NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Feature
From Grothendieck to Naor

Interview
Ragni Piene

Obituary
Maryam Mirzakhani

European
Mathematical
Society

March 2018
Issue 107
ISSN 1027-488X

Rome, venue of the EMS Executve Committee Meeting, 23–25 March 2018

Scan the QR code to go to the Newsletter web page:
http://euro-math-soc.eu/newsletter

European Mathematical Society

Newsletter No. 107, March 2018

ISSN 1027-488X

Published by the
EMS Publishing House
ETH-Zentrum SEW A21
CH-8092 Zürich, Switzerland.
homepage: www.ems-ph.org

For advertisements and reprint permission requests contact: newsletter@ems-ph.org

EMS Executive Committee

President

Prof. Pavel Exner
(2015–2018)
Doppler Institute
Czech Technical University
Břehová 7
11519 Prague 1
Czech Republic
e-mail: ems@ujf.cas.cz

Vice-Presidents

Prof. Volker Mehrmann
(2017–2018)
Institut für Mathematik
TU Berlin MA 4–5
Strasse des 17. Juni 136
10623 Berlin
Germany
e-mail: mehrmann@math.TU-Berlin.DE

Prof. Armen Sergeev
(2017–2020)
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8
119991 Moscow
Russia
e-mail: sergeev@mi.ras.ru

Secretary

Prof. Sjoerd Verduyn Lunel
(2015–2018)
Department of Mathematics
Utrecht University
Budapestlaan 6
3584 CD Utrecht
The Netherlands
e-mail: s.m.verduynlunel@uu.nl

Treasurer

Prof. Mats Gyllenberg
(2015–2018)
Department of Mathematics
and Statistics
University of Helsinki
P. O. Box 68
00014 University of Helsinki
Finland
e-mail: mats.gyllenberg@helsinki.fi

Ordinary Members

Prof. Nicola Fusco
(2017–2020)
Dip. di Matematica e Applicazioni
Complesso Universitario di
Monte Sant' Angelo
Via Cintia
80126 Napoli
Italy
e-mail: n.fusco@unina.it

Prof. Stefan Jackowski
(2017–2020)
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa
Poland
e-mail: sjack@mimuw.edu.pl

Prof. Vicente Muñoz
(2017–2020)
Departamento de Geometria y
Topologia
Facultad de Matematicas
Universidad Complutense de
Madrid
Plaza de Ciencias 3
28040 Madrid
Spain
e-mail: vicente.munoz@mat.ucm.es

Prof. Beatrice Pelloni
(2017–2020)
School of Mathematical &
Computer Sciences
Heriot-Watt University
Edinburgh EH14 4AS
UK
e-mail: b.pelloni@hw.ac.uk

Prof. Betül Tanbay
(2017–2020)
Department of Mathematics
Bogazici University
Bebek 34342 Istanbul
Turkey
e-mail: tanbay@boun.edu.tr

EMS Secretariat

Ms Elvira Hyvönen
Department of Mathematics
and Statistics
P. O. Box 68
(Gustaf Hällströmin katu 2b)
00014 University of Helsinki
Finland
Tel: (+358) 2941 51141
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dr. Richard H. Elwes
School of Mathematics
University of Leeds
Leeds, LS2 9JT
UK
e-mail: R.H.Elwes@leeds.ac.uk

EMS Agenda

2018

13–15 April
Meeting of Presidents, Maynooth University, Dublin, Ireland

23–24 June
EMS Council, Prague, Czech Republic

29–30 July
IMU General Assembly, São Paulo, Brazil

EMS Scientific Events

2018

3–6 April
British Congress of Mathematics Education, Warwick, UK

4–6 April
Probability, Analysis and Dynamics '18,
University of Bristol, UK

21–22 May
Nonlinear Analysis and the Physical and Biological Sciences,
Edinburgh, UK

28–31 May
Emil Artin International Conference,
Yerevan State University, Yerevan, Armenia

30 May –3 June
Dynamic Equations on Time Scales, Będlewo, Poland

9–15 June
42nd Summer Symposium in Real Analysis,
Steklov Mathematical Institute and Herzen University,
Saint-Petersburg, Russia

11–14 June
British Mathematical Colloquium 2018,
University of St Andrews, UK

11–15 June
International Conference on Complex Analysis, Potential
Theory and Applications, University College Dublin, Ireland

11–15 June
3-manifolds and Geometric Group Theory,
CIRM, Marseille, France

11–15 June
Secondary and Delocalized Index Invariants,
University of Copenhagen, Denmark

9–14 July
Young African Scientists in Europe (YASE), Toulouse, France

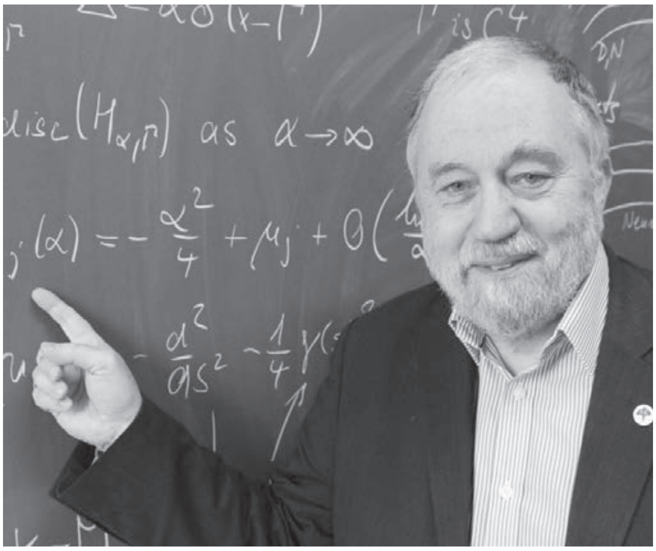
23–27 July
11th European Conference on Mathematical and Theoretical
Biology (ECMTB 2018), Lisbon, Portugal

1–9 August
ICM 2018
Rio Centro Convention Center, Rio de Janeiro, Brazil

Editorial – Message from the President

Pavel Exner, President of the EMS

Dear EMS members, dear friends,



Another calendar page turns, and another year heads for the history books. We may have to wait a decade for the next prime numbered year, but many mathematical and mathematics-related events will occur much sooner. To begin with, the year we are entering has been declared The Year of Mathematical Biology, and I think this is a good omen. We are all now convinced that mathematics is omnipresent, but this truth has dawned in different ways in different fields. It took over two centuries from its first successful marriage with physics, before mathematics started to be applied seriously to biological and social systems. The explosive growth, over the last six or seven decades, of our knowledge of biological mechanisms provides strong motivation to focus on questions which mathematicians and biologists could address together.

There will be many events over the year, such as the European Conference on Mathematical and Theoretical Biology in Lisbon (the programme of which includes the annual lecture jointly organized by the EMS and the Bernoulli Society), the EMS Joint Mathematical Weekend in Joensuu, summer schools, and a great deal more.

The year 2018 will be also important in the life of our society. As happens every second year, the EMS Council (our highest authority) will meet to discuss both our achievements and our next goals. The June Council Meeting in Prague will need to elect the new EMS president, to make decisions about the society's budget, and to settle other important questions related to the EMS's mission and smooth running.

It is satisfying to note that the number of EMS members, both individual and corporate, continues to grow steadily on average. The conditions in which we live are not all the same, of course, and it is encouraging indeed when the national society of a country plagued by serious political and economic problems once again fulfils its membership duties. (They could serve as an example to other corporate members whose approach is more – shall we say – relaxed.) The EMS is not a rich society even compared to some of our members, to say nothing of our partners overseas, but we are doing well financially and are delighted to be able to support more summer schools, conferences, distinguished speakers, and other activities than ever before.

As usual, the turn of the year brings a renewal of our standing committees, the backbone of the society's work. In some, changes are minimal, in others substantial (in part due to the eight-year cap on committee service). This year, the Applied Mathematics and Ethics Committees will undergo the biggest changes, with at least a half of their membership renewed. Let me take this opportunity to thank all departing committee members for their hard work, and to wish all newcomers success and satisfaction in working towards common goals.

Our gratitude is owed also to all those who work for the broader European mathematical community, in a wide variety of roles including for EU-MATHS-IN, EuDML, on the boards of mathematical journals and research centres, prize committees, and in a multitude of other ways.

The coming year also heralds exciting mathematical events worldwide, principally the International Congress in Mathematicians at the beginning of August in Rio de Janeiro, to which we are looking forward. By that time, we will also know the location of the 2022 ICM. As representatives of all mathematicians on our continent, we express no preference on the competition between Paris and Saint Petersburg, but we have no doubt that either choice will lead to a wonderful meeting. We are glad that, either way, after sixteen years the congress will return to the continent of its birth.

We may be European patriots, but at the same time we do not forget that there is a single world of mathematics, and we continue working to make connections all over the globe. The EMS has recently completed our list of cooperation agreements with major mathematical societies on other continents, by signing an agreement with the Chinese Mathematical Society. We hope it will lead to exciting joint ventures.

The Chinese note brings to mind yin and yang, and with cooperation naturally comes competition. To give one example, let me mention zbMATH. We are pleased that FIZ Karlsruhe, our partner in this enterprise, has had its financial support renewed, guaranteeing that the healthy competition with MathSciNet, beneficial for the whole mathematical community, will continue. We have just signed a new agreement fixing the EMS's involvement in the future development of zbMATH, and we are ready to work on further improvements of this great reference tool.

A year ago, I mentioned here the worrying state of the world, and I have to say that the situation has not improved – rather the opposite, with no need to list all the neuralgic points of the globe. We can do little to influence those political tectonic processes, but it is important to preserve and strengthen the esprit de corps, and to oppose the regular calls to ostracize some or other part of our community. A good example of such an attitude was the Second Caucasian Conference which convened (after a one-year delay) in the east of Turkey. I thank those who attended despite pressure they faced at home, and I hope this tradition will continue.

Having said that, I wish all of you good health and a lot of interesting mathematics in 2018.

Farewells within the Editorial Board of the EMS Newsletter

With the December 2017 issue, **Ramla Abdellatif, Eva Miranda** and **Olaf Teschke** ended their editorship of the Newsletter. We express our deep gratitude for all the work they have carried out with great enthusiasm and competence, and thank them for contributing to a friendly and productive atmosphere.
Three new members have rejoined the Editorial Board in January 2017. It is a pleasure to welcome Jean-Bernard Bru, Gemma Huguet and Octavio Paniagua Taboada, introduced below.

New Editors Appointed

Jean-Bernard Bru is an Ikerbasque research professor at both the Mathematics Department of the University of the Basque Country (UPV/EHU) and at the Basque Center for Applied Mathematics (BCAM) in Bilbao (Spain). He started his career as an independent researcher in 1999 with a PhD in mathematical physics from the University of Aix-Marseille II (France). Before settling in the Spanish Basque Country in 2009, Bru taught and carried out research in several places: the Mathematics Department of the University of California at Davis (USA), the School of Theoretical Physics (D.I.A.S.) in Dublin (Ireland), the Mathematics Department of Johannes Gutenberg-University Mainz (Germany) and the Physics University of Vienna (Austria). The bulk of his research ranges from mathematical analysis of the many-body problem to operator algebras, stochastic processes, differential equations and convex and functional analysis. As leader of the Quantum Mechanics Group at the BCAM, his general objective is to develop new, mathematically rigorous methods to investigate quantum many-body systems at and near equilibrium. He is a guarantor researcher of a 4-year renewable Severo Ochoa project associated with the BCAM. He has participated in many conferences, has given a number of lectures and has made multiple research visits to universities across Armenia, Brazil, Europe and the USA.
For more details, see
http://www.ikerbasque.net/es/jean-bernard-bru and
http://www.bcamath.org/en/people/jbru

Gemma Huguet received her PhD in 2008 at the Universitat Politècnica de Catalunya. She has held postdoctoral positions at the Centre de Recerca Matemàtica (Barcelona) and New York University (NYU), including a Courant Instructor position at the Courant Institute of Mathematical Sciences (NYU). She has had several research stays at the University of Texas at Austin, the Mathematical Biosciences Institute at Ohio State University and the Institute for Mathematics and its Applications (Minnesota). She has been a Juan de la Cierva research fellow and, since 2016, she has been a Ramon y Cajal researcher at the Departament de Matemàtiques of the Universitat Politècnica de Catalunya.

Her research interests are in dynamical systems and applications to biology, particularly neuroscience. She combines analytical and numerical techniques to study the role of invariant objects and their connections in the organisation of dynamics. Her main contributions lie in the area of Arnold diffusion and synchronisation of oscillators. She has also collaborated with several experimental groups in projects on biomathematics and computational neuroscience. She has been a member of scientific and organisation committees of conferences and workshops and she has been involved in several activities for engaging women in mathematics.

Octavio Paniagua Taboada Octavio Paniagua Taboada is an editor for zbMATH at the Berlin office of FIZ Karlsruhe. He received a bachelor's degree and a Master's in mathematics from the National Autonomous University of Mexico (UNAM) and a PhD in mathematics from the Université Paris-Sud XI, Orsay, France. His doctoral thesis was "Spectral decomposition of orthogonal groups and Arthur's conjectures" under the supervision of Professor Laurent Clozel. He occupied postdoctoral positions at the Georg-August-Universität Göttingen and the Philipps-Universität Marburg. He has been an editor of zbMATH since January 2014.

The EMS Gordin Prize

The EMS Gordin Prize has been established to honour the memory of Mikhail Gordin. It will be awarded at the International Vilnius Conference on Probability Theory and Mathematical Statistics in July 2018 to a junior mathematician from an Eastern European country who works in probability or dynamical systems.

The award consists of a cash prize of US $4000 and travel funds up to $1000 to support the laureate, chosen by a committee appointed by the EMS, in attending the conference.

Nominations have to be sent to the EMS Office, ems-office@helsinki.fi, no later than 30 April 2018.

Annales Henri Lebesgue

Xavier Caruso (Université de Rennes, France), Dominique Cerveau (Université de Rennes, France), Sébastien Gouëzel (Université de Nantes, France), Xhensila Lachambre (Université de Rennes, France), Nicolas Raymond (Université de Rennes, France) and San Vũ Ngọc (Université de Rennes, France)

Translated from the French La Gazette des Mathématiciens ***155*** *by P.D. Hislop (University of Kentucky, Lexington, KY, USA), with the authorisation of the* Gazette.

Of fleeting and eternal mathematics

Mathematical theorems, their proofs and ideas that give them life do not belong to anyone, not even their authors. Upon writing this, one can easily imagine the small smile appearing on the mouth of the reader, as if to betray the beginning of a small resistance to this statement. These somewhat exaggerated statements, however, are able to open the way to reflection. It is in the same way that from an indistinguishable block of stone, certain sculptors of Antiquity were able to carve beautiful and graceful figures rivalling nature. One has to mention the story of a certain Cypriot Pygmalion, who created such a lifelike sculpture and who loved it with such passion that Venus gave her life.[1] There are also many legends in which men assume the character of demiurges,[2] who give life to shapeless and inert objects. We remember, for example, the wise men Deucalion and Pyrrha, saved from the flood by Jupiter, who recreated humanity by throwing stones (probably clay stones) behind them.[3]

Deucalion and Pyrrha, Rubens (1636), Prado Museum.

Beyond the symbolism of these stories, it is in clay that the first mathematical calculations and the first recitation of these myths (Eastern and Western) were written, as if the authors were like the artisans and mythical creators. Of course, the Ancients not only wrote on tablets: the texts from Antiquity mention that the geometers drew their figures on sand in order to remember their reasoning and to transmit their ideas to future generations. In this manner, Socrates led a slave to publicly solve the problem of the duplication of a square.[4] But, dear reader, perhaps you did not choose to read this article in order that we snare you in Ovid's *Metamorphoses* or tell you about Platonic reminiscences. What remains of the sand that anchored the geometers' arguments in the moment or of the clay tablets of the scribes that were supposed to preserve their works?

A giant with feet of clay[5]

A little in the Platonic spirit that, worn down by time, joins desire and forgetfulness, let's leave antiquity and together leap over the centuries to the present. Chalk replaced sand and university amphitheatres and specialised schools welcome assemblies of students. Classes and recitations, at the core so fleeting, fight regularly against the forgetfulness and safeguard the fabulous sum of knowledge acquired since Antiquity. This knowledge calls for our responsibility: the question of scientific memory and its diffusion is urgent. But what has become of the clay tablets? Not so long ago, works of mathematics were exclusively published on paper. Perhaps, dear reader, you yourself have lingered in mathematics libraries and wandered from aisle to aisle in search of some elusive mathematical theorem? Perhaps you have sat in a comfortable chair, an article in one hand and a pen in the other, secretly charmed by this precious pleasure? Little by little, mathematical works have been digitalised. From now on, these works haunt many public and private servers; they are immediately accessible and are no longer weighed down by paper. Of course, they have not become pure spirits and printing them has not yet become a spiritual endeavour. They are still material and most of them are lodged in the servers of the commercial publishers that, by convention, we'll call Elsa and Sponz.

This commercialisation exerts a continuous financial pressure on public institutions (laboratories, research centres, universities, etc.) serving science. Elsa and Sponz only care about the preservation of knowledge as an afterthought: they decide our needs to satisfy theirs. In this way, access to works of mathematics is not only for a fee. It is also submitted, for example, to the rule of the bouquet of journals: in order to access one journal, we must also access a collection of other journals that we might not desire. A research centre might want a bouquet of roses and tulips but the obscure florists require it to add some daisies, dandelions and, sometimes, an entire

1 Ovid: *Metamorphoses*, Book X, 243.
2 From the Greek δεμοξ (people) and εργον (work): originally meaning artisan, now creator.
3 Ovid: *Metamorphoses*, Book I, 325.
4 Plato: *Meno*, 80d.
5 Book of Daniel, verses 2.31–2.45, Bible.

haystack. Where is the scientific coherence in that strategy?[6] The more that we consider these practices normal, the less we find them astounding and, all the while, we see considerable sums of money leave the budgets of research centres each year.

The birth of the Annales Henri Lebesgue

Despite the fact that many colleagues regret this situation, many don't know how to change these publishing practices. They remark, however, that the authors of many of these articles are very often financed by public research agencies and that the editors and referees donate their work for free. How can one imagine that the fruits of this work are the source of private profits when the fruits depend on the public funds that finance the authors, editors and referees?

This question is all the more gripping when public means of distribution and conservation are readily available for mathematics articles. The recently-founded Centre Mersenne[7] is, in effect, able to furnish all the services necessary for the publication of mathematics articles: establishing a website for the journal, standardisation, distribution and archiving of articles. To summarise: this can be done with public funds and for far less cost.

It is in this context that the *Annales Henri Lebesgue* came to life. For more than two years, researchers in the west of France have worked to create the journal. In the beginning, to be honest, the *Annales* was only a vague and elusive idea. However, the sentiment that these ideas incarnate: open access, free publication and high standards, led them to return to these conversations with new vigour. The Centre Henri Lebesgue[8] aided in nourishing these ideas, moulding them and giving them structure.

Several colleagues have been contacted in order to constitute a strong and motivated editorial board. These colleagues were enthusiastic to participate in this community movement of mathematicians, supported by the Centre National de la Recherche Scientifique. The positive responses were overwhelming and there was a fear there would be too many editors! Zealous colleagues installed the Open Journals Systems[9] and adapted it to the needs of a mathematics journal. The new journal was legally registered and a graphic artist designed a website for article submission. The *Annales Henri Lebesgue*[10] became a real and independent journal.

Readership and editorial committee

The *Annales Henri Lebesgue* is a general mathematics journal, completely electronic, that strives to publish high-quality articles. It is freely accessible to all. Although the initiative was born in the west of France, the diverse editorial committee represents many fields of mathematics. Roughly half of the committee is composed of mathematicians from other regions, the majority of whom are foreigners. Of course, this new journal will not resolve all the problems of for-profit publishing by itself. It will join the collection of mathematics journals that have reasonable publishing practices.[11] The editorial committee will be renewed regularly in order to both involve other mathematicians as well as to cover, over time, a large spectrum of mathematical fields, taking into account the broadness of mathematics.

Publish your papers in a free-access journal!

The *Annales Henri Lebesgue* is accessible and open to all, from advanced graduate students to experienced researchers. Many might hesitate to send a good paper to a newly-established journal whose reputation is not yet fully established. One might wonder if papers published in this journal will enjoy immediate recognition. One would be surprised, however, by the growing enthusiasm of mathematicians, especially young mathematicians, for these editorial initiatives and by their desire to be associated with the journal and these initiatives. In creating this journal, we are responding to this desire in offering them a journal worthy of their best papers. So, it is without hesitation and with enthusiasm that we ask mathematicians to give life to the *Annales Henri Lebesgue*.

In fact, good reputations, for the most part, do not spring from the thigh of Jupiter: it is necessary to attract high-quality works and important that a serious editorial board is open to their evaluation. The research papers, in a certain sense, are more important than the journals themselves. Quality papers do not need journals to be well-written or to have an important scientific value. On the other hand, they need the care of the editorial board and quality referees. It is the work of these people who make, over time, the reputation of the journal. This idea has been key in the discussions around the creation of the *Annales Henri Lebesgue*.

Mathematicians have the means to supervise the totality of the publication process and to participate in a coherent editorial policy. The Annales Henri Lebesgue are among the clay stones that we wish to leave behind us. Contribute to giving them life!

Xavier Caruso [xavier.caruso@normalesup.org] works at the Mathematic Institute of Rennes (IRMAR), France. His field of interest is number theory and especially p*-adic numbers (specifically* p*-adic Galois representations and explicit computations with* p*-adic objects).*

[6] One may consult the article by F. Hélein (*La Gazette des Mathématiciens* **147**) in which the author considers this question.
[7] The Centre Mersenne provides comprehensive scientific publishing infrastructure, and is a joint project of the CNRS and Université Grenoble Alpes: http://www.centre-mersenne.org/en/mersenne/
[8] https://www.lebesgue.fr/fr.
[9] https://pkp.sfu.ca/ojs/.
[10] https://annales.lebesgue.fr/index.php/AHL/.
[11] Non-exhaustive lists may be found at http://cedram.org/ or http://www.emsph.org/journals/journals.php.

Dominique Cerveau [dominique.cerveau@univ-rennes1.fr] works at the University of Rennes, France. His research topics concern complex geometry and holomorphic foliations. He is the Chief Editor of the Annales Henri Lebesgue.

Sébastien Gouëzel [sebastien.gouezel@univ-nantes.fr] works at the Jean Leray Institute (Nantes), France. His research focuses on the interactions between dynamical systems and probability theory.

Xhensila Lachambre [xhensila.lachambre@univ-rennes1.fr] works at the Mathematics Institute of Rennes (IRMAR), France. She is the Editorial Secretary of the Annales Henri Lebesgue.

Nicolas Raymond [nicolas.raymond@univ-rennes1.fr] works at the Mathematics Institute of Rennes (IRMAR), France. His research focuses on semiclassical spectral theory with magnetic fields.

San Vũ Ngọc [san.vu-ngoc@univ-rennes1.fr] works at the Mathematic Institute of Rennes (IRMAR), France. He is interested in microlocal analysis, symplectic geometry, integrable systems and mathematical physics.

From Grothendieck to Naor: A Stroll Through the Metric Analysis of Banach Spaces

Gilles Godefroy (Institut de Mathématiques de Jussieu – Paris Rive Gauche, Paris, France)

In July 1954, Alexandre Grothendieck writes the introduction to his "cours relativement complet sur la théorie des espaces vectoriels topologiques, ou plus précisément sur la partie de la théorie qui peut être considérée comme le prolongement direct et l'aboutissement des idées de S. Banach", published in Sao Paulo under the title "espaces vectoriels topologiques". He states that Banach's theory has not really been surpassed in its essential results, which are the applications of the Baire and Hahn-Banach theorems. He mentions, though, a seminary on the "most recent developments in tensorial topological analysis", without specifying that these are due to him and that they represent a real surpassing of Banach's ideas and those of his school. In fact, these fundamental results had been published by Grothendieck the year before, also in Sao Paulo, in his famous Résumé [6]. Rather than being a "straightforward continuation of S. Banach's ideas", they offer a radically different point of view, even if we had to wait until 1968 for the importance of the Résumé to be internationally recognised, thanks to an article by Joram Lindenstrauss and Alexander Pelczynski. This article marks the beginning of the study of (metric, finite-dimensional, combinatorial) rigid structures in functional analysis, some examples of which can be seen below. A nonlinear component has recently been introduced in this field of research, which was motivated, in particular, by questions from computer science and where many young talents have obtained outstanding results over the past 15 years. Among this new generation, Assaf Naor plays a central role and the reader will notice that his work constitutes the thread of this note.

So why should we try to embed metric spaces into one Banach space or another? What importance can the numerical value of the Grothendieck constant possibly have? We do not ask these questions out of mere intellectual curiosity. They are indeed a way toward discoveries. Let us see how.

1 The Ribe programme

The theorem of M. I. Kadec (1967) states that any separable Banach space of infinite dimension is homeomorphic to the Hilbert space. This result was extended to the non-separable case in 1981 by H. Torunczyk, who showed that two Banach spaces of the same density character are homeomorphic (where the density character of a space is the minimum of the set of cardinals of dense subsets). The topological theory of Banach spaces is thus trivial, in a sense. However, these first results fail to provide information when we consider applications that are not supposed to be linear but respect all or part of the metric structure, and can force the isomorphism; the classical theorem of Mazur-Ulam, for instance, states that any surjective isometry between Banach spaces is affine. In 1976, Martin Ribe published a very interesting theorem, which states that two uniformly homeomorphic Banach spaces have the same local structure, that is, the same subspaces of finite dimension up to an isomorphism constant. This means that if there exists a bijection f between two Banach spaces X and Y such that f and f^{-1} are both uniformly continuous then there exists a constant $C > 0$ such that for any subspace of finite dimension $E \subset X$, there exists a subspace $F \subset Y$ such that F is C-isomorph to E (so there exists a linear isomorphism T from E to F such that $\|T\|.\|T^{-1}\| \leq C$) and conversely when X and Y are swapped. In simpler words, X and Y have the same subspaces of finite dimension. The local structure of a Banach space is thus a uniform invariant.

A quantitative form of the theorem of Ribe was given by Bourgain in 1987. To express it, we need the following notation, which will be used throughout this note. If (M, d_M) and (N, d_N) are two metric spaces and if $f : M \to N$ fulfils

$$a d_M(x, y) \leq d_N(f(x), f(y)) \leq A d_M(x, y) \tag{1}$$

for any pair $(x, y) \in M^2$, the quantity $A/a = D(f)$ is said to be the distortion of f. If there exists such a function f, we say that M bi-Lipschitz embeds into N. In this case, we write

$$c_N(M) = \inf\{D(f);\ f : M \to N \text{ satisfies (1)}\}$$

and, of course, $c_N(M) = +\infty$ if there is no such function f. In the particular case where $N = L_p$, endowed with its usual norm, we simply write

$$c_{L_p}(M) = c_p(M).$$

The cases $p = 2$ and $p = 1$ will be particularly important. Using this notation, Bourgain's theorem of discretisation reads as follows. There exists an absolute constant $C > 0$ such that if $\epsilon > 0$, Y is a normed space, X is a normed space of dimension n and N is a δ-lattice of X with

$$\delta < e^{-(n/\epsilon)^{Cn}}$$

then $c_Y(N) \geq (1 - \epsilon)c_Y(X)$. We recall that N is a δ-lattice of X if for any $x \in X$, one has $\inf\{\|x - y\|;\ y \in N\} \leq \delta$. So, if a sufficiently fine lattice of X embeds bi-Lipschitzly into Y, the same applies for the whole space X. The Ribe theorem follows, since a uniformly continuous map defined on a normed space becomes Lipschitz when restricted to a uniformly discrete lattice with a quantitative control. The theorem of discretisation suggests the existence of finite metric spaces, which represent an obstruction to a local property of Banach spaces.

Ribe's theorem gives a start to the Ribe programme, in the terminology of Joram Lindenstrauss and Jean Bourgain: given a local property (p) of Banach spaces, find a property (P) of metric spaces M that coincides with (p) when M is a Banach space. The Ribe theorem states that this is possible, in principle, but will only be useful if the property (P) is as simple and canonical as possible. The Ribe programme aims to transfer the properties of the structured field of Banach spaces to the larger class of metric spaces. It allows us to study metric spaces using our knowledge and intuition on the geometry of Banach spaces. Assad Naor is an eminent expert on this approach, which has turned out to be remarkably efficient when studying metric spaces, enabling us to find applications that might not have been discovered without the Ribe programme. The metric spaces hide rich structures that we are able to discover when we consider the right properties (P) that derive from the local properties of Banach spaces.

Thus, the purpose of the Ribe programme is in particular the following. Given a local property (p), find a good definition of a property (P) of the metric spaces and, once it is defined, prove that when the metric space in question is a Banach space, (P) reduces to (p). If this approach is interesting for Banach spaces, it turns out to be the key to problems about metric spaces, which at first sight have no relation to normed spaces. Our main reference about the Ribe programme is [13] and the reader may also refer to the lecture by Keith Ball at the Bourbaki seminar [1].

A Banach space X is said to be of type p, where $1 \leq p \leq 2$, if there exists $C > 0$ such that

$$2^{-n} \sum_{\epsilon_i = \pm 1} \| \sum_{i=1}^{n} \epsilon_i x_i \|_X \leq C \left(\sum_{i=1}^{n} \|x_i\|_X^p \right)^{1/p}$$

for all vectors $x_1, x_2, \dots x_n$ in X. The triangular inequality yields that every space is of type 1, so type p appears as a strong triangular inequality (modulo a randomisation). The inequalities of Khintchine show that no Banach space can be of type $p > 2$. On the other hand, a Banach space is of cotype q, with $2 \leq q < +\infty$, if one has

$$\left(\sum_{i=1}^{n} \|x_i\|_X^q \right)^{1/q} \leq C 2^{-n} \sum_{\epsilon_i = \pm 1} \| \sum_{i=1}^{n} \epsilon_i x_i \|_X$$

Again, the inequalities of Khintchine show that the cotype of every space is bounded below by 2. The spaces l_p ($1 \leq p < +\infty$) are of type $\inf(p, 2)$ and of cotype $\sup(p, 2)$. The theorem of Kwapien (1972) states that a Banach space X is isomorphic to a Hilbert space if and only if X is of type 2 and of cotype 2.

From a geometric point of view, the definition of type can be seen as an inequality between the lengths of the diagonals of a parallelepiped and the lengths of their edges, which extends the Euclidean identity of the parallelogram. The nonlinear version, given by Per Enflo, is the following. A geometric cube of a metric space M is a subset of M indexed by $\{-1, 1\}^n$. A diagonal is a pair $(x_\epsilon, x_{-\epsilon})$ and an edge is a pair (x_ϵ, x_δ), where ϵ and δ differ by only one coordinate. Then, M has, by definition, metric type p if one has

$$2^{-n} \sum \text{diagonals} \leq C \left(2^{-n} \sum (\text{edges})^p \right)^{1/p}.$$

It is clear that a Banach space X of metric type p also has type p, and an inequality given by Gilles Pisier in 1986 shows (almost) the converse: if X is of type p then it is of metric type $p - \epsilon$ for all $\epsilon > 0$. Gilles Pisier also showed, in 1973, that a Banach space X has a non-trivial type $p > 1$ if and only if it does not contain uniformly the spaces l_1^n (so, $\mathbf{R}^n$ equipped with the norm $\| \,.\, \|_1$). In other words, X only has the trivial type $p = 1$ if and only if there exists $C > 0$ such that for any n, there exists a subspace E_n of X that is C-isomorph to l_1^n. The corresponding metric result was shown in 1986 by J. Bourgain, V. Milman and H. Wolfson: a metric space M has a type $p > 1$ if and only if it does not contain uniformly bi-Lipschitz copies of Hamming cubes $H_n = (\{-1, 1\}^n, \| \,.\, \|_1)$. So, the uniform presence of Hamming cubes in a Banach space X (i.e. $\sup_n [c_X(H_n)] < \infty$) is the metric obstruction to the non-trivial type for X.

It turns out to be difficult to find a good definition of the metric cotype but the problem was solved by Manor Mendel and Assaf Naor in 2008: a metric space M is of cotype q if there exists $C > 0$ such that for every n, there exists k such that for every function $f : \mathbb{Z}_{2k}^n \to M$, one has

$$\sum_{j=1}^{n} \sum_{x \in \mathbb{Z}_{2k}^n} d_M(f(x + ke_j), f(x))^q$$
$$\leq Ck^q / 3^n \sum_{\epsilon \in \{-1,0,1\}^n} \sum_{x \in \mathbb{Z}_{2k}^n} d_M(f(x + \epsilon, f(x))^q,$$

where e_j stands for the element of $\mathbb{Z}_{2k}^n$ that takes the value 1 at position j and 0 elsewhere. With this definition, a Banach space X is of cotype q if and only if it has metric cotype q, leading to a metric analogue to the theorem of Bernard Maurey and Gilles Pisier (1976) saying that a Banach space is of cotype $q < +\infty$ if and only if it does not uniformly contain the spaces l_∞^n. M. Mendel and A. Naor deduced from these considerations a very general dichotomy theorem.

Theorem 1. *Let $\mathcal{F}$ be a family of metric spaces. Then, exactly one of the following assertions is true:*
(i) *For every finite metric space F and every $\epsilon > 0$, there exists $M \in \mathcal{F}$ such that $c_M(F) \leq 1 + \epsilon$.*
(ii) *There exist $\alpha > 0$ and $K > 0$ such that for any integer n, there exists a metric space M_n with n points such that for all $N \in \mathcal{F}$, one has $c_N(M_n) \geq K(log n)^\alpha$.*

In other terms, if a family $\mathcal{F}$ is not quasi-isometrically universal for the finite metric spaces then spaces of cardinality n will show it with a distortion that grows, at least, like a power of $log(n)$.

