NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Feature
Friable Integers
Interactions of Probability and Number Theory

Interviews
Sir Michael Atiyah
Michel Waldschmidt
Bernadette Faye

Societies
The Spanish Society for Applied Mathematics

European Mathematical Society

March 2019
Issue 111
ISSN 1027-488X

Sir Michael Atiyah

European Mathematical Society

Newsletter No. 111, March 2019

The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

ISSN 1027-488X

Published by the
EMS Publishing House
ETH-Zentrum SEW A21
CH-8092 Zürich, Switzerland
homepage: www.ems-ph.org

For advertisements and reprint permission requests contact: newsletter@ems-ph.org

Editorial Team

Scan the QR code to go to the Newsletter web page:
http://euro-math-soc.eu/newsletter

EMS Executive Committee

President

Prof. Volker Mehrmann
(2019–2022)
Technische Universität Berlin
Sekretariat MA 4-5
Straße des 17. Juni 136
10623 Berlin, Germany
e-mail: mehrmann@math.tu-berlin.de

Vice-Presidents

Prof. Armen Sergeev
(2017–2020)
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8
119991 Moscow
Russia
e-mail: sergeev@mi.ras.ru

Prof. Betül Tanbay
(2019–2022)
Department of Mathematics
Bogazici University
Bebek 34342 Istanbul
Turkey
e-mail: tanbay@boun.edu.t

Secretary

Prof. Sjoerd Verduyn Lunel
(2015–2022)
Department of Mathematics
Utrecht University
Budapestlaan 6
3584 CD Utrecht
The Netherlands
e-mail: s.m.verduynlunel@uu.nl

Treasurer

Prof. Mats Gyllenberg
(2015–2022)
Department of Mathematics
and Statistics
University of Helsinki
P.O. Box 68
00014 University of Helsinki
Finland
e-mail: mats.gyllenberg@helsinki.fi

Ordinary Members

Prof. Jorge Buescu
(2019–2022)
Department of Mathematics
Instituto Superior Técnico
Av. Rovisco Pais
1096 Lisbon Codex, Portugal
e-mail: jbuescu@math.ist.utl.pt

Prof. Nicola Fusco
(2017–2020)
Dip. di Matematica e Applicazioni
Complesso Universitario di
Monte Sant' Angelo
Via Cintia
80126 Napoli
Italy
e-mail: n.fusco@unina.it

Prof. Stefan Jackowski
(2017–2020)
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa
Poland
e-mail: sjack@mimuw.edu.pl

Prof. Vicente Muñoz
(2017–2020)
Departamento de Algebra,
Geometría y Topología
Universidad de Málaga
Campus de Teatinos, s/n
29071 Málaga
Spain
e-mail: vicente.munoz@uma.es

Prof. Beatrice Pelloni
(2017–2020)
School of Mathematical &
Computer Sciences
Heriot-Watt University
Edinburgh EH14 4AS
UK
e-mail: b.pelloni@hw.ac.uk

EMS Secretariat

Ms Elvira Hyvönen
Department of Mathematics
and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
00014 University of Helsinki
Finland
Tel: (+358) 2941 51141
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dr. Richard H. Elwes
School of Mathematics
University of Leeds
Leeds, LS2 9JT
UK
e-mail: R.H.Elwes@leeds.ac.uk

EMS Agenda

2019

22–23 March
EMS Executive Committee Meeting, Berlin, Germany

23–24 March
EMS Meeting of the Presidents, Berlin, Germany

EMS Scientific Events

2019

1–5 April
Imaging and Machine Learning
Institut Henri Poincaré, Paris, France

16–18 April
International Meeting on Applied Mathematics & Evolution
(IMAME 2019)
La Rochelle, France

25–28 June
Barcelona Analysis Conference (BAC19)
Barcelona, Spain

9–13 July
SIAM Conference on Applied Algebraic Geometry
Bern, Switzerland

15–19 July
ICIAM 2019
Valencia, Spain

22–26 July
30th International Workshop on Operator Theory and
its Applications – IWOTA 2019
Lisbon, Portugal

29 July–2 August
British Combinatorial Conference 2019
Glasgow, UK

26–29 August
Caucasian Mathematics Conference (CMC-III)
Rostov-on-Don, Russian Federation

2020

5–11 July
ICCOPT 2019 – 6th International Conference on Continuous
Optimisation
Berlin, Germany

Editorial – From the New President

Volker Mehrmann, President of the EMS

Dear colleagues,

I wish to address a warm welcome to all members of the EMS, and I want to thank the EMS council for their trust in electing me the new president. Let me begin by thanking the past president, Pavel Exner, for his tireless work for the society in the past four years and also before that. Having served on the Executive Committee for six years and as vice president of the EMS for two years I can really appreciate his great efforts, and as I said when we had our last EC meeting in Barcelona, I am really hoping to count on him for many future activities connected with the EMS and also in dealing with the high politics in Brussels and in the countries of the member societies. I am looking forward to working for the society and hope I can live up to the high standards that Pavel has set. I am sure that this ambition will be well supported by the Executive Committee and the other highly active committees of the EMS.

The EMS is a very well-functioning society with a clear focus on improving the advancement of mathematics in all its aspects of research, education, and advancement of young scientists. This includes improvements in diversity, the popularization of mathematics in society, and proper scientific conduct in research, publication, or evaluation of scientific quality. Another important topic in a continent as highly diverse as Europe is to improve opportunities for mathematical research in economically less favoured countries. In all these areas the EMS has strongly active committees whom I would like to thank already for their past and future work for the society and the mathematical community as a whole.

Although the society is doing well, there is always room for improvement and there are major challenges on the horizon; some of them were addressed in the last editorial by Pavel Exner, but they are so important that I will mention them again.

Mathematical research and education in Europe is highly unbalanced across the continent. This is mostly due to the economic imbalance in Europe but also to the different appreciation of mathematics in different parts of Europe. In general, research funding for mathematics in the EU is too low and extremely unbalanced on a national but also European level. The whole community has to work together to change this. We cannot accept that the European Commission thinks that mathematical research funding via the ERC is sufficient, while other funding streams treat mathematics rather badly. This criticism should not diminish the great support by the ERC, but rather should address the neglect of mathematics in other EU funding programs. However, we should also be aware that in the past this has partly been our own fault. As mathematicians in leading positions or in receipt of important awards, we should speak out for the whole community and appreciate not only the top success stories in mathematics but also the great research work in not-so-prominent areas of mathematics, and proudly advertise it. To achieve this goal it is necessary that we communicate better between the different fields of mathematics (be they more theoretical or more applied) and that we treat each other with more respect than in the past. The mathematics community should work strongly together in lobbying for the advancement of mathematics in Europe: this includes participation in open consultations, or in the EU commission in discussions with ministries and stakeholders on a national level.

A second issue where we should become more active is the advancement of young mathematicians. Currently, the job opportunities in academia and industry for students who finish with a masters or PhD degree are good in highly industrialized European countries, but are generally very unevenly distributed across the whole of Europe. To change this, a major effort within the mathematics community is needed. We should prepare more mathematicians for academia and industry. One way the EMS can help is to sponsor more summer schools on important research topics. However, the community should become active and send in more applications (currently, most EMS-sponsored events have very few applicants) and we should also make an effort to increase the international visibility of our young colleagues.

Mathematical publications are an important part of our activities. The great work of the EMS Publishing House has to be continued after the retirement of the Managing Director Thomas Hintermann but also many new developments have to be taken into consideration. The EMS PH will change its status in the coming weeks and become a limited company owned by the EMS, and a new managing director will be hired (see the advertisement below). In view of the fact that publishing is becoming more and more electronic, the current publishing models will have to be rethought, and this will be a major activity for the EMS in the coming years. This also concerns the current open-access strategies of the EU and the member states, which will have a huge effect on mathematical publishing in the future. This topic must be a major issue in our discussions with our national governments and with the EU, and we should be prepared to participate in these activities with strong quality control to achieve a sustainable solution. Another issue that is going in a similar direction is the FAIR (Findable, Accessible, Interoperable, Reusable) principle for the availability of mathematical data, where the EMS will have to take a significant role.

Applied and industrial mathematics is strong in some countries but very weak in others, and the EMS together with ECMI (the European Consortium for Mathematics in Industry) has founded the EU-MATHS-IN network (https://www.eu-maths-in.eu/EUMATHSIN/), which has the mission of leveraging the impact of mathematics on innovations in key technologies by enhanced communication and information exchange among the involved stakeholders on a European level. It will act as facilitator, translator, educator, and link between the various players and their communities in Europe.

Furthermore, interdisciplinary communication and cooperation with other sciences still needs to be improved. Last year this cooperation was intensified within the Year of Mathematical Biology, but with other fields of science these links certainly need stronger attention.

Finally, we are all complaining that mathematical education is getting worse in Europe. The whole community has to make major efforts to avoid further degradation of standards and to improve not only general mathematical education, but also that of highly talented young people. To achieve this it is in my opinion essential to involve more young scientists in the bodies of the community, to support initiatives by young researchers, and to increase the popularity of mathematics in schools, universities, and society as a whole. Please participate in the survey launched by the education committee (http://euro-math-soc.eu/news/19/02/15/survey-ems-education-committee).

I am looking forward to four years of working together with you for the well-being of mathematics in Europe.

Volker Mehrmann received his Diploma in mathematics in 1979, his PhD in 1982, and his habilitation in 1987 from the University of Bielefeld, Germany. He spent research years at Kent State University in 1979–1980, at the University of Wisconsin in 1984–1985, and at the IBM Research Center in Heidelberg in 1988–1989. After spending the years 1990–1992 as a visiting full professor at the RWTH Aachen, he was a full professor at TU Chemnitz from 1993 to 2000. Since then he has been a full professor of mathematics at TU Berlin.

He is a member of acatech (the German academy of engineering) and the Academia Europaea, he was the president of GAMM (the International Association of Applied Mathematics and Mechanics), vice president of the European Mathematical Society (EMS), chair of MATHEON (the Mathematics for Key Technologies Research Center), and chair of the Einstein Center ECMath in Berlin. He has been president of the EMS since January 2019.

He is a SIAM Fellow, has held an ERC Advanced Grant, and was also a member of the ERC Mathematics Panel. He is an editor of several journals in numerical analysis and editor-in-chief of Linear Algebra and its Applications. His research interests are numerical mathematics/scientific computing, applied and numerical linear algebra, control theory, and the theory and numerical solution of differential-algebraic equations.

Farewell within the Editorial Board of the EMS Newsletter

With issue N°110 (December 2018), our copy-editor **Chris Nunn** ended his editorship of the Newsletter. Chris was an extraordinary member of our Board. In a way, he is a part of the EMS Newsletter history. Chris was engaged by the EMS as a copy-editor in 2005. From issue N°57 until N°110 he checked the British spelling and grammar used in the EMS Newsletter.

We express our deep gratitude to Chris for all the work he has carried out with great enthusiasm, competence and patience for almost half of the previous existence of the Newsletter. We thank Chris for contributing to a friendly and productive atmosphere that we shall keep in our magazine, and we wish him the best for his future.

Sir Vaughan F. R. Jones to Deliver a Public Lecture at 8ECM

Tomaž Pisanski (University of Primorska, Koper, Slovenia)

We are delighted to announce that Fields medallist Sir Vaughan F. R. Jones FRS will deliver a public lecture at the 8th European Congress of Mathematics.

Sir Vaughan F. R. Jones is a New Zealand-born mathematician based in the USA, renowned worldwide for his remarkable work on von Neumann algebras and knot polynomials. He was awarded a Fields medal in 1990 for his discoveries in the mathematical study of knots – including an improvement on the Alexander polynomial (now called the Jones polynomial) – working from an unexpected direction with origins in the theory of von Neumann algebras, an area of analysis already much developed by Alain Connes. These discoveries led to the solution of a number of classical problems in knot theory, and to increased interest in low-dimensional topology.

His work on polynomial invariants of knots also had remarkable implications in the field of molecular biology, where new insight was gained into how DNA can remove the tangles that result when replication and cell division firstly duplicates the DNA and subsequently has to pull the chromosomal mass into different cells. The result represents a landmark in modern mathematics whose ramifications still remain to be fully explored.

Professor Jones is currently Professor Emeritus at the University of California, Berkeley, where he has been on the faculty since 1985, as well as Stevenson Distinguished Professor of Mathematics at Vanderbilt University (from 2011). He is also a Distinguished Alumni Professor at the University of Auckland.

We look forward to welcoming Professor Jones at the 8th European Congress of Mathematics in Portorož, Slovenia!

Further details will be posted on the 8ECM website, please visit www.8ecm.si for more news.

Tomaž Pisanski (Tomaz.Pisanski@upr.si) is the President of the SDAMS and a professor of mathematics and computer science at the University of Primorska. He is the chair of the organising committee of 8ECM. His research interests include various aspects of discrete mathematics. He is the co-author of a book on configurations.

Friable Integers: An Overview

Cécile Dartyge (Université de Lorraine, Vandœuvre-lès-Nancy, France)

1 Introduction

A friable (or smooth) integer is an integer without large prime factors. More precisely, let $P^+(n)$, $P^-(n)$ denote respectively the largest and the smallest prime factor of an integer n.[1] For $y \geqslant 2$, an integer n is said to be y-friable if $P^+(n) \leqslant y$ and y-sifted if $P^-(n) > y$. The term 'friable' reflects the possibility of splitting these integers into small divisors – the small prime factors. When the friable parameter y is very small, these integers can be represented as products of divisors with good size control. This flexibility in the factoring of friable integers is at the origin of several recent breakthroughs in analytic number theory. Another fundamental observation is that every integer n has a unique decomposition of the form $n = ab$ where a is y-friable and b is y-sifted. The structure of the sifted part b resembles that of a prime number while the friable part a is supposed to behave more like a standard integer. This simple idea is often an efficient starting point in the study of sums appearing in analytic number theory problems, as described, for example, in Section 5.1.

Research on friable integers really started less than a hundred years ago. Since the 1980s, the theory has been actively developing not only for its own interest, but also for the multiple applications available.

Let $S(x, y)$ denote the set of the y-friable integers not exceeding x and let $\Psi(x, y)$ be its cardinality. In the first part of this survey, we provide an account of the various methods developed for estimating $\Psi(x, y)$ according to the relative sizes of x and y and we briefly describe some results related to multiplicative properties of friable integers. Next we depict the role of friable integers in certain factoring algorithms. In the last part, we present various number theoretic-problems for which friable integers led to significant advances.

This presentation is far from exhaustive. Our aim is mainly to provide an overview of the role of friable integers in different areas from number theory. One will find more complete presentations in the impressive survey papers of Hildebrand and Tenenbaum [13], Granville [11], Pomerance [17]. We furthermore recommend the book by Crandall and Pomerance [5] and those by Tenenbaum [18,19] which are unavoidable references respectively in algorithmic and analytic number theory.

Throughout this text the letter p with or without subscript denotes a prime number and (a, b) is the g.c.d. of the integers a and b. To simplify notation, we write $\ln_k(t)$ the kth iterated Napierian logarithm of the positive real number t, so that, in particular, for $k = 2$, $\ln_2 t = \ln\ln t$.

