construction was given by Markus (1970). Further results about the structure of nonhereditarily complete systems were obtained by N. Nikolski, L. Dovbysh and V. Sudakov (1977). For an extensive survey of spectral synthesis and hereditary completeness, the reader is referred to [13].

Interesting examples of hereditarily (in)complete systems also appear in papers by D. Larson and W. Wogen (1990), E. Azoff and H. Shehada (1993) and A. Katavolos, M. Lambrou and M. Papadakis (1993) in connection with reflexive operator algebras.

2 Spectral synthesis for exponential systems

Let $e_\lambda(t) = e^{i\lambda t}$ be a complex exponential. For $\Lambda = \{\lambda_n\} \subset \mathbb{C}$, we consider

$$\mathcal{E}(\Lambda) = \{e_\lambda\}_{\lambda\in\Lambda}$$

as a system in $L^2(-a, a)$. The series $\sum_n c_n e_{\lambda_n}$ are often referred to as *nonharmonic Fourier series*, in contrast to “harmonic” orthogonal series. A good introduction to the subject can be found in [26].

Exponential systems play a most prominent role in analysis and its applications. Geometric properties of exponential systems in $L^2(-a, a)$ were one of the major themes of 20th century harmonic analysis. Let us briefly mention some of the milestones of the theory.

(i) *Completeness of exponential systems.*
This basic problem was studied in the 1930–1940s by N. Levinson and B. Ya. Levin. One of the most important contributions is the famous result of A. Beurling and P. Malliavin (1967), who gave an explicit formula for the radius of completeness of a system $\mathcal{E}(\Lambda)$ in terms of a certain density of Λ. By the radius of completeness of $\mathcal{E}(\Lambda)$, we mean

$$r(\Lambda) = \sup\{a > 0 : \mathcal{E}(\Lambda) \text{ is complete in } L^2(-a, a)\}.$$

A new approach to these (and related) problems and their far-reaching extensions can be found in [9, 10, 17, 18].

(ii) *Riesz bases of exponentials.*
The first results about Riesz bases of exponentials, which go back to R. Paley and N. Wiener (1930s), were of perturbative nature. Assume that the frequencies λ_n are small perturbations of integers,

$$\sup_{n\in\mathbb{Z}} |\lambda_n - n| < \delta.$$

Paley and Wiener showed that $\mathcal{E}(\Lambda)$ is a Riesz basis in $L^2(-\pi,\pi)$ if $\delta = \pi^{-2}$. It was a longstanding problem to find the best possible δ; finally, M. Kadets (1965) showed that the sharp bound is $1/4$. However, it was clear that one cannot describe all Riesz bases in terms of "individual" perturbations. A complete description of exponential bases in terms of the Muckenhoupt (or Helson–Szegö) condition was given by B. S. Pavlov (1979) and was further extended by S. V. Hruschev, N. K. Nikolski and B. S. Pavlov (see [11] for a detailed account of the problem). Yu. Lyubarskii and K. Seip (1997) extended this description to the L^p-setting. These results revealed the connection of the problem to the theory of singular integrals.

(iii) *Exponential frames* (*sampling sequences*).
A system $\{x_n\}$ in a Hilbert space H is said to be a frame if there exist positive constants A, B such that

$$A\sum_n |(x,x_n)|^2 \le \|x\|^2 \le B\sum_n |(x,x_n)|^2,$$

i.e. a generalised Parceval identity holds. Here, we omit the requirement of minimality to gain in "stability" of the reconstruction; there is a canonical choice of coefficients so that the series $\sum_n c_n x_n$ converge to x. If a frame $\{x_n\}$ is minimal then it is a Riesz basis.

Exponential frames were introduced by R. Duffin and A. C. Schaeffer (1952), while their complete description was obtained relatively recently by J. Ortega–Cerdà and K. Seip [22]; this solution involves the theory of de Branges spaces of entire functions, which is to be discussed below. For an extensive review on exponential frames on disconnected sets, see a recent monograph by A. Olevskii and A. Ulanovskii [21].

Synthesis up to codimension 1

The *spectral synthesis* (or *hereditary completeness*) problem for exponential systems was also a longstanding problem in nonharmonic Fourier analysis. Let $\mathcal{E}(\Lambda)$ be a complete and minimal system of exponentials in $L^2(-a,a)$ and let $\{\tilde e_\lambda\}$ be the biorthogonal system. It was shown by R. Young (1981) that the biorthogonal system $\{\tilde e_\lambda\}$ is always complete.

Problem 6. Is it true that any complete and minimal system of exponentials $\{e_\lambda\}_{\lambda\in\Lambda}$ in $L^2(-a,a)$ is hereditarily complete, i.e. any function $f \in L^2(-a,a)$ belongs to the closed linear span of its "harmonics" $(f,\tilde e_\lambda)e_\lambda$?

This question was answered in the negative by the authors jointly with Alexander Borichev [2]. Surprisingly, it turned out, at the same time, that spectral synthesis for exponential systems always holds up to at most one-dimensional defect.

Theorem 7. There exists a complete and minimal system of exponentials $\{e_\lambda\}_{\lambda\in\Lambda}$, $\Lambda\subset\mathbb{R}$, in $L^2(-\pi,\pi)$ that is not hereditarily complete.

Thus, in general, there exists no linear summation method for nonharmonic Fourier series $\sum_{\lambda\in\Lambda}(f,\tilde e_\lambda)e_\lambda$ associated to a complete and minimal exponential system.

It is worth mentioning that "bad" sequences Λ can be regularly distributed, e.g. be a bounded perturbation of integers: in Theorem 7 one can choose a uniformly separated sequence Λ so that $|\lambda_n - n| < 1$, $n\in\mathbb{Z}$.

Theorem 8. If the system of exponentials $\{e_\lambda\}_{\lambda\in\Lambda}$ is complete and minimal in $L^2(-a,a)$ then, for any partition $\Lambda = \Lambda_1\cup\Lambda_2$, $\Lambda_1\cap\Lambda_2 = \emptyset$, the corresponding mixed system has defect at most 1, that is,

$$\dim\left(\{e_\lambda\}_{\lambda\in\Lambda_1}\cup\{\tilde e_\lambda\}_{\lambda\in\Lambda_2}\right)^\perp \le 1.$$

It turns out that incomplete mixed systems are always highly asymmetric. Given a complete and minimal system $\mathcal{E}(\Lambda)$ in $L^2(-\pi,\pi)$, it is natural to expect that "in the main" Λ is similar to $\mathbb{Z}$. This is indeed the case. As a very rough consequence of more delicate results (such as the Cartwright–Levinson theorem), let us mention that Λ always has density 1:

$$\lim_{r\to\infty}\frac{n_r(\Lambda)}{2r} = 1,$$

where $n_r(\Lambda)$ is the usual counting function, $n_r(\Lambda) = \#\{\lambda\in\Lambda, |\lambda|\le r\}$. Analogously, one can define the upper density $D_+(\Lambda)$:

$$D_+(\Lambda) = \limsup_{r\to\infty}\frac{n_r(\Lambda)}{2r}.$$

Theorem 9. Let $\Lambda\subset\mathbb{C}$, let the system $\mathcal{E}(\Lambda)$ be complete and minimal in $L^2(-a,a)$ and let the partition $\Lambda = \Lambda_1\cup\Lambda_2$ satisfy $D_+(\Lambda_2) > 0$. Then, the mixed system $\{e_\lambda\}_{\lambda\in\Lambda_1}\cup\{\tilde e_\lambda\}_{\lambda\in\Lambda_2}$ is complete in $L^2(-a,a)$.

Theorem 9 shows that there is a strong asymmetry between the systems of reproducing kernels and their biorthogonal systems. The completeness of a mixed system may fail only when we take a sparse (but infinite!) subsequence Λ_1.

We conclude this subsection with one open problem.

Problem 10. Given a hereditarily complete system of exponentials, does there exist a linear summation method for the corresponding nonharmonic Fourier series?

Translation to the entire functions setting

A classical approach to a completeness problem is to translate it (via a certain integral transform) to a uniqueness problem in some space of analytic functions. In the case of exponentials on an interval, the role of such a transform is played by the standard Fourier transform $\mathcal{F}$,

$$(\mathcal{F}f)(z) = \frac{1}{2\pi}\int_{-a}^{a} f(t)e^{-izt}dt.$$

By the classical Paley–Wiener theorem, $\mathcal{F}$ maps $L^2(-\pi,\pi)$ unitarily onto the space

$$\mathcal{PW}_a = \{F - \text{ entire},\ F\in L^2(\mathbb{R}),\ |F(z)|\le Ce^{a|z|}\}.$$

Paley–Wiener space $\mathcal{PW}_a$ (also known as the space of bandlimited functions of bandwidth $2a$) plays a remarkably important role in signal processing.

The Fourier transform maps exponentials in $L^2(-a,a)$ to the cardinal sine functions:

$$(\mathcal{F}e_{\bar\lambda})(z) = k_\lambda(z) = \frac{\sin a(z-\bar\lambda)}{\pi(z-\bar\lambda)}.$$

Note that the functions k_λ are the *reproducing kernels* in Paley–Wiener space $\mathcal{PW}_a$, i.e. the function k_λ generates the evaluation functional at the point λ:

$$F(\lambda) = (F, k_\lambda), \qquad F \in \mathcal{PW}_a.$$

In particular, the orthogonal expansion $f = \sum_{n\in\mathbb{Z}} c_n e^{int}$ in $L^2(-\pi,\pi)$ becomes

$$F(z) = \sum\nolimits_{n\in\mathbb{Z}} c_n \frac{\sin \pi(z-n)}{\pi(z-n)}, \quad c_n = F(n), \tag{3}$$

the classical Shannon–Kotelnikov–Whittacker sampling formula.

Moreover, this translation makes it possible to find an explicit form of the biorthogonal system, which is not possible when staying inside $L^2(-a,a)$. Let $\{k_\lambda\}_{\lambda\in\Lambda}$ be a complete and minimal system in $\mathcal{PW}_a$. Its biorthogonal system may then be obtained from one function G_Λ known as the generating function of Λ. This is an entire function with the zero set Λ, which can be defined by the formula

$$G_\Lambda(z) = \lim_{R\to\infty} \prod_{\lambda\in\Lambda,\,|\lambda|<R} \left(1-\frac{z}{\lambda}\right),$$

with the properties that $G_\Lambda \notin \mathcal{PW}_a$ (otherwise this would be a contradiction to completeness) but $\frac{G_\Lambda}{z-\lambda} \in \mathcal{PW}_a$ for any $\lambda \in \Lambda$ by the minimality of the system $\{k_\lambda\}_{\lambda\in\Lambda}$. The biorthogonal system is then given by

$$g_\lambda(z) = \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z-\lambda)}. \tag{4}$$

Thus, with any function $F \in \mathcal{PW}_a$, we can associate two (formal) Fourier series expansions:

$$F \in \mathcal{PW}_a \sim \sum_{\lambda\in\Lambda} c_\lambda \frac{\sin a(z-\bar\lambda)}{\pi(z-\bar\lambda)}, \qquad c_\lambda = (F, g_\lambda),$$

$$F \in \mathcal{PW}_a \sim \sum_{\lambda\in\Lambda} F(\lambda) \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z-\lambda)}.$$

The first series is an expansion with respect to cardinal sine functions while the second one is a Lagrange-type interpolation series.

Our results on exponential systems can be reformulated for reproducing kernels of Paley–Wiener space: *for any complete and minimal system $\{k_\lambda\}_{\lambda\in\Lambda}$ and any partition $\Lambda = \Lambda_1 \cup \Lambda_2$,*

$$\dim\left(\{k_\lambda\}_{\lambda\in\Lambda_1} \cup \{g_\lambda\}_{\lambda\in\Lambda_2}\right)^\perp \le 1 \tag{5}$$

but the defect 1 is possible.

3 Spectral synthesis in de Branges spaces and applications

Preliminaries on de Branges spaces

The theory of Hilbert spaces of entire functions was created by L. de Branges at the end of the 1950s and the beginning of the 1960s. It was the main tool in his famous solution of the direct and inverse spectral problems for two-dimensional canonical systems. These are second order ODEs that include, as particular cases, the Schrödinger equation on an interval, the Dirac equation and Krein's string equation. For the general theory of de Branges spaces, we refer to the original monograph by de Branges [8]; for information relating to inverse problems, see [23, 24].

De Branges spaces proved to be highly nontrivial and are interesting objects from the point of view of function theory. Surprisingly, they appear to be unavoidable in substantially different branches of analysis, for example:

- Polynomial approximations on the real line.
- Orthogonal polynomials and random matrix theory (see, for example, [15]).
- Model (backward shift invariant) subspaces of Hardy space: $K_\Theta = H^2 \ominus \Theta H^2$, where H^2 is Hardy space and Θ is an inner function (for a discussion of this relation, see, for example, [9]).
- Functional models for different classes of linear operators [5, 12].

and even

- Analytic number theory, the Riemann Hypothesis and Dirichlet L-functions [14].

There are equivalent ways to introduce de Branges spaces: an axiomatic approach or a definition in terms of the generating Hermite–Biehler entire function. Here, we will not go into details, instead confining ourselves to an equivalent representation of de Branges spaces via *spectral data*. This representation of a de Branges space in terms of a weighted Cauchy transform (which can already be found in the work of de Branges) turns out to be a very useful tool; it relates the study of de Branges spaces with singular integral operators (see, for example, [6]).

Let $T = \{t_n\}_{n\in N} \subset \mathbb{R}$ be an increasing sequence (one-sided or two sided, the index set being a subset of $\mathbb{Z}$) such that $|t_n| \to \infty$, $|n| \to \infty$. Let $\mu = \sum_n \mu_n \delta_{t_n}$ be a measure supported by T such that $\sum_n (t_n^2+1)^{-1}\mu_n < \infty$. Consider the class of entire functions

$$\mathcal{H} = \left\{F : \ F(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z-t_n}\right\}, \tag{6}$$

where A is some (fixed) entire function that is real on $\mathbb{R}$ and vanishes exactly on T, and $\{c_n\} \in \ell^2$.

Set $\|F\|_{\mathcal{H}} = \|\{c_n\}\|_{\ell^2}$. With this norm, $\mathcal{H}$ is a reproducing kernel Hilbert space. It is an axiomatic de Branges space and any de Branges space can be represented in this way.

We call the pair (T,μ) the *spectral data* for space $\mathcal{H}$. Of course, formally, the space also depends on the choice of the entire function A but spaces with the same spectral data and different functions A are canonically isomorphic.

Example 11. If $T = \mathbb{Z}$, $\mu_n = 1$ and $A(z) = \sin \pi z$ then $\mathcal{H} = \mathcal{PW}_\pi$. The corresponding representation of the elements of $\mathcal{H}$ coincides with the Schannon–Kotelnikov–Whittaker sampling formula (3).

An important feature of de Branges spaces is that they have the so-called *division property*: if a function f in a de Branges space $\mathcal{H}$ vanishes at some point $w \in \mathbb{C}$ then the function $f(z)/(z-w)$ also belongs to $\mathcal{H}$. Another characteristic property of a de Branges space is existence of *orthogonal bases of reproducing kernels*. It is clear from the definition that the functions $A(z)/(z-t_n)$ form an orthogonal basis in $\mathcal{H}$ and are reproducing kernels of $\mathcal{H}$ up to normalisation.

Spectral synthesis problem and its solution

Let $\{k_\lambda\}$ be a complete and minimal system of reproducing kernels in a de Branges space $\mathcal{H}$. As in the Paley–Wiener space, its biorthogonal system is given by formula (4) for some appropriate generating function G_Λ. However, in contrast to the Paley–Wiener case, the biorthogonal system in general need not be complete (Baranov, Belov, 2011).

We will say that a de Branges space has the *spectral synthesis property* if any complete and minimal system of reproducing kernels with the complete biorthogonal system (this assumption is included) is also hereditarily complete, i.e. all mixed systems are complete. In [3], the following problem is addressed.

Problem 12. Which de Branges spaces have the spectral synthesis property? If a space does not have the spectral synthesis property, what is the possible size of the defect for a mixed system?

Why is hereditary completeness of reproducing kernels in de Branges spaces an interesting and significant topic? There are several motivations for that:

- Relation to exponential systems and nonharmonic Fourier series as discussed above.
- N. Nikolski's question: whether there exist nonhereditarily complete systems of reproducing kernels in model spaces $K_\Theta = H^2 \ominus \Theta H^2$? Note that de Branges spaces form an important special subclass of model spaces.
- Spectral synthesis for rank one perturbations of self-adjoint operators (see Subsection 3).

In [3], a complete description of de Branges spaces with the spectral synthesis property was obtained. To state it, we need one more definition. An increasing sequence $T = \{t_n\}$ is said to be *lacunary* (or *Hadamard lacunary*) if

$$\liminf_{t_n\to\infty} \frac{t_{n+1}}{t_n} > 1, \qquad \liminf_{t_n\to-\infty} \frac{|t_n|}{|t_{n+1}|} > 1,$$

i.e. the moduli of $|t_n|$ tend to infinity at least exponentially.

Theorem 13. Let $\mathcal{H}$ be a de Branges space with spectral data (T, μ). Then, $\mathcal{H}$ has the spectral synthesis property if and only if one of the following conditions holds:

(i) $\sum_n \mu_n < \infty$.

(ii) The sequence $\{t_n\}$ is lacunary and, for some $C > 0$ and any n,

$$\sum_{|t_k|\le|t_n|} \mu_k + t_n^2 \sum_{|t_k|>|t_n|} \frac{\mu_k}{t_k^2} \le C\mu_n. \qquad (7)$$

Note that condition (3) implies that the sequence of masses μ_n also grows at least exponentially.

Now we turn to the problem of the size of the defect (i.e. the dimension of the complement to a mixed system). It turns out that one can construct examples of systems of reproducing kernels with large or even infinite defect.

Theorem 14. For any increasing sequence $T = \{t_n\}$ with $|t_n| \to \infty$, $|n| \to \infty$, and for any $N \in \mathbb{N} \cup \{\infty\}$, there exists a measure μ such that, in the de Branges space with spectral data (T, μ), there exists a complete and minimal system of reproducing kernels $\{k_\lambda\}_{\lambda\in\Lambda}$ whose biorthogonal system $\{g_\lambda\}_{\lambda\in\Lambda}$ is also complete but, for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$,

$$\dim\left(\{g_\lambda\}_{\lambda\in\Lambda_1} \cup \{k_\lambda\}_{\lambda\in\Lambda_2}\right)^\perp = N.$$

The key role in this construction plays the balance between the "spectrum" $\{t_n\}$ and the masses $\{\mu_n\}$. If $\sum_n \mu_n = \infty$, but there exists a subsequence t_{n_k} of T such that $\sum_k t_{n_k}^{2N-2}\mu_{n_k} < \infty$, then, in the corresponding de Branges space, one has mixed systems with any defect up to N. Conversely, the estimate $\mu_n \ge |t_n|^{-M}$ for some $M > 0$ and all n implies an estimate from the above in terms of M for the defect.

Spectral theory of rank one perturbations of compact self-adjoint operators

Let A be a compact self-adjoint operator in a separable Hilbert space H with simple point spectrum $\{s_n\}$ and trivial kernel. In other words, A is the simplest infinite-dimensional operator one can imagine, diagonalisable by the classical Hilbert–Schmidt theorem. Surprisingly, the spectral theory of rank one perturbations of such operators is already highly nontrivial.

For $a, b \in H$, consider the rank one perturbation L of A,

$$L = A + a \otimes b, \qquad Lf = Af + (f, b)a, \qquad f \in H.$$

For example, one may obtain examples of rank one perturbations changing one boundary condition in a second order differential equation.

In [5], the spectral properties of rank one perturbations are studied via a functional model. Several similar functional models for rank one perturbations (or close classes of operators) have been developed, e.g. by V. Kapustin (1996) and G. Gubreev and A. Tarasenko (2010). Let us present the idea of this model without going into technicalities.

For $t_n = s_n^{-1}$, consider a de Branges space $\mathcal{H}$ with spectral data (T, μ), where μ is some measure supported by T. Let G be an entire function such that $G \notin \mathcal{H}$ but $G(z)/(z-w) \in \mathcal{H}$ whenever $G(w) = 0$. This means that the function G has growth just slightly larger than is possible for the elements of $\mathcal{H}$. Assume also that $G(0) = 1$. Consider the linear operator

$$(MF)(z) = \frac{F(z) - F(0)G(z)}{z}, \qquad F \in \mathcal{H}. \qquad (8)$$

It is easily seen that M is a rank one perturbation of a compact self-adjoint operator with spectrum $t_n^{-1} = s_n$. The functional model theorem from [5] proves that any rank one perturbation of A is unitary equivalent to a model operator M of the form (8) for some de Branges space $\mathcal{H}$ and function G. Therefore, while the spectrum $T = \{t_n\}$ is fixed, the masses μ_n and the function G are free parameters of the model.

It is easy to see that the eigenfunctions of M are of the form $G(z)/(z-\lambda)$ for $\lambda \in Z_G$, where $Z(G)$ denotes the zero set of G. The point spectrum of M is thus given by $\{\lambda^{-1} : \lambda \in Z_G\}$. Multiple zeros of G correspond to generalised eigenvectors but we assume, for simplicity, that G has simple zeros only. Thus, the system of eigenfunctions of the rank one perturbation L is unitary equivalent to a system of the form $\{g_\lambda\}_{\lambda\in\Lambda}$, as in (4), while eigenfunctions of L^* (which is also a rank one perturbation of A) correspond to a system $\{k_\lambda\}_{\lambda\in\Lambda}$ of reproducing kernels in $\mathcal{H}$.

Thus, we relate the spectral properties of rank one perturbations to geometric properties of systems of reproducing kernels (in view of the symmetry, we interchange the roles of L and L^*):

- Completeness of $L \Longleftrightarrow$ completeness of a system of reproducing kernels $\{k_\lambda\}$ in $\mathcal{H}(E)$.
- Completeness of $L^* \Longleftrightarrow$ completeness of the system biorthogonal to the system of reproducing kernels.
- Spectral synthesis for $L \Longleftrightarrow$ hereditary completeness of $\{k_\lambda\}_{\lambda\in\Lambda}$, i.e. for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the system $\{k_\lambda\}_{\lambda\in\Lambda_1} \cup \{g_\lambda\}_{\lambda\in\Lambda_2}$ is complete in $\mathcal{H}$.

The results of Subsection 3 lead to a number of striking examples for rank one perturbations of compact self-adjoint operators. These examples show that the spectral theory of such perturbations is a rich and complicated subject that is far from being completely understood.

Theorem 15 (Baranov, Yakubovich)**.** For any compact self-adjoint operator A with simple spectrum, there exists its rank one perturbation $L = A + a \otimes b$ such that:

(i) Ker L = Ker $L^* = 0$ and L is complete but the eigenvectors of L^* span a subspace with infinite codimension.

(ii) L and L^* are complete but L admits no spectral synthesis with infinite defect (i.e. there exists an L-invariant subspace E such that the generalised vectors of L that belong to E have infinite codimension in E).

For more results about completeness and spectral synthesis of rank one perturbations, see [5]. One may also ask for which compact self-adjoint operators A there exists a rank one perturbation that is a Volterra operator (i.e. the spectrum can be destroyed by a rank one perturbation). This problem was solved in [4], where it was shown that the spectrum $\{s_n\}$ is "destructible" if and only if $t_n = s_n^{-1}$ form the zero set of an entire function of some special class introduced by M. G. Krein (1947).

4 Spectral synthesis in Fock-type spaces

Classical Bargmann–Fock space

Fock-type spaces form another important class of Hilbert spaces of entire functions. In contrast to de Branges spaces, where the norm is defined as an integral over the real axis (with respect to some continuous or discrete measure), in Fock-type spaces the norm is defined as an area integral. Classical Fock space $\mathcal{F}$ (also known as Bargmann, Segal–Bargmann or Bargmann–Fock space) is defined as the set of all entire functions F for which

$$\|F\|^2_{\mathcal{F}} := \frac{1}{\pi}\int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < \infty,$$

where m stands for the area Lebesgue measure. This space (as well as its multi-dimensional analogues) plays a most prominent role in theoretical physics, serving as a model of the phase space of a particle in quantum mechanics. It also appears naturally in time-frequency analysis and Gabor frame theory. There is a canonical unitary map from $L^2(\mathbb{R})$ to $\mathcal{F}$ (the Bargmann transform), which plays a role similar to that of the Fourier transform in the Paley–Wiener space setting. Note that, clearly, all functions in $\mathcal{F}$ are of order at most 2 and satisfy the estimate $|F(z)| \le C\exp(\pi|z|^2/2)$ (which can be slightly refined).

The reproducing kernels of $\mathcal{F}$ are the usual complex exponentials, $k_\lambda(z) = e^{\pi\bar{\lambda}z}$. Moreover, the Bargmann transform of the phase-space shift of the Gaussian, i.e. of the function $e^{2\pi i\eta t}e^{-\pi(t-\xi)^2}$, coincides up to normalisation with $e^{\pi\lambda z}$, $\lambda = \xi + i\eta$. Thus, geometric properties (e.g. spectral synthesis) of the phase-space shifts of the Gaussian are equivalent to the corresponding properties of the exponentials in Fock space.

Radial Fock-type spaces

Considering radial weights differing from the Gaussian weight, one obtains a wide class of Hilbert spaces of entire functions. Namely, for a continuous function $\varphi : [0,\infty) \to (0,\infty)$, we define the radial Fock-type space as

$$\mathcal{F}_\varphi = \Big\{F \text{ entire} : \|F\|^2_{\mathcal{F}_\varphi} := \frac{1}{\pi}\int_{\mathbb{C}} |F(z)|^2 e^{-\varphi(|z|)} dm(z) < \infty\Big\}.$$

We always assume that $\log r = o(\varphi(r))$, $r \to \infty$, to exclude finite-dimensional spaces.

Any Fock-type space is a reproducing kernel Hilbert space. It was shown by K. Seip [25] that in classical Fock space there are no Riesz bases of reproducing kernels. Recently, A. Borichev and Yu. Lyubarskii [7] showed that Fock-type spaces with slowly growing weights $\varphi(r) = (\log r)^\gamma$, $\gamma \in (1,2]$, have Riesz bases of reproducing kernels corresponding to real points and, thus, can be realised as de Branges spaces with equivalence of norms (this is clear from the representation of de Branges spaces via their spectral data). Moreover, $\varphi(r) = (\log r)^2$ is in a sense the sharp bound for this phenomenon. Namely, it is shown in [7] that if $(\log r)^2 = o(\varphi(r))$, $r \to \infty$, and φ has a certain regularity then $\mathcal{F}_\varphi$ has no Riesz bases of reproducing kernels.

In view of the examples above, one may ask which de Branges spaces can be realised as radial Fock-type spaces, that is, there is an area integral norm that is equivalent to the initial de Branges space norm. Surprisingly, it turns out that this class of de Branges spaces exactly coincides with the class of de Branges spaces (ii) with the spectral synthesis property in Theorem 13.

Theorem 16 (Baranov, Belov, Borichev)**.** Let $\mathcal{H}$ be a de Branges space with spectral data (T,μ). Then, the following statements are equivalent:

(i) There exists a Fock-type space $\mathcal{F}_\varphi$ such that $\mathcal{H} = \mathcal{F}_\varphi$.

(ii) $\mathcal{H}$ is rotation invariant, that is, the operator $R_\theta : f(z) \mapsto f(e^{i\theta}z)$ is a bounded invertible operator in $\mathcal{H}$ for some (all) $\theta \in (0,\pi)$.

(iii) The sequence T is lacunary and (7) holds.

Thus, in the space $\mathcal{F}_\varphi$ with $\varphi(r) = (\log r)^\gamma$, $\gamma \in (1,2]$, any complete and minimal system of reproducing kernels is hereditarily complete. We also mention that Riesz bases in some de Branges spaces with lacunary spectral data have been described by Yu. Belov, T. Mengestie and K. Seip [6].

Synthesis in Fock space and the Newman–Shapiro problem

Now we turn to the case of classical Fock space $\mathcal{F}$. Though it has no Riesz bases of reproducing kernels, there exist many complete and minimal systems of reproducing kernels. The two-dimensional lattice $\mathbb{Z}+i\mathbb{Z}$ plays for Fock space a role similar to the role of the lattice $\mathbb{Z}$ for Paley–Wiener space $\mathcal{PW}_\pi$. In particular, if $\Lambda = (\mathbb{Z}+i\mathbb{Z}) \setminus \{0\}$ then $\{k_\lambda\}_{\lambda\in\Lambda} = \{e^{\pi\bar{\lambda}z}\}_{\lambda\in\Lambda}$ is a complete and minimal system, whose generating function is the Weierstrass sigma-function (up to the factor z). The second author proved (2015) the following Young-type theorem for Fock space.

Theorem 17 (Belov)**.** For any complete and minimal system of reproducing kernels (i.e. exponentials) in $\mathcal{F}$, its biorthogonal system is also complete.

On the other hand, the first author recently proved that classical Fock space has no spectral synthesis property. Equivalently, this means that there exist nonhereditarily complete systems of phase-space shifts of the Gaussian in $L^2(\mathbb{R})$. At the same time, there are good reasons to believe that there exists a universal upper bound for the defects of mixed systems. The proofs of these results are to appear elsewhere.

Theorem 18 (Baranov)**.** There exist complete and minimal systems $\{e^{\pi\bar{\lambda}z}\}_{\lambda\in\Lambda}$ of reproducing kernels in $\mathcal{F}$ that are not hereditarily complete, that is, for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the mixed system $\{e^{\pi\bar{\lambda}z}\}_{\lambda\in\Lambda_1} \cup \{g_\lambda\}_{\lambda\in\Lambda_2}$ is not complete in $\mathcal{F}$.

Moreover, this example of a nonhereditarily complete system of reproducing kernels in $\mathcal{F}$ admits the following reformulation. Given a function $G \in \mathcal{F}$, let us denote by $\mathcal{R}_G$ the subspace of $\mathcal{F}$ defined as

$$\mathcal{R}_G = \{GF : GF \in \mathcal{F}, F - \text{entire}\}.$$

Thus, $\mathcal{R}_G$ is the (closed) subspace in $\mathcal{F}$ that consists of functions in $\mathcal{F}$ that vanish at the zeros of G with appropriate multiplicities. The example of Theorem 18 shows that there exists $G \in \mathcal{F}$ such that $z^nG \in \mathcal{F}$ for any $n \geq 1$ and $\overline{\text{Span}}\{z^nG : n \in \mathbb{Z}_+\} \neq \mathcal{R}_G$.