We have, thus, a convenient metric approach of type and cotype: let us remark on the latter that a Banach space X has a trivial cotype $(+\infty)$ if and only if it contains bi-Lipschitzly all the locally finite metric spaces [2] with a distortion bounded from above by a universal constant. We recall that a Banach space X is said to be super-reflexive when every space Y with the same local structure as X (that is, uniformly the same subspaces of finite dimension) is reflexive. The metric characterisation of super reflexivity was given by J. Bourgain in 1986: let T_n^k be the k-regular tree of height n equipped with the geodesic distance. Then, a Banach space X is super reflexive if and only if, for every $k \geq 3$, one has

$$\lim_{n \to +\infty} c_X(T_n^k) = +\infty.$$

Bourgain also showed that, for a fixed k, the quantity $c_2(T_n^k)$ is of order $\sqrt{log\ n}$. This leads naturally to the project of characterising the quantitative properties of the norms in metric terms, since the theorem of Enflo-Pisier states that a space is super reflexive if and only if it admits an equivalent norm that is uniformly convex and/or uniformly smooth, with the modulus of convexity and/or smoothness being controlled by a power of the parameter. J. R. Lee, M. Mendel, A. Naor and Y. Peres showed that the existence of a uniformly convex norm with a module in ϵ^q on X was equivalent to the Markov type q (a metric notion introduced by Keith Ball) but, so far, there has been no metric characterisation of the spaces for which there exists a uniformly smooth norm with module in η^p. We remark, along these lines, that if two Banach spaces X and Y contain Lipschitz-isomorphic lattices (which is the case when X and Y are uniformly homeomorphic) and if X has an *asymptotically* uniformly smooth norm with asymptotic modulus of power type p then Y will have, for all $\epsilon > 0$, such a norm with an asymptotic modulus of power type $(p-\epsilon)$ and this $\epsilon > 0$ disappears if X and Y are Lipschitz-isomorphic ([4] and Theorem 3.2 in [5]).

The most important result of the local theory of Banach spaces is undoubtedly the Dvoretzky theorem. Its role in the Ribe programme as well as the ideas it inspired are so important that we dedicate an entire chapter to it.

2 The nonlinear versions of the Dvoretzky theorem

In 1961, Aryeh Dvoretzky positively solved a conjecture formulated by Grothendieck in 1956, establishing the following fundamental theorem. Let n be an integer and $\epsilon > 0$. There exists an integer $N = N(n, \epsilon)$ such that if X is a normed space of dimension N, there exists a linear map $T : l_2^n \to X$ such that $\|T\|.\|T^{-1}\| < 1 + \epsilon$ (where, of course, T^{-1} is defined on the range of T). In other words, every normed space of sufficiently large dimension contains almost spherical sections. Works by T. Figiel, J. Lindenstrauss, V. Milman and Yehoram Gordon (1985) prove more precisely that there exists a universal constant c such that if $0 < \epsilon < 1$ and $n \in \mathbb{N}$, one can use $N(n, \epsilon) = exp[cn\epsilon^{-2}]$.

Together with the Ribe programme, the Dvoretzky theorem suggests the following conjecture. Given a finite metric space M, there exists a large subset S of M such that $c_2(S)$ is small; therefore, S embeds into the Euclidian space with controlled distortion. The publications of Assaf Naor, some of them in collaboration with Manor Mendel, precisely establish this conjecture, which paved the way to many applications. Our main reference on this subject is [10]. This work proves the central role that is played by ultrametric spaces, which will be described below.

A metric space M is said to be ultrametric if, for all x, y and z in M, one has $d(x,z) \leq \sup[d(x,y), d(y,z)]$. If a finite set M is ultrametric, the relation R, defined by xRy if $d(x, y) < diam(M)$, is an equivalence relation. When applying this remark to every equivalence class (which is itself ultrametric) and iteratively, one can identify M with the leaves of a tree equipped with the geodesic distance. It follows, in particular, that a finite ultrametric space and, more generally, a compact ultrametric space M embeds isometrically into a Hilbert space: $c_2(M) = 1$. The main result of the work by Mendel and Naor is the theorem of the ultrametric skeleton [11], which reads as follows.

Theorem 2. *For every $\epsilon > 0$, there exists $c_\epsilon \in [1, +\infty)$ such that: for any compact metric space M and any probability measure μ on M, there exists a compact subset S of M and a probability measure ν supported by S such that S embeds into an ultrametric space with distortion at most $9/\epsilon$ and, for every $(x, r) \in M \times [0, +\infty)$, one has $\nu(B(x, r) \cap S) \leq (\mu(B(x, c_\epsilon r)))^{1-\epsilon}$.*

This ubiquity of ultrametric spaces, which was discovered through the Ribe programme, bears numerous consequences. Here is a first corollary.

Corollary 1. *For any $\epsilon > 0$ and any integer n, every finite metric space M of cardinality n contains a subset S of cardinality at least $n^{1-\epsilon}$ embedded in an ultrametric space with distortion at most $(9/\epsilon)$.*

We now apply the theorem to a uniform probability μ on M and to $r = 0$. As ν is a probability on the subspace S, there exists $x \in S$ such that $\nu(\{x\}) \geq 1/|S|$. But $\nu(\{x\}) \leq \mu(\{x\})^{1-\epsilon} = 1/n^{1-\epsilon}$. Therefore, $|S| \geq n^{1-\epsilon}$.

Assaf Naor and Terence Tao have proven that in corollary 1 we can replace $(9/\epsilon)$ by a bound $D(\epsilon)$ that tends toward 2 when ϵ tends toward 1. This is an aspect of the "phase transition at distortion 2", discovered by Y. Bartal, N. Linial, M. Mendel and A. Naor, of the maximum size of the approximately Euclidean subset of a metric space of cardinality n, which, when one crosses the distortion 2, passes from a power of n to $log(n)$. Assaf Naor and the above co-authors have also shown that there exist metric spaces of cardinality n for which the corollary leads to an optimal result of existence of subsets that are embeddable into the Euclidean space. Therefore, up to a universal factor, the best way to find an approximately Euclidean subset is to find an approximately ultrametric one.

The corollary below uses a non-trivial probability measure μ.

Corollary 2. *For any $\epsilon \in (0, 1)$ and any $\alpha \in (0, +\infty)$, every metric compact M with its Hausdorff dimension greater than α contains a closed subset with a Hausdorff dimension bigger than $(1 - \epsilon)\alpha$ that is embedded into an ultrametric space with distortion at most $(9/\epsilon)$.*

In order to deduce this corollary from Theorem 2, we use a Frostman measure μ on M, that is, a measure such that $\mu(B(x, r)) \leq Cr^\alpha$ for all couples (x, r), and then it can be easily proved that S is convenient. The dimension of S is, again, optimal, even for the approximately Euclidean subsets. A beautiful application of this corollary, given by T. Keleti, A. Mathe and O. Zindulka, is that if K is a metric compact with Hausdorff dimension bigger than $n \in \mathbb{N}$, there exists a Lipschitz surjection of K onto $[0, 1]^n$.

Theorem 2 is linked to the theorem of majorising measures by Michel Talagrand [16]. We recall that if X is a metric space and $\mathcal{P}_X$ the set of probability measures on X, the functional $\gamma_2(X)$ of Fernique–Talagrand is defined by the formula

$$\gamma_2(X) = \inf_{\mu \in \mathcal{P}_X} \sup_{x \in X} \int_0^{+\infty} \sqrt{-log\Big(\mu(B(x, r))\Big)} dr.$$

An important step in the proof is to construct in every finite metric space X a subset S that embeds (with an absolute upper bound on the distortion) into an ultrametric space, such that one has $\gamma_2(X) \leq C\gamma_2(S)$. It turns out that Theorem 2 allows us to prove the theorem of majorising measures, by describing it as the result of an integration on r of pointwise estimates. We can see the link between the two theorems through the role of the trees with orthogonal branches, which represent the Gaussian processes with independent increments and also allow the Euclidean embedding of ultrametric spaces.

Corollary 1 has, in addition, applications to theoretical computer science. This is not surprising, since an essentially Euclidean set constitutes a field where linear algebra and the numerous algorithms it contains can display all their power. Besides, we know that Naor, who is today at the Department of Mathematics at Princeton University, was formerly a member of *Microsoft Research*. Let us turn to the *approximate distance oracle*. A metric space with n points is completely determined by the distances between points, that is, by $n(n-1)/2$ pieces of data. However, the triangular inequality shows that those data are not independent. There exists a certain redundancy, which invites us to search for an essential subset of distances that will allow the estimation of all the others with a given precision. Corollary 1 allowed M. Mendel and A. Naor to prove the existence of a "constant query time oracle" for the approximate distances as follows.

Corollary 3. *Let D be strictly greater than* 1. *Any metric space M of n points can be preprocessed in a* time $O(n^2)$, *in a way so as to stock a number $O(n^{1+O(1/D)})$ of data such that, given $(x, y) \in M^2$, one obtains, in a uniformly bounded time, a number $E(x, y)$ that satisfies $d(x, y) \leq E(x, y) \leq Dd(x, y)$.*

The importance of this result (which was improved quantitatively in 2014 by S. Chechik, using similar methods) lies in the fact that the query time is bounded by a universal constant and therefore depends neither on D nor on n. At the price of a certain distortion D, it is possible to control both the query time and the size of the data.

3 Grothendieck inequality and combinatorial optimisation

In his article from 1953 that founds the metric theory of tensor products and the modern theory of continuous linear operators between Banach spaces [6], Alexandre Grothendieck showed the following inequality. There exists a constant K_G such that, if S_H refers to the unit sphere of the Hilbert space (usually denoted H), one has, for all integers n and m and any real matrix (a_{ij}),

$$\sup\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle x_i, y_j\rangle;\ (x_i), (y_j) \subset S_H\right\}$$
$$\leq K_G \sup\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\epsilon_i\delta_j;\ \epsilon_i = \pm 1, \delta_j = \pm 1\right\}. \quad (2)$$

In other words, if we replace the Hilbert space on the left by the set of real numbers, the supremum is of the same order. There exists a complex version of this inequality, which differs from the real version only by the value of the constant K_G.

In his seminal article, Grothendieck applies this inequality to the linear operators, proving, among many other results, that every operator T from L_∞ to L_1 factors through a Hilbert space, i.e. that there exist operators $A : L_\infty \to H$ and $B : H \to L_1$ such that $T = BA$. An essential reference regarding applications of the Grothendieck inequality is Gilles Pisier's article from 2012 [15].

The link between the Grothendieck inequality and combinatorial optimisation, for which article [7] is our main reference, is established by the interpretation of inequality (2) as a vectorial relaxation of the estimate of a supremum. Given a sufficiently regular compact convex set K of semidefinite positive symmetric matrices $k \times k$ and a matrix (c_{ij}), one can, in polynomial time, compute the maximum of the quantity

$$\sum_{i=1}^{k}\sum_{j=1}^{k} c_{ij}x_{ij}$$

on all matrices (x_{ij}) belonging to K. The computation of the left side of (2) is an example of this technique (called *semidefinite programming*, our reference being [14]) and can therefore be done in polynomial time in k with arbitrary precision. This is consequently the case for the right side of (2), up to the constant K_G, of course. The right side is the norm of the matrix (a_{ij}), considered as a linear map from l^n_∞ to l^m_1.

Following Noga Alon and Assaf Naor in their discovery, we investigate the link between inequality (2) and the estimation of the *cut norm* of an $m \times n$ matrix $A = (a_{ij})$, defined by

$$||A||_{cut} = \max_{S,T} |\sum_{i\in S, j\in T} a_{ij}|,$$

where the maximum is taken over the subsets $S \subset \{1, \dots, m\}$ and $T \subset \{1, \dots n\}$. Let B be the matrix of size $(m+1)\times(n+1)$ obtained by attaching to A a column and a line, in such a way that all lines and all columns of B are of zero sum. A quite simple direct calculation proves that

$$||A||_{cut} = \frac{1}{4}||B||_{\infty\to 1}.$$

The semidefinite programming allows one to calculate the quantity $||B||_{\infty\to 1}$ in polynomial time, up to the constant K_G. The same holds for the norm $||A||_{cut}$ and there exists, accordingly, an algorithm of polynomial time that computes a quantity $\alpha(A)$ such that

$$||A||_{cut} \leq \alpha(A) \leq C||A||_{cut}, \quad (3)$$

with $C = K_G$. However, unless $P = NP$, such an algorithm does not exist if $C < 13/12$, and P. Raghavendra and D. Steurer have proven that modulo the combinatorial conjecture named (UGC) (*unique games conjecture*), the Grothendieck constant is optimal for the existence of an algorithm of polynomial time that provides $\alpha(A)$ satisfying (3). We will return to the numerical value of the Grothendieck constant below.

The regularity lemma by Szemeredi states informally that every combinatorial graph G (a finite set V of vertices pairwise linked or not by edges forming a set $E \subset V^2$) can be partitioned into a controlled number of subsets that interact pseudo-randomly. More precisely, if X and Y are disjoint subsets of V, one denotes by $e(X, Y) = |(X \times Y) \cap E|$ the cardinal of the set of edges that join X and Y. If $\epsilon > 0$ and $\delta > 0$, we say that the pair (X, Y) is (ϵ, δ)-regular if, once $S \subset X$ and $T \subset Y$ satisfy $|S| \geq \delta|X|$ and $|T| \geq \delta|Y|$, one has

$$\left|\frac{e(S,T)}{|S|.|T|} - \frac{e(X,Y)}{|X|.|Y|}\right| \leq \epsilon.$$

This quasi-uniformity of the quantities $e(S,T)/|S|.|T|$ means that the pair (X, Y) is pseudo-random, that is to say, close to a bipartite graph where every pair $(x, y) \in X \times Y$ is joined independently by an edge with probability $e(X, Y)/|X|.|Y|$. In order to construct Szemeredi partitions in polynomial time, the important step is to determine in polynomial time if a pair (X, Y) is close to being (ϵ, δ)-regular. For this purpose, N. Alon and A. Naor consider, given two disjoint subsets X and Y of cardinality n of V, the matrix $A = (a_{xy})$ indexed by $(X \times Y)$, such that

$$a_{xy} = 1 - \frac{e(X,Y)}{|X|.|Y|}$$

if $(x, y) \in E$ and

$$a_{xy} = -\frac{e(X,Y)}{|X|.|Y|}$$

otherwise. Therefore, the matrix A is the difference of the adjacency matrix of the graph G restricted to $X \times Y$ and a matrix whose entries are the probabilities that two vertices of X and Y are connected. It can easily be verified that if (X, Y) is not (ϵ, δ)-regular then $||A||_{cut} \geq \epsilon\delta^2 n^2$. The algorithm for the computation of the cut norm (up to K_G) in polynomial time allows one to decide in polynomial time if (X, Y) is (ϵ, δ)-regular or to find a pair of parts of X and Y that prove it is not (for other values of ϵ and δ). This also requires a polynomial time for the procedure of *rounding*, which we describe below.

Our goal is to find the choices of signs (ϵ_i) and (δ_j), whose existence is guaranteed by inequality (2) and which fulfil this inequality. To achieve this, we will apply a method developed by Jean-Louis Krivine in 1977. Let f and g be two measurable odd functions from $\mathbb{R}^k$ to $\{-1, 1\}$. Let G_1 and G_2 be two independent arbitrary Gaussian vectors in $\mathbb{R}^k$. For $t \in (-1, 1)$, we define

$$H_{f,g}(t) = \mathbb{E}\left[f\left(\frac{1}{\sqrt{2}}G_1\right)g\left(\frac{t}{\sqrt{2}}G_1 + \frac{\sqrt{1-t^2}}{\sqrt{2}}G_2\right)\right].$$

Under a simple condition, an analytic method allows one to determine a scalar $c(f, g)$ such that if $(x_i), (y_j)$ are vectors of norm 1 in (2), there exist unit vectors $(u_i), (v_j)$ in $\mathbb{R}^{m+n}$ such that for all $1 \leq i \leq m$ and $1 \leq j \leq n$, one has

$$\langle u_i, v_j\rangle = H_{f,g}^{-1}\left(c(f,g)\langle x_i, y_j\rangle\right).$$

Now, let G be a random $k \times (m + n)$ matrix whose coefficients are independent standard Gaussian variables. If one sets

$$\epsilon_i = f\left(\frac{1}{\sqrt{2}}Gu_i\right),\ \delta_j = g\left(\frac{1}{\sqrt{2}}Gv_j\right),$$

we find

$$\mathbb{E}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\epsilon_i\delta_j\right] = c(f,g)\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\langle x_i, y_j\rangle.$$

This identity produces the rounding algorithm in polynomial time to produce a conveniant choice of signs for (2), also applicable to the (ϵ, δ)-regular couples of Szemeredi. Besides that, it allows one to bound K_G, since it proves that for every pair (f, g), one has $c(f, g)^{-1} \geq K_G$. Using $f = g = sign(x)$, Jean-Louis Krivine has shown by this method that $K_G \leq \pi/(2log(1 + \sqrt{2}))$, and this is the best possible result when considering the odd functions defined on $\mathbb{R}$. However, M. Braverman, M. Makarychev, K. Makarychev and Assaf Naor proved in 2011 that if $f = g$ corresponds to the partition of the plane on both sides of the graph of the polynomial $y = c(x^5 - 10x^3 + 15x)$ with a well chosen $c > 0$, the resulting estimation is better than the one given by Krivine. This solved a problem that had been open since 1977. So, the current estimates for the Grothendieck constant are:

$$\frac{\pi}{2}e^{\eta_0^2} = 1.676\ldots \leq K_G < \pi/\left(2log(1+\sqrt{2})\right) = 1.782\ldots,$$

where $\eta_0 = 0.25573\ldots$ is the only solution to the equation

$$1 - 2\sqrt{\frac{2}{\pi}}\int_0^\eta e^{-z^2/2}dz = \frac{2}{\pi}e^{-\eta^2}.$$

This lower bound was obtained in 1991 by J. A. Reeds in an unpublished work.

M. Braverman, M. Makarychev, K. Makarychev and Assaf Naor have conjectured that the best Krivine scheme in dimension 2 corresponds to two distinct odd partitions f and g, where f is the following "tiger fur":

It is likely that Krivine schemes in dimension $k \geq 3$ will lead to finer bounds on the Grothendieck constant.

Let us follow the thoughts of Assaf Naor and his coauthors on the combinatorial aspects of inequality (2). Let G be a graph with n vertices denoted by $\{1, \ldots n\}$ and $E \subset \{1, \ldots n\}^2$ the set of its edges. The Grothendieck constant of G, denoted by $K(G)$, is the smallest constant K such that

$$\sup_{(x_i)\subset S_H}\sum_{(i,j)\in E} a_{ij}\langle x_i, x_j\rangle \leq K\sup_{\epsilon_i=\pm1}\sum_{(i,j)\in E} a_{ij}\epsilon_i\epsilon_j \qquad (4)$$

holds true for any real matrix $A = (a_{ij})$. The Grothendieck inequality (2) is the particular case of (4) that corresponds to the bipartite graphs (i.e. of chromatic number 2) and, as a consequence,

$$K_G = \sup_n\{K(G);\ G \text{ bipartite graph with } n \text{ vertices}\}.$$

Additionally, if K_n stands for the complete graph with n vertices, the corresponding Grothendieck constant is of order $log(n)$. The Grothendieck constant of a graph G is clearly related to the combinatorics of G and has, as such, its own interest. On the other hand, the right term in (4) is relevant to statistical physics: if G weighted by the matrix A represents the possible interactions of n particles affected by a spin $\epsilon_i = \pm1$

then the total energy generated by these particles in the system in the Ising model of the spin glass is $E = -(\sum_{(i,j)\in E} a_{ij}\epsilon_i\epsilon_j)$. A configuration of the spins $(\epsilon_i) \in \{-1,1\}^n$ represents a *ground state* if it minimises the total energy.

So, finding a ground state of the spin glass corresponds to maximising the right term in (4). It is known that this can be done in polynomial time for the planar graphs and that it is an NP-complete problem if G is a grid in dimension 3. However, as the quantity on the left side in (4) is related to semidefinite programming, the right side can be calculated in polynomial time up to a factor $K(G)$.

We would like to close this section by mentioning the work of M. Charikar and A. Wirth on the case of the graphs $G = (V, E)$ with their edges weighted by 1 or -1 if the vertices are considered to be "similar" or "different", the absence of an edge meaning that no judgement is given regarding the similarity of the corresponding vertices. The problem to be solved is to split the graph G in a manner that maximises the number of similarities between the members of the same subset of the partition as well as the number of differences between members of different subsets. The bottleneck of the construction is to obtain a partition in two subsets, which again means maximising the right side in (4) and consequently requires the same techniques.

4 Extensions of Lipschitz functions

Let M be a metric space and S a non-empty subset of M. A formula given by MacShane (1934) proves that any Lipschitz function $f : S \to \mathbb{R}$ extends to a function with the same Lipschitz constant on M. In order to obtain such an extension, all we need to do is to set

$$\overline{f}(m) = \inf\{f(s) + Lip(f)d(m,s);\ s \in S\}.$$

As we know, a function $f : M \to N$ between two metric spaces is said to be Lipschitz if there exists $C > 0$ such that $d_N(f(x), f(y)) \leq C d_M(x,y)$ for all $(x,y) \in M^2$, and the minimum of the constants C for which these inequalities are fulfilled is called the Lipschitz constant of f and is denoted $Lip(f)$.

MacShane's formula is a very useful tool but has two shortcomings: firstly, it is not linear in f and secondly, it essentially uses the order structure on $\mathbb{R}$. It is therefore not directly usable for the similar question of the extension of Lipschitz functions with values in a Banach space. A positive result (contemporary to MacShane's formula) is Kirszbraun's theorem, which states that if S is a subset of a Hilbert space H, every Lipschitz map from S to H can be extended with the same Lipschitz constant to H. However, Joram Lindenstrauss proved in 1963 that this result cannot be extended to Banach spaces, even if we allow extensions with arbitrary Lipschitz constants. This more general frame provides specific problems, since a "point by point" extension technique and Zorn's lemma will not lead to a result if the Lipschitz constant explodes during construction. In his PhD (directed by J. Lindenstrauss, himself a student of Dvoretzky), Assaf Naor showed, in particular, that a Lipschitz map from a subset S of a Hilbert space H to a Banach space X can't generally extend Lipschitzly to H, even if, for instance, $X = L_4$. The Kirszbraun theorem is, thus, essentially optimal. However, on the positive side, A. Naor, Y. Peres, O. Schramm and S. Sheffield established the nonlinear version of a classical result by Bernard Maurey: every Lipschitz function of a subset S of a 2-uniformly smooth space X in a 2-uniformly convex space Y extends Lipschitzly to the space X.

This theorem is only one of the results on Lipschitz extensions obtained by Assaf Naor. Our reference for the rest of this section will be his article from 2005 with James R. Lee [9]. One of the difficulties we need to overcome in relation to the extensions is the transition from a local to a global extension. To achieve this, one of the ideas of Lee and Naor was to use partitions of unity that are subordinated to a random covering, and to obtain the desired regularity by taking the average. This analytic approach will be combined with some particular decompositions taken from theoretical computer science and adapted here to infinite sets, respecting the conditions for measurability.

In the following, we would like to present more precisely the least technical concept from this approach: the *gentle* partitions of unity. If M is a metric space, S a closed subset of M and (Ω, μ) a measured space, and if $K > 0$, a function $\Psi : \Omega \times M \to (0, +\infty)$ is a K-gentle partition of unity relative to S if the following applies:

(i) For every $x \in M\backslash S$, the function $\omega \to \Psi(\omega, x)$ is measurable and $\int_\Omega \Psi(\omega, x) d\mu(\omega) = 1$.
(ii) If $x \in S$, we have $\Psi(\omega, x) = 0$ for every $\omega \in \Omega$.
(iii) There exists a Borel function $\gamma : \Omega \to S$ such that, for every $(x,y) \in M^2$,

$$\int_\Omega d(\gamma(\omega), x).|\Psi(\omega, x) - \Psi(\omega, y)| d\mu(\omega) \leq K d(x,y).$$

If f is a Lipschitz function defined on S on values in a Banach space Z, we set, for any $x \in M\backslash S$,

$$\overline{f}(x) = \int_\Omega f(\gamma(\omega))\Psi(\omega, x) d\mu(\omega)$$

and $\overline{f}(x) = f(x)$ if $x \in S$. It can easily be seen that $\overline{f}$, which of course extends f, is Lipschitz and fulfils $Lip(\overline{f}) \leq 3KLip(f)$. Note that the extension $\overline{f}$ obtained by this formula depends linearly upon f.

The problem is now to establish the existence of gentle partitions. Lee and Naor prove that, referring to the subset S, being *doubling* is a sufficient condition. We know that a metric space M is said to be doubling, with doubling constant $\lambda(M)$, if any ball of M is contained in the union of $\lambda(M)$ balls of half radius. They then prove the following.

Theorem 3. *There exists a universal constant $C > 0$ such that if M is a doubling metric space, of doubling constant $\lambda(M)$, Y a metric space that contains M isometrically and Z a Banach space then every Lipschitz function $f : M \to Z$ extends to a Lipschitz function $\overline{f} : Y \to Z$ such that $Lip(\overline{f}) \leq C log(\lambda(M)) Lip(f)$.*

This theorem generalises and unifies previous results of W. B. Johnson, J. Lindenstrauss and G. Schechtman. If M is a finite metric space of cardinality n then $log(\lambda(M)) = O(log(n))$. Furthermore, if M is a subset of a space $\mathbb{R}^d$ then $Log(\lambda(M)) = O(d)$. Note that, for spaces of cardinality n, Lee and Naor prove that there exists an extension that satisfies $Lip(\overline{f}) \leq C(log(n)/log(log(n)))Lip(f)$.

James R. Lee and Assaf Naor also show that, in Theorem 3, one can replace the doubling space M by a subset of a Riemann surface of genus g, and then one has $Lip(\overline{f}) \leq C(g+1)Lip(f)$. On the other hand, recall that a graph G excludes a graph H if one cannot find H based on G by iterating the following operations: removing an edge or collapsing it by identifying the two vertices joined by that edge. Lee and Naor then prove that if a graph G excludes the complete graph of n vertices K_n then $G = M$ satisfies the conclusion of Theorem 3 with $Lip(\overline{f}) \leq Cn^2 Lip(f)$. Thus, the trees (that is, the graphs that exclude K_3) uniformly satisfy Theorem 3, which is a theorem due to Jiri Matoušek. Furthermore, the planar graphs (which exclude K_5) satisfy the same conclusion, up to a constant. Interestingly, Theorem 3 also allows one to prove that if M is a doubling metric space, the free space $\mathcal{F}(M)$ on M has Grothendieck's bounded approximation property [8].

5 Lipschitz embeddings

We begin this section with a closer look at the doubling metric spaces M, for which exists a constant $\lambda(M)$ such that any ball in M is contained in the union of $\lambda(M)$ balls of half radius. Patrice Assouad obtained, in 1983, the following noteworthy result. Let (M, d) be a doubling space. For any $\alpha \in (0, 1)$, one denotes by M_α the space M equipped with the distance d^α defined by $d^\alpha(x, y) = [d(x, y)]^\alpha$. Then, there exists $N(\alpha)$ such that the space M_α embeds bi-Lipschitzly into $l_2^{N(\alpha)}$, i.e. into $\mathbb{R}^{N(\alpha)}$ endowed with the Euclidean distance. The operation that consists of replacing a distance d by d^α with $0 < \alpha < 1$ is called *snowflaking*. Assouad's theorem was improved in 2012 by Assaf Naor and Ofer Neiman as follows. If $K > 0$ and $\epsilon \in (0, 1/2)$, there exist $N = N(K)$ and $D = D(K, \epsilon)$ such that if M is a K-doubling metric space then $M_{1-\epsilon}$ embeds bi-Lipschitzly into $l_2^{N(K)}$ with distortion $D(K, \epsilon)$. Compared to the original result by Assouad, the improvement is that the dimension N does not depend on ϵ anymore.

Another groundbreaking contribution to the theory of Lipschitz embeddings was made by Jean Bourgain in 1986: any finite metric space of cardinality $|M|$ embed into the Hilbert space with a distortion $D = O(log(|M|))$. It is convenient to compare this result with the theorem by Fritz John, which states that the Banach-Mazur distance of a normed space of dimension n to l_2^n is bounded from above by $\sqrt{n}$. The example of the cubes $M = \{0, 1\}^k$ might suggest that the metric analogue of the dimension of a finite space M is $log(|M|)$, and the expected estimation would be $O(\sqrt{log(|M|)})$. The upper bound given by Bourgain is, however, optimal, which follows, in particular, from an essentially nonlinear phenomenon: the existence of expanding graphs. Assaf Naor and his collaborators have given further examples from harmonic analysis, as well as a noteworthy link to Assouad's theorem: any finite metric space M of doubling constant $\lambda(M)$ embeds into the Hilbert space with a distortion $D = O(\sqrt{log(\lambda(M))log(|M|)})$. Bourgain's theorem then follows from the trivial bound $\lambda(M) \leq |M|$. There is an important lemma by Bill Johnson and Joram Lindenstrauss on dimension reduction that we would like to mention: if M is a finite subset of the Hilbert space H and $\epsilon > 0$, there exists a bi-Lipschitz map $f : M \to l_2^n$ (which is, in fact, the restriction of a linear map of H in H) of distortion $D(f) < 1 + \epsilon$, with $n = O(log(|M|)/\epsilon^2))$. In a joint work published in 2010, Bill Johnson and Assaf Naor showed that a Banach space X that fulfils the conclusion of this lemma is very close to a Hilbert space, without necessarily being itself Hilbertian. In particular, L^p only fulfils the conclusion if $p = 2$.

Following Assaf Naor's contribution to the international congress in Hyderabad in 2010 [12], we now examine the connections between these embedding theorems, the Heisenberg group and the algorithms for solving the *Sparsest Cut problem*. These connections are relevant to the geometric theory of groups, for which our reference is Etienne Ghys' Bourbaki lecture [3] and, of course, the work of M. Gromov. We denote by $\mathbb{H}$ the Heisenberg group considered as the space $\mathbb{R}^3$ endowed with the structure of a non-commutative group described by

$$(a, b, c).(a', b', c') = (a + a', b + b', c + c' + ab' - ba').$$

We denote by $\mathbb{H}(\mathbb{Z})$ its discrete subgroup $\mathbb{Z}^3$. If we equip $\mathbb{H}(\mathbb{Z})$ with the word distance related to a finite family of generators (equivalent to the Carnot-Carathéodory distance), we obtain a doubling metric space. It follows from the theorem of differentiability by Pierre Pansu that this doubling space does not bi-Lipschitz embed into the Hilbert space, which leads to the necessity of *snowflaking* in Assouad's theorem. More generally, Pansu's theorem implies that $\mathbb{H}(\mathbb{Z})$ does not embed into L_p with $1 < p < +\infty$ but is not applicable to possible embeddings in L_1.

A metric space (M, d) is said to be *of negative type* if its snowflaking $M_{1/2} = (M, d^{1/2})$ embeds isometrically into the Hilbert space. The space L_1 is of negative type. The group $\mathbb{H}(\mathbb{Z})$, equipped with an equivalent distance, is also of negative type. We now describe the relationship between these particular metric spaces and the Sparsest Cut problem.

Let C and D be two symmetric functions $\{1, \ldots n\} \times \{1, \ldots n\} \to (0, +\infty)$, named "capacity" and "demand". If S is a non-empty subset of $\{1, \ldots n\}$, we set

$$\Phi(S) = \frac{\sum_{i=1}^n \sum_{j=1}^n C(i, j)|\mathbf{1}_S(i) - \mathbf{1}_S(j)|}{\sum_{i=1}^n \sum_{j=1}^n D(i, j)|\mathbf{1}_S(i) - \mathbf{1}_S(j)|}.$$

We now seek to estimate the quantity $\Phi^*(C, D) = \min_S \Phi(S)$, which is the smallest possible ratio of capacity/demand for cuts between S and its complement, hence the name Sparsest Cut. The particular case where $C(i, j) = 1$ if i and j are joined by an edge and $C(i, j) = 0$ otherwise, and where $D = 1$, is the problem of isoperimetry of G: find a subset S such that the relation of the cardinal of its border (i.e. the edges between S and $V \backslash S$) to its cardinal is the smallest possible. It is known that the computation of $\Phi^*(C, D)$ is NP-complete. Under the combinatorial conjecture (UGC), this is even the case for the estimation of $\Phi^*(C, D)$ up to a constant: we cannot expect to find a "Grothendieck constant" in this case as we did in Section 3. However, a similar approach is used in Naor's article.

A first step is to use an extreme ray argument to prove the equation

$$\Phi^*(C, D) = \min_{(f_i) \subset L_1} \frac{\sum_{i=1}^n \sum_{j=1}^n C(i, j)\|f_i - f_j\|_1}{\sum_{i=1}^n \sum_{j=1}^n D(i, j)\|f_i - f_j\|_1}. \tag{5}$$

We can now consider the minimisation problem in the larger space of all metrics on $\{1, \ldots, n\}$, which leads to a problem of linear programming, solvable in polynomial time. But according to Bourgain's theorem, a minimising distance will be at a distance controlled by $log(n)$ of a Euclidean distance, which is embeddable in L_1 (which isometrically contains l_2 via a sequence of independent Gaussian variables). We then deduce from (5) that we have solved the Sparsest Cut problem in polynomial time, up to a factor $log(n)$. A look into the proof of Bourgain's theorem also shows us a way of obtaining a set S that realises a small cut.