1 With the conventions $P^+(1) = 1$ and $P^-(1) = \infty$.

2 How to count friable integers?

2.1 The Dickman function

According to the sizes of y and x, several methods allow approximating $\Psi(x, y)$. Most of the estimates produced depend on the ratio

$$u = \frac{\ln x}{\ln y}\cdot$$

Dickman proved in 1930 that for all fixed $u > 0$, a positive proportion of integers less than x are $x^{1/u}$-friable:

$$\lim_{x\to+\infty} \frac{\Psi(x, x^{1/u})}{x} = \varrho(u)$$

where ϱ, the Dickman function, is the only continuous function on $\mathbb{R}^+$, differentiable on $]1, +\infty[$, and satisfying the delay differential equation

$$u\varrho'(u) = \varrho(u - 1)$$

with initial condition $\varrho(u) = 1$ for $0 \leqslant u \leqslant 1$. This function $\varrho(u)$ corresponds to the probability of an integer less than x being $x^{1/u}$-friable. The Dickman function also appears in another context with no connection to friable integers: if $\{U_n\}_{n=1}^{\infty}$ is a sequence of independent random variables uniformly distributed in $[0, 1]$, then the series $Y = U_1+U_1U_2+U_1U_2U_3+\cdots$ converges almost surely to a random variable distributed according to an absolutely continuous law with density $e^{-\gamma}\varrho$, where γ is the Euler constant.[2]

Such emergence of delay differential equations is not specific to the study of friable integers. It is genuinely linked to sieve methods. Let us, traditionally, note $\Phi(x, y)$ the number of y-sifted integers not exceeding x. Then a similar process leads to proving that the probability for an integer to be y-sifted is $\omega(u)$ (see for example [19], Chapter III.6), where $\omega(u)$ is the Buchstab function. This function is defined by $\omega(u) = 1/u$ for $1 \leqslant u \leqslant 2$ and $(u\omega(u))' = \omega(u - 1)$, with continuity at $u = 2$. Similar functions show up, for example, in the Rosser–Iwaniec sieve or the Jurkat–Richert sieve.

The reader will find in [19, Chapter III.5] a very precise study of the Dickman function. It is rapidly decreasing as u tends to $+\infty$, as shown by the Hildebrand–Tenenbaum estimate ([13, Corollary 2.3]), valid for $u \geqslant 1$:

$$\varrho(u) = \exp\left\{-u\left(\ln u + \ln_2(u+2) - 1 + O\Big(\frac{\ln_2(u+2)}{\ln(u+2)}\Big)\right)\right\}.$$

The function ϱ has been implemented in several pieces of mathematical software, for example Sage. The following values are given in [11]:

$$\varrho(2) \approx 3.07 \times 10^{-2}, \varrho(5) \approx 3.55 \times 10^{-4},$$
$$\varrho(10) \approx 2.77 \times 10^{-11}, \varrho(20) \approx 2.46 \times 10^{-29},$$
$$\varrho(50) \approx 6.72 \times 10^{-97}, \text{ etc.}$$

2 So $\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} 1/n - \ln N$.

2.2 $\Psi(x,y)$ for large y, through functional equations

This initial result by Dickman has been improved by de Bruijn. The starting idea is as follows: for $z \geqslant y$, an integer n counted in $\Psi(x,z)$ is either y-friable or of type mp with $P^+(m) \leqslant p$, and the prime p lies in $]y,z]$. We deduce the *Buchstab identity*:

$$\Psi(x,y) = \Psi(x,z) - \sum_{y<p\leqslant z} \Psi\Big(\frac{x}{p}, p\Big) \qquad (1 < y \leqslant z \leqslant x). \quad (1)$$

This leads to an iteration with (obvious) initial condition

$$\Psi(x,y) = \lfloor x \rfloor \quad (y \geqslant x).$$

Since $x/p \leqslant p$ when $\sqrt{x} \leqslant p \leqslant x$, we immediately get $\Psi(x/p, p) = \lfloor x/p \rfloor$ under this assumption. Applying this to (1) yields an asymptotic formula for $y > x^{1/2}$. Inserting this new formula back into (1), we get an estimate for $\Psi(x,y)$ when $y > x^{1/3}$ and so on. This functional approach leads to the uniform estimate for $x \geqslant y \geqslant 2$ (see for example Theorem III.5.8 of [18]):

$$\Psi(x,y) = x\varrho(u) + O\Big(\frac{x}{\ln y}\Big).$$

This formula loses accuracy for "large" values of u. For example, when $u \geqslant \ln_2 y$, the main term $x\varrho(u)$ is dominated by the error term $O(x/\ln y)$. Given the present state of zero-free regions of the Riemann zeta function, the limit of de Bruijn's method is actually the range

$$y > \exp\big((\ln x)^{5/8+\varepsilon}\big), \quad (2)$$

for any fixed $\varepsilon > 0$. The range of de Bruijn's approximation to $\Psi(x,y)$ has been improved by Hildebrand through another functional equation:

$$\Psi(x,y)\ln x = \int_1^x \Psi(t,y)\frac{\mathrm{d}t}{t} + \sum_{\substack{p^k\leqslant x\\ p\leqslant y}} \Psi\Big(\frac{x}{p^k}, y\Big)\ln p. \quad (3)$$

This is derived by evaluating in two different ways the sum

$$S = \sum_{n\in S(x,y)} \ln n.$$

First, Abel summation provides

$$S = \Psi(x,y)\ln x - \int_1^x \Psi(t,y)\frac{\mathrm{d}t}{t},$$

then the additivity of the logarithm $\ln n = \sum_{p^k|n} \ln p$ furnishes the second term of the right-hand side of (3). An advantage of this formula is keeping the friable parameter y constant, so that $\Psi(x,y)$ appears as a mean value of itself in only one variable. This makes the regularization arising from the iterations more efficient. Hildebrand proved that, for all fixed $\varepsilon > 0$, the formula

$$\Psi(x,y) = x\varrho(u)\Big\{1 + O\Big(\frac{\ln(u+1)}{\ln y}\Big)\Big\} \quad (4)$$

holds uniformly in a range larger than (2)

$$\exp((\ln_2 x)^{5/3+\varepsilon}) \leqslant y \leqslant x. \quad (5)$$

This region (5) is closely related to the error term of the prime number theorem. Any progress on this error term implies a corresponding improvement on (5). Actually, Hildebrand proved that the Riemann hypothesis[3] is satisfied if and only if (4) holds in the region $y \geqslant (\ln x)^{2+\varepsilon}$. In the same range (5), Saias obtained an estimate for $\Psi(x,y)$ which is more precise than (4), the main term $x\varrho(u)$ being replaced by a more involved expression $\Lambda(x,y)$, already present in de Bruijn's article. Nevertheless, it is possible to obtain asymptotic formulas for $\ln(\Psi(x,y)/x)$ in wider domains. Very precise formulas may be found for example in [4], [13], or [19] . A consequence of these estimates is that for all fixed $0 < \varepsilon < 1$, and uniformly for $u \leqslant y^{1-\varepsilon}$, we have

$$\Psi(x,y) = xu^{-(1+o(1))u}, \quad (6)$$

as y and u tend to $+\infty$, . This formula gives an idea of the order of magnitude of $\Psi(x,y)$. We deduce for example that for fixed $\alpha > 1$, $\Psi(x,(\ln x)^\alpha) = x^{1-1/\alpha+o(1)}$.

2.3 Geometric method for small values of y

When y gets smaller than a power of $\ln x$, one must proceed in a different way to obtain an asymptotic formula. We observe that $\Psi(x,y)$ is the number of solutions $(m_p)_{p\leqslant y}$ in nonnegative integers of the inequality $\prod_{p\leqslant y} p^{m_p} \leqslant x$. Taking logarithms, this condition becomes $\sum_{p\leqslant y} m_p \ln p \leqslant \ln x$. We are thus counting integer points inside a polytope of $\mathbb{R}^{\pi(y)}$, where $\pi(y)$ is the number of prime numbers not exceeding y. This approach is efficient for very small values of y. Ennola proved in this way that for $2 \leqslant y \leqslant \sqrt{\ln x}$,

$$\Psi(x,y) = \frac{1}{\pi(y)!}\prod_{p\leqslant y}\frac{\ln x}{\ln p}\Big\{1 + O\Big(\frac{y^2}{(\ln x)(\ln y)}\Big)\Big\}. \quad (7)$$

La Bretèche and Tenenbaum [4] improved on this result (range for y, quality of the error term) by employing a mixed approach resting on the residue theorem and the saddle-point method, as described in the next paragraph.

2.4 The saddle-point method

Formula (4) provides an approximation to $\Psi(x,y)$ by a regular function but this is not the case for (7) which depends on $\pi(y)!$. What happens in the domain not covered by these two estimations, that is, $\sqrt{\ln x} \leqslant y \leqslant \exp((\ln\ln x)^{5/3+\varepsilon})$?

This question has been solved by Hildebrand and Tenenbaum. They gave an estimate for $\Psi(x,y)$ by using a third approach: the saddle-point method. The indicator function of y-friable integers is a multiplicative function. Let $\zeta(s,y)$ be the associated Dirichlet series:

$$\zeta(s,y) := \sum_{P^+(n)\leqslant y}\frac{1}{n^s} = \prod_{p\leqslant y}\Big(1 - \frac{1}{p^s}\Big)^{-1}.$$

By Perron's formula, ([19, Chapter II.2]), we can represent $\Psi(x,y)$ in the following way for all $\alpha > 0$ and $x \notin \mathbb{N}$:

$$\Psi(x,y) = \frac{1}{2i\pi}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,y)x^s\frac{\mathrm{d}s}{s}.$$

The optimal choice for α corresponds to the saddle point of the integrand, that is, the unique solution of the equation

$$-\frac{\zeta'(\alpha,y)}{\zeta(\alpha,y)} = \ln x, \text{ with } -\frac{\zeta'(\alpha,y)}{\zeta(\alpha,y)} = \sum_{p\leqslant y}\frac{\ln p}{p^\alpha - 1}. \quad (8)$$

Hildebrand and Tenenbaum obtained the following formula, valid uniformly for $x \geqslant y \geqslant 2$:

$$\Psi(x,y) = \frac{x^\alpha\zeta(\alpha,y)}{\alpha\sqrt{2\pi\varphi_2(\alpha,y)}}\Big\{1 + O\Big(\frac{1}{u} + \frac{\ln y}{y}\Big)\Big\},$$

3 The Riemann ζ function is defined by $\zeta(s) = \sum_{n\geqslant 1} 1/n^s$ for $s \in \mathbb{C}$ with real part strictly larger than 1. It has an analytic continuation, also denoted ζ, on $\mathbb{C} \setminus \{1\}$ into a function with a simple pole at $s = 1$. The Riemann hypothesis asserts that the real parts of all non-trivial zeros of ζ are equal to $1/2$.

where $\varphi_2(s, y)$ is the second derivative in s of $\ln \zeta(s, y)$.

A priori this formula provides only little directly exploitable information since the parameter α defined by (8) looks rather mysterious. However, by the prime number theorem Hildebrand and Tenenbaum obtained precise asymptotic estimates for $\alpha = \alpha(x, y)$. This enabled them to recover, and actually extend, the results of Hildebrand mentioned in the previous section. Moreover, they determined very precise estimates for the local behaviour for $\Psi(x, y)$. For example, provided y tends to $+\infty$, they proved that $\alpha(x, y) = o(1)$ if and only if $y \leqslant (\ln x)^{1+o(1)}$ where now the "$o(1)$" refers to x. A consequence is that the asymptotics $\Psi(2x, y) \sim \Psi(x, y)$ holds if, and only if, $y \leqslant (\ln x)^{1+o(1)}$.

Very recently, La Bretèche and Tenenbaum [4] obtained new estimates for $\Psi(x, y)$ in the critical range $1 \leqslant y \leqslant (\ln x)^{1+o(1)}$ elucidating completely the behaviour of this function in this region. Their results highlight some important discontinuities of $\Psi(x, y)$ when y is a prime number of size $o((\ln x)^{2/3}(\ln_2 x)^{1/3})$. The extremely friable integers are not smooth.

3 Some properties of friable integers

Before setting out the applications in cryptography and in other problems of analytic number theory, we address the following question: In what respect do friable integers behave like ordinary integers?

We will employ several criteria frequently used in analytic number theory: distribution in short intervals, in arithmetic progressions, and restricted mean values of multiplicative functions. The last paragraph in this section is devoted to the Turán–Kubilius inequality, an important tool in probabilistic number theory.

3.1 Distribution in short intervals

We expect that friable integers are evenly distributed in short intervals, that is,

$$\frac{\Psi(x + z, y) - \Psi(x, y)}{z} \sim \frac{\Psi(x, y)}{x} \tag{9}$$

in a large range for x, y, z. Hildebrand obtained an estimate of type (9) in the domain (5) and provided the interval length z satisfies $xy^{-5/12} \leqslant z \leqslant x$. Hildebrand and Tenenbaum established asymptotic estimates in larger domains in y but for longer intervals $]x, x + z[$.

For shorter intervals, Friedlander and Lagarias proved that there exists a constant $c > 0$ such that for all fixed $\alpha > 0$ and $\beta > 1-\alpha-c\alpha(1-\alpha)$, the interval $[x, x+x^\beta]$ contains a positive proportion of x^α-friable integers. Other results on the distribution of friable integers in short intervals appear in [13] and [11]. Very recently, Matomäki and Radziwiłł [15] achieved spectacular progress: they showed that for all $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that, for large enough x, the interval $[x, x + C(\varepsilon)\sqrt{x}]$ contains at least $\sqrt{x}/(\ln x)^4$ x^ε-friable integers. We will see in Section 5 that exhibiting friable integers in such small intervals is an important step in several factoring algorithms. For example, this is the case for the quadratic sieve.

3.2 Friable integers in arithmetic progressions

We denote by $\Psi(x, y; a, q)$ the number of y-friable integers not exceeding x and congruent to a modulo q and we let $\Psi_q(x, y)$ stand for the cardinality of y-friable integers $\leqslant x$ and coprime to q. When $(a, q) \neq 1$ and q is y-friable, this cardinality is equal to $\Psi(x/d, y; a/d, q/d)$ where we have put $d = (a, q)$. It is thus sufficient to restrict to the case $(a, q) = 1$. Assuming good distribution in the invertible classes modulo q, we expect that, for $(a, q) = 1$,

$$\Psi(x, y; a, q) \sim \frac{\Psi_q(x, y)}{\varphi(q)}, \tag{10}$$

where φ is the Euler totient function: $\varphi(n) = \#(\mathbb{Z}/n\mathbb{Z})^*$. Here again there are two goals: to find estimates in domains as large as possible in terms of both q and y. In fact, determining the main term $\Psi_q(x, y)$ is already a difficult problem.

In this direction, one can find very nice results in the literature such as the works of Fouvry and Tenenbaum [9], Granville [10], and recently the articles by Harper [12] and Drappeau [8]. Harper proved that (10) holds provided the ratio $\ln x/\ln q$ tends to ∞ and $q \leqslant y^{4\sqrt{e}-\varepsilon}$, $y \geqslant y_0(\varepsilon)$.

However the condition $\ln x/\ln q \to \infty$ is very restricting: it doesn't cover the case $q \approx x^\alpha$ even for very small $\alpha > 0$. This obstacle can be circumvented by considering the distribution on average over q. Indeed, Harper proved that, for suitable $c > 0$, relation (10) is satisfied for all $(\ln x)^c \leqslant y \leqslant x$ and almost all $q \leqslant \sqrt{\Psi(x, y)}(\ln x)^{-7/2}$. Furthermore, Drappeau obtained a similar result on average for $q < x^{3/5-\varepsilon}$ but with a weaker control of the uniformity in the classes a modulo q and for a friability range of type $(\ln x)^c \leqslant y \leqslant x^{c'}$ where $c' > 0$ is a very small constant.

3.3 Multiplicative functions and friable integers

In analytic number theory, one often faces the problem of estimating sums of the type

$$\Psi_f(x, y) := \sum_{n \in S(x,y)} f(n)$$

where f is a multiplicative function such that $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Some typical examples are the τ function which furnishes the number of divisors, Dirichlet characters, the Möbius function which will be defined in Section 5.1, $r(n)$ the number of representations of n as a sum of two squares and $\varrho_P(n)$ the number of roots of P modulo n for a given polynomial $P \in \mathbb{Z}[X]$. When f is an oscillating function such that $\sum_{n\leqslant x} f(n) = o(x)$, as for example the Möbius function or Dirichlet characters, one aims to obtain the largest possible region in y for which $\Psi_f(x, y) = o(\Psi(x, y))$. When the function f is non-negative, one hopes for asymptotic formulas. In applications, the values $f(p)$ are often close on average to a real number κ: $\kappa = 2$ for $f = \tau$, $\kappa = 1$ for $f = \varrho_P$ if P is irreducible, etc.