We stated this result to compare it with a longstanding problem in Fock space that has a similar form. This problem was posed in the 1960s by D. J. Newman and H. S. Shapiro [19], who were motivated by an old paper of E. Fisher (1917) on differential operators. Assume that a function G from $\mathcal{F}$ is such that $e^{wz}G \in \mathcal{F}$ for any $w \in \mathbb{C}$, that is, its growth is smaller than the critical one. One may define on the linear span of all exponentials the (unbounded) multiplication operator $M_GF = GF$. It is natural to expect that the adjoint of M_G will then be given by an infinite order differential operator $G^*(\frac{\partial}{\partial z})$, where $G^*(z) = \overline{G(\bar{z})}$. Newman and Shapiro showed that the positive answer to this question is equivalent to the positive solution of the following problem.

Problem 19. Let $G \in \mathcal{F}$ be such that $e^{wz}G \in \mathcal{F}$ for any $w \in \mathbb{C}$. Is it true that

$$\overline{\text{Span}}\{e^{wz}G : w \in \mathbb{C}\} = \mathcal{R}_G\ ?$$

Newman and Shapiro showed that the equality holds in the case where G is a linear combination of exponential monomials. In our example in Theorem 18, however, the function G admits multiplication by polynomials in Fock space but not multiplication by the exponents. Thus, the spectral synthesis problem of Newman and Shapiro remains open.

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Diffusion, Optimal Transport and Ricci Curvature for Metric Measure Spaces

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Lower Ricci curvature bounds play a crucial role in several deep geometric and functional inequalities in Riemannian geometry and diffusion processes. Bakry–Émery [8] introduced an elegant and powerful technique, based on commutator estimates for differential operators and so-called Γ*-calculus, to derive many sharp results. Their curvature-dimension condition has been further developed by many authors, mainly in the framework of Markov diffusion modelled on weighted Riemannian manifolds, with relevant applications to infinite dimensional problems.*

A new synthetic approach relying on entropy and optimal transport has been more recently introduced by Lott, Sturm and Villani. It relies structurally on the notions of distance and measure and can therefore be used to extend the curvature-dimension condition to the general nonsmooth setting of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$*. Among its many beautiful properties, the synthetic approach is stable with respect to measured Gromov convergence.*

The equivalence of the two points of view can be directly proved in a smooth differential setting but it is a difficult task in a general metric framework, when explicit calculations in local charts are hard (if not impossible) to justify.

We will try to give a brief and informal introduction to the two approaches and show how the Otto variational interpretation of the Fokker-Planck equation and the theory of metric gradient flows has provided a unifying point of view, which allows one to prove their equivalence for arbitrary metric measure spaces. As a byproduct, by combining Γ*-calculus and optimal transport techniques, an impressive list of deep results in Riemannian geometry and smooth diffusion have a natural counterpart in the nonsmooth metric measure framework.*

1 Ricci curvature and Γ-calculus in a smooth setting

In order to introduce and explain the basic notions we will deal with, we consider a smooth, complete and connected d-dimensional Riemannian manifold $(\mathbb{M}, \mathrm{g})$ endowed with its Riemannian distance d_g and a reference measure $\mathfrak{m} = \mathrm{e}^{-V}\mathrm{Vol}_\mathrm{g}$ that is absolutely continuous with respect to the Riemannian volume form Vol_g; its density is associated to the smooth potential $V : \mathbb{M} \to \mathbb{R}$. The metric tensor g induces a norm $|\nabla f|_\mathrm{g}$ of the gradient of a smooth function $f : \mathbb{M} \to \mathbb{R}$, given in local coordinates by $|\nabla f|^2_\mathrm{g} = \sum_{i,j} \mathrm{g}^{ij}\partial_i f \partial_j f$.

The combination with the reference measure $\mathfrak{m}$ gives rise to the quadratic energy form

$$\mathcal{E}(f) := \int_\mathbb{M} |\nabla f|^2_\mathrm{g} \,\mathrm{d}\mathfrak{m}, \quad \mathcal{E}(f, g) := \int_\mathbb{M} \langle \nabla f, \nabla g\rangle_\mathrm{g} \,\mathrm{d}\mathfrak{m}, \tag{1}$$

and to the second order differential operator $\mathrm{L} = \Delta_\mathrm{g} - \langle \nabla V, \cdot\rangle_\mathrm{g}$ satisfying the integration-by-parts formula

$$\mathcal{E}(f, g) = -\int_\mathbb{M} f\, \mathrm{L}g \,\mathrm{d}\mathfrak{m}.$$

L reads in local coordinates as

$$\mathrm{L}f = \mathrm{e}^W \sum_{i,j} \partial_i(\mathrm{e}^{-W}\, \mathrm{g}^{ij}\partial_j f), \quad W = V + \frac{1}{2}\log\det(\mathrm{g}^{ij}),$$

and generates a semigroup $(\mathsf{P}_t)_{t\ge0}$ through the solution $f_t = \mathsf{P}_t f$ of the diffusion equation

$$\partial_t f_t = \mathrm{L} f_t \quad \text{in } [0, \infty) \times \mathbb{M}, \quad f_0 = f. \tag{2}$$

Whenever $f_0 \in \mathrm{C}^\infty_c(\mathbb{M})$ is smooth with compact support, f is a classical C^∞ solution to (2) and one can extend L from $\mathrm{C}^\infty_c(\mathbb{M})$ to a self-adjoint operator in $L^2(X, \mathfrak{m})$ generating a Markov semigroup.

Example 1. Choosing $\mathbb{M} = \mathbb{R}^d$ with the Euclidean metric and $V \equiv 0$, one gets the Laplace operator and the corresponding heat flow. More general choices of V yield the drift-diffusion operator and the weighted Dirichlet form

$$\mathrm{L}f = \Delta f - \nabla V \cdot \nabla f, \quad \mathcal{E}(f) := \int_{\mathbb{R}^d} |\nabla f|^2\, \mathrm{e}^{-V}\,\mathrm{d}x;$$

the Gaussian space $(\mathbb{R}^d, |\cdot|, \mathfrak{m}_G)$ corresponds to the Gaussian measure $\mathfrak{m}_G$ associated to $V_G(x) := \frac{1}{2}|x|^2 + \frac{d}{2}\ln(2\pi)$ and to the Ornstein-Uhlenbeck operator $\mathrm{L}f = \Delta f - x \cdot \nabla f$.
The usual elliptic second order operators in divergence form

$$\mathrm{L}f = \sum_{i,j} \partial_i(\mathrm{g}^{ij}\partial_j f), \quad \mathcal{E}(f) = \int_{\mathbb{R}^d} \sum_{i,j} \mathrm{g}^{ij}\partial_i f \partial_j f \,\mathrm{d}x,$$

are associated to the uniformly elliptic metric tensor g^{ij} and to the potential $V := -\frac{1}{2}\log\det(\mathrm{g}^{ij})$ by the expression of the volume measure in local coordinates.
The Laplace-Beltrami operator Δ_g and the energy form

$$\mathrm{L}f = \Delta_\mathrm{g} f, \quad \mathcal{E}(f) = \int_\mathbb{M} |\nabla f|^2_\mathrm{g} \,\mathrm{dVol}_\mathrm{g},$$

in a Riemannian manifold corresponds to the choice $V \equiv 0$.

It is interesting to note that the operator L encodes the information concerning the metric tensor, which can be reconstructed by the commutation identity yielding the Γ-tensor

$$\Gamma(f,g) := \frac{1}{2}\Big(\mathrm{L}(fg) - (\mathrm{L}f)\,g - f\,\mathrm{L}g\Big), \quad \Gamma(f) := \Gamma(f,f),$$

since one easily gets

$$\Gamma(f,g) = \langle \nabla f, \nabla g\rangle_{\mathrm{g}}, \quad \Gamma(f) = |\nabla f|^2_{\mathrm{g}}.$$

Γ also characterises the class of the Lipschitz functions (and thus the Riemannian distance), since

$$\Gamma(f) \le L^2 \quad \Rightarrow \quad f \text{ is } L\text{-Lipschitz}. \tag{3}$$

Bakry–Émery [8, 9] introduced a further geometric tensor, called Γ_2, obtained by an iterated commutation:

$$\Gamma_2(f) := \frac{1}{2}\mathrm{L}\Gamma(f) - \Gamma(f, \mathrm{L}f). \tag{4}$$

Thanks to the Bochner-Lichnerowicz identity, the Γ_2 tensor can be expressed by the following remarkable formula, involving, in a crucial way, Ricci curvature and the Hessian of V:

$$\Gamma_2(f) = \left|\nabla^2 f\right|^2_{\mathrm{g}} + \mathrm{Ric}_{\mathrm{g}}(\nabla f, \nabla f) + \nabla^2_{\mathrm{g}} V(\nabla f, \nabla f).$$

According to Barky-Émery, the weighted Riemannian manifold $(\mathbb{M}, \mathsf{d}_{\mathrm{g}}, \mathfrak{m})$ satisfies the *curvature-dimension condition* $\mathrm{BE}(K,N)$, $K \in \mathbb{R}$, $N \in [1,\infty]$ if, for every smooth function f,

$$\Gamma_2(f) \ge K\,\Gamma(f) + \frac{1}{N}\left|\mathrm{L}f\right|^2. \tag{5}$$

Example 2. When $V \equiv 0$, the $\mathrm{BE}(K,N)$ condition is equivalent to $\mathrm{Ric}_{\mathrm{g}} \ge K\mathrm{g}$ and $d \le N$. In particular, Euclidean space $\mathbb{R}^d$ satisfies the $\mathrm{BE}(0,d)$ condition; the d-dimensional unit sphere $\mathbb{S}^d$ (resp. hyperbolic space $\mathbb{H}^d$) is the reference model for $\mathrm{BE}(d-1,d)$ (resp. $\mathrm{BE}(-(d-1),d)$).
When a general potential V is involved, (5) also reflects the convexity of V: in the simplest Euclidean case of $\mathbb{R}^d$ and $N=\infty$, (5) is equivalent to $\nabla^2 V \ge KI$. In particular, Gaussian space $(\mathbb{R}^d, |\cdot|, \mathfrak{m}_G)$ satisfies the $\mathrm{BE}(1,\infty)$ condition.

2 Two equivalent formulations of the curvature-dimension condition

The $\mathrm{BE}(K,N)$ condition has many deep and beautiful functional and geometric consequences (some of them are listed at the end of this paper in Section 7). Here we focus on two relevant (and seemingly far-reaching) aspects.

Pointwise gradient estimates for Markov diffusion

A first application concerns the behaviour of the semigroup $(\mathsf{P}_t)_{t\ge 0}$ associated to (2).

Theorem 3 ([8, 45])**.** The weighted manifold $(\mathbb{M}, \mathsf{d}_{\mathrm{g}}, \mathfrak{m})$ satisfies the curvature-dimension condition $\mathrm{BE}(K,N)$ if and only if, for every smooth function f with compact support and for every $t \ge 0$,

$$\mathrm{e}^{2Kt}\,|\nabla \mathsf{P}_t f|^2_{\mathrm{g}} + \frac{2}{N}E_{2K}(t)\,|\mathrm{L}\mathsf{P}_t f|^2 \le \mathsf{P}_t |\nabla f|^2_{\mathrm{g}}, \tag{6}$$

where

$$E_\lambda(t) := \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}s = \begin{cases} \lambda^{-1}(\mathrm{e}^{\lambda t}-1) & \text{if } \lambda \ne 0, \\ t & \text{if } \lambda = 0. \end{cases}$$

In the flat Euclidean case $\mathbb{M} = \mathbb{R}^d$, $V \equiv 0$, (6) with $N=\infty$ simply follows by Jensen's inequality and the commutation property of the heat equation $\nabla \mathsf{P}_t f = \mathsf{P}_t(\nabla f)$. In the general case, (6) reflects the commutator bounds coded in (5); the simplest situation is provided by the $\mathrm{BE}(K,\infty)$ case, when (6) follows (at least formally – see, for example, [9, Sec. 3.2.3]) by the monotonicity property of the quantity

$$\Lambda(s) := \mathsf{P}_s(\Gamma(\mathsf{P}_{t-s} f)), \quad 0 \le s \le t,$$

which satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}s}\Lambda(s) = 2\mathsf{P}_s(\Gamma_2(\mathsf{P}_{t-s}f)) \ge 2K\mathsf{P}_s(\Gamma(\mathsf{P}_{t-s}f)) = 2K\,\Lambda(s).$$

Brunn–Minkowski and Prékopa–Leindler inequalities

A second instance of application of curvature bounds concerns the curved version of the celebrated Brunn–Minkowski inequality

$$\mathrm{Vol}((1-\vartheta)A + \vartheta B) \ge \left(\mathrm{Vol}(A)\right)^{1-\vartheta}\left(\mathrm{Vol}(B)\right)^{\vartheta}, \quad \vartheta \in [0,1],$$

for an arbitrary couple of Borel sets $A, B \subset \mathbb{R}^d$; here, $(1-\vartheta)A + \vartheta B = \{(1-\vartheta)a + \vartheta b : a \in A, b \in B\}$.

In order to state it in a Riemannian manifold, it is convenient to denote by $Z_\vartheta(a,b)$, $a, b \in \mathbb{M}$, the set of interpolating points $x \in \mathbb{M}$ satisfying

$$\mathsf{d}_{\mathrm{g}}(a,x) = (1-\vartheta)\mathsf{d}_{\mathrm{g}}(a,b), \ \mathsf{d}_{\mathrm{g}}(x,b) = \vartheta\mathsf{d}_{\mathrm{g}}(a,b), \quad \vartheta \in [0,1].$$

Theorem 4 ([17])**.** The weighted manifold $(\mathbb{M}, \mathsf{d}_{\mathrm{g}}, \mathfrak{m})$ satisfies the curvature-dimension condition $\mathrm{BE}(K,\infty)$ if and only if, for every $\vartheta \in [0,1]$ and Borel functions $f, f_0, f_1 : \mathbb{M} \to [0,\infty)$ satisfying

$$f(x) \ge \exp\Big(-\frac{K}{2}\vartheta(1-\vartheta)\mathsf{d}^2_{\mathrm{g}}(a,b)\Big) f_0(a)^{1-\vartheta}\, f_1(b)^{\vartheta} \tag{7a}$$

whenever $a, b \in \mathbb{M}$ and $x \in Z_\vartheta(a,b)$, it holds that

$$\int_{\mathbb{M}} f\,\mathrm{d}\mathfrak{m} \ge \left(\int_{\mathbb{M}} f_0\,\mathrm{d}\mathfrak{m}\right)^{1-\vartheta}\left(\int_{\mathbb{M}} f_1\,\mathrm{d}\mathfrak{m}\right)^{\vartheta}. \tag{7b}$$

In particular, if $K \ge 0$, we have

$$\mathfrak{m}(Z_\vartheta(A,B)) \ge \left(\mathfrak{m}(A)\right)^{1-\vartheta}\left(\mathfrak{m}(B)\right)^{\vartheta}, \tag{8}$$

where

$$Z_\vartheta(A,B) := \{x \in Z_\vartheta(a,b) \quad \text{for some } a \in A,\ b \in B\}.$$

(8) admits the refined N-dimensional version,

$$\mathfrak{m}(Z_\vartheta(A,B))^{1/N} \geq \tau_{K,N}^{1-\vartheta}(\delta)\mathfrak{m}(A)^{1/N} + \tau_{K,N}^{\vartheta}(\delta)\mathfrak{m}(B)^{1/N},$$

where $\tau_{K,N}(\cdot)$ are suitable distortion coefficients only depending on K and N and δ is the minimal (resp. maximal) distance between the points of A and B if $K \geq 0$ (resp. $K < 0$).

We shall see that the optimal transport point of view provides a nice unifying interpretation of (6) and (7a,b), which will only depend on the metric d_g and on the reference measure $\mathfrak{m}$ defined on $\mathbb{M}$, without referring to its differential structure. The basic idea is to lift the geometric properties of $\mathbb{M}$ to the space of Borel probability measures $\mathcal{P}(\mathbb{M})$: (2) can be interpreted as a gradient flow in $\mathcal{P}(\mathbb{M})$ and an interpolation of sets as in (8) becomes a geodesic in $\mathcal{P}(\mathbb{M})$. In both cases, the curvature-dimension condition can be characterised by the behaviour of the entropy functional along these two classes of curves.

3 Optimal transport and the geometry of probability measures

Optimal transport provides a natural way to introduce a geometric distance between probability measures, which reflects the properties of d_g in $\mathbb{M}$. We introduce it in the more general framework of a complete and separable metric space (X, d).

We call $\mathcal{P}_2(X)$ the space of Borel probability measures with finite quadratic moment: every $\mu \in \mathcal{P}_2(X)$ satisfies

$$\int_X \mathsf{d}^2(x, x_o)\,\mathrm{d}\mu(x) < \infty$$

for some (and thus any) reference point $x_o \in X$.

For a given couple of measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, we consider the collection $\mathrm{Plan}(\mu_0, \mu_1)$ of all the *transport plans* or *couplings* between μ_0, μ_1, i.e. measures $\boldsymbol{\mu} \in \mathcal{P}(X \times X)$ with marginals μ_0, μ_1, thus satisfying

$$\boldsymbol{\mu}(A \times X) = \mu_0(A), \quad \boldsymbol{\mu}(X \times B) = \mu_1(B)$$

for every Borel subset $A, B \subset X$. The squared Kantorovich-Rubinstein-Wasserstein distance $\mathrm{W}_\mathsf{d}(\mu_0, \mu_1)$ (Wasserstein distance, for short) is then defined as

$$\mathrm{W}_\mathsf{d}^2(\mu_0,\mu_1) := \min_{\boldsymbol{\mu}\in\mathrm{Plan}(\mu_0,\mu_1)} \int_{X\times X} \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\boldsymbol{\mu}(x_0,x_1). \quad (9)$$

Equation (9) is an important example of the class of *optimal transport problems,* where the squared distance function $\mathsf{d}^2(x_0, x_1)$ in (9) is replaced by a general cost $\mathsf{c} : X \times X \to \mathbb{R}$.

W_d is a distance on $\mathcal{P}_2(X)$ inducing the topology of weak convergence with quadratic moments, i.e. convergence of all the integrals $\mu \mapsto \int_X \phi\,\mathrm{d}\mu$ whenever $\phi : X \to \mathbb{R}$ is continuous with at most quadratic growth. $(\mathcal{P}_2(X), \mathrm{W}_\mathsf{d})$ is a complete and separable metric space and it inherits other useful properties from (X, d) such as compactness or existence of geodesics. We refer the interested reader to a number of books [3, 44, 39].

The dynamical formulation of Benamou–Brenier

The Wasserstein distance W_d enjoys two other important *dynamical* characterisations, which play a crucial role in the geometric formulation of the properties discussed in Section 2.

According to the first one, which is due to Benamou–Brenier [10], the Wasserstein distance between $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{M})$ in the Riemannian manifold $\mathbb{M}$ can be evaluated by minimising the action

$$\mathrm{W}_\mathsf{d}^2(\mu_0,\mu_1) = \min \int_0^1 \int_{\mathbb{M}} |\boldsymbol{v}|_\mathrm{g}^2\,\mathrm{d}\mu_t\,\mathrm{d}t \quad (10a)$$

among all the weakly continuous solutions $(\mu, \boldsymbol{v})$ of the continuity equation

$$\partial_t\mu_t + \mathrm{div}_\mathrm{g}(\mu_t\boldsymbol{v}_t) = 0, \quad t \in [0,1], \quad (10b)$$

connecting μ_0 to μ_1. Equation (10b) has to be intended in duality with smooth test functions, i.e.

$$\int_0^1 \int_{\mathbb{M}} \Big(\partial_t\zeta(x,t) + \langle\nabla\zeta(x,t), \boldsymbol{v}(x,t)\rangle_\mathrm{g}\Big)\,\mathrm{d}\mu_t\,\mathrm{d}t = 0,$$

for every $\zeta \in \mathrm{C}_c^\infty(\mathbb{M} \times (0,1))$.

The Benamou–Brenier representation (10a,b) can be further extended to Lipschitz (or even absolutely continuous) curves $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(\mathbb{M})$: they can be characterised as solutions to the continuity equation (10b) with a Borel vector field $\boldsymbol{v}$ satisfying

$$\int_X |\boldsymbol{v}_t(x)|_\mathrm{g}^2\,\mathrm{d}\mu_t(x) = \lim_{h\to 0} \frac{\mathrm{W}_\mathsf{d}^2(\mu_t,\mu_{t+h})}{h^2} \quad (11)$$

for a.e. $t \in (0,1)$. The limit on the right side of equation (11) has a natural interpretation as the squared metric velocity $|\dot\mu_t|_{\mathrm{W}_\mathsf{d}}^2$ of μ at the time t, whose integral

$$\int_0^1 |\dot\mu_t|_{\mathrm{W}_\mathsf{d}}\,\mathrm{d}t = \int_0^1 \Big(\int_{\mathbb{M}} |\boldsymbol{v}_t|_\mathrm{g}^2\,\mathrm{d}\mu_t\Big)^{1/2}\,\mathrm{d}t$$

expresses the length of the curve in $\mathcal{P}_2(X)$.

Duality and Hamilton–Jacobi equations

The second characterisation of W_d is a dynamical version of the dual Kantorovich formulation

$$\frac{1}{2}\mathrm{W}_\mathsf{d}^2(\mu_0,\mu_1) = \sup_{\phi\in\mathrm{C}_b(X)} \int_X \mathsf{Q}_1\phi(y)\,\mathrm{d}\mu_1(x) - \int_X \phi(x)\,\mathrm{d}\mu_0(x)$$

shared with all optimal transport problems. Here, Q_1 denotes the inf-convolution

$$\mathsf{Q}_1\phi(y) := \inf_{x\in X} \frac{1}{2}\mathsf{d}^2(x,y) + \phi(x)$$

and it is the value at time $t = 1$ of the Hopf-Lax evolution $(\mathsf{Q}_t)_{t\geq 0}$, defined by

$$\mathsf{Q}_t\phi(y) := \inf_{x\in X} \frac{1}{2t}\mathsf{d}^2(x,y) + \phi(x), \quad \phi \in \mathrm{C}_b(X).$$

When (X, d) is a *length space*, i.e. for every couple of points $x, y \in X$ and $\varepsilon > 0$, there exists an ε-midpoint $z_\varepsilon \in X$ satisfying

$$\max\big(\mathsf{d}(x, z_\varepsilon), \mathsf{d}(z_\varepsilon, y)\big) \le (1/2 + \varepsilon)\,\mathsf{d}(x, y),$$

then $(\mathsf{Q}_t)_{t\ge 0}$ is a semigroup in $\mathrm{Lip}_b(X)$, the space of bounded and Lipschitz real functions defined in X. In the Euclidean case $X = \mathbb{R}^d$, $\phi_t(\cdot) := \mathsf{Q}_t\phi$ is the unique viscosity solution of the Hamilton–Jacobi equation

$$\partial_t\phi_t(x) + \frac{1}{2}|\nabla\phi_t(x)|^2 = 0, \quad \lim_{t\downarrow 0}\phi_t(x) = \phi(x),$$

and an analogous property holds in the Riemannian setting. In general metric spaces, one can give a metric interpretation to the quantity $|\nabla\phi|$ by considering the *local slope* or *local Lipschitz constant* of a map $\phi : X \to \mathbb{R}$,

$$|\nabla\phi|(x) := \limsup_{y\to x}\frac{|\phi(y) - \phi(x)|}{\mathsf{d}(y, x)}. \tag{12}$$

It is possible to prove [4] that, for every $x \in X$, $\phi_t(\cdot) := \mathsf{Q}_t\phi$ satisfies

$$\partial_t\phi_t(x) + \frac{1}{2}|\nabla\phi_t|^2(x) = 0 \tag{13}$$

for every $t \in (0, \infty)$ with at most countably many exceptions. A further regularisation in time [2] shows that

$$\frac{1}{2}\mathrm{W}_{\mathsf{d}}^2(\mu_0, \mu_1) = \sup\int_X \phi_1(y)\,\mathrm{d}\mu_1(y) - \int_X \phi_0(x)\,\mathrm{d}\mu_0(x), \tag{14a}$$

where the supremum runs among all the regular subsolutions $\phi \in \mathrm{C}^1([0, 1]; \mathrm{Lip}_b(X))$ of (13), i.e. solving

$$\partial_t\phi_t + \frac{1}{2}|\nabla\phi_t|^2(x) \le 0 \quad \text{in } X \times [0, 1]. \tag{14b}$$

4 Wasserstein distance and lower curvature bounds in smooth Riemannian manifolds

The two equivalent formulations of the curvature-dimension condition $\mathrm{BE}(K, \infty)$ in Section 2 have nice counterparts in terms of the Kantorovich-Rubinstein-Wasserstein distance. In order to keep the exposition simpler, we just focus on the case $N = \infty$.

Pointwise gradient estimates and Wasserstein contraction

The first formulation relies on the Kuwada duality result [34], which exploits the dual dynamic representation formula (14a,b) and deals with a couple of dual maps: $\mathsf{P} : \mathrm{C}_b(X) \to \mathrm{C}_b(X)$ linear and continuous and $\mathsf{P}^* : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying

$$\int_X \mathsf{P}\phi\,\mathrm{d}\mu = \int_X \phi\,\mathrm{d}(\mathsf{P}^*\mu) \quad \text{for every } \varphi \in \mathrm{C}_b(X),\ \mu \in \mathcal{P}(X).$$

Kuwada's duality states that in length metric spaces and under minimal assumptions, P preserves Lipschitz functions and satisfies the pointwise bound

$$|\nabla\mathsf{P}\phi| \le L\,\mathsf{P}|\nabla\phi| \quad \text{for every } \phi \in \mathrm{Lip}_b(X)$$

if and only if P^* is a L-Wasserstein contraction, i.e.

$$\mathrm{W}_{\mathsf{d}}(\mathsf{P}^*\mu, \mathsf{P}^*\nu) \le L\,\mathrm{W}_{\mathsf{d}}(\mu, \nu) \quad \text{for every } \mu, \nu \in \mathcal{P}_2(X).$$

In the case of the Markov semigroup $(\mathsf{P}_t)_{t\ge 0}$ introduced in Section 1 in a Riemannian setting, one gets the following theorem [38, 42].

Theorem 5. The semigroup $(\mathsf{P}_t)_{t\ge 0}$ satisfies the pointwise bound (6) of Theorem 3 with $K \in \mathbb{R}$, $N = \infty$, if and only if the semigroup $(\mathsf{P}_t^*)_{t\ge 0}$, defined on $\mathfrak{m}$-absolutely continuous measures $\mu = \varrho\mathfrak{m}$ by $\mathsf{P}_t^*\mu := (\mathsf{P}_t\varrho)\mathfrak{m}$, admits a unique extension to $\mathcal{P}_2(X)$ satisfying the contraction property

$$\mathrm{W}_{\mathsf{d}}(\mathsf{P}_t^*\mu, \mathsf{P}_t^*\nu) \le \mathrm{e}^{-Kt}\,\mathrm{W}_{\mathsf{d}}(\mu, \nu) \quad \text{for every } \mu, \nu \in \mathcal{P}_2(X). \tag{15}$$

Even more refined estimates hold in the case $N < \infty$.

Notice that one of the advantages of (15) with respect to (6) and to the differential formulation of the $\mathrm{BE}(K, \infty)$ condition given by (5) relies on the weaker regularity requirement of its formulation: it involves probability measures and the Markov semigroup $(\mathsf{P}_t)_{t\ge 0}$ but avoids local differentiability structures. This is one of the recurrent themes of pushing the geometric information coded into lower Ricci curvature bounds toward general metric measure spaces.

Prékopa–Leindler inequality and convexity of the entropy

A second crucial interpretation provides a link between the weighted Prékopa–Leindler inequality (7a,b), the geometric notion of geodesics in metric spaces and the entropy functional.

Geodesics in a metric space $(\mathcal{X}, \mathsf{d}_\mathcal{X})$ are length-minimising curves $(\mathsf{x}_\vartheta)_{\vartheta\in[0,1]}$ in $\mathcal{X}$ satisfying

$$\mathsf{d}_\mathcal{X}(\mathsf{x}_{\vartheta_0}, \mathsf{x}_{\vartheta_1}) = (\vartheta_1 - \vartheta_0)\mathsf{d}_\mathcal{X}(\mathsf{x}_0, \mathsf{x}_1), \quad 0 \le \vartheta_0 \le \vartheta_1 \le 1.$$

A real functional $\Phi : D(\Phi) \subset \mathcal{X} \to \mathbb{R}$ is called *geodesically K-convex* if every couple of points $x_0, x_1 \in D(\Phi)$ can be connected by a geodesic $(\mathsf{x}_\vartheta)_{\vartheta\in[0,1]}$ along which

$$\Phi(\mathsf{x}_\vartheta) \le (1 - \vartheta)\Phi(x_0) + \vartheta\Phi(x_1) - \frac{K}{2}\vartheta(1 - \vartheta)\mathsf{d}^2(x_0, x_1). \tag{16}$$

Since $(\mathcal{P}_2(X), \mathrm{W}_{\mathsf{d}})$ is a metric space, we can also consider geodesics at the level of probability measures: these are curves $(\mu_\vartheta)_{\vartheta\in[0,1]}$ in $\mathcal{P}_2(X)$ satisfying

$$\mathrm{W}_{\mathsf{d}}(\mu_{\vartheta_0}, \mu_{\vartheta_1}) = (\vartheta_1 - \vartheta_0)\mathrm{W}_{\mathsf{d}}(\mu_0, \mu_1), \quad 0 \le \vartheta_0 \le \vartheta_1 \le 1. \tag{17}$$

In a pioneering paper, McCann [36] pointed out the role and the interest of geodesic convexity of suitable integral functionals in $\mathcal{P}_2(X)$ as the *relative entropy*,

$$\mathrm{Ent}(\mu) := \begin{cases} \int_X \varrho\log\varrho\,\mathrm{d}\mathfrak{m} & \text{if } \mu = \varrho\mathfrak{m},\ \varrho|\log\varrho| \in L^1(X, \mathfrak{m}),\\ +\infty & \text{otherwise.}\end{cases}$$

As a beautiful example, Theorem 4 admits a nice reformulation in terms of its geodesic K-convexity.