However, there is an even better way, which is to use the fact that the metric of L_1 is of negative type. Let $M(C, D)$ be defined as the minimum of the quantity $\sum_{i=1}^{n}\sum_{j=1}^{n} C(i, j)d_{ij}$ under the constraint that $\sum_{i=1}^{n}\sum_{j=1}^{n} D(i, j)d_{ij} = 1$ and d_{ij} is a metric of negative type. As in Section 3, we can calculate in polynomial time the quantity $M(C, D)$ by semidefinite programming. It is clear that $M(C, D) \leq \Phi^*(C, D)$. Equation (5) additionally shows that $\Phi^*(C, D) \leq C_n M(C, D)$, where we set

$$C_n = \sup\{c_1(\{1, \ldots, n\}, d);\ d \text{ a metric of negative type}\}$$

and this inequality is exactly optimal. Thus, in order to approximately solve the Sparsest Cut problem in polynomial time, we are led to examine the embedding of finite metric spaces in L_1. Assaf Naor proves (together with S. Arora and J. R. Lee) that $c_1(M) \leq c_2(M) = O(log(|M|)^{1/2+o(1)})$ if M is of negative type but also (together with J. Cheeger and B. Kleiner) that $c_1(M_n) \geq (log(n))^c$ for a certain $c > 0$ if M_n is the subset $\{1, \ldots, n\}^3$ of $\mathbb{H}(\mathbb{Z})$. The quantity $o(1)$ that appears in the first result is an artefact of the proof, controlled by $(logloglogn)/(loglogn)$, and it is likely it can be eliminated; this is at least possible in the particular case of the isoperimetric problem. Work (since 2014) of Vincent Lafforgue and Assaf Naor suggested the existence of finite sets such that the constant c in the second result should be the optimal value $1/2$, and this was indeed shown by A. Naor and R. Young in 2017, using the 5-dimensional Heisenberg group. Hence, the bi-Lipschitz embedding of metric spaces of negative type in L_1 gives a solution to the Sparsest Cut problem in polynomial time, up to a proportion $O(\sqrt{logn})$.

We now come to the end of our portrait of recent work on metric theory of normed spaces but would like to invite the reader to consult the original articles. To conclude, let us frame the current approach in its historical context. In the past century, existence theorems in analysis have mostly been obtained by more or less constructive topological methods, relying, for instance, on compactness or completeness, or on fine combinatorial results. After that, probabilistic methods have been powerful tools, which have established, by the use of random methods, the existence of numerous objects that were not provided by explicit construction. The spectacular explosion of our computational power invites us no longer to seek non-constructive existence theorems that are unsatisfactory for those who need to apply them. We are now able to determine if and how an algorithm can provide the mathematical object for which we search, in an optimal or almost optimal manner. All of the highlighted contributions clearly belong to that third generation. Let us keep up with recent progress: the 21st century has only just begun.

Bibliography

[1] K. Ball, *The Ribe Programme*, Séminaire Bourbaki, Vol. 2011–2012, Exposé 1047, Astérisque 352 (2013).

[2] F. Baudier, G. Lancien, *Embeddings of locally finite metric spaces into Banach spaces*, Proc. AMS 136 (2008), 1029–1033.

[3] E. Ghys, *Les groupes hyperboliques*, Séminaire Bourbaki, Vol. 1989-90, Exposé 722, Astérisque 189–190 (1990).

[4] G. Godefroy, N. J. Kalton, G. Lancien, *Szlenk indices and uniform homeomorphisms*, Trans. AMS, 353 (2001), 3895–3918.

[5] G. Godefroy, G. Lancien, V. Zizler, *The non-linear geometry of Banach spaces after Nigel Kalton*, Rocky Mountain J. of Maths 44 (2014), 1529–1583.

[6] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bull. Soc. Mat. Sao-Paulo 8 (1953), 1–79.

[7] S. Khot, A. Naor, *Grothendieck-type inequalities in combinatorial optimization*, Comm. Pure Appl. Math. 65 (2012), 992–1035.

[8] G. Lancien, E. Pernecká, *Approximation properties and Schauder decompositions in Lipschitz-free spaces*, J. Funct. Anal. 264 (2013), 2323–2334.

[9] J. R. Lee, A. Naor, *Extending Lipschitz functions via random metric partitions*, Invent. Math. 160 (2005), 59–95.

[10] M. Mendel, A. Naor, *Ramsey partitions and proximity data structures*, J. Eur. Math. Soc. 9 (2007), 253–275.

[11] M. Mendel, A. Naor, *Ultrametric skeletons*, Proc. Nat. Acad. Sci. USA, 111, 48 (2013), 19251–19255.

[12] A. Naor, *L_1 embeddings of the Heisenberg group and fast estimation of graph isoperimetry*, Proceedings of the ICM, Hyderabad 2010, Vol. III, 1549–1575.

[13] A. Naor, *An introduction to the Ribe program*, Japan J. Math. 7 (2012), 167–233.

[14] P. Pansu, *Difficulté d'approximation (d'après Khot, Kindler, Mossel, O'Donnell, …)*, Séminaire Bourbaki, Vol. 2011–2012, Exposé 1045, Astérisque 352 (2013).

[15] G. Pisier, *Grothendieck's theorem, past and present*, Bull. AMS 49 (2012), 237–323.

[16] M. Talagrand, *Regularity of Gaussian processes*, Acta Mathematica 159 (1987), 99–149.

Gilles Godefroy [godefroy@math.jussieu.fr] is directeur de recherches at C.N.R.S. and a member of the Institut de Mathématiques de Jussieu (Paris). His research field is functional analysis and, in particular, the geometry of Banach spaces.

This is the English translation of the French article "De Grothendieck à Naor : une promenade dans l'analyse métrique des espaces de Banach", published in *La Gazette des Mathématiciens*, Société Mathématique de France (No. 151, January 2017). The author thanks J.-B. Bru and M. Gellrich Pedra for the English translation. The EMS Newsletter is thankful to *La Gazette des Mathématiciens* for authorisation to republish this text.

Interview with Ragni Piene

Ulf A. Persson (Chalmers University of Technology, Göteborg, Sweden)

Ragni Piene has a cold and is too tired to go to the department. She graciously invites me to conduct the interview in her home and prepares an omelette for us both for lunch. Through her window, one can see the neighbouring house where she grew up. She has, as they say, returned to her roots. Now, the interview can start; we sit facing each other, with the divided omelette, of which, incidentally, I was given the lion's share, on our plates in front of us, ready to be attacked as well.

From the flyer "Welcome to UiO", University of Oslo, 2013. Photo: Faculty of Mathematics and Natural Sciences, University of Oslo.

Ulf Persson: So let us start from the beginning. How did it all start with mathematics?
Ragni Piene: I always tell people how my father used to sketch, with his ski-pole in the snow, equations for me and my brother to solve when we were spending Easter vacation at Spidsbergseter near the Rondane mountains.

What kind of problems and how old were you?
It was simple equations in two unknowns. You cannot do anything fancy in the snow with a ski-pole. And I guess I was nine or ten, and my brother was two years older.

Your father was a mathematician?
Yes and no. He was a mathematical educator; he did not do research in mathematics, if that is what you mean by a mathematician. After completing his degree in Oslo –with excellent grades – he studied abroad: in Copenhagen, Paris and Göttingen, where he followed courses by Hilbert, Courant, Weyl, Herglotz,… He attended the ICM in Oslo in 1936, in Amsterdam in 1954 and in Edinburgh in 1958, and also the Scandinavian mathematics congresses. He was, for many years, an editor of the *Norsk Matematisk Tidsskrift* and its successor, *Nordisk Matematisk Tidsskrift*.

So your father was the decisive influence?
When it comes to mathematics, he was not the only influence. I had very good mathematics teachers. The first was a woman, during my first years at school, and she encouraged me a lot. But it is true my father has been the major influence in my life, much more than my mother. And his influence was not confined to mathematics; it was a general cultural influence, involving art, books, literature, music and politics. My mother was a psychoanalyst, and I was never much interested in psychology. My father's mathematical interest was far more congenial to me.

You were best in the class in mathematics during your school years?
I probably was in the beginning but, at around the age of 13, compulsory schooling ended in Norway and I entered what was called the "realskole" and then the "gymnasium", modelled, as the traditional school system was in Scandinavia, on the German system. Then, competition was much tougher but, yes, I was always among the very best. I did not get the highest mark in maths in the final exam as I made some trivial mistake. To get the highest mark, you had to be perfect.

What did you like at school and what did you not like?
During the first compulsory years at school, schoolwork did not interest me that much; it was sports and handicraft – I liked to knit and sew. It was not until I entered "Katedralskolen"[1] that school became serious. That school, known as "Katta", was an elite school. In order to enrol, you had to have very good grades or – since it was a school run by the state, not the city – come from outside Oslo, which implied that there were children of government and parliament members. In my class, there were the sons of two consecutive prime ministers. I was never particularly studious (that was looked down upon) and I often did my assignments at the very last moment.

I know that attitude well from Swedish schools at the time. To work hard to get good grades was looked down upon; you should get them anyway without effort, otherwise you might as well forget it.
I attended the branch called "reallinjen", which focused on mathematics and physics – in addition to branches in modern languages and classical languages, there was another one in science more attuned to natural sciences such as chemistry and biology.

[1] Literally the Cathedral school, a common name for higher level schools in Scandinavia, reflecting the close connection between the church and education in traditional times – incidentally, this was the school Niels Henrik Abel attended.

But the latter did not interest you?
No, it did not. And, to be honest, I was not interested in physics either. We had a bad teacher. Now, in retrospect, I regret that I do not know any physics.

What about astronomy?
That was not a subject at school.

But astronomy is something that appeals to all mathematically inclined children, I would think. It is filled with numbers. One of the first books I read was on astronomy and I soaked in all the vital data of the planets. Also, becoming aware of the large distances filled me with a dreadful sense of vertigo.
That I can very well understand. I, too, was somewhat scared of all that empty space and that horror was somehow connected with the concept of infinity, which got to be very tangible.

After school, you started to study mathematics at university. Surely you had to study other subjects too?
Yes, you had to study two subjects in addition to your major one, which in my case, needless to say, was mathematics.

And physics was not an option.
Definitely not. Statistics, on the other hand, was a natural option. My father loved statistics but I found I did not. One thing was clear: I did not want to become a teacher. My father would no doubt have loved it. He did, after all, devote his life to the education of teachers but I wanted to get out of the rut – many neighbours and friends of my parents were teachers. So I needed to choose a subject not taught at school and philosophy was one such that appealed to me. I read Kant's critique of reason in German…

…so the German you learned in school came in handy…
…and we did Wittgenstein, reading his *Tractatus* very carefully (so carefully that we only had time to read part of it).

And what did you think of it?
I was puzzled and could not quite take it seriously. But what really excited me was logic. We studied Gödel among others. In fact, when I returned to mathematics, logic was my choice. But I was disappointed. It all appeared so contrived to me.

What saved you from logic?
I was told by a friend to look out for a new course on a topic called algebraic geometry, to be taught by a certain Audun Holme who had just returned from the States. It was rumoured to be very exciting stuff.

What was your next step?
I wanted to go to Paris; this was clearly where all the action was. And there was a tradition in Norway, especially in Oslo, to go abroad for advanced study, since there were no PhD programmes at the time.

Karl Egil Aubert, as an example, had gone to Paris for his doctorate and Paris would be the destination of choice for a whole generation of algebraists in Norway. Arnfinn Laudal had been there. So had Dan Laksov and many others. This was the 1960s and there was the excitement of Grothendieck, the Bourbaki seminars, you name it.

So how did you go about it?
I applied for a French state scholarship. I got a lot of help to do it; local expertise and experience was readily available.

Your advisor was Michel Raynaud – how was it to work with him?
At that time, at least, he was very formal and forbidding, I guess in the classical French tradition. Appointments to see him in his office had to be made a long time in advance.

Who else did you study with?
I took courses from Giraud, Demazure and Verdier.

But Raynaud was your main contact?
I had to present something to him to get my diploma. Not actual research – we read Shafarevich's *Lectures on minimal models and birational transformations of two dimensional schemes* and then had to exhibit our understanding in front of Raynaud.

A kind of oral.
I read Shafarevich along with Philippe Lelédy and Renée Elkik – she was really the star, so smart, on top of it all, even generalising results by Michael Artin. Philippe was more on my level.

I recall that she and Boutot visited Harvard for a year or term when I came there in 1971. It must have been shortly thereafter. So how did your French get along?
You keep nagging me about my French all the time. Sure, in the beginning when I was with people, I sat there, saying nothing. The French I took at school had been called oral French but we had had little opportunity to speak French. Our teacher loved songs and we had to learn a lot of French songs.

But you did not burst into song to break out of your silence. You are not the singing type, I presume.
I am definitely not the singing type. I just sat and listened and absorbed and gradually I started making out what they were actually saying and acquiring a vocabulary until I was able to break out of my linguistically imposed shell. I also tried to read French books but I never acquired a proficiency that could be compared to my English. I can sometimes fake it by availing myself of some French mannerism but French is a difficult language, especially when it comes to writing.

Most French cannot write properly themselves…
You mean that the rules for written French are made intentionally arcane to preserve a distinction between the educated and the rest. That could well be true.

What else did you do in Paris? Was mathematics the dominant aspect of your visit or was being in Paris and abroad what most excited you?
I would say both – the one thing required the other. As to other things, I was politically interested, needless to say from the perspective of the far left…

That was very fashionable at the time: sixty-eight and all that – yet another American import starting with the Civil Rights movement, morphing into the Antiwar movement, resulting in student revolts, which spread…
This is probably an accurate analysis. Campus unrest in the States predated May 1968 in Paris. But you should understand that my engagement was not just fashion, as I fear it turned out to be with many others. In Paris, I participated in demonstrations; once, we were trapped inside the Jussieu campus and only those with a Jussieu ID were allowed out. I was terrified of being caught and sent back to Norway, forfeiting my French scholarship. Luckily, it resolved itself at the last minute; I do not recall exactly how.

So you only spent a year in Paris?
Unfortunately. I would have loved to have had an extension of the scholarship but that was not possible at the time, so I returned to Oslo.

I recall that there was a big conference on algebraic geometry in Oslo that Summer in 1970 but I only found out in retrospect.
You were not informed! Nowadays, that would be unheard of, the way information is spread so effectively.

Maybe too effectively. So, what did you do in Oslo in those years?
I was politically very active; in retrospect, I realise that I was simply being exploited. It took a lot of time and energy, so finishing my Master's thesis (on a topic suggested by Raynaud) took longer than it should.

Eventually, you left Oslo and went to MIT to do a PhD, with Steven Kleiman as your advisor. When you began to work with Kleiman, it meant that you had to start from scratch again?
I would not say "from scratch" but it certainly involved a reorientation from the kind of algebraic geometry I had been doing up to then.

Could you elaborate?
Already, Hilbert, in his famous lecture in Paris 1900, had addressed the issue of a rigorous foundation for Schubert's enumerative calculus. Enumerative geometry was of no interest at the time for the French school dominated by Grothendieck but a revival of classical geometry took place in the States in the early 1970s and, at a famous meeting in Kalamazoo, Hilbert's problems were considered anew and how to update them. It "fell upon" Kleiman to rework Schubert calculus and enumerative geometry. I was enlisted and my task was to learn about Chern classes and absorb the new intersection theory for singular varieties developed by Fulton and MacPherson.

At the conference in honour of Mireille Martin-Deschamps, Université de Versailles, 2010. Photo: Laure Frerejean.

My work was then cut out for me to study the classical authors and translate them into a modern setting, giving new and rigorous proofs with modern techniques and especially to treat singular varieties.

I recall Mumford reporting how you were climbing up on the giant bust of Mittag-Leffler in the library of that institute in order to get hold of the volumes of Baker hidden above him. Was Baker your favourite author?
I certainly studied him carefully, my eyes being opened up to the riches of late 19th and early 20th century geometry, which had gone out of fashion, as the story you relate indicates.

So how did it go?
I must admit that I had serious doubts initially. To be honest, this was the first time I had been serious about mathematics; my sojourn in Paris had all been fun in a way and now I was older, more mature and responsible. Did I have what it took? I have always been a good sleeper but now, for the first time, I started being plagued by insomnia. It did pass but, in retrospect, I realise the kind of pressures I was living under. I pulled through though.

I guess according to the oft quoted Nietzschean doctrine that what does not kill you strengthens you. Could you give a taste of what you were doing?
My thesis was about general Plücker formulas for singular varieties. This included a study of polar varieties and their rational equivalence classes. For a nonsingular variety, these polar classes can be expressed in terms of the Chern classes (and vice versa) – by extending the definition of polar varieties to singular varieties, one obtains Chern classes for singular varieties, the so-called Chern–Mather classes. By taking degrees, I obtained numerical formulas, thus giving rigorous proofs, as well as generalisations, of classical formulas.

And in this vein, you have continued?
Yes. There have been a lot of bundles of principal parts (or jet bundles), dual varieties, Gauss maps – also in positive characteristics. A main interest over the last few

years has been joint work with Kleiman on the enumeration of singular curves on surfaces. There have, of course, been other topics, like Hilbert schemes and, more recently, toric varieties, and many other collaborators.

What is your motivation for being a mathematician, except keeping on doing what you have been successful at? Have you ever, for whatever reason, considered applied mathematics?
Actually, many years ago, my mother's friend, who was a professor of medicine, talked to me about biomathematics, which he considered an emerging field. That turned out to be true but I was not particularly interested in biology and did not think I could do anything useful. But I have actually participated in several EU networks on geometric modelling and I will give an invited lecture at the SIAM Conference on Applied Algebraic Geometry in Atlanta this coming August.

But you are a pure mathematician at heart. If you had not done mathematics, you would not have done science?
Definitely not. I would be doing something in the humanities – archaeology, linguistics,…

So, in your opinion, mathematics is a humanistic rather than natural science?
Mathematics is pursued by many different people with very different temperaments. Among them, there certainly are those who have a more engineering type of attitude to mathematics, seeing it in terms of computations, or just as the language of Nature (as famously proclaimed by Galileo).

But your attitude is more that of an artist? You have artists in your family?
On the walls here, you see paintings by my paternal grandfather. As you can tell, he was a professional painter.

Your father did not follow in the footsteps of your grandfather?
In fact, they did not get along very well, I learned later in life. My grandfather was very religious and my father was an avowed atheist.

I think we are digressing. Could you help me get back on track again?
Why don't you ask me about women and mathematics?

That is a good idea. Let us get straight to the heart of it. Do you feel, as a woman, oppressed by your male colleagues?
No. I never felt "oppressed". But there are subtle issues, some coming from always being a minority, and some women mathematicians have had real problems.

So what is the issue? What is the point of forming an association of women mathematicians at all?
Is that not obvious?

With Pierre Cartier and Aline Bonami at the Wagah Border ceremony, on the occasion of a conference at the Abdul Salam School of Mathematical Sciences in Lahore. Photo: Paul Vaderlind.

If so, could you elaborate? It is not obviously obvious but it may turn out to be.
I will do my best. For one thing, there are very few women in mathematics and I do not think it is a healthy situation. When I came to Paris from Oslo, I was surprised, as well as delighted and encouraged, that there were so many women among my fellow students. Still, during my entire career as a student, I was never once taught a maths course by a woman. Isn't that remarkable?

But you told me that your favourite teacher in mathematics was a woman.
Now you are being silly. This was in elementary school. Women have always been encouraged to become teachers but the issue is not teaching.

I am expected to be provocative.
That is quite another thing. When I met a few fellow women mathematicians, I felt an instant rapport. Most of them I liked very much; we seemed to have so much in common and we became friends for life.

I have always claimed that those who suffer most from the lack of women in mathematics are male mathematicians, most of whom would gladly exchange many of their male colleagues for women. But I gather it is not the concern for male mathematicians that motivates you.
Women mathematicians suffer from the imbalance more than male mathematicians.

But I guess we can agree that the imbalance is due to the choice of women who elect not to become mathematicians not primarily because of the imbalance but because they do not really care that much for mathematics. And one should not force women to become mathematicians if they do not take the initiative themselves.
This is your opinion. I think that more women could have "liked to become mathematicians". Real role models have been missing. Why did you become inspired to become a mathematician?

The short answer is* Men of Mathematics *by Bell, a book I read in my early teens.

There, you see! I think Emmy Noether is the only woman included in that book…

You are wrong. You must not have read Bell. The one woman treated at length appears in the chapter on Weierstrass and that is Sofia Kovalevskaya, or Kowalewski as he calls her. Bell ends with Cantor and Poincaré so no modern mathematics. Hilbert was still alive when he wrote…

I guess you are right but my point remains unaffected. When I came to MIT, one of the first books I saw in the bookstore was *Women of Mathematics* – it had just been published and was, needless to say, much thinner than Bell's.

I see your point. But what exactly do you object to in a male dominated field?

Needless to say, there exist male mathematicians who resent women, who do not think that women are smart enough to do mathematics or have what it takes.

But I grant you that they tend to be marginal and that I myself have usually been treated with the utmost respect and goodwill.

But goodwill and bending backwards can easily become patronising. These days, when a woman is being honoured in some way or another, there is always the suspicion that it is being done symbolically, not on her own merits but because she is a woman. That is demeaning.

If there would be more of a balance, those problems would not appear. There would, in particular, be no need for "Women in mathematics" and, as you point out, mathematicians are very well equipped to judge performances objectively, unlike in many other academic disciplines. But there is a long way to go. When I came to Paris, as I told you, I was so pleasantly surprised and expected further progress but unfortunately the trend has been broken. That is sad but I think one should not be discouraged. When I have been interviewed by the media, in Norway or abroad, to the extent I have a special message I want to convey, it is simple:

"mathematics + women = true".

I feel I have a duty to be available, to show by personal example. Of course I want to be in control. I recall an incident that angered me a lot. It happened just after I left MIT. A brochure appeared with my face on it and the caption "MIT – a place for women". They did that without consulting me at all. The picture did not identify me, which might be both an extenuating and an aggravating factor but, above all, they should have asked my permission.

But would you have given it if they had asked?

Probably not. MIT was not a place for women at all. In the maths department, there were a few women graduate students but not a single woman on the tenured faculty. To be fair, I should say that the brochure was part of an effort to promote women in science – as was a visiting assistant professorship for a woman. The first to hold this position was Dusa McDuff, which is how I got to meet her.

You have been active in administration, in recent years spectacularly so. How did it all start?

I do not understand what you mean by spectacular?

You did ask before, if you remember, whether I continued doing the same thing. In the mid-1980s, I had a son – this was a very important thing in my life and maybe changed my perspectives somewhat.

Perhaps, as a result, I felt ready to take on responsibilities at the university, acting at one time as Vice-Dean of the Faculty of Mathematics and Natural Sciences.

As to the spectacular aspect, I guess you refer to the General Assembly of the IMU in Shanghai in 2002.

Please continue.

In 2002, the term of a Scandinavian member of the Executive Committee ended and Erling Størmer, a colleague here in Oslo, thought it would be a good idea that he was replaced by a Scandinavian and suggested Uffe Hagerup, a distinguished Danish mathematician working in operator algebras. But Hagerup declined and my name, I presume as a second choice, was put on the ballot. There was an election at the assembly and I became a member.

I remember I was there and I voted for you, not as a woman or a Scandinavian. You should perhaps point out that this was not just a formality; many names were on the ballot.

Well, I must admit that I found it all very gratifying.

Being elected to the Executive Committee is based on mathematical merit and this is not always compatible with administrative acumen. I figure that you, as a woman of capability and common sense, found yourself a mission there.

It is true that some members are less active and capable than others. Being a very good mathematician does not necessarily mean that you are inept at such tasks. The president during my first term – John Ball – was very efficient and capable and I really enjoyed working with him. I was also lucky that his successor László Lovász was cut from the same cloth.

We forgot to recall that you were the very first woman to become a member of the prestigious Executive Committee; that must have caused a stir.

This is true, and I was interviewed by the magazine *New Scientist* because of it.

Then, after my first term, there were more women on the committee and, later, Ingrid Daubechies became the first female president ever. I like to think that I, or rather my example, broke ground.

What does it entail working on the Executive Committee? Is it just about arranging the ICM?

Arranging the ICM is the main business, for obvious reasons; it is then when everything comes together. But there are many other issues also. At the general assemblies, many topics are discussed and it is the responsibility of the committee to see that the resolutions are implemented. And then there are many sub-committees and

The 2014 Abel Prize ceremony. Photo: The Norwegian Academy of Science and Letters.

commissions – I was involved with the one on mathematics in the developing world.

I got to work with Herb Clemens, who was a pioneer and who has always been very involved in promoting mathematics in the developing world and being attuned to the particular problems that face them.

What is your opinion of the ICMs? Are they too big? Do they fill any function whatsoever?
They are big; that cannot be denied. In the old days, they could still take group pictures of all the participants.

I have a special relation to the ICMs, apart from having been a member of the Executive Committee. The first I "attended" was Edinburgh in 1958 – my father brought the family along. He was a member of the ICMI, the educational commission of the IMU, from 1954 to 1958 and gave an invited talk at the ICM in Amsterdam in 1954 on "School mathematics for Universities and for life" – so, indeed, I have been exposed to it all since childhood.

He also took the family to Moscow in 1966 but sadly he was taken ill and hospitalised – a precursor of what would happen two years later when he died.

I know you think mathematical education has less and less to do with mathematics nowadays…

…or education for that matter.
You said it. But to return to the ICMs. Having been, for so many years, actively involved with them, I am hardly in a position to dismiss them. I do think they play a very important role…

…apart from giving out the Fields Medals?
Yes, apart from giving out the Fields Medals. It has to do with the community of mathematicians and the unity of mathematics. The medals have maybe become too important?

But they give mathematics public exposure.
I think one exaggerates that. True, the Fields Medal has appeared in some movies, notably "Good Will Hunting" but, seriously, how many people outside mathematics do you think know about the Fields Medal? I would think that even people in applied mathematics, non-academics I should add, are not, in general, aware of it.

The first Fields Medals were awarded in Oslo in 1936, to Ahlfors and Douglas, as you know. Douglas was actually present at the congress but not when the medals were presented. Maybe he overslept or he may not even have been notified and did something else. It was not such a big deal back then. The whole congress was not built around it; it was just a side-issue to provide encouragement to young mathematicians to keep up the good work.

As I recall you putting it once, the Fields Medal bestows greatness, rather than just confirming it, as most other prizes tend to do.

A question that has been raised is whether the names should be announced ahead of the congress.

That would be stupid and against the special aura that surrounds the drama of the medals.
There are leaks of course and sometimes they get it all wrong. But most people are kept in the dark until the very end, so the dramaturgy actually works.

I see that you are getting tired and you want to go back to bed and sleep. I will not keep you much longer. But I cannot refrain from asking you about your involvement with the Abel Prize, especially since I recall that you had some scepticism concerning maths prizes in the past.
I was involved in the Abel Prize from the very beginning and I later chaired the committee for four years. That was very interesting but I obviously cannot disclose any details, let alone secrets, if that is what you are hoping for. Lately, my work has been, together with Helge Holden, to edit and publish, every five years, the books on the Abel Prize laureates. You know those books; you have yourself reviewed them for the EMS Newsletter.

I see that you are really getting very exhausted. I will leave, I promise. Thank you very much for allowing us access to talk to you at such length.
As they say, the pleasure is entirely mine. Good night!

Ulf Persson received his PhD at Harvard in 1975 under David Mumford. His dissertation was entitled "Degenerations of Algebraic Surfaces". Persson's professional publications have been almost exclusively in algebraic geometry and especially on surfaces. He is inordinately proud of having introduced the notion of the 'geography of surfaces', where the notion of 'geography' has caught on in other contexts. Persson has been based in Sweden since 1979 but did many stints as a visitor to a variety of American universities during the 1980s. In recent years his activities have widened. He founded the Newsletter of the Swedish Mathematical Society during his presidentship and has been its main editor for most of the time since then. He has also been an editor of the EMS Newsletter. He is fond of conducting somewhat idiosyncratic interviews with mathematicians, some of them appearing in this newsletter but the more extreme appearing in the Newsletter of the Swedish Mathematical Society. As is not unusual for people who are aging, he has picked up his youthful interest in philosophy and has published a book and an article on Popper.

History of Mathematics in the Netherlands: Where to find it

Danny Beckers (Vrije Universiteit, Amsterdam, The Netherlands)

It is only somewhat exaggerated to state that the history of Dutch mathematics can be learned through the history of the Royal Dutch Mathematical Society. The society is, at least, a good place to start if you want to find out more about the history of Dutch mathematics. The Amsterdam University Library, Department of Special Collections (http://bijzonderecollecties.uva.nl/en), together with the Holland State Archive (http://noord-hollandsarchief.nl/bronnen/archieven) in Haarlem, keep most of the collection of the society. The first collection, in the centre of Amsterdam, holds the library of the society. The second holds the archive of the society, which has been kept rather meticulously up to date by the librarians of the society.

Before going into detail about what you may or may not find in these two locations about the history of Dutch mathematics, first we will cover some details about the history of this peculiar society. It was founded in 1778 and it is from those days that it inherits its motto "untiring labour overcomes all". Both the society and some of its active members have helped to commemorate and keep alive the history of Dutch mathematics. The society started as a small, local initiative by some well-to-do Amsterdam citizens, who were concerned about the economic state of their country. They made a living in mathematics, that is, they earned money thanks to the results of calculations, measurements and their knowledge of maths, in the capacities of surveyor, accountant, engineer and teacher. They thought that the Dutch Golden Age (the years of economic growth and colonial expansion following the successful revolt against the Spanish king) had resulted from superior mathematical knowledge – the state of which they found lacking in their days. Convinced of the value of maths for their respective professions and even more for the wellbeing of their country, they wanted to spread knowledge of the subject to as many people as possible. In fact, this small Amsterdam-based society was one of many societies at the time that were founded by like-minded people in various places [Beckers 1999].

This small-scale Amsterdam Society gradually changed its character. In Europe, these societies were quite common. Most of them disappeared during the 19th century when mathematics as a formal academic discipline arose and the idea of pure mathematics became the epitome of maths. This particular Amsterdam society was different because it sought the cooperation of academics to strive for solid standards in maths education. Thus, when these academics in the late 19th century strove for a professional organisation of their own, the society was the obvious place to start. Contrasting with other European countries, where national societies were founded in the second half of the 19th century, the Dutch, with a very small academic base, already had their organisational structure firmly established: by 1875, it was mostly the academic mathematicians who could and would spare the time to actually keep the society alive [Beckers 2001].

The concerns of the society are beautifully reflected in its publication policy [Alberts and Beckers 2010]: around 1800 publications, consisting mostly of textbooks and a somewhat megalomaniac project of translating Montucla's *Histoire des Mathématiques* in four volumes [Montucla 1784–1804]. At irregular intervals, journal quires were distributed, mostly containing exercises and solutions and sometimes a paper on a mathematical subject. From 1875 onwards, there was a steady flow in the society's journal, from then on called the *Nieuw Archief voor Wiskunde*. Exercises were still published but the journal started publishing short mathematical research papers. It became increasingly difficult to keep teachers involved. By 1900, the society published the *Nieuw Archief*, an international review journal, the *Revue Sémestrielle*, and a journal containing series of exercises and solutions. By that time, accountants had already left the society, insurance calculators were organising themselves within a separate society with their own journal and maths teachers wanted their own journal. At the time, the board of the society, anxious that the teachers might also leave the society, decided to fund the teachers' initiative. From this, and other teacher journals, in the 1920s emerged the Dutch Maths Teachers Association [Blom 1998]. Until the 1920s, the society managed to keep its international position by balancing the needs of the teachers and academics nationally whilst, at an international level, using the advantage of being a small country that no one feared and actively reconciling and looking after everyone's interests. This worked until the great depression. From then onwards, the society had to discontinue many of its efforts. Only the *Nieuw Archief* was continually published.

After the Second World War, the society briefly sparkled again, for example organising the International Mathematical Conference in Amsterdam in 1954. But in the new structures the society and its members helped build, it soon became just one of the national players. The Mathematisch Centrum (Mathematical Centre), founded in Amsterdam in 1946, for example, would be leading the way in how mathematical research was going to be organised and funded in the Netherlands [Alberts 1998]. This centre would launch the Netherlands into the computer age, bluffing their way in by presenting a computer that only worked once [Alberts and Van Vlijmen 2017].

It built quite a name for itself in computer science, harbouring famous people such as Adriaan van Wijngaarden (1916–1987) and Edsger Dijkstra (1930–2002). For that reason, it changed its name in 1983, and nowadays the Mathematisch Centrum is known as the Centrum voor Wiskunde en Informatica (CWI, Centre for Maths and Computer Science).

Meanwhile, the Dutch Mathematical Society dwindled. Teachers had virtually left it and the Dutch research community (who wanted to know who was getting appointed or retired and where) were better off with the Dutch Teachers Association, which at least published many of the public lectures in its journal. In contrast, the *Nieuw Archief voor Wiskunde* tried to keep up its pre-war appearance of an international research journal. At the end of the 20th century, the society re-invented itself. From 2000 onwards, its journal was restyled and, since then, has functioned as a journal informing all Dutch speaking mathematicians – in the broad sense of the word (http://www.nieuwarchief.nl/serie5/index.php). It received royal status in 2003.

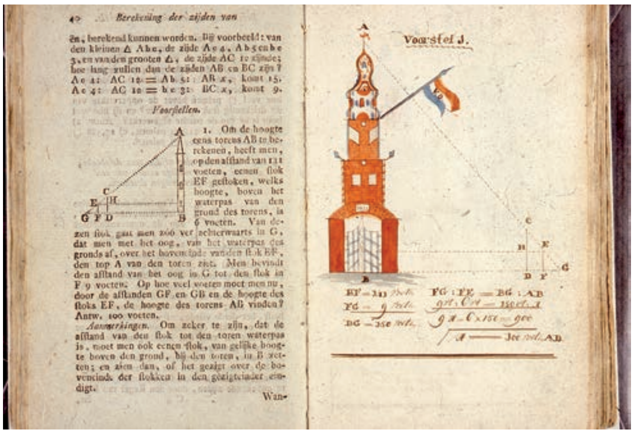
Let's return to the two places mentioned at the beginning where you can find the collection of the society. Special collections in Amsterdam, apart from the book collection, contain absolutely stunning 17th and 18th century manuscripts on navigation, recreational mathematics and accounting. They were collected by the 18th and 19th century members of the society and were intended to illustrate the glory of the Dutch Golden Age of mathematical knowledge. It also contains the manuscripts of these early members, which are just as interesting. For example, among the manuscripts that one of the early members of the society donated is an absurd recreational mathematics exercise from a book by Heinrich Meissner. It resulted in a polynomial of degree 28 with huge coefficients. To avoid mistakes, two of the early members of the society calculated and recalculated all the steps in the process and checked the results against each other before actually writing their result on a 2.5 metre long leaf of paper. The 1907 version of the catalogue of the society's library, also containing the list of manuscripts, is available online via the library catalogue (or at https://www.wiskgenoot.nl/sites/default/files/afbeeldingen/watbiedt/Publicaties/SysCatBoekerijWG1907.pdf).