Under such hypotheses Tenenbaum and Wu [20] proved formulas of the type

$$\Psi_f(x, y) = C_\kappa(f)x\varrho_\kappa(u)\ln(y)^{\kappa-1}(1 + E(x, y)),$$

where ϱ_κ is the fractional convolution power of order κ of the Dickman function,[4] $C_\kappa(f)$ is a convergent Eulerian product

4 The function ϱ_κ is continuous on $]0, \infty[$, differentiable on $[1, \infty[$, and satisfies the delay equation $u\varrho_\kappa'(u) + (1 - \kappa)\varrho_\kappa(u) + \kappa\varrho_\kappa(u - 1) = 0$ for $u > 1$, with initial condition $\varrho_\kappa(u) = u^{\kappa-1}/\Gamma(\kappa)$ for $0 < u \leqslant 1$.

depending on f and κ, and $E(x, y)$ is an error term, which for brevity we do not define here. Under very general conditions, we have $E(x, y) = o(1)$ in ranges analogous to the Hildebrand region (5) for $\Psi(x, y)$.

3.4 The Turán–Kubilius inequality

In analytic number theory, one frequently needs to evaluate the *normal* behaviour of a given arithmetic function, i.e., the behaviour for almost all integers n, or again on a set of integers having natural density 1.[5] In particular, one would like to know whether there exists a function g with any prescribed regularity such that $|f(n) - g(n)|$ is very small for almost all integers n. In this case we say that g is a normal order of f. A famous example is the Hardy and Ramanujan theorem stating that $g(n) = \ln_2 n$ is a normal order for either of the functions $\omega(n)$ and $\Omega(n)$, giving the total number of prime factors of n, counted with or without multiplicity.

In many instances, a good candidate for g is the mean value of f, that is,

$$g(N) = E_N(f) = \frac{1}{N}\sum_{n\leqslant N} f(n).$$

We enter the domain of probabilistic number theory. An often efficient method consists in evaluating the variance

$$V_N(f) = E_N(|f(n) - E_N(f)|^2)$$

and applying the Bienaymé–Chebyshev inequality. We need however to be able to evaluate this variance and approximate the mean.

The Turán–Kubilius inequality provides a bound for the variance of additive functions. An arithmetic function h is said to be additive if $h(mn) = h(m)+h(n)$ whenever $(m, n) = 1$. The functions $\ln n$, $\omega(n)$, $\Omega(n)$, are prototypes of additive functions. Such functions are determined by their values on prime powers. The reader will find in [19] a detailed construction of the probabilistic model that can be attached to an additive function. Approximating the probability that an integer is divisible by p^k and not by p^{k+1} by $1/p^k - 1/p^{k+1} = (1-1/p)p^{-k}$, it is reasonable to expect that, for additive f, the mean value $E_N(f)$ is close to $E_N^*(f) := \sum_{p^\nu\leqslant N}(1 - 1/p)f(p^\nu)/p^\nu$. A corresponding approximation of the variance becomes

$$V_N^*(f) = \frac{1}{N}\sum_{1\leqslant n\leqslant N}\left|f(n) - E_N^*(f)\right|^2.$$

The Turán–Kubilius inequality states that, uniformly for all additive, complex-valued f, we have

$$V_N^*(f) \leqslant \left\{4 + O\left(\sqrt{\frac{\ln\ln N}{\ln N}}\right)\right\}B_N(f)^2,$$

where $B_N(f)^2$ is the corresponding approximation of the second moment:

$$B_N(f)^2 = \sum_{p^\nu\leqslant N}\frac{|f(p^\nu)|^2}{p^\nu}\left(1 - \frac{1}{p}\right).$$

As an immediate application we get a quantitative version of the Hardy–Ramanujan theorem quoted above. La Bretèche and Tenenbaum [3] extended this inequality to friable integers. For $p \leqslant y$, the probability that a friable integer is exactly divisible by p^k is close to $(1 - 1/p^\alpha)p^{-k\alpha}$ where α is the saddle point defined by (8). The expectation, variance, and second order moment associated to this probabilistic model for friable integers are respectively

$$E_N^*(f, y) = \sum_{p^\nu\in S(N,y)}\frac{f(p^\nu)}{p^{\alpha\nu}}\left(1 - \frac{1}{p^\alpha}\right),$$

$$V_N^*(f, y) = \frac{1}{\Psi(N, y)}\sum_{n\in S(N,y)}|f(n) - E_N^*(f, y)|^2,$$

$$B_N(f, y)^2 = \sum_{p^\nu\in S(N,y)}\frac{|f(p^\nu)|^2}{p^{\alpha\nu}}\left(1 - \frac{1}{p^\alpha}\right).$$

La Bretèche and Tenenbaum proved that there exists an absolute constant $C > 0$ such that for all $2 \leqslant y \leqslant N$, we have

$$V_N^*(f, y) \leqslant CB_N(f, y)^2.$$

Furthermore, they established in the same optimal range the more precise inequality

$$V_N^*(f, y) \ll V(Z_f),$$

where $V(Z_f)$ is the variance of the probabilistic model Z_f associated to the additive function f on the set of friable integers. This theorem has very nice consequences for properties of friable integers. In this survey, we will present only one but the reader will find other interesting applications in [3]. Let us denote by $\{p_j(n)\}_{1\leqslant j\leq\omega(n)}$ the increasing sequence of the prime factors of an integer n. A very surprising fact is that, for almost all integers n, the order of magnitude of $p_j(n)$ depends only on j: for almost all $n \leqslant x$, we have $\ln_2(p_j(n)) \sim j$ for $J_x \leqslant j \leqslant \omega(n)$, where J_x is any function tending to infinity with x.

The friable Turán–Kubilius inequality provides a normal order for these quantities $p_j(n)$ when n runs through friable integers. La Bretèche and Tenenbaum proved that "small" prime factors of friable integers behave similarly to those of ordinary integers but that, after a certain critical index, one observes an increasing compression phenomenon. In particular, for almost all $n \in S(x, y)$ and $y \leqslant (\ln x)^{1+o(1)}$, J_x as above, we have $p_j(n) = p_j^{1+o(1)}$ for $J_x \leqslant j \leqslant \omega(n)$, where p_j is the jth prime number. Since, by the prime number theorem, $p_j \sim j\ln j$, the situation is thus very different from that of generic, normal integers.

4 Applications to algorithmic number theory and cryptography

Security of a variety of public-key cryptographic systems relies on the difficulty of factoring integers whose prime factors are large. For example, the public key in the RSA system is an integer N that is the product of two "large" prime factors, i.e., $N = pq$. Decoding is equivalent to determining the prime factors p and q.

Friable integers play a prominent role in a number of factoring algorithms, in particular in the process of finding the above p and q. Friable integers are also needed in the discrete logarithm problem and in some primality testing. In this section we provide a general idea of their use in some of these algorithms. We have selected situations that can be described

5 A subset $A \subset \mathbb{N}$ is of natural density 1 if

$$\lim_{N\to\infty}\frac{1}{N}\#\{n \leqslant N : n \in A\} = 1.$$

with little mathematical background only. In particular we will not evoke the factoring algorithms and the primality testing based on elliptic curves which are nevertheless most frequently used.

The role of the friable integers in these algorithms is analogous to the one that will be depicted in this section even if the context is different. We refer for example to Pomerance's survey in the International Mathematical Congress in Zurich in 1994 [17] and to the book by Crandall and Pomerance [5], which are important references in this context.

4.1 The quadratic sieve

The quadratic sieve was devised by Pomerance at the beginning of the eighties. We aim to find the prime factors of an integer n that we know to be composite. The starting idea is that if a and b are two integers such that $a \not\equiv \pm b \pmod n$ and $a^2 \equiv b^2 \pmod n$ then $(a-b, n)$ will be a non-trivial divisor of n. The challenge is thus to determine such integers a and b.

The first step of the quadratic sieve consists in considering the y-friable values of the polynomial $Q(t) = t^2 - n$ when t is close to $\sqrt{n}$, that is, for $|t - \sqrt{n}| < n^{\varepsilon}$ with some small $\varepsilon > 0$. For such t, $Q(t)$ will be small: $|Q(t)| \leqslant (2 + n^{-1/2+\varepsilon})n^{1/2+\varepsilon} < 3n^{1/2+\varepsilon}$. It is natural to expect that these values have multiplicative properties resembling those of the integers in the interval $]-3n^{1/2+\varepsilon}, 3n^{1/2+\varepsilon}[$. Why do we need so many friable values $Q(t)$? We write as $(t_1, Q(t_1)), \dots, (t_N, Q(t_N))$ the pairs for which $Q(t_i)$ is friable. The second idea of Pomerance is that if we have at least $N \geqslant \pi(y)+1$ such y-friable integers then we can build a square with these $Q(t_i)$. This can be seen with a linear algebraic argument. The factoring of each $Q(t_i)$ is of the shape $Q(t_i) = \prod_{p \leqslant y} p^{v_p(Q(t_i))}$ where $v_p(a)$ denotes the p-adic valuation of a. We can associate to each $Q(t_i)$ a vector in $\mathbb{F}_2^{\pi(y)}$ whose coordinates are the $v_p(Q(t_i)) \mod 2$, $p \leqslant y$. Since $N > \pi(y)$, the number of vectors is strictly bigger than the dimension. Thus these vectors are not independent, and there exists $J \subset \{1, \dots, N\}$ such that $\sum_{j \in J} v_p(Q(t_j)) \equiv 0 \mod 2$ for all $p \leqslant y$, in other words such that $\prod_{j \in J} \prod_{p \leqslant y} p^{v_p(Q(t_j))}$ is a square denoted by v^2. We will get

$$\Big(\prod_{j \in J} t_j\Big)^2 \equiv \prod_{j \in J} (t_j^2 - n) \equiv v^2 \pmod n.$$

If $\prod_{j \in J} t_j \not\equiv \pm v \pmod n$, $(v - \prod_{j \in J} t_j, n)$ is a non-trivial divisor of n detected with friable integers. Using (6) and under the assumption that the values of the previous polynomial behave like generic integers in the interval $]-3n^{1/2+\varepsilon}, 3n^{1/2+\varepsilon}[$, Pomerance proved that the complexity of the quadratic sieve is $L(n)^{1+o(1)}$ with $L(n) = \exp(\sqrt{\ln n \ln_2 n})$; $L(n)^{\sqrt{2}/2}$ is in fact the optimal friable limit for this algorithm.

4.2 The number field sieve

The quadratic sieve is usually used for factoring integers with less than one hundred digits. For larger integers the number field sieve, to which this section is devoted, is available.

Let n be a number to factorise. We first determine a polynomial f of degree $d \geqslant 2$ such that $f(m) \equiv 0 \pmod n$ for some integer m close to $n^{1/d}$. For example, for $m = \lfloor n^{1/d} \rfloor$, we can form the polynomial f with the expansion of n in base m: $n = m^d + c_{d-1}m^{d-1} + \cdots + c_1 m + c_0$, where the digits c_j lie between 0 and $m-1$. Next, we consider the polynomial $f(X)$. If it is reducible, we immediately obtain a factor of n. We can thus assume this does not hold. Let $\vartheta \in \mathbb{C}$ be a root of $f(X)$. We try to find a set S of pairs of coprime integers (a, b) such that

$$\prod_{(a,b) \in S} (a - b\vartheta) = \gamma^2, \quad \prod_{(a,b) \in S} (a - bm) = v^2 \tag{11}$$

for some $\gamma \in \mathbb{Z}[\vartheta]$, $v \in \mathbb{Z}$. The process to derive such squares is similar to the one described in the case of the quadratic sieve. The first step consists in finding pairs (a, b) such that $(a - bm)$ and $N(a - b\vartheta)$ are friable where N is the norm on $\mathbb{Q}(\vartheta)$.

Assuming again that the values of the considered polynomials behave like random integers, Buhler, H. Lenstra, and Pomerance proved that the complexity of the number field sieve is $\ll \exp(c(\ln n)^{1/3}(\ln_2 n)^{2/3})$, this bound being achieved for polynomials of degree $d \sim \left(\frac{3 \ln n}{\ln_2 n}\right)^{1/3}$. These results are based in particular on conjectures on the distribution of friable integers in short intervals and in polynomial sequences. Currently these conjectures are out of reach, especially in the case of the number field sieve where the degree of the optimal polynomial is very high.

These last years, much research has been devoted to this subject. We saw above that binary forms of type

$$F(a, b) = (a - bm)N(a - b\vartheta)$$

play an important role in the number field sieve. For general binary forms $F \in \mathbb{Z}[X_1, X_2]$, Balog, Blomer, Tenenbaum, and the author established some lower bounds for

$$\Psi_F(x, y) = \Big|\{1 \leqslant a, b \leqslant x : P^+(F(a, b)) \leqslant y\}\Big|$$

when $y \geqslant x^{\alpha_F + \varepsilon}$ where α_F depends on the structure of F. In the case of irreducible binary forms, the exponent

$$\alpha_F = \deg F - 2$$

is admissible. Lachand [14] obtained asymptotic formulas valid in domains with $y = x^{o(1)}$ when f is a cubic or a product of linear terms (with an explicit expression of the previous y-exponent "o(1)" in the cubic case).

4.3 The discrete logarithm problem

The discrete logarithm is used in many cryptographic protocols. Let p be a large prime number, g a generator of $\mathbb{F}_p^*$, and $t \in \mathbb{F}_p^*$. The discrete logarithm problem[6] consists in determining ℓ such that $g^\ell = t$. We then write $\ell = \log_g t$. We start out by selecting those powers g^m having a y-friable representative. If we can find sufficiently many such powers, a linear algebra argument will enable us to determine the discrete logarithms of the primes $q \leqslant y$. After this stage, we consider the products $g^m t$ where m is a random integer. If one of the $g^m t$ is y-friable, thus of type $g^m t = \prod_{i=1}^r q_i^{a_i}$, with all the $q_i \leqslant y$, we will deduce that $\log_g t = -m + \sum_{i=1}^r a_i \log_g(q_i)$.

5 Applications of friable integers to analysis and number theory

On many occasions, friable integers opened new perspectives in problems that had remained out of reach for decades. In this section we briefly expose their use in various contexts.

6 We can work in a more general context by replacing $\mathbb{F}_p^*$ with a cyclic group.

5.1 The prime number theorem, Daboussi's theorem for multiplicative functions

In the first half of the last century, many mathematicians were convinced that an elementary proof of the prime number theorem was not possible; the qualification 'elementary' means here using only the usual tools of real analysis, and among other things avoiding complex analysis.

It was a huge surprise when Erdős and Selberg provided in 1949 an elementary but rather difficult proof of the prime number theorem.

In 1984, Daboussi [6] gave a very elegant proof by using friable integers. Let μ denote the Möbius function. This function is defined in the following way: $\mu(n) = 0$ if n is divisible by the square of a prime number, otherwise $\mu(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n. A classical result in number theory asserts that the prime number theorem is equivalent to the formula

$$M(x) := \sum_{n\leqslant x} \mu(n) = o(x). \tag{12}$$

One of the ideas of Daboussi is to represent $M(x)$ in terms of sums of the Möbius function over friable integers, viz.

$$M(x, y) := \sum_{n\in S(x,y)} \mu(n).$$

Writing $n = ab$ where a is y-friable and b is y-sifted, we arrive at the formula

$$M(x) = \sum_{P^-(b)>y} \mu(b)M(x/b, y).$$

The other steps follow a more natural progression than the initial proof of Erdős and Selberg. The main ingredients are very simple estimates on sifted and on friable integers, the crucial point being the upper bound of some kind of mean value of the $M(x, y)$.