Theorem 6 ([37, 18])**.** The Prékopa–Leindler inequality (7a, b) holds in the weighted Riemannian manifold $(\mathbb{M}, \mathsf{d}_\mathrm{g}, \mathfrak{m})$ if and only if the functional Ent is geodesically K-convex in $\mathcal{P}_2(\mathbb{M})$. In this case, for every $(\mu_\vartheta)_{\vartheta\in[0,1]}$ satisfying (17), we have

$$\mathrm{Ent}(\mu_\vartheta) \le (1 - \vartheta)\mathrm{Ent}(\mu_0) + \vartheta\mathrm{Ent}(\mu_1) - \frac{K}{2}\vartheta(1 - \vartheta)\mathrm{W}_{\mathsf{d}}(\mu_0, \mu_1). \tag{18}$$

Gradient flows in $\mathcal{P}_2(X)$: convexity and contraction

The two remarkable properties highlighted by Theorems 5 and 6 can be better understood by the Jordan-Otto interpretation of the semigroup P^* as the gradient flow of the entropy functional in Wasserstein space $\mathcal{P}_2(\mathbb{M})$. There are (at least) three different ways to justify this interpretation. The first one combines the Benamou–Brenier result with the De Giorgi notion of curves of maximal slope [3], the second one is related to the so-called *JKO/minimising movement* variational scheme [20, 31, 3] and the last one captures the energy-distance interaction of convex functions in Euclidean space.

Curves of maximal slope.

One starts from the basic remark that solutions in $\mathbb{R}^d$ to the gradient flow equation

$$\mathsf{x}'(t) = -\nabla\Phi(\mathsf{x}(t)) \tag{19}$$

for a smooth function $\Phi : \mathbb{R}^d \to \mathbb{R}$ can be characterised by the maximal rate decay of Φ, in the sense that along a general smooth curve y, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathsf{y}(t)) = -\nabla\Phi(\mathsf{y}(t))\cdot\mathsf{y}'(t) \geq -|\nabla\Phi(\mathsf{y}(t))|\,|\mathsf{y}'(t)|, \tag{20}$$

whereas along any solution of (19), we have precisely

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathsf{x}(t)) = -|\nabla\Phi(\mathsf{x}(t)|\,|\mathsf{x}'(t)| = -|\nabla\Phi(\mathsf{x}(t))|^2 = -|\mathsf{x}'(t)|^2, \tag{21}$$

an identity that is, in fact, equivalent to (19).

In order to mimic the above argument at the Wasserstein level, we first observe that, along any smooth solution $\mu_t = \varrho_t\mathfrak{m}$ of the continuity equation (10b), it is possible to compute the time derivative of the entropy by an integration by parts, obtaining

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}(\mu_t) &= \int_{\mathbb{M}}\langle\nabla\log\varrho_t, \boldsymbol{v}_t\rangle\,\mathrm{d}\mu_t\\ &\geq -\Big(\int_{\mathbb{M}}|\nabla\log\varrho_t|_{\mathsf{g}}^2\,\mathrm{d}\mu_t\Big)^{1/2}\Big(\int_{\mathbb{M}}|\boldsymbol{v}_t|_{\mathsf{g}}^2\,\mathrm{d}\mu_t\Big)^{1/2}.\end{aligned}$$

Thus, the rate of decay of Ent is bounded from below by the product of the Wasserstein velocity of the curve (11) and the square root of the Fisher information functional

$$\begin{aligned}\mathrm{F}(\mu) &= \int_{\varrho>0}|\nabla\log\varrho|_{\mathsf{g}}^2\,\mathrm{d}\mu_t = \int_{\varrho>0}\frac{|\nabla\varrho|_{\mathsf{g}}^2}{\varrho}\,\mathrm{d}\mathfrak{m}\\ &= 4\int_{\mathbb{M}}|\nabla\sqrt{\varrho}|_{\mathsf{g}}^2\,\mathrm{d}\mathfrak{m}, \quad \mu = \varrho\mathfrak{m},\end{aligned} \tag{22}$$

which plays the same role as $|\nabla\Phi|$ in (20). On the other hand, along the solution $\mu_t = \mathsf{P}_t^*\mu = (\mathsf{P}_t\varrho)\mathfrak{m}$ induced by (2), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}(\mu_t) = -\mathrm{F}(\mu_t) = -|\dot{\mu}_t|_{\mathrm{W}_{\mathsf{d}}}^2, \tag{23}$$

which corresponds to (21).

The minimising movement approach.

A second point of view (and, in fact, the original one adopted by Jordan–Kinderlehrer–Otto [31]) concerns a variational scheme that can be used to construct gradient flows in general metric spaces [20, 3]. It is strongly related to the implicit Euler scheme for (19), which suggests approximating the values of the solution x at the discrete points $k\tau$, $k \in \mathbb{N}$, of a uniform grid of step $\tau > 0$ by the solutions $\mathsf{X}_\tau^k \approx \mathsf{x}(k\tau)$ of the recursive discrete scheme

$$\frac{1}{\tau}(\mathsf{X}_\tau^k - \mathsf{X}_\tau^{k-1}) + \nabla\Phi(\mathsf{X}_\tau^k) = 0, \quad k = 1, 2, \cdots, \tag{24}$$

starting from an approximation $\mathsf{X}_\tau^0 \approx \mathsf{x}(0)$. One can select solutions to (24) by choosing X_τ^k among the minimisers of

$$\mathsf{X} \mapsto \frac{1}{2\tau}|\mathsf{X} - \mathsf{X}_\tau^{k-1}|^2 + \Phi(\mathsf{X}).$$

At the level of measures, replacing the Euclidean distance in $\mathbb{R}^d$ by the Wasserstein distance in $\mathcal{P}_2(X)$ and the function Φ by the relative entropy, one eventually obtains the following scheme.

$$\begin{aligned}&\text{Given } M_\tau^0 := \mu = \varrho\mathfrak{m}, \text{ find } M_\tau^k \in \mathcal{P}_2(X) \text{ by minimising}\\ &M \mapsto \frac{1}{2\tau}\mathrm{W}_{\mathsf{d}}^2(M, M_\tau^{k-1}) + \mathrm{Ent}(M), \quad k = 1, 2, \cdots.\end{aligned}$$

Defining $M_\tau(t)$ as the piecewise constant interpolant of the values M_τ^k in each interval $((k-1)\tau, k\tau]$, one can prove, under very general assumptions [31, 3, 4], that the family M_τ converges locally uniformly in $\mathcal{P}_2(X)$ to the solution $\mu_t = \mathsf{P}_t^*\mu = (\mathsf{P}_t\varrho)\mathfrak{m}$ associated to (2).

Gradient flows and evolution variational inequalities.

In the Euclidean framework, one can evaluate the derivative of the squared distance function of a solution x of (19) from any given point $y \in \mathbb{R}^d$, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|\mathsf{x}(t) - y|^2 = \langle\nabla\Phi(\mathsf{x}(t), y - \mathsf{x}(t)\rangle,$$

and then use the K-convexity inequality

$$\langle\nabla\Phi(x), y - x\rangle \leq \Phi(y) - \Phi(x) - \frac{K}{2}|x - y|^2$$

to obtain the Evolution Variational Inequality (EVI_K)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|\mathsf{x}(t) - y|^2 \leq \Phi(y) - \Phi(\mathsf{x}(t)) - \frac{K}{2}|\mathsf{x}(t) - y|^2 \quad \forall y \in \mathbb{R}^d. \tag{25}$$

It is not hard to see that (25), in fact, characterises the solutions to (19). Since (25) just involves the ambient distance and the driving functional Φ, one is tempted to use it for a possible definition of gradient flows in arbitrary metric spaces [3].

Definition 7. Let $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ be a metric space and let $\Phi : D(\Phi) \subset \mathcal{X} \to \mathbb{R}$ be a given l.s.c. functional. A semigroup $(\mathsf{S}_t)_{t\geq 0}$ in $D(\Phi)$ is an EVI_K-flow of Φ if, for every $x \in D(\Phi)$, the curve $\mathsf{x}_t := \mathsf{S}_t x$ is locally Lipschitz in $(0, \infty)$ and solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}_{\mathcal{X}}^2(\mathsf{x}(t), y) \leq \Phi(y) - \Phi(\mathsf{x}(t)) - \frac{K}{2}\mathsf{d}_{\mathcal{X}}^2(\mathsf{x}(t), y) \quad \forall\, y \in \mathcal{X}.$$

In general metric spaces, the existence of an EVI_K-flow is a much stronger requirement than the simple energy-dissipation identity (21): it encodes both the K-convexity of Φ and a sort of infinitesimal Riemannian behaviour of the distance d_X [19].

Theorem 8. Suppose that $(\mathsf{S}_t)_{t\ge 0}$ is an EVI_K flow for Φ on (X, d_X). Then, S is a K-exponential contraction, i.e.

$$\mathsf{d}_X(\mathsf{S}_t x, \mathsf{S}_t y) \le \mathrm{e}^{-Kt}\mathsf{d}_X(x, y).$$

If, moreover, any couple of points $x, y \in D(\Phi)$ can be joined by a geodesic in $\overline{D(\Phi)}$ then Φ is strongly K-geodesically convex.

Taking the above result and Otto heuristics concerning the Riemannian character of the Wasserstein distance into account, it is not completely surprising that Theorems 5 and 6 can be obtained as a consequence of the EVI_K characterisation of the Markov semigroup P^* in $\mathcal{P}_2(\mathbb{M})$. In fact, the weighted Riemannian manifold $(\mathbb{M}, \mathsf{d}_g, \mathfrak{m})$ satisfies the curvature-dimension $\mathrm{BE}(K,\infty)$ condition if and only if the semigroup $(\mathsf{P}^*_t)_{t\ge 0}$ is the EVI_K flow of the entropy functional in $\mathcal{P}_2(X)$: for every $\mu \in D(\mathrm{Ent})$, the curve $\mu_t = \mathsf{P}^*_t\mu$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathrm{W}^2_{\mathsf{d}}(\mu_t, \nu) \le \mathrm{Ent}(\nu) - \mathrm{Ent}(\mu_t) - \frac{K}{2}\mathrm{W}^2_{\mathsf{d}}(\mu_t, \nu) \qquad (26)$$

for every $\nu \in D(\mathrm{Ent})$.

5 Metric measure spaces, Gromov convergence and the Lott–Sturm–Villani curvature-dimension condition

Let us now address the question of how to extend the previous results to a general metric measure space $(X, \mathsf{d}, \mathfrak{m})$, i.e. a complete and separable metric space (X, d) endowed with a non-negative Borel measure $\mathfrak{m}$, which we assume here to be with full support, satisfying the growth condition

$$\mathfrak{m}(B_r(x_o)) \le a\,\mathrm{e}^{b\,r^2}, \quad a, b \in \mathbb{R}_+. \qquad (27)$$

This general class of spaces naturally arises when lower Ricci curvature bounds are considered, thanks to the following remarkable result of Gromov (see, for example, [30]) and to the deep contributions by Cheeger-Colding [14, 15, 16].

Theorem 9 (Gromov compactness)**.** Let $(\mathbb{M}^h, \mathsf{d}^h_g, \mathfrak{m}^h)$, $h \in \mathbb{N}$, be a sequence of weighted Riemannian manifolds with uniformly bounded diameter and satisfying a uniform $\mathrm{BE}(K, N)$ condition, for some $K, N \in \mathbb{R}$ independent of h. Then, there exist a limit metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and a subsequence $n \mapsto h(n)$ such that $(\mathbb{M}^{h(n)}, \mathsf{d}^{h(n)}_g, \mathfrak{m}^{h(n)})$ converges to $(X, \mathsf{d}, \mathfrak{m})$ under the measured Gromov convergence.

Perhaps the simplest way to introduce measured Gromov convergence for normalised (i.e. $\mathfrak{m}(X) = 1$) metric measure spaces is to resort to another beautiful theorem of Gromov [30], characterising the equivalence class of metric measure spaces up to measure-preserving isometries. It relies on the notion of cylindrical metric functionals of the form

$$\varphi^\star[(X, \mathsf{d}, \mathfrak{m})] := \int_{X^n} \varphi\big(\mathsf{d}(x_i, x_j)^n_{i,j=1}\big)\,\mathrm{d}\mathfrak{m}^{\otimes n}(x_1, \cdots, x_n), \quad (28)$$

where $n \in \mathbb{N}$ and $\varphi \in \mathrm{C}_b(\mathbb{R}^{n\times n})$.

It is clear that if two metric measure spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$, $i = 1, 2$, are isomorphic, i.e. there exists an isometry $\iota : X_1 \to X_2$ preserving distances and volumes

$$\mathsf{d}_2(\iota(x), \iota(y)) = \mathsf{d}_1(x, y), \quad \mathfrak{m}_1(\iota^{-1}(A)) = \mathfrak{m}_2(A)$$

for every couple of points $x, y \in X_1$ and every Borel set $A \subset X_2$, then $\varphi^\star[(X_1, \mathsf{d}_1, \mathfrak{m}_1)] = \varphi^\star[(X_1, \mathsf{d}_2, \mathfrak{m}_2)]$ for every cylindrical functional $\varphi^\star$ as in (28). The Gromov reconstruction theorem guarantees the converse property: if two normalised metric measure spaces are indistinguishable by all the cylindrical functionals then they are isomorphic. It justifies the following definition [30, 28] (see [40, 44, 27] for other equivalent approaches and for the relation with measured Gromov-Hausdorff convergence).

Definition 10 (Measured Gromov convergence)**.** A sequence of normalised metric measure spaces $(X^h, \mathsf{d}^h, \mathfrak{m}^h)$, $h \in \mathbb{N}$, converges to a limit metric measure space $(X, \mathsf{d}, \mathfrak{m})$ if, for every cylindrical functional $\varphi^\star$ as in (28), we have

$$\lim_{h\to\infty} \varphi^\star[(X^h, \mathsf{d}^h, \mathfrak{m}^h)] = \varphi^\star[(X, \mathsf{d}, \mathfrak{m})].$$

In view of Theorem 9, it is natural to look for a synthetic definition of lower Ricci curvature bounds for general metric measure spaces that is stable under measured Gromov convergence, a programme that has been outlined in [14, Appendix 2]. Such a definition has been independently introduced by Sturm [40, 41] and Lott–Villani [35], starting from the smooth characterisation given by Theorem 6.

Definition 11 (The Lott–Sturm–Villani $\mathrm{CD}(K,\infty)$ condition)**.** A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{CD}(K,\infty)$ condition if the entropy functional Ent is geodesically K-convex in $(\mathcal{P}_2(X), \mathrm{W}_{\mathsf{d}})$: every couple $\mu_0, \mu_1 \in D(\mathrm{Ent})$ can be connected by a geodesic $(\mu_\vartheta)_{\vartheta\in[0,1]}$ satisfying (17) along which (18) holds.

A similar but technically more complicated notion can be introduced in the case of a finite dimension upper bound $N < \infty$ (and we do not distinguish here between CD, CD^* or CD^e classes of spaces). Besides its intrinsic geometric structure, just involving the notion of distance (through Wasserstein geodesics) and measure (through the entropy functional), a crucial feature of the above definition is its stability with respect to measured Gromov convergence: if $(X^h, \mathsf{d}^h, \mathfrak{m}^h)$, $h \in \mathbb{N}$, is a sequence of $\mathrm{CD}(K, N)$ metric measure spaces converging to $(X, \mathsf{d}, \mathfrak{m})$ in the measured Gromov topology then $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{CD}(K, N)$ metric measure space.

It is possible to prove that $\mathrm{CD}(K, N)$ metric measure spaces enjoy many of the geometric properties that are a consequence of the curvature-dimension condition in the smooth Riemannian setting (see the final remarks below).

Definition 11, however, captures only part of the information coded in the Riemannian formalism, since geodesic K-convexity of the entropy functional is also shared by Finsler (non-Riemannian) geometries: perhaps the simplest example is given by the space $(\mathbb{R}^d, \|\cdot\|, \mathrm{Vol})$, where the distance is induced by an arbitrary norm $\|\cdot\|$ (not necessarily Hilbertian) and the measure is the usual Lebesgue one.

On the other hand, in many examples where a specific Dirichlet energy form and a corresponding Markov semigroup can be constructed, one may adopt the Bakry–Émery point of view to characterise a curvature-dimension condition. In this direction, Gromov wrote [29, page 85]: *"There is another option for the abstract theory of Ricci ≥ 0, where instead of the metric one emphasizes the heat flow (diffusion), but at this stage it is unclear whether the two approaches are equivalent and if not which one is better for applications."*

A relevant question is to single out a stronger condition for general metric measure spaces that is still stable under measured Gromov convergence and is capable of reproducing the pointwise gradient estimates (6) along a suitably adapted version of the heat flow. This would have the great advantage of combining both the tools from Γ-calculus and optimal transport and hopefully extending to the non-smooth framework many deep results and techniques available in the Riemannian setting. As explained in Section 4, the Wasserstein gradient flow of the entropy functional provides a unifying point of view for both the approaches and keeps the basic feature of a pure geometric formulation in terms of distance and measure.

6 RCD(K,∞) spaces and the link between BE and CD

The Cheeger energy and its L^2-gradient flow

A first step in the direction of extending the Bakry–Émery approach to the setting of metric measure spaces concerns the construction of a canonical energy form, the so called *Cheeger energy* [13], and of the related evolution semigroup.

The Cheeger energy can be obtained by a relaxation procedure from the functional $\int_X |\nabla f|^2(x)\,\mathrm{d}\mathfrak{m}(x)$, initially defined on bounded Lipschitz functions and involving the local slope introduced in (12).

Definition 12 (The Cheeger energy)**.** For every $f \in L^2(X,\mathfrak{m})$, we define

$$\mathrm{Ch}(f) := \inf\Big\{ \liminf_{n\to\infty} \frac{1}{2}\int_X |\nabla f_n|^2\,\mathrm{d}\mathfrak{m} : \; f_n \in \mathrm{Lip}_b(X),\; f_n \xrightarrow{L^2} f\Big\}, \tag{29}$$

with proper domain $D(\mathrm{Ch}) := \{f \in L^2(X,\mathfrak{m}) : \mathrm{Ch}(f) < \infty\}$.

It is possible to prove that Ch is a convex, 2-homogeneous, lower semicontinuous functional in $L^2(X,\mathfrak{m})$ with a dense domain. For every $f \in D(\mathrm{Ch})$, there exists at least one optimal sequence $(f_n)_n \subset \mathrm{Lip}_b(X)$ converging to f in $L^2(X,\mathfrak{m})$ and realising the infimum in (29): the corresponding slopes $|\nabla f_n|$ converge strongly in $L^2(X,\mathfrak{m})$ to a unique limit that is called the *weak gradient of f* and is denoted by $|\nabla f|_w$. The map $f \mapsto |\nabla f|_w$ is 1-homogeneous and subadditive, enjoys some natural calculus rules [4, 26] and represents Ch by the formula

$$\mathrm{Ch}(f) = \frac{1}{2}\int_X |\nabla f|_w^2\,\mathrm{d}\mathfrak{m},$$

which can also be useful to define the Fisher information $\mathrm{F}(\mu)$ of a nonnegative measure $\mu = \varrho\mathfrak{m}$ as in (22):

$$\mathrm{F}(\mu) = 8\,\mathrm{Ch}(\sqrt{\varrho}\,) = \int_{\varrho>0} \frac{|\nabla\varrho|_w^2}{\varrho}\,\mathrm{d}\mathfrak{m}, \quad \mu = \varrho\mathfrak{m}. \tag{30}$$

Even if Ch is not a quadratic form, it is still possible to use convex analysis to define the nonlinear Laplacian $-\Delta_X f$ as the element of minimal L^2-norm of its L^2-subdifferential, consisting of all the functions $\xi \in L^2(X,\mathfrak{m})$ satisfying the variational inequality

$$\int_X \xi(g-f)\,\mathrm{d}\mathfrak{m} \leq \mathrm{Ch}(g) - \mathrm{Ch}(f) \quad \text{for every } g \in D(\mathrm{Ch}).$$

It is a remarkable result of the theory of gradient flows in Hilbert spaces that, for every $f \in L^2(X,\mathfrak{m})$, there exists a unique locally Lipschitz curve $(f_t)_{t>0}$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \Delta_X f_t \quad \text{for a.e. } t>0, \quad \lim_{t\downarrow 0} f_t = f. \tag{31}$$

The map $\mathsf{P}_t : f \mapsto f_t$ defines a continuous semigroup of contractions in $L^2(X,\mathfrak{m})$; by the specific property of Ch, $(\mathsf{P}_t)_{t\geq 0}$ can also be extended to a semigroup of contractions in every $L^p(X,\mathfrak{m})$, preserving positivity, mass and constants.

It is then possible to prove, in many cases, that the semigroup $(\mathsf{P}_t)_{t\geq 0}$ coincides with the Wasserstein gradient flow of the entropy functional (as a curve of maximal slope and as a limit of the minimising movement variational scheme): this important identification holds, in particular, for the whole class of CD(K,∞) metric measure spaces [24, 4].

Theorem 13. If $(X,\mathsf{d},\mathfrak{m})$ is a CD(K,∞) space then, for every $\mu = \varrho\mathfrak{m} \in \mathcal{P}_2(X)$ with $\mathrm{Ent}(\mu) < \infty$, the curve $\mu_t = (\mathsf{P}_t\varrho)\mathfrak{m}$ is locally Lipschitz in $\mathcal{P}_2(X)$ and the map $t \mapsto \mathrm{Ent}(\mu_t)$ is locally Lipschitz and satisfies (23); it is, moreover, the limit of the minimising movement scheme (30). Conversely, any locally Lipschitz curve $\mu_t = \varrho_t\mathfrak{m}$, $t \geq 0$, in $\mathcal{P}_2(X)$ solving (23) satisfies $\varrho_t = \mathsf{P}_t\varrho_0$.

Quadratic Cheeger energies and a metric setting for the Bakry–Émery approach

If one looks for a good metric framework where the Bakry–Émery approach can be applied, there are at least two essential properties: the linearity of P_t and the link between distance and energy. Since P_t is originally defined as the gradient flow of the Cheeger energy in Hilbert space $L^2(X,\mathfrak{m})$, it is not surprising that the linearity of P_t is related to the quadraticity of Ch; as a byproduct, it induces a nice connection between weak gradients and Γ-calculus.

Theorem 14 (Cheeger energy, Dirichlet forms and Γ)**.** The semigroup $(\mathsf{P}_t)_{t\geq 0}$ is linear if and only if the Cheeger energy is quadratic, i.e. for every $f, g \in D(\mathrm{Ch})$,

$$\mathrm{Ch}(f+g) + \mathrm{Ch}(f-g) = 2\,\mathrm{Ch}(f) + 2\,\mathrm{Ch}(g). \tag{32}$$

In this case, $\mathcal{E}(f,g) := \mathrm{Ch}(f+g) - \mathrm{Ch}(f) - \mathrm{Ch}(g)$ is a strongly local Dirichlet form, whose Γ-tensor coincides with the weak gradient and whose generator L coincides with Δ_X:

$$\Gamma(f) = |\nabla f|_w^2 \text{ if } f \in D(\mathrm{Ch}), \quad \mathrm{L}f = \Delta_X f \text{ if } f \in D(\Delta_X).$$

In the metric setting, a nice collection of smooth functions where the pointwise differential formulation of the Bakry–Émery condition (5) can be stated is lacking; however, it is possible to give a suitable weak formulation that is still equivalent to the pointwise gradient estimate (6): e.g. the $\mathrm{BE}(K,\infty)$ condition is equivalent to asking that the map

$$s \mapsto \mathrm{e}^{-2Ks}\int_X \Gamma(\mathsf{P}_{t-s}f)\mathsf{P}_s\phi\,\mathrm{d}\mathfrak{m}$$

is increasing in $(0,t)$ for every $f \in D(\mathrm{Ch})$ and every nonnegative $\phi \in L^\infty(X,\mathfrak{m})$.

Concerning the link with distance, the very definition of Cheeger energy shows that every bounded L-Lipschitz function f satisfies

$$\Gamma(f) = |\nabla f|_w^2 \le L \quad \mathfrak{m}\text{-a.e.} \tag{33}$$

In order to infer geometric properties on (X,d) from the energy form, it is natural to ask that every function $f \in D(\mathrm{Ch})$ satisfying (33) admits an L-Lipschitz representative.

The $\mathrm{RCD}(K,\infty)$ condition and the entropic EVI_K-flow

Summarising the discussion above for a general metric measure space with a quadratic Cheeger energy, it is possible to ask for the Lott–Sturm–Villani $\mathrm{CD}(K,\infty)$ condition or for the Bakry–Émery condition $\mathrm{BE}(K,\infty)$. It turns out that these are, in fact, equivalent and can be unified by the notion of EVI_K flow [4, 5, 6].

Theorem 15. For a general metric measure space $(X,\mathsf{d},\mathfrak{m})$, the following properties are equivalent:

(1) The Cheeger energy is quadratic according to (32) (and thus $(\mathsf{P}_t)_{t\ge 0}$ is linear) and $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{CD}(K,\infty)$ space.

(2) The Cheeger energy is quadratic according to (32), every function satisfying (33) is L-Lipschitz and the Bakry–Émery condition holds (in a suitably weak formulation).

(3) The entropy functional Ent admits a EVI_K flow according to Definition (7).

This result leads to the following definition.

Definition 16 (The Riemannian curvature-dimension condition)**.** A metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathrm{RCD}(K,\infty)$ condition if one of the equivalent properties of Theorem 15 is satisfied.

The theorem above has been remarkably extended to the case of the finite dimension condition by Erbar-Kuwada-Sturm [21] by introducing a suitable notion of $\mathrm{EVI}_{K,N}$ flow for the entropy power functional

$$\mathrm{H}_N(\mu) := \exp\Big(-\frac{1}{N}\mathrm{Ent}(\mu)\Big).$$

A different approach, using Rény entropies in the original formulation of the Lott–Sturm–Villani condition, has also been developed by [1]. A crucial result due to the formulation in terms of entropy and Wasserstein distance is the following stability property with respect to measured Gromov convergence [6, 27].

Theorem 17. If $(X^h,\mathsf{d}^h,\mathfrak{m}^h)$, $h \in \mathbb{N}$, is a sequence of $\mathrm{RCD}(K,N)$ metric measure spaces converging to $(X,\mathsf{d},\mathfrak{m})$ in the measured Gromov topology then $(X,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}(K,N)$ metric measure space. Moreover, if the diameters of X^h are uniformly bounded and $\lambda_k(\mathrm{L}^h)$, $k \in \mathbb{N}$, are the ordered sequences of eigenvalues of the compact operator L^h, we have

$$\lim_{h\to\infty}\lambda_k(\mathrm{L}^h) = \lambda_k(\mathrm{L}) \quad \text{for every } k \in \mathbb{N}.$$

7 Applications

It is really difficult to give even a partial account of the ongoing and striking developments of the metric theory of CD and RCD spaces. Both are sufficiently flexible and strong to guarantee a series of structural geometric results: among them, we quote the tensorisation property, the global-to-local and local-to-global characterisations of the CD/RCD conditions and the development of a nice first and second order calculus [26].

We now recall some of the most important geometric and functional analytic estimates (often stated in particular exemplifying cases) that can be derived for a general metric measure space $(X,\mathsf{d},\mathfrak{m})$. We start from the properties valid for all $\mathrm{CD}(K,N)$ spaces, where the recent results of Cavalletti-Mondino [11, 12] solve a series of important open problems and show the power of the optimal transport approach (in the RCD framework, they can also be deduced by Γ-calculus tools – see [9]).

Bishop–Gromov inequality: For $x_0 \in X$, the map

$$r \mapsto \frac{\mathfrak{m}(B_r(x_0))}{\int_0^r s^{K,N}(t)\,\mathrm{d}t} \quad \text{is nonincreasing,}$$

where $s^{K,N}$ is the function providing the measure of the spheres in the model space of Ricci curvature K and dimension N [44].

Bonnet–Myers diameter estimate: If $K > 0$ then the diameter of X is bounded by $\pi\sqrt{(N-1)/K}$.

Spectral gap and Poincaré inequality: If $K > 0$ then

$$\int_X (f-\bar f)^2\,\mathrm{d}\mathfrak{m} \le \frac{N-1}{NK}\int_X |\nabla f|_w^2\,\mathrm{d}\mathfrak{m}, \quad \bar f = \int_X f\,\mathrm{d}\mathfrak{m},$$

and a sharp inequality also holds for L^p with $p \ne 2$ [12].

Log–Sobolev and Talagrand inequalities: If $K > 0$ and $\mathfrak{m}(X) = 1$ then [12]

$$\frac{KN}{2(N-1)}\mathrm{W}_{\mathsf{d}}^2(\mu,\mathfrak{m}) \le \mathrm{Ent}(\mu) \le \frac{N-1}{2KN}\mathrm{F}(\mu).$$

Sharp Sobolev inequalities: If $K > 0$, $N > 2$, $2 < p \le 2^\star := 2N/(N-2)$ then [12]

$$\|f\|_{L^p}^2 \le \|f\|_{L^2}^2 + \frac{(p-2)(N-1)}{KN}\int_X |\nabla f|_w^2\,\mathrm{d}\mathfrak{m}.$$

Levy–Gromov inequality: If $\mathfrak{m}(X) = 1$, $\mathrm{diam}(X) = D$ and $A \subset X$ with perimeter $P(A) < \infty$ then

$$P(A) \ge \mathcal{I}_{K,N,D}(\mathfrak{m}(A)),$$

where $\mathcal{I}$ is a suitably defined model isoperimetric profile for the parameters K, N, D (such as the N-dimensional sphere, when N is an integer and $K > 0$). The case $N = \infty$ holds in RCD spaces [9, Cor. 8.5.5], [7].

Let us now consider the specific case of $\mathrm{RCD}(K, N)$ spaces, where $(\mathsf{P}_t)_{t\geq 0}$ is a linear Markov semigroup associated to a Markov process and second order calculus tools can also be developed [22].

Li-Yau and Harnack inequalities: If $K \geq 0$ and $N < \infty$ then [9, Cor. 6.7.6]

$$\mathrm{L}(\log \mathsf{P}_t f) \geq -\frac{N}{2t} \quad t > 0,$$
$$\mathsf{P}_t f(x) \leq \mathsf{P}_{t+s} f(y) \left(\frac{t+s}{t}\right)^{N/2} \mathrm{e}^{\mathsf{d}^2(x,y)/2}.$$

The splitting theorem [25]: If $K \geq 0$, $N \in [2, \infty)$ and X contains a line, i.e. there exists a map $\gamma : \mathbb{R} \to X$ such that $\mathsf{d}(\gamma(s), \gamma(t)) = |t - s|$ for every $s, t \in \mathbb{R}$, then $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to the product of $\mathbb{R}$ (with Euclidean distance and the usual Lebesgue measure) and a $\mathrm{RCD}(0, N-1)$ space.