The Archive in Haarlem contains, among other things, the minutes of the board. The very first book of minutes has, in a rather embarrassing way, been destroyed in an argument among the early board members [Engelsman 1978] but all the others have been preserved. Many other things one might expect to find in a society's archive are also there: correspondence and reports from its various committees and information about its finances and the library.

What would you miss if you restricted yourself to the society? Well, quite a lot, really. But it is easy to use this story about the society as a way to fill in the gaps and, in many cases, you may already find yourself in the right place – or just around the corner. From the history of the society, you can work out what is missing: academic mathematics before the society became of national importance; all mathematics predating the founding of the Dutch Republic; all history of mathematics education from the 1920s onwards; and, last but not least, the history of the new institutions arising after World War II.

We start with the easiest one. Of the new institutions arising after World War II, the archives of both ZWO (the funding agency that was started after the war to regulate the financing of pure scientific research) and the

The Library of the Royal Dutch Mathematical Society is held by the Special Collections Department of Amsterdam University. It contains many publications by the members of the society but it also has a huge collection on Dutch Golden Age maths books, such as the *Wisconstighe Gedachtenissen* (1604–1608) by Simon Stevin and publications by Frans van Schooten, Jr., and Christiaan Huygens. During the 19th century, the society systematically started collecting all important mathematical publications, so one can find everything by Gauss and others. The collection was freely accessible to members until the early 20th century and since then it is accessible to everybody. Since 1894, it has steadily been growing, since the society was supporting the publication of the review journal, which made its library the accumulation point for all math publications in the world. Moreover, the library contains many manuscripts, among them several 17th and 18th century manuscript versions of important textbooks on navigation by Klaas de Vries and Hendrik Gietermaker but also a stunning 15th century manuscript on root extraction, a beautiful 18th century manuscript on the construction of sundials and many 19th century manuscripts by society members, including lecture notes and other scribbles. Some, like the one in the picture, are simply solutions to textbook exercises but contain beautifully coloured drawings. The picture beautifully reflects the interests of the society members around 1800.



Manuscript with colored drawings. Excercises and solutions in a textbook from the library of the society. Bijzondere Collecties Universiteitsbibliotheek UvA Amsterdam.

Mathematical Centre (the present-day CWI) are also in Haarlem. So you may find yourself in the right spot. The personal union that existed in the early years between the society and the new institutions made them preserve their archives in the very same spot.

Slightly more difficult is mathematics predating the founding of the Dutch Republic (1581). The material consists mostly of textbooks and learning texts on commercial and ordinary arithmetic and some elementary geometry. It is scattered around the globe in various libraries and (private) collections but has been made accessible by Kool [Kool 1999].

Regarding the history of mathematics education, the most important archives happen to be in Haarlem too. There, the archive of the Dutch Maths Teachers Association is kept. Archives of several important people in Dutch maths education are there as well. A valuable source for information is the journal of the Dutch Society for Mathematics Teachers, which has recently been digitised and made available online by the society (https://archief.vakbladeuclides.nl/). You can access it from Haarlem but also from anywhere in the world. The journal has existed since 1924 and is a great source because many Dutch mathematicians kept in touch with maths teachers through this journal.

Last but not least, a difficult point is finding archives of Dutch mathematicians. Let's start with the good news: for some of these archives, Haarlem is the place. The personal collections of famous mathematicians such as Hans Freudenthal (1905–1990) [La Bastide 2006], David van Dantzig (1894–1959) [Alberts 2000], N.G. De Bruijn (1918–2012) and several others are preserved there. But additional material and archives not deposited in Haarlem are more difficult to trace. After 1850, if archives were preserved, the mathematical society will at least have made an effort to keep them safe for future generations. You will, at least, find the name of your subject in the archive or the library of the society, which might provide a hint as to where to look for more information. In general, there are two places to look for archive material: either in the archive of the university where the individual was appointed or in the archive of the organisation that appointed them. As an example of the latter, between 1815 and 1968, some of the universities were state funded so appointments and sometimes more archive material can be found in the Archive of the Home Department (since 1918, the Ministry of Education) in The Hague. Other universities were privately funded or paid by the municipality so one can find more material in the archive of the funding organisation – usually kept in a municipal archive. Furthermore, to keep things interesting, after 1850, there were chairs (also at the state universities) that were funded by private organisations. Moreover, before 1850, there were quite a number of people outside academic circles who contributed enormously to Dutch mathematics.

Here are a few examples. Leyden University was founded in 1575. Soon after, a curriculum for mathematics in the vernacular was laid down. It was intended for the training of engineers that could help in the war, in fortress building and water works [Krüger 2014]. Other universities, with mathematicians teaching there, were founded in the late 16th and early 17th centuries; both Latin and vernacular curricula were taught. Some of the surviving lesson plans and syllabi are in the possession of university libraries and these have been partly digitised. Most university libraries have made policies (although most of these have been discontinued due to budget cuts) to collect papers of other important scientists as well. For example, the scientific notes by Christiaan Huygens can be found at Leyden University. Another early 19th century example is the Utenhove Collection of the Utrecht University Library. This mathematically interested (and well versed!) baron donated his entire library to the University of Utrecht and it has been kept there ever since.

If a mathematician worked at a particular university, you are likely to find their papers at that very place. Not all of these notes are in the libraries where you might expect them, though. For example, part of the lecture notes and papers from the Leyden mathematician Frans van Schooten, Jr., (1615–1660; editor of the Latin translation of Descartes' *Géométrie*, published 1649) can be found at the Groningen University Library [Dopper 2014]. The papers of the Frisian Academy at Franeker, which no longer exists, are scattered across the globe [Dijkstra 2012]. Many of these particular papers have meanwhile been digitised (see http://facsimile.ub.rug.nl/cdm) but since this is, in general, not (yet) the case, it is necessary to visit the individual library catalogues to see what the various institutes hold. More recently, the lecture notes of Eindhoven Technical University Library have been made available online (https://www.win.tue.nl/doc/AntiekeWiskundeDictaten/). Among these, for example, are the handwritten, photocopied lecture notes of Edsger Dijkstra on formal programming that were used there until 1990.

Most universities will, moreover, hold archive material on their former employees. Sometimes these will only consist of a track record but a few letters or a description of their library are often present. Sometimes entire libraries, including manuscripts, are kept by university libraries. Special collections at the Amsterdam University Library have already been mentioned but there are several of these kinds of collection that might be of interest. The Bierens de Haan Collection of Leyden University Library is another example, in this case collected by one of the former professors of mathematics at the university.

University libraries in the Netherlands are all committed to having their collections searchable on the web. Most of the material, although not always manuscript collections (entirely), is thereby accessible through Worldcat. That doesn't necessarily imply that they also make their collections available online. The Dutch Royal Library in The Hague hosts, with the cooperation of many university libraries and other institutions, a digital library Delpher (http://www.delpher.nl), focused on books, journals and newspapers. Among those, many things of interest to the historian of maths are to be found (the journal *Nieuw Archief voor Wiskunde* of the Dutch Mathematical Society is due to appear there completely in 2018) but, as an archive source, the transcription of radio bulletins broadcast between 1937 and 1984 is most noteworthy. Unfortunately, the search options are only available in Dutch. Another online initiative is by the Dutch Royal Academy of Sciences. The website of its history institute supports several digitising initiatives, covering many books and manuscripts of Dutch mathematicians (https://www.huygens.knaw.nl/resources/?lang=en).

If all the above doesn't help, one has to start digging in municipal archives. Even if you have found a lot about a mathematician, it might be a good idea to check out such a place. Although some of the smaller municipalities in the Netherlands are closing their archives, many of them still support their own archive service. If they have discontinued their archive then the papers are stored in one of the archives still active in the region. If you are lucky, the municipality where you want to be has registered its archives with the central website for archive searches (www.archieven.nl/en). Many have but unfortunately not all of them. The Haarlem collections may be found through this service. Municipal archives will usually contain archive material on people or organisations of particular interest to the municipality, including local (mathematical) societies and clubs that were founded during the late 18th and early 19th centuries. So, the Leiden Municipal Archive preserves what is left of the Leyden Mathematical Society Mathesis Scientiarum Genitrix (1785). The Groningen Municipal Archive preserves the archive of Minerva, a technical school that held local mathematics contests during the early 19th century. Mathematicians naturally participated in these kinds of activities, just as they may have served on the boards of local organisations whose archives are kept in these places.

Finally, it is worth mentioning the collections of a few museums. The history of science museum Boerhaave in Leyden not only has a great collection of mathematical instruments, including Napier rods, measuring instruments and a rather intriguing object from the early history of the Dutch Mathematical Society [Engelsman 2004], but it also holds an archive with a number of letters and personal objects from noteworthy mathematicians, such as W. Kapteyn (1849–1927). It also includes letters from internationally renowned figures like Albert Einstein and Paul Ehrenfest.

The Amsterdam Scheepvaart Museum holds a beautiful collection of nautical instruments – including interactive working explanations – besides a room filled with maps. At the school museum in Dordrecht, you will find all kind of educational objects, as well as a huge collection of textbooks. Mathematical puzzles and games can be found in Teylers' Museum at Haarlem.

Since the late 1990s, a special committee assigned by the Royal Dutch Mathematical Society has taken up the task of systematically collecting and preserving the archives of distinguished mathematicians. This Commissie Persoonlijke Archieven Wiskundigen (CPAW; Personal Archives of Mathematicians Committee), a permanent commission of the Dutch Mathematical Society, helps to preserve archive material, both actively and passively. It convenes once or twice each year and its members actively approach retired mathematicians of renown to ask them to consider donating their archives. The active collaboration of the committee with the archive at Haarlem has secured the storage of the archives of several scholars of renown [Alberts, Koetsier and Bolten 2003]. Some of the archives are kept in storage in Amsterdam at the CWI. There is a list available at https://www.wiskgenoot.nl/persoonlijke-archieven-van-wiskundigen. Although it does take time and money to make these archives accessible and, indeed, some of the archives are not yet open to the public, inquiries about particular archives can always be made through the committee.

Archives at the committee's disposal are usually ordered by the former owner, the advantage being that relatively many archives then end up in the hands of the committee. The obvious disadvantage is that the archives clearly only consist of what the owner wants to be remembered for. Being a historian, one should always be aware that archives never speak for themselves and even obvious facts should be checked against other data, if possible, or should be completed with other findings. The ordering of the Freudenthal Archive is a beautiful case in point. The archive was clearly a way for the former owner to emphasise his lifelong interest in mathematics education and it clouds the fact that he was much more occupied with mathematics until the late 1960s. Every letter and snippet about the Dutch Committee for Modernising Maths Education is there, suggesting that Freudenthal was actively involved from the beginning. One needs to visit the Dutch State Archives in The Hague, where the minutes of this committee are kept, to find out that he never attended a meeting and only started actively contributing in 1967 [Beckers 2016].

You may have noticed that archive material concerning the history of mathematics in the Netherlands is scattered across the country in a variety of institutions. So, despite (or as a result of) having read the above, if you don't agree with the idea that the questions of your par-

Maquette by one of the early members of the Dutch mathematical society. See [Engelsman 2004]. Collection Museum Boerhaave, Leyden, inv.nr V10256.

ticular interest are best approached through the collections of the Royal Dutch Mathematical Society or the collections that have been assembled with the help of one of the society's committees, you will certainly have to agree that it does bring quite some advantages in travelling arrangements to start from there!

References

A website with an introduction, bibliographical overview and various sources is maintained by Prof. Dr. Jan Hogendijk, University of Utrecht, http://www.jphogendijk.nl/.

Alberts 1998: Gerard Alberts, *Jaren van berekening. Toepassingsgerichte initiatieven in de Nederlandse wiskundebeoefening 1945–1960*, Amsterdam: AUP (1998).

Alberts 2000: Gerard Alberts, *Twee geesten van de wiskunde. Biografie van David van Dantzig*, Amsterdam: CWI (2000).

Alberts, Koetsier and Bolten 2003: Gerard Alberts, Teun Koetsier en Godelieve Bolten, Snippers met formules. Archieven van wiskundigen, in: *Nieuw Archief voor Wiskunde* (V) 4 (2003), 46–50.

Alberts and Beckers 2010: Gerard Alberts en Danny Beckers, Wiskunstige Verlustiging. De tijdschriften die het genootschap wel en niet uitgaf, 1770-2000, in: *Nieuw Archief voor Wiskunde* (V) 11 (2010), 20–26.

Alberts and Van Vlijmen 2017: Gerard Alberts en Bas van Vlijmen, *Computerpioniers. Het begin van het computertijdperk in Nederland*, Amsterdam: AUP (2017).

La Bastide 2006: Sacha la Bastide – van Gemert, "Elke positieve actie begint met critiek". *Hans Freudenthal en de didactiek van de wiskunde*, Hilversum: Verloren (2006).

Beckers 1999: Danny Beckers, "Mathematics our Goal!" Dutch mathematical societies around 1800, in: *Nieuw Archief voor Wiskunde* (IV) 17 (1999), 465–474.

Beckers 2001: Danny Beckers, "Untiring labor overcomes all!" The Dutch Mathematical Society in comparison to its various counterparts in Europe, in: *Historia Mathematica* 28 (2001), 31–47.

Beckers 2016: Danny Beckers, Wiskundige voor een betere wereld. Hans Freudenthal (1905-1990), in: Vittorio Bussato, Mineke van Essen en Willem Koops (red.), *Zeven Grondleggers van de Onderwijskunde*, Amsterdam: Bert Bakker (2016), 113-158.

Blom 1998: Klaske Blom, *Van de akten van bekwaamheid*, Utrecht (1998), http://www.jphogendijk.nl/projects/ActenvanBekwaamheid.pdf.

Dijkstra 2012: Arjen Dijkstra, *Between academics and idiots. A cultural history of mathematics in the Dutch province of Friesland (1600–1700)*, Enschede (2012).

Dopper 2014: Jantien Dopper, *A life of learning in Leiden. The mathematician Frans van Schooten (1615–1660)*, Utrecht (2014).

Engelsman 1978: S. B. Engelsman, Het Wiskundig Genootschap en eerste secretaris Strabbe, in: R.H. de Jong, T.W.M. Jongmans and P.H. Krijgsman (eds.), *Tweehonderd jaar Onvermoeide Arbeid*, Amsterdam: Wiskundig Genootschap (1978), 9–19.

Engelsman 2004: Steven Engelsman, Het monument van oprichting, in: *Nieuw Archief voor Wiskunde* (V) 5 (2004), 129–130.

Kool 1999: Marjolein Kool, *"Die conste vanden getale". Een studie over Nederlandstalige rekenboeken uit de vijftiende en zestiende eeuw*, Hilversum: Verloren (1999).

Krüger 2014: Jenneke Krüger, *Actoren en factoren achter het wiskundecurriculum sinds 1600*, Utrecht (2014).

Montucla 1784-1804: Jean Etienne Montucla, *Historie der wiskunde*, Amsterdam: erven F. Houttuyn (1782-1804) [Four volumes, translation by A. B. Strabbe of French original].

Danny Beckers (d.j.beckers@vu.nl) teaches history of mathematics and history of computer science at Vrije Universiteit Amsterdam. His research interests are in the history of 19th and 20th century mathematics and science education.

Maryam Mirzakhani (1977–2017)

Anton Zorich (Université Paris-Diderot, Paris, France)

> *... je dirai quelques mots sur toi, mais je ne te gênerai point en insistant avec lourdeur sur ton courage ou sur ta valeur professionnelle. C'est autre chose que je voudrais décrire ... Il est une qualité qui n'a point de nom. Peut-être est-ce la "gravité", mais le mot ne satisfait pas. Car cette qualité peut s'accompagner de la gaieté la plus souriante ...*
>
> Antoine de Saint-Exupéry

> *You have to ignore low-hanging fruit, which is a little tricky. I am not sure if it is the best way of doing things, actually – you are torturing yourself along the way. But life is not supposed to be easy.*
>
> Maryam Mirzakhani

On 14 July 2017, Maryam Mirzakhani died. Less than three years earlier, she had received the Fields Medal *"for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces"*, becoming the first woman to win the Fields Medal. She was often the first. For example, together with her friend Roya Beheshti, she was the first Iranian girl to participate in the International Mathematical Olympiad. She won two gold medals: in 1994 and in 1995. Despite all the glory, Maryam always remained extremely nice, friendly, modest and not the least bit standoffish. Meeting her at a conference, you would, at first glance, take her for a young postdoc rather than a celebrated star. She worked hard, mostly "keeping low profile" (using her own words). Kasra Rafi, Maryam's friend since school years, said about her: "*Everything she touched she made better*". This concerned things much broader than just mathematics.

Maryam was born and grew up in Tehran with a sister and two brothers. In one of her rare interviews (given on the demand of the Clay Mathematics Institute at the end of her Clay Research Fellowship), she said: *"My parents were always very supportive and encouraging. It was important for them that we have meaningful and satisfying professions but they did not care as much about success and achievement."* After passing a severe entrance test, Maryam entered the Farzanegan school for girls in Tehran. Having completed her undergraduate studies in Sharif University in Tehran in 1999, she came to Harvard University for graduate studies and received her PhD degree in 2004. The results of her thesis were astonishing for everybody, including Maryam's doctoral advisor C. McMullen: Maryam had discovered beautiful ties between seemingly very different geometric counting problems. In particular, she had discovered how the count of closed non-self-intersecting geodesics on hyperbolic surfaces is related to the Weil–Petersson volumes of the moduli spaces of bordered hyperbolic surfaces. As an application, Maryam had found an alternative proof of Witten's celebrated conjecture first proved by M. Kontsevich.

Maryam in CIRM, Luminy, 2008
(Picture by F. Labourie)

Amazing thesis

I cannot adequately describe the full depth of Maryam's amazing thesis. Maryam's perceptive insight on how to use dynamics in geometric problems would, unfortunately, remain invisible. Hopefully, this brief description will give an idea of the key actors and of the interplay between them.

Simple closed geodesics. Moduli spaces.

A closed curve on a surface is called *simple* if it does not have self-intersections. Closed geodesics on a hyperbolic surface usually do have self-intersections. Indeed, since the classical works of Delsarte, Hubert and Selberg, it is known that the number of closed geodesics of length at most L on a hyperbolic surface grows with the rate e^L/L when the bound L grows. However, Mary Rees and Igor Rivin showed that the number $N(X, L)$ of *simple* closed geodesics of length at most L grows only polynomially in L. Maryam went further and obtained striking results on this more subtle count of simple closed hyperbolic geodesics.

Let us start with a concrete example of a family of hyperbolic surfaces. Consider a configuration of six distinct points on the Riemann sphere $\mathbb{C}\mathrm{P}^1$. Using an appropriate holomorphic automorphism of the Riemann sphere, we can send three out of six points to, say, 0, 1 and ∞. There is no more freedom: any further holomorphic automorphism of the Riemann sphere fixing 0, 1 and ∞ is already the identity transformation. Hence, the three remaining points serve as three independent complex parameters in the space of configurations $\mathcal{M}_{0,6}$ of six points on the Riemann sphere, considered up to a holomorphic diffeomorphism.

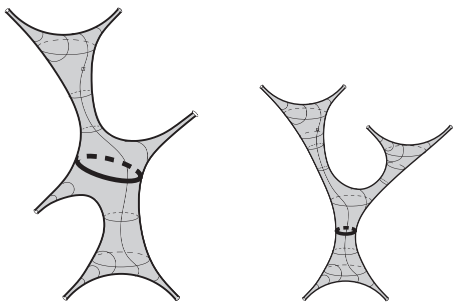


Figure 1. Schematic picture of hyperbolic spheres with cusps

By the uniformisation theorem, complex structures on a surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points, so the moduli space $\mathcal{M}_{0,6}$ can also be seen as the family of hyperbolic spheres with six cusps.

Deforming the configuration of points on $\mathbb{C}\mathrm{P}^1$, we can drastically change the shape of the corresponding hyperbolic surface, making it quite symmetric or, on the contrary, creating very long and very narrow bottlenecks between parts of the surface.

The space $\mathcal{M}_{g,n}$ of configurations of n distinct points on a smooth closed orientable Riemann surface of genus $g \geq 2$ is even richer. While the sphere admits only one complex structure, a surface of genus $g \geq 2$ admits a complex $(3g-3)$-dimensional family of complex structures. As in the case of the Riemann sphere, complex structures on a smooth surface with marked points are in natural bijection with hyperbolic metrics of constant negative curvature with cusps at the marked points. For genus $g \geq 2$, one can let $n = 0$ and consider the space $\mathcal{M}_g = \mathcal{M}_{g,0}$ of hyperbolic surfaces without cusps.

Theorem 1 (Mirzakhani, 2008)**.** For any hyperbolic surface X in the family $\mathcal{M}_{g,n}$, the number of simple closed geodesics has exact polynomial asymptotics:

$$\lim_{L\to+\infty} \frac{N(X,L)}{L^{6g-6+2n}} = const(X)\,,$$

where the constant $const(X)$ admits explicit geometric interpretation, and the power of the bound L in the denominator is the dimension $\dim_{\mathbb{R}} \mathcal{M}_{g,n} = 6g-6+2n$ of the corresponding family of hyperbolic surfaces.

Now, I would like to describe a discovery of Maryam that I find particularly beautiful and, at first glance, even difficult to believe. Let us return to hyperbolic spheres with six cusps, as in Figure 1. A simple closed geodesic on a hyperbolic sphere separates the sphere into two components. We either get three cusps on each of these components (as in the left picture in Figure 1) or two cusps on one component and four cusps on the complementary component (as in the right picture in Figure 1). Hyperbolic geometry excludes other partitions. Denote the numbers of such specialised simple closed geodesics by $N_{3+3}(X,L)$ and by $N_{2+4}(X,L)$ respectively. We have $N_{3+3}(X,L) + N_{2+4}(X,L) = N(X,L)$.

Maryam proved that the asymptotic frequency of simple closed geodesics of each topological type is well defined for every hyperbolic surface and computed it. In our example, Maryam's computation gives the following proportions:

$$\lim_{L\to+\infty} \left(N_{3+3}(X,L) : N_{2+4}(X,L)\right) = 4:3\,.$$

Isn't it astonishing: the asymptotic frequency of simple closed geodesics of a given topological type is one and the same for *any* hyperbolic surface $X \in \mathcal{M}_{0,6}$ no matter how exotic the shape of the particular hyperbolic surface is!

The result of M. Mirzakhani is, of course, much more general than this particular example. There is a finite number of equivalence classes of simple closed curves on a topological surface of genus g with n punctures, considered up to a homeomorphism of the surface. M. Mirzakhani proved that the asymptotic frequency of simple closed geodesics of each type on any hyperbolic surface X in $\mathcal{M}_{g,n}$ is well defined and is one and the same for all X in $\mathcal{M}_{g,n}$. Maryam expressed any such asymptotic frequency in terms of the *intersection numbers* of moduli spaces. In this way, Maryam described geometric properties of individual hyperbolic surfaces in terms of geometry and topology of the ambient moduli spaces.

We shall come back to intersection numbers when discussing Maryam's proof of Witten's conjecture.

Weil–Petersson volumes of moduli spaces

Now, consider several closed hyperbolic geodesics simultaneously. Assume that they have neither self-intersections nor intersections between each other. Cutting the initial hyperbolic surface by such a collection of simple closed geodesics, we get several *bordered* hyperbolic surfaces with geodesic boundary components.

We denote by $\mathcal{M}_{g,n}(b_1,\ldots,b_n)$ the moduli space of hyperbolic surfaces of genus g with n geodesic boundary components of lengths $b_1,\ldots,b_n$. By convention, the zero value $b_i = 0$ corresponds to a cusp of the hyperbolic metric, so the moduli space $\mathcal{M}_{g,n}$ considered in the previous section corresponds to $\mathcal{M}_{g,n}(0,\ldots,0)$ in this more general setting.

A hyperbolic *pair of pants* (as in Figure 2) is by far the most famous bordered hyperbolic surface. Topologically, a pair of pants is a sphere with three holes. It is known that for any triple of nonnegative numbers $(b_1,b_2,b_3) \in \mathbb{R}^3_+$, there exists a hyperbolic pair of pants $P(b_1,b_2,b_3)$ with geodesic boundaries of given lengths, and that such a hyperbolic pair of pants is unique (we always assume that the boundary components of P are numbered). It is also known that two geodesic boundary components γ_1,γ_2 of any hyperbolic pair of pants P can be joined by a single geodesic segment $\nu_{1,2}$ orthogonal to both γ_1 and γ_2 (see Figure 2). Thus, every geodesic boundary component γ of any hyperbolic pair of pants might be endowed with a canonical distinguished point. The construction can be extended to the situation when both remaining bound-

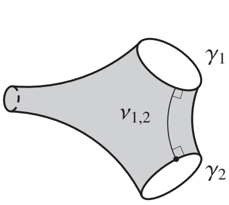


Figure 2. Hyperbolic pair of pants

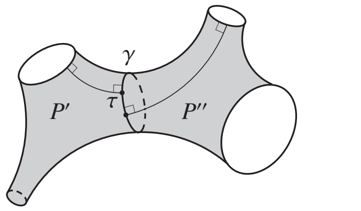


Figure 3. Twist parameter τ responsible for gluing together two hyperbolic pairs of pants

ary components of the pair of pants are represented by cusps.

Two hyperbolic pairs of pants $P'(b'_1, b'_2, \ell)$ and $P''(b''_1, b''_2, \ell)$ sharing the same length $\ell > 0$ of one of the geodesic boundary components can be glued together (see Figure 3). The hyperbolic metric on the resulting hyperbolic surface Y is perfectly smooth and the common geodesic boundary of P' and P'' becomes a simple closed geodesic γ on Y.

Recall that each geodesic boundary component of any pair of pants is endowed with a distinguished point. These distinguished points record how the pairs of pants P' and P'' are twisted, with respect to each other, when we glue them together by a common boundary component (see Figure 3). Hyperbolic surfaces $Y(\tau)$ corresponding to different values of the twist parameter τ in the range $[0, \ell[$ are generically not isometric.

In a similar way, any hyperbolic surface X of genus g with n geodesic boundary components admits a decomposition in hyperbolic pairs of pants glued along simple closed geodesics $\gamma_1, \ldots, \gamma_{3g-3+n}$. It is clear from what was said above that we can vary all $3g - 3 + n$ lengths $\ell_{\gamma_i}(X)$ of the resulting simple closed geodesics γ_i on X and vary the twists $\tau_{\gamma_i}(X)$ along them to obtain a deformed hyperbolic metric. The resulting collection of $2 \cdot (3g - 3 + n)$ real parameters serve as local *Fenchel–Nielsen coordinates* in the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$.

By the work of W. Goldman, each space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ carries a natural closed non-degenerate 2-form ω_{WP} called the *Weil–Peterson symplectic form*. S. Wolpert proved that ω_{WP} has a particularly simple expression in Fenchel–Nielsen coordinates, that is, no matter what pants decomposition we choose, we get

$$\omega_{WP} = \sum_{i=1}^{3g-3+n} d\ell_{\gamma_i} \wedge d\tau_{\gamma_i}\,.$$

The wedge power ω^n of a symplectic form on a manifold M^{2n} of real dimension $2n$ defines a volume form on M^{2n}. The volume $V_{g,n}(b_1, \ldots, b_n)$ of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ with respect to the volume form $\frac{1}{(3g-3+n)!}\,\omega_{WP}^{3g-3+n}$ is called the *Weil–Petersson* volume of the moduli space $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$; it is known to be finite.

To give an account of Mirzakhani's work on Weil–Petersson volumes, we start with the identity of G. McShane.

Theorem (G. McShane, 1998)**.** Let $f(x) = (1 + e^x)^{-1}$ and let X be a hyperbolic torus with a cusp. Then,

$$\sum_{\gamma} f(\ell_\gamma(X)) = \frac{1}{2}\,,$$

where the sum is taken over all simple closed geodesics γ on X, and $\ell_\gamma(X)$ is the length of the geodesic γ.

This identity is, in some sense, a miracle: though the length spectrum of simple closed geodesics is different for different hyperbolic tori with a cusp, the sum above is identically $\frac{1}{2}$ for any X in $\mathcal{M}_{1,1}$. Ten years after McShane's breakthrough, M. Mirzakhani was asked to present his result at the seminar of her scientific advisor Curt McMullen. Preparing the talk, Maryam discovered a remarkable generalisation of McShane's identity to hyperbolic surfaces of any genus with any number of boundary components.

Let us discuss why such identities are relevant to the Weil–Petersson volumes of the moduli spaces. Integrating the right side of McShane's identity over the moduli space $\mathcal{M}_{1,1}$ with respect to the Weil–Petersson form, one obviously gets $\frac{1}{2}\operatorname{Vol}\mathcal{M}_{1,1}$. It is less obvious that the integral of the sum on the left side admits a geometric interpretation as the integral of f over a certain natural cover $\mathcal{M}^*_{1,1}$ of the initial moduli space $\mathcal{M}_{1,1}$. This cover is already much simpler than the original moduli space: it admits global coordinates in which the integral of f can be easily computed.

Mirzakhani's more general identity does not immediately yield the volume. However, cutting the initial surface by simple closed geodesics involved in her identity and developing the idea of averaging over all possible hyperbolic surfaces, Mirzakhani got a recursive relation for the volume $V_{g,n}(b_1, \ldots, b_n)$ in terms of volumes of simpler moduli spaces. These relations allowed Maryam to prove the following statement and to compute the volumes explicitly.

Theorem 2 (Mirzakhani, 2007)**.** The volume $V_{g,n}(b_1, \ldots, b_n)$ is a polynomial in $b_1^2, \ldots, b_n^2$; namely, we have:

$$V_{g,n}(b_1, \ldots, b_n) = \sum_{d_1+\cdots+d_n \le 3g-3+n} C_{d_1,\ldots,d_n} \cdot b_1^{2d_1} \ldots b_n^{2d_n}, \quad (1)$$

where $C_{d_1,\ldots,d_n} > 0$ lies in $\pi^{6g-6+2n-2(d_1+\cdots+d_n)} \cdot \mathbb{Q}$.

Simple recursive formulae for volumes in genera $0, 1, 2$ were found earlier by P. Zograf. Precise asymptotics of volumes for large genera were recently proved by M. Mirzakhani and P. Zograf up to a multiplicative constant conjecturally equal to $\frac{1}{\sqrt{\pi}}$, which still resists a rigorous evaluation.

Witten's conjecture

The family of all complex lines passing through the origin in $\mathbb{C}^{n+1}$ forms the complex projective space $\mathbb{C}\mathrm{P}^n$. This space carries the natural *tautological line bundle*: its fiber over a "point" $[\mathcal{L}] \in \mathbb{C}\mathrm{P}^n$ is the line $\mathcal{L}$ considered as a vector space. Any complex line bundle ξ over a compact manifold M can be induced from the tautological bundle by an appropriate map $f_\xi : M \to \mathbb{C}\mathrm{P}^n$ (for a sufficiently large n depending on M). The second cohomology of the complex projective space $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \simeq \mathbb{Z}$ has a distinguished generator c_1. The pullback $c_1(\xi) = f_\xi^* c_1 \in H^2(M; \mathbb{Z})$ is called the *first Chern class* of the line bundle ξ.

We have already used a natural bijective correspondence between hyperbolic metrics of constant negative curvature with n cusps and complex structures endowed with n distinct marked points $x_1, \ldots, x_n$ on a closed smooth surface of genus g. In this section, we use this latter interpretation of the moduli space $\mathcal{M}_{g,n}$.

Consider the cotangent space $\mathcal{L}(C, x_i)$ to the Riemann surface C at the marked point x_i. Varying $(C, x_1, \ldots, x_n)$ in $\mathcal{M}_{g,n}$, we get a family of complex lines $\mathcal{L}(C, x_i)$ parameterised by the points of $\mathcal{M}_{g,n}$. This family forms a line bundle $\mathcal{L}_i$ over the moduli space $\mathcal{M}_{g,n}$. This *tautological line bundle* $\mathcal{L}_i$ extends to the natural *Deligne–Mumford compactification* $\overline{\mathcal{M}}_{g,n}$ of the initial moduli space. The space $\overline{\mathcal{M}}_{g,n}$ is a nice compact complex orbifold so, for any $i = 1, \ldots, n$, one can define the first Chern class $\psi_i := c_1(\mathcal{L}_i)$. Recall that cohomology has a ring structure so, taking a product of k cohomology classes of dimension 2 (as the first Chern class), we can integrate the resulting cohomology class over a compact complex manifold of complex dimension k. In particular, for any partition $d_1 + \cdots + d_n = 3g - 3 + n$ of $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$ into the sum of nonnegative integers, one can integrate the product $\psi_1^{d_1} \cdot \cdots \cdot \psi_n^{d_n}$ over the orbifold $\overline{\mathcal{M}}_{g,n}$. By convention, the "*intersection number*" (or the "correlator" in a physical context) is defined as

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \,. \tag{2}$$

As always, when there are plenty of rational numbers indexed by partitions or such, it is useful to wrap them into a single generating function. The resulting generating function for correlators (2) is really famous. For physicists, it is the *free energy of two-dimensional topological gravity*. In mathematical terms, E. Witten conjectured in 1991 a certain recursive formula for the numbers (2) and interpreted this recursion in the form of KdV differential equations satisfied by the generating function. The conjecture caused an explosion of interest in the mathematical community: a single formula interlaced quantum gravity, enumerative algebraic geometry, combinatorics, topology and integrable systems.

The first proof of Witten's conjecture was discovered by M. Kontsevich, who used metric ribbon graphs as a "combinatorial model" of the moduli space to express the intersection numbers (2) as a sum over 3-valent ribbon graphs. Maryam Mirzakhani suggested an alternative proof. She ingeniously applied techniques of symplectic geometry to the moduli spaces of bordered Riemann surfaces $\mathcal{M}_{g,n}(b_1, \ldots, b_n)$ discussed in the previous section. Maryam recognised the intersection numbers (2) in the coefficients $C_{d_1,\ldots,d_n}$ from formula (1) for the Weil–Petersson volumes $V_{g,n}(b)$ (up to a routine normalisation factor). This allowed Maryam to reduce the recurrence relations for the intersection numbers contained in Witten's formula to recurrence relations for the volumes $V_{g,n}(b)$ discussed above and thus prove Witten's conjecture.