This limpid process of sifted–friable factorization combined with convolution[7] methods may be used to produce a new proof of the following theorem due to Daboussi: if f is a multiplicative function (i.e., $f(mn) = f(m)f(n)$ when m and n are coprime) with modulus at most 1 then for all real irrational α, we have

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n) \exp(2i\pi n\alpha) = 0.$$

5.2 Friable integers and Waring's problem

Waring's problem consists in determining, given an integer $k \geqslant 2$, the smallest integer s such that all natural numbers can be represented as a sum of s kth powers. In a slightly weaker version one only asks that all sufficiently large enough integers are representable. In this latter version the number s is traditionally denoted by $G(k)$. For example, Linnik showed that $G(3) \leqslant 7$, and Davenport proved that $G(4) = 16$.

The circle method is a direct approach in which one expresses the number $R(n)$ of representations of any integer n as sum of s kth powers by a Cauchy or a Fourier integral:

$$R(n) = \int_0^1 F(\alpha)^s \exp(-2i\pi n\alpha)\,\mathrm{d}\alpha,$$

$$\text{with } F(\alpha) = \sum_{m\leqslant n^{1/k}} \exp(2i\pi m^k\alpha).$$

The main contribution to this integral arises from the so-called *major arcs* corresponding to those α close to a rational number with small denominator. Traditionally, the set of these α is denoted by $\mathcal{M}$, and its complement $\mathfrak{m} =]0, 1[\smallsetminus\mathcal{M}$ is called the set of *minor arcs*.

An important feature of this method consists in showing that the contribution from the minor arcs is negligible. One can use bounds of type

$$\int_{\mathfrak{m}} |F(\alpha)|^s \,\mathrm{d}\alpha \leqslant \max_{\alpha\in\mathfrak{m}} |F(\alpha)|^{s-2\ell} \int_0^1 |F(\alpha)|^{2\ell}\,\mathrm{d}\alpha,$$

with $2\ell \leqslant s$. The integral on the right-hand side then corresponds to the number of solutions of the Diophantine equation

$$x_1^k + \cdots + x_\ell^k = y_1^k + \cdots + y_\ell^k \quad (1 \leqslant x_i, y_i \leqslant n^{1/k}).$$

By restricting some variables to be friable, Wooley [21] settled functional inequalities between the number of solutions of a Diophantine equation in ℓ friable variables x_i, y_i and the number of solutions of an equation in $\ell-1$ variables. The process is very complicated and not suitable for a short account, however Wooley obtained in this way many important new results, in particular that $G(k) \leqslant k \ln k + k \ln_2 k + O(k)$ as $k \to \infty$. The best previously know bound was $\leqslant (2 + o(1))k \ln k$.

5.3 Friable summation and Davenport identities

Duffin and then Fouvry and Tenenbaum [9] introduced a new summation method, called friable summation or P-summation. It consists of ordering the indexes of a series according to their largest prime factor and then studying convergence as the friable parameter approaches infinity. This leads to the following definition: a series $\sum \alpha_n$ is said to have friable sum α (or to be P-convergent to α) if

$$\sum_{P^+(n)\leqslant y} \alpha_n = \alpha + o(1) \quad (y \to +\infty).$$

A series already summable for the friable method is called *regular* if it is convergent in the usual sense and its friable sum α is equal to its ordinary sum:

$$\lim_{y\to+\infty} \sum_{P^+(n)\leqslant y} \alpha_n = \alpha = \lim_{N\to+\infty} \sum_{n=1}^{N} \alpha_n.$$

It may happen that the friable sum exists but does not coincide with the usual sum. Theorem 11 of [9] provides infinitely many examples. While, for $\vartheta \in \mathbb{R} \smallsetminus \mathbb{Z}$, the series $\sum_{n\geqslant1} \exp(2i\pi n\vartheta)/n$ converges (in the usual sense) to

$$\log\Big(\frac{1}{1 - \exp(2i\pi\vartheta)}\Big),$$

where, here and in the sequel to this paragraph, the complex logarithm is understood as the principal determination, Fouvry and Tenenbaum proved that, for all rationals $\vartheta = a/q$ with $(a, q) = 1$, $1 \leqslant a < q$ the friable sum exists and has value

$$\lim_{y\to\infty} \sum_{P(n)\leqslant y} \frac{\exp(2i\pi na/q)}{n} = \log\Big(\frac{1}{1 - \exp(2i\pi a/q)}\Big) + \frac{\Lambda(q)}{\varphi(q)},$$

where Λ is the Von Mangoldt function, defined by $\Lambda(q) = \ln p$ if $q = p^k$, $\Lambda(q) = 0$ otherwise.

Friable summation avoids the Gibbs phenomenon and can be employed to solve arduous questions in analysis such as

7 The convolution product of two arithmetic functions is defined in Section 5.3.

the problem of Davenport's identities that we will describe now.

The Dirichlet convolution product of two arithmetic functions u, v is defined by the formula

$$u * v(n) = \sum_{d|n} u(d)v(n/d) \qquad (n \geqslant 1).$$

The identity element for this composition law in the ring of arithmetic functions is often denoted by δ, so that $\delta(n) = 1$ for $n = 1$ and $\delta(n) = 0$ for $n \geqslant 2$. We denote by $\mathbf{1}$ the constant function equal to 1 for all positive integers. We then have the fundamental formula $\delta = \mu * \mathbf{1}$.

Let $B_1(t)$ be the first Bernoulli function, defined by $B_1(t) = \{t\} - 1/2$ if $t \notin \mathbb{Z}$ and $B_1(t) = 0$ for $t \in \mathbb{Z}$, where $\{t\}$ denotes the fractional part of t. It coincides everywhere with its Fourier series:

$$B_1(\vartheta) = -\sum_{k \geqslant 1} \frac{\sin(2\pi k\vartheta)}{\pi k}.$$

With this equation, we formally obtain for two arithmetic functions f and g such that $f = g * \mathbf{1}$, the beautiful identity

$$\sum_{m \geqslant 1} \frac{f(m)}{\pi m} \sin(2\pi m\vartheta) + \sum_{n \geqslant 1} \frac{g(n)}{n} B_1(n\vartheta) = 0. \qquad (13)$$

This led Davenport to formulate the following problem: given f and g, determine those real numbers ϑ such that relation (13) holds. This is a very difficult task and we do not provide a general answer.

Davenport proved that, in the case $(f, g) = (\delta, \mu)$, identity (13) is satisfied for all $\vartheta \in \mathbb{R}$. However, his argument does not work in the emblematic cases $(f, g) = (\ln, \Lambda)$, $(\tau, \mathbf{1})$, where τ is the divisor counting function. It was only sixty years later that these cases were solved by La Bretèche and Tenenbaum [2]. One of the crucial ingredients of their work is the application of friable summation which is remarkably well adapted to this problem. They proved that for $(f, g) = (\ln, \Lambda)$, (13) holds for all real ϑ, but, in the case $(f, g) = (\tau, \mathbf{1})$, they gave a criterion in terms of the continued fraction expansion of the irrational ϑ for the validity of (13). The case of the powers of convolution of $\mathbf{1}$ has then been handled by B. Martin.

5.4 Small gaps between prime numbers

We end this survey with an account of spectacular progress obtained using friable integers: the works of Zhang [22] and Maynard [16] on small gaps between the prime numbers. Zhang caused a sensation in 2013[8] by showing that there are infinitely many prime numbers $p \neq q$ such that

$$|p - q| \leqslant 70\,000\,000.$$

After this breakthrough the upper bound was reduced several times, in particular by the Polymath project. At the end of this same year there was another spectacular result: Maynard announced that this bound could be reduced to 600. The current value is 246. The twin prime conjecture, which corresponds to infinitely many prime gaps equal to 2, seems less inaccessible than ten years ago.

A key ingredient of Zhang's proof is a result on the average distribution of the primes in arithmetic progressions with friable moduli. This friable structure enables one to consider sets of integers not exceeding x and satisfying some congruence conditions modulo integers larger than $\sqrt{x}$; this was crucial in Zhang's approach to prove bounded gaps between infinitely many prime numbers.

Very recently Régis de la Bretèche [1] wrote a fascinating article for the *Gazette des Mathématiciens* on the cooperative project Polymath around the breakthroughs by Zhang [22] and Maynard [16] on this subject. The interested reader can look at [1] (in French) for more details of this wonderful progress. Undoubtedly, very beautiful mathematics is yet to be discovered along the paths of friable integers.

8 The corresponding article appeared in 2014.

Acknowledgements

I am very grateful to Pierrick Gaudry, Martine and Hervé Queffélec, Anne de Roton, Gérald Tenenbaum, and to the two anonymous referees for their meticulous rereading and for all their remarks related to the French version in the *Gazette des Mathématiciens* [7]. I would also like to express warm thanks to Irène Marcovici and to Gérald Tenenbaum for all their help for this English version.

Bibliography

[1] R. de la Bretèche. Petits écarts entre les nombres premiers et polymath: une nouvelle manière de faire de la recherche en mathématiques? *SMF Gaz. des Math.*, 140:19–31, 2014.

[2] R. de la Bretèche and G. Tenenbaum. Séries trigonométriques à coefficients arithmétiques. *J. Anal. Math. 92*, 92:1–79, 2004.

[3] R. de la Bretèche and G. Tenenbaum. Entiers friables : inégalité de Turán–Kubilius et applications. *Invent. Math.*, 159:531–588, 2005.

[4] R. de la Bretèche and G. Tenenbaum. Une nouvelle approche dans la théorie des entiers friables. *Compos. Math.*, 153:453–473, 2017.

[5] R. Crandall and C. Pomerance. *Prime numbers, a computational perspective*, volume 4ème édition. Springer, 546 pp., 2001.

[6] H. Daboussi. Sur le théorème des nombres premiers. *C. R. Acad. Sc. Paris, Série I*, 298(8):161–164, 1984.

[7] C. Dartyge. Entiers friables : un tour d'horizon. *SMF Gaz. des Math.*, 156:29–39, 2018.

[8] S. Drappeau. Théorèmes de type Fouvry–Iwaniec pour les entiers friables. *Compos. Math.*, 151:828–862, 2015.

[9] E. Fouvry and G. Tenenbaum. Entiers sans grand facteur premier en progressions arithmétiques. *Proc. London Math. Soc. (3)*, 63:449–494, 1991.

[10] A. Granville. Integers, without large prime factors, in arithmetic progressions. I. *Acta Math.*, 170:255–273, 1993.

[11] A. Granville. Smooth numbers: computational number theory and beyond. *Algorithmic number theory, MSRI Proceedings*, 44:267–323, 2008.

[12] A. J. Harper. On a paper of K. Soundararajan on smooth numbers in arithmetic progressions. *J. Number Theory*, 132(1):182–199, 2012.

[13] A. Hildebrand and G. Tenenbaum. Integers without large prime factors. *Journal de Théorie des Nombres de Bordeaux*, 5(2):411–484, 1993.

[14] A. Lachand. *Entiers friables et formes binaires*. Thèse, Université de Lorraine, 2014.

[15] K. Matomäki and M. Radziwiłł. Multiplicative functions in short intervals. *Ann. of Math. (2)*, 183(3):1015–1056, 2016.

[16] J. Maynard. Small gaps between primes. *Ann. of Math. (2)*, 183(1):383–413, 2015.

[17] C. Pomerance. The role of smooth sumbers in number theoretic algorithms. *Proceedings of the international congress*

of mathematiciens, Zurich Switzerland 1994, 5(2):411–422, 1995.
[18] G. Tenenbaum. *Introduction à la théorie analytique et probabiliste des nombres*, volume 4ème édition. coll. Échelles, Belin, 592 pp., 2015.
[19] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*, volume Third edition. Graduate Studies in Mathematics 163, Amer. Math. Soc., 2015.
[20] G. Tenenbaum and J. Wu. Moyennes de certaines fonctions multiplicatives sur les entiers friables. *J. Reine Angew. Math.*, 564:119–166, 2003.
[21] T. D. Wooley. Large improvements in Waring's problem. *Ann. of Maths.*, 135:131–164, 1992.
[22] Y. Zhang. Bounded gaps between primes. *Ann. of Math. (2)*, 179(3):1121–1174, 2014.

Cécile Dartyge [cecile.dartyge@univ-lorraine.fr] is maîtresse de conférences in the Institut Elie Cartan, Université de Lorraine, France. Her actual research interests are in analytic number theory.

Some Recent Interactions of Probability and Number Theory

Corentin Perret-Gentil (Université de Montréal, Canada)

Around eighty years after its birth, the field of probabilistic number theory continues to see very interesting developments. On the occasion of a thematic program on the subject that took place last May in Montréal, we give a brief survey of a (far from exhaustive) selection of recent advances.

1 Introduction

While many number-theoretical questions are statistical in nature, *probabilistic number theory* is usually understood as the use of probabilistic techniques or ideas in number theory [44], analogously to what analysis is to analytic number theory. The association of the two latter was notably proved fruitful in Hadamard and de la Vallée Poussin's 1896 proof of the prime number theorem

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty), \tag{1}$$

that gives an asymptotic for the number $\pi(x)$ of prime numbers $p \le x$.

The origins of probabilistic number theory can be traced back to Turán's new proof [47], in 1934, of the result by Hardy and Ramanujan from 1917 on the normal order of ω, the "number of prime divisors" function: for any $\varepsilon > 0$,

$$|\omega(n) - \log\log n| \le (\log\log n)^{1/2+\varepsilon}$$

for almost all integers n (that is, the proportion of integers $n \le N$ such that this does not hold goes to 0 as $N \to \infty$). Turán's argument can be seen as based on Chebyshev's inequality

$$\mathrm{Prob}\left(|X - \mathbb{E}(X)| \ge \alpha\sqrt{\mathrm{Var}(X)}\right) \le 1/\alpha^2,$$

for any $\alpha > 0$ and X a random variable with finite expected value $\mathbb{E}(X)$ and variance $\mathrm{Var}(X)$. However, as Elliott reports [11, Ch. 12], Turán did not realise it before a letter from the probabilist Mark Kac. It is relevant to recall here that, at the time of Turán's paper, Kolmogorov's axiomatisation of probability theory had just been published, in 1933.

Still according to [11], Kac asked Turán whether he could compute higher asymptotic moments of ω, maybe suggesting that ω had a Gaussian limiting probability distribution. Using the concept of independent random variables and the central limit theorem,[1] Erdős and Kac ([12], 1940) proved this and strengthened the Hardy–Ramanujan theorem by showing that

$$f(n) = \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \qquad (n \in \mathbb{N})$$

has a standard normal limiting probability distribution, i.e.,

$$\frac{|\{n \le x : f(n) \le z\}|}{x} \xrightarrow{x\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}dt =: \Phi(z)$$

for any $z \in \mathbb{R}$. Along with the Erdős–Wintner theorem (1939) on limiting distributions of additive functions on the integers, this can be seen as the beginning of probabilistic number theory. We refer the reader to W. Schwarz's survey [44] for an account of the main developments that followed.

Herein, we would like to give a brief survey of a (far from exhaustive) selection of recent works that use concepts such as martingales, suprema of Gaussian and log-correlated processes, orderings of weakly correlated random variables, normal approximations, large deviation estimates, comparison inequalities, and random Fourier series, to obtain significant results or insights in number theory.

1 A proof using the method of moments was given by Halberstam in 1955.

2 Random multiplicative functions

We recall that the Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined by $\mu(n) = (-1)^{\omega(n)}$ if n is square-free and $\mu(n) = 0$ otherwise. It is multiplicative, and the prime number theorem (1) is equivalent to

$$M_\mu(x) := \sum_{n\le x} \mu(n) = o(x) \quad \text{and} \quad \sum_{n\le x} \frac{\mu(n)}{n} = o(1)$$

as $x \to \infty$. The summatory function $M_\mu(x)$ is called Mertens' function, and it turns out that the Riemann hypothesis is equivalent to $M_\mu(x) = O(x^{1/2+\varepsilon})$ for any $\varepsilon > 0$. On the other hand, we have

$$\overline{\lim_{x\to\infty}} M_\mu(x)/\sqrt{x} > 0 \quad \text{and} \quad \underline{\lim_{x\to\infty}} M_\mu(x)/\sqrt{x} < 0,$$

and an unpublished conjecture of Gonek states that

$$\overline{\lim_{x\to\infty}} \frac{M_\mu(x)}{\sqrt{x}(\log\log\log x)^{5/4}} = \pm A$$

for some constant $A > 0$ (see [40]).