The maximal diameter theorem [32]: If $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{RCD}(N, N+1)$ condition with $N > 0$ and there exists points $x, y \in X$ such that $\mathsf{d}(x, y) = \pi$ then $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to the spherical product of $[0, \pi]$ and a $\mathrm{RCD}(N-1, N)$ space with diameter less than π.

Volume-to-metric cones [23]: If $K = 0$, there exists $x_o \in X$ such that $\mathfrak{m}(B_r(x_o)) = (R/r)^N \mathfrak{m}(B_r(x_o))$ for some $R > r > 0$ and the sphere centred at x_0 of radius $R/2$ contains at least three points then the ball $B_R(x_o)$ is locally isometric to the ball $B_R(y_o)$ of the cone Y built over an $\mathrm{RCD}(N-2, N-1)$ space.

We conclude this brief review by noting that there have been some recent striking applications to time-dependent metric measure spaces and Ricci flows [43, 33].

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Claude Shannon: His Work and Its Legacy[1]

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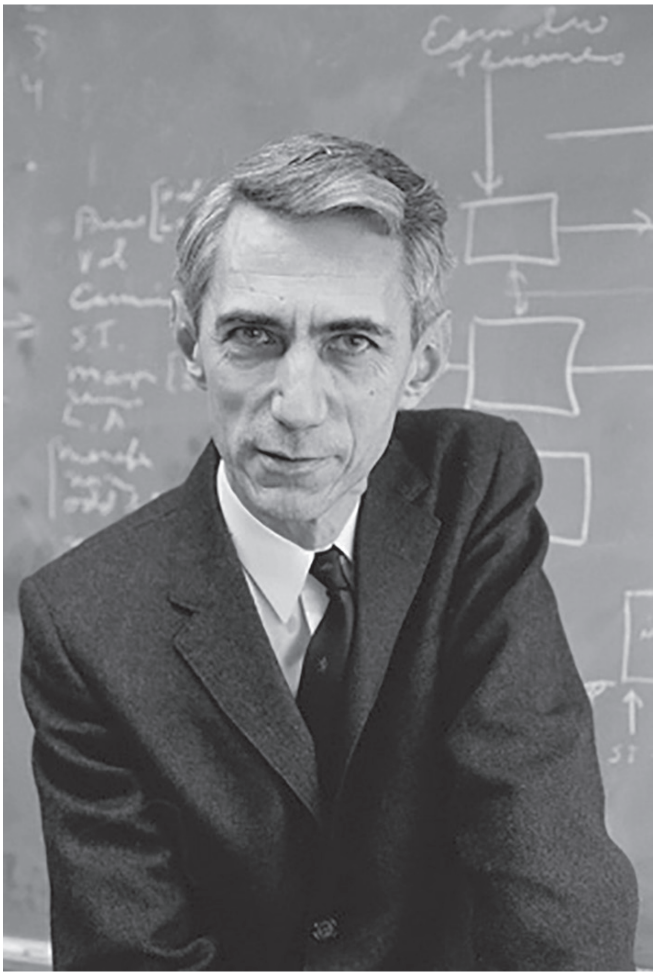
The year 2016 marked the centennial of the birth of Claude Elwood Shannon, that singular genius whose fertile mind gave birth to the field of information theory. In addition to providing a source of elegant and intriguing mathematical problems, this field has also had a profound impact on other fields of science and engineering, notably communications and computing, among many others. While the life of this remarkable man has been recounted elsewhere, in this article we seek to provide an overview of his major scientific contributions and their legacy in today's world. This is both an enviable and an unenviable task. It is enviable, of course, because it is a wonderful story; it is unenviable because it would take volumes to give this subject its due. Nevertheless, in the hope of providing the reader with an appreciation of the extent and impact of Shannon's major works, we shall try.

To approach this task, we have divided Shannon's work into 10 topical areas:

- Channel capacity
- Channel coding
- Multiuser channels
- Network coding
- Source coding
- Detection and hypothesis testing
- Learning and big data
- Complexity and combinatorics
- Secrecy
- Applications

We will describe each one briefly, both in terms of Shannon's own contribution and in terms of how the concepts initiated by Shannon have influenced work in the intervening decades. By necessity, we will take a minimalist approach in this discussion. We offer apologies for the many topics and aspects of these problems that we must necessarily omit.



Channel capacity

By Shannon's own characterisation: "The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point." The channel is the medium – wire, cable, air, water, etc. – through which that communication occurs. Often, the channel transmits information in a way that is noisy or imperfect. The notion that truly reliable communication is possible even in the face of noise and the demonstration that a channel has an inherent maximal rate at which it can reliably deliver information are, arguably, Shannon's most important contributions to the field. These concepts were first expounded in his foundational 1948 paper. He developed these ideas further throughout the 1950s and even into the 1960s, examining the capacity of particular channels, looking at the effects of feedback and other features of existing communication networks and also, because capacity in his vision is an asymptotic quantity, looking at ways in which that asymptote is achieved.

Since Shannon's original work, the notion of capacity has evolved in several directions. For example, the traditional notion of capacity has been generalised to remove many of Shannon's original simplifying assumptions. In addition, the notion of capacity has been expanded to capture other notions of communication. For example, identification capacity was introduced to measure the

[1] This paper was adapted from a talk presented at The Bell Labs Shannon Conference on the Future of the Information Age, Murray Hill, NJ, 28–29 April 2016, celebrating the occasion of the Shannon centennial.

capacity when one is only interested in knowing when a message is present, not necessarily what is in the message, computation capacity was introduced to measure how many computations are possible in certain circumstances, and so on. Capacity has also been applied to many types of channels that have emerged since Shannon's day. Examples include quantum channels, which include both quantum as well as classical notions of transmission and noise, fading channels, which model signal attenuation in wireless transmissions, and, most famously, multiple-antenna channels, which form the basis of modern wireless broadband communications. Even more recently, Shannon's asymptotic concept of capacity, which relies on the ability to use a channel an unlimited number of times, has been examined in a finite-blocklength setting, where only a limited number of channel uses is considered; the finite-blocklength constraint is relevant to modern, delay-constrained applications such as multimedia communications.

Channel capacity has been an enduring concept. Even today, almost seven decades later, we are still using the notion of capacity to think about how communication channels behave. We have every expectation that it will continue to be an important concept well into the future.

Channel coding

In his 1948 paper, Shannon showed that, for any communication rate less than capacity, one can communicate with arbitrarily small error probabilities. In Shannon's paradigm, reliability is achieved through channel coding: transmitters protect signals against errors by introducing redundancy into each message before transmission, and receivers apply their knowledge of the type of redundancy employed to improve their probability of correctly determining the intended message from the channel output. The idea of adding redundancy to a signal was not new but, prior to Shannon, many communications engineers thought that achieving arbitrarily small error required more and more redundancy, therefore necessarily forcing the rate of transmission to zero. The idea that an arbitrarily small probability of error could be achieved with some constant rate of transmission therefore flew in the face of conventional wisdom at the time of its introduction.

Shannon's notion of channel coding initiated a tremendous amount of research and spawned entire subfields within the field of information theory. In particular, a significant amount of fundamental work went on in the 1950s through to the 1980s, when some of the very basic codes and decoding algorithms that we still use today were developed. Notable examples include algebraic codes, such as the Reed-Solomon family of channel codes that form the basis of codes used in modern storage media, and the Viterbi sequential decoding algorithm, which has found an astonishing array of applications, including its use in essentially every mobile phone in use today. The developments of more recent times have been no less impressive. In the 1990s, turbo codes were discovered, which, together with corresponding iterative decoding ideas, revolutionised the field of data transmission. This was followed quickly by another revolution, namely space-time coding. These ideas have driven a lot of what has happened in practice since that time, including the revival of the near-capacity-achieving low-density parity-check codes and the introduction of multiple-input multiple-output (MIMO) systems. These advances have enabled modern high-capacity data communication systems. And, of course, there have been many other key developments, including fountain and Raptor codes, polar codes, etc. In recent times, we have also seen a resurgence of some of the earlier ideas related to areas such as cloud storage and other distributed storage applications. So, channel coding provides a further example of a very early idea of Shannon's that has played a critical role in driving what is happening in technology today.

Multiuser channels

Shannon introduced the notions of channel coding and capacity in a very simple communication setting in which a single transmitter sends information to a single receiver. The techniques that he used to analyse channels in this setting are applicable well beyond this simple "point-to-point" communication model. Multiuser channel models generalise point-to-point channel models by incorporating multiple transmitters, multiple receivers, multi-directional flow of information or some combination of these features.

The generalisation from point-to-point channels to multiuser channels shows up in Shannon's own work as early as the 1950s. In his 1956 paper, Shannon generalised his network model from the point-to-point scenario to channels incorporating feedback; the goal in that work was to understand when feedback from the receiver to the transmitter increases the rate at which the transmitter can send to the receiver. That work employed two notions of capacity: the capacity achievable with an asymptotic notion of reliability and the capacity achievable with perfect reliability. In the former, information delivery is considered reliable if the probability of error can be made arbitrarily small. In the latter, information delivery is considered reliable only if the probability of error can be made to equal zero for a sufficiently large number of channel uses.

In 1960, Shannon generalised the network further by considering two-way channels. Two-way channels differ from point-to-point channels with feedback in that the point-to-point channel with feedback has only a single message travelling from the transmitter to the receiver while the two-way channel has messages travelling from each node to the other. The 1960 paper also mentions a channel in which a pair of transmitters sends information through a shared medium to a single receiver; that channel would today be called a "multiple access channel". The 1960 paper mentions future work to appear on this topic; while no such paper is found in the literature, it is clear that Shannon was thinking about generalisations beyond two-communicator models.

Starting in the late 1960s, multiuser channels became an important area for information theory research. Research on feedback investigated the improved trade-offs between rate and error probability achievable through feedback. Research on two-way channels yielded improved upper and lower bounds on achievable rate regions. A wide array of new channel models were developed, including multiple access channels (in which multiple transmitters send information to a single receiver), broadcast channels (in which a single transmitter sends possibly distinct information to multiple receivers), relay channels (in which a single transmitter sends information to a single receiver with the aid of a relay that can both transmit and receive information but has no messages of its own to transmit) and interference channels (in which the transmissions of multiple transmitters interfere at the multiple receivers with which they are trying to communicate).

Generalisations of Shannon's channel model are not limited to increasing the number of transmitters or receivers in the networks. Other generalisations include compound channels, which capture channels with unknown or varying statistics, wiretap channels, which model channels with eavesdroppers, and arbitrarily varying channels, which capture channels under jamming. Joint source-channel coding has also been a major topic in the multiuser communication literature. While the optimality of separation between source and channel coding holds for the point-to-point scenario studied by Shannon, it does not hold in general and a good deal of work has gone into understanding when such separation is optimal and how to achieve optimal performance when it is not.

While interest in multiuser channels waxes and wanes over time due to the difficulty of the problems, the massive size and huge importance of modern communication networks makes multiuser information theory an important area for continuing research.

Network coding

The examples given above of multiuser channels are typically used to model wireless communication environments. But wireless networks are not the only multiuser communication networks. After all, Shannon's work was itself originally inspired by communication networks like the wireline phone and telegraph networks of his day, each of which connected vast numbers of users over massive networks of wires. The modern field of network coding studies such networks of point-to-point channels. Typically, the point-to-point channels in these models are assumed to be noiseless, capacitated links. The field of network coding began with questions about the capacity of network coding networks. In some scenarios, notably the case of multicast network coding, the capacity is known, and efficient algorithms are available for achieving those bounds in practice. However, for most networks, the network coding capacity remains incompletely understood.

Given the difficulty of solving the general network coding problem, a variety of special cases have been considered. One of these is the family of index coding networks. Unlike general network coding networks, index coding networks are networks in which only one node in the network has an opportunity to code. It has been shown that if one could solve all index coding networks then that would provide a means of solving all network coding networks as well. That is, any network coding instance can be represented by an index coding instance whose solution would give you a solution to the original network coding problem.

In addition to work on network coding capacity, there has also been quite a bit of work on network code design, as well as work on the relationship between networks of capacitated links and the corresponding networks of noisy channels that they are intended to model. Results in this domain demonstrate that the capacity of a network of noisy channels is exactly equal to the capacity of the network coding network achieved by replacing each channel by a noiseless, capacitated link of the same capacity. Thus, Shannon's channel capacity fully characterises the behaviour of noisy, memoryless channels at least insofar as they affect the capacity of the networks in which they are employed.

Other questions considered in the domain of network coding include network error correction, secure network coding, network coding in the presence of eavesdroppers, network coding techniques for distributed storage and network coding for wireless applications with unreliable packet reception.

Source coding

Source coding, also called data compression, refers to the efficient representation of information. Shannon's work introduces two types of source coding to the literature: lossless source coding, in which the data can be reconstructed from its description either perfectly or with a probability of error approaching zero, and lossy source coding, in which greater efficiency in data representation is obtained through the allowance of some level of inaccuracy or "distortion" in the resulting reproduction. While Shannon's 1948 paper famously discusses both source coding and channel coding and is often described as the origination point for both ideas, Shannon first posed the lossy source coding problem in an earlier communication.

Shannon's 1948 paper sets a lot of highly influential precedents for the field of lossless source coding. It introduces the now-classical approach to deriving upper bounds on the rates needed to reliably describe a source and gives a strong converse to prove that no better rates can be achieved. It also includes both the ideas of fixed- and variable-length codes, that is, codes that give the same description length to all symbols, and codes that give different description lengths to different symbols. Arithmetic codes, which remain ubiquitous to this day, have their roots in this paper. The notions of entropy, entropy rate, typical sequences and many others also come from the 1948 paper.

The 1948 paper also looks at lossy source coding, describing the optimal trade-off between rate and distortion in lossy source description and intuitively explaining its derivation. In a 1959 paper, Shannon revisits the trade-off between rate and distortion in lossy source coding, giving more details of the proof, coining the term "rate distortion function" to describe that bound and presenting more examples of solutions of the rate distortion function for different sources.

Since then, there has been a lot of work in both lossless and lossy source coding. Much of the work in the 1950s through the 1970s looked at detailed proofs and extensions of the original ideas. Extensions include model generalisations to allow sources with memory, nonergodic sources, and so on. Advances were also made in practical code design for both lossless and lossy source coding. Huffman developed his famous source coding algorithm, which is still in use. Tunstall codes looked at coding from variable-length blocks of source symbols to fixed-length descriptions. Arithmetic codes were further developed for speed and performance. Algorithms for designing fixed and variable-rate vector quantizers were also introduced.

In the years that followed, a lot of work was done on universal source coding and multi-terminal source coding. Universal source codes are data compression algorithms that achieve the asymptotic limits promised by Shannon without requiring a priori knowledge of the distribution from which the source samples will be drawn. Results on universal source coding include code designs for both lossless and lossy universal source coding and analyses of code performance measures such as the rate at which a code's achievable rate (and, in the case of lossy coding, distortion) approaches the optimal bound. Like multiuser channel codes, multi-terminal source codes are data compression algorithms for networks with multiple transmitters of information, multiple receivers of information or both. Examples include the work of Ahlswede and Körner on source coding with coded side information and the work of Slepian and Wolf on distributed source coding networks, where source coded descriptions are sent by independent encoders to a single decoder.

In addition to advances in the theory of optimal source codes and their performance, there has been much research and development aimed at building and standardising lossless and lossy source codes for a variety of communication applications. These algorithms are critical parts of many of the data-rich applications that are becoming increasingly ubiquitous in our world.

Detection and hypothesis testing

Another field in which Shannon's influence has been felt has been that of signal detection and hypothesis testing. Although one might not normally think of Shannon in this context, he worked directly on signal detection in some of his very early work in 1944, in which he explored the problem of the best detection of pulses, deriving the optimal maximum a posteriori probability (MAP) procedure for signal detection; his work was one of the earliest expositions of the so-called "matched filter" principle. Also, by revealing the advantages of digital transmission in communications, he showed the importance of these fields to communication theory in general. And further, he expounded the idea that there is an optimal sampling rate for digitising signals through the famous Nyquist-Shannon sampling theorem.

These ideas have motivated quite a bit of work in subsequent years and to the present day. For example, channel decoding is, in essence, hypothesis testing with large numbers of hypotheses, and some famous results from this area have been developed within the context of sequence detection, including the Viterbi algorithm, noted above, and Forney's maximum likelihood sequence detector. Related to these developments is multiuser detection, which is also motivated by data detection in multiple-access communications, and the closely related problem of data detection in MIMO systems. Distributed detection, which is a problem motivated by wireless sensor networking, is also a successor to these ideas. And, returning to the sampling theorem, one of the major trends today in signal processing is compressed sensing, which exploits signal sparsity to go well beyond Nyquist-Shannon sampling to capture the essence of a signal with far fewer samples. So, again, although we might not think of Shannon as being a progenitor of this field, these connections show that his work has had a major influence either directly or as a motivator.

Machine learning

Another topic that is very much in evidence today is that of machine learning and its role in big data applications. Shannon was an early actor in the application of machine learning ideas – in 1950, he wrote one of the earliest chess-playing computer programs and, in 1952, he developed "Theseus", the famous maze-solving mouse. Of course, we have come a very long way in machine learning since those early contributions, driven by ever more powerful computers. For example, many games have been conquered: checkers in the 1950s, chess with Deep Blue in the 1990s, Jeopardy with Watson in 2011 and go with AlphaGo in 2016. And, of course, there have been many fundamental developments in learning and related tasks, such as neural networks and decision trees, and also graphical models, which have played a major role in channel decoding. These developments are behind contemporary developments such as deep learning and self-driving cars. So, again, Shannon was an early pioneer of a field that has turned out to be a very important part of modern technology and science.

Complexity and combinatorics

Quite a bit of Shannon's work related to and influenced the fields of complexity and combinatorics. Shannon's Master's thesis, perhaps his most famous work next to the 1948 paper, drew a relationship between switching circuits and Boolean algebra. His 1948 paper also introduced many tools that continue to be useful to combinatorial applications. Shannon's 1956 paper on zero-error capacity revisits the capacity problem – shifting the approach from a probabilistic perspective with asymptotic guarantees of reliability to a combinatoric perspective in which reliability requires the guaranteed accurate reproduction of every possible message that can be sent by the transmitter.

Over time, information theory has been and continues to be used for a variety of applications in the complexity and combinatorics literature. Results include generalisations of tools originally developed by Shannon in the probabilistic framework to their combinatoric alternatives. An example is Shannon's typical set, which captures a small set of sequences of approximately equal individual probability that together capture a fraction of the total probability approaching 1. This set generalises to more combinatoric alternatives such as that used in the method of types. Fano's inequality, entropy space characterisations, and a variety of other tools from information theory likewise play a role in the combinatorics and complexity literature.

The field of communication complexity also draws upon information theory tools. Concentration inequalities are another example of areas that sit at that boundary between combinatorics and information theory, bringing in tools from both of these communities to solve important problems. One can also find many examples in the literature involving bounding various counting arguments using information theory tools.

Cyber security

Cyber security is another extremely important aspect of modern technology that has its roots, at least in terms of its fundamentals, in Shannon's work. In particular, he established an information theoretic basis for this field in his 1949 paper (in turn based on earlier classified work), in which he addressed the question of when a cipher system is perfectly secure in an information theoretic sense. In this context, he showed the very fundamental result that cipher systems can only be secure if the key – that is, the secret key that is shared by sender and receiver and used to create an enciphered message – has at least the same entropy as the source message to be transmitted. Or, in other words, he showed that only one-time pads are perfectly secure in an information theoretic context.

Shannon's work was allegedly motivated by the SIGSALY system, which was used between Churchill and Roosevelt to communicate by radio telephone in World War II and which made use of one-time pads provided through physical transport of recordings of keys from Virginia to London. Most cyber security systems today, of course, do not use one-time pads. In fact, almost none do. Rather, they use smaller bits of randomness, expand that into a key and use computational difficulty to provide security. Nevertheless, the fundamental thinking comes from Shannon. Public key cryptosystems, of course, were not invented by Shannon but they are basically part of the legacy of looking at cyber security, or secret communications, from a fundamental point of view.

Another major advance in information theoretic characterisations of security was Wyner's introduction of the wire-tap channel in 1975, which gets away from a shared secret and uses the difference in the physical channels, from the transmitter to a legitimate receiver and to an eavesdropper, to provide data confidentiality. This setting introduces the notion of secrecy capacity, which is defined as the maximum rate at which a message can be transmitted reliably to the legitimate receiver while being kept perfectly secret from the eavesdropper. Wyner's work was extended by Csiszár and Körner to the broadcast channel with confidential messages, which is a model that has driven considerable research since, particularly in the recent development of wireless physical layer security, which makes use of radio physics to provide a degree of security in wireless transmission. The 1990s notion of common randomness as a source of distilling secret keys for use in cipher systems also has its roots in information theory and is another basis for wireless physical layer security.

So, again, we see another very important field of contemporary technology development influenced by Shannon's work.

Applications

While Shannon originally developed information theory as a means of studying the problems of information communication and storage, ideas from his work were very quickly taken up by other fields. In 1956, Shannon wrote about this phenomenon in an article titled "The Bandwagon," where he warned of "an element of danger" in the widespread adoption of information theory tools and terminology. In that article, he noted his personal belief that "information theory will prove useful in these other fields" but also argued that "establishing of such applications is not a trivial matter of translating words to a new domain, but rather the slow tedious process of hypothesis and experimental verification".

Today, information theory is used in a wide variety of fields. Biology and finance are two major examples of fields where people are starting to apply information theoretic tools: in one case to study how biological systems transmit and store information and in the other to model long-term behaviour of markets and strategies for maximising performance in such markets. Applications also exist in fields like linguistics, computer science, mathematics, probability, statistical inference and so on.

Concluding remarks

While one can barely skim the surface of Shannon's work and legacy in an article such as this, it should be clear that his genius has benefitted modern science and engineering, and thereby society, in countless ways. We hope that this very brief overview will inspire continuing interest in Shannon and his work and continuing interaction across the boundaries of the many distinct fields that share tools, philosophies, and interests with the field of information theory.

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Visions for Mathematical Learning: The Inspirational Legacy of Seymour Papert (1928–2016)

Celia Hoyles and Richard Noss (UCL Knowledge Lab, University College, London, UK)

Seymour Papert, who died on 31 July, was a mathematician with two PhDs in pure mathematics, from the University of Witwatersrand, South Africa, and the University of Cambridge, UK. He was a founder of artificial intelligence with Marvin Minsky at MIT, a psychologist working alongside Jean Piaget, a political activist against apartheid and, on a personal level, a wonderful cook and loyal friend. Since his death, the web has been awash with reminiscences and detailed accounts of his intellectual contributions, not only to the fundamental subjects in which he was the undisputed leader but also to the field of education, to a scholar who believed and showed that the computer, or at least the very carefully crafted use of the computer, could introduce young and old alike to the joys and power of mathematics and mathematical thinking.

In this short article, we have selected four pieces of work that directly impacted on the mathematics education field and community. Significantly, these are among his less well-known lectures and papers and we hope that, by airing them, the realisation of Papert's vision of a new kind of learnable mathematics may be one step closer.

1980: Keynote in ICME Berkeley, USA

Seymour gave one of the four plenaries at ICME 1980. Sadly, as far as we can tell, there was no transcript produced of Seymour's remarks. We are, however, grateful to Jeremy Kilpatrick (who attended the talk) for pointing us to a 1980 book edited by Lynn Steen and Don Albers, which includes a 4-page synopsis of Seymour's talk.[1]

Apparently, Seymour was inspirational. From the abstract, we know that he began:

> *"We are at the beginning of what is the decade of mathematics education. Not just in how children learn, but what they learn: we will see dramatic changes in what children learn; we will see subject matters that formerly seemed inaccessible or difficult even at college level learned by young children; we will see changes in where learning takes place, and in the process of learning itself."*

[1] https://books.google.cz/books?id=zcq9BwAAQBAJ&pg=PA12&lpg=PA12&dq=%22.

Seymour Papert and Celia Hoyles.

Seymour Papert and Richard Noss.

Even at this early stage, some 30 years before the presence of computers in school became commonplace, Seymour was addressing the question of epistemology, the 'what' of mathematics education – a theme that permeated his writings and speeches ever since.

1986: Keynote at the Tenth Conference of the International Group for the Psychology of Mathematics Education (PME 10) in London, UK

The title of Seymour's talk was "Beyond the Cognitive: the Other Face of Mathematics".[2] This talk was again inspirational and maybe a little controversial. He began by stating how he

> *"shared with Piaget the heuristic value that trying as hard as one can to understand as much as one can of children's mathematics and mathematicians' mathematics in the same categories. Doing so can illuminate both sides" (p. 1).*

How right he was – as so many of us have now experienced in our own work. Seymour argued for a greater importance to be accorded to the affective side of mathematics; remember that this keynote was 30 years ago, when mathematics education research was firmly grounded in the cognitive paradigm. In particular, Seymour noted how some people tended to identify with mathematical objects: a precursor of the hugely influential movement 'embodied mathematics'? "Do you observe the mathematical scene in your head or are you in it?" he asks (p. 2). And then the punchline that we will never forget: he showed how the "Euclidean propositions can be seen in a different light as special cases of turtle theorems" (p. 3), thus illustrating beautifully how a geometry that starts with the intuitions of body movement rather than abstract points and lines can be no less rigorous but considerably more inviting.

1996: Launching a new journal: the International Journal of Computers for Mathematics Learning (IJCML)

In 1996, Seymour became the founding editor of a new journal, IJCML. In the first issue, he undertakes "An Exploration in the Space of Mathematics Educations". This brilliant contribution begins memorably:

> *A mathematical metaphor frames the intentions of this paper. Imagine that we know how to construct an N-dimensional space, ME, in which each point represents an alternative mathematics education – or ame – and each dimension a feature, such as a component of content, a pedagogical method, a theoretical or ideological position. Each "reform" of mathematics education introduces new points and each fundamental idea a new dimension. Thus, if one considers a particular point (an ame) in ME, among its many "coordinates" are a (metaphorical) measure that runs from informal to formal and another that runs from instructionist to constructivist. In the paper I shall define seven more such oppositional principles that have not been recognized in the past as structuring choices in mathematics education. (Papert, 1996)*

The reader will not miss the daring and imaginative style of this metaphor. The article focuses on how the medium of expression can make any specific 'ame' seem 'natural', again using elementary geometric examples as illustration. But Seymour argues:

> *"…there is no doubt that in general much more can be done at an elementary level with dynamic than with algebraic characterizations of curves".*

[2] http://dailypapert.com/wp-content/uploads/2015/07/BeyondTheCognitive.pdf

Recall that this was a decade or so before dynamic geometry became widespread! In 2004, Seymour took up the theme of the mediation of knowledge more generally in a little-known but, for us, highly significant speech to open the London Knowledge Lab, where we both work. Take a look at https://mediacentral.ucl.ac.uk/Play/3004.

2006: Opening keynote to ICMI 17 Study Conference, Technology Revisited, Vietnam

In July 2002, the ICMI Executive Committee launched the 17th ICMI Study, called "Technology Revisited", the title reflecting the fact that the very first ICMI Study, held in Strasbourg in 1985, focused on the influence of computers and informatics on mathematics and its teaching. The Programme Committee wanted the Study Conference to be opened by a scholar with vision, experience and stature in the fields of mathematics, mathematics education and technology. We chose Seymour and to our delight he accepted by return of email. The tone of his emails became more and more excited as the conference approached. In his talk, Seymour spoke to the title '*30 Years of Digital Technologies in Mathematics Education and the Future*', using the recently prototyped and revolutionary '100 dollar laptop' (renamed the 'XO') to present his talk. He argued that, with full and easy access to computers, we face the challenge to consider not only how existing knowledge can be addressed in technology-enhanced ways but also that we should reserve at least 10% of our time and energy to consider what new types of mathematical knowledge and practices might emerge as a result. His accident the next day was a most terrible shock to us and to all the participants, and the conference struggled to continue after this tragedy, even as Seymour struggled in hospital. The best tribute we could think of was to try to keep the spirit of his ambition alive throughout the meeting by asking for participants to consider 'Seymour's 10%' in all their sessions and their subsequent papers. (Adapted from Hoyles, C., & Lagrange, J. B., 2010)

We hope that this short piece will keep Seymour's vision and struggle alive.

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Richard Noss is a founding member of the Logo and Mathematics Education Group inspired by Seymour. He became a professor of mathematics education at the Institute of Education in 1996. He was the founder and director of the London Knowledge Lab – exploring the future of learning with digital technologies – for its first decade. Richard co-authored (with Celia Hoyles) 'Windows on Mathematical Meanings: Learning Cultures and Computers' in 1996.

Do We Create Mathematics or Do We Gradually Discover Theories Which Exist Somewhere Independently of Us?

Vladimir L. Popov (Steklov Mathematical Institute, Moscow, Russia), Editor of EMS Newsletter

The headline question is a phrase from the article "Mathematics: Art and Science" by A. Borel, which is the English translation of the text of his lecture delivered (in German) in Munich in 1981. I remembered about this article in October 2016 when I saw the title "Discoveries, not inventions – Interview with Ernest B. Vinberg" of an article for publication in the EMS Newsletter, No. 102, December 2016 [2]. The very title clearly implies a definite answer to this question but it turns out that, in the interview, the issue of whether mathematicians explore something that exists independently of them or something they have invented is not discussed. However, as is shown in the article by A. Borel, this issue is, in fact, deeper than it may seem and contains items for discussion that might engage readers of the EMS Newsletter. Because of this, the mentioned article by A. Borel is reprinted below. There is also another reason to do this: in addition to a discussion of the formulated question, in this article, A. Borel also discusses other principal issues of a general nature related to mathematics, such as the problem of its relationship with the natural and applied sciences. The severity of the statements on these issues does not abate with time: to the citations in the reprinted paper by A. Borel, one could add the well known viewpoint of V. Arnold (and see also the recent interview with the Fields Medallist S. Novikov [3]). These are the items for discussion that might engage readers of the EMS Newsletter. Note that the reprinted article by A. Borel is not his only public statement on the subject (see [1]).

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[1] A. Borel, On the place of mathematics in culture (Faculty lecture given at the Institute for Advanced Study, Oct. 16, 1991), *Œuvres, Vol. IV, 1983–1999*, Springer, 419–440.