Echoes

Geometer Kasra Rafi, Maryam's school friend, says the following about her studies at Harvard: "*She faced the same challenges as the rest of us but she moved through them much more quickly. And not that everything in her life was perfect. When she finished her amazing thesis, she asked herself: 'And what if all of this is wrong … ?' She had the fears that everybody has, she felt all the anxiety, but she managed to pass through it way more quickly than you can imagine …*"

Maryam's thesis was not wrong. It is a true masterpiece. The proofs are neither very technical nor particularly complicated. However, they insightfully put together tools and ideas from many different areas of contemporary dynamics and geometry. Reading the three relatively short papers describing this work gives the same euphoric feeling as listening to your favourite piece of music, reading your best-loved poet or gazing upon your preferred painting. Re-reading these papers might echo something you had been thinking about and reveal a simple and unexpected solution.

For my collaborators and me, it has already happened several times: Maryam's papers are full of beautiful ideas that are still at the stage of being absorbed by the mathematical community. For example, the proportion 4 : 3 for asymptotic frequencies of simple closed geodesics on a hyperbolic sphere with six cusps was very recently verified and confirmed by M. Bell and S. Schleimer, who used train tracks in their experiments. V. Delecroix, E. Goujard, P. Zograf and I recently proved that square-tiled surfaces of different combinatorial types have the same asymptotic frequencies as those discovered by Maryam for the corresponding simple closed hyperbolic geodesics.

Slow thinker

Having defended her PhD thesis, Maryam Mirzakhani got a prestigious Clay Mathematics Institute Research Fellowship. (Note that three out of four 2014 Fields Medallists are also former Clay Research Fellows.) In the same interview that I mentioned above, she said about this period of time: *"It was a great opportunity for me; I spent most of my time at Princeton, which was a great experience. The Clay Fellowship gave me the freedom to think about harder problems, travel freely and talk to other mathematicians. I am a slow thinker and have to spend a lot of time before I can clean up my ideas and make progress. So I really appreciate that I didn't have to write up my work in a rush."* What Maryam called "slowness" is actually "depth" or a kind of quality that Saint-Exupéry fails to describe in one word. In 2008, at the age of 31, Maryam Mirzakhani became a full professor at Stanford University, where she worked ever since.

To mention just one of Mirzakhani's numerous results of this period, I have to say a word about the *earthquake flow* introduced by Thurston. Given a non-self-intersecting closed geodesic on a hyperbolic surface, you can cut the surface by the geodesic, twist the two sides of the cut with respect to each other and reglue the cuts to get a new hyperbolic surface as in Figure 3. Having the imagination of Bill Thurston, you can twist a hyperbolic surface X along a closed subset of X, formed as a disjoint union of simple geodesics (such a subset is called a *hyperbolic lamination*). Moreover, Thurston defined a continuous family of simultaneous twists on the large space $\mathcal{ML}_{g,n}$ of all *measured geodesic laminations* on all hyperbolic surfaces. For many years, the properties of the resulting earthquake flow were completely enigmatic. In particular, it was not known whether it had any dense trajectories.

One more time Maryam Mirzakhani discovered unexpected ties between seemingly different objects. In some sense, she recognised in Thurston's earthquake flow the much more familiar and better understood horocycle flow on the moduli space of quadratic differentials. More precisely, she established a *measure isomorphism* between the two flows with respect to the corresponding natural invariant measures. Some important applications of this theorem were obtained instantly; some were recognised very recently – a decade later. I am sure that more will appear in the future. Mathematics is a slow science (in the same sense that Maryam was a "slow thinker").

If you ever saw Maryam attend a lecture in a large auditorium like in MSRI, she was always standing behind the back row of seats. It was neither impatience nor extravagance. I have never seen the slightest trace of a capricious "genius Olympiad kid" in Maryam: she simply had serious health problems with her back, which she never manifested otherwise. She would later mention with humour and irony that "serious" might become very relative.

Magic Wand theorem

In addition to having brilliant ideas, Maryam worked hard, as she worked on the Magic Wand theorem. From a dynamical point of view, the moduli space of holomorphic differentials can be viewed as a "homogeneous space with difficulties". I am citing Alex Eskin, who knows both facets very well: how the dynamics on the moduli space might mimic the homogeneous dynamics in some situations and how deep the difficulties might be.

The rigidity theorems, including and generalising the theorems proved by Marina Ratner at the beginning of the 1990s, show why homogeneous dynamics is so special. (Sadly, Marina Ratner also died just a week before Maryam.) General dynamical systems usually have some very peculiar trajectories living in very peculiar fractal subsets. Such trajectories are rare but there are still plenty of them. In particular, the question of identification of all (versus almost all) orbit closures or of all invariant measures has no sense for most dynamical systems: the jungle of exotic trajectories is too large. In certain situations, this diversity creates a major difficulty: even when you know plenty of fine properties of the trajectory launched from almost every starting point, you have no algorithm to check whether the particular initial condition you are interested in is generic or not. Ergodic theory is aimed at responding to statistical questions but might become completely powerless in the study of specific initial data.

The situation in homogeneous dynamics is radically different. In certain favourable cases, one can prove that *any* orbit closure is a nice homogeneous space, *any* invariant measure is the corresponding Haar measure, etc. This kind of rigidity allows one to obtain fantastic applications to number theory, developed, in particular, by J. Bourgain, E. Lindenstrauss, G. Margulis and T. Tao (to mention only Fields Medallists out of an extremely impressive list of celebrated scientists working in this area).

For several decades, it was not clear to what extent the dynamics of $SL(2,\mathbb{R})$-action on the moduli space of Abelian and quadratic differentials resembles homogeneous dynamics. For Alex Eskin, who came to dynamics in the moduli space from homogeneous dynamics, it was, probably, the main challenge for 15 years. Maryam Mirzakhani joined him in working on this problem around 2006. She was challenged by the result of her scientific advisor Curt McMullen, who had solved the problem in the particular case of genus two, ingeniously reducing it to the homogeneous case of genus one. After several years of collaboration, the first major part of the theorem, namely the measure classification for $SL(2,\mathbb{R})$-invariant measures, was proved. We forced Alex Eskin to announce it at the conference in Bonn in the Summer of 2010.

To illustrate the importance of this theorem, I cite what Artur Avila said about this result of Eskin and Mirzakhani to S. Roberts for the *New Yorker* article in memory of Maryam: *"Upon hearing about this result, and knowing her earlier work, I was certain that she would be a front-runner for the Fields Medals to be given in 2014, so much so that I did not expect to have much of a chance."* I do not think that Maryam thought much about the Fields Medal at this time (several years later she took the email message from Ingrid Daubechies announcing that she had received the Fields medal as a joke and just ignored it) but she certainly knew how important the theorem was. For the last few years, basically every paper in my domain has used the Magic Wand theorem in one way or another.

However, it took Alex Eskin and Maryam Mirzakhani several more years of extremely hard work to extend their result, proving the rigidity properties not only for the group $SL(2,\mathbb{R})$ of all 2×2 matrices with unit determinant but also for its subgroup of upper-triangular 2×2 matrices (which is already *amenable*). The difference might seem insignificant. However, exactly this difference is needed for the most powerful version of the Magic Wand theorem. Part of the theorem concerning orbit closures was proved in collaboration with Amir Mohammadi; an important complement was proved by Simion Filip.

Suppose, for example, that you have to study billiards in a rational polygon (that is, in a polygon with angles that are rational multiples of π). What mathematical object can be more simple-minded and down-to-earth than a rational triangle? However, the only known efficient approach to the study of billiards in rational polygons consists of the following. Applying symmetries over the sides of the polygon, unfold your billiard trajectory into a closed surface. The billiard trajectory gets unfolded into a non-self-intersecting winding line on this closed *translation surface*. This nice trick is called Katok–Zemlyakov construction.

Consider, for example, the triangle with angles $\frac{3\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{4}$. It is easy to check that a generic billiard trajectory moves in one of eight directions at any time. We can unfold the triangle to a regular octagon glued from eight copies of the triangle. Identifying the opposite sides of the octagon, we get a closed surface of genus two endowed with a flat metric. There is no

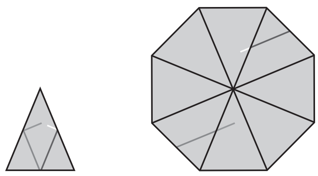


Figure 4. Billiard trajectories unfold into leaves of surface foliation

contradiction with Gauss–Bonnet theorem, since our flat metric has a conical singularity: all vertices of the octagon are glued into one point with the cone angle 6π. Geodesics on the resulting flat surface correspond to unfolded billiard trajectories.

It is convenient to incorporate the direction of the straight-line foliation into the structure of the translation surface and, turning the resulting polygon, place the distinguished direction into a vertical position. Acting on such "polarised" octagons with linear transformations of the plane, we get other octagons with sides distributed into pairs, where sides in each pair are parallel and have equal lengths. From any such polygon, we can glue a translation surface. Having a translation surface, we can unwrap it to a polygon in many ways (see Figure 5). This gives an idea of why the $GL(2,\mathbb{R})$-action on the space of translation surfaces is anything but easy to study.

Now, touch the resulting translation surface with the Magic Wand theorem to get the closure of its $GL(2,\mathbb{R})$-orbit in the moduli space of all translation surfaces sharing the same combinatorial geometry as the initial surface. According to the Magic Wand theorem, the orbit closure is a very nice orbifold that would provide you with plenty of fine information about the initial state. Is it not like getting a Cinderella Pumpkin Coach? One of the last works of Maryam Mirzakhani, performed in collaboration with Alex Wright, proves that despite the fact that the translation surface obtained after unfolding a rational triangle has plenty of symmetries, the corresponding orbit closure is often as large as it can be: it coincides with the entire ambient moduli space of translation surfaces. (The triangle considered above, however, gives a small orbit closure.)

The proof of the Magic Wand theorem is a titanic work, which absorbed numerous recent fundamental developments in dynamical systems; most of these developments do not have any direct relation to moduli spaces. I still do not understand how they managed to do it. Very serious technical difficulties appeared at every stage of the project, not to mention that in the four years between 2010 and 2014 Maryam gave birth to a daughter and managed to overcome the first attack of cancer. Since then, I believed that Maryam could do everything.

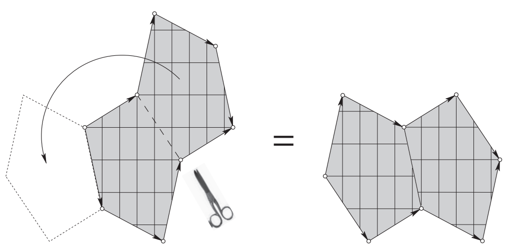


Figure 5. Polygonal patterns of the same translation surface

I cannot help telling a story that is symbolic to me. After M. Mirzakhani received the Fields Medal, I was asked by the "Gazette of the SMF" to write an article about the Magic Wand theorem and to ask Maryam for her picture to include in the article. The photograph that I received from Maryam was unexpected for a scientific paper: a three-year-old girl was holding two balloons of sophisticated shape (Riemann surfaces) almost as big as the girl herself.

However, the picture seemed to me absolutely appropriate. It perfectly represented my own image of Maryam; I was just surprised that she would suggest such a picture herself. Maryam carried through her entire life the curiosity and imagination that are so natural for children but which, unfortunately, are lost by most grown-ups.

Then came the next email: "Oops, sorry Anton, you got a picture of my daughter :-)" I had taken Anahita for Maryam.

Curt McMullen has a story related to Anahita that occurred during the ICM laudation. Presenting Maryam's work to thousands of people, Curt was nervous, asking himself how Maryam, sitting in the front row, might perceive his description of her accomplishments. During the talk, he realised that Maryam was spending most of the time tending to Anahita sitting on her knees.

In the Autumn of 2016, I learned that the illness had come back. But I also knew for sure that Maryam was doing her best to stay with her daughter and with her family as long as possible. I was not the only one to believe that Maryam could do what no other human can do. But by admiring someone's outstanding courage, we cannot expect that person to produce miracles.

If you want to learn more about Maryam as a personality, read the article "A Tenacious Explorer of Abstract Surfaces", written by Erica Klarreich for *Quanta* Magazine, and watch the Stanford Memorial recorded on October 2017 on YouTube. Maryam's husband, Jan Vondrák, her shield and support, said at the memorial: "*I want to say to the young people who are asking questions 'What would Maryam say?' that though she was a role model, it does not mean that you have to be exactly like her ... You have to find your own path. You have to find what you love. You have to find what you are good at and what is meaningful to you. And if you do it well then you would have made Maryam happy.*"

"A light was turned off today," wrote Firouz Naderi, announcing Maryam's death. Both Maryam's work and her personality inspired and encouraged many people all over the world – women and men. Maryam's light will be kept inside us.

Anton Zorich (anton.zorich@gmail.com) works at University Paris Diderot and frequently visits the Center of Advanced Studies at Skoltech. His current research interests are mostly focused on interactions between geometry and dynamics in moduli spaces.

This paper contains extracts of the author's publications in *Notices of the AMS*, **62:11** and in *Gazette des Mathématiciens*, **154**.

A Mathematical Polyglot

Elisabetta Strickland (University of Rome Tor Vergata, Italy), EMS-WIM Committee

On 13 August 2014, when Maryam Mirzakhani entered the main hall of the COEX, a gigantic conference centre in Seoul, South Korea, the Opening Ceremony of the 27th ICM (the International Congress of Mathematicians) was about to start. The 5000 people sitting there were all waiting for the names of the four Fields Medallists to be announced and all the women in the audience were incredibly excited, as rumours had been circulating that for the first time a woman would be among the winners and that that woman would be Maryam. As a fortunate circumstance, she was sitting close to me, as the delegates to the IMU General Assembly, which had taken place three days before in Gyeongju, had been positioned close to the Fields Medal winners. I really couldn't resist reaching over to her. I noticed immediately her grey-blue eyes and her interesting expression: she didn't show any outward emotion; she obviously had an unwavering self-confidence and, at the same time, a fundamental humility. In my excitement, I asked her if she sensed the strong feelings about the meaning of her presence there for all the women mathematicians in the hall. I noticed that she was moved by my words and she simply and gracefully shook my hand and smiled, whispering "Thank you".

Then, her name was officially announced with those of the other three Medal winners: Artur Avila, Manjul Bhargava and Martin Hairer. A novel feature of the ceremony were short films about the winners, produced by Jim Simmons, the American mathematician, hedge fund manager and philanthropist, who, through his foundation, supports projects in mathematics and in research in general. We could all see on the screen Maryam in her home in Palo Alto, California, kneeling with a felt-tip pen in her hand, doodling on vast white sheets of drawing paper unrolled on the floor of her room, to the amusement of her toddler Anahita, who believed she was painting instead of drawing Riemann surfaces.

I'll never forget the roar that went up in the hall when her name was announced. Tears came to our eyes and we stood up screaming in joy because, all of a sudden, it was true: for the first time after so many editions of the ICM since 1897, a woman had received the Fields Medal, the most coveted award in mathematics. The citation by the IMU Fields Medal Committee was "for her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces".

Maryam was 37 years old at the time. Even if some of us had been informed that she was undergoing therapy for breast cancer, we dismissed the thought from our minds; this was a time for optimism. Unfortunately, three years later we learned that her body hadn't succeeded in remission from the disease and the cancer had spread to her bones. In July 2017, she died.

Since that magic day in Seoul, she did some fantastic work in the world of billiard tables, a profound and

Maryam Mirzakhani in Seoul, together with the other Fields Medal winners: Artur Avila, Manjul Bhargava and Martin Hairer. Author: Gert-Martin Greuel. Source: Archives of the Mathematisches Forschunginstitut Oberwolfach.

modern mathematical subject. She developed this work together with Alex Eskin and partly in collaboration with Amir Mohammadi. The crowning result of this work, by some considered one of the most important of the decade, is now known as the "Magic Wand theorem".

Reading her biography, one finds out that until mathematics attracted her, she planned to become a novelist. It was only later in her school career that she discovered that she achieved a special understanding of mathematics, thanks to an encouraging teacher, so she switched her interest to science. In the Iran of her childhood, books cost very little and she read so many that later her maths had a literary tinge, as if the problems she studied were the characters of a fascinating plot. She just had to do her best to know them and work them out.

Maybe her early passion for literature was also the reason for her being, by her own admission, a "slow" mathematician: her husband, Jan Vondrak, an electrical engineer and theoretical computer scientist at IBM Almaden Research Center in San José, California, joked about her steady approach to all areas of her life. He used to tell a story that went back to their graduate student days, she at Harvard, he at the Massachusetts Institute of Technology. They used to go jogging together and while he would initially run faster and keep ahead, eventually he would tire, while her slow but steady pace meant she would overtake him and arrive first.

She was so smart in school that she entered the Tehran Farzanegan School, an institution for educationally gifted girls. Then, she was the first girl, together with her friend and schoolmate Roya Beheshti, to represent her country in the Mathematics Olympic Games, winning gold medals for two successive years, in 1994 in Hong

Kong and in 1995 in Toronto. After graduating from Sharif University in Tehran, she followed a path taken by many Iranian students, as she left for the United States for postgraduate studies and in 2004 obtained her PhD at Harvard, under the doctoral supervision of Curtis McMullen, a fellow Fields Medal winner.

She obtained a job in Princeton immediately after and then moved to Stanford in 2008 as a full professor. This means that she had a first class career far from her country but was always grateful to her native Iran, as she experienced first-hand how the education and careers of women were encouraged. Being not only the first woman to win the Fields Medal but the first Iranian, she became a celebrity in her country, so much so that, after her death, the media gave up portraying her with a headscarf, showing instead an Iranian woman with short hair, her head uncovered. This meant that officials allowed editors to ignore Iran's strict dress code for female pictures to mark her death. She remained a heroine for many of her fellow countrymen, including President Hassan Rouhani, who released a condolence message about her "unique brilliance and contribution to scientific progress of Iranian women".

While Maryam was not easily disappointed and was always confident in herself, when she was told by email that she had been awarded the Fields Medal, she ignored the message, assuming it was a joke. She resisted pressure to be a role model because she believed that many other women were also doing great things in mathematics. But she believed that discouragement is a real problem for female mathematicians, as the peak years of mathematical productivity often coincide with the time in life that women give birth and care for small children. So, when she won the medal, she expressed the hope that her award would encourage young female mathematicians.

Stanford University had to work hard on press releases in order to explain her specialisation in the geometry and dynamics of complex curved surfaces, an abstract field that reads like an obscure foreign language to non-mathematicians. The Stanford press release states that she worked on "moduli spaces, Teichmüller theory, hyperbolic geometry, ergodic theory and symplectic geometry", an astonishing variety of fields. This is like stating that she was a mathematical polyglot. In addition to their beauty and complexity, her results in these fields will help physicists and cosmologists to investigate the fundamental nature of the universe.

After her death, Stanford President Marc Tessier-Lavigne said that "Maryam has gone far too soon but her impact will live on for the thousands of women she inspired to pursue mathematics and science". We can only agree with these words.

Elisabetta Strickland is a full professor of algebra at the Department of Mathematics of the University of Rome "Tor Vergata". She was Deputy President of the National Institute of Advanced Mathematics (INdAM) from 2007 to 2015. Since 2014, she has been a member of the Women in Mathematics Committee (WIM) of the European Mathematical Society. She is also a co-founder of the Gender Interuniversity Observatory GIO over the state universities in Rome. Since 2016, she has been an Ambassador of Italy on the Committee of Women in Mathematics (CWM) of the International Mathematical Union. In 2013, she was awarded the "Excellent Women in Rome" Prize from the Capitoline Administration.

Pi Day: An International Festival of Mathematics

Elena Berardini (Aix-Marseille Université, Marseille, France), Joël Cohen (Université Paris Sud, Orsay, France), Guillaume Geoffroy (Aix-Marseille Université, Marseille, France) and Annamaria Iezzi (University of South Florida, Tampa, USA)

We, as mathematicians, may not have a patron saint but we do have a favourite number: it goes by the name of "π", "pi", "the circle constant" or "approximately 3.14". This number is so dear to us that we have reserved a special button for it on most calculators and we have created a holiday in its honour: 14 March. This tradition comes from the United States, where this date is written 3.14 and where it was celebrated for the first time in 1988, at the Exploratorium of San Francisco.

Every year, Pi Day is an opportunity for all mathematicians, geeks and science enthusiasts around the world to get together and celebrate science, not to mention the excuse to eat pie.

Eager to get in on this pie-eating action, a group of PhD students in Marseille with a lot of enthusiasm, energy and perhaps a dash of irrationality imported the concept across the ocean. Their original idea was to blend education and entertainment together into one. What started out in 2013 as a small gathering at a PhD student seminar with pies soon became a bigger event that attracted thousands of people from the general public. Thus was born the Pi Day Association, which now travels across France showcasing its own mathematical musicals.

A mathematical musical!

Since 2015, our association has written and produced a new show every year, blending various elements of music, entertainment and education together. In 2016, we reworked the formula and combined the elements into a musical about the life and scientific discoveries of Archimedes. The show was a great success that attracted nearly 800 spectators and received the d'Alembert Prize of the French Mathematical Society. In 2017, we were back with an even bigger project: a brand new musical entitled *From Marseille to Vegas*. We gave three (not quite π) performances: one in Paris (Théâtre des Variétés) on 14 March, one in Marseille (Le Silo) on 16 March and one in Lyon (Le Transbordeur) on 19 March.

From Marseille to Vegas is a story about the misadventures of four PhD students. Having grown tired and disenchanted with academic life, they dream up the crazy idea of trying their luck at the casinos in Las Vegas using their mathematical knowledge. Can they play chance with mathematics? This musical humorously addresses the themes of probability theory, the life and status of PhD students and the role of mathematics and science in society.

Each performance was accompanied by three short talks given by experienced popularisers of mathematics. The speakers and their topics came from different backgrounds but they all revolved around the general theme of randomness. We learnt, among other things, how to cheat at slot machines, how to tame crowds with mathematical formulas and how Leonardo Fibonacci liked his pasta. We spent some time wondering if statistics really mean anything and if it is really such a small, small world after all.

Before the performance and during the break, members of the audience could take part in mathematical activities organised by students. The third event, in Lyon, also featured the traditional pie contest: participants were invited to bring their pies, share them with the rest of the spectators and try to win the best prizes.

A successful bet for mathematics

In total, the three events attracted almost 2,000 spectators of all ages, who came to learn, think, dream, laugh and sing to the sound of mathematics: quite a success for such a "boring" science!

In case you were unlucky enough to miss the event, you can catch up on our website: www.piday.fr. Stay tuned for our next musical, slated to premiere on March 14, 2019, and to be performed across France. If you want to support us, join us in our new project or if you would like the show to come to your city, do not hesitate to contact us! We are looking forward to meeting you for a new series of irrational, transcendental, but probably normal events!

Elena Berardini is a PhD student in mathematics at Aix-Marseille Université and is treasurer of the association.
Joël Cohen is a lecturer at Université Paris-Sud and is vice-president of the association.
Guillaume Geoffroy is a PhD student in mathematics at Aix-Marseille Université and is president of the association.
Annamaria Iezzi is a postdoctoral researcher in mathematics at the University of South Florida and is secretary of the association.
Together, they organised Pi Day 2017 in Paris, Lyon and Marseille. In particular, Joël and Guillaume wrote the musical and coordinated the artistic committee, and Annamaria and Elena were in charge of coordinating the actual realisation of the project, from funding to communication and stage set.
None of this would have been possible without an extensive team of volunteers and many sponsors: the names of all the culprits are listed on www.piday.fr.

Fondation Sciences Mathématiques de Paris (FSMP)

Gaël Octavia (FSMP, Paris, France)

Paris – a fantastic place for doing mathematics

Paris represents the largest concentration of mathematicians in the world, with a scientific spectrum that is unequalled anywhere else: whatever question you ask in any field of mathematical sciences, there is an expert in Paris able to work on it! Paris also hosts nearly a third of the students enrolled on a mathematics Master's in France.

History of the FSMP

In 2006, a call for projects was launched by the French government in order to create Thematic Networks for Advanced Research, to promote French research and give it more international visibility and attractiveness. Parisian mathematicians and computer scientists saw this as an opportunity to provide an official framework and adequate administrative structure for an outstanding scientific community and the largest centre of mathematical sciences in the world! The Fondation Sciences Mathématiques de Paris was finally born on 21 December 2006 (https://www.sciencesmaths-paris.fr/).

Over the past few years, the FSMP has consolidated its sources of funding to ensure its sustainability. It has carried forward several projects with success and has received financial support from the French Government, the Région Ile-de-France (the Paris area), the City of Paris and, since 2017, the European Horizon 2020 programme, along with the Marie Sklodowska-Curie COFUND project MathInParis. Furthermore, the FSMP raises funds and receives patronage from individuals and companies.

The FSMP's launch day at the Collège de France. © UPMC, Pierre Kitmacher

The FSMP's network

The FSMP was founded by University Pierre-et-Marie-Curie, University Paris-Diderot, CNRS and Ecole Normale Supérieure. It included, from its inception, the four chairs of mathematics of the College de France. In the years that followed its creation, other scientific partners joined and strengthened the network: Inria and the Universities of Paris-Dauphine, Paris-Descartes, Paris 1 Panthéon-Sorbonne, Paris 13 and Paris Sciences et Lettres (teams from the Ecole des Mines, the Observatoire de Paris and EHESS).

The FSMP presently federates 13 laboratories in mathematical sciences and 23 Inria teams. It gathers together more than 1,800 researchers (amongst them 900 permanent posts), including four Fields Medallists, 20 members of the French Academy of Science and many recipients of national and international awards. It is a world leader in mathematical research. Although the Shanghai ranking should be considered with caution, let us recall, in mathematics, that the first university outside the USA is University Pierre-et-Marie-Curie (ranked 3rd) and that among the top 100 are also Ecole Normale Supérieure and the Universities of Paris-Dauphine, Paris-Diderot and Paris 13.

Federating the mathematical community in the Paris area and in France

The FSMP has succeeded in bringing together all the players in mathematical and theoretical computing research in Paris and its northern suburbs. Following its example, similar unifying structures have been founded in the southern suburbs (Fondation Mathématique Jacques Hadamard), the east (Fédération de Recherche Bézout) and the west (Paris-Seine). These structures are scientific partners of the FSMP. Together with Institut Henri Poincaré as well, they organise scientific events and carry out joint projects. This efficient and highly structured network, which the FSMP initiated and heavily contributed to develop, is a great asset in the fight against fragmentation, helping maintain the scientific unity of the mathematical community in the Paris area.

The FSMP also sets up actions to help in federating the French community of mathematicians, such as developing a reference web portal to post all postdoctoral job offers in France and make them more visible and, with other institutions, commissioning a report showing the impact of mathematics on the French economy and society.

The FSMP's programmes

The FSMP runs and finances four flagship programmes dedicated to outstanding researchers, promising young mathematicians and brilliant students from all over the world:

- The Paris Graduate School of Mathematical sciences (PGSM), which consists of one- or two-year scholarships for Master's students. This programme was pioneered in France. It was immediately a huge success with an average of 300 applications per year, imposing a need for very strong selection. In the framework of the PGSM, the FSMP has set up special programmes in partnership with embassies in order to promote exchanges with their countries: thus, PGSM-Chile, PGSM-Cuba, PGSM-Iran and PGSM-Romania have emerged, allowing students from these countries to come and follow their Master's degrees in Paris. The programme presently welcomes about 20 students every year.

PGSM welcome day. © FSMP

- The doctoral programme, which provides doctoral allowances and academic and administrative support to PhD students. This programme has 30 laureates each year.
- The postdoctoral programme, which offers funding and hosting to around 20 talented young researchers every year, for one- or two-year positions. In France, it was the first programme of its kind, conceived in accordance with international selection standards. It has been very successful since 2007, with an average of 150 applications each year. Selection is extremely hard. With this programme, the FSMP competes with the best Anglo-Saxon universities.
- Excellence Chairs, which offer scientific stays of four to twelve months in laboratories of Paris for top-rank senior researchers or junior researchers with an exceptional profile, in order to create lasting collaborations. The programme welcomes one to three laureates each year, who give courses, organise seminars and participate in many scientific activities during their stay. The laureates receive an attractive salary and additional support to finance students, invite researchers and organise workshops. Researchers among the most renowned in their fields, such as Edward Frenkel, Ron DeVore, Hélène Esnault and the Fields Medallist Timothy Gowers, have been laureates of this programme.

These four main programmes cover all levels of research and training in mathematics, from Master's degree to high-level research activities with international visibility.

The FSMP also finances smaller programmes such as invitations of foreign researchers to its laboratories, funding of scientific stays abroad for PhD students and the Math C2+ programme, which consists of giving a taste of mathematics to high school students through lectures, laboratory visits, workshops and courses focused on research approaches.

Integration of PhD laureates

The FSMP is very concerned with the professional integration of young doctors in mathematics, the PhD degree still being less valued by companies in France when compared to Anglo-Saxon countries. The FSMP has set up several actions to help them learn more about the job market and the opportunities offered to them, not just in the academic world but also in the economic and industrial worlds, and help them optimise their job searching: e.g. *how to write a good resume* and *how to be effective during a job interview* workshops, conferences and meetings with potential employers.

Courses and conferences

The FSMP participates in the organisation of many courses, conferences and workshops involving members of its laboratories or laureates of its programmes. Its commitment and support relate to the scientific aspects as well as the logistics and communications aspects.

Among the annual events organised by the FSMP, "*Mathématiques en mouvement*" ("*Mathematics in motion*") is a conference where students (bachelor's, preparatory classes and Master's) are shown the prodigious diversity of research in mathematics through accessible lectures, round tables, posters exhibitions and meetings with young mathematicians – sometimes PhD students.

All lectures are filmed and are available on the FSMP's website.

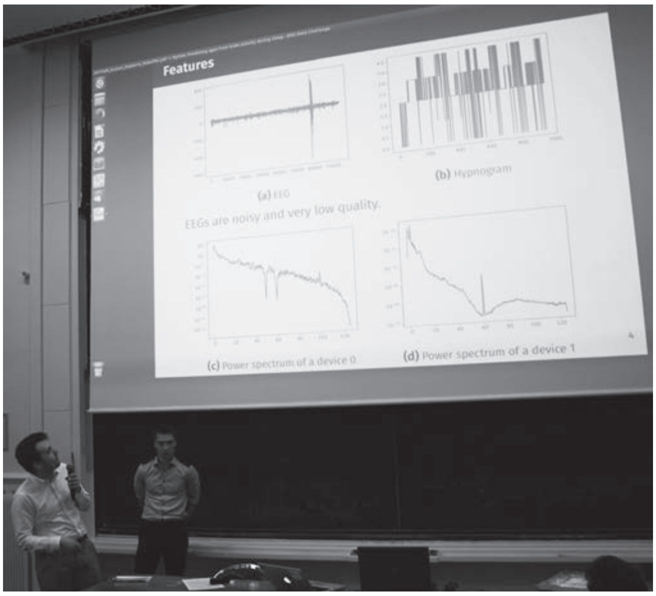
Collaboration with the industrial world

Strengthening collaboration between academic mathematicians and the industrial world is one of the FSMP's core missions. The FSMP addresses this in a number of ways.

It promotes relationships between academics and business researchers, co-finances industrial theses, offers continuous training in mathematics and computer science and helps set up doctoral-consulting contracts with companies.

Every year, the FSMP organises with a new industrial partner (EADS, Areva, Huawei…) the conference "*Horizon Maths*", which brings together academic mathematicians and R&D researchers in companies on a theme chosen according to the scientific interest of the partner.

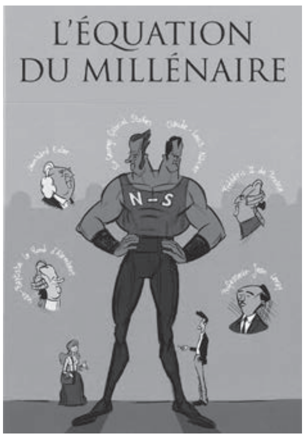
The FSMP encourages incubation of start-ups by its young graduates in mathematics (Master's or PhD). It supports Challenge Data, a machine learning competition that offers to students the chance to solve problems of data classification, regression and prediction proposed by start-ups and companies, with real data (medical data, images, sounds, marketing data, internet searches, etc.), giving rise to meetings and creating opportunities.



Students solving industrial problems at Challenge Data. @ FSMP

Actions for the general public

Disseminating mathematical culture outside the community of specialists, and particularly to the general public, is also a mission of the FSMP. This is addressed through various types of media, including short popular articles, video interviews of researchers published on its website, blogs and a mathematical calendar every year.



The FSMP is co-editor of *Mathématiques, l'explosion continue*, a brochure that presents 25 subjects of applied mathematics in 25 accessible articles. It also created and published *L'Equation du millénaire* (*The Millennium Equation*), a comic strip about the Navier–Stokes equations.

The FSMP also participates in and supports the holding of flagship events of scientific dissemination, such as the "Salon de la Culture et des Jeux Mathématiques", where 25,000 visitors, including many children and teenagers, come each year to discover mathematics in a playful way.

The FSMP takes part each year in the "Fête de la Science", organising the animation "Raconte-moi ta thèse!" ("Tell me about your thesis!"), a *speed meeting* between Parisian PhD students and the general public.

It supported financially and contributed to the dissemination of the exhibitions *Regards in spaces in dimension 3* and *Espace Imaginaires, motifs and mirages* designed by Pierre Berger (University Paris-13), a researcher who offers an artistic approach to geometry.

The FSMP also supports the biennial Parity Day, which examines the presence of women in mathematics.