As a heuristic for sums of multiplicative functions such as $M_\mu(x)$, Wintner ([48], 1944) studied sums of *Rademacher random multiplicative functions* f: for $f(p)$ a sequence of independent identically distributed (iid) random variables indexed by primes, uniform on $\{-1, 1\}$, we let

$$f(n) := \begin{cases} \prod_{p\mid n} f(p), & n \text{ square-free}, \\ 0 & \text{otherwise}. \end{cases}$$

Wintner showed that for any $\varepsilon > 0$,

$$M_f(x) := \sum_{n\le x} f(n) \ll_\varepsilon x^{1/2+\varepsilon} \qquad \text{almost surely}$$

as $x \to \infty$, most recently improved by Lau–Tenenbaum–Wu ([37], 2013) to $M_f(x) \ll_\varepsilon \sqrt{x}(\log\log x)^{2+\varepsilon}$ almost surely, which is to be compared with the law of the iterated logarithm

$$\overline{\lim_{n\to\infty}} \frac{\sum_{i=1}^n X_i}{\sqrt{2n\log\log n}} = 1 \qquad \text{almost surely,}$$

when X_i are iid with mean 0 and variance 1 (e.g., if no multiplicative structure were imposed on f).

On the other hand, Halász ([20], 1982) showed that there exists a constant $B > 0$ such that

$$M_f(x) \neq O\Big(\sqrt{x} e^{-B\sqrt{\log\log x \log\log\log x}}\Big) \qquad \text{almost surely.} \quad (2)$$

Alternatively, one may also define $f(p)$ to be uniform on the unit circle in $\mathbb{C}$, giving rise to *Steinhaus random multiplicative functions*.

Martingales and normal distributions of $M_f^{(k)}(x)$

Given the considerations above about a random multiplicative function f, it would be interesting to obtain information on the limiting distribution of $M_f(x)$ as $x \to \infty$. A simpler object is obtained by restricting the sum to integers having a fixed small number $k \ge 1$ of prime factors, that is,

$$M_f^{(k)}(x) := \sum_{\substack{n\le x \\ \omega(n)=k}} f(n).$$

When k is small, there is not as much multiplicative dependency among the values of f in the sum, and the problem may be more manageable than for the full sum $M_f(x)$. For example, when $k = 1$, the limiting distribution of $M_f^{(1)}(x)/\sqrt{x}$ is standard normal by the central limit theorem.

In 2009, Hough [29] showed

Theorem 1 ([29]). *For f Rademacher, $z \in \mathbb{R}$, and $k = k(x) = o(\log\log\log x)$, we have a normal limiting distribution:*

$$\mathbb{P}\left(\frac{M_f^{(k)}(x)}{\mathbb{E}\Big(M_f^{(k)}(x)^2\Big)} \le z \right) \xrightarrow{x\to\infty} \Phi(z). \qquad (3)$$

The proof proceeds, classically, by the method of moments. An important idea is that if n has few prime factors, then it should have a large one.

In 2013, Harper [23] significantly extended the range allowed for k by another method:

Theorem 2 ([23]). *For f Rademacher, $z \in \mathbb{R}$, and $k = k(x) = o(\log\log x)$, the limiting normal distribution* (3) *still holds.*

His idea, starting from an insight of Blei and Janson ([3], 2004), is to identify a *martingale difference sequence* and apply the central limit theorem for those due to McLeish ([39], 1974). Indeed, we can decompose $M^{(k)}(x) = \sum_{p\le x} M_p^{(k)}(x)$, where

$$M_p^{(k)}(x) := f(p) \sum_{\substack{n\le x/p \\ \omega(n)=k-1 \\ P(n)<p}} f(n),$$

for $P(n)$ the largest prime factor of n. By the linearity of expectation, it follows that

$$\mathbb{E}(M_p^{(k)}(x) \mid f(\ell) \quad (\ell < p \text{ prime})) = 0,$$

so that $\big(M_p^{(k)}(x)\big)_p$ is a martingale difference sequence with respect to the filtration $\big(\sigma(\{f(\ell) : \ell \le p \text{ prime}\})\big)_p$. Theorem 2 is then reduced to verifying the hypotheses of McLeish's result, which amounts to number-theoretical estimates that constitute most of the paper.

Using a version of Stein's method for normal approximation developed by Chatterjee ([9], 2008), Chatterjee and Soundararajan ([10], 2012) obtained a similar result for sums of f in short intervals.

Note that the range $k = \omega(n) = o(\log\log n)$ of Theorem 2 falls just short of the size of a typical integer given by the Erdős–Kac theorem (see Section 1). One may wonder how large k may be while keeping a normal limiting distribution (3). In the same article [23], Harper also gave the following negative result:

Theorem 3 ([23]). *Let $0 < \varepsilon < A$. The limiting normal distribution* (3) *does* not *hold if $\varepsilon \log\log x \le k = k(x) \le A\log\log x$.*

This is proved by showing that the expectation of a thresholded second moment does not converge to what it should, through a conditioning argument that allows a good estimation.

Lower bounds for suprema of Gaussian processes, and omega results for $M_f(x)$

Another interesting result by Harper ([21], 2013) is the following strong improvement to Halász's negative result (2):

Theorem 4 ([21]). *For f Rademacher and $\varepsilon > 0$, we have*

$$M_f(x) \neq O(\sqrt{x}(\log\log x)^{-2.5+\varepsilon}) \qquad \textit{almost surely.}$$

The main input is new general bounds for suprema of Gaussian processes. Indeed, Halász's proof of (2) shows that almost certain lower bounds on $|M_f(x)|$ can be obtained from lower bounds on

$$\sup_{t\geq 1} \exp\left(S(t,x,f) - \log t - \log\log(t+2)/2\right),$$

where

$$S(t,x,f) := \sum_{p\leq x} f(p)\frac{\cos(t\log p)}{p^{1/2+1/\log x}}.$$

Using a multivariate central limit theorem, $f(p)$ can be replaced by a sequence $g(p)$ of independent standard Gaussians. Moreover, t can essentially be assumed to lie in a finite set T.

To analyse the resulting process $(S(t,x,g))_{t\in T}$, Harper develops general lower bounds for upper tail probabilities

$$\mathbb{P}(\max_{t\in T} Z(t) \geq u), \qquad \text{with } Z(t) \text{ jointly standard normal,}$$

which can be non-trivial even when u is of moderate size, while existing results require u to be very large. The strategy is to first decompose and condition the probability, and then to apply several comparison inequalities, along with the known distribution of the maximum of a Brownian motion. This yields lower bounds on the resulting probabilities that depend on the correlations between the $Z(t)$, which are estimated in the case of the process above.

Moments of random multiplicative functions

Using the results of Harper [21] mentioned in the previous section, Harper, Nikeghbali, and Radziwiłł ([27], 2015) obtained lower bounds on the moments

$$N_f(x,k) := \mathbb{E}\left|M_f(x)\right|^k$$

of $M_f(x)$, improving on results of Bondarenko–Seip ([6], 2016):

Theorem 5 ([27]). *For f Rademacher or Steinhaus, as $x\to\infty$,*

$$N_f(x,1) \gg \sqrt{x}(\log\log x)^{-3+o(1)}. \tag{4}$$

In particular, for $k\in[0,1]$, $N_f(x,2k) \gg x^k(\log\log x)^{-6+o(1)}$.

A first application of the lower bounds from [21] gives that (4) holds for infinitely many x, and a more delicate argument yields the theorem.

The authors also compute certain moments asymptotically,[2] relying on a general result of La Bretèche (2001) on mean values of multiplicative functions:

Theorem 6 ([27]). *Let $k\geq 1$ be an integer. There exist explicit constants $C_k, D_k > 0$ such that, as $x\to\infty$ we have the following:*

1. *For f Steinhaus,*

$$N_f(x,2k) \sim C_k x^k (\log x)^{(k-1)^2}. \tag{5}$$

2. *For f Rademacher and $k\geq 3$,*

$$\mathbb{E}\left(M_f(x)^k\right) \sim D_k x^{k/2}(\log x)^{k(k-3)/2}.$$

From Theorems 5 and 6, they make the following guesses for the remaining moments:

2 This was also obtained independently by Granville and Soundararajan (unpublished), and by Heap and Lindqvist (2016).

Conjecture 7 ([27]). *For f Steinhaus and $k\in\mathbb{R}^+$, there exists a constant $C_k>0$ such that*
- *if $k\geq 1$, the asymptotic* (5) *still holds;*
- *if $k\in[0,1]$, then $N_f(x,2k)\sim C_k x^k$ as $x\to\infty$.*

In particular, the case $k=1$ would disprove Helson's conjecture ([28], 2010) that $M_f(x)$ exhibits more than square-root cancellation:

Conjecture 8 ([28]). *For f Steinhaus, as $x\to\infty$, $N_f(x,1) = o(\sqrt{x})$.*

This conjecture would be surprising from the point of view of a number-theoretical model, but it can be motivated as follows (see also [27, pp. 2–3], [28]): by definition of f, the statement is equivalent to

$$\lim_{T\to\infty}\int_0^T \left|\sum_{n\leq x} n^{-it}\right| dt = o(\sqrt{x}).$$

A first insight is that the inner sum is a multiplicative analogue of the Dirichlet kernel $\sum_{n\leq x} e^{2\pi int}$, whose L^1 norm on $[0,2\pi]$ is $\ll \log x$. A second one is that Bondarenko, Heap, and Seip ([5], 2015) showed that

$$\lim_{T\to\infty}\int_0^T \left|\sum_{n\leq x} n^{-1/2-it}\right| dt \ll (\log x)^{1/4+o(1)},$$

which is also stronger than square-root cancellation.

In two recent preprints [24,25], Harper announced explicit formulas for all the moments $\mathbb{E}|M_f(x)|^k$, with $k\in\mathbb{R}^+$ and f Rademacher or Steinhaus. In particular, for f Steinhaus,

$$N_f(x,1) \asymp \sqrt{x}(\log\log x)^{-1/4},$$

which (for $k=1$ and f Steinhaus) proves Helson's conjecture 8 and disproves Conjecture 7.

The first step in the computation of the moments is a careful passage to Euler products, reducing to the consideration of expected values of the form

$$\mathbb{E}\left(\left[\int_{-1/2}^{1/2} |F_x(1/2+it)|^2 dt\right]^{k/2}\right), \quad F_x(s) = \prod_{p\leq x}\left(1-\frac{f(p)}{p^s}\right)^{-1},$$

where the random variables $(\log|F_x(1/2+it)|)_{|t|\leq 1/2}$ are approximately Gaussian and have logarithmic covariance structure. Writing

$$\left|F_x(1/2+it)\right|^2 = e^{2h(t)} \quad \text{with} \quad h(t) = \log\left|F_x(1/2+it)\right|,$$

the second step draws links to *critical multiplicative chaos* to analyse these random Euler products.

3 Maximum of the zeta function on bounded intervals

Little is known about the maximum modulus

$$M(T) := \max_{0\leq t\leq T} |\zeta(1/2+it)|$$

of the Riemann zeta function on an initial interval of the critical line. The Lindelöf hypothesis (hence the Riemann hypothesis) implies that

$$M(T) \ll \exp\left(C\frac{\log T}{\log\log T}\right) \qquad \text{as } T\to\infty$$

for some constant $C > 0$. A conjecture of Farmer, Gonek, and Hughes ([13], 2007) states that

$$M(T) = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log T \log\log T}\right). \qquad (6)$$

Alternatively, one may also consider bounded intervals, for which Fyodorov, Hiary, and Keating ([17,18]) have proposed, based on numerical evidence and links with statistical mechanics the following conjecture:

Conjecture 9. *For t sampled uniformly in $[0, T]$,*

$$\max_{\substack{u\in\mathbb{R}\\ |u-t|\le 1}} \log|\zeta(1/2+iu)| = \log\log T - \frac{3}{4}\log\log\log T + X_T,$$

where X_T is a random variable converging weakly to an explicit distribution as $T \to \infty$.

Note that by Selberg's central limit theorem ([45], 1946), for $t \in [0, T]$ uniform,

$$\frac{\log|\zeta(1/2+it)|}{\sqrt{(1/2)\log\log T}}$$

converges in law to a standard normal random variable. Moreover, the maximum of independent normal random variables with mean 0 and variance $\frac{1}{2}\log\log T$ is

$$\log\log T - \frac{1}{4}\log\log\log T + O(1),$$

which is a summand $-\frac{1}{2}\log\log\log T$ away from Conjecture 9; the former would account for the non-independence. There is a multivariate version of Selberg's theorem by Bourgade ([7], 2010), with logarithmic correlations, but this is not enough to make this heuristic rigorous (see [2, p. 4]).

From the analysis of log-correlated random processes (more particularly *branching random walks*), Arguin, Belius, Harper, Radziwiłł, and Soundararajan have recently progressed towards Conjecture 9, or an analogue in a random model. This will be the subject of the following sections.

Supremum of log-correlated Gaussian random variables and leading-order term for a random model

As for conjecture (6) of Farmer–Gonek–Hughes, Conjecture 9 is based on modelling $|\zeta(1/2+is)|$ by the characteristic polynomial of a random unitary matrix.

In 2013, Harper [22] obtained the leading term $\log\log T$ for an analogous model based on random Euler products. The motivation for the model is the following, adapted from an argument by Soundararajan based on the work of Selberg:

Proposition 10 ([22])**.** *Under the Riemann hypothesis, for $T \ge 1$ large enough, there exists $H \subset [T, T+1]$ of relative measure ≥ 0.99 such that for all $t \in H$,*

$$\log|\zeta(1/2+it)| = \mathrm{Re}\left(\sum_{p\le T}\frac{1}{p^{1/2+it}}\frac{\log(T/p)}{\log T}\right) + O(1).$$

Since $(p^{-it})_{p \text{ prime}}$, for $t \in [0, T]$ uniform, converges as T goes to ∞, in the sense of finite distributions, to $(U_p)_{p\text{ prime}}$ for U_p iid uniform on the unit circle, this suggest the model

$$\tilde{M}_1(T) := \max_{h\in[0,1]}\sum_{p\le T}\frac{\mathrm{Re}(U_p p^{-ih})}{p^{1/2}} \qquad (7)$$

for $\max_{h\in[0,1]}\log|\zeta(1/2+i(t+h))|$ when t is uniform in $[0, T]$. Harper's main result is then essentially the following:

Theorem 11 ([22])**.** *As $T \to \infty$, $\tilde{M}_1(T) = (1+o_P(1))\log\log T$, where $o_P(1)$ stands for convergence to 0 in probability.*

More precisely, it is also shown that the second-order term should lie between $-2\log\log\log T$ and $-\frac{1}{4}\log\log\log T$. The upper bounds use tail bounds for sums of independent random variables of Talagrand ([46], 1995), while the lower bound uses the bounds from [21] mentioned in Section 2.

Branching random walks and Conjecture 9 for the model

Through a connection with branching random walks, Arguin, Belius, and Harper ([2], 2017) managed to obtain the second-order term for the model (7), thus improving Theorem 11:

Theorem 12 ([2])**.** *As $T \to \infty$, we have*

$$\tilde{M}_1(T) = \log\log T - \frac{3}{4}\log\log\log T + o_P(\log\log\log T),$$

where the error converges to 0 in probability when divided by $\log\log\log T$.