[2] A. Fialowski, J. Hilgert, B. Ørsted, V. Salnikov, Discoveries, not inventions – Interview with Ernest B. Vinberg, *EMS Newsletter* 102 (2016), 31–34.

[3] E. Kudryavtseva, The breakup of the mandatory knowledge had happen – Interview with the Fields medalist Sergei Novikov, 19.12.2016, http://www.kommersant.ru/doc/3169063 (in Russian).

Mathematics: Art and Science

A. Borel

Published with the permission of Springer Science and Business Media: extracts from pp. 9–17, The Mathematical Intelligencer*, Vol. 5, No. 4, 1983, doi:10.1007/BF03026504.*

The Mathematical Intelligencer *Editor's note: Apart from some minor changes, the following article is a translation of the text of a lecture delivered, in German, at the Carl Friedrich von Siemens Stiftung, Munich, on May 7, 1981, and, in a slightly modified form, as the first of three "Pauli-Vorlesungen", on February 1, 1982, at the Federal School of Technology, Zurich.*

The Intelligencer *requested permission from the author to publish a translation of the text (translated by Kevin M. Lenzen). We supplied the translation which the author checked and modified. We wish to thank him for his considerable help in improving the original translation.*

The German text of this lecture was published by the C. F. v. Siemens Stiftung (Mathematik: Kunst und Wissenschaft, Themenreihe XXXIII). *We are grateful for their permission to publish this English language version.*

Ladies and Gentlemen,

It is a great honour to be invited to address you here but one which is fraught with difficulties. First, there is a rather natural reluctance for a practicing mathematician to philosophise about mathematics instead of just giving a mathematical talk. As an illustration, the English mathematician G. Hardy called it a "melancholy experience"to write *about* mathematics rather than just prove theorems! However, had I not surmounted that feeling, I wouldn't be here, so I need not dwell on it any more. More serious difficulties arise from the fact that there are mathematicians and non-mathematicians in the audience. Whether one should conclude from this that my talk is best suited for an empty audience is a question that every one of you will have answered within the next hour and therefore needs no further elaboration. The difficulty brought about by the presence of mathematicians here is that it makes me aware (almost painfully aware) that, in fact, everything about my topic has already been said, all arguments have already been presented and pros and cons argued: mathematics is only an art, or only a science, the queen of sciences, merely a servant of science or even art and science combined. The very subject of my address, in Latin *Mathesis et Ars et Scientia Dicenda*, appeared as the third topic in the defence of a dissertation in the year 1845. The opponent claimed it was only art but not science [1]. It has occasionally been maintained that mathematics is rather trivial, almost tautological, and as such certainly unworthy of being regarded either as art or as science [2]. Most arguments can be supported by many references to outstanding mathematicians. It is even possible sometimes, by selective citation, to attribute widely different opinions to one and the same mathematician. So I would like to emphasise at the outset that the professional mathematicians assembled here are unlikely to hear anything new.

If I turn to the non-mathematicians, however, I encounter a much bigger, almost opposite problem: my task is to say something about the essence, the nature, of mathematics. In so doing, however, I cannot assume that the object of my statements is common knowledge. Of course, I can presuppose a certain familiarity with Greek mathematics, Euclidean geometry, for example, perhaps the theory of conic sections, or even the rudiments of algebra or analytical geometry. But they have little to do with the object of present-day mathematical research. Starting from this more or less familiar ground, mathematicians have gone on to develop ever more abstract theories, which have less and less to do with everyday experience, even when they later find important applications in the natural sciences. The transition from one level of abstraction to the next has often been very difficult even for the best mathematicians and it represented, in their time, an extremely bold step. I couldn't possibly give a satisfactory survey of this accumulation of abstractions upon abstractions and of their applications in just a few minutes. Still, I would feel quite uncomfortable simply to philosophise about mathematics without saying anything specific on its contents. I would also like to have a small supply of examples at hand to be able to illustrate general statements about mathematics or the position of mathematics with respect to art and the natural sciences. I shall therefore attempt to describe, or at least to give an idea of, some such steps.

In doing so, I will not be able to define precisely all my terms and I don't expect full understanding by all. But that is not essential. What I want to communicate is really just a feeling for the nature of these transitions, perhaps even for their boldness and significance in the history of thought. And I promise not to spend any more than 20 minutes doing so.

A mathematician often aims for general solutions. He enjoys solving many special problems with a few general formulae. One can call this economy of thought or laziness. An age-old example is the solution to a second-degree equation, say

$$x^2 + 2\,bx + c = 0.$$

Here, b and c are given real numbers. We are looking for a real number x that will satisfy this equation. For centuries, it has been known that x can be expressed in terms of b and c by the formula

$$x = -b \pm \sqrt{b^2 - c}.$$

If $b^2 > c$, we can take the square root and get two solutions. If $b^2 = c$ then $x = -b$ is said to be a double solution. If $b^2 < c$, however, then we cannot take the square root and we maintain, at least at the beginning secondary school level, that there is no solution.

In the 16th century, similar formulas were devised for third- and even fourth-degree equations, such as the equation

$$x^3 + ax + b = 0.$$

I won't write the formula out but it contains square roots and cube roots – so-called radicals. An extremely interesting phenomenon was discovered that came to be called the *casus irreducibilis*. If this equation has three distinct real solutions and we apply the formula, which in principle allows one to compute them, then we meet square roots of negative numbers; at the outset, these are meaningless. If we ignore the fact that they don't exist, however, and are not afraid to compute with them then they cancel out and we get the solutions, provided we carefully follow certain formal rules. In short, starting from the given real numbers a, b, we arrive at the sought for ones by using "nonreal numbers". The square roots of negative numbers were called "imaginary numbers"to distinguish them from the real numbers and controversies raged as to whether it was actually legitimate to use such nonreal numbers. Descartes, for example, did not want to have anything to do with them. Only around the year 1800 was a satisfactory solution – satisfactory for some at least – to this problem found. The real numbers are embedded in a bigger system consisting of the points of the plane, i.e. pairs of real numbers, between which one defines certain operations that have the same formal properties as the four basic operations in arithmetic. The real numbers are identified with the points on the horizontal axis and the square roots of negative numbers with those on the vertical axis. One then began to speak of complex (or imaginary) numbers. Formally, we can use these mathematical objects almost as easily as the real numbers and can obtain solutions that are sometimes real, sometimes complex. For the second-degree equation mentioned earlier, we can now say that there are two complex solutions if $b^2 < c$.

To a certain extent, this is, of course, merely a convention but it wasn't easy to grant these complex numbers the same right to existence as real numbers and not to regard them as a mere tool for arriving at real numbers. There was no strict definition of real numbers back then but the close connection between mathematics and measurement or practical computation gave real numbers a certain reality, in spite of the difficulties with irrational and negative numbers. It wasn't the same with complex numbers, however. That was a step in an entirely new direction, bringing a purely intellectual creation to the fore. As mathematicians became used to this new step, they began to realise that many operations performed with functions, such as polynomials, trigonometric functions, etc., still made sense when complex values were accepted as arguments and as values. This marked the beginning of complex analysis or function theory. As early as 1811, the mathematician Gauss pointed out the necessity of devising such a theory for its own sake:

> *The point here is not practical utility; rather, for me, analysis is an independent science which would lose an extraordinary amount of beauty and roundness by discriminating against those fictitious quantities [3].*

Apparently, even he did not foresee the practical relevance complex analysis was later to achieve, as in the theories of electricity or aerodynamics, for example.

But that is not the end of it. Allow me, if you will, to mention two further steps toward greater abstraction. Let us return to our second-degree equation. One can now say that it has, in general, two solutions that may be complex numbers. Similarly, an equation of the n-th degree has n solutions if one accepts complex numbers. From the 16th century on, people wondered whether there was also a general formula that would express the solutions of an equation of degree at least five from the coefficients by means of radicals. It was finally proved to be impossible. One proof (chronologically the third) was given by the French mathematician E. Galois within the framework of a more general theory, which was not understood at the time and subsequently forgotten. Some 15 years later, his work was rediscovered and understood only with great difficulty by a very few, so new was his viewpoint. Given an equation, Galois considered a certain set of permutations of the roots and showed that certain properties of this set of permutations are decisive. That was the beginning of an independent study of such sets of permutations, which later came to be known as Galois groups. He showed that an equation is solvable by means of radicals only when the groups involved belong to a certain class: namely, the solvable groups, as they came to be called. The theorem mentioned earlier, regarding equations of degree at least five, is then a consequence of the fact that the group associated to a general equation of the n-th degree is solvable only when $n = 1, 2, 3, 4$ [4]. The important properties of such groups, for instance to be solvable, are actually independent of the nature of the objects to be permuted and this led to the idea of an "abstract group" and to theorems of great significance, applicable in many areas of mathematics. But, for many years, this appeared to be nothing more than pure and very abstract mathematics. As a mathematician and a physicist were discussing the curriculum for physics at Princeton University around the year 1910, the physicist said they could no doubt leave out group theory, for it would never be applicable to physics [5]. Not 20 years later, three books on group theory and quantum mechanics appeared and, since then, groups have been fundamental in physics as well.

The following will serve as a final example. I said earlier that we can consider complex numbers to be points in the plane. An Irish mathematician, N.R. Hamilton, wondered whether one could define an analogue of the four basic operations among the points of three-dimensional space, thus forming an even more comprehensive number system. It took him about 10 years to find the answer: it is not possible in three-dimensional space but it is in four-dimensional space. We do not need to try to imagine just what four-dimensional space is here. It is simply a figure of speech for quadruples of real numbers instead of triples or pairs of real numbers. He called these new numbers quaternions. He did, however, have to do without one property of real or complex numbers, which, up until then, had been taken for granted: commutativity in multiplication, i.e. $a \times b = b \times a$. He also showed that calculus with quaternions had applications in the mathematical treatment of questions in physics and mechanics. Later, many other algebraic systems with a noncommutative product were defined, notably matrix algebras. This also appeared to be an entirely abstract form of mathematics, without connections to the outside world. In 1925, however, as Max Born was thinking about some new ideas of W. Heisenberg's, he discovered that the most appropriate formalism for expressing them was none other than matrix algebra, and this suggested that physical quantities be represented by means of algebraic objects that do not necessarily commute. This led to the uncertainty principle and was the beginning of matrix quantum mechanics and of the assignment of operators to physical quantities, which is at the basis of quantum mechanics [6].

With this last example, I shall conclude my attempts to describe some mathematical topics. The examples are, of course, extremely incomplete and not at all representative of all areas of mathematics. They do have two properties in common, however, which I would like to emphasise since they are valid in a great many cases. First of all, these developments lead in the direction of ever greater abstraction, further and further away from nature. Second, abstract theories developed for their own sake have found important applications in the natural sciences. The suitability of mathematics to the needs of the natural sciences is, in fact, astonishingly great (one physicist spoke once of the "unreasonable effectiveness of mathematics"[7]) and is worthy of a far more detailed discussion than I can afford to enter into here.

The transition to ever greater abstraction is not to be taken for granted, as you may have gathered from Gauss' quotation. Mathematics was originally developed for practical purposes such as bookkeeping, measurements and mechanics; even the great discoveries of the 17th century, such as infinitesimal and integral calculus, were, at first, primarily tools for solving problems in mechanics, astronomy and physics. The mathematician Euler, who was active in all areas of mathematics and its applications – including shipbuilding – also wrote papers on pure number theory and, more than once, felt the need to explain that it was as justified and important as more practically oriented work [8]. Mathematics was, from the very beginning, of course, a kind of idealisation but, for a long time, was not as far removed from reality or, more precisely, from our perception of reality as in the examples mentioned earlier. As mathematicians went further in this direction, they became increasingly aware that a mathematical concept has a right to existence as soon as it has been defined in a logically consistent manner, without necessarily having a connection with the physical world, and that they had the right to study it even when there seemed to be no practical applications at hand. In short, this led more and more to "Pure Mathematics"or "Mathematics for Its Own Sake".

But if one leaves out the controlling function of practical applicability, the question immediately arises as to how one can make value judgments. Surely not all concepts and theorems are equal; as in George Orwell's *Animal Farm*, some must be more so than others. Are there then internal criteria that can lead to a more or less objective hierarchy? You will notice that the same basic question can be asked about painting, music or art in general. It thus becomes a question of aesthetics. Indeed, a usual answer is that mathematics is, to a great extent, an art, an art whose development has been derived from, guided by and judged according to aesthetic criteria. For the layperson, it is often surprising to learn that one can speak of aesthetic criteria in so grim a discipline as mathematics. But this feeling is very strong for the mathematician, even though it is difficult to explain. What are the rules of this aesthetic? Wherein lies the beauty of a theorem, of a theory? Of course, there is no single answer that will satisfy all mathematicians but there is a surprising degree of agreement, to a far greater extent, I think, than exists in music or painting.

Without wishing to maintain that I can explain this fully, I would like to attempt to say a bit more about it later. At the moment, I shall content myself with the assertion that the analogy with art is one with which many mathematicians agree. For example, G. H. Hardy was of the opinion that if mathematics has any right to exist at all then it is only as art [9]. Our activity has much in common with that of an artist: a painter combines colours and forms, a musician tones, a poet words, and we combine ideas of a certain sort. The painter E. Degas wrote sonnets from time to time. Once, in a conversation with the poet S. Mallarmé, he complained that he found writing difficult even though he had many ideas, indeed an overabundance of ideas. Mallarmé answered that poems were made of words, not ideas [10]. We, on the other hand, work primarily with ideas.

This feeling of art becomes even stronger when one thinks of how a researcher works and progresses. One should not imagine that the mathematician operates entirely logically and systematically. He often gropes about in the dark, not knowing whether he should attempt to prove or disprove a certain proposition, and essential ideas often occur to him quite unexpectedly, without him even being able to see a clear and logical path leading to them from earlier considerations. Just as with composers and artists, one should speak of inspiration [11].

Other mathematicians, however, are opposed to this view and maintain that an involvement with mathematics without being guided by the needs of the natural sciences is dangerous and almost certainly leads to theories that may be quite subtle and may provide the mind with a peculiar pleasure but which represent a kind of intellectual mirror that is completely worthless from the standpoint of science or knowledge. For example, the mathematician J. von Neumann wrote in 1947:

> *As a mathematical discipline travels far from its empirical sources, or still more, if it is second and third generation only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely* l'art pour l'art … *there is a great danger that the subject will develop along the line of least resistance … will separate into a multitude of insignificant branches…*
> *In any event … the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less directly empirical ideas [12].*

Still others have taken a more intermediate stance: they fully recognise the importance of the aesthetic side of mathematics but feel that it is dangerous to push mathematics for its own sake too far. Poincaré, for example, wrote:

> *In addition to this, it provides its disciples with pleasures similar to painting and music. They admire the delicate harmony of the numbers and the forms; they marvel when a new discovery opens up to them an unexpected vista; and does the joy that they feel not have an aesthetic character even if the senses are not involved at all? …*
> *For this reason, I do not hesitate to say that mathematics deserves to be cultivated for its own sake, and I mean the theories which cannot be applied to physics just as much as the others [13].*

But a few pages further on, he returns to this comparison and adds:

> *If I may be allowed to continue my comparison with the fine arts then the pure mathematician who would forget the existence of the outside world could be likened to the painter who knew how to combine colours and forms harmoniously but who lacked models. His creative power would soon be exhausted [14].*

This denial of the possibility of abstract painting strikes me as especially noteworthy since we are in Munich, where, not much later, an artist would concern himself quite deeply with this question (namely, Wassily Kandinsky). It was sometime in the first decade of this century that he suddenly felt, after looking at one of his own canvases, that the subject can be detrimental to the painting in that it may be an obstacle to direct access to forms and colours: that is, to the actual artistic qualities of the work itself. But, as he wrote later [15], "a frightening gap"(*eine erschreckende Tiefe*) and a mass of questions confronted him, the most important of which was: "What should replace the missing subject?"Kandinsky was fully aware of the danger of ornamentation, of a purely decorative art, and wanted to avoid it at all costs. Contrary to Poincaré, however, he did not conclude that painting without a real subject had to be fruitless. In fact, he even developed a theory of the "inner necessity"and "intellectual content"of a painting. Since about 1910, as you know, he and other painters in increasing numbers have dedicated themselves to so-called abstract or pure painting, which has little or nothing to do with nature.

If one does not want to admit an analogous possibility for mathematics, however, then one will be led to a conception of mathematics that I would like to summarise as follows. On the one hand, it is a science because its main goal is to serve the natural sciences and technology. This goal is actually at the origin of mathematics and is constantly a wellspring of problems. On the other hand, it is an art because it is primarily a creation of the mind and progress is achieved by intellectual means, many of which issue from the depths of the human mind and for which aesthetic criteria are the final arbiters. But this intellectual freedom to move in a world of pure thought must be governed, to some extent, by possible applications in the natural sciences.

However, this view is really too narrow; in particular, the final clause is too limiting and many mathematicians have insisted on complete freedom of activity. First of all, as has already been pointed out, many areas of mathematics that have proved important for applications would not have been developed at all if one had insisted on applicability from the beginning. In spite of the above quotation, von Neumann himself pointed this out in a later lecture:

> *But still a large part of mathematics which became useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful: and there were no general indications that it even would be so … This is true of all science. Successes were largely due to forgetting completely about what one ultimately wanted, or whether one wanted anything ultimately, in refusing to investigate things which profit, and in relying solely on guidance by criteria of intellectual elegance…*
> *And I think it extremely instructive to watch the role of science in everyday life, and to note how in this area the principle of* laissez faire *has led to strange and wonderful results [16].*

Secondly, and for me more importantly, there are areas of pure mathematics which have found little or no application outside mathematics but which one cannot help viewing as great achievements. I am thinking, for example, of the theory of algebraic numbers, class field theory, automorphic functions, transfinite numbers, etc.

Let us return to the comparison with painting once again and take as "subject"the problems that are drawn from the physical world. Then, we see that we have painting drawn from nature as well as pure or abstract painting.

This comparison is, however, not yet entirely satisfactory, for such a description of mathematics would not encompass all its essential aspects, in particular its coherence and unity. Indeed, mathematics displays a coherence that I feel is much greater than in art. As a testimony to this, note that the same theorem is often proved independently by mathematicians living in widely separated locations or that a considerable number of papers have two, sometimes more, authors. It can also happen that parts of mathematics that have been developed completely independently of one another suddenly demonstrate deep connections under the impact of new insights. Mathematics is, to a great extent, a collective undertaking. Simplifications and unifications maintain the balance with unending development and expansion; they display again and again a remarkable unity even though mathematics is far too large to be mastered by a single individual.

I think it would be difficult to account fully for this by appealing solely to the criteria mentioned earlier: namely, subjective ones like intellectual elegance and beauty, and consideration of the needs of natural sciences and technology. One is then led to ask whether there are criteria or guidelines other than those. In my opinion, this is the case and I would now like to complete the earlier description of mathematics by looking at it from a third standpoint and adding another essential element to it. In preparation for this, I would like to digress, or at least apparently digress, and take up the question: 'Does mathematics have an existence of its own? Do we create mathematics or do we gradually discover theories which exist somewhere independently of us?' If this is so, where is this mathematical reality located?

It is, of course, not absolutely clear that such a question is really meaningful. But this feeling – that mathematics somehow, somewhere, pre-exists – is widespread. It was expressed quite sharply, for example, by G.H. Hardy:

> *I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations", are simply our notes of our observations. This view has been held, in one form or another, by many philosophers of high reputation, from Plato onwards… [17].}*

If one is a believer then one will see this pre-existent mathematical reality in God. This was actually the belief of Hermite, who once said:

There exists, if I am not mistaken, an entire world which is the totality of mathematical truths, to which we have access only with our mind, just as a world of physical reality exists, the one like the other independent of ourselves, both of divine creation [18].

It wasn't too long ago that a colleague explained in an introductory lecture that the following question had occupied him for years: 'Why has God created the exceptional series?'

But a reference to divine origin would hardly satisfy the nonbeliever. Many do, however, have a vague feeling that mathematics exists somewhere, even though, when they think about it, they cannot escape the conclusion that mathematics is exclusively a human creation.

Such questions can be asked of many other concepts such as state, moral values, religion, etc., and would probably be worthy of consideration all by themselves. But for want of time and competence, I shall have to content myself with a short and possibly oversimplified answer to this apparent dilemma by agreeing with the thesis that we tend to posit existence on all those things that belong to a civilization or culture in that we share them with other people and can exchange thoughts about them. Something becomes objective (as opposed to "subjective") as soon as we are convinced that it exists in the minds of others in the same form as it does in ours and that we can think about it and discuss it together [19]. Because the language of mathematics is so precise, it is ideally suited to defining concepts for which such a consensus exists. In my opinion, that is sufficient to provide us with a {\it feeling} of an objective existence, of a reality of mathematics similar to that mentioned by Hardy and Hermite above, regardless of whether it has another origin, as Hardy and Hermite maintain. One could speculate forever on this last point, of course, but that is actually irrelevant to the continuation of this discussion.

Before I elaborate on this, I would like to note that similar thoughts about our conception of physical reality have been expressed. For example, Poincaré wrote:

Our guarantee of the objectivity of the world in which we live is the fact that we share this world with other sentient beings…
That is therefore the first requirement of objectivity: that which is objective must be common to more than one spirit and as a result be transmittable from one to the other… [20]

And Einstein:

By the aid of speech, different individuals can, to a certain extent, compare their experiences. In this way, it is shown that certain sense perceptions of different individuals correspond to each other, while for other sense perceptions no such correspondence can be established. We are accustomed to regard as real those sense perceptions which are common to different individuals, and which therefore are, in a measure, impersonal [21].

Now back to mathematics. Mathematicians share an intellectual reality: a gigantic number of mathematical ideas, objects whose properties are partly known and partly unknown, theories, theorems, solved and unsolved problems, which they study with mental tools. These problems and ideas are partially suggested by the physical world; primarily, however, they arise from purely mathematical considerations (such as groups or quaternions to go back to my earlier examples). This totality, although stemming from the human mind, appears to us to be a natural science in the normal sense, such as physics or biology, and is for us just as concrete. I would actually maintain that mathematics not only has a theoretical side but also an experimental one. The former is clear: we strive for general theorems, principles, proofs and methods. That is the theory. But, in the beginning, one often has no idea of what to expect and how to continue, and one gains understanding and intuition through experimentation, that is, through the study of special cases. First, one hopes to be led in this way to a sensible conjecture and, second, perhaps to stumble upon an idea that will lead to a general proof. It can also happen, of course, that certain special cases are of great interest in themselves. That is the experimental side. The fact that we operate with intellectual objects more than with real objects and laboratory equipment is actually not important. The feeling that mathematics is, in this sense, an experimental science is also not new.

Hermite, for example, wrote to L. Königsberger around 1880:

The feeling expressed at that point in your letter where you say to me: "The more I think about all these things, the more I come to realise that mathematics is an experimental science like all other sciences." This feeling, I say, is also my feeling [22].

Traditionally, these experiments are carried out in one's head (or with pen and paper) and for this reason I have spoken of mental tools. I should add, however, that for about 20 years, real apparatuses, namely, electronic computers, have been playing an increasing role. They have actually given this experimental side of mathematics a new dimension. This has advanced to the extent that one can already see important, reciprocal and fascinating interactions between computer science and pure mathematics.

The word "science"in my title now takes a broader meaning: it refers not only to the natural sciences, as it did earlier, but also – and this to a much greater extent – to the conception of mathematics itself as an experimental and theoretical science or, I would venture to say, as a *mental* natural science, as a natural science of the intellect, whose objects and modes of investigations are all creations of the mind.

This makes it somewhat easier for me to speak of motivation and aesthetics. If one does not want to take applications in the natural sciences as a yardstick, one is still not thrown back upon mere intellectual elegance. There still remain almost practical criteria: namely, applicability in mathematics itself. The consideration of this mathematical reality, the open problems, the structure, needs and connections among various areas, already indicates possibly fruitful, valuable directions and allows the mathematician to orient himself and attach relative values to problems as well as to theories. Often a test for the value of a new theory is whether it can solve old problems. *De facto*, this limits the freedom of a mathematician, in a way which is comparable to the constraints imposed on a physicist, who after all doesn't choose at random the phenomena for which he wants to construct a theory or to devise experiments. Many examples show that mathematicians have often been able to foresee how certain areas of mathematics will develop and which problems should be taken up and probably quickly solved. Rather often, statements about the future of mathematics have proved true. Such predictions are not perfect but they are successful enough to indicate a difference from art. Analogous relatively successful forecasts about the future of painting, for example, hardly exist at all.

I don't want to go too far in this. However, I suggested the concept of mathematics as a mental natural science as *one of three elements*, not as the whole. On the one hand, I don't want to overlook the importance of the interactions between mathematics and the natural sciences. First, it is a common saying that all disciplines in the natural sciences must strive for a mathematical formulation and treatment – indeed, that a discipline achieves the status of a science only when this has been carried out. Thus, it is surely important that mathematicians try to help in this way. Second, it is doubtless a great achievement to formulate and treat complicated phenomena mathematically, and the new problems that are thereby introduced represent an enrichment for mathematics. One need only think of probability. I only mean that it is simply not necessary to put the idea of applicability in the foreground in order to do valuable mathematics. The history of mathematics shows that many outstanding achievements came from mathematicians who weren't thinking at all about external applications and who were led by purely mathematical considerations. And as has already been mentioned and illustrated, these contributions often found important applications in the natural sciences or in engineering, often in completely unforeseen ways.

On the other hand, I don't want to say that one can foresee everything completely rationally. Actually, this isn't the case even in the natural sciences, especially since one often does not know in advance which experiments will prove interesting. Outstanding mathematicians have also been wrong and have sometimes, precisely in the name of applicability within mathematics, termed fruitless, idle or even dangerous, new ideas that later proved fundamental. The freedom not to consider practical applications, which von Neumann demanded for science as a whole, must also be demanded within mathematics.

One could object that this analogy between mathematics and the natural sciences overlooks one essential difference: in the natural sciences or in technology, one often encounters problems that one has to solve in order to advance at all. In the world of mathematical thought, one has still *de jure* the freedom to put aside apparently unsolvable, overly difficult problems and turn to other, more manageable ones and maybe, in fact, follow the path of least resistance, just as von Neumann had feared. Wouldn't that be a temptation for a mathematician who defines mathematics as "the art of finding problems that one can solve"? Interestingly enough, I heard this definition from a mathematician whose works are especially remarkable because they treated so many problems which seemed quite special at the time but which later proved fundamental and whose solutions opened up new paths, namely, Heinz Hopf.

It cannot be denied, however, that sometimes paths of least resistance are indeed followed, leading to trivial or meaningless work. It can also happen that a successful school later falls into a sterile period and then even, at worst, exerts a harmful influence. Remarkably enough, however, an antidote always comes along, a reaction that eliminates these mistaken paths and fruitless directions. Up until now, mathematics has always been able to overcome such growth diseases and I am convinced that it will always do so, as long as there are so many talented mathematicians. It is very odd, however. Many of us have this feeling of a unity in mathematics but it is dangerous to prescribe overly precise guidelines in the name of our conception of it. It is more important that freedom reigns, despite occasional misuse. Why this is so successful cannot be fully explained. If one thinks of Hopf, for example, one can, to a certain extent, see rational criteria in his choice of problems: they were, for instance, often the first special cases of a general problem for which known methods of proof were not applicable. He was, of course, aware of this. But that doesn't explain everything. He probably didn't always foresee how influential his work would become; and, most likely, he did not worry about it. It is simply a part of the talent of a mathematician to be drawn to "good" problems, i.e. to problems that turn out to be significant later, even if it is not obvious at the time he takes them up. The mathematician is led to this partly by rational, scientific observations and partly by sheer curiosity, instinct, intuition or purely aesthetic considerations. Which brings me to my final subject: the aesthetic feeling in mathematics.

I have already mentioned the idea of mathematics as an art, a poetry of ideas. With that as a starting point, one would conclude that, in order for one to appreciate mathematics, to enjoy it, one needs a unique feeling for intellectual elegance and beauty of ideas in a very special world of thought. It is not surprising that this can hardly be shared with non-mathematicians: our poems are written in a highly specialised language, the mathematical language; although it is expressed in many of the more familiar languages, it is nevertheless unique and translatable into no other language; and unfortunately, these poems can only be understood in the original. The resemblance to an art is clear. One must also have a certain education for the appreciation of music or painting, which is to say one must learn a certain language.

I have long agreed with such opinions and analogies. Without changing my fundamental position with regard to mathematics, I would nonetheless like to reformulate them somewhat in the direction of my previous statements. I believe that our aesthetics are not always so pure and esoteric but also include a few more earthly yardsticks such as meaning, consequences, applicability, usefulness – but within the mathematical science. Our judgment of a theorem, a theory or a proof is also influenced by this but it is often simply equated to the aesthetic. I would like to try to explain this using Galois' theory mentioned earlier. This theory is generally treasured as one of the most beautiful chapters in mathematics. Why? First, it solved a very old and, at that time, most important question about equations. Second, it is an extremely comprehensive theory that goes far beyond the original question of solvability by radicals. Third, it is based on only a few principles of great elegance and simplicity, which are formulated within a new framework with new concepts that demonstrate the greatest originality. Fourth, these new viewpoints and concepts, especially the concept of a group, opened new paths and had a lasting influence on the whole of mathematics.

You will notice that of these four points only the third is a truly aesthetic judgment, and one about which one can have one's own opinion only when one understands the technical details of the theory. The others have a different character. One could make similar statements about theories in any natural science. They have a greater objective content, and a mathematician can have his own opinion about them even if he doesn't fully grasp the technical details of the theory. For the purpose of this discussion, I have separated these four elements but normally I would not always do so explicitly, and all four contribute to the impression of beauty. I do think that, in this respect, this example is fairly typical: what we describe as aesthetic is actually often a fusion of different views. For example, I would naturally find a method of proof more beautiful if it found new and unexpected applications, although the method itself hadn't changed. It may have become more important but in and of itself not more beautiful. Since all this takes place within mathematics itself, it will hardly help the non-mathematician penetrate our aesthetic world. I hope, however, that it will help him find more plausible the fact that our so-called aesthetic judgments display a greater consensus than in art, a consensus that goes far beyond geographical and chronological limitations. In any case, I regard this as being a major factor. But once again, I must avoid taking this too far. It is a question of degree, not an absolute difference. An aesthetic judgment on the work of a composer or a painter also draws on external factors such as influence, predecessors and the position of the work with relation to other works, even if it is to a lesser extent. On the other hand, there are differences of opinion and fluctuations in time in the evaluation of mathematical works, though not to such a strong degree, I would add. All these nuances need a good deal of explanation, which I cannot go into here for lack of time.