In 2010, none of the major French media sent journalists to Hyderabad to cover the ICM, during which the French mathematicians Cédric Villani and Ngo Bau Chau received the Fields Medal (unlike other nations and even though France was assured of getting at least one Fields Medal). To compensate for this absence, the FSMP produced a blog (which was, at that point, unique), relating the congress day-by-day and often taken up by French journalists left at home, letting the French public know more about the ICM. Four years later, the idea had caught on and several other bloggers shared their experiences of the Seoul Congress (ICM 2014, at which the French-Brazilian mathematician Artur Avila received a Fields Medal).

A look to the future

Over the last few years, the FSMP has become a key player in mathematical research and training in Paris, in France and internationally. Its significant contribution to the vitality, excellence and attractiveness of the laboratories in its area will continue into the future by drawing outstanding foreign researchers and brilliant foreign students into France and by strengthening the links between research in the mathematical sciences and the world of economics. There are still challenges ahead but the FSMP is willing and able to take advantage of all opportunities to make research in mathematical sciences more successful and contribute to developing and maintaining excellence in its network.

Graduated from Télécom Sud-Paris in 2001, Gaël Octavia was firstly an information systems engineer and then became a scientific journalist. From 2002 to 2008 she works as sub-editor for Tangente, a magazine of mathematical content for the public in general. In February 2008, she joins the FSMP as Communication Manager. She is also a playwright and novelist.

The Turkish Mathematical Society

Betül Tanbay (Bogaziçi University, Istanbul, Turkey) and Attila Aşkar (Koç University, Istanbul, Turkey)

The Turkish Mathematical Society (TMD) is a Turkish organisation dedicated to the development of mathematics in Turkey. Its members are either individual mathematicians living in Turkey or Turkish mathematicians living abroad, adding up to more than 800. The society seeks to serve mathematicians particularly in universities, research institutes and other forms of higher education. In Turkey, there are more than a hundred mathematics departments, with around 2,000 faculty members, 30,000 undergraduates and 6,500 graduate students. In 2016, Turkish mathematicians published 2,300 articles in international media covered by SCI and secured an h-factor of 96. This is representative of the annual productivity.

The Turkish Mathematical Society was founded in 1948 by eminent researchers of Istanbul University and Istanbul Technical University. The governing body of the TMD is its General Assembly, consisting of all its full members. The General Assembly meets every two years and appoints the Executive Committee members, who are responsible for the running of the society. The TMD became a full member of the IMU in 1960 and was raised to Group II in 2016. It became a member of the EMS in 2008 and joined MASSEE in 2014. The society is located in Istanbul and has also, since 1992, had a branch in Ankara.

View from the terrace of the building of the TMD offices.

For the last 30 years, an annual symposium has been held in different cities and universities of the country. Each year, hundreds of academicians attend these symposia to collaborate over research and benefit from lectures presented by researchers who are selectively invited by a Scientific Advisory Committee. Through the organisation of special sessions, young researchers are given the opportunity to present their theses, giving them a professional platform for discussing their ideas.

The announcement of the 22nd National Symposium organised in the Nesin Mathematical Village.

The society takes part in the organisation of the Caucasian Mathematics Conference (CMC) and, in August 2017, the second CMC was held in Van, Turkey. The TMD is an institutional member of the Silkroad Mathematical Center in Beijing and its president is a member of the Steering Committee.

CMC-II, Van, 22 August 2017.

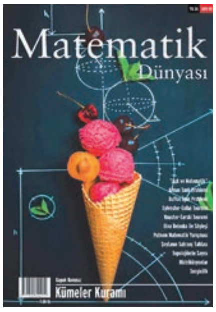
The only financial source of the TMD used to be the modest membership fees. In the last four years, through a project called MAD (Matematik Araştırma Dostları, i.e. Friends of Mathematical Research), a group of civil societies and individuals, convinced of the significance of a mathematical society, has been formed and has allowed the TMD to support more than 40 conferences, symposia, workshops, summer schools and youth activities in mathematics in the country (http://tmd.org.tr/mad-2014-17-raporu/). In particular, the MAD-Youth Fund, again donated by a civil society, has supported five youth workshops at four universities attended by hundreds of undergraduates from throughout Turkey.

A cooperation with the Institute of Oberwolhfach and the Istanbul Center for Mathematical Sciences (IMBM) has allowed the TMD to initiate the IMAGINARY expositions in Turkey: 12 expositions attended by over 30,000 visitors in nine cities. The project is still ongoing and the goal is to exhibit IMAGINARY in a few more cities every year.

The announcement of the Diyarbakir IMAGINARY exhibition in November 2015.

The TMD offers four fellowships a year to undergraduates in mathematics, with the goal of encouraging talented youth as well as bringing the importance of this field to the attention of society at large.

A popular quarterly mathematics magazine *Matematik Dünyası* has been published since 1991. It sells over



An issue of the popular magazine *Matematik Dünyası*.

10,000 copies per issue and is very popular with high school and undergraduate students. The magazine is designed to encourage students of all ages to pursue activities and careers in the mathematical sciences. Successive issues are dedicated to specific subjects covering contemporary advancements in mathematics.

Professional associations come about when a critical mass forms in a community. Leaving aside the individual efforts of exceptional talent in the distant past, we can see the beginning of organised efforts with the foundation of the Royal School of Naval Engineering in 1773, during the time of the Ottoman Empire. A significant turning point for formal mathematics education in Turkey was in 1830, when the "Madrassa" was transformed into a "University", later to be called Imperial University in 1900. With the energies and modernisation efforts of the new Turkish Republic in 1923, the formation of Istanbul University and Istanbul Technical University followed in 1933.

There have been two western interactions pertaining to the sciences in general and mathematics in particular. The first was the famous Einstein letter to the Turkish President Atatürk in 1933. This letter described the difficulties regarding academic freedom in Europe (with perhaps worse things to follow) and that there were many European scientists and scholars interested in coming to Turkish universities. Indeed, during the period 1933-1945, more than 200 German scholars (of which 15 were involved with mathematics and science) joined Turkish universities. Among the mathematicians, von Mises, Prager and Horninger left the most positive legacies. The second interaction was the report by Courant and Born on the formation of an Institute of Mathematics à la Göttingen in Istanbul. It is sad that the recommendations of this report were followed only partially and the necessary funds were not allocated.

With the efforts of the new generation of mathematicians, educated mostly in Western Europe with the support of special fellowships from the young Turkish Republic, a critical mass was forming in Turkey, resulting in the formation of the Turkish Mathematical Society in 1948. One of the first presidents of the TMD was Cahit Arf, whose picture and equation on the 10 lira banknote makes us proud and happy.

The 10 lira banknote, with Cahit Arf's photo and the Arf equation.

The TMD is also proud to have been an institution capable of renewal and continuity at the same time. Past board members still actively work for the society, whilst many young researchers have become members of the board. Over the past 20 years, the society has put particular importance on international relations and it has been represented at most meetings and assemblies of organisations such as the IMU and the EMS. We believe in the importance of global mathematical cooperation for a healthy and peaceful development of humanity.

The last four presidents of the TMD from left to right: Betül Tanbay (2010–2016), Ali Ülger (2008–2010), Attila Aşkar (2016–), Tosun Terzioğlu (1989–2008).

Attila Aşkar is currently the president of the Turkish Mathematical Society and a professor of applied mathematics at Koç University in Turkey, where he served as president from 2001 to 2009. He held academic appointments at Boğaziçi, Brown, Princeton and Paris VI Universities, the Max-Planck Institute in Göttingen and the Royal Institute of Technology in Stockholm. He received the Junior Scientist Award and the Science Award of the National Research Council, the Information Age Award of the Ministry of Culture and was elected to the National Academy of Sciences of Turkey. Attila Aşkar received his engineering diploma from Istanbul Technical University in 1966 and his PhD from Princeton University in 1969.

Betül Tanbay is the first woman president of the Turkish Mathematical Society and a professor in functional analysis at Boğaziçi University in Istanbul, where she has served as senate member, vice-provost for foreign affairs and chairwoman. She has held visiting positions at the Universities of California, Santa Barbara and Berkeley, Kansas, Pennsylvania State, Paris VI and Bordeaux. She was founder and first co-director of the Istanbul Center for Mathematical Sciences, a member of the executive and advisory boards of the Feza Gürsey Institute, a scientific council member of IméRA, a project director of the doctoral and postdoctoral network grants of the National Research Council, a committee member of IMU-CWM, EMS-Raising Public Awareness, EMS-Ethics and currently EMS-Executive. Betül Tanbay received her undergraduate degree from ULP, Strasbourg, in 1982 and graduate degrees from UC Berkeley in 1989.

ICMI Column

Jean-Luc Dorier (Université de Genève, Switzerland)

The 2017 Felix Klein and Hans Freudenthal Awards

The ICMI is proud to announce the seventh recipients of the *Klein and Freudenthal Awards.*

The *Felix Klein and Hans Freudenthal Awards*, presented in each of the odd-numbered years since 2003, are two prizes created by the ICMI for recognising outstanding achievement in mathematics education research. They respectively honour a lifetime achievement (*Felix Klein Award*, named after the first president of the ICMI – 1908 until 1920) and a major cumulative programme of research (*Hans Freudenthal Award*, named after the eighth president of the ICMI – 1967 until 1970). By paying tribute to outstanding scholarship in mathematics education, these awards serve not only to encourage the efforts of others but also to contribute to the development of high standards for the field through the public recognition of exemplars. Each award consists of a medal and a certificate, accompanied by a citation. They have a character similar to that of an honorary university degree. At the International Congress on Mathematical Education (ICME), the awardees are honoured during the opening ceremony. Furthermore, the awardees are invited to present special lectures (ICMI Award Lectures) at the congress. The *Felix Klein and Hans Freudenthal Awards* are selected by an anonymous award committee of distinguished international scholars. The jury for the 2017 awards was chaired by Professor Anna Sfard, Haifa University, Israel.

We give some key biographical elements below; full citations of the work of the two 2017 medallists can be found at: https://www.mathunion.org/icmi/awards/icmi-awards.

The following table gives a list of all the previous awardees since the creation of the medals in 2003:

	Felix Klein Award	*Hans Freudenthal Award*
2003	Guy Brousseau	Celia Hoyles
2005	Ubiratan d'Ambrosio	Paul Cobb
2007	Jeremy Kilpatrick	Anna Sfard
2009	Gilah Leder	Yves Chevallard
2011	Alan Schoenfeld	Luis Radford
2013	Michèle Artigue	Frederick Leung
2015	Alan Bishop	Jill Adler

Deborah Loewenberg Ball, William H. Payne Collegiate Professor in Education and an Arthur F. Thurnau Professor in the University of Michigan, USA receives the 2017 Felix Klein Award.

The *Felix Klein Award 2017* is awarded to **Professor Deborah Loewenberg Ball** in recognition of her outstanding contributions and her leadership role in deepening our understanding of the complexities of teaching mathematics and in improving the practice of teaching and of teacher education. These achievements are grounded in Deborah Ball's firm belief that research and the practice of teaching are co-constitutive and must always be developed in tandem. Early in her life, Deborah Ball, at that time an exceptionally talented elementary school mathematics teacher, set out to investigate what was involved in the work of teaching children mathematics "for understanding". Her intention was to uncover the work in order to support the learning of teaching practice. Ever since then, her ambition has been to contribute in a substantial way to the project of improving ways in which mathematics teachers support their students' learning. This goal gave rise to two lines of work, both of them combining research with development in the domain of teacher education. The first strand, in which the research element came first, has been generating studies revolving around the question of what mathematical knowledge is required for teaching learners. In the second line of work, related to the practice of education in a more immediate way, the development of innovative teacher preparation programmes has been combined with research, through which Deborah Ball has been trying to gain a better grasp of the moment-to-moment dilemmas with which teachers grapple in the classroom.

The first of these pursuits gave rise to the theory of MKT, Mathematical Knowledge for Teaching, the kind of knowledge that requires competence in both everyday and academic mathematical discourses but is not identical to either. In her multiple studies, Deborah Ball and her colleagues have been able to identify many unique features of MKT and then to corroborate the conjecture about a correlation between teachers' competence in this special brand of mathematics and the achievements of their students.

The second, newer strand of Deborah Ball's work is focused on *TeachingWorks*, a national organisation she established at the University of Michigan to help in improving teachers' preparation and to define "a professional threshold for entry to teaching". The mission of the institute is to identify "high-leverage" teaching practices, that is, those recurring elements of teachers' classroom activities that are central to what Deborah Ball terms "the work of teaching".

Deborah Ball has been an elementary classroom teacher before and during her studies at Michigan State University, which she completed in 1988 with a PhD in mathematics education. Upon graduation, she joined Michigan State University and, in 1996, she was recruited to the University of Michigan to develop the mathematics education group. She has been teaching at the University of Michigan ever since and also spent over a decade serving as Dean of the School of Education there. She has played multiple leadership roles, not only within the community of mathematics education but also within that of education at large, and not only within the United States but internationally. With more than 30 years of outstanding achievements in mathematics education research and development, Deborah Ball is a most distinguished member of the mathematics education community and a highly deserving recipient of the 2017 Felix Klein Award.

Terezinha Nunes, Professor Emeritus of Educational Studies at the University of Oxford, UK receives the 2017 Hans Freudenthal Award.

The *Hans Freudenthal Award 2017* is awarded to **Professor Terezinha Nunes** for her outstanding contribution to our understanding of mathematical thinking, its origins and development. For more than 35 years now, she has been researching children's mathematical learning taking place in formal and informal settings. The results of her numerous, exemplarily designed studies combine into an insightful, consistent and comprehensive story of the emergence and evolution of mathematical thinking. This constantly developing account has been inspiring the work of mathematics education researchers and informing mathematics teachers' practices all over the world. It has had a major impact on both what we know about children's learning of mathematics and on how we know and think about it.

Terezinha Nunes' research has been immensely innovative and influential from its earliest stages. In one of her first studies, she documented the mathematical skills of young Brazilian street vendors, who, although almost unschooled and incapable of executing paper-and-pencil arithmetic tasks, proved impressively proficient in complex money transactions. Her later research on the development of mathematical thinking, conducted in Brazil and the UK, spans multiple mathematical topics, from additive and multiplicative reasoning to fractions, variables, randomness and probability. She has studied children's logical reasoning and its role in the learning of mathematics, as well as problem solving and the way mathematics is being used in science.

While forging her stories on children's thinking about numbers, Terezinha Nunes has been transforming her own thinking as a researcher. She has come a long way from being a traditionally trained clinical psychologist, with research firmly grounded in Piaget's ideas about human development, to being inspired by cultural psychology and the work of Vygotsky and his followers to at least the same extent. Her tendency for bridging apparent opposites and bringing aspects that are separate together also finds its expression in her attempts to improve the practice of teaching mathematics.

Terezinha Nunes began her studies in psychology in her native Brazil and earned her Master's and PhD degrees at City University of New York (1975 and 1976 respectively). She began her academic career in Brazil at the Federal University of Minas Gerais and the University of Pernambuco. Later, she moved to the United Kingdom, where she taught at the Institute of Education, University of London, Oxford Brookes University and, since 2005, the University of Oxford. She is now a professor emerita at the University of Oxford and a fellow of Harris Manchester College, Oxford. Throughout her career, she has completed tens if not hundreds of studies, most of which were conducted in Brazil and in the UK. An exceptionally prolific writer, she has authored or co-authored more than a dozen books and almost 200 journal papers, book chapters and encyclopedia entries in English and Portuguese. An ardent team player and highly appreciated teacher, Terezinha Nunes has been an inspiration to her colleagues and to her many students. As an outstanding researcher driven by an insatiable passion for knowing and one who has made a paramount contribution to mathematics education and is likely to continue adding substantial insights for years to come, Terezinha Nunes is an eminently deserving recipient of the Hans Freudenthal Award for 2017.

Discussion Document ICMI Study 24

School mathematics curriculum reforms: Challenges, changes and opportunities

Co-Chairs: Yoshinori Shimizu (Japan, yshimizu@human.tsukuba.ac.jp) & Renuka Vithal (South Africa, vithalukzn@gmail.com)

Read the full discussion document at: http://www.human.tsukuba.ac.jp/~icmi24/discussiondocument.

School mathematical reforms have taken place in many countries around the world in the recent past. Although contexts vary significantly, much could be learnt from deeper and more substantial reflections and research about different aspects of these reforms.

Reforms have been large-scale, involving education systems as a whole, at a national, state, district or regional level, in which mathematical curricula, standards and frameworks have been developed and implemented. Changes have taken place at all levels of mathematics in the school educational system from pre-primary through to senior secondary level.

School mathematics reforms are often conducted with changes in all aspects of the curriculum: mathematics content, pedagogy, teaching and learning resources (e.g. texts and technologies), and assessment and examinations.

This ICMI study topic invokes not only questions about changes in curriculum design but – with force – questions about the implementation of these changes across an educational system. A curriculum reform will be influential or have impact insofar as it can be implemented and sustained. What has functioned (or not) at the time of implementing a curricular change? What are the limitations? How have resources (e.g. textbooks and technology) influenced the reforms and their enactment? How must large-scale teacher preparation be conducted to achieve the reform goals? How do diverse social, economic, cultural and national contexts condition the nature and extent of curricular reforms, especially teacher expectation, attitudes and beliefs, and the social and cultural backgrounds of students? How are assessments of students' learning influential in curriculum reforms? An ICMI Study offers an opportunity to provide a synthesis and meta-analysis of different aspects of school mathematics reforms historically, geographically and globally.

The overarching question of this ICMI Study is: to explore the school mathematics curriculum reforms that have been or are taking place, especially at a meta, macro or system level; and to learn about the many different aspects of mathematics curriculum reforms from past experiences, to specify the current status and issues in reforms worldwide, and to identify possible directions for the future of school mathematics.

The following five themes are selected for the study to address the research questions.

A. Learning from the past: driving forces and barriers shaping mathematics curriculum reforms.
B. Analysing school mathematics curriculum reforms for coherence and relevance.
C. Implementation of reformed mathematics curricula within and across different contexts and traditions.
D. Globalisation and internationalisation, and their impacts on mathematics curriculum reforms.
E. Agents and processes of curriculum design, development and reforms in school mathematics.

Each of these selected themes is aligned with a group of specific questions to be addressed in the study.

ICMI Study 24 on school mathematics curriculum reforms is planned to provide a platform for teachers, teacher educators, researchers and policymakers around the world to share research, practices, projects and analyses. Although these reports will form part of the programme, substantial time will also be allocated for collective work on significant problems in the topic, which will eventually form parts of a study volume. As in every ICMI Study, ICMI Study 24 is built around an international study conference and directed toward the preparation of a published volume.

The study conference will take place in the Tsukuba International Congress Center, Tsukuba, Japan, and will be hosted by the University of Tsukuba. The conference will take place 26-30 November 2018, with an opening reception on the evening of Sunday 25 November 2018.

As is usual practice for ICMI Studies, participation in the study conference will be by invitation only for the main/corresponding authors of the submitted contributions that are accepted.

The International Programme Committee for ICMI Study 24 invites submissions of contributions of several kinds, including: research papers related to school mathematics curriculum reform issues; theoretical, cultural, historical and epistemological essays (with deep connection to curriculum reforms); discussion and position papers analysing curriculum policy and practice issues; synthesis and meta-analysis reports on empirical studies; reviews of curriculum reform efforts, especially at macro levels; and papers on comparative studies in curriculum reform initiatives.

30 April 2018: Submissions must be made online no later than 30 April 2018 but earlier if possible.
30 June 2018: Papers will be reviewed, decisions will be made about invitations to the conference and notifications of these decisions will be sent to the corresponding/main authors by the end of June.

Information about registration, visa applications, costs and details of accommodation can be found on the ICMI Study 24 website: http://www.human.tsukuba.ac.jp/~icmi24/.

Members of the International Programme Committee

The members of the International Programme Committee are:

Yoshinori Shimizu (Japan, yshimizu@human.tsukuba.ac.jp) and Renuka Vithal (South Africa, vithalukzn@gmail.com) (Co-Chairs)
Angel Ruiz (Costa Rica, ruizz.angel@gmail.com)
Al Cuoco (USA, acuoco@edc.org)
Marianna Bosch (Spain, marianna.bosch@iqs.url.edu)
Soheila Gholamazad (Iran, soheila_azad@yahoo.com)
Will Morony (Australia, wmorony@aamt.edu.au)
Yan Zhu (China, yzhu@kcx.ecnu.edu.cn)
Ferdinando Arzarello, ICMI liaison member (Italy, ferdinando.arzarello@unito.it) and
Abraham Arcavi, ex-officio member as ICMI Secretary-General (Israel, abraham.arcavi@weizmann.ac.il).

ERME Column

Orly Buchbinder (University of New Hampshire, USA), Jason Cooper (Weizmann Institute of Science, Rehovot, Israel), Gabriel Stylianides (University of Oxford, UK) and Kirsten Pfeiffer (National University of Ireland, Galway)

ERME Topic Conferences

European Society for Research in Mathematics Education (ERME) Topic Conferences (ETCs) are organised on a specific research theme or themes related to the work of thematic working groups at CERME conferences. Their aim is to extend the work of the group or groups in specific directions, with clear value for the mathematics education research community. Three new ETCs have recently been announced:

- *ETC4 on Classroom-based Research on Mathematics and Language*, Dresden, Germany, 22–24 March 2018.
- *ETC5 on Mathematics Education in the Digital Age* (MEDA), Copenhagen, Denmark, 5–7 September 2018.
- *ETC6 on University Mathematics Education*, Kristiansand, Norway, 5–7 April 2018.

More details can be found on the ERME website (http://www.mathematik.uni-dortmund.de/~erme/index.php?slab=erme-topic-conferences).

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME) holds a biennial conference (CERME), at which research is presented and discussed in Thematic Working Groups (TWGs). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME's Thematic Working Group 1 – Argumentation and Proof

Group leaders: Gabriel Stylianides, Orly Buchbinder and Kirsten Pfeiffer

Thematic Working Group 1 focuses on a topic that is at the very core of mathematics: argumentation and proof. This working group has been represented at CERME from its inception in 1998 and has been growing stronger ever since. At the recent CERME10 conference, which took place in Dublin, Ireland, contributors from 18 countries, across four continents, presented 27 full papers and one poster.

The constantly growing interest in TWG1 reflects the importance that researchers in the field of mathematics education attribute to argumentation and proof for students' learning of mathematics. The papers contributed to this group spanned a wide range of topics and a multitude of methodological approaches. One of the central issues addressed in TWG1 has been the nature of proof and its relationship to argumentation. Participants examined this topic from mathematical, historical, epistemological and theoretical perspectives. These discussions helped bring to the surface both the diversity of positions and their common grounds, and contributed to the emergence of theoretical tools for designing and for researching the teaching and learning of proof.

Another recurring theme of TWG1 has been the complex connection between logic and linguistics in argumentation and proof. Topics in this theme touch upon the value and usefulness of explicit instruction of mathematical logic for fostering proof competencies, such as the writing and the comprehension of proofs. Since logical competence has implications for many advanced mathematical topics, it is critically relevant for both researchers and educators at the tertiary level. Many of the TWG1 papers that examined the relationship between language and logic focused on identifying aspects and situations that are likely to create discontinuities or support between language and logic. The group discussions have been greatly enriched through multi-linguistic and multicultural membership of the group participants, who also brought to the fore the influence of sociocultural contexts on the teaching and learning of proof.

The research reported in TWG1 aims to improve our understanding of argumentation and proof and to enhance its teaching and learning at all levels, from kindergarten to undergraduate – including teacher preparation. Thus, there has been increasing interest among the participants of TWG1 in topics such as the design of curricular materials (e.g. tasks, textbooks and courses) and assessment and classroom intervention studies relating to argumentation and proof. Some of these studies have examined the integration of argumentation and proof into specific subjects such as geometry, (abstract) algebra, calculus and real analysis, while other studies have focused on teaching practices that promote – or inhibit – argumentation and proof in the classroom.

Over the years, the inclusive and collaborative atmosphere of this group has contributed to fruitful research collaborations beyond the CERME meetings, resulting in special issues of international journals, books and other types of scholarship. TWG1 continues to be a rich platform for researchers to present and discuss a plethora of topics on argumentation and proof, which seem to be ever more relevant for the members of the mathematics education community.

Orly Buchbinder is an assistant professor of mathematics education at the University of New Hampshire, USA. Her research explores teaching and learning of reasoning and proving at the secondary level.

Jason Cooper is a research fellow at the University of Haifa's Faculty of Education. He is also a researcher at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including roles of advanced mathematical knowledge in teaching and contributions of research mathematicians to the professional development of teachers. He has been a member of the ERME board since 2015.

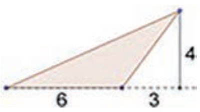
Gabriel Stylianides is a professor of mathematics education at the University of Oxford (UK) and is also a fellow of the University of Oxford's Worcester College. His research focuses on issues related to the meaningful engagement of students at all levels of education (including university students) in fundamental mathematical practices – notably reasoning and proving, problem solving and algebraic thinking.

Kirsten Pfeiffer is a lecturer of mathematics and the manager of the mathematics and statistics support centre at the National University of Ireland, Galway. Her research focuses on the teaching and learning of mathematics at university level, with a special interest in the teaching of mathematical proof and the role of mathematics support centres from a sociocultural point of view.

Mathematicians and Primary School Teachers Learning From Each Other

Jason Cooper and Abraham Arcavi (both Weizmann Institute of Science, Rehovot, Israel)

1 MATHEMATICIAN1: How do I prove the formula for the area of triangles that look like this? How can I convince myself that it's half the base times the altitude?



[Teachers work in small groups for approximately 5 minutes]

2 MATHEMATICIAN1: Does anyone object to a hint?

3 TEACHER: No hints!!!

4 MATHEMATICIAN1:Questions like this can fascinate kids. This is what's fun in math. Not calculating areas.

[5 minutes later]

5 TEACHER: If you don't solve it now, I'll work on it all night.

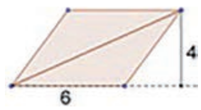
6 MATHEMATICIAN1: The big triangle's area is 6+3 – its base – times 4 over 2. I know this from what we showed previously about right-angled triangles. Now, we take away the smaller triangle: 3 times 4 over 2. We can do algebra. $S = \frac{(6+3)\times 4}{2} - \frac{3\times 4}{2} = \frac{(6+3-3)\times 4}{2}$.

You can actually see the algebraic trick visually; the area of this is… [pause] Well, I guess you can't really see it. We show it by algebraic proof.

7 TEACHER: But I want to do it as something tangible.

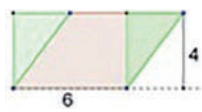
8 MATHEMATICIAN1: I don't see how you can do that.

9 TEACHER: Look. If I take two congruent triangles, I can form a parallelogram. Its area is exactly the rectangle's – the base multiplied by its altitude.



10 MATHEMATICIAN1: Very nice! But you're assuming you know the formula for the area of a parallelogram

11 TEACHER: But I can move from the parallelogram to a rectangle. Cut here and move it to here, and I get a rectangle.

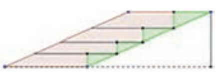


12 MATHEMATICIAN1: Will that work for any parallelogram?

13 TEACHER: Of course!

14 MATHEMATICIAN1: Are you sure? Even if the base is very small and the side is very long?

15 MATHEMATICIAN2: In this case, you can slice the parallelogram. If you slice it thinly enough, each parallelogram will be of the "right" kind. I just thought of it now. If I hadn't been here, I'd never have thought of it!



This transcript is taken from a professional development course for practising primary school teachers (grades

3-6) in Israel, which was initiated and run by mathematicians. The idea of mathematicians being involved in the professional development of teachers should not appear far-fetched to the readers of this newsletter; Felix Klein believed that "the whole sector of mathematics teaching, from its very beginnings at elementary school right through to the most advanced level research, should be organized as an organic whole" (Klein 1923, p. 24). From this perspective, it is natural to assume that university mathematicians should have an important role in the professional development of primary school teachers, though in Israel, and in many other contexts, the involvement of mathematicians in primary school mathematics is rare. There are significant differences between the mathematics taught and practised in universities and in primary schools and, though research mathematicians generally have some experience of teaching mathematics, their university experience may be of limited relevance in the context of teaching in primary school. Hyman Bass, a former president of the American Mathematical Society, who has become extensively involved in teacher education, suggests that "Mathematics for Teaching" is best seen as a field of applied mathematics and that "the first task of the mathematician who wishes to contribute in this area is to understand sensitively the domain of application, the nature of its mathematical problems, and the forms of mathematical knowledge that are useful and usable in this domain" [Bass 2005]. However, it is far from obvious how mathematicians should go about engaging in Mathematics for Teaching and gaining such a "sensitive understanding" of its issues.

The aim of this article is to describe, by means of two representative examples, how this meeting of mathematicians and teachers can create opportunities not only for primary school teachers to learn mathematics but also for mathematicians to learn some Mathematics for Teaching. We will highlight some differences between the two communities' perspectives on teaching and learning mathematics and show how these differences provide a springboard for mathematical and pedagogical discussions, which create opportunities for mutual learning – from and with each other. For a more comprehensive account of professional development, the reader is referred to the first author's unpublished doctoral dissertation (Cooper, 2016) and to published work (Cooper & Karsenty, 2016; Pinto & Cooper, 2017; Cooper & Arcavi, 2013).

Background on the course

In 2009, Raz Kupfermann, a professor of mathematics at the Hebrew University of Jerusalem, approached the Ministry of Education with an initiative to undertake a professional development course (henceforth PD) for practising primary school teachers. He had taken an interest in mathematics teaching at his children's school and sensed that teachers might benefit from the perspective of mathematicians in order to deepen their understanding of the content they teach. Thus, he suggested a for-credit course that would focus on a deep understanding of primary school content. This was an unusual stance since, in Israel (as in many other countries), primary school PD is usually taught by experienced teachers and is pedagogically oriented, attending to issues such as textbook selection, "best practices" for teaching particular topics, methods for managing heterogeneity in classrooms, where to find high quality supplemental tasks, etc. The assumption that underlies this stance is that primary school content is straightforward and does not warrant special attention. The ministry officials were happy to offer PD that focuses on the mathematics of primary school and consented to have it run by mathematicians. Teacher feedback was so positive that a group of graduate students were recruited to address the high enrolment in the following years.

In 2010, Kupfermann approached the two authors (separately), sensing that the PD might benefit from the involvement of mathematics educators. This involvement evolved into the first author's doctoral project, under the guidance of the second author. Data was collected in the 2011-2012 school year, in which 100 teachers participated in six separate groups, each of which was co-taught by two graduate students at the Hebrew University (for the most part, PhD students of mathematics) under the aegis of Kupfermann. The first author was a participant observer, witnessing and contributing to the planning of PD lessons and debriefing the instructors after the lessons. We took field notes and audio recordings from all the lessons (ten 3-hour lessons for each of the six groups), audio recordings of meetings and interviews with the instructors, teacher expectation questionnaires (at the outset) and feedback questionnaires (after each lesson). When possible, teachers were also interviewed. The doctoral dissertation was a multiple case study of episodes from 10 of the 60 recorded lessons. Here, we focus on a short section from one such episode, which was not included in the dissertation.

Analysis of the excerpt – What was going on?

In this section, we take a close look at some of the utterances from the excerpt and hypothesise as to the parties' underlying perspectives (implicit or even tacit) on the nature of mathematical activity and on its teaching and learning in primary school.

A recurring theme in the PD was the instructors' attempts to present the learning of mathematics as a sense-making activity, knowing that the Ministry of Education's curriculum, and its implementation in textbooks, may lead teachers to over-emphasise procedural aspects of the subject. The Ministry of Education requires that 5th grade students should "calculate the area of polygons, including obtuse-angle triangles" [The Pedagogical Secretariat of the Israeli Ministry of Education, 2009A]. Most textbooks explain the formula for a triangle's area, first for right-angled triangles (half a rectangle) and then for interior altitudes (dropping an altitude to separate the triangle into two right-angled triangles). Though students are expected to apply the formula for all triangles, students, and for the most part teachers as well, do not usually question why the formula should hold when the altitude is external. Raising this issue in the PD carries some implicit ideas about mathematics and its teaching and learning, which we propose as MP1 (Mathematician-Principle-1), based on U1 and U4:[1]

MP1. Mathematics is about making sense of formulas, not just about applying them. Teachers should know why the formulas that they teach "work". Perhaps students should as well.

This sentiment is shared by Lockhart, who used the same problem of making sense of the formula for the area of a triangle in his well-known "Mathematician's Lament" (2009).

It is evident that the teachers were highly engaged in this investigation, at first refusing to receive any hints ("No hints!!!" – U3) yet eventually demanding resolution ("If you don't solve it now, I'll work on it all night." – U5). However, the teachers were not quite satisfied with the mathematician's explanation ("I want to do it as something tangible." – U7), implying that:

TP1. Explanations for students should be tangible.

It is difficult to say exactly what would count as a tangible explanation for teachers. In this context, based on familiar explanations for the formula when the altitude is internal, and also on U9 in the excerpt, it seems to be related to scissor-congruence, namely, a tangible explanation is one that can be demonstrated by cutting the triangle along straight lines and rearranging its parts (possibly duplicated). This notion of tangibility is grounded in the teacher's experience – knowing what kinds of activities primary school students engage in and what kinds of mathematical reasoning they find to be more (or less) appropriate. For a research mathematician, there may be little difference between adding and subtracting areas; however, for primary school students and teachers, it is much more natural to see a triangle as being *made up* of two right-angled triangles than it is to see a triangle's area as the difference between two areas. For the latter, one must envision a third triangle that is not part of the problem.

Another unexpected difficulty had to do with what Mathematician1 called an "algebraic trick" (in fact, an application of the distributive property) to obtain the formula for the triangle's area. Algebraic manipulation is not part of the primary school curriculum; students may be familiar with the distributive property but if so then only through appropriate visual mediation. For example, 2-digit multiplication, which relies strongly on the distributive property, is often mediated as the area of a rectangle, where the distributive property is represented by sectioning a rectangle (e.g. 3×27 is represented by two joined rectangles, one representing 3×20 and the other representing 3×7). However, Mathematician1 had to concede that in the case of the *triangle's* area, his "algebraic trick" (i.e. the distributive property) did not have an obvious visual representation (U6, U8).