Let us assume that $\log T = 2^n$ for some integer $n \ge 1$, and for every $h \in [0, 1]$, let $X_n(h) = \sum_{p\le e^{2^n}}\mathrm{Re}(U_p p^{-ih})p^{-1/2}$. One can compute the covariances explicitly and check that for $h, h' \in [0, 1]$,

$$\mathbb{E}(X_n(h)X_n(h')) \approx \frac{1}{2}\begin{cases}(-1)\log|h-h'| & : |h-h'| \ge 2^{-n},\\ n\log 2 & : |h-h'| < 2^{-n}.\end{cases}$$

Decomposing

$$X_n(h) = \sum_{i=0}^{n} Y_i(h) \quad\text{with}\quad Y_i(h) := \sum_{2^{i-1}<p\le 2^i}\mathrm{Re}(U_p p^{-ih})p^{-1/2}$$

and letting $h \wedge h' = \lfloor -\log(|h-h'|)/\log 2\rfloor$, we have that $Y_i(h)$ and $Y_i(h')$ are almost perfectly correlated with variance $\sigma^2 \approx \log(2)/2$ if $i \le h \wedge h'$, and almost perfectly correlated if $i > h \wedge h'$. Moreover, the variation of the $X_n(h)$ is captured by 2^n equally spaced $h \in [0, 1]$.

This is similar to a branching random walk (or branching Brownian motion) on a binary tree of depth n, where iid Gaussian random variables Y_i with mean 0 and variance σ^2 are attached to every edge, and each of the 2^n leaves is associated with the unique random walk on the edges from the root. Indeed, $X_n(h)$ and $X_n(h')$ would correspond to leaves with lowest common ancestor at height $h \wedge h'$ (see Figure 1).

Bramson ([8], 1978) determined the maximum of a branching Brownian motion to be roughly

$$cn - \frac{3\sigma^2}{2c}\log n, \quad\text{where}\quad c = \sigma\sqrt{2\log 2}.$$

When $\sigma^2 = \log(2)/2$, this gives precisely the leading- and subleading-order terms predicted by Conjecture 9.

Working on this analogy, Arguin, Belius, and Harper obtained Theorem 12 using a method of Kistler ([32], 2015) to

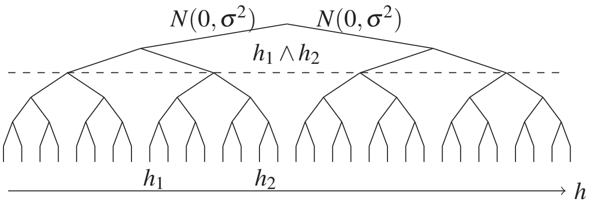


Figure 1. Branching Brownian motion.

handle processes like $X_n(h)$ that may not have an exact tree structure and where the Y_i may not be exactly Gaussian. Two particular pieces of information required by the method are large deviation estimates and Berry–Esseen approximations (to compare the Y_i to Gaussians). We refer the reader to [2, pp. 5–7] for a detailed sketch of the strategy.

Branching random walks and leading-order term for the actual zeta function

In collaboration with Bourgade, Radziwiłł, and Soundararajan, Arguin and Belius ([1], 2016) successfully adapted these ideas to obtain the leading-order term in Conjecture 9, that is, for the actual zeta function:[3]

Theorem 13 ([1])**.** *For $\varepsilon > 0$ and t uniform in $[T, 2T]$,*

$$\left| \max_{\substack{u\in\mathbb{R}\\|u-t|\le 1}} \log|\zeta(1/2+iu)| - \log\log T \right| < \varepsilon$$

almost surely, as $T \to \infty$.

The implied upper bound follows from a Sobolev-type inequality and known bounds on the moments of ζ. For the lower bound, the first step is to reduce to understanding the maximum of finite Dirichlet series $\sum_{p\le X} p^{-\sigma+iu}$, where $\sigma \approx 1/2$; this is done using ideas from the alternative proof by Radziwiłł and Soundararajan ([41], 2015) of Selberg's central limit theorem. Then, an approximate branching random walk is identified and studied with Kistler's method, as in [2] with the model arising from Proposition 10.

4 Biases in prime number races

The prime number theorem in arithmetic progressions, a quantitative version of Dirichlet's theorem on primes in arithmetic progressions, states that if $q \ge 1$ and $a \in (\mathbb{Z}/q)^\times$, then

$$\pi(x,q,a) := \frac{|\{p \le x : p \equiv a \pmod q\}|}{\pi(x)} \sim \frac{1}{\varphi(q)}$$

as $x \to \infty$, where $\pi(x)$ is the number of primes $p \le x$. In other words, primes equidistribute in admissible congruence classes.

In particular, if $a_1, a_2 \in (\mathbb{Z}/q)^\times$, then $\frac{\pi(x,q,a_1)}{\pi(x,q,a_2)} \to 1$. One may wonder whether $\pi(x,q,a_1) > \pi(x,q,a_2)$ for infinitely many x, and if the two orderings are equally likely. Chebyshev observed in 1853 that, surprisingly, we have $\pi(x,4,3) > \pi(x,4,1)$ most of the time.

More generally, for $n, q \ge 2$ fixed and

$$\boldsymbol{a} \in \mathcal{A}_n(q) := \{\boldsymbol{a} \in ((\mathbb{Z}/q)^\times)^n : a_1, \ldots, a_n \text{ distinct}\},$$

we can study the $x \ge 2$ such that the ordering

$$\pi(x,q,a_1) > \pi(x,q,a_2) > \cdots > \pi(x,q,a_n) \tag{8}$$

holds (a "Shanks–Rényi prime number race").

In a breakthrough work, Rubinstein and Sarnak ([43], 1994) showed that, conditionally on the general Riemann hypothesis (GRH) and the $\mathbb{Q}$-linear independence of non-negative imaginary parts of non-trivial zeros of Dirichlet L-functions (conjecture LI), the ordering (8) happens for infinitely many x, actually for a positive logarithmic density[4]

$$\delta(\boldsymbol{a},q) := \lim_{X\to\infty} \frac{1}{\log X} \int_2^X \delta_{(8)\text{ holds}} \frac{dx}{x}.$$

Under these conjectures, they confirm Chebyshev's observation by showing that $\delta(3,1,4) = 0.9959\ldots$ In general, they give an explicit expression for the densities $\delta(\boldsymbol{a},q)$, a criterion for the symmetry of the density function (in which cases the races are unbiased, i.e., $\delta(\boldsymbol{a},q) = 1/n!$ for all $\boldsymbol{a} \in \mathcal{A}_n(q)$), and show that the biases dissolve as $q \to \infty$, that is,

$$\lim_{q\to\infty} \max_{\boldsymbol{a}\in\mathcal{A}_n(q)} |\delta(\boldsymbol{a},q) - 1/n!| = 0 \tag{9}$$

when $n \ge 2$ is fixed. To do so, the main step is the following:

Theorem 14 ([43])**.** *As $X \to \infty$,*

$$(x \in [2,X]) \mapsto \left(\frac{\log x}{\sqrt{x}}(\varphi(q)\pi(x,q,a_i) - \pi(x))\right)_{1\le i\le n}$$

has limiting distribution given by the random vector

$$X_{q,a_1,\ldots,a_n} = (X(q,a_1),\ldots,X(q,a_n)), \quad \textit{where}$$

$$X(q,a) = -C_q(a) + \sum_{\substack{\chi \pmod q\\ \chi\ne\chi_0}} \sum_{\gamma_\chi>0} \frac{2\operatorname{Re}(\chi(a)U(\gamma_\chi))}{\sqrt{1/4+\gamma_\chi^2}},$$

$$C_q(a) = -1 + \sum_{\substack{b^2\equiv a \pmod q\\ 1\le b\le q}} 1,$$

for $U(\gamma_\chi)$ iid on the unit circle in $\mathbb{C}$ and γ_χ running over the non-negative zeros of $L(1/2+i\gamma_\chi,\chi)$.

We refer the reader to [19] for a survey of subsequent results.

In the remainder of this section, we will direct our attention to recent advances in *prime number races with many contestants*, that is, when n is allowed to grow to infinity with q, instead of being fixed. All the results stated will be conditional on the GRH and LI conjectures.

Prime races with many contestants

Feuerverger and Martin ([14], 2000) conjectured that (9) still holds when $n = n(q) \to \infty$, $n \le \varphi(q)$, i.e., the biases still dissolve.

In 2012, Lamzouri [35] obtained a first uniform version of (9) in a certain range:

Theorem 15 ([35])**.** *If $2 \le n \le \sqrt{\log q}$ and $\boldsymbol{a} \in \mathcal{A}_n(q)$, then*

$$\delta(\boldsymbol{a},q) = \frac{1}{n!}\left(1 + O\left(\frac{n^2}{\log q}\right)\right).$$

Note that the second summand of $X(q,a)$ above is given as a weighted sum of independent random variables, and $C_q(a)$ can essentially be ignored. The idea behind Theorem 15 is to approximate $X_{q,a_1,\ldots,a_n}$ as a multivariate normal random variable (through a quantitative central limit theorem), with an estimated covariance matrix, and to then directly estimate the resulting density function.

Concerning larger ranges of n, an unpublished conjecture of Ford and Lamzouri states that there should actually be a transition when $n = (\log q)^{1+o(1)}$:

3 The same result was obtained independently at the same time by J. Najnudel, under the Riemann hypothesis.

4 However, the natural density does not exist, disproving a conjecture by Knapowksi and Turán.

Conjecture 16. *Let $\varepsilon > 0$, $q \geq 2$ be large enough, and $n = n(q)$.*

1. *If $2 \leq n \leq (\log q)^{1-\varepsilon}$, then* (9) *holds.*
2. *If $(\log q)^{1+\varepsilon} \leq n \leq \varphi(q)$, then there are extreme biases, namely there exist $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}_n(q)$ such that*

$$n!\delta(\boldsymbol{a}, q) \to 0 \qquad \text{and} \qquad n!\delta(\boldsymbol{b}, q) \to \infty.$$

A stronger version of the first part of the conjecture was proved last year by Harper and Lamzouri ([26], 2018):

Theorem 17 ([26])**.** *If $2 \leq n \leq (\log q)/(\log\log q)^4$, then* (9) *holds. More precisely, for any $\boldsymbol{a} \in \mathcal{A}_n(q)$,*

$$\delta(\boldsymbol{a}, q) = \frac{1}{n!}\left(1 + O\left(\frac{n(\log n)^4}{\log q}\right)\right).$$

This follows the same strategy as Theorem 15 above (performing the normal approximation with a 2009 result by Reinert and Röllin [42]), but with better estimates for the covariances through harmonic analysis.

Orderings of weakly correlated normal random variables and the leader in prime races

From there, we can also study the "leader" in a prime number race, i.e., for $\boldsymbol{a} \in \mathcal{A}_n(q)$, the logarithmic density $\delta_1(\boldsymbol{a}, q)$ of the $x \geq 2$ such that

$$\pi(x, q, a_1) > \pi(x, q, a_2), \ldots, \pi(x, q, a_n). \tag{10}$$

An application of Theorem 17 shows that $\delta_1(\boldsymbol{a}, q) \to 1/n$ if $2 \leq n = n(q) = o(\log q/(\log\log q)^4)$, i.e., each contestant has an equal chance of being the leader in this range. However, Harper and Lamzouri showed that this can be significantly extended with more involved arguments:

Theorem 18 ([26])**.** *We have $\delta_1(\boldsymbol{a}, q) \to 1/n$ as soon as $2 \leq n = n(q) \leq \varphi(q)^{1/32}$.*

After the normal approximation and the estimation of the covariances, this is derived from a general result on the ordering of weakly correlated jointly normal random variables that Harper and Lamzouri establish:

Theorem 19 ([26])**.** *For $n \geq 2$ and $\varepsilon > 0$, let $X_1, \ldots, X_n$ be jointly normal random variables, each with mean 0 and variance 1. Let $r_{i,j} = \mathbb{E}(X_i X_j)$ denote the covariances, and assume that $|r_{i,j}| \leq \varepsilon$ whenever $i \neq j$. Then*

$$\left|P\Big(X_1 > \max_{2 \leq i \leq n} X_i\Big) - \frac{1}{n}\right| \ll_\varepsilon n^{-100} + n^{-1.99} \sum_{2 \leq i \leq n} |r_{1,i}| + n^{-2.99} \sum_{2 \leq i < j \leq n} |r_{i,j}|.$$

If the X_i were independent, then the probability would be exactly $1/n$. An important input in the proof of Theorem 19 is the use of the normal comparison result of Li–Shao ([38], 2002) to compare $X_1, \ldots, X_n$ to independent normal random variables. However, as the probabilities may be small with respect to rather large bounds on the covariances (as is the case for Theorem 18), this alone may yield a trivial bound. To overcome this, the authors note that if the X_i were independent, then

$$\max_{2 \leq j \leq n} X_j = \sqrt{(2 - o(1))\log n} \quad \text{with high probability,}$$

while

$$P\Big(X_1 > \sqrt{(2 - o(1))\log n}\Big) = \frac{1}{n^{1-o(1)}}.$$

In other words, the small probability $1/n$ that X_1 is the leader is mostly caused by the event that X_1 is large enough to be so. Conditioning on the latter gives bounds which are more achievable, and using Slepian's comparison inequality (which is single sided unlike that of Li–Shao, but always non-trivial) allows one to conclude the argument.

Using similar ideas, Harper and Lamzouri also obtain estimates for the logarithmic density $\delta_k(\boldsymbol{a}, q)$ (for $k < n$) of the $x \geq 2$ such that

$$\pi(x, q, a_1) > \cdots > \pi(x, q, a_k) > \max_{k+1 \leq j \leq n} \pi(x, q, a_j). \tag{11}$$

Normal approximation and extreme biases

In the direction of the second part of Conjecture 16, Harper and Lamzouri ([26], 2018) gave one of the first results (along with work by Fiorilli [15]) where biases do *not* dissolve asymptotically:

Theorem 20 ([26])**.** *Let $\varepsilon > 0$. There exists a constant $c_\varepsilon > 0$ such that if $\varphi(q)^\varepsilon \leq n \leq \varphi(q)$, there exists $\boldsymbol{a} \in \mathcal{A}_n(q)$ with*

$$\delta(\boldsymbol{a}, q) < (1 - c_\varepsilon)/n!.$$

The idea is to get biases for $\delta_k(\boldsymbol{a}, q)$, which then gives biases for $\delta(\boldsymbol{a}, q)$ by summing over permutations of the components of $\boldsymbol{a}$. The ordering (11) corresponds to the ordering

$$X(q, a_1) > \cdots > X(q, a_k) > \max_{k+1 \leq j \leq n} X(q, a_j) \tag{12}$$

of the random variables in Theorem 14. If this holds, then $X(q, a_1), \ldots, X(q, a_k)$ (after renormalization) should all be larger than $\approx \sqrt{2\log n}$ with high probability, as in the previous section. If $(a_1, \ldots, a_k) \in \mathcal{A}_k(q)$ is chosen so that $X(q, a_1), \ldots, X(q, a_k)$ have (maximum) pairwise correlations $\approx -\frac{\log 2}{\log q}$, this introduces a bias of size $\approx \frac{k\log n}{\log q}$ inside the exponential of the density function, after normal approximation as in Theorem 17.

However, note that the biases in Theorem 20 are always close to and smaller than 1, unlike the extreme biases predicted by Conjecture 16. In a recent preprint, Ford, Harper, and Lamzouri [16] improve on this by showing that the second part of the conjecture holds, actually as soon as $n/\log q$ goes to ∞. More precisely,

Theorem 21 ([16])**.** *There exists an absolute constant $C > 0$ such that if $1 \ll n \leq \varphi(q)$, there exist $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}_n(q)$ with*

$$\delta(\boldsymbol{a}, q) \leq \exp\left(-\frac{\min(n, \varphi(q)^{1/50})}{C\log q}\right)\frac{1}{n!},$$
$$\delta(\boldsymbol{b}, q) \geq \exp\left(\frac{\min(n, \varphi(q)^{1/50})}{C\log q}\right)\frac{1}{n!}.$$

This follows the same strategy as the one sketched above for Theorem 20, with two main improvements: to get extreme biases, the parameter k is allowed to grow to the order of magnitude of n instead of being fixed; to get small and large biases, the situation (12) is replaced by a slightly different one. One of the issues that arises is that the typical Berry–Esseen-type errors in the normal approximation of $X_{q,a_1,\ldots,a_n}$ are too large with respect to the main term. To overcome this, the authors develop a multivariate "moderate deviation" estimate for sums of independent random variables, from the Lindeberg replacement strategy.