In the limited amount of time at my disposal, it would, of course, be easier to make only sweeping short statements about mathematics. But unfortunately, or fortunately, just as in other human undertakings to which many people have contributed over many centuries, mathematics refuses to let itself be described by just a few simple formulas. Almost every general statement about mathematics has to be qualified somehow. One exception, perhaps the only one, might be this statement itself. I hope I have, at least, given the impression that mathematics is an extremely complex creation, which displays so many essential traits in common with art and experimental and theoretical sciences that it has to be regarded as all three at the same time, and thus must be differentiated from all three as well.

I am aware that I have raised more questions than I have answered, treated too briefly those I have discussed and not even touched upon some important ones, such as the value of this creation. One can, of course, point to innumerable applications in the natural sciences and in engineering, many of which have a great influence on our daily life, thereby establishing a social right to existence for mathematics. But I must confess that, as a pure mathematician, I am more interested in an assessment of mathematics in itself. The contributions of the various mathematicians meld into an enormous intellectual construct, which, in my opinion, represents an impressive testimony to the power of human thinking. The mathematician Jacobi once wrote that "the only purpose of science is to honour the human mind"[23]. I believe that this creation does indeed do the human mind great honour.

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Notes

1 The dissertation was by L. Kronecker, see *Werke*, 5 Vol., Teubner, Leipzig, 1895–1930, Vol. 1, p. 73. The opponent was G. Eisenstein. The source I am aware of because the name and the opinion of the opponent is a footnote by E. Lampe to a lecture by P. du Bois-Reymond, "Was will die Mathematik und was will der Mathematiker?", published posthumously by E. Lampe in *Jahresbericht der Deutschen Mathematiker-Vereinigung* 19 (1910), 190–198.

2 For a discussion of a number of such opinions, see A. Pringsheim, "Ueber den Wert und angeblichen Unwert der Mathematik", *Jahresbericht der Deutschen Mathematiker-Vereinigung* 13 (1904), 357–382.

3 Letter to F. W. Bessel, 18 November 1811. See G. F. Auwers Verlag, *Briefwechsel zwischen Gauss und Bessel*, Leipzig 1880, p. 156.

4 Actually, the beginnings of group theory can already be traced to some earlier work, notably by Lagrange, which was, in part, familiar to Galois. The latter's standpoint was, however, so general and abstract and, in addition, so sketchily described that it was assimilated only slowly. For historical information on the theory of equations and the beginnings of group theory, see, for example, N. Bourbaki, *Eléments d'histoire des mathématiques*, Hermann ed., Paris, 1969, third and fifth articles.

5 F.J. Dyson, "Mathematics in the physical sciences", *Scientific American* 211, September (1964), 129–146.

6 See B.L. van der Waerden's historical introduction in "Sources in Quantum Mechanics", *Classics of Science*, Vol. 5, Dover Publications, New York, 1967, especially pp. 36–38. Also see Dirac's remarks on the introduction of non-commutativity in quantum mechanics in *loc. cit.* [7].

7 E. P. Wigner, "The unreasonable effectiveness of mathematics in the natural sciences", *Communications on Pure and Applied Mathematics* 13 (1960), 1–14.
Among the many aspects of this interaction, the one that appears most remarkable to me is that the mathematical formalism sometimes leads to basic, new and purely physical ideas. One well known example is the discovery of the positron. In 1928, P.A.M. Dirac set up quantum mechanic relativistic equations for the movement of the electron. These equations also allowed a solution with the same mass as the electron but with the opposite electrical charge. All attempts to explain these solutions satisfactorily, or to eliminate them by some suitable modification of the equation, were unsuccessful. This led Dirac eventually to conjecture the existence of a particle with the necessary properties, which was later established by Anderson. For this, see P. A. M. Dirac, "The development of quantum theory"(J. R. Oppenheimer Memorial Prize acceptance speech), Gordon and Breach, New York, 1971.
A newer and even more comprehensive example would be the use of irreducible representations of the special unitary group SU(3) in three complex variables, which led to the so-called "eightfold way". One of the first successes of this theory was quite striking, namely, the discovery of the particle Ω^-: nine baryons were assigned, through consideration of two of their characteristic quantum numbers, to nine points of a very specific mathematical configuration consisting of 10 points in a plane [the 10 weights of an irreducible 10-dimensional representation of SU(3)]; this led M. Gell'man to conjecture that there should also be a particle corresponding to the tenth point, which would then possess certain well-defined properties. Such a particle was observed some two years later. A further development along these lines led to the theory of "quarks". For the beginnings of this theory, see F.J. Dyson, *loc. cit.* [5] and M. Gell'man and Y. Ne'eman, *The Eightfold Way*, W.A. Benjamin, New York, 1964.

8 See a number of papers in L. Euler's *Opera Omnia*, especially I.2, 62–63, 285, 461, 576; I.3, 5.2. I want to thank A. Weil for pointing this out to me. Here is an example (translated from Latin by Weil), *loc. cit.* pp. 62–63, published in 1747:
"Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. Moreover, even if the proof of some proposition does not appear to have any present use, it usually turns out that the method by which this problem has been solved opens the way to the discovery of more useful results.
"Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. Actually, far from being useless, this theory is of no little use even in analysis. Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import."

9 G.H. Hardy, *A Mathematician's Apology*, Cambridge University Press, 1940; new printing with a foreword by C.P. Snow, pp. 139–140.

10 P. Valery, *Degas, danse, dessin*, A. Vollard éd., Paris, 1936; *Œuvres* II, La Pléiade, Gallimard éd., Paris, 1966, pp. 1163–1240, especially pp. 1207–1209.

11 The following excerpt from a letter from C.F. Gauss to Olbers, written on 3 September 1805, shortly after Gauss had solved a problem (the "sign of the Gaussian Sums") he had been working on for years, can serve as an example:
"Finally, just a few days ago, success – but not as a result of my laborious search but only by the grace of God I would say. Just as it is when lightning strikes, the puzzle was solved; I myself would not be able to show the threads which connect that which I knew before, that with which I had made my last attempt, and that by which it succeeded."See Gauss, *Gesammelte Werke*, Vol. 10_I, pp. 24–25. Here one must also mention H. Poincaré's description of some of his fundamental discoveries on automorphic functions. H. Poincaré, "L'invention mathematique" in *Science et Méthode*, E. Flammarion éd., Paris, 1908, Chap. III.

12 J. v. Neumann, "The mathematician" in Robert B. Heywood, *The Works of the Mind*, University of Chicago Press, 1947, pp. 180–187. *Collected Works*, 6 Vol., Pergamon, New York, 1961, Vol. I, pp. 1–9.

13 H. Poincaré, *La Valeur de la Science*, E. Flammarion, Paris, 1905, Chap. 5, p. 139. Actually, this chapter is the printed version of a lecture that Poincaré delivered at the First International Congress of Mathematicians, Zurich, 1897.

14 *Loc. cit.* [13], p. 147.

15 W. Kandinsky, *Rückblick 1901–1913*, H. Walden ed., 1913. New printing by W. Klein Verlag, Baden–Baden, 1955. See pp. 20–21.

16 J. v. Neumann, "The role of mathematics in the science and in society", address to Princeton Graduate Alumni, June 1954. See *Collected Works*, 6 Vol., Pergamon, New York, 1961, Vol. VI, pp. 477–490.

17 See G.H. Hardy, *loc. cit.* [9], pp. 123–124.

18 G. Darboux, "La vie et l'Œuvre de Charles Hermite", *Revue du mois*, 10 January 1906, p. 46.

19 See L. White, "The locus of mathematical reality: An anthropological footnote", *Philosophy of Science* 14 (1947), 189303; also in J.R. Newman, *The World of Mathematics*, 4 Vol., Simon and Schuster, New York, 1956, Vol. 4, pp. 2348–2364.

20 H. Poincaré, *loc. cit.* [13], p. 262.

21 A. Einstein, *Vier Vorlesungen über Relativitätstheorie*, held in May 1921 at Princeton University, Fr. Vieweg und Sohn, Braunschweig, 1922, p. 1. English translation in: *The Meaning of Relativity*, Princeton University Press, Princeton, 1945.

22 See L. Königsberger, "Die Mathematik eine Geistes- oder Naturwissenschaft?", *Jahresbericht der Deutschen Mathematiker-Vereinigung* 23 (1914), 1–12.

23 In a letter of 2 July 1830 to A.M. Legendre, see C.G.J. Jacobi, *Gesammelte Werke*, G. Riemer, Berlin, 1881–1891, Vol. 1, pp. 453–455. Since this statement is sometimes misquoted, we prefer to give here its original context:
"Mais M. Poisson n'aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous fait des reproches, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. II est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain et que sous ce titre une question de nombres vaut autant qu'une question du système du monde."

Results of a Worldwide Survey of Mathematicians on Journal Reform

Cameron Neylon (Curtin University, Perth, Australia), David M. Roberts (University of Adelaide, Australia) and Mark C. Wilson (University of Auckland, New Zealand)

1 Introduction

Scholarly communication is in a state of ferment. The shift over the last few decades from print to digital dissemination has set off a wide range of movements for change, from the radical to the more modest and incremental. At the centre of many of these debates is the move toward wider access to the research literature. Mathematics occupies an unusual place in these debates, being simultaneously radical in the degree of uptake of new approaches such as arXiv.org for rapid dissemination prior to peer review but also highly conservative in terms of the move to online-only journals and wide access models more generally.

Several surveys have examined the opinions of researchers generally (most recently Tenopir et al. [6] – see also their literature review – Taylor & Francis [5] and Solomon [4] and issues of access to funding (Solomon and Björk [2], Björk & Solomon [1]) and Dallmeier-Tiessen et al. [3] but few have focused on the views of mathematicians specifically. We sought to understand how those engaged in mathematical research viewed the importance of enhancing access to the mathematics research literature and their interest in a wider range of innovations, including changes to peer review and publication practice. We also aimed to get feedback from the mathematics community on specific issues they saw with mathematical journals.

1.1 Methodology

An online survey instrument was made available via Google Forms from 12 April 2016 and submissions were initially solicited through personal emails, social media and research mathematics mailing lists (including DMANET, the Australian Mathematical Society and the European Mathematical Society — note that the American Mathematical Society declined to advertise it). In order to increase the number of responses, we made a second wave of approaches to recent authors in mathematics journals, societies and mathematics departments worldwide.[1]

The survey cannot be taken as representing the general opinion of mathematicians because we have no information about who responded – full anonymity was promised to participants. However, we are confident that we reached a broad cross-section of the community. Of respondents, in the last three years, 33% have acted as an editor for a mathematics journal, 93% have authored a paper and 86% have acted as a referee.

The survey addressed general questions of desire for change, specific issues and the association of specific factors with journal prestige. Questions on prestige were framed in two different ways. In one set, respondents were asked how they personally associate specific factors with the prestige of journals. In the second set, they were asked how the community associate those same factors with the prestige of journals. This allows us to identify consistent differences between individual (self-reported) views and the assumptions those same individuals have about community views. All data, including a copy of the survey itself and raw and processed responses, and the code used for processing, are available at https://figshare.com/projects/Survey_of_mathematical_publishing/16944.

2 Results

We closed the survey when it reached exactly 1000 responses, on 28 August 2016.[2]

2.1 Demographics

Respondents self-reported as PhD student (10.5%), postdoc (15.5%), tenure-track (7%), tenured (57%) and other (emeritus, librarian, etc.) (10%). Surveys in Europe and North America of career stages of researchers give very different results for the distribution of career stages. The respondent distribution is not inconsistent with these other surveys but we cannot show that the respondents are demographically representative. Geographical representation was dominated by Europe (54%) and North America (25%). Other respondents selected Oceania (11%), Asia (6%), South America (4%) and Africa (0.5%) as locations.

2.2 Appetite for change

On a five point scale from 1 being "the status-quo is completely acceptable" and 5 being "almost all [journals] need serious work", 78% of respondents selected 3, 4 or 5. Amongst respondents, there is a strong desire for

[1] Authors' email addresses were extracted from issues of the following journals in the years 2014–16: Acta Appl. Math., Acta Inf., Acta Math. Sin. (Engl. Ser.), Adv. Comput. Math., BIT, Calc. Var. Partial Differential Equations, Comput. Math. Organ. Theory, Comput. Math. Model., Funct. Anal. Appl., Graphs Combin., Invent. Math., J. Algebraic Combin., J. Engrg. Math., J. Math. Sci. (N.Y.), J. Theoret. Probab., Manuscripta Math., Monatsh. Math., Numer. Math., Potential Anal., Probab. Theory Related Fields, Statist. Papers, Theoret. and Math. Phys. Mathematics departments were chosen with no particular plan from universities in China, Czechia, Israel, Japan, Sweden, Turkey, Azerbaijan, Iran and South Africa.

[2] Thanks to Ben Rohrlach for additional exploratory analysis and help with R.

change. Free text answers describing the major perceived problems revealed serious concerns that suggest systemic issues: almost 200 journals from 57 publishers were mentioned by name as needing serious improvement. These ranged from journals at large commercial publishers and university presses to small Open Access journals that do not charge an Article Processing Charge (APC), over the whole spectrum of prestige. Table 1 gives a classification of the stated issues into main categories (from the 466 respondents who named a journal). Of particular concern is the number of respondents who had concerns with the quality of peer review. For example, 126 journals or publishers were named as being unsatisfactory in the time taken for refereeing or the time taken from acceptance to publication.

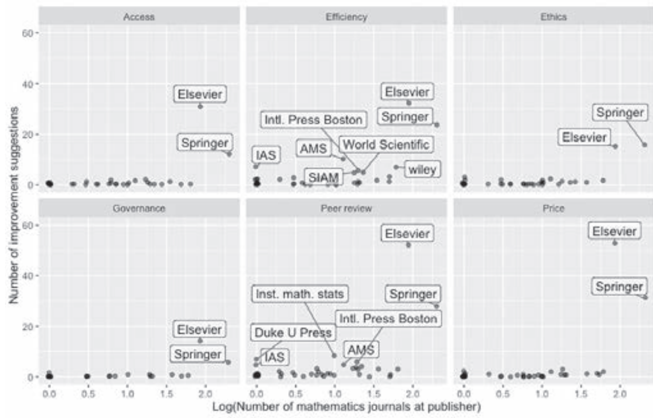

Figure 1: Number of suggestions per publisher, by category and portfolio size (small horizontal jitter added for clarity).

Table 1: Distribution of free-form comments by area where improvement is needed.

Issue	N	%
peer review quality	139	30
efficiency	115	24
price	101	21
other quality	83	17
access	72	15
ethics	35	7
governance	27	6
unclear	15	3

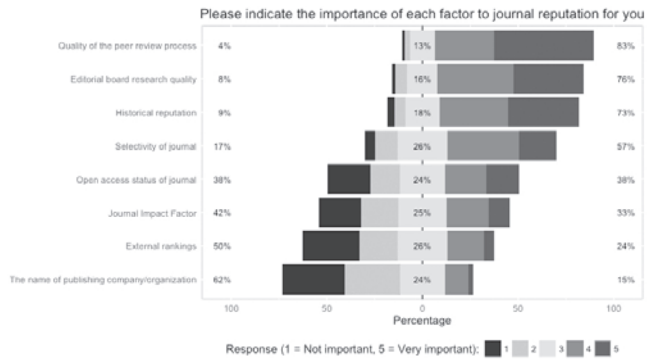

Figure 2: Stacked diverging bar chart of Likert scale responses.

On this question, those who had acted as editors did not differ substantially from those who had not. To protect anonymity, the survey did not ask which journals editors worked for but with over 330 editors this sample must include many associated with traditionally run journals.

Figure 1 plots the suggestions for each publisher in each category by the size of the mathematics journal portfolio (or rather, by $\log_{10}$ of the number of mathematics journals to account for the two orders of magnitude range: 1–202 journals). Any publisher with at least five journals suggested is labelled. One would expect publishers with larger mathematics portfolios to garner more criticism but there is essentially little trend among publishers excluding Elsevier and Springer. Even though Elsevier publishes less than half as many mathematics journals as Springer, its journals get more suggestions for improvements in all categories but one.

2.3 Which attributes of journals contribute to journal prestige?

A diversity of studies continue to show that journal reputation or prestige is an important factor for authors in selecting a journal. In two sets of questions, we asked respondents how important they thought specific aspects were for journal reputation and how important they thought those same aspects were for the community's view of reputation. Results are summarised in Figure 2.

The most important factor for respondents was the quality of peer review (median rank 5). This was followed by the reputation of editors and historical reputation, and selectivity (median 4), then Journal Impact Factor (JIF), Open Access status and external rankings (median 3). The publisher had the lowest median ranking (2), with a mode of 1.

When we asked for the respondents' assessment of the importance of these factors in the community's view, a striking pattern emerged, as shown in Figure 3. For factors that might be considered as traditional markers of prestige (publisher, external rankings, JIF), respondents believe they matter more to the community than they do to themselves. That is, respondents tend to believe themselves less influenced by such "external" factors than the community. For other "traditional" markers (editors' reputation, historical reputation, degree of selectivity), this was less pronounced but the tendency is in the same direction.

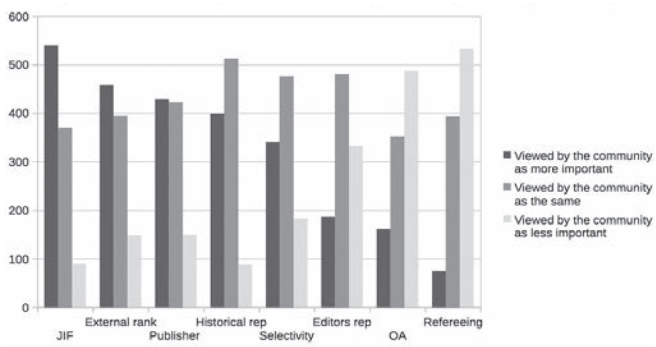

Figure 3: Respondents' beliefs about community opinion on issues.

When asked about Open Access (OA), respondents implied strongly that it was more important to them than the community. Combined together, this shows that our respondents believe their colleagues to be more influenced by traditional markers and less interested in OA than they are. These differences matter. Change is risky. If mathematicians are pessimistic about their colleagues' desire for change then working for change is much less appealing. It is one thing for the status quo to be supported by peer pressure but it appears it may be supported by the *perception* of peer pressure.

Finally, the difference between personal and community views on the importance of the peer review process was both striking and disturbing. By a strong margin, most respondents view the quality of peer review as more important to themselves than they believe it is to the community. If this is true beyond our sample, it is concerning because it suggests that individuals do not see the community as a whole as driven by high standards. While this is potentially a result of sample bias, further investigation of this finding should be carried out.

2.4 Changing practice

If there is change, what should it look like? When asked to rate the importance of elements of journal publishing, high ethical standards and timely and thorough peer review were rated the most important (median 5). All other factors (Open Access, low cost of publication, non-profit status, transparent costings, community control and use of modern internet technologies) had a median ranking of 4. The most frequent ranking (mode) was 5 for all of these questions, apart from low cost. Perhaps more informatively, there is greater distribution in responses for those lower ranked priorities. In terms of the specifics of change, editors are less keen on Open Access than non-editors. This may be related to their having a substantially stronger view that author payments for publication are unacceptable (see Section 2.5).

In terms of new practices, almost a quarter of respondents supported open peer review as a default (with opt-out) and half supported post publication review with moderated comments and commenter identities revealed. Nearly half supported the publication of anonymous referee reports, suitably presented, to help readers. Free-form responses were also allowed and, of the 53 constructive suggestions made, 11 mentioned double-blind refereeing. Editors were clearly less favourable towards open review (26% vs. 38%) and community election of editors (31% vs. 43%) than non-editors. Interestingly, editors were slightly more supportive of banning monetary payments to editors (45% vs. 41%) and of editor term limits (31% vs. 29%).

2.5 Funding of increased access

Because mathematics is a discipline with relatively little funding and therefore has limited discretionary resources, it is commonly believed that there is a strong aversion to author publication charges (APCs). However, opinions on APCs were split, with (roughly) a quarter believing them unacceptable in principle, a quarter saying they should be paid by library consortia and a quarter saying they were "OK if they are sufficiently low". Respondents were, however, united on one issue. Only 2% believed that they were "not a problem, and competition in the journal market will take care of them".

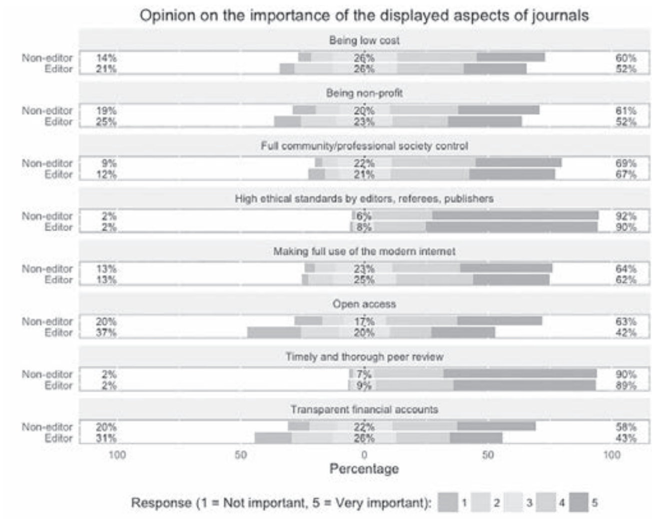

Figure 4: Importance of journal aspects: editors and non-editors.

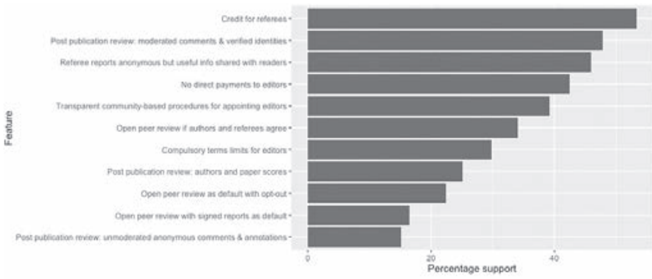

Figure 5: Support for new practices.

3 Discussion

Overall, we interpret these results as showing that respondents are strongly in favour of change in the publishing system but pessimistic about the support the efforts for such change would get from their colleagues. There is strong support for high(er) ethical standards and high quality peer review, and substantial support for rather radical changes to the way journals operate. These issues are also the subject of serious concerns raised in free-text answers. Editors and publishers should take note of these concerns, alongside the demand for greater transparency in editor selection and editorial processes. On several of these issues, editors' views diverge from that of the community and this should be a subject of some concern. However, there is substantial agreement between editors and non-editors on many issues.

When asked what should happen if efforts by editors to reform a journal are blocked by the publisher, over half of respondents favoured resigning to join a better journal (29%) or to create a new one (32%). Only a very small proportion (4.5%) favoured settling for the status quo. For this set of respondents at least, the appetite for change is there and community support for bold moves by editors on behalf of the community is strong.

To our knowledge, no previous study has sought to compare the views of individuals with their views of the community. Although it may reflect a sampling bias, it is striking that respondents to this survey show a strong tendency to claim views that are more aligned with change than those they believe the community hold, particularly on Open Access and traditional measures of prestige and quality. This is in sharp contrast to their views on peer review, where there appears to be pronounced scepticism on the importance other members of the community place on the quality of peer review.

3.1 How is Europe different?

We recalculated some of the results for the subset of data in which the respondent indicated they work in Europe. One difference observed is that European respondents were somewhat keener on Open Access than non-Europeans (the distribution of European answers stochastically dominated the non-European). In terms of demographics, there were more PhD students and fewer editors in the European respondent set than the non-European but the differences were not very large. However, we have not delved into this issue rigorously and leave it to our European colleagues to analyse our publicly available data.

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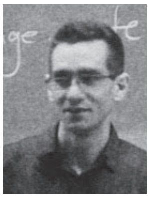

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M&MoCS – International Research Center on Mathematics and Mechanics of Complex Systems

Francesco dell'Isola (University of Rome "La Sapienza", Italy and International Research Center on Mathematics and Mechanics of Complex Systems, L'Aquila, Italy), Luca Placidi (International Telematic University Uninettuno, Italy and International Research Center on Mathematics and Mechanics of Complex Systems, L'Aquila, Italy) and Emilio Barchiesi (University of Rome "La Sapienza", Italy and International Research Center on Mathematics and Mechanics of Complex Systems, L'Aquila, Italy)

The Centre

The International Research Center for Mathematics & Mechanics of Complex Systems (M&MoCS) is a research centre of the University of L'Aquila. The centre was established in 2010 by the Dipartimento di Ingegneria delle Strutture, delle Acque e del Terreno (DISAT), the Dipartimento di Matematica Pura e Applicata (DMPA) of the University of L'Aquila and the Dipartimento di Strutture of Roma Tre University, with the financial and logistic aid of Fondazione Tullio Levi-Civita. The current director of the centre is Francesco dell'Isola, who is a full professor of mechanics of solids at "La Sapienza" University of Rome. The previous director of the centre was Angelo Luongo, who is a full professor of mechanics of solids at the University of L'Aquila.

The mission of the centre is the development and dissemination of scientific knowledge. In order to pursue these goals, the centre: (a) conducts and coordinates research activities; (b) promotes initiatives for the enhancement of scientific liaison between researchers in mathematical fields and researchers in solid and fluid mechanics, operating both in Italy and abroad; (c) promotes, supports and organises highly qualified educational activities, such as training, Master's and doctorate courses; (d) encourages the promotion of mathematics and mechanics of complex systems through publications, conferences, seminars and exhibitions; and (e) carries out consultancy and research activities for organisations and institutions.

Workshops and summer schools

From 2011 to 2013, the centre offered the Sperlonga Summer Schools on Mechanics and Engineering Sciences: courses and seminars, organised with an interdisciplinary attitude, aimed at introducing young scientists to present-day developments in mechanics at the interface with mathematics, physics, materials science, biology and engineering. Lectures were complemented by discussion sessions to foster lively interactions amongst participants. Moreover, the centre, in collaboration with the CNRS International Associate Laboratory Coss&Vita, the Paris Federation of Mechanics Labs and the GDR MeGe (French Research Network), has organised, since its foundation, many workshops in Arpino, Alghero and Catania. Since 2014, the centre has organised two EUROMECH colloquia, and a third one "Generalized and microstructured continua: new ideas in modeling and/or applications to structures with (nearly-)inextensible fibers" is planned for 3-8 April 2017 in Arpino (Italy). In the framework of a joint effort with Warsaw University of Technology and among the many seminars organised at the centre, an introductory course in analytical continuum mechanics and computational mechanics is offered every year. Many other events are scheduled throughout the year.

To find out more information, please visit the website http://memocs.univaq.it.

Research and facilities

Research activities carried out at the centre are directed toward the formulation of computationally tractable mathematical models to predict phenomena occurring in complex systems and address their numerical solution. Experimental research is also being carried out at the centre facilities. The research mainly concerns: variational and optimisation methods, gamma convergence, homogenisation techniques for periodic media, mechanics of fluids and solids, vibration control by means of piezoelectric actuators, composite materials, landmine detection, biomechanics of growing tissues, fluid dynamics and transport phenomena, kinetic theory, vibrations and waves in continuous and multi-phase media, plasticity, damage mechanics, continuum mechanics, stability and control of structures, identification of materials and mechanical systems, dynamical systems and bifurcation theory, fluid dynamics models for the analysis of traffic flows and the social sciences, and numerical differential modelling of the mechanical and electromagnetic response of biological materials and nano-structures.

In the following subsections, we briefly summarise the main activities of the experimental laboratories.

Laboratory of Materials and Structures Testing

The activities of the Laboratory of Materials and Structures Testing concern experiments on materials and structures for the purpose of consultancy, applied research and teaching. The aim of the laboratory is to provide the construction industry with a diagnosis of the state of degradation of civil works, thus providing an assessment of the residual life of structures.

Laboratory of Structures and Smart Materials

The Laboratory of Structures and Smart Materials is actively engaged in the study and prototyping of smart structures. The research group addresses the issue of mechanical structure vibration damping using piezoelectric transducers coupled with electronic systems and the research is directed toward linear and nonlinear control of structural dynamics. Applications considered include the design of soundproofing systems, wing and blade flutter control, the identification of structural damage and the design of smart systems able to self-monitor the evolution of their constitutive parameters. Uncertainty modelling in inhomogeneous structures with unknown inhomogeneity and stimulated by piezoelectric actuators and the analysis of metals subject to the action of external loads and induced structural change (such as anisotropy and strength of material) are also investigated.

Naval Structures and Onboard Instrumentation Laboratory

The Naval Structures and Onboard Instrumentation Laboratory (officially known as Laboratorio Strutture Navali e Strumentazioni di Bordo) conducts research on prototypes of ship structures (surface and underwater vehicles). The research activity is focused on the areas of control and vibration damping in the field of marine structures and stability of ship structures subject to the action of fluid waves and impact. The laboratory is directly involved in the development and realisation of the project SEALAB. The SEALAB project includes the construction of an experimental surface marine mobile station, functioning as a test bed for marine technologies. It is intended to develop, validate, refine and eventually patent new design solutions, devices and innovative systems in the field of marine engineering. At the same time, SEALAB aims to bring together, in a single project, the efforts and skills of university and industry research. Although SEALAB was born with the goal of providing an experimental platform to develop new solutions, the vehicle being developed can be used in an HSU (high-speed unmanned) version as a coastal patrol in autonomous and/or high speed (close to 200 km/h) remote driving. It could potentially be employed in civil defence or for the coast guard, etc.

Vibrations Laboratory

The Vibrations Laboratory conducts research in the fields of smart structures, vibration control, noise generation and transmission, and structural integrity monitoring. In particular, research focuses on: (a) non-destructive fresco integrity diagnosis with Doppler vibrometer scanning lasers; (b) development of acoustic-vibrators for the location of historically significant fresco and plaster defects; (c) development of lighter and cheaper flexible robotic systems; (d) development of smart structures, equipped with piezoelectric sensors such as thin plates or panels for noise and vibration control; (e) finite element modelling for the design of structures in the civil and industrial sectors, such as wood laminated orthotropic structures; and (f) energy analysis of buildings.