[1] U1, U2, etc. indicate the numbered utterances from the lesson excerpt.

A teacher suggested a very nice alternate explanation (forming a parallelogram from two congruent triangles), yet it raised some concerns on the part of the mathematician. Do we already "know" the area of a parallelogram (U10)? Does the argument address the general case or are we making some tacit assumptions (U12)? Though not voiced explicitly, perhaps he was concerned with circular logic, wondering whether the formula for the area of a parallelogram might later be proven based on the area of a triangle – apparently a more basic mathematical object. These concerns imply the following.

MP2. Since mathematical knowledge is built on previous knowledge, it is important to be clear about what is known and what is not known at any particular time, taking care to avoid circular logic.

MP3. Mathematical arguments should be general, i.e. valid for all cases.

Regarding MP2, we note that the mathematicians in the PD tended to use the words "what is known" differently from the teachers. For the mathematicians, the meaning was usually epistemic, a mathematical kind of knowing, referring to what has already been shown to be true. Teachers, on the other hand, often took a cognitive/pedagogical approach to "knowing", considering what their students had encountered in the past, regardless of how rigorously it had been justified. For example, teachers realise that students "know" halves well before the topic of fractions is introduced in school.

Both the mathematicians in this transcript drew on their mathematical experience. Mathematician1, concerned with MP3 (arguments should be general), spontaneously generated an example (U14) for which the teacher's argument would fail. This is the same example that the Gestalt theorist Max Werthheimer (1959) posed in a class where the teacher had taught a method for calculating the area of a parallelogram that follows the reasoning presented in U11. In that classroom, the teacher called the skewed parallelogram "a queer figure", which her students naturally could not deal with. Yet, for Mathematician1, a method that does not address such "queer" figures is inadequate. Mathematician2 suggested an argument to overcome this deficiency, reducing the ill-behaved parallelogram to a disjoint union of well-behaved parallelograms.

Rapprochement – what was learned

The mathematicians and the teachers had different agendas regarding the PD. The mathematicians were concerned primarily with mathematical content (the area of a triangle) and meta-content (mathematics as sense-making), whereas the teachers had students and teaching in mind, perhaps looking out for an activity to use in their own classroom. In this section, we discuss the opportunities for learning that this episode afforded, for both the teachers and the mathematicians in creating a space where the agendas of both communities combined around a mathematical investigation, allowing productive communication and reflection on each other's points of view.

Perhaps the most noteworthy aspect of this episode is the engagement and enthusiasm on the part of the teachers. Many of them had not previously had opportunities to experience the challenge and excitement of mathematical investigation and discovery and were thus unlikely to create such opportunities for their students. The concern they voiced regarding the tangible nature of explanations suggests that at least some of them were considering the possibility of taking this activity to their own classes, thus coming closer in their teaching to principle MP1, whereby doing mathematics is primarily a sense-making activity. This sense-making has some rules: the teachers needed to apply principle MP2 and be explicit about how they "know" the area of a parallelogram. Furthermore, they needed to take care that this "knowing" applies to all parallelograms. This short episode can be seen as a demonstration of what the university perspective on mathematics is about and how it can be applied to reveal the relevance of this kind of mathematics for primary school.

The mathematicians were provided with an opportunity to reflect on the tangible and visual nature of geometric proofs invoked by a primary school teacher. Mathematician1 seems to have been quite surprised to discover that his application of the distributive rule was not represented visually in his sketch (U6) and was a bit too quick in resigning to the necessity of an abstract proof ("I don't see how you can do [something tangible]"). It was a teacher, drawing on her own mathematical knowledge *for teaching*, who suggested a proof that is more appropriate for primary school, avoiding the necessity to imagine the difference between two areas. The mathematicians accepted this constraint and drew on their own expertise (perhaps taking inspiration from notions of "slicing" that they have encountered in the context of Cavalieri's principle or in the context of integration) to complete a tangible yet general proof. Mathematician2's comment is revealing – his "discovery" of a proof was a result of his interaction with the teachers – he would not have come to realise it on his own.

The plot thickens: volume of a pyramid

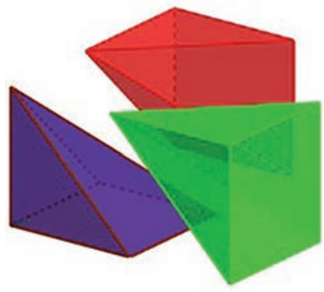
In one of the following PD sessions, the same mathematicians decided to address a related topic – the volume of a pyramid. We describe this lesson more briefly, contrasting it with the case of the triangle.

According to the Ministry of Education's guidelines for sixth grade, the volume of a pyramid – one third the volume of a prism with the same base and altitude – "will be deduced by filling hollow prisms and pyramids with water or sand" [The Pedagogical Secretariat of the Israeli Ministry of Education, 2009B]. Teachers are instructed to demonstrate[2] that it takes three pyramids to fill a prism with the same base and altitude as the pyramid. We may wonder whether such a demonstration satisfies principle MP1. Though they agreed that such a demonstration is better than simply providing the formula, the mathematicians did not see it as a mathematical justification.

How can primary school teachers and students "make sense" of the formula and understand why it "works"? Can the formula be explained in a manner that will be "tangible"? The volume of a pyramid (one third the volume of a prism with the same base and altitude) appears to be a natural extension of the area of a triangle (one half the area of a rectangle with the same base and altitude) but, for some reason, 1/2 is replaced by 1/3. And, indeed, some teachers were expecting a tangible explanation that follows the case of the triangle, perhaps breaking the pyramid into disjoint parts that can be duplicated and rearranged as a prism, and asked why the 3-dimensional case should be qualitatively different from the planar case.

The mathematicians knew that there is, in fact, a deep difference. They were familiar with Hilbert's third question from 1900: "Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled to yield the second?", and also with Dehn's negative answer to this question, showing that there exist tetrahedra with equal base area and equal altitude (hence equal volume) that have different Dehn invariants and thus are not scissor-congruent. Given this state of affairs, together with the teachers' expectation for a tangible explanation (TP1), a new notion of tangibility was called for.

The mathematicians opted for a demonstration but not the one suggested by the Ministry. They decided to focus on a special case: three identical "right-angle" pyramids that can be arranged to form a cube. They led the teachers through an activity that involved solving the tangible puzzle of constructing a cube from pyramids. This, too, is only a demonstration, not an explanation of "why" the formula holds in general and certainly not a proof of the general case. Yet, it has some special features – both mathematical and pedagogical – that made it more appealing for the mathematicians than the Ministry's suggestion.

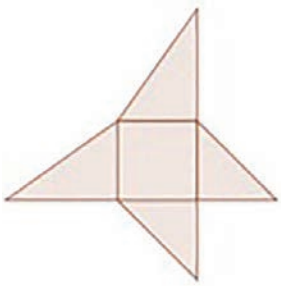


Cube comprised of 3 pyramids.

- Pedagogical: It suggests a classroom-ready activity for students – teachers can use it, perhaps with some modifications, to actively engage students in their own classrooms, as opposed to the demonstration advocated by the Ministry of Education, where students passively observe the teacher. The activity could even include constructing the pyramids from printouts of their nets.
- Mathematical: Having teachers calculate the dimensions of the net of the pyramid, ostensibly in order to prepare handouts for a classroom activity, provided an opportunity for them to engage with some relatively advanced mathematics (from the perspective of primary school teachers) that they don't often deal with:

[2] Perhaps a more appropriate word than "deduce" from the Ministry's guidelines.

the Pythagorean theorem, in an authentic problem-solving setting.
- Mathematical: It is not difficult to prove that a cube can be dissected into three disjoint pyramids, assuming familiarity with the cube's rotational symmetries; rotations around the cube's diagonal map the three pyramids onto each other.



Net of a pyramid.

- Mathematical[3]: This special case can, in principle, be extended to the general case, relying on the following observations.
  - A general pyramid can be approximated by a collection of disjoint pyramids, each having a square base and all having a common vertex. Thus, it is sufficient to prove the volume formula for pyramids with square bases.
  - Applying Cavalieri's principle, it can be shown that all pyramids with congruent bases and equal altitudes have the same volume, since their planar sections parallel to the base have the same area. Thus, without loss of generality, it can be assumed that the edge from the pyramid's vertex to its base is perpendicular to the base.
  - Given the base, a pyramid's volume is proportional to its altitude. This, too, is a consequence of Cavalieri's principle: if a pyramid's altitude is multiplied by a factor of k, keeping the base unchanged, and the pyramid is then dilated by a factor of 1/k (multiplying its volume by a factor of $(1/k)^3$), the result is, by Cavalieri's principle, a pyramid whose volume is $(1/k)^2$ times the volume of the original, since each section has $(1/k)^2$ times the area of the respective section in the original. The conclusion is that the first transformation (multiplying the altitude by a factor of k) must have multiplied the volume by a factor of k.

  Although some of these observations may be a bit difficult to comprehend, they have simple and familiar versions for triangles in the plane, which can help make sense of the 3-dimmensional case.

In this demonstration, the mathematicians had found a way to reconcile their principle of making sense of formulas (MP1) with the teachers' principle of tangibility (TP1), using their knowledge of advanced mathematics (Dehn invariant) to propose a demonstration that is not only pedagogically appropriate but also mathematically valid, without seriously compromising their principle of generality (MP3).

In conclusion

A number of research mathematicians have taken an active interest in pre-college mathematics education and, in particular, in teacher preparation and professional development. Some have shared their experience with other mathematicians (e.g. the March 2011 issue of the Notices of the American Mathematical Society), reporting that elementary mathematics is surprisingly interesting and engaging. In this article, we have tried to add to this body of work and to exemplify the mutual benefits of a meeting of mathematicians and primary school teachers in professional development. In such a meeting, the teachers clearly have much to learn from the mathematicians. What is less obvious is that the mathematicians may also have much to learn, not only about teaching mathematics in primary school but also about the nature of the mathematics that is taught and learnt. In this meeting, both sides were committed to their own agenda; the mathematicians were guided by universal principles of mathematics that should not be "watered down", even in primary school, and the teachers were guided by their expectation that the professional development should be relevant for their teaching, taking into account the children's ways of thinking and doing. Both parties remained true to their agendas, while at the same time listening sensitively to the other. As a result, the teachers had the opportunity to engage in mathematical investigation and discovery in the context of the content they teach, while the mathematicians had the opportunity to develop Mathematics for Teaching, along with a sensitivity for the nuances of mathematics as it is taught and learnt in primary school. On the way, they co-developed some new insights into teaching some mathematics that is often overlooked in primary school.

We have limited our discussion to two related episodes, yet they are representative of the whole of the PD. The first author's doctoral dissertation [Cooper 2016], in which 10 cases were analysed in depth, addressed no more than 10% of the data collected in a single year of this ongoing project, which was incredibly rich in opportunities for learning similar to the ones described above. This setting can serve not only as an opportunity for teachers' professional development but also as a way to address Bass' call for mathematicians, who wish to contribute in this area of school mathematics, to come to "understand sensitively the domain of application, the nature of mathematical problems, and the forms of mathematical knowledge that are useful and usable" [Bass 2005] in the context of primary school mathematics education. We have shown not only how such sensitivity can be developed but also how it can contribute to making mathematicians' expertise relevant for primary school teachers.

[3] Based, in part, on personal communication with Hyman Bass.

References

Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin (New Series) of the American Mathematical Society,* 42(4), 417–430.

Cooper, J. (2016). Mathematicians and primary school teachers learning from each other. Unpublished doctoral dissertation. Retrieved from https://stwww1.weizmann.ac.il/wp-content/uploads/2016/09/Cooper-dissertation-finalAppendix.pdf.

Cooper, J., & Karsenty, R. (2016). Can teachers and mathematicians communicate productively? The case of division with remainder. *The Journal of Mathematics Teacher Education* (First Online 17 October 2016). doi:10.1007/s10857-016-9358-7.

Cooper, J., & Arcavi, A. 2013. Mathematicians and elementary school mathematics teachers – meetings and bridges. In Y. Li, & J. N. Moschkovich (Eds.), *Proficiency and beliefs in learning and teaching mathematics – Learning from Alan Schoenfeld and Günter Törner* (pp. 179–200). Rotterdam: Sense Publishers.

Klein, F. (1923). Göttinger Professoren. Lebensbilder von eigener Hand. 4. *Mitteilungen des Universitätsbundes Göttingen*, 5(1), 11–36.

Lockhart, P. (2009). *A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form.* Bellevue Literary Press.

Pinto, A., & Cooper, J. (2017). In the pursuit of relevance – Mathematicians designing tasks for elementary school teachers. *International Journal of Research in Undergraduate Mathematics Education*, 3(2), 311–337. doi:10.1007/s40753-016-0040-3.

The Pedagogical Secretariat of the Israeli Ministry of Education. (2009 A). Details of the curriculum in mathematics for grade 5. Retrieved from Mathematics for primary school for all sectors: http://meyda.education.gov.il/files/mazkirut_pedagogit/matematika/tochnyotlemud/kita5.pdf.

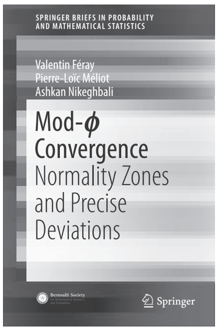
The Pedagogical Secretariat of the Israeli Ministry of Education. (2009 B). Details of the curriculum in mathematics for grade 6. Retrieved from Mathematics for primary school for all sectors: http://meyda.education.gov.il/files/mazkirut_pedagogit/matematika/tochnyotlemud/kita6.pdf.

Wertheimer, M. (1959). *Productive Thinking* (enlarged edition). New York: Harper (first edition published 1945).

Jason Cooper is a research fellow at the University of Haifa's Faculty of Education. He is also a researcher at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including roles of advanced mathematical knowledge in teaching and contributions of research mathematicians to the professional development of teachers. He has been a member of the ERME board since 2015.

Abraham Arcavi is the incumbent of the Lester B. Pearson Professorial Chair at the Department of Science Teaching of the Weizmann Institute of Science, Israel. He works on the teaching and learning of mathematics, curriculum development and professional development of mathematics teachers. At present (2016-2020) he is the Secretary General of the International Commission on Mathematical Instruction (ICMI).

Book Reviews



Valentin Féray, Pierre-Loïc Méliot and Ashkan Nikehgbali

Mod-ϕ Convergence
Normality Zones and Precise Deviations

Springer, 2016
152 p.
ISBN 978-3-319-46821-1
ISBN 978-3-319-46822-8 (eBook)

Reviewer: Mark Podolskij (Aarhus, Denmark)

Since the birth of probability theory, weak convergence has played a major role in the field. One of the most celebrated examples of weak convergence is the central limit theorem. If $(Y_i)_{i\geq 1}$ is a sequence of independent, identically distributed random variables with mean μ and variance $\sigma^2 > 0$ then it holds that

$$\lim_{n\to\infty} \mathbb{P}\Big(n^{-1/2}\sum_{i=1}^{n}\sigma^{-1}(Y_i-\mu)\leq z\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}\exp(-y^2/2)dy \qquad \text{for all } z\in\mathbb{R},$$

where the right side is the probability distribution of a standard normal random variable. However, in many situations, proving weak convergence of a given sequence of random variables $(X_n)_{n\geq 1}$ does not suffice to obtain the desired result and a much finer analysis of the probability distribution of X_n is required. Two of the most prominent examples include the *Edgeworth expansion* and the *large deviations principle*.

This book presents a concise study of *mod-ϕ convergence*, which is a new concept of describing the fluctuations of a sequence $(X_n)_{n\geq 1}$. The notion of mod-ϕ convergence first appeared in a slightly different form in the work of E. Kowalski and A. Nikeghbali in the context of analytic number theory and random matrix theory (see [2, 3]). Intuitively speaking, it can be described as follows. Assume that a given sequence of random variables $(X_n)_{n\geq 1}$ admits the decomposition

$$X_n = G_n + Z_n, \tag{1}$$

where G_n and Z_n are independent, G_n has a normal distribution with mean 0 and variance $t_n\to\infty$ and Z_n converges weakly to a random variable Z. In the terminology of the book, this example is a prototypical case of mod-Gaussian convergence. In other words, the dominating part G_n is Gaussian and obviously $X_n/\sqrt{t_n}$ converges weakly to a standard normal distribution but the decomposition offers more insight into the distributional properties of X_n. This particular example of mod convergence can be extended to an arbitrary, *infinitely divisible* distribution ϕ. Recall that a probability distribution ϕ is called infinitely divisible if for each $n\in\mathbb{N}$ there exists a probability distribution ϕ_n such that $\phi = \phi_n^{\star n}$ (the n-fold convolution of ϕ_n with itself). The formal definition of mod-ϕ convergence is introduced as follows.

Definition. Let $(X_n)_{n\geq 1}$ be a sequence of random variables and let $\varphi_n(z) = \mathbb{E}[\exp(zX_n)]$ denote its moment generating function, which is assumed to exist on a strip $S_{(a,b)} = \{z : \operatorname{Re}(z)\in(a,b)\}$ with $-\infty\leq a<b\leq\infty$. We say that $(X_n)_{n\geq 1}$ *converges mod-ϕ on $S_{(a,b)}$ with parameter* (t_n,ψ) if the convergence

$$\lim_{n\to\infty}\exp\left(-t_n\eta(z)\right)\varphi_n(z) = \psi(z)$$

holds locally uniformly on $S_{(a,b)}$, $t_n\to\infty$ and ϕ is an infinitely divisible distribution with moment generating function $\eta(z) = \int_{\mathbb{R}}\exp(zx)\phi(dx)$.

Obviously, in setting (1), the mod-Gaussian convergence with parameter (t_n,ψ) holds, where ψ is the moment generating function of the limiting random variable Z. Since the definition of mod-ϕ is based upon a moment generating function, it is rather natural that *cumulants* are important objects in the analysis. To give a simple example, let us again consider a sequence $(Y_i)_{i\geq 1}$ of independent, identically distributed random variables with mean 0 and variance σ^2. Assume that the cumulants $\kappa_3,\ldots,\kappa_{\nu-1}$ of Y_1 are zero and $\kappa_\nu\neq 0$ for some $\nu\geq 3$. Then, the following statement holds (see Example 2.1.2 in the book).

Lemma. The sequence of random variables $X_n = n^{-1/\nu}\sum_{i=1}^{n}Y_i$ converges in the mod-Gaussian sense with $t_n = \sigma^2 n^{(\nu-2)/\nu}$ and $\psi(z) = \exp(\kappa_\nu z^\nu/\nu!)$.

The authors demonstrate a great variety of applications of mod-ϕ convergence. Chapters 3 and 4 focus on the so-called *normality zones* that describe the range of scalings for which the normal approximation is valid. One of the most interesting applications of mod-ϕ convergence is the precise version of the Ellis-Gärtner theorem demonstrated in Chapter 6. The Ellis-Gärtner theorem is a key result of large deviations theory, which investigates the lower and upper bounds for log-probabilities $\log\mathbb{P}(X_n\in t_nA)$, where A is a Borel set. It turns out that, when X_n converges in the mod-ϕ sense, similar precise bounds can be deduced for the probability $\mathbb{P}(X_n\in t_nA)$ itself.

Quite naturally, when the moment generating function of X_n is available in closed form, it is easier to prove mod-ϕ convergence. Some of these situations are discussed in Chapter 7. For instance, the authors present the asymptotic theory for the number $\omega(n)$ of distinct prime divisors of $n\in\mathbb{N}$. The celebrated Erdős–Kac central limit theorem [1] states that $(\omega(n)-\log\log n)/\sqrt{\log\log n}$ converges weakly to the standard normal distribution. Applying the expansions from [4], the authors show the following result (see Proposition 7.2.3).

Theorem. The sequence $(\omega(n))_{n\geq 1}$ converges in the mod-Poisson sense with $t_n = \log\log n$ and $\psi(z) = \lambda_0(\exp(z))$, where the function λ_0 is defined by

$$\lambda_0(x) = \frac{1}{\Gamma(x)} \prod_p \left(1 + \frac{x}{p-1}\right)\left(1 - \frac{1}{p}\right)^x$$

and the product is taken over all primes p.

This result is extended to other arithmetic functions in Proposition 7.2.11. The remaining Chapters 8–11 demonstrate several applications of mod-ϕ convergence to random graphs and random partitions.

This book is very well written and the analytic arguments are easy to follow for a reader with a sound background in probability theory. Currently, the concept of mod-ϕ convergence is rather exotic in the literature but this is exactly what makes it so exciting to explore its properties and applications! As this book mostly focuses on probabilistic topics, I would also recommend reading the papers of E. Kowalski, A. Nikeghbali and their coauthors for interesting applications in analytic number theory. In future, the notion of mod-ϕ convergence might also play an important role for problems in mathematical statistics.

References

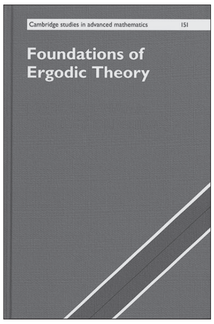
[1] P. Erdős and M. Kac (1940): The Gaussian law of errors in the theory of additive number theoretic functions. *American Journal of Mathematics*, 62, 738–742.

[2] J. Jacod, E. Kowalski and A. Nikeghbali (2011): Mod-Gaussian convergence: new limit theorems in probability and number theory. *Forum Mathematicum*, 23(4), 835–873.

[3] E. Kowalski and A. Nikeghbali (2010): Mod-Poisson convergence in probability and number theory. *International Mathematics Research Notices* 18, 3549–3587.

[4] G. Tenenbaum (1995): *Introduction to analytic and probabilistic number theory*. Volume 46 of Cambridge Studies in Advanced Mathematics, Cambridge University Press.

Mark Podolskij [mpodolskij@math.au.dk] is professor of probability and mathematical statistics at Aarhus University, Denmark. His research interests are in limit theorems for stochastic processes, stochastic analysis, Stein's method, high dimensional probability, non-parametric statistics, and their applications in economics and physics.



Marcelo Viana and Krerley Oliveira

Foundations of Ergodic Theory

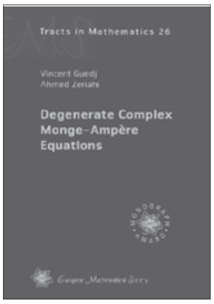
Cambridge University Press, 2016
xvi, 530 p.
ISBN 978-1-107-12696-1

Reviewer: Thomas B. Ward

The Newsletter thanks zbMATH and Thomas B. Ward for the permission to republish this review, originally appeared as Zbl 1369.37001.

This interesting book covers the basic machinery of ergodic theory, with an emphasis on how ergodic theory may be used to study dynamical systems and the thermodynamic formalism. The origins of the text lie in various courses aimed at undergraduates, and that is visible in several ways. There are more than 400 well-constructed and informative exercises. Great care has been taken to avoid onerous pre-requisites and in some cases more than one proof is given where that will aid understanding. Finally the text is carefully written with both detailed proofs and thoughtful explanations of key concepts. The first chapter gives a brief overview of recurrence in the context of ergodic theory and topological dynamics, along with some key early examples. This chapter already has some non-trivial concepts and results, including the Poincaré recurrence theorem, Kac's theorem, and Birkhoff multiple recurrence. Here – as elsewhere in the book – more advanced results are described and referenced without proof, making it easy for an interested reader to go further (in this case one of Furstenberg's multiple recurrence results presented as a measure-theoretic analogue of Birkhoff multiple recurrence). The second chapter covers the existence of invariant measures, and introduces some of the basic functional-analytic tools and the Koopman operator. Again some more advanced results appear right away, including van der Waerden's theorem on arithmetic progressions in a finite colouring of the natural numbers and the Furstenberg correspondence linking Szemerédi's theorem to the earlier (stated) multiple recurrence result. Again interesting asides are included that point at more advanced results like the Green–Tao theorem. The third chapter covers the basic pointwise (Birkhoff) and mean (von Neumann) ergodic theorems, with the former deduced from the more general sub-additive ergodic theorem introduced here with an eye on applications later. Again some less conventional results at this level are brought in, including the Furstenberg–Kesten theorem and the Oseledets multiplicative ergodic theorem and the related concept of Lyapunov exponents. The fourth chapter introduces ergodicity and some equivalent formulations of it, along with many examples of ergodicity and the basic properties arising. With the emphasis later on applications to dynamical systems, smooth examples and the notion of conservative systems are also introduced. Some of the more advanced material touched on here includes KAM theory, elliptic periodic points, the Poincaré–Birkhoff fixed point theorem, geodesic flows, Anosov systems and billiards. The fifth chapter is a short technical interlude, with ergodic decomposition deduced from Rokhlin's disintegration theorem. Conditional expectation and conditional measures are constructed here. The sixth chapter studies unique ergodicity and minimality, including Furstenberg's proof of Weyl's polynomial equidistribution result. Among some more advanced results, a very brief sketch of Furstenberg's construction of a minimal real-analytic diffeomorphism of the 2-torus preserving Lebesgue measure for which Lebesgue measure is not ergodic and an outline of the construction of Haar measure on a Lie group are provided. Chapters seven and eight further develop basic ergodic theory with the same pattern of basic material (mixing, decay of correlations, measurable and spectral equivalence, discrete and Lebesgue spectrum) together with more examples and some brief mentions of more advanced topics (including conditions for isomorphism at the level of measure algebras to imply isomorphism via a measure-preserving map, interval exchanges, the Chacon map). Chapters nine and ten introduce measure-theoretic and topological entropy respectively, up to the variational principle. More advanced topics include the full ergodic decomposition of entropy due to Jacobs, the Shannon–McMillan–Breiman theorem, pressure and the variational principle, and again several further topics are mentioned including the local entropy theory of Brin and Katok and the Margulis–Ruelle inequality. The last two chapters see the theory developed applied to study expanding maps, thermodynamic formalism, the Ruelle and Livšic theorems, decay of correlations, and dimension of conformal repellers. This is an excellent addition to the literature on ergodic theory, and is ideally suited to a substantial course for strong students with relatively modest prerequisites. It has an attractive mixture of carefully developed topics, and the student will readily access more advanced materials after this solid and well-written foundation.

Tom Ward is Deputy Vice-Chancellor for Student Education and a professor of mathematics at the University of Leeds. He is the author of several monographs, including "Heights of polynomials and entropy in algebraic dynamics" with Graham Everest, "Ergodic theory with a view towards Number Theory" with Manfred Einsiedler and most recently "Functional analysis, spectral theory, and applications" with Manfred Einsiedler.



Vincent Guedj and Ahmed Zeriahi

Degenerate Complex Monge–Ampère Equations

European Mathematical Society, 2017
xxiv, 472 p.
ISBN 978-3-03719-167

Reviewer: Dan Coman

The Newsletter thanks zbMATH and Dan Coman for the permission to republish this review, originally appeared as Zbl 1373.32001.

This comprehensive monograph gives an excellent exposition of pluripotential theory in Euclidean space and on compact complex manifolds, with emphasis on the solutions to complex Monge–Ampère equations and on important applications to complex geometry. It is organized in four parts and sixteen chapters.

The first part is a self-contained presentation of pluripotential theory on domains in $\mathbb{C}^n$. Potential theory in several complex variables, or briefly pluripotential theory, deals with the study of plurisubharmonic functions and positive closed currents. The theory provides powerful tools that led to significant advances in complex analysis, geometry and dynamics. The book starts by developing the basic properties of harmonic, subharmonic, plurisubharmonic functions in Chapter 1, and of positive closed currents in Chapter 2. A central role in pluripotential theory is played by the complex Monge–Ampère operator which associates to a suitable plurisubharmonic function u the positive measure $(dd^c u)^n$. It provides a generalization to higher dimensions of the Laplacian of a subharmonic function in $\mathbb{C}$. Chapters 3–5 give a nice account of the Bedford–Taylor theory on defining the complex Monge–Ampère operator for locally bounded plurisubharmonic functions, the Monge–Ampère capacity and its applications, and the solution to various Dirichlet problems for the Monge–Ampère equation. In Chapter 6 the authors show that the general method of viscosity can be used to solve Monge-Ampère equations on domains in $\mathbb{C}^n$, by developing the corresponding notions and techniques in the complex case.

In the second part the authors extend the notions and results of pluripotential theory from the local setting to that of compact Kähler manifolds. In this case, the plurisubharmonic functions, which by the maximum principle must be constant, are replaced by the quasi-plurisubharmonic (qpsh) ones. Important topics from complex geometry are reviewed without proof in Chapter 7, while the class of qpsh functions and the corresponding envelopes and capacities are discussed in Chapters 8–9. Chapter 10 deals with the definition and properties of the complex Monge–Ampère operator acting on suitable classes of unbounded qpsh functions. Of special importance is the class $\mathcal{E}$ on which the complex Monge–Ampère operator can be defined by taking advantage of the compact setting, as well as several subclasses of qpsh functions of finite energy. A new phenomenon is that the class $\mathcal{E}$ contains unbounded qpsh functions for which the complex Monge–Ampère operator cannot be locally defined by the methods of Bedford and Taylor, or, more generally, Błocki and Cegrell.

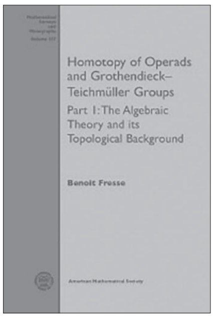
The culmination of the book is the third part, where degenerate complex Monge–Ampère equations are solved on compact Kähler manifolds by various techniques. A variational approach is presented in Chapter 11 where the complex Monge–Ampère equations under consideration appear as the Euler–Lagrange equations of certain functionals acting on finite energy classes of qpsh functions. The viscosity approach developed in the local setting in Chapter 6 is employed in the compact setting in Chapter 13. The main difficulty in solving such equations comes from the lack of smoothness of the solutions, so weak solutions have to be considered and new tools have to be developed. The partial regularity of solutions is studied in Chapters 12 and 14, where it is shown that they are Hölder continuous, and in some cases smooth, away from a divisor.

The book ends by giving in Part 4 several important applications to complex geometry of the results developed so far. Chapter 15 deals with the study of canonical metrics in Kähler geometry, the Calabi–Yau theorem, the construction of Kähler–Einstein metrics, and the Riemannian structure of the space of Kähler metrics. Chapter 16 treats the existence of singular Kähler–Einstein metrics on mildly singular varieties which are important in the Minimal Model Program.

It is worthwhile to note that this book is an extension of the lecture notes of a graduate course given by the authors at Université Paul Sabatier in Toulouse, France. Hence every chapter ends with a section of nice exercises.

This monograph covers a great deal of modern topics in several complex variables and complex geometry and gives a wide array of interesting recent applications. It will undoubtedly be a great resource for current researchers and graduate students interested in pluripotential theory and complex geometry.

Dan Coman is a professor of mathematics at Syracuse University in Syracuse, New York. He earned his Ph.D. in 1997 from the University of Michigan, Ann Arbor, under the direction of Professor John Erik Fornaess. His research interests are in several complex variables, pluripotential theory, and their applications to complex geometry and dynamics.



Benoit Fresse

Homotopy of Operads and Grothendieck–Teichmüller Groups Part 1: The Algebraic Theory and its Topological Background

American Mathematical Society, 2017
xlvi, 532 p.
ISBN 978-1-4704-3481-6

Reviewer: Steffen Sagave

The Newsletter thanks zbMATH and Steffen Sagave for the permission to republish this review, originally appeared as Zbl 1373.55014.

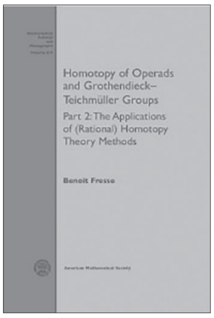
The present monograph is the first in a series of two volumes on the interplay between the theory of operads, rational homotopy theory, and Grothendieck–Teichmüller groups. The second part is reviewed in [Zbl 1375.55007 (*see below*)].

Operads were originally introduced in the 1970s by May and Boardman–Vogt. Their first application was a recognition principle stating that after group completion, any space with an action of the little n-disks operad has the homotopy type of an n-fold loop space. In subsequent years, operads have found applications outside of algebraic topology. An instance of relevance for the book under review is the solved Deligne conjecture about the action of an E_2-operad on Hochschild cochains. The author outlines that its applications to the Kontsevich formality theorem and deformation quantizations were a motivation for the main new result of the present work, which provides an identification of the homotopy groups of the homotopy automorphism space of the rationalization of the little 2-disks operad, formed in the category of topological operads. More precisely, the author shows that this space has trivial homotopy groups in degrees above 1, that its fundamental group is $\mathbb{Q}$, and most notably that its group of path components is $\mathrm{GT}(\mathbb{Q})$, the rational pro-unipotent version of the Grothendieck–Teichmüller group that has been introduced by V. G. Drinfel'd [*Algebra Anal.* 2, No. 4, 149–181 (1990; Zbl 0718.16034)].

The two volumes comprise more than 1200 pages of text. The enormous length of this work is due to the fact that the author included careful introductions to the subjects that are touched on along his way to the main result. While some of this is just a review of known notions and results, there are also substantial new contributions, like the rational homotopy theory for operads developed in the second volume. Despite the length of these two books, the material is arranged in a very user friendly way as the volumes, parts, chapters, and sections start with meaningful and valuable introductions.

The present first volume primarily provides background about the theory of operads that is needed for the formulation and for the proof of the main theorem, which is postponed to the second volume. The book begins by reviewing the definition of operads, both in the unitary and the non-unitary case. The author stresses the fact that operads can take values in symmetric monoidal categories where the tensor product is not required to be compatible with colimits and uses this observation to define Hopf operads as operads in counitary commutative coalgebras. In the next sections, the author introduces the important class of E_n-operads. Based on unpublished work by Fiedorowicz, he presents a method to recognize E_2-operads by using a suitable notion of braided operads. After a review of Hopf algebras and Malcev completion of groups, a Malcev completion for groupoids and operads is set up. These notions are then used to give a first operadic description of the pro-unipotent Grothendieck-Teichmüller group as the group of operad automorphisms of the Malcev completion of a suitable operad of parenthesized braids. The volume ends with appendices on trees and free operads and on the cotriple resolution of operads.