5 Random Fourier series and paths of partial exponential sums

We conclude with works on exponential sums that also involve some deep results from algebraic geometry.

An exponential sum of fundamental interest in number theory is the Kloosterman sum

$$\mathrm{Kl}_{2,p}(a) = \frac{1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i(ax + \overline{x})}{p}\right) \quad (a \in \mathbb{F}_p^\times).$$

These are real numbers, and Weil's proof of the Riemann hypothesis for curves over finite fields (1948) shows that they lie in $[-2, 2]$. In practice, partial Kloosterman sums

$$\mathrm{Kl}_{2,p}(a,t) = \frac{1}{\sqrt{p}} \sum_{0 \le x \le tp} \exp\left(\frac{2\pi i(ax + \overline{x})}{p}\right) \in \mathbb{C},$$

for $t \in [0, 1)$, are just as interesting. When t is not an integer multiple of $1/p$, let us replace $\mathrm{Kl}_{2,p}(a, t)$ by a linear interpolation of the values at the two closest integers. For every $a \in \mathbb{F}_p^\times$, this gives a continuous path $t \mapsto \mathrm{Kl}_{2,p}(a, t)$ made of straight lines, that ends at $\mathrm{Kl}_{2,p}(a) \in [-2, 2]$ (see Figure 2). In the 1980s, such paths of partial exponential sums were studied by Lehmer, Dekking–Mendès France, Loxton, and Deshouillers. For the Kloosterman sum, with the uniform measure on $\mathbb{F}_p^\times$, we get a stochastic process

$$(K_p(t))_{t \in [0,1)},$$

whose limiting distribution (as $p \to \infty$) was recently studied by Kowalski and Sawin ([33], 2014). To do so, they define the random Fourier series

$$K(t) = \sum_{h \in \mathbb{Z}} \frac{\exp(2\pi i h z) - 1}{2\pi i h} \mathrm{ST}_h \qquad (t \in [0, 1]),$$

where $(\mathrm{ST}_h)_{h \in \mathbb{Z}}$ are independent random variables distributed according to the Sato–Tate measure $\frac{1}{2\pi}\sqrt{4 - x^2}$ on $[-2, 2]$.

Their main result is the following:

Theorem 22 ([33])**.** 1. *$K(t)$ converges almost surely and in law, taking symmetric partial sums. The limit, as a random function, is almost surely continuous. For any $t \in [0, 1)$, one has $\mathbb{E}(K(t)) = 0$ and $\mathrm{Var}(K(t)) = t$.*
2. *In the sense of convergence of finite distributions,*

$$(K_p(t))_{t \in [0,1)} \xrightarrow[fd]{p \to \infty} (K(t))_{t \in [0,1)}.$$

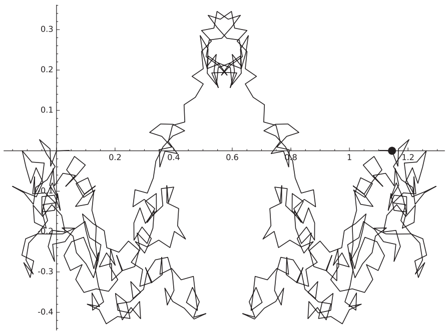


Figure 2. Partial sums of the Kloosterman sum $\mathrm{Kl}_{2,547}(1)$**.**

The proof of the second part uses the method of moments. The latter are estimated asymptotically from the work of Katz [30, 31], relying in particular on Deligne's generalisation of the Riemann hypothesis over finite fields.

One could also ask for the stronger result of convergence in law as $C([0, 1))$-valued random variables. It turns out that this is linked to important conjectures on short exponential sums. Using Prokhorov's theorem (that is, by checking Kolmogorov's tightness criterion), Kowalski and Sawin show the convergence in law unconditionally for Birch sums

$$\mathrm{Bi}_p(a,t) = \frac{1}{\sqrt{p}} \sum_{n \le t} \exp\left(\frac{2\pi i(ax + x^3)}{p}\right) \qquad (a \in \mathbb{F}_p^\times)$$

and for a two-dimensional domain variant of $\mathrm{Kl}_{2,p}$.

This yields interesting applications, such as bounds for the probability of large values of partial Kloosterman sums and partial Birch sums, which is analogous to the recent results of Bober, Goldmakher, Granville, and Koukoulopoulos ([4], 2018) for Dirichlet characters. For example, we have the following theorem:

Theorem 23 ([33])**.** *For $A > 0$, let*

$$L(A) = \lim_{p \to \infty} \frac{|\{a \in \mathbb{F}_p^\times : \max_{0 \le t < p} |\mathrm{Bi}_p(a,t)| > A\}|}{p - 1}.$$

There exists a constant $c > 0$ such that for any $A > 0$,

$$c^{-1} \exp(-\exp(Ac)) \le L(A) \le c \exp(-\exp(A/c)).$$

This follows from Theorem 22, with elementary arguments for the lower bound, and general tail bounds on sums of martingale difference sequences for the upper bound (see also Section 2).

Theorem 23 was very recently improved in a preprint by Lamzouri [36], in particular by obtaining upper and lower bounds in a uniform range for A with respect to p, and of roughly the same order of magnitude. The lower bound holds similarly for Kloosterman sums, while the upper bounds is conditional on certain bounds on short Kloosterman sums.

Finally, the support of the random Fourier series $K(t)$ was computed by Kowalski and Sawin in a subsequent work ([34], 2017), with further arithmetic applications to Kloosterman sums having all their partial sums small.

Acknowledgements
The author thanks Lucile Devin, Javier Fresán, Adam Harper, and Youness Lamzouri for helpful comments on the manuscript.

Bibliography

[1] Louis-Pierre Arguin, David Belius, Paul Bourgade, Maksym Radziwiłł, and K. Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure and Applied Math.*, 2016.

[2] Louis-Pierre Arguin, David Belius, and Adam J. Harper. Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.*, 27(1):178–215, 2017.

[3] Ron Blei and Svante Janson. Rademacher chaos: tail estimates versus limit theorems. *Ark. Mat.*, 42(1):13–29, 2004.

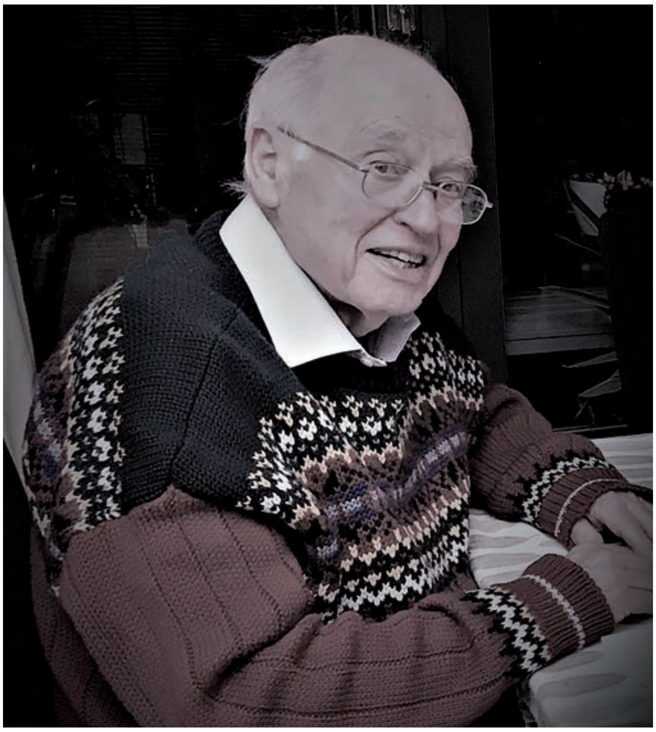
[4] Jonathan Bober, Leo Goldmakher, Andrew Granville, and Dimitris Koukoulopoulos. The frequency and the structure of large character sums. *J. Eur. Math. Soc.*, 2018.

[5] Andriy Bondarenko, Winston Heap, and Kristian Seip. An inequality of Hardy-Littlewood type for Dirichlet polynomials. *J. Number Theory*, 150:191–205, 2015.
[6] Andriy Bondarenko and Kristian Seip. Helson's problem for sums of a random multiplicative function. *Mathematika*, 62(1):101–110, 2016.
[7] Paul Bourgade. Mesoscopic fluctuations of the zeta zeros. *Probab. Theory Related Fields*, 148(3-4):479–500, 2010.
[8] Maury D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.
[9] Sourav Chatterjee. A new method of normal approximation. *Ann. Probab.*, 36(4):1584–1610, 2008.
[10] Sourav Chatterjee and Kannan Soundararajan. Random multiplicative functions in short intervals. *Int. Math. Res. Not. IMRN*, 2012(3):479–492, 2012.
[11] Peter D. T. A. Elliott. *Probabilistic number theory. II, Central limit theorems*, volume 240 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin–New York, 1980.
[12] Paul Erdős and Mark Kac. The Gaussian law of errors in the theory of additive number theoretic functions. *Amer. J. Math.*, 62:738–742, 1940.
[13] David W. Farmer, Steven M. Gonek, and Christopher P. Hughes. The maximum size of L-functions. *J. Reine Angew. Math.*, 609:215–236, 2007.
[14] Andrey Feuerverger and Greg Martin. Biases in the Shanks–Rényi prime number race. *Experiment. Math.*, 9(4):535–570, 2000.
[15] Daniel Fiorilli. Highly biased prime number races. *Algebra Number Theory*, 8(7):1733–1767, 2014.
[16] Kevin Ford, Adam J. Harper, and Youness Lamzouri. Extreme biases in prime number races with many contestants. 2017. Preprint, arXiv:1711.08539.
[17] Yan V. Fyodorov, Ghaith A. Hiary, and Jonathan P. Keating. Freezing transition, characteristic polynomials of random matrices, and the riemann zeta function. *Phys. Rev. Lett.*, 108, 2012.
[18] Yan V. Fyodorov and Jonathan P. Keating. Freezing transitions and extreme values: random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(2007), 2014.
[19] Andrew Granville and Greg Martin. Prime number races. *Amer. Math. Monthly*, 113(1):1–33, 2006.
[20] Gábor Halász. On random multiplicative functions. In *Hubert Delange colloquium (Orsay, 1982)*, volume 83 of *Publ. Math. Orsay*, pages 74–96. Univ. Paris XI, Orsay, 1983.
[21] Adam J. Harper. Bounds on the suprema of Gaussian processes, and omega results for the sum of a random multiplicative function. *Ann. Appl. Probab.*, 23(2):584–616, 2013.
[22] Adam J. Harper. A note on the maximum of the Riemann zeta function, and log-correlated random variables. 2013. arXiv:1304.0677.
[23] Adam J. Harper. On the limit distributions of some sums of a random multiplicative function. *J. Reine Angew. Math.*, 2013(678), 2013.
[24] Adam J. Harper. Moments of random multiplicative functions, I: low moments, better than squareroot cancellation, and critical multiplicative chaos. 2017. Preprint, arXiv:1703.06654.
[25] Adam J. Harper. Moments of random multiplicative functions, II: high moments. 2018. Preprint, arXiv:1804.04114.
[26] Adam J. Harper and Youness Lamzouri. Orderings of weakly correlated random variables, and prime number races with many contestants. *Probab. Theory Related Fields*, 170(3-4):961–1010, 2018.
[27] Adam J. Harper, Ashkan Nikeghbali, and Maksym Radziwiłł. A note on Helson's conjecture on moments of random multiplicative functions. In Carl Pomerance and Michael Th. Rassias, editors, *Analytic Number Theory*, pages 145–169. Springer International Publishing, Cham, 2015.
[28] Henry Helson. Hankel forms. *Studia Math.*, 198(1):79–84, 2010.
[29] Bob Hough. Summation of a random multiplicative function on numbers having few prime factors. *Math. Proc. Cambridge Philos. Soc.*, 150(2):193–214, 2011.
[30] Nicholas M. Katz. *Gauss sums, Kloosterman sums, and monodromy Groups*, volume 116 of *Annals of Math. Studies*. Princeton University Press, 1988.
[31] Nicholas M. Katz. *Exponential sums and differential equations*, volume 124 of *Annals of Mathematical Studies*. Princeton University Press, 1990.
[32] Nicola Kistler. Derrida's random energy models. From spin glasses to the extremes of correlated random fields. In *Correlated random systems: five different methods*, volume 2143 of *Lecture Notes in Math.*, pages 71–120. Springer, 2015.
[33] Emmanuel Kowalski and William F. Sawin. Kloosterman paths and the shape of exponential sums. *Compos. Math.*, 152(7), 2016.
[34] Emmanuel Kowalski and William F. Sawin. On the support of the Kloosterman paths. 2017. Preprint, arXiv:1709.05192.
[35] Youness Lamzouri. The Shanks-Rényi prime number race with many contestants. *Math. Res. Lett.*, 19(3), 2012.
[36] Youness Lamzouri. On the distribution of the maximum of cubic exponential sums. 2018. Preprint, arXiv:1802.09701.
[37] Yuk-Kam Lau, Gérald Tenenbaum, and Jie Wu. On mean values of random multiplicative functions. *Proc. Amer. Math. Soc.*, 141(2):409–420, 2013.
[38] Wenbo V. Li and Qi-Man Shao. A normal comparison inequality and its applications. *Probability Theory and Related Fields*, 122(4):494–508, April 2002.
[39] Don L. McLeish. Dependent central limit theorems and invariance principles. *Ann. Probability*, 2:620–628, 1974.
[40] Nathan Ng. The distribution of the summatory function of the Möbius function. *Proc. London Math. Soc. (3)*, 89(2):361–389, 2004.
[41] Maksym Radziwiłł and K Soundararajan. Selberg's central limit theorem for $|\log\zeta(1/2+it)|$. *L'Enseignement Mathématique*, 2018. To appear, arXiv:1509.06827.
[42] Gesine Reinert and Adrian Röllin. Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition. *The Annals of Probability*, 37(6):2150–2173, November 2009.
[43] Michael Rubinstein and Peter Sarnak. Chebyshev's bias. *Experiment. Math.*, 3(3):173–197, 1994.
[44] Wolfgang Schwarz. Some highlights from the history of probabilistic number theory. In *Probability and number theory—Kanazawa 2005*, volume 49 of *Adv. Stud. Pure Math.*, pages 367–419. Math. Soc. Japan, Tokyo, 2007.
[45] Atle Selberg. Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvid.*, 48(5):89–155, 1946.
[46] Michel Talagrand. The missing factor in Hoeffding's inequalities. *Ann. Inst. H. Poincaré Probab. Statist.*, 31(4):689–702, 1995.
[47] Paul Turán. On a Theorem of Hardy and Ramanujan. *J. London Math. Soc.*, 9(4):274–276, 1934.
[48] Aurel Wintner. Random factorizations and Riemann's hypothesis. *Duke Math. J.*, 11:267–275, 1944.

Corentin Perret-Gentil [corentin.perretgentil@gmail.com] received his PhD at ETH Zürich in 2016, and is now a postdoctoral fellow at the Centre de recherches mathématiques in Montréal, Canada, working in analytic number theory.