Humanitarian Demining Laboratory (HDL)

The main aim of the Humanitarian Demining Laboratory (HDL) is to develop new anti-personnel mine detection devices for humanitarian demining. Experimental activity is being carried out on a promising original active thermal technology based on localised heating pulses and temperature sensing. With the aim of developing a multi-sensor platform employing data fusion and collaborating robotic agents, vibrometric/acoustic and GPR techniques are also employed. For experimental purposes, a computer-controlled cart that can move over a sand box while holding a heater and other instrumentation has been realised. In the sand box, two accurate low-metal-content mine surrogates of different materials are hidden, together with another object. An outdoor "minefield" has also recently been realised.

Functional Multiscale Metamaterials and Smart Systems Lab

The activities carried out in the Functional Multiscale Metamaterials and Smart Systems Lab range from numerical modelling to experiments on micro- and nano-structural and functional materials. The materials have many applications, from biomaterials to energy harvesting. The laboratory has equipment for electron, ionic and atomic force microscopy (including advanced technologies for spectroscopy and nanoindentation) for the structural characterisation of micro- and nano- innovative materials.

MEMOCS Journal

The International Research Center for the Mathematics and Mechanics of Complex Systems has founded the homonymous journal Mathematics and Mechanics of Complex Systems, abbreviated to MEMOCS, for the benefit of the community of researchers in mechanics and mathematics. MEMOCS is peer-reviewed, indexed in all major databases and free to both authors and readers. It publishes articles from diverse scientific fields with a specific emphasis on mechanics. Articles must rely on the application or development of rigorous mathematical methods. Indeed, the journal is intended to foster a multidisciplinary approach to knowledge firmly based on mathematical foundations. It will serve as a forum where scientists from different disciplines meet to share a common, rational vision of science and technology. It is intended to support and divulge research whose primary goal is to develop mathematical methods and tools for the study of complexity. The journal also fosters and publishes original research in related areas of mathematics of proven applicability, such as variational methods, numerical methods and optimisation techniques. Besides their intrinsic interest, such treatments can become heuristic and epistemological tools for further investigations and provide methods for deriving predictions from postulated theories. Papers focusing on and clarifying aspects of the history of mathematics and science are also welcome. All methodologies and points of view, if rigorously applied, are considered.

To find out more information, please visit the website http://memocs.org.

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Francesco dell'Isola received cum laude his degree in physics at the University of Naples "Federico II" in 1986. At the same university, in 1992, he received a PhD in mathematical physics with a thesis on the rational thermodynamics of nonmaterial bidimensional continua. Since 2006, he has been a full professor of solid and structural mechanics at "La Sapienza" University of Rome. Professor dell'Isola has authored more than 150 papers in international journals.

Luca Placidi graduated cum laude in physics at the University of Naples "Federico II" in 2001 and in engineering at the Virginia Polytechnic Institute in 2002. He received a PhD in 2004 from the Technical University of Darmstadt and a second one in 2006 from "La Sapienza" University of Rome. He has authored five books and more than 40 papers in journals. Since 2011, he has been an assistant professor at the International Telematic University Uninettuno.

Emilio Barchiesi completed cum laude his MSc in mathematical engineering at the University of L'Aquila (Italy) in 2016, defending a thesis on the numerical identification of mathematical models for the description of engineering fabrics. He is currently pursuing his PhD studies in theoretical and applied mechanics at "La Sapienza" University of Rome. His main research interests lie in homogenisation theory, computational mechanics, higher gradient continua and variational methods.

Join the London Mathematical Society (LMS)

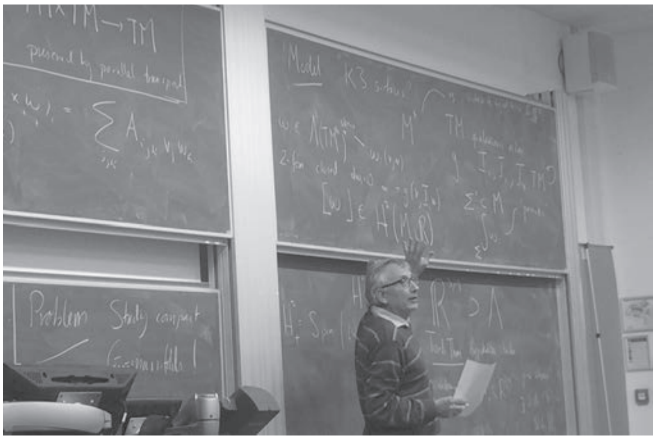
Elizabeth Fisher (London Mathematical Society, London, UK)

About the LMS

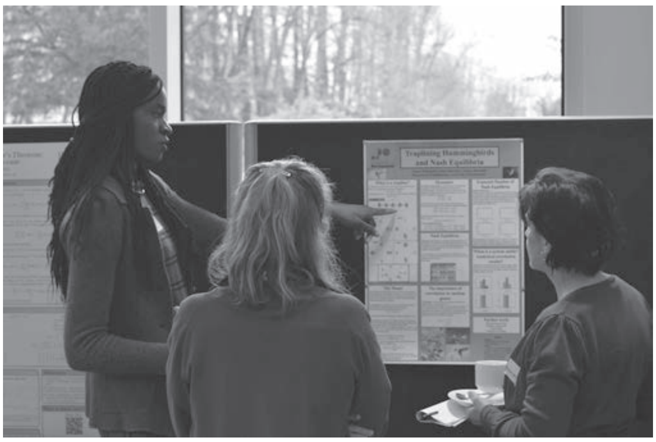
As a UK-wide learned society for mathematics, the London Mathematical Society (LMS) advances, disseminates and promotes mathematical knowledge, both nationally and internationally. Its activities include:

- Grants to support conferences and collaborative research between UK based and non-UK based mathematicians, e.g. Research in Pairs (https://www.lms.ac.uk/grants/research-pairs-scheme-4).
- Training opportunities aimed at Young Mathematicians and Early Career Researchers, e.g. LMS-CMI Research Schools: https://www.lms.ac.uk/events/lms-cmi-research-schools.
- Prestigious prizes to recognise achievements in mathematics research, with nine Whitehead Prize winners going on to win EMS Prizes: https://www.lms.ac.uk/prizes.
- Thirteen international peer-reviewed journals, seven of which are in collaboration with other learned societies and institutions, and two book series (lecture notes and student texts): https://www.lms.ac.uk/publications.
- Scientific lectures and meetings for research mathematicians, including meetings at the ECM and the ICM: https://www.lms.ac.uk/events/society-meetings.
- Representation of mathematics research and education to Government and other national policymakers and sponsors: https://www.lms.ac.uk/policy/policy-consultations.
- Participation in international mathematical initiatives and promoting the discipline more widely, e.g. supporting mathematics in Africa through Mentoring African Research in Mathematics (MARM) and grants to conferences at the African Mathematics Millennium Science Initiative (AMMSI): https://www.lms.ac.uk/grants/international-grants#Africa.

LMS Members at the Annual General Meeting, London, November 2016.



Simon Donaldson gives a talk at the South West & South Wales Regional Meeting, Bath, December 2016.



Participants at the LMS Women in Mathematics Day, Edinburgh, April 2016.

Speakers at the Research School, Belfast, September 2016.

Further information about recent LMS activities can also be found in the Society's 2015–16 Annual Review: www.lms.ac.uk/sites/lms.ac.uk/files/files/files/reports/LMS%20Annual%20review%202016%20web.pdf.

The LMS and the European Mathematical Society

The LMS has been a member society of the EMS since 1990. The collaboration between the LMS and the EMS has included:

- Hosting the EMS Ethics Committee in 2013 and both the EMS Executive Committee and the EMS Applied Mathematics Committee in 2014.
- Travel Grants to European Congresses of Mathematics.
- Participation of LMS delegates at the EMS Council Meetings.

This partnership was recently celebrated at a Joint Meeting held at the University of Birmingham in September 2015 to mark the societies' respective anniversaries: 25 years for the EMS and 150 years for the LMS.

The LMS is pleased to have reciprocal agreements with 20 mathematical societies. Any non-UK based EMS members thinking of joining the LMS are advised to check the list to see if their society is included when applying: https://www.lms.ac.uk/membership/membership-categories#Reciprocity.

How to join

The LMS welcomes applications via its online form: https://www.lms.ac.uk/membership/online-application, and applicants are formally elected to membership at one of the society meetings of the LMS.

Do visit the website for full details on the application process (https://www.lms.ac.uk/membership/how-join) and the current fees (https://www.lms.ac.uk/sites/lms.ac.uk/files/Membership/Subscription%20Rates%20and%20Notes%202016-17%20updated.pdf).

Benefits of LMS membership include:

- Membership of a vibrant, national and international mathematics community.
- Networking opportunities.

LMS Immediate Past President Terry Lyons FRS (Oxford), EMS-LMS Meeting Organiser Chris Parker (Birmingham) and EMS President Pavel Exner (Academy of Sciences of the Czech Republic) at the Joint LMS-EMS Anniversary Meeting, Birmingham, September 2015.

LMS President Simon Tavaré (Cambridge) welcomes new LMS members at the 7ECM in Berlin, July 2016

Membership of the LMS

The LMS has a membership of 2,800 members that make up a vibrant international mathematical community, with 20% of the LMS membership based outside the UK, including 234 members based in other European countries.

Membership of the LMS falls into three categories:

- *Ordinary membership*: for academic staff, mathematicians in other occupations and all those with a deep commitment to mathematics and an interest in mathematical research.
- *Associate membership*: for undergraduates, postgraduates and early career mathematicians who are within three years of completing their PhD.
- *Reciprocity membership*: for those not normally resident in the UK and also members of some overseas mathematical societies.

- Opportunities to influence national policy.
- Full voting rights in society elections – your chance to shape the future of the LMS.
- A complimentary monthly newsletter – available in print and online.
- Regular members-only LMS e-Updates.
- Opportunities to attend events hosted by the society.
- Free online subscriptions to the Bulletin of the London Mathematical Society, the Journal of the London Mathematical Society, the Proceedings of the London Mathematical Society and Nonlinearity (published jointly with the Institute of Physics).
- Members discount on other selected LMS publications: 25% discount on the LMS Lecture Note Series and 25% discount on LMS Student Texts.
- Use of the Verblunsky Members' Room at De Morgan House, Russell Square, London.

- Use of University College London Library, where the society's library is housed.
- The opportunity to sign the LMS Members' Book, which dates back to 1865 when the society was founded and which contains signatures of members throughout the years, including Augustus De Morgan, Henri Poincaré, G. H. Hardy and Mary Cartwright.

Contact the LMS

We would be pleased to hear from you. For queries about membership, do email us at membership@lms.ac.uk and for regular news from the LMS, follow us on Twitter @ LondMathSoc.

And what about Brexit?

Following the Brexit vote, the LMS would like to express its support and solidarity with the 31% of the UK mathematical academic community from other EU states. We would like to reassure our EU friends and colleagues, wherever they are based, that we are keen to maintain our close mathematical and professional relationships with the rest of Europe, however the Brexit process proceeds. The EU referendum statement from the Council for the Mathematical Sciences (CMS), of which the LMS is a member, is also available at https://www.lms.ac.uk/news-entry/05072016-1201/council-mathematical-sciences-eu-referendum-statement.

EMS President Pavel Exner and LMS Vice-President John Greenlees at the UK Parliamentary Links Day with Science, June 2016.

Elizabeth Fisher
LMS Membership & Activities Officer

ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

Call for nominations for the 2017 ICMI Felix Klein and Hans Freudenthal Awards

Since 2003, the International Commission on Mathematical Instruction (ICMI) has awarded biannually two awards to recognise outstanding accomplishments in mathematics education research: the Felix Klein Medal and the Hans Freudenthal Medal (http://www.mathunion.org/icmi/activities/awards/introduction/).

The Klein and Freudenthal Awards Committee consists of a Chair (Professor Anna Sfard), who is nominated by the President of the ICMI, and five other members who remain anonymous until their terms have come to an end. The committee is currently entering the 2017 cycle of selecting awardees and welcomes nominations for the two awards from individuals or groups of individuals in the mathematics education community.

Practical information can be seen at http://www.mathunion.org/icmi/activities/awards/call-for-awards-2017/.

All nominations must be sent no later than 15 April 2017.

The USA–Finland Workshop

A few months ago, the U.S. National Commission on Mathematics Instruction in collaboration with the University of Helsinki held a Workshop on Supporting Mathematics Teachers and Teaching in the United States and Finland. The bilateral meeting of U.S. and Finnish mathematics educators was held on 1–2 August at the University of Helsinki in Helsinki, Finland, and was attended by 30 experts from both nations and approximately 70 international experts virtually. The workshop was sponsored by Åbo Akademi University, Högskolestiftelsen i Österbotten, the National Science Foundation and Svensk-Österbottniska samfundet.

Videos of the workshop sessions, presentations (in PDF format for download) and background readings are NOW available at http://sites.nationalacademies.org/PGA/biso/ICMI/PGA_173314.

The 69th CIEAEM Conference

The 69th Conference of the *Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques* (International Commission for the Study and Improvement of Mathematics Teaching) will take place on 15–19 July 2017 at the Freie Universität Berlin, Germany. The theme of the event is "Mathematisation: social process & didactic principle". The Call for Papers was distributed in December 2016. For more details, see http://www.cieaem.org/.

ICME 14 and ICME 15

The next International Congress on Mathematical Education ICME 14 will be held in Shanghai on 12–19 July 2020 (see http://www.icme14.org/ and watch out for upcoming announcements and further information).

The ICMI is calling for intentions to bid for ICME 15 (to be held in July 2024). Mathematics education and/or mathematics associations of potential host countries are invited to consider hosting the congress.

Important dates:

- Preliminary declaration of intent to present a bid to act as host for ICME 15 should be received by the Secretary-General of the ICMI (abraham.arcavi@weizmann.ac.il) by 1 December 2017.
- Firm bids should reach the Secretary-General by 1 November 2018 (in 12 hard copies).

For further details please see http://www.mathunion.org/icmi/conferences/icme-international-congress-on-mathematical-education/icme-15-2024/.

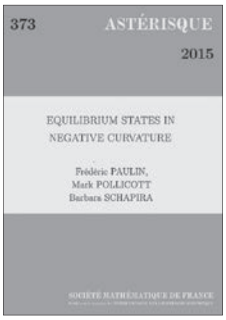
Subscribing to the ICMI Newsletter

Most of the previous material was first published in the ICMI Newsletter.
There are two ways of subscribing to ICMI News:

1. Visit www.mathunion.org/index.php using a Web browser and go to the "Subscribe" button to subscribe to ICMI News online.
2. Send an email to icmi-news-request@mathunion.org with the Subject-line: "Subject: subscribe".

All previous issues can be seen at http://www.mathunion.org/pipermail/icmi-news.

Book Reviews



Frédéric Paulin, Mark Pollicott and Barbara Schapira

Equilibrium States in Negative Curvature

Astérisque, 2015
289 p.
ISBN 978-2-85629-818-3

Reviewer: Katrin Gelfert

The Newsletter thanks zbMATH and Karin Gelfert for the permission to republish this review, originally appeared as Zbl 1347.37001.

This book studies Gibbs measures (or states) for the geodesic flows of negatively curved Riemannian manifolds. Its framework allows, in particular, the removal of the compactness assumption on the manifold. The authors consider a Hölder continuous function (also called a potential) F defined on the unit tangent bundle T^1M and prove many results on the existence, uniqueness and finiteness of the Gibbs measure of F. This very comprehensive monograph not only includes detailed proofs and background material but also highlights possible further developments and open problems.

Gibbs measures have a thermodynamic origin and were introduced for hyperbolic dynamical systems by *Ya. G. Sinai* [Usp. Mat. Nauk 27, No. 4(166), 21–64 (1972; Zbl 0246.28008)], *R. Bowen* [Equilibrium states and the ergodic theory of Anosov diffeomorphisms. 2nd revised ed. Berlin: Springer (2008; Zbl 1172.37001)] and *D. Ruelle* [Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics. 2nd edition. Cambridge: Cambridge University Press. (2004; Zbl 1062.82001)], revealing their intimate relation with symbolic dynamical systems over finite alphabets. In the case of the non-wandering set of the geodesic flow (and, in particular, when M is compact), this reduction to a symbolic system allows a detailed analysis of Gibbs (equilibrium) measures and the (weighted) distribution of periodic orbits. No such coding theory that does not lose geometric information is known in the non-compact case. The authors circumvent this difficulty by geometrically constructing and studying Gibbs measures in this general case.

The guiding objects are three numerical invariants associated to the weighted dynamics. The *critical exponent* is defined by means of the fundamental group Γ of M acting by isometries on the universal cover $\widetilde{M}$ of M and measures the exponential growth rate of the orbit points of the group weighted by the (lift to $T^1\widetilde{M}$ of the) potential F. The *Gurevich pressure* is the exponential growth rate of the values of the potential along periodic orbits of increasing period (the possibility of non-compact manifolds requires a restriction to periodic orbits that meet a given compact set, though it is shown that the resulting rate does not depend on the choice of set). The *topological pressure* (in other contexts also called the *variational pressure*) is defined as the supremum of the measure-theoretic entropy plus the averaged potential with respect to the measure, where the supremum is taken over all invariant Borel probability measures. A measure realising the supremum is called an *equilibrium state* for the potential. It is shown that all these quantities are well defined and, moreover, that all three of them agree (which extends [*A. Manning*, Ann. Math. (2) 110, 567–573 (1979; Zbl 0426.58016)], [*D. Ruelle*, Bol. Soc. Bras. Mat. 12, No. 1, 95–99 (1981; Zbl 0599.58038)] and [*W. Parry*, Lect. Notes Math. 1342, 617–625 (1988; Zbl 0667.58056)] in the case when M is compact). See Chapters 3, 4 and 6.

The book develops a Patterson–Sullivan theory for Gibbs measures (Chapter 3). Here, the *Gibbs measure* m_F of F is defined through *Mohsen*'s extension [Ann. Sci. Éc. Norm. Supér. (4) 40, No. 2, 191–207 (2007; Zbl 1128.58008)] of the Patterson–Sullivan construction [*S. Patterson*, Acta Math. 136, 241–273 (1976; Zbl 0336.30005) and *D. Sullivan*, Publ. Math., Inst. Hautes Étud. Sci. 50, 171–202 (1979; 0439.30034)]. In the case when M is compact, they recover, up to some scalar multiple, for $F=0$, (the Patterson-Sullivan construction of) the Bowen–Margulis measure and, for F being the *(un) stable Jacobian* (that is, the pointwise exponential growth rate of the Jacobian of the geodesic flow restricted to the strong (un)stable manifold), the Liouville measure. One fundamental result of the book is the *Variational Principle*: in the case when the Gibbs measure of F is finite then this (properly normalised) measure is the unique equilibrium state for F; otherwise, there exists no equilibrium state.

Under the hypotheses that the Gibbs measure m_F is finite and mixing, Dirac masses on an orbit weighted by the potential are asymptotically equidistributed toward the product of Patterson densities on the sphere at infinity $\partial_\infty\widetilde{M}$, which in turn provides counting asymptotics of orbit points. This type of result translates to results on the equidistribution of periodic orbits and on counting of periodic orbits (which, in the compact case, are due to *R. Bowen* [Am. J. Math. 94, 413–423 (1972; 0249.53033)]); see Chapter 9. The authors show the equivalence of the following properties:

- The Poincaré series of (Γ, F) diverges at the critical exponent.
- The conical limit set of Γ has positive Patterson measure.
- The action of Γ on $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$ is ergodic and conservative.

- The geodesic flow is ergodic and conservative with respect to the Gibbs measure of F (Chapter 5).

Geometric criteria for the finiteness and the mixing property of m_F are also discussed (Chapter 8). One particular and very natural focus is on the case when F is the (un) stable Jacobian (Chapter 7).

Further investigations concern the ergodic theory of the strong unstable foliation of T^1M (Chapter 10). Again, under the hypotheses that m_F is finite and mixing, there exists an (up to scalar multiple) unique family of transverse measures for the strong unstable foliation that is quasi-invariant for the holonomy map (taking into account the potential F), which enables unique ergodicity results. This extends work by *T. Roblin* [Mém. Soc. Math. Fr., Nouv. Sér. 95, 96 p. (2003; Zbl 1056.37034)] and builds essentially on work by the last author [Propriétés ergodiques du feuilletage horosphérique d'une variété à courbure négative. Orléans: Univ. Orléans (PhD Thesis) (2003)].

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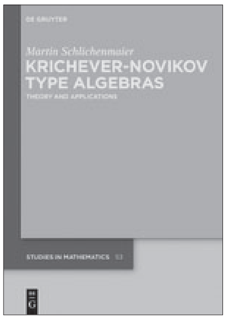

Martin Schlichenmaier

Krichever-Novikov Type Algebras. Theory and Applications

De Gruyter, 2014
xv, 360 p.
ISBN 978-3-11-027964-1

Reviewer: Alice Fialowski

The Newsletter thanks zbMATH and Alice Fialowski for the permission to republish this review, originally appeared as Zbl 1347.17001.

The goal of this book is to acquaint the reader with Krichever–Novikov type infinite dimensional Lie algebras, to study their properties and to highlight the most important applications.

The book is excellent for studying the topic. It has 14 chapters, an extended bibliography and an index. The only prerequisites are the theories of Lie algebras and Riemann surfaces. To make the presentation self-contained, Chapter 1 recalls the basics of Lie algebras, superalgebras, Virasoro and current algebras, and Riemann surfaces of genus $g \geq 0$. In Chapter 2, algebraic structures of specific meromorphic objects on a compact Riemann surface are studied. Such algebras are called Krichever–Novikov type if they are holomorphic outside a fixed set of points. They form an important class of infinite dimensional Lie algebras, which are far from being understood in general. Krichever–Novikov type algebras nicely combine geometric and algebraic properties.

These algebras are, in general, not graded, which is often useful for dealing with infinite dimensional algebras. Instead, as Krichever and Novikov observed, a weaker concept 'almost grading' can be introduced, which makes the necessary constructions possible. The definition is given in Chapter 3. The original almost-graded structure is given for the 2-point situation and the author extends it to multi-point cases. This chapter also defines almost grading for the type of algebras introduced earlier.

Chapter 4 deals with some technical but useful details of almost grading, like the proof of existence of an almost grading. For this, the author uses Riemann–Roch type arguments to show the existence of certain basis elements that give the almost-graded structure. In Chapter 5, there are explicit expressions for the homogeneous basis elements.

The following chapters deal with representations of algebras. Firstly, in Chapter 6, central extensions of Krichever–Novikov type algebras are studied: their relation to Lie algebra cohomology and their construction for geometrically induced Lie algebras. Central extensions appear naturally in quantisations of classical field theories. Chapter 7 studies fermionic Fock space representations or, equivalently, semi-infinite wedge representations of these algebras. One can also obtain a representation of a certain central extension of the full algebra of differential operators with the help of a regularisation procedure. As a technical tool, infinite dimensional matrix algebras are used.

In Chapter 8, the author shows that the semi-infinite wedge form also comes with the representation of an algebra that has a Clifford algebra-like structure. The corresponding field theory system is called the b–c system. In a mathematical context, the operators b and c are defined via anticommutators. In this chapter, arbitrary representation spaces of these field operators are considered and the energy momentum tensor is defined. Its "modes" define a representation of an almost-graded central extension of a vector field algebra. The 2-point and the multi-point situations are discussed.

Chapter 9 contains the detailed study of higher genus current algebras and their central extensions. The almost-graded central extensions are classified. These algebras correspond to gauge symmetries. As examples of representations, the Verma modules and fermionic Fock space representations are introduced. Chapter 10 presents the Sugawara construction for arbitrary genus in the multi-point situation. This construction relates gauge symmetry to conformal symmetry, which is realised by vector field algebras and their central extensions.

The last four chapters cover the main applications of the theory. Chapter 11 presents the author's and O. Sheinman's global operator approach to Wess–Zumino–Novikov–Witten models and the Knizhnik–Zamolodchikov connection. Their approach uses global objects, which are Krichever–Novikov algebras and their representations. A crucial point is that a certain subspace of the Krichever–Novikov algebra of vector fields can be identified with tangent directions on the moduli space of the geometric data. One can define conformal blocks and also, with the global Sugawara construction, the higher genus multi-point Knizhnik–Zamolodchikov connection. It should be mentioned that this global construction (so far) only works over an open dense subset of the moduli space. The approach of other authors provides a valid theory for the compactified moduli space.

Another application of Krichever–Novikov type algebras is related to deformations. This is the material of Chapter 12 and the results have been investigated jointly with A. Fialowski. It turns out that Witt/Virasoro algebra deforms into elliptic vector field Krichever–Novikov algebras. These families are locally nontrivial, in spite of the fact that Witt/Virasoro algebra is formally rigid, because the second cohomology group with values in the adjoint module is trivial. The same is true of current algebras. The point is that, in the infinite dimensional situation, the vanishing cohomology space does not imply that the algebra is rigid in the geometric sense.

Chapter 13 introduces Lax operator algebras, which are a new class of global current type algebra. They are related to integrable systems. In these algebras, additional singularities are allowed. It is possible for such Lax operator algebras to introduce almost grading and classify associated almost-graded central extensions. The construction works so far for $gl(n)$, $sl(n)$, $so(n)$, $sp(2n)$ and G_2. In the last chapter, there are further developments and related subjects, mainly noted through references.

The book should convince the reader that, beside Krichever–Novikov type algebras being mathematically very interesting infinite dimensional geometric examples, they are important in conformal field theory, integrable systems, deformations and many other topics.

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Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

God created the natural numbers. The rest is the work of man.

Leopold Kronecker (1823–1891)

The column *Solved and Unsolved Problems* will continue presenting six proposed problems and two open problems, as has been done over recent years. The set of proposed and open problems in each issue will be devoted to a specific field of mathematics. In every issue featuring this column, solutions will be presented to the proposed problems from the previous issue along with the names of solvers. Possible progress toward the solution of any of the open problems proposed in this column will also be featured. The goal of the *Solved and Unsolved Problems* column is to provide a series of intriguing proposed problems and open problems ranging over several areas of mathematics. Effort will also be made to present problems of an interdisciplinary flavour.

The column in this issue is devoted to number theory. As is well known, number theory is one of the oldest and most vibrant areas of pure mathematics. Over the last few decades, it has also found important applicability in various scientific domains such as cryptography, coding theory, theoretical computer science and even nuclear physics and quantum information theory.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

171. Prove that every integer can be written in infinitely many ways in the form

$$\pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2$$

for some choices of signs + and −.

(Dorin Andrica, Babeşş Bolyai University, Cluj-Napoca, Romania)

172. Show that, for every integer $n \geq 1$ and every real number $a \geq 1$, one has

$$\frac{1}{2n} \leq \frac{1}{n^{a+1}} \sum_{k=1}^{n} k^a - \frac{1}{a+1} < \frac{1}{2n}\left(1 + \frac{1}{2n}\right)^a.$$

(László Tóth, University of Pécs, Hungary)

173. Let $c_n(k)$ denote the Ramanujan sum, defined as the sum of kth powers of the primitive nth roots of unity. Show that, for any integers n, k, a with $n \geq 1$,

$$\sum_{d|n} c_d(k) a^{n/d} \equiv 0 \pmod{n}.$$

(László Tóth, University of Pécs, Hungary)

174. Prove, disprove or conjecture:

1. There are infinitely many primes with at least one 7 in their decimal expansion.
2. There are infinitely many primes where 7 occurs at least 2017 times in their decimal expansion.
3. There are infinitely many primes where at most one-quarter of the digits in their decimal expansion are 7s.
4. There are infinitely many primes where at most half the digits in their decimal expansion are 7s.
5. There are infinitely many primes where 7 does not occur in their decimal expansion.

Note. Let p be a prime. Then, the decimal expansion of $1/p$ is often called the "decimal expansion of p".

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

175. Show that there is an infinite sequence of primes $p_1 < p_2 < p_3 < \cdots$ such that p_2 is formed by appending a number in front of p_1, p_3 is formed by appending a number in front of p_2 and so on. For example, we could have $p_1 = 3$, $p_2 = 13$, $p_3 = 313$, $p_4 = 3313$, $p_5 = 13313, \ldots$. Of course, you might have to add more than one digit at a time. Find a bound on how many digits you need to add to ensure it can be done.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

176. Consider all pairs of integers x, y with the property that $xy - 1$ is divisible by the prime number 2017. If three such integral pairs lie on a straight line on the xy−plane, show that both the vertical distance and the horizontal distance of at least two of such three integral pairs are divisible by 2017.

(W. S. Cheung, Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong)

II Two new open problems (on ζ-functions) by Preda Mihăilescu, Mathematisches Institut, Göttingen, Germany

Let K be a number field, let $I(K)$ denote the set of integral ideals of K, including the trivial ideal $1 = \mathcal{O}(K)$, let $P(K) \subset I(K)$ denote the principal ideals and let $C(K)$ be the ideal class group of K. Denote by $N_K = N$ the absolute norm $\mathbf{N}_{K/\mathbb{Q}}$ and let $d = [K : \mathbb{Q}]$. The Dedekind ζ-function of K is

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I} \frac{1}{|N_K \mathfrak{a}|^s}. \quad (1)$$

If $K = \mathbb{Q}$ then

$$\zeta_K(s) = \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

is the Riemann ζ-function. More precisely, the Dirichlet series above define the respective ζ-functions on the half plane $\mathbb{H}_1 = \{s \in \mathbb{C} : \Re(s) > 1\}$, on which the series are absolutely convergent. They have a pole at $s = 1$ and it is proved by means of Mellin transforms that they have an analytic continuation to $\mathbb{C}$, with no other singularity except for $s = 1$. The properties of the $\zeta_K(s)$ have been investigated by a series of classical mathematicians, including Dirichlet, Weierstraß and Hecke. We refer to Lang's Algebraic Number Theory [La], Chapters V–VIII, for a review of the classical results on $\zeta_K(s)$. Let $\mathfrak{K} \in C(K)$ be a class.

The counting function $J(\mathfrak{K}, t)$ for $t \in \mathbb{R}_{>1}$ plays an essential role in this context. It counts the number of ideals $\mathfrak{a} \in I(K) \cap \mathfrak{K}$ that have norm less than t. This is done by choosing some fixed ideal $\mathfrak{b} \in \mathfrak{K}^{-1}$ and counting the number of ideals $(\alpha) \in P(K) \cap \mathfrak{b}$ that have norm bounded by $|N(\alpha)| < t|N(\mathfrak{b})|$. It is an elementary fact (proved in [La]) that these ideals are in one-to-one correspondence to the ideals of $\mathfrak{K}$ with norm less than t. Let $E = O^\times(K)$ be the global units; certainly, for $\alpha \in O(K)$, the principal ideal $(\alpha) \in P(K)$ is generated by any element of the orbit αE of α under the action of the units by multiplication. We are thus reduced to the problem of counting orbits of numbers $\alpha \in \mathfrak{b}$ under the action of the units. Here enters the geometry of numbers. For details of the classical estimates, we refer the reader to any detailed deduction of the classical results in any book on algebraic number theory that also treats analytical results – the account of Lang is a possible example.