In total, this volume provides a clear and comprehensive introduction to the theory of operads and some of its applications, and it should indeed achieve the author's aim "to be accessible to a broad readership of graduate students and researchers interested in the applications of operads".



Benoit Fresse

Homotopy of Operads and Grothendieck–Teichmüller Groups Part 2: The Applications of (Rational) Homotopy Theory Methods

American Mathematical Society, 2017
xxxv, 704 p.
ISBN 978-1-4704-3482-3

Reviewer: Steffen Sagave

The Newsletter thanks zbMATH and Steffen Sagave for the permission to republish this review, originally appeared as Zbl 1375.55007.

The present monograph is the second of a series of two volumes on the interplay of the theory of operads, rational homotopy theory, and Grothendieck–Teichmüller groups. The first volume has been reviewed in [Zbl 1373.55014 (*see above*)], and we refer to that review for a brief motivation of this work.

The main result proven in this book is the identification of the pro-unipotent Grothendieck–Teichmüller group GT($\mathbb{Q}$) as the group of homotopy automorphisms of the rationalization of the little 2-disks operad. To this end, the author develops a rational homotopy theory of operads.

The volume under review begins with an account of abstract homotopy theory. The author reviews the definitions of model categories and different approaches to mapping spaces in model categories. This is followed by a review of rational homotopy theory and related results about dg-algebras, simplicial algebras and cosimplicial algebras. Based on this, the author develops model category structures for various flavors of operads. This facilitates him to set up a rational homotopy theory of operads. One main step in this program is the definition of a Hopf cochain dg-cooperad associated with an operad in simplicial sets. The general theory is then applied to E_n-operads. The author presents an explicit model for the rationalization of an E_2-operad from joint work with Willwacher. The case of the little 2-disks operad that is most important for the main result of the present book is studied in more detail, and one model for its rationalization given here is the operad in simplicial sets obtained by applying the classifying space functor to an operad of chord diagrams introduced in the first volume.

In the last part of this volume, the author establishes his main result about the identification of the homotopy automorphism space of the rationalization of the little 2-disks operad. To achieve this, he uses cotriple resolutions of operads to construct a Bousfield–Kan type spectral sequence for the computation of mapping spaces between operads. The E_2-terms of this spectral sequence can be described in terms of cotriple cohomology, and the passage to a suitable Kozul dual operad reduces the problem to a computation with a Gerstenhaber operad. This last step uses results from a preprint of E. Getzler and J.D.S. Jones} ["Operads, homotopy algebra and iterated integrals for double loop spaces", Preprint, arXiv:hep-th/9403055]. After the proof of the main theorem, there is an outlook on recent other results on mapping spaces of operads. The book ends with an appendix on cofree cooperads and bar duality of operads.

This book provides a very useful reference for known and new results about operads and rational homotopy theory and thus provides a valuable resource for researchers and graduate students interested in (some of) the many topics that it covers. As it is the case for the first volume, careful introductions on the various levels of the text help to make this material accessible and to put it in context.

Steffen Sagave is an assistant professor at Radboud University Nijmegen, The Netherlands. He received his Ph.D. and habilitation from the University of Bonn. His research interests are algebraic topology and its interactions with algebraic K-theory and derived algebraic geometry.

Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.

John von Neumann (1903–1957)

The column in this issue is devoted to fundamentals of mathematical analysis.

Mathematical analysis (or simply analysis) is an enormous field and arguably one of the most central in all of mathematics, appearing in the most abstract of research as well as an extremely wide range of applicable areas like physics, engineering, finance, sociology and biology, to name just a few.

In mathematics, in principle, one can study two categories of structures and phenomena: *discrete*[1] and *continuous*. Generally speaking, the study of the continuous lies at the heart of analysis. The origin of analysis as an independent field of mathematics traces back to the 17th century, with the discovery of the differential by Isaac Newton, and Gottfried Wilhelm Leibniz playing a central role in its genesis. We must note, though, that several important mathematical concepts of analysis were introduced even earlier. For example, the concept of an *integral* traces back to Eudoxus (ca. 390–337 BC) and Archimedes (ca. 287–212 BC). Some of the central generative discoveries of analysis arose from the effort to answer fundamental questions in disciplines such as astronomy, optics and engineering, as well as from the effort to determine mathematical methods for the calculation of areas, volumes, centres of gravity, etc., for both theoretical and practical applications.

Since analysis, as mentioned above, is a vast field of mathematics with several subfields, we shall devote future individual columns to subfields like real analysis, complex analysis, harmonic analysis, etc.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

187. Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 0}$ be sequences such that $a_n > 0$, $b_n > 0$ and $c_n > 0$ for $n \geq 1$ and:

(G1) $c_0 = 0$ and c_n is increasing,

(G2) $c_{n+1} - c_n$ is decreasing for $n \geq 0$,

(G3) $c_k\left(\frac{a_{k+1}}{a_k} - 1\right) \geq c_n\left(\frac{b_{n+1}}{b_n} - 1\right)$ for $1 \leq k < n$.

Given a function f, let

$$A_n = \frac{1}{c_{n-1}} \sum_{k=1}^{n-1} f\left(\frac{a_k}{b_n}\right), \qquad n \geq 2.$$

Then, if f is real, convex increasing and non-negative on an interval $[D, E]$ that includes all the points $\frac{a_k}{b_n}$ for $k < n$, prove that A_n increases with n.

(Shoshana Abramovich, University of Haifa, Israel)

188. For a function $f : \mathbb{R} \to \mathbb{R}$ and a positive integer n, we denote by f^n the function defined by $f^n(x) = (f(x))^n$.

(a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a function that has an antiderivative then $f^n : \mathbb{R} \to \mathbb{R}$ satisfies the intermediate value property for any $n \geq 1$.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that has an antiderivative and for which $f^n : \mathbb{R} \to \mathbb{R}$ has no antiderivatives for any $n \geq 2$.

(Dorin Andrica, Babesş Bolyai University, Cluj-Napoca, Romania)

189.

(a) Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of continuous real-valued functions on a compact metric space X that converges pointwisely to a continuous function f. Show that the convergence must be uniform.

(b) Show by a counterexample that the compactness of X in (a) is necessary.

(c) Determine whether (a) remains valid if the sequence $\{f_n\}_{n=1}^{\infty}$ is not monotone.

(W. S. Cheung, University of Hong Kong, Pokfulam, Hong Kong)

190. Let $\{a_n\}$ be a sequence of positive numbers. In the ratio test, we know that the condition

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

is not sufficient to determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent. For example, if $a_n = 1/n$ then

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} = 1 - \frac{1}{n+1} = 1 - \frac{n+1}{(n+1)^2}$$

and if $a_n = 1/n^2$ then

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = 1 - \frac{2n+1}{(n+1)^2}.$$

Hence, the coefficient a in the expression $1 - \frac{an+1}{(n+1)^2}$ plays an important role in the convergence of $\sum a_n$. In this question, we would like to study it more closely.

Let a be a non-negative real number and let $\{a_n\}$ be a sequence with $a_n > 0$, satisfying

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{an+1}{(n+1)^2} \tag{1}$$

for all $n \geq n_0 := [|2 - a|] + 1$, where $[x]$ is the integral part of x.

(i) Show that if $a > 0$ then

$$\lim_{n\to\infty} a_n = 0.$$

If $a = 0$, for any $\lambda > 0$, find an example such that

$$\lim_{n\to\infty} a_n = \lambda.$$

(ii) Show that if $a > 1$ then

$$\sum_{n=1}^{\infty} a_n$$

is convergent. Is this still true when $a = 1$?

(Stephen Choi and Peter Lam, Simon Fraser University, Burnaby B.C., Canada)

191. Show that for any $a, b > 0$, we have

$$\frac{1}{2}\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2.$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

192. Let $a, b, c, d \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n\to\infty} \begin{pmatrix} \cos\frac{a}{n} & \sin\frac{b}{n} \\ \frac{c}{n} & \cos\frac{d}{n} \end{pmatrix}^n.$$

(Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania)

II Open Problems: Some questions related to the Heisenberg uncertainty principle, by Larry Guth

(Massachusetts Institute of Technology, Department of Mathematics, Cambridge, Massachusetts, USA)

The Heisenberg uncertainty principle is a fundamental idea in Fourier analysis. It loosely says that f and $\hat{f}$ cannot both be concentrated into small regions. When I was a student, I thought of the Heisenberg uncertainty principle as a single inequality, which was proven almost a hundred years ago. But this idea that f and $\hat{f}$ cannot both be concentrated into small regions can be made precise in many ways. So there are many cousins of the Heisenberg uncertainty principle. In a recent paper, Jean Bourgain and Semyon Dyatlov proved a striking new variant of the Heisenberg uncertainty principle called the fractal uncertainty principle – see [BD] for the original paper and [D] for an expository survey paper. After looking at that paper and talking with Semyon, I have been wondering about different variations of the Heisenberg uncertainty principle. I think there is probably a great deal that we don't know yet about the Heisenberg uncertainty principle and here are some questions in that spirit.

Some of the questions seem cleanest in the setting of functions on $\mathbb{Z}_N$ – the integers modulo N. Suppose that

$$f : \mathbb{Z}_N \to \mathbb{C}.$$

Recall that $\hat{f} : \mathbb{Z}_N \to \mathbb{C}$ is defined by

$$\hat{f}(m) = \frac{1}{N}\sum_{n\in\mathbb{Z}_N} f(n) e^{-2\pi i \frac{mn}{N}}.$$

Then, we have the Fourier inversion theorem

$$f(n) = \sum_{m\in\mathbb{Z}_N} \hat{f}(m) e^{2\pi i \frac{mn}{N}}.$$

Suppose that f is supported in a set $X \subset \mathbb{Z}_N$ and $\hat{f}$ is supported in $Y \subset \mathbb{Z}_N$. What can we say about X and Y? One classical version of the Heisenberg uncertainty principle says that $|X||Y| \ge N$. On the other hand, if $|X| + |Y| > N$ then there is always a non-zero function f so that f is supported in X and $\hat{f}$ is supported in Y. The set of all such functions is a linear subspace of $L^2(\mathbb{Z}_N)$ defined by $2N - |X| - |Y|$ equations, and if $|X| + |Y| > N$ then the dimension of this subspace is at least 1. In the case when N is prime, we have a complete characterisation. The result was proven independently by Tao [T], by Biro and by Meshulam (see [T] for more references).

Theorem 1 Suppose that N is prime and $X, Y \subset \mathbb{Z}_N$. Then, there is a non-zero function f with the support of f in X and the support of $\hat{f}$ in Y if and only if $|X| + |Y| > N$.

On the other hand, if N is composite then the situation is quite different because of the subgroups of $\mathbb{Z}_N$. For instance, if $N = M^2$ is a square and if f is the characteristic function of the multiples of M then the support of f has cardinality M and the support of $\hat{f}$ is also a subgroup of cardinality M. This saturates the bound $|X||Y| \ge N$.

193*. For each N, give a complete characterisation of possible pairs $X, Y \subset \mathbb{Z}_N$ admitting a non-zero function f with the support of f in X and the support of $\hat{f}$ in Y.
Theorem 1 gives the result when N is prime. The square of a prime could be a good next case. I believe the state of the art is in a paper of Meshulam [M].
It would also be interesting to prove more quantitative versions of the Heisenberg uncertainty principle. Instead of asserting that there is no function f so that f is supported in X and $\hat{f}$ is supported in Y, it would be nice to say that there is no function f so that f is concentrated in X and $\hat{f}$ is concentrated in Y. For any sets $X, Y \subset \mathbb{Z}_N$, define

$$H(X,Y) := \max_{f:\|f\|_{L^2(\mathbb{Z}_N)}=1} \|f\|_{L^2(X)}\|\hat{f}\|_{L^2(Y)}.$$

Based on the examples above, we expect that $H(X, Y)$ could be big in cases related to subgroups of $\mathbb{Z}_N$ and probably also in cases related to approximate subgroups of $\mathbb{Z}_N$ such as arithmetic progressions. It would be interesting to better understand what happens in other cases that are far from these. One class of examples is random examples.

194*. Suppose $0 < \alpha < 1$ and suppose that $X, Y \subset \mathbb{Z}_N$ are independent random subsets chosen uniformly among all subsets of cardinality N^α. Estimate the expected value of $H(X, Y)$.
For this question, I think it would even be interesting to find a conjecture.
In additive combinatorics, there are several ways of saying that a set X is far from being an approximate subgroup. One such way uses the idea of additive energy. Recall that the energy of X is defined by

$$E(X) := \left|\{(x_1, x_2, x_3, x_4) \in X^4 : x_1 + x_2 = x_3 + x_4\}\right|.$$

If X is a subgroup of $\mathbb{Z}_N$ then there is a unique choice of x_4 for each x_1, x_2, x_3 and so $E(X) = |X|^3$, which is the maximum possible value of $E(X)$. On the other hand, if X is a random subset of $\mathbb{Z}_N$ of cardinality N^α then $E(X) \sim |X|^3 N^{\alpha-1} + |X|^2$.

195*. Given $|X|, |Y|, E(X), E(Y)$, what is the maximum possible size of $H(X, Y)$?
There are a lot of parameters in this question, so let me highlight one particular case that seems interesting to me.

196*. Suppose that $|X| = |Y| \leq 2N^{1/2}$ but $E(X), E(Y) \leq |X|^{2.1}$. Estimate the maximum possible value of $H(X, Y)$.

References

[BD] J. Bourgain and S. Dyatlov, Spectral gaps without the pressure condition, arXiv:1612.09040.

[D] S. Dyatlov, Control of eigenfunctions on hyperbolic surfaces: An application of fractal uncertainty principle, arXiv:1710.08762.

[M] R. Meshulam, An uncertainty inequality for finite abelian groups. *European J. Combin.* **27** (2006), 63–67.

[T] T. Tao, An uncertainty principle for cyclic groups of prime order. *Math. Res. Lett.* **12** (2005), 121–127.

III Solutions

179. Let $p = p_1p_2\cdots p_n$ and $q = q_1q_2\cdots q_n$ be two permutations. We say that they are *colliding* if there exists at least one index i so that $|p_i - q_i| = 1$. For instance, 3241 and 1432 are colliding (choose $i = 3$ or $i = 4$), while 3421 and 1423 are not colliding. Let S be a set of *pairwise colliding* permutations of length n. Is it true that $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

Solution by the proposer. Yes. Let $p \in S$, let $q \in S$ and let p' (resp. q') denote p (resp. q) modulo 2. As p and q are colliding, there is no index i so that $p'_i = q'_i$. Therefore, if we consider all elements of S modulo 2, we get a set of $|S|$ *different* vectors of length n that have only zeros and ones as coordinates, and in which the number of zeros is $\lfloor n/2 \rfloor$. The number of such vectors is $\binom{n}{\lfloor n/2 \rfloor}$, hence that number is an upper bound for $|S|$. This proof is due to János Körner and Claudia Malvenuto.

Remark: It is not known if the presented upper bound is optimal, though empirically it is for $n \leq 7$. □

Also solved by Mihaly Bencze (Brasov, Romania), Souvik Dey (Kolkata, India), Jim K. Kelesis (Athens, Greece), Panagiotis T. Krasopoulos (Athens, Greece), Alexander Vauth[2] *(Lübbecke, Germany).*

180. Let us say that a word w over the alphabet $\{1, 2, \cdots, n\}$ is *n-universal* if w contains all $n!$ permutations of the symbols $1, 2, \ldots, n$ as a subword, not necessarily in consecutive positions. For instance, the word 121 is 2-universal as it contains both 12 and 21, while the word 1232123 is 3-universal. Let $n \geq 3$. Does an n-universal word of length $n^2 - 2n + 4$ exist?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

Solution by the proposer. Yes. Write down n copies of 1 and, between two consecutive copies of 1, insert any permutation of the set $\{2, 3, \cdots, n\}$. This is a string of $n^2 - n + 1$ entries. Call the strings between two consecutive copies of 1 *segments*. Now remove $n - 3$ entries as follows. Let $1 < i < n - 1$. Moving left to right, remove an entry j from the ith segment. Then, in segment $i - 1$, move the entry j of that segment into the last position. In segment $i + 1$, move the entry j of that segment into the first position. The result is a string with the desired length that contains all permutations of length n. This construction is due to S. P. Mohanty.

Remark: While this construction is optimal for $n \leq 7$, it is not optimal in general. For $n \geq 13$, one can construct an n-universal word of length $\lceil n^2 - \frac{7n}{3} + \frac{19}{3} \rceil$. □

Also solved by Mihaly Bencze (Brasov, Romania), Jim K. Kelesis (Athens, Greece), Sotirios E. Louridas (Athens, Greece), Socratis Varelogiannis (Paris, France).

181. Given natural numbers m and n, let $[m]^n$ be the collection of all n-letter words, where each letter is taken from the alphabet $[m] = \{1, 2, \ldots, m\}$. Given a word $w \in [m]^n$, a set $S \subseteq [n]$ and $i \in [m]$, let $w(S, i)$ be the word obtained from w by replacing the j^{th} letter with i for all $j \in S$. The Hales–Jewett theorem then says that for any natural numbers m and r, there exists a natural number n such that every r-colouring of $[m]^n$ contains a monochromatic combinatorial line, that is, a monochromatic set of the form $\{w(S, 1), w(S, 2), \ldots, w(S, m)\}$ for some $S \subseteq [n]$. Show that for $m = 2$, it is always possible to take S to be an interval in this theorem, while for $m = 3$, this is not the case.

(David Conlon, Mathematical Institute, University of Oxford, Oxford, UK)

Solution by the proposer. For the $m = 2$ case, consider the following $r + 1$ words of length r:

$$\begin{aligned} w_0 &= 111\ldots11, \\ w_1 &= 111\ldots12, \\ w_2 &= 111\ldots22, \\ &\vdots \\ w_{r-1} &= 122\ldots22, \\ w_r &= 222\ldots22. \end{aligned}$$

That is, w_i is 1 for the first $r - i$ letters and 2 from then on. By the pigeonhole principle, since there are $r + 1$ words but only r colours, two of these words, say w_i and w_j with $i < j$, receive the same colour. But then, taking $S = [r - j + 1, r - i]$, we see that $w_i = w_i(S, 1)$ and $w_j = w_i(S, 2)$, as required.

For $m = 3$, given a word w, let $n(w)$ be the number of consecutive pairs of letters in w that differ from one another and let χ be the 3-colouring of the words in $[3]^n$ given by $\chi(w) = n(1w1) \pmod 3$, where $1w1$ is the $(n + 2)$-letter word formed by adding a single 1 before and after w. Suppose now that the words in $[3]^n$ have been coloured with χ and there is a monochromatic combinatorial line defined by a word $w \in [3]^n$ and an interval $S \subseteq [n]$. Suppose also that the letter in w that immediately precedes S is a, while the letter that immediately follows S is b (note that we added the dummy 1's above so that these are always defined). If now, for example, $a = 1$ and $b = 2$, it is easy to check that $\chi(w(S, 1)) \neq \chi(w(S, 3))$, since changing $\ldots 1|11\ldots1|2\ldots$ to $\ldots 1|33\ldots3|2\ldots$ adds one to the number of consecutive pairs of letters that differ from one another. Therefore, this case cannot occur. Similarly, one can easily verify that none of the other possible choices of a and b can occur. Therefore, S cannot have been an interval. □

Also solved by Mihaly Bencze (Brasov, Romania), Socratis Varelogiannis (Paris, France).

182. (A) Let $A_1, A_2, \ldots$ be finite sets, no two of which are disjoint. Must there exist a finite set F such that no two of $A_1 \cap F$, $A_2 \cap F, \ldots$ are disjoint?
(B) What happens if all of the A_i are the same size?

(Imre Leader, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK)

Solution by the proposer. (A) No. Just make the A_i meet "further and further to the right". For example, take the sets

$$\{2,4,5\},\ \{1,3,5,6\},\ \{2,4,6,7\},\ \{1,3,5,7,8\},\ \{2,4,6,8,9\},\ \ldots$$

(B) There does have to be such a set. We fix one set A in our family and group the other sets according to how they intersect A – so we write $F(I)$ for the sets in our family that intersect A equal I (for each non-empty subset I of A). More precisely, let us write $G(I)$ for the family formed by each set in $F(I)$ but with I removed. So, to be done, we would like that for each I and J (disjoint subsets of A), there exists a finite set on which all of $G(I)$ meet all of $G(J)$.

This looks a lot like the original statement. So, instead, we prove the stronger statement "for any r and s, if we have some r-sets and some s-sets and each of the r-sets meets each of the s-sets then there is a finite set on which each r-set meets each s-set". And the above argument does prove this, by induction on, say, $r + s$. □

Also solved by John N. Daras (Athens, Greece), Souvik Dey (Kolkata, India), Jean Moulin-Ollagnier (Palaiseau, France), Alexander Vauth (Lübbecke, Germany).

183. The following is from the 2012 Green Chicken maths contest between Middlebury and Williams Colleges. A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \ldots, 2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus, 30 and 1593 are connected by an edge as 3 divides each but 30 and 49 are not. The colouring number of a graph is the smallest number of colours needed so that each vertex is coloured *and* if two vertices are connected by an edge then those two vertices are not coloured the same. The Green Chicken says the colouring number of this graph is at most 9. Prove he is wrong and find the correct colouring number.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

Solution by the proposer. The colouring number is at least 10, as the vertices 2, 4, 8, 16, 32, …, $1024 = 2^{10}$ are all connected to each other, and thus we need at least 10 colours. Why? This is a complete graph with 10 vertices, and its colouring number is 10. As this subgraph of our graph has colouring number 10, the entire graph has colouring number at least 10.

We can get a very good lower bound easily. Instead of looking at powers of 2, we can look at the even numbers. There are 1006 even numbers and each even number is connected to every other. Thus, we have a complete graph with 1006 vertices, implying the colouring number is at least 1006.

It's easy to see the colouring number is at most $2012 - \pi(2012) + 1$, where $\pi(2012)$ is the number of primes at most 2012. Why? We can colour all the primes the same colour, as none are connected to any other. That's our plus 1; the $2012 - \pi(2012)$ comes from a trivial bounding, using a different colour for each remaining vertex.

Interestingly, our lower bound is the answer: the colouring number is 1006. To see this, choose 1006 colours and colour each even number with one of these colours, never using the same colour twice. Note we *have* to do this, as no two even numbers can share a colour. We are left with colouring the odd numbers 3, 5, 7, 9, …, 2011. We colour the vertex $2k+1$ with the colour of vertex $2k$. Note $2k+1$ and $2k$ can't share a factor d greater than 1 and are thus not connected. (If they shared a factor, it would have to divide their difference, which is 1.) Since vertex $2k$ is the only vertex that has the colour we want to use for vertex $2k + 1$, we see that we have a valid colouring. We showed the colouring number must be at least 1006; since we've found a colouring that works with 1006 colours, we know this must be the answer. □

Also solved by Mihaly Bencze (Brasov, Romania), Jim K. Kelesis (Athens, Greece), Panagiotis T. Krasopoulos (Athens, Greece), Alexander Vauth (Lübbecke, Germany), Socratis Varelogiannis (Paris, France).

184. There are n people at a party. They notice that for every two of them, the number of people at the party that they both know is *odd*. Prove that n is an odd number.[3]

(Benny Sudakov, Department of Mathematics, ETH Zürich, Zürich, Switzerland)

Solution by the proposer. Let G be a graph whose vertices are the people at the party and two are connected if they know each other. Then this graph has the property that every pair of vertices have an odd number of common neighbours. Let N_v be the set of neighbours of some vertex v and let $G[N_v]$ be the subgraph of G induced by this set. Then, all degrees of $G[N_v]$ are odd, since these are exactly the number of common neighbours that v has with its neighbours. Since the sum of the degrees of the vertices in $G[N_v]$ is twice its number of edges (an even number) we have that $|N_v|$ is even. Therefore, all vertices in G have even degree.

Consider now the adjacency matrix A of G over the field with two elements (addition and multiplications are modulo 2). This is an n by n symmetric matrix whose rows and columns are indexed by the vertices of G and $a_{uv} = 1$ if u, v are adjacent and 0 otherwise. Note that the sum of the columns of A is 0 (modulo 2), since every row has an even number of 1's. Therefore, A does not have full rank. Consider $B = A^2$. It is easy to check that the diagonal of B consists of the degrees of the vertices (modulo 2) of G and b_{uv} is the number of common neighbours of u and v (modulo 2). Therefore, A^2 has 0's on the diagonal and 1's everywhere else. When n is even, such a matrix has full rank, since it has a non-zero determinant (which is easy to compute). Since A does not have full rank, neither does A^2. This implies that n is odd. □

Also solved by Mihaly Bencze (Brasov, Romania), Jim K. Kelesis (Athens, Greece), Jean Moulin-Ollagnier (Palaiseau, France).

Notes

1. For the *"Solved and Unsolved Problems" column devoted to discrete mathematics, the reader is referred to Issue 105, September 2017, EMS Newsletter, p. 55.*
2. Alexander Vauth made the interesting remark that this problem has also been affirmatively proven in the paper: Körner, J., and Malvenuto, C., *Pairwise colliding permutations and the capacity of infinite graphs*, SIAM Journal on Discrete Mathematics, 20(1) (2006), 203-212.
3. This problem appeared a long time ago in the Tournament of the Towns.

We would like you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *probability theory*.

The American Mathematical Society is excited to announce its acquisition of the Mathematical Association of America's book publishing program. The high-quality mathematics titles and textbooks developed and edited by *MAA Press* are now published as an imprint of the AMS Book Program.

MAA PRESS An Imprint of the AMERICAN MATHEMATICAL SOCIETY

COMMON SENSE MATHEMATICS

Second Edition

Ethan Bolker & Maura Mast

Emphasises common sense and common knowledge in approaching real problems through popular news items and finding useful mathematical tools and frames with which to address those questions. Most quantitative reasoning texts are arranged by mathematical topics to be mastered. This book looks at real life stories that can be best understood with careful reading and a little mathematics.

MAA Textbooks, Vol. 33

Feb 2018 326pp 9781939512109 Hardback €66.00

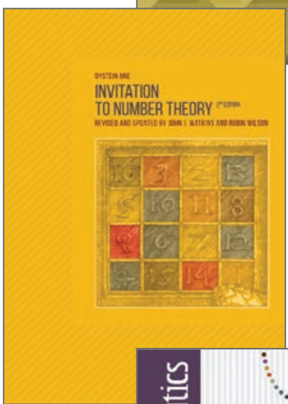
EXERCISES IN (MATHEMATICAL) STYLE

John McCleary

What does style mean in mathematics? Style is both how one does something and how one communicates what was done. In this book, the author investigates the worlds of the well-known numbers, the binomial coefficients. The author follows the example of Raymond Queneau's *Exercises in Style*, offering the reader 99 stories in various styles.

Anneli Lax New Mathematical Library, Vol. 48

Feb 2018 290pp 9780883856529 Paperback €48.00



INVITATION TO NUMBER THEORY

Second Edition

Oystein Ore

Number theory is the branch of mathematics concerned with the counting numbers, 1, 2, 3,... and their multiples and factors. But in spite of their simplicity, you will meet a multitude of topics in this book: magic squares, cryptarithms, finding the day of the week for a given date, constructing regular polygons, pythagorean triples, and many more.

Anneli Lax New Mathematical Library, Vol. 49

Feb 2018 134pp 9780883856536 Paperback €48.00

PORTAL THROUGH MATHEMATICS

Journey to Advanced Thinking

O. A. Ivanov & Robert G. Burns

A collection of puzzles and problems mostly on topics relating to secondary mathematics. The problems and topics are fresh and interesting and frequently surprising. The book is organised into 29 themes, each a topic from algebra, geometry or calculus and each launched from an opening puzzle or problem. There are also excursions into number theory, solid geometry, physics and combinatorics.

Anneli Lax New Mathematical Library, Vol. 47

Feb 2018 318pp 9780883856512 Paperback €55.00

Free delivery at **eurospanbookstore.com/maa**

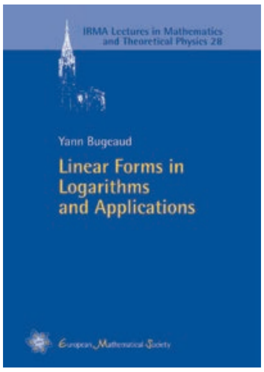
CUSTOMER SERVICES:
Tel: +44 (0)1767 604972
Fax: +44 (0)1767 601640
Email: eurospan@turpin-distribution.com

FURTHER INFORMATION:
Tel: +44 (0)20 7240 0856
Fax: +44 (0)20 7379 0609
Email: info@eurospangroup.com

AMS AMERICAN MATHEMATICAL SOCIETY

New books published by the European Mathematical Society

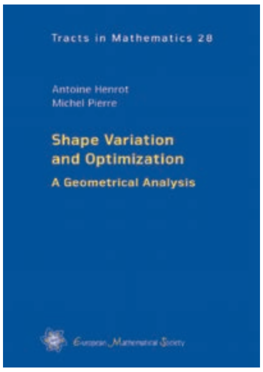
Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.



Yann Bugeaud (Université de Strasbourg, France)
Linear Forms in Logarithms and Applications (IRMA Lectures in Mathematics and Theoretical Physics, Vol. 28)

ISBN 978-3-03719-183-5. 2018. 240 pages. Softcover. 17 x 24 cm. 38.00 Euro

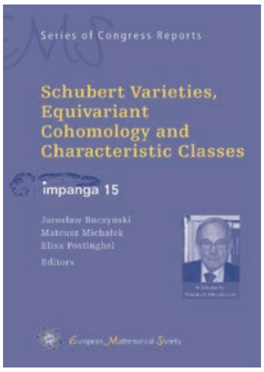
The aim of this book is to serve as an introductory text to the theory of linear forms in the logarithms of algebraic numbers, with a special emphasis on a large variety of its applications. We wish to help students and researchers to learn what is hidden inside the blackbox 'Baker's theory of linear forms in logarithms' (in complex or in p-adic logarithms) and how this theory applies to many Diophantine problems, including the effective resolution of Diophantine equations, the *abc*-conjecture, and upper bounds for the irrationality measure of some real numbers.
Written for a broad audience, this accessible and self-contained book can be used for graduate courses (some 30 exercises are supplied). Specialists will appreciate the inclusion of over 30 open problems and the rich bibliography of over 450 references.



Antoine Henrot (Université de Lorraine, Vandœuvre-lès-Nancy, France) and Michel Pierre (ENS Cachan Bretagne, Bruz, France)
Shape Variation and Optimization. A Geometrical Analysis (EMS Tracts in Mathematics)

ISBN 978-3-03719-178-1. 2018. 379 pages. Hardcover. 17 x 24 cm. 68.00 Euro

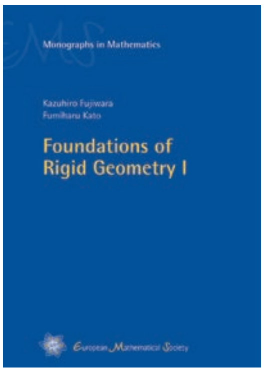
Optimizing the shape of an object to make it the most efficient, resistant, streamlined, lightest, noiseless, stealthy or the cheapest is clearly a very old task. But the recent explosion of modeling and scientific computing have given this topic new life. Many new and interesting questions have been asked. A mathematical topic was born – shape optimization (or optimum design).
This book provides a self-contained introduction to modern mathematical approaches to shape optimization, relying only on undergraduate level prerequisite but allowing to tackle open questions in this vibrant field. The analytical and geometrical tools and methods for the study of shapes are developed. In particular, the text presents a systematic treatment of shape variations and optimization associated with the Laplace operator and the classical capacity. Emphasis is also put on differentiation with respect to domains and a FAQ on the usual topologies of domains is provided. The book ends with geometrical properties of optimal shapes, including the case where they do not exist.



Schubert Varieties, Equivariant Cohomology and Characteristic Classes (EMS Series of Congress Reports)
Jarosław Buczyński (Polish Academy of Sciences and University of Warsaw, Poland), Mateusz Michałek (Polish Academy of Sciences, Warsaw, Poland and Max Planck-Institute, Leipzig, Germany) and Elisa Postinghel (Loughborough University, UK), Editors

ISBN 978-3-03719-182-8. 2017. 354 pages. Hardcover. 17 x 24 cm. 78.00 Euro

IMPANGA stands for the activities of Algebraic Geometers at the Institute of Mathematics, Polish Academy of Sciences, including one of the most important seminars in algebraic geometry in Poland. The topics of the lectures usually fit within the framework of complex algebraic geometry and neighboring areas of mathematics.
This volume is a collection of contributions by the participants of the conference IMPANGA15, organized by participants of the seminar, as well as notes from the major lecture series of the seminar in the period 2010–2015. Both original research papers and self-contained expository surveys can be found here. The articles circulate around a broad range of topics within algebraic geometry such as vector bundles, Schubert varieties, degeneracy loci, homogeneous spaces, equivariant cohomology, Thom polynomials, characteristic classes, symmetric functions and polynomials, and algebraic geometry in positive characteristic.



Kazuhiro Fujiwara (Nagoya University, Japan) and Fumiharu Kato (Tokyo Institute of Technology, Japan)
Foundations of Rigid Geometry I (EMS Monographs in Mathematics)

ISBN 978-3-03719-135-4. 2017. 863 pages. Hardcover. 16.5 x 23.5 cm. 108.00 Euro

Rigid geometry is one of the modern branches of algebraic and arithmetic geometry. It has its historical origin in J. Tate's rigid analytic geometry, which aimed at developing an analytic geometry over non-archimedean valued fields. Nowadays, rigid geometry is a discipline in its own right and has acquired vast and rich structures, based on discoveries of its relationship with birational and formal geometries.
In this research monograph, foundational aspects of rigid geometry are discussed, putting emphasis on birational and topological features of rigid spaces. Besides the rigid geometry itself, topics include the general theory of formal schemes and formal algebraic spaces, based on a theory of complete rings which are not necessarily Noetherian. Also included is a discussion on the relationship with Tate's original rigid analytic geometry, V.G. Berkovich's analytic geometry and R. Huber's adic spaces. As a model example of applications, a proof of Nagata's compactification theorem for schemes is given in the appendix. The book is encyclopedic and almost self-contained.