Briefly, the numbers of the field K have two representations in $\mathbb{R}^{r+1}$, with $r = r_1 + r_2 - 1$ the Dirichlet rank of the units. The first representation is $\mu : \mathbb{K}^\times \to \mathbb{R}^{r+1}$ via $x \mapsto (|\sigma_i(x)|^{\delta_i})_{i=1}^{r+1}$, with $(\sigma_i)_{i=1}^{r_1}$ an enumeration of the real embeddings of K and $(\sigma_j)_{j=r_1+1}^{r+1}$ an enumeration of representatives of pairs of complex conjugate embeddings; the exponents are $\delta_i = 1$ for real embeddings and $\delta_j = 2$ for complex embeddings. The map μ is continued by an additive one $\lambda : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$, defined by $\lambda(\mu(x))_k = \log(|\mu(x)_k|)$ for $k = 1, 2, \ldots, r$ and $\lambda(\mu(x))_{r+1} = |N(x)|^{1/d}$. The fundamental classical result deduced by investigating $J(\mathfrak{K}, t)$ under these maps is

$$J(\mathfrak{K}, t) = \rho_K t + O(t^{1-1/d}). \tag{2}$$

The constant ρ_K is completely determined in terms of the data of the field, which are Δ, R, w – the discriminant, the regulator and the number of roots of unity of the field respectively. It is independent of $\mathfrak{K}$ and its value is, with these notations,

$$\rho_K = \frac{2^{r+1}\pi^{r_2} R}{w\sqrt{\Delta}}.$$

The order of magnitude of the error term is determined by a crude argument involving the fact that the fundamental domain $D(1) \subset \mathbb{R}^{r+1}$ used for estimating $J(\mathfrak{K}, t)$ is Lipschitz-parametrisable. One can rephrase the formula above by stating that there certainly exists some constant $\gamma_K(\mathfrak{K})$, depending only on K and possibly also on the class $\mathfrak{K}$, such that

$$|J(\mathfrak{K}, t) - \rho_K t| \leq \gamma_K(\mathfrak{K}) \cdot t^{1-1/d}, \quad \text{for } t > \Delta.$$

It is important to choose a lower bound for t in order to obtain an accurate order of magnitude but the bound Δ chosen in our definition is not stringent. One may expect, for reasons discussed in the Remarks below, that these constants are quite small. However, the present methods of estimates, which have only recently been worked out by van Order and Murty [MO] to the effect of obtaining explicit bounds on $\gamma_K(\mathfrak{K})$, yield excessively large values for the bound. We shall make the definition of our constant uniform to make it independent of the class and then state our first problem, which is a conjecture. We define the constant γ_K by

$$\gamma_K := \inf_{t > \sqrt{\Delta}, \mathfrak{K} \in C(K)} \left\{ \gamma \in \mathbb{R}_{>0} : |J(\mathfrak{K}, t) - \rho_K t| \leq \gamma \cdot t^{1-1/d} \right\}. \tag{3}$$

We define the surface of the units as follows: for a fundamental system of units $u_i \in E(K)$, we let $S(\vec{u})$ be the surface of the fundamental parallelepiped of the lattice spanned by the vectors $w_i := \lambda(\mu(u_i))$. The surface $S(E(K)) = \inf_{\vec{u}} S(\vec{u})$, the infimum over all the fundamental systems of units of K.

177*. We keep the notation introduced above, in particular the notation in (2) and (3).

(i) Prove that $\gamma_K = c_1 R^{a_1} \Delta^{a_2} + c_2 S(E)^{b_1} \cdot \Delta^{b_2}$, with constants $c_1, c_2 > 0$ and powers $a_1, a_2, b_1, b_2 \in \mathbb{Q}$, which do not depend on the extension degree d.

(ii) Prove that there is an additional constant $0 < C < 1$ such that

$$|J(\mathfrak{K}, t) - \rho_K t| > C\gamma_K t^{1-1/d}$$

for all $t > \Delta$.

We continue our investigation of the counting function J for arbitrary number fields K with a problem on the geometry of numbers. For a given class $\mathfrak{K}$, one can consider the lattices $L_\mathfrak{a}$ spanned by some ideal $\mathfrak{a} \in \mathfrak{K}$ as a $\mathbb{Z}$-module in Minkowski space. We are interested in determining how close such a lattice can come to orthonormal lattices, if we allow $\mathfrak{a}$ to take all the ideals in $\mathfrak{K}$ as its value. The following definition will introduce quantitative measures for the "distance" of a lattice to an orthonormal one. Let $\Lambda \subset \mathbb{R}^n$ be a full lattice, let $(v_i)_{i=1}^n \subset \mathbb{R}^n$ be a spanning set of generators and let A_v be the matrix with these vectors as columns. Let the Euclidean norm of a matrix $B = (b_{i,j})_{i,j=1}^n$ be the norm $\|B\| = \sqrt{\sum_{i,j} b_{i,j}^2}$ and let B^T denote the transpose. Then we define the *orthonormality defect* of this base by

$$\omega_v(\Lambda) = \inf_{\lambda \in \mathbb{R}_+} \|A \cdot A^T - \lambda I\|.$$

The orthonormality defect of the lattice is defined by $\omega(\Lambda) := \inf_v \omega_v(\Lambda)$, the infimum being over all bases of Λ.

Now, let a class $\mathfrak{K} \subset O(K)$ be fixed and $\mathfrak{b} \in \mathfrak{K}^{-1} \cap I(K)$ be any integral ideal. The image of $\mathfrak{b}$ under the map μ is a lattice $L_\mathfrak{b} \subset \mathbb{R}^{r+1}$. Let $s_\mathfrak{b} = |N(\mathfrak{b})|^{1/d}$ and normalise the lattice to $L'_\mathfrak{b} = L_\mathfrak{b}/s_\mathfrak{b}$, a lattice of volume one. The orthonormality defect of $\mathfrak{b}$ is naturally given by $\omega(\mathfrak{b}) = \omega(L'_\mathfrak{b})$. For our counting function, the choice of $\mathfrak{b}$ is arbitrary. We may multiply $\mathfrak{b}$ by field elements (not necessarily integral) and obtain ideals of the same class. This leads to defining the orthonormality defect of the class $\mathfrak{K}$ by

$$\omega(\mathfrak{K}) = \inf_{\mathfrak{b} \in \mathfrak{K}^{-1} \cap I(K)} \omega(\mathfrak{b}). \tag{4}$$

The defect of the class $\mathfrak{K}$ is thus defined by means of ideals in $\mathfrak{K}^{-1}$. The second problem concerns orthonormality defects of classes.

178*.

(i) Find an optimal estimate for the orthonormality defect $\omega(\mathfrak{K})$ of a class $\mathfrak{K} \in C(K)$.

(ii) Prove or disprove that the radii verify an ultrametric inequality

$$\omega(\mathfrak{K} \cdot \mathfrak{K}') \leq \max(\omega(\mathfrak{K}), \omega(\mathfrak{K}')) .$$

Remarks: The Riemann[1] zeta function $\zeta(z)$ has a Laurent expansion in a neighbourhood of its simple pole at $z = 1$:

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (z-1)^n, \tag{5}$$

where γ_n are the Stieltjes constants

$$\gamma_n = \lim_{n\to\infty} \left(\sum_{k=1}^{m} \frac{\ln^n k}{k} - \frac{\ln^{n+1} m}{m+1} \right), n = 0, 1, \ldots \tag{6}$$

Clearly, γ_0 is the Euler-Mascheroni constant and note that all the terms of the sequence $(\gamma_n)_{n\geq 0}$ are Euler-Mascheroni type constants. Here are the first decimals of γ_n for $n = 0, 1, 2, 3, 4, 5$.

$$\gamma_0 = 0.5772156649\ldots, \gamma_1 = -0.0728158454\ldots,$$
$$\gamma_2 = 0.0096903631\ldots, \gamma_3 = 0.0020538344\ldots,$$
$$\gamma_4 = 0.0023253700\ldots, \gamma_5 = 0.0007933238\ldots.$$

An elementary proof of the expansion (1) can be obtained by the Euler–Maclaurin summation formula. In the paper [AT], formula (1) and some asymptotic evaluations were obtained by using the Laplace transform. The behaviour of these constants suggests that the error term in (2) might be small, despite our present incapacity of finding appropriate estimates – hence the relevance of these two research problems.

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III Solutions

163. Find all positive integers m and n such that the integer

$$a_{m,n} = \underbrace{2\ldots2}_{m \text{ time}}\underbrace{5\ldots5}_{n \text{ time}}$$

is a perfect square.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. We have $a_{1,1} = 25 = 5^2$ and $a_{2,1} = 225 = 15^2$. In the first step, we will show that if $a_{m,n}$ is a perfect square then $n = 1$. We can write

$$\begin{aligned} a_{m,n} &= 2(10^{m+n-1} + \cdots + 10^n) + 5(10^{n-1} + \cdots + 1) \\ &= 2 \cdot 10^n \cdot \frac{10^m - 1}{9} + 5 \cdot \frac{10^n - 1}{9}. \end{aligned}$$

Therefore, the relation $a_{m,n} = x^2$ is equivalent to

$$2 \cdot 10^{m+n} + 3 \cdot 10^n - 5 = (3x)^2. \tag{7}$$

If $n \geq 2$, it follows that $3x$ is divisible by 5, hence $x = 5x_1$ for some positive integer x_1. Replacing in equation (7), we get the equation

$$2 \cdot 2^{m+n} \cdot 5^{m+n-1} + 3 \cdot 2^n \cdot 5^{n-1} - 1 = 5(3x_1)^2,$$

which is not possible.

Now, we will prove that for $m \geq 3$ the integer $a_{m,1} = \underbrace{2\ldots2}_{m \text{ time}}5$ is not a perfect square. For $n = 1$, equation (7) is equivalent to

$$2 \cdot 10^{m+1} + 25 = (3x)^2,$$

that is,

$$2 \cdot 10^{m+1} = (3x - 5)(3x + 5).$$

It follows that $3x - 5 = 2^a \cdot 5^b$ and $3x + 5 = 2^{m+2-a} \cdot 5^{m+1-b}$, where a and b are non-negative integers, hence

$$2^{m+2-a} \cdot 5^{m+1-b} - 2^a \cdot 5^b = 10, \tag{8}$$

that is,

$$2^{m+1-a} \cdot 5^{m-b} - 2^{a-1} \cdot 5^{b-1} = 1. \tag{9}$$

We consider the following cases for equation (9).

Case 1: $a = 1$. We obtain $2^m \cdot 5^{m+1-b} - 5^{b-1} = 1$. If $b = 1$ then it follows that $5^m = 2$, which is not possible for $m \geq 1$.

If $b = m$, we obtain $2^m - 5^{m-1} = 1$, which is not possible because $5^{m-1} > 2^m$ for $m \geq 1$.

Case 2: $a = m+1$. It follows that $5^{m-b} - 2^m \cdot 5^{b-1} = 1$. If $b = 1$, we get $5^{m-1} - 2^m = 1$, which is not possible because $5^{m-1} > 2^{m+1} > 2^m + 1$ when $m \geq 1$.

If $b = m$, we obtain $2^{m+2} \cdot 5^m = 0$, which is not possible.

In conclusion, the only solutions are $m = 1, n = 1$ and $m = 2, n = 1$. □

Also solved by Panagiotis T. Krasopoulos (Athens, Greece), Hans J. Munkholm, Ellen S. Munkholm (University of Southern Denmark, Odense, Denmark), F. Plastria (BUTO-Vrije Universiteit Brussel), José Hernández Santiago (Morelia, Michoacan, Mexico)

164. Prove that every power of 2015 can be written in the form $\frac{x^2+y^2}{x-y}$, with x and y positive integers.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. We have $2015 = 5 \cdot 13 \cdot 31$. Because 31 is congruent to 3 modulo 4, it follows that 31 divides both x and y, etc. We get $x = 31^n x_1, y = 31^n y_1$ and replace in the equation to obtain $x_1^2 + y_1^2 = 5^n \cdot 13^n(x_1 - y_1)$. But $5 \cdot 13 = 65 = 8^2 + 1^2$, hence $5^n \cdot 13^n = (8^2 + 1^2)^n = a^2 + b^2$, where we can assume that $a > b$. The equation is equivalent to $(x_1 + y_1)^2 + (x_1 - y_1)^2 - 2 \cdot 5^n \cdot 13^n(x_1 - y_1) + (5^n \cdot 13^n)^2 = (5^n \cdot 13^n)^2$, that is,

$$(x_1 + y_1)^2 + (5^n \cdot 13^n - x_1 + y_1)^2 = (5^n \cdot 13^n)^2.$$

The last equation is Pythagorean and we select solutions as

$$5^n \cdot 13^n = a^2 + b^2,\ x_1 + y_1 = a^2 - b^2,\ 5^n \cdot 13^n - x_1 + y_1 = 2ab\,,$$

where a and b are positive integers such that

$$5^n \cdot 13^n = (8^2 + 1^2)^n = a^2 + b^2 \ \text{ and } \ a > b\,.$$

It follows that

$$x_1 = a^2 - ab = a(a - b), y_1 = ab - b^2 = b(a - b)\,.$$

Finally, it follows that the equation is solvable and has solution

$$(x, y) = (31^n a(a - b), 31^n b(a - b))\,.$$

For example, for $n = 1$, we have $a = 8, b = 1$, hence we get the solution to the reduced equation modulo 31, $(x_1, y_1) = (8(8 - 1), 1(8 - 1)) = (56, 7)$. Finally, it follows that the equation is solvable and has solution

$$(x, y) = (31 \cdot 56, 31 \cdot 7) = (1736, 217)\,.$$

□

Also solved by Mihály Bencze (Brasov, Romania), Panagiotis T. Krasopoulos (Athens, Greece), Hans J. Munkholm, Ellen S. Munkholm (University of Southern Denmark, Odense, Denmark), F. Plastria (BUTO-Vrije Universiteit Brussel)

165. Find the smallest positive integer k such that, for any $n \geq k$, every degree n polynomial $f(x)$ over $\mathbb{Z}$ with leading coefficient 1 must be irreducible over $\mathbb{Z}$ if $|f(x)| = 1$ has not less than $[\frac{n}{2}] + 1$ distinct integral roots.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer. Suppose $f(x)$ is a degree n polynomial with leading coefficient 1 such that $|f(x)| = 1$ has at least $[\frac{n}{2}] + 1$ distinct integral roots. Assume that $f(x)$ is reducible, say, $f(x) = g(x)h(x)$, with $\deg g \leq \deg h$. Clearly we have $\deg g \leq [\frac{n}{2}]$.
Suppose $|f(x_i)| = 1$ for $i = 1, \ldots, m$ with $m \geq [\frac{n}{2}] + 1$, where $x_i \in \mathbb{Z}$ are distinct. We have $g(x_i) = \pm 1$ for all $i = 1, \ldots, m$. Without loss of generality, assume that $g(x_i) = 1$ for $1 \leq i \leq \ell$, $g(x_j) = -1$ for $\ell + 1 \leq j \leq m$, and $\ell \geq \frac{m}{2}$.
Then,

$$g(x) - 1 = (x - x_1)(x - x_2) \cdots (x - x_\ell)P(x)$$

for some polynomial $P(x)$. Observe that

$$\ell \leq \deg g \leq \left[\frac{n}{2}\right] < \left[\frac{n}{2}\right] + 1 \leq m\,.$$

Since

$$g(x_j) = -1 \qquad \forall\, \ell + 1 \leq j \leq m\,,$$

we have

$$(x_j - x_1)(x_j - x_2) \cdots (x_j - x_\ell)P(x_j) = -2 \qquad \forall\, \ell + 1 \leq j \leq m\,,$$

and so

$$(x_j - x_1)(x_j - x_2) \cdots (x_j - x_\ell)|2 \qquad \forall\, \ell + 1 \leq j \leq m\,. \qquad (*)$$

If $\ell \geq 4$, $(x_j - x_1)(x_j - x_2) \cdots (x_j - x_\ell)$ is a product of 4 or more distinct non-zero integers and so its absolute value is ≥ 4 and cannot divide 2. Hence $\ell \leq 3$.

If $\ell = 3$, (*) reduces to

$$(x_j - x_1)(x_j - x_2)(x_j - x_3)|2\,.$$

Observe that there can be at most one $a \in \mathbb{Z}$ satisfying

$$(a - x_1)(a - x_2)(a - x_3)|2\,.$$

Thus, we must have $m = 3$ or 4.
If $\ell \leq 2$, since $\ell \geq \frac{m}{2}$, we also have $m \leq 4$.

Since $m \geq [\frac{n}{2}] + 1$, we have $n \leq 7$.

This shows that, for any $n > 7$, if $|f(x)| = 1$ has not less than $[\frac{n}{2}] + 1$ distinct integral roots then $f(x)$ is irreducible.

Finally, observe that $k = 7$. In fact, for $n = 7$, the function $f(x)$ defined by

$$h(x) = 1 + x(x - 3)(x - 2)(x - 1)$$
$$g(x) = 1 + x(x - 3)(x - 1)$$
$$f(x) = g(x)h(x)$$

is reducible, whereas $|f(x)| = 1$ when $x = 0, 1, 2, 3$. So k cannot be made smaller. □

Also solved by Mihály Bencze (Brasov, Romania), F. Plastria (BUTO-Vrije Universiteit Brussel)

166. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing (f not necessarily continuous). If $f(0) > 0$ and $f(100) < 100$, show that there exists $x \in \mathbb{R}$ such that $f(x) = x$.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer. Define $A := \{x \in [0, 100] : f(x) \geq x\}$. Since $0 \in A$, $A \neq \phi$, let $a := \sup A$. Clearly, $a < 100$. For any $\varepsilon > 0$, there exists $x \in A$ such that $a - \varepsilon < x \leq a$. Hence,

$$a - f(a) \leq a - f(x) < x + \varepsilon - f(x) < \varepsilon\,.$$

As $\varepsilon > 0$ is arbitrary, we have $a \leq f(a)$.

Suppose $f(a) - a = \delta > 0$. Then, for any $x \in (a, a + \delta) \cap [0, 100]$, x does not belong to A and, by the monotonicity of f, we have

$$f(x) \geq f(a) = a + \delta > f(a + \delta) \geq f(x)\,,$$

which is absurd. Thus $f(a) = a$. □

Also solved by A. M. Encinas (Universitat Politècnica de Catalunya, Spain), Laurent Moret-Bailly (IRMAR, Université de Rennes 1, France), F. Plastria (BUTO-Vrije Universiteit Brussel, Belgium).

167. Show that, for any $a, b > 0$, we have

$$\frac{1}{2}\left(1 - \frac{\min\{a, b\}}{\max\{a, b\}}\right)^2 \leq \frac{b - a}{a} - \ln b + \ln a \leq \frac{1}{2}\left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2. \qquad (10)$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

Solution by the proposer. Integrating by parts, we have

$$\int_a^b \frac{b - t}{t^2} dt = \frac{b - a}{a} - \ln b + \ln a \qquad (11)$$

for any $a, b > 0$.

If $b > a$ then

$$\frac{1}{2}\frac{(b - a)^2}{a^2} \geq \int_a^b \frac{b - t}{t^2} dt \geq \frac{1}{2}\frac{(b - a)^2}{b^2}. \qquad (12)$$

If $a > b$ then

$$\int_a^b \frac{b - t}{t^2} dt = -\int_b^a \frac{b - t}{t^2} dt = \int_b^a \frac{t - b}{t^2} dt$$

and

$$\frac{1}{2}\frac{(b - a)^2}{b^2} \geq \int_b^a \frac{t - b}{t^2} dt \geq \frac{1}{2}\frac{(b - a)^2}{a^2}. \qquad (13)$$

Therefore, by (12) and (13), we have for any $a, b > 0$ that

$$\int_a^b \frac{b - t}{t^2} dt \geq \frac{1}{2}\frac{(b - a)^2}{\max^2\{a, b\}} = \frac{1}{2}\left(\frac{\min\{a, b\}}{\max\{a, b\}} - 1\right)^2$$

and

$$\int_a^b \frac{b - t}{t^2} dt \leq \frac{1}{2}\frac{(b - a)^2}{\min^2\{a, b\}} = \frac{1}{2}\left(\frac{\max\{a, b\}}{\min\{a, b\}} - 1\right)^2.$$

By the representation (11), we then get the desired result (10). □

Also solved by Panagiotis T. Krasopoulos (Athens, Greece), John N. Lillington (Wareham, UK), F. Plastria (BUTO-Vrije Universiteit Brussel)

168. Let $f : I \to \mathbb{C}$ be an n-time differentiable function on the interior $\mathring{I}$ of the interval I, and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on $\mathring{I}$. Show that, for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in \mathbb{R}\setminus\{0, 1\}$, we have the representation

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(a)(x-a)^k + (-1)^k \lambda f^{(k)}(b)(b-x)^k\right] + S_{n,\lambda}(x,a,b), \quad (14)$$

where the remainder $S_{n,\lambda}(x,a,b)$ is given by

$$S_{n,\lambda}(x,a,b) := \frac{1}{n!}\left[(1-\lambda)(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds + (-1)^{n+1}\lambda(b-x)^{n+1}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds\right]. \quad (15)$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

Solution by the proposer. Using Taylor's representation with the integral remainder, we can write the following two identities:

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt \quad (16)$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k + \frac{(-1)^{n+1}}{n!}\int_x^b f^{(n+1)}(t)(t-x)^n\,dt \quad (17)$$

for any $x, a, b \in \mathring{I}$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$, that

$$\int_c^d h(t)\,dt = (d-c)\int_0^1 h((1-s)c+sd)\,ds.$$

Therefore,

$$\int_a^x f^{(n+1)}(t)(x-t)^n\,dt = (x-a)\int_0^1 f^{(n+1)}((1-s)a+sx)(x-(1-s)a-sx)^n\,ds = (x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds$$

and

$$\int_x^b f^{(n+1)}(t)(t-x)^n\,dt = (b-x)\int_0^1 f^{(n+1)}((1-s)x+sb)((1-s)x+sb-x)^n\,ds = (b-x)^{n+1}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds.$$

The identities (16) and (17) can then be written as

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{n!}(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds \quad (18)$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k + (-1)^{n+1}\frac{(b-x)^{n+1}}{n!}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds. \quad (19)$$

Now, if we multiply (18) by $(1-\lambda)$ and (19) by λ and add the resulting equalities, a simple calculation yields the desired identity (14) with the reminder from (15). □

Also solved by Mihály Bencze (Brasov, Romania), Panagiotis T. Krasopoulos (Athens, Greece), John N. Lillington (Wareham, UK)

Remark 1. Note that Problems 155 and 159 were also solved by John N. Lillington (Poundbury, Dorchester, UK)

Remark 2. K. P. Hart noted that the answer to problem 157 can be found in the article by Freudenthal and Hurewicz from 1936, https://eudml.org/doc/212824.

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, and by email to michail.rassias@math.uzh.ch.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Discrete Mathematics*.

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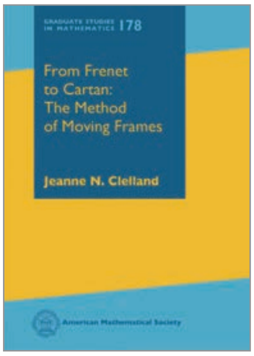

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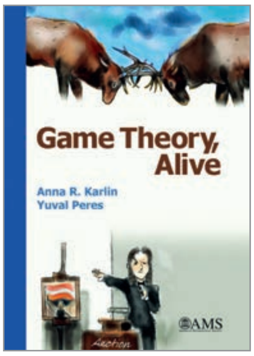

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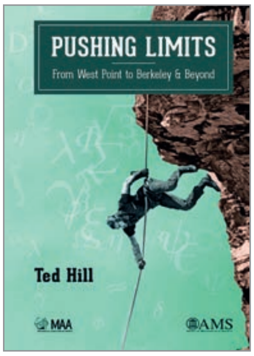

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Ted Hill, Georgia Tech

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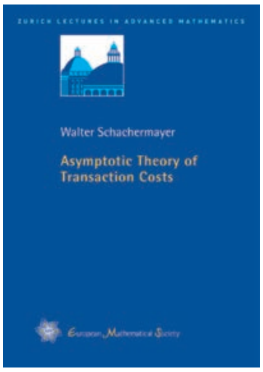
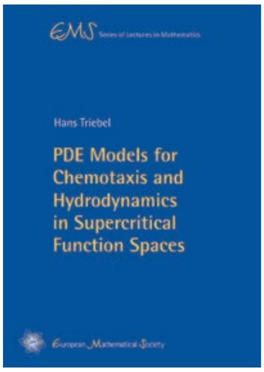


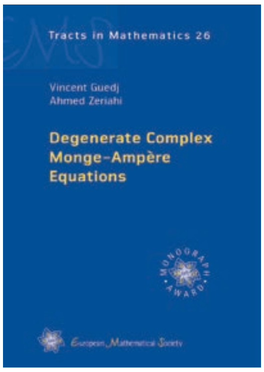
Walter Schachermayer (Universität Wien, Austria)
Asymptotic Theory of Transaction Costs (Zürich Lectures in Advanced Mathematics)
ISBN 978-3-03719-173-6. 2017. 160 pages. Hardcover. 17 x 24 cm. 34.00 Euro

A classical topic in Mathematical Finance is the theory of portfolio optimization. Robert Merton's work from the early seventies had enormous impact on academic research as well as on the paradigms guiding practitioners.
One of the ramifications of this topic is the analysis of (small) proportional transaction costs, such as a Tobin tax. The lecture notes present some striking recent results of the asymptotic dependence of the relevant quantities when transaction costs tend to zero.
An appealing feature of the consideration of transaction costs is that it allows for the first time to reconcile the no arbitrage paradigm with the use of non-semimartingale models, such as fractional Brownian motion. This leads to the culminating theorem of the present lectures which roughly reads as follows: for a fractional Brownian motion stock price model we always find a shadow price process for given transaction costs. This process is a semimartingale and can therefore be dealt with using the usual machinery of mathematical finance.



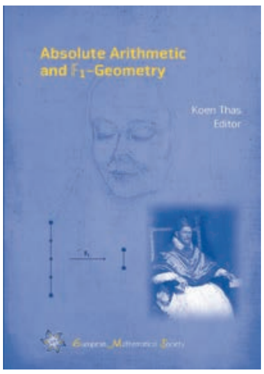
Hans Triebel (University of Jena, Germany)
PDE Models for Chemotaxis and Hydrodynamics in Supercritical Function Spaces (EMS Series of Lectures in Mathematics)
ISBN 978-3-03719-171-2. 2017. 138 pages. Hardcover. 17 x 24 cm. 32.00 Euro

This book deals with PDE models for chemotaxis (the movement of biological cells or organisms in response of chemical gradients) and hydrodynamics (viscous, homogeneous, and incompressible fluid filling the entire space). The underlying Keller–Segel equations (chemotaxis), Navier–Stokes equations (hydrodynamics), and their numerous modifications and combinations are treated in the context of inhomogeneous spaces of Besov–Sobolev type paying special attention to mapping properties of related nonlinearities. Further models are considered, including (deterministic) Fokker–Planck equations and chemotaxis Navier–Stokes equations.
These notes are addressed to graduate students and mathematicians having a working knowledge of basic elements of the theory of function spaces, especially of Besov-Sobolev type and interested in mathematical biology and physics.



Vincent Guedj and Ahmed Zeriahi (both Université Paul Sabatier, Toulouse, France)
Degenerate Complex Monge–Ampère Equations (EMS Tracts in Mathematics, Vol. 26)
ISBN 978-3-03719-167-5. 2017. 496 pages. Hardcover. 17 x 24 cm. 88.00 Euro

Complex Monge–Ampère equations have been one of the most powerful tools in Kähler geometry since Aubin and Yau's classical works, culminating in Yau's solution to the Calabi conjecture. A notable application is the construction of Kähler-Einstein metrics on some compact Kähler manifolds. In recent years degenerate complex Monge–Ampère equations have been intensively studied, requiring more advanced tools.
The main goal of this book is to give a self-contained presentation of the recent developments of pluripotential theory on compact Kähler manifolds and its application to Kähler–Einstein metrics on mildly singular varieties. After reviewing basic properties of plurisubharmonic functions, Bedford–Taylor's local theory of complex Monge–Ampère measures is developed. In order to solve degenerate complex Monge–Ampère equations on compact Kähler manifolds, fine properties of quasi-plurisubharmonic functions are explored, classes of finite energies defined and various maximum principles established. After proving Yau's celebrated theorem as well as its recent generalizations, the results are then used to solve the (singular) Calabi conjecture and to construct (singular) Kähler–Einstein metrics on some varieties with mild singularities.
The book is accessible to advanced students and researchers of complex analysis and differential geometry.



Absolute Arithmetic and $\mathbb{F}_1$-Geometry
Koen Thas (University of Gent, Belgium), Editor
ISBN 978-3-03719-157-6. 2016. 404 pages. Hardcover. 17 x 24 cm. 68.00 Euro

It has been known for some time that geometries over finite fields, their automorphism groups and certain counting formulae involving these geometries have interesting guises when one lets the size of the field go to 1. On the other hand, the nonexistent field with one element, $\mathbb{F}_1$, presents itself as a ghost candidate for an absolute basis in Algebraic Geometry to perform the Deninger–Manin program, which aims at solving the classical Riemann Hypothesis.
This book, which is the first of its kind in the $\mathbb{F}_1$-world, covers several areas in $\mathbb{F}_1$-theory, and is divided into four main parts – Combinatorial Theory, Homological Algebra, Algebraic Geometry and Absolute Arithmetic. Topics treated include the combinatorial theory and geometry behind $\mathbb{F}_1$, categorical foundations, the blend of different scheme theories over $\mathbb{F}_1$ which are presently available, motives and zeta functions, the Habiro topology, Witt vectors and total positivity, moduli operads, and at the end, even some arithmetic.
Each chapter is carefully written by experts, and besides elaborating on known results, brand new results, open problems and conjectures are also met along the way. The diversity of the contents, together with the mystery surrounding the field with one element, should attract any mathematician, regardless of speciality.