Last Interview with Sir Michael Atiyah

John Alexander Cruz Morales (Universidad Nacional de Colombia, Bogotá, Colombia)



Last November I visited Edinburgh for one week, thanks to an invitation from Sir Michael Atiyah to discuss mathematical physics and, in particular, some of his latest projects and ideas. The weather, cold and rainy, was not the best for my taste but I really enjoyed walking around the city, going from my residence at the Glendale Guest House to Sir Michael's flat or his office at the School of Mathematics at the University of Edinburgh in order to get involved in a mathematical conversation with him. It was a wonderful week. We talked in the mornings, and then in the afternoons I received emails from Sir Michael summarizing our previous conversations and preparing the next one. We met every single day during my visit and the energy and enthusiasm Sir Michael showed will always be one of my best mathematical memories. I also gave a talk at the colloquium in the School of Mathematics thanks to Sir Michael's suggestion and I was able to appreciate again his sharp and deep mathematical insights in the form of comments on my presentation. The last day of my visit was December 1st and Sir Michael agreed to record our last conversation as an interview. That conversation is the one I am reproducing here. It was an informal and warm chat. This interview contains his last words to me, which I am sharing with the reader. After that, I got a couple of emails from him before receiving the sad news of his death and then this small interview gained more significance. I had already admired the mathematician but now, thanks to my visit, I also admire the man: a humble man full of generosity and kindness.

JACM: We have the dichotomy between inventing and discovering in mathematics. So, what do you think about this? Do we discover mathematics or invent mathematics?
MA: Mathematical truths are in our world. Our invention is to select from all the possible mathematical truths the ones that are really interesting. Now, how and why do we see that something is interesting? Because it is beautiful, it has a nice structure, it produces human emotion. A machine can not teach us to understand beauty. Newton said he was like somebody who goes to the beach and picks a nice pebble. There are million of pebbles on the beach but which one do you pick up? You pick up the ones that are nice, shiny and beautiful and thus you discover that pebble, you invent it. When you discover a mathematical truth you are inventing it because you find it and you choose one because it is more beautiful than the others. So, invention is selection. I select things that appeal to me. I am an artist. You have all the possible musical notes you can write. Why should I write some of those? There are million of possibilities of musical notes. Taking some is composing, is invention; we regard this as invention. All the possibilities are there. Inventing a symphony is a human creation. So, what is the difference in creating a beautiful piece of music and creating a beautiful mathematical theorem? I think it is the same.

When you think about mathematics, how do you choose a problem? What are your criteria for choosing a mathematical problem?
Well, everybody has their own criteria. Each mathematician has his own taste. But, of course, many mathematicians have the same views. Good mathematicians tend to appreciate each other even if they have different tastes. We choose a problem which we think it is interesting if it has features that appeal to us. It must be hard. You will see some beauty and some form emerging from it, like an artist. It must be deep; you can put a lot of adjectives but at the end it is difficult to describe. It is the decision of the individual artist or mathematician and that is personal. You can give a hundred of reasons but this does not define it completely. All you can do is to say that you view beauty in what you see. But, how to define beauty? How to define a good piece of mathematics? You can do it by illustrations. Poincaré duality is a beautiful piece of mathematics; you have to illustrate it by examples.

So, it is clear that beauty is an important criterion for you.
Absolutely! But we all have different kinds of beauties..

Who are the mathematicians that had an influence on your work? Who are your mathematical heroes?

That is easy. From the past, Archimedes and Newton. In the more recent times Hermann Weyl was my hero. He had interests in everything in mathematics; he knew about beauty. Hermann Weyl is the mathematician that I have identified with most. Of course there are others. Poincaré I like very much and Abel who died very young. But Hermann Weyl is the closest one who had most influence on me.

Hermann Weyl was interested in the interactions between physics and mathematics like you . So, why is that interaction so important for you?

Until the 19th century nobody made a distinction. Was Newton a mathematician or a physicist? It is almost impossible to say. Newton was a great mathematician and a great physicist, no question. They did not distinguish. Later on mathematics became specialized; knowledge increased and then mathematics and physics seemed quite different, but that is not true. In physics there is a part to do with understanding experimental results and that is important science. Science has both experimental and theoretical sides. The theoretical side is like mathematics, the experimental side is different, but all mathematics is involved by taking things on experiments and making their understanding better by developing beautiful theories: Maxwell equations and so on. They are beautiful pieces of mathematics that showed to be very powerful in physics. All of physics has evolved by looking at all the data in experiments and then you find a simple way to organize it and that simple way of organizing is mathematics. It is ridiculous to separate mathematics and physics and it is also ridiculous to separate one knowledge from the other, for example to separate physics and chemistry or to separate physics and biology.

In many places you have pointed out the importance of mathematical dreams. What is the role of the dreams in mathematics?

Well, the main part of mathematics is to me not when you finish writing a proof , that is the last part. The main part is the initial ideas you want to develop. They are dreams. They may come at night when you are dreaming or during the day time; they are visions. You see the landscape, see some beautiful structure you do not know what it is going to be. You are dreaming when you are trying to capture some of this and then you have a plan. The proof of a theorem is not the beginning, it is the end. So, you start with a dream and end with a theorem. That is the natural process. Dreams are very important. Birkar, who got a Fields medal in Rio, said in his lecture that I told him once that a mathematician without a dream is not a mathematician. I think this is inspiring.

In fact, in your ICM lecture in Rio you talked about arithmetic physics as a research program. Might you summarise your vision regarding this program?

Yes. If you look at mathematics and physics they overlap in different areas; sometimes in geometry, sometimes in algebra, sometimes in number theory. For example, one very concrete case is the things called modular forms which arose in number theory. On the other hand, modular forms appear in physics as partition functions. Very often they are the same ones, and you wonder why. My aim is to find a natural framework which explains all this. There should be some natural way in which arithmetic, geometry, algebra and physics are all involved.

I appreciate a lot that you gave me some of your time for having this conversation. Let me make a final question. What would be your advice for a young mathematician?

My first advice for a young mathematician is that you have to be passionate. If you do it because you think you will get some money or you get a good job, forget it! There are easier ways to do that. And it is very hard work: you struggle and a lot of the time you are frustrated. The only way you succeed is if you are passionate. If you are passionate, then you will go a long way. The second thing is listen to your elders. Take their experience, but be your own person. Follow your own instinct because that way if you succeed, you will be unique. If you only follow what your teachers tell you, you will only repeat what they did. Listen to your teachers, go to lectures, read books; with that knowledge ask: what I want to do now. What you need is passion, persistence and risking things for searching for the beauty. If you do that, you have some chance to succeed. And do not be afraid to open your mind, talk to people. Interacting with other people, you get ideas. You need a balance between your passion and your family. You need a balanced life.

John Alexander Cruz Morales {jacruzmo@unal.edu.co] is a Colombian assistant professor in the mathematics department at Universidad Nacional de Colombia. His research interests are algebraic geometry and mathematical physics. He is also interested in the history and philosophy of mathematics.

Interview with Michel Waldschmidt

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

Michel Waldschmidt is a French mathematician, whose professional work has centred on number theory, especially dealing with transcendental numbers. But here he is interviewed about his passionate involvement with mathematics in the Third World, as reflected in his vice-presdency of CIMPA (Centre international de mathématiques pure et appliquées) during the years 2005–2009. He is also a member of the EMS committee CDC (Committee for Developing Countries).

UP: You have travelled a lot?
MW: One could say so: I have in fact taught in about seventy different countries.

So you knew Michael Passare?
Of course I did. So sad that it came to such a bad end in Oman. It was the heart and not a fall as we first understood.

Correct.
He was a passionate traveller: he visited 152 countries.

Yes, he was very serious about the game. Ultimately he would have wanted to visit all countries, I guess.
He almost achieved it. He was such a nice guy. He was also a member of the Committee for Developing Countries of the European Mathematical Society. I remember in April 2008 flying from Vienna to Venezia for a meeting of CDC EMS at ICTP thinking that it would just be a short haul to Trieste. I arrived at eight o'clock in the evening and enquired about the next train connection to Trieste and was told that there was none until the next morning! Then Michael showed up with two colleagues from Sweden and asked if I was going to the meeting in Trieste. Sure, I said. 'We have rented a car, why don't you join us,' he said. Just like that. That was very nice.

You also travelled a lot in Africa.
This is true, but Asia is my main territory. Mainly South East Asia. This year I already visited Africa three times; but I have the most experience with Asia.

What is it that you and your committee want to do?
First, the CDC of the European Mathematical Society is not my committee, now I am only an associate member, but true, I have been very active in it since the turn of the millennium and I have been the chair. I am also very much involved in the activities of the CIMPA (Centre international de mathématiques pures et appliquées), of which I have been vice-president. These are two examples of institutions doing remarkably efficient work with extremely little money.

So you are the right guy to ask?
At least when it comes to South East Asia.

Sorry for the interruption…
…no problem. What we want to do is to contribute helping our colleagues from less developed countries to improve the level of mathematics there. We give lectures at the master's level, we organize workshops and research schools, we help creating networks. One of our strategies is to set up good research centres (called Emerging Research Centres of Excellence, ERCE) to which people also from neighbouring countries can travel and learn, and not have to go to the West, because this is not sustainable.

You mean just to educate a few people in the West for them to either stay in the West or to return to nothing at home is not doing anything at all?
It is better than nothing, but it is far from our goal. When we meet students who wish to get a PhD in mathematics they invariably enquire about funds for going to the West…

…this might be good for the individuals but not for the country?
It may be good for both. We tell them that it is much more efficient for them to go to an institution of international level in a neighbouring country than to go through the hassle of such a drastic relocation. And as you point out, we want to do this primarily for the country, but we mind also for the individuals. As an example, the CDC of EMS supported a young mathematician from Indonesia who went to ASSMS Lahore to pursue his studies up to his PhD. This is a success story and we wish we had more funds to repeat it.

But individuals are concrete entities while countries are abstract ones, and I am sure that in some cases you cannot but prioritize an actual individual and his life and career.
As I see it, it is ultimately a question of justice. People like you and me had the privilege of growing up in functioning and prosperous Western democracies with good school systems and were able to follow our inclinations and become mathematicians. People all over the world should have that opportunity. We witness a fantastic

development of higher education in all countries. Each country should have mathematicians who participate in international activities like ICM and are full members of the mathematics community, connected with mathematicians all around the world. Even in countries like Papua New Guinea or Bhutan, there are mathematicians who are eager to have international contacts. So you see ultimately it is about people as individuals. By helping the countries we help people.

And not the other way around.
Both are closely connected.

Still, you cannot just build a research institution and invite eager people like you from abroad to spend a few months lecturing. Sure it will be fun for you, as is testified by your many trips, and I would love to do it myself, but does it leave any traces? Is it not like throwing a stone into water, there is big splash, and there are some waves. But the stone sinks to the bottom and soon the water is as smooth again as it was before, and no one could suspect anything at all.
I hope it is not as bad as that. We start a program when we are asked by colleagues from the country to do so – we answer their demand. I certainly believe that our efforts are not wasted but that some traces remain. In fact we have discovered some very good students in different countries and helped them, so at least on the individual level it has not been in vain.

But we are talking about sustainable changes.
You are right. In many of the developing countries, mathematics is in fact very weak. There may be universities, where mathematics is being taught, even PhDs given, but there is no real mathematics done, nothing new produced, and that's what research in mathematics is all about after all. Just imagine writing a PhD thesis which contains nothing new! The problem is that the people at the universities are isolated, often ignorant and, to be honest, for many of them, not particularly bright. Fortunately there are some remarkably motivated individuals; these are the colleagues with whom we are working. I am amazed to see how devoted they are, working in very difficult conditions. They do a tremendous job, they deserve respect, and we want to help them as much as possible. But they are not the majority. Teachers, even at universities, are badly paid and as a consequence most bright young people seek other careers. In fact some colleagues from local universities are not happy with us because we expose their shortcomings and render them superfluous, so in fact this explains that we often do not get any official support.

So that is my point: you need to develop a mathematical infrastructure and that means having a good school system at all levels, from the elementary ones up to high school.
That is very true, but it will take time, perhaps a generation or so. All that we can do is to apply a top-down approach and hope that our good influence will trickle down. By improving university teachers and hence gradually also high-school teachers down to the village, we will get a positive feedback.

The educational system has two purposes. One is to look for and identify real talent in order to encourage and help; the other is to improve the general public, meaning to impart literacy and a respect for learning and culture. Those two aims are not necessarily conflicting, on the contrary; but if one has to be chosen, I would say that the first is the most important. Gauss's sadistic teacher did at least recognise Gauss's genius, and thereby doing mankind a great favor. The history of science is filled with such stories (but of course we only know about the success stories) and it is not clear that our modern Western educational system would be up to the challenge (in fact I know stories indicating the opposite).
In France we have a successful combination of the elitist system and the egalitarian one. We still maintain institutions such as École Normale Supérieure and at the same time we have the ambition that at least 90 percent of the population should get a high-school education. And I would say it works well. It shows that with mass education you can maintain the same standards as with the more elitist ones of the past.

Why should such a large percentage of the general population be subjected to a formal education they cannot profit from in an intellectual way? But this is now considered as a right on a par with clean water and medical care. The responsibility of learning no longer lies with the student but the teacher, just as it lies with the surgeon not the patient, when it comes to an operation.
We are diverging a bit from the purpose of our conversation, but I agree. It is a problem of actually finding the good students. It happens that we find some promising students; before taking them to some Western institutions, we need to make sure that they have some independence of mind. In some cases it turns out that they have only learned things by heart and have no idea what it really means. Student easily memorise the axioms of topology say, but when asked questions about open sets on the real line they are stymied.

What is the point of memorizing mathematics! Whatever gave them the idea?
No point at all of course. But, I guess, in many subjects, such as history, memorization is important and inescapable.

Not even in history is it that important and there is such a thing as understanding in history as well. Things you learn that excite and stimulate you, you need not commit to memory, they become part of you automatically…
…as in mathematics.

Very true. But when it comes to languages, learning poems by heart is supposed to be very good. After all language is a kind of motor skill. But now there is so much emphasis on so-called creativity and critical thinking,

and modern students are supposed to excel in that, while when it comes down to it, most students feel very comfortable with just memorizing, forget about creativity, as you discover whenever you give an exam with interesting problems. For many people, not the least in the Third World, education is merely a means of getting on in the world, and you do what it takes, be it to memorise meaningless facts. It is just part of a ritual.
What are you leading up to? I thought you were supposed to ask the questions and leave me to elaborate the answers.

Bear with me. Be patient!
I am trying my best, but there are limits.

My point is simply that what you need is an educational system in which the teachers have the right intellectual attitude. In particular, a math teacher would never think of memorizing mathematics but love the subject and have some talent as well. Forget about so-called teaching skills, they are just spurious, what counts is attitude and personality. In the West the traditional educational system is dismantled, the prestige of teachers has taken a nosedive, and they are now badly paid as a result. In fact until the seventies, members of the parliament in Sweden, were not supposed to be paid more than high-school teachers.
Really!

So I have been told. Ironically I think it might be easier to create a good educational system in the developing world than to reverse the trend of degeneration in the West. After all, there are a lot of poor, talented people who can be induced to take teaching as a career and earn prestige by that. Once such a system is in place, you have a foundation to build on.
You do not leave me much left to say. But I'll do my best. I agree with you, and there are countries, thought of as poor and backwards, with an excellent educational system. I would, in particular, point to Iran. And as we all know they produce excellent mathematicians without our help. Often in CIMPA research schools when some students come from Iran they are the best (and the next ones are from Vietnam). India also has excellent mathematical centers. After its independence, India decided to have the atomic bomb: a politician, Nehru, a scientist, Homi Bhabha, and an industry, Tata, created the Tata Institute of Fundamental Research in Bombay in order to acquire the necessary scientific level in physics, and this included the need to have mathematicians of the highest international level. They succeeded. I visited TIFR for the first time in 1976 and now I go to India about twice every year.

What about South Korea? In 1980 or so, only one research paper was published, and now!
I do not know much of the details about the rise of mathematics in Korea, I should as I have an in-law from Korea. It is true that the government supported strongly the development of mathematics in order to reach the highest level in technology. As a matter of fact, nowadays Korea is one of the leading countries in mathematics; the previous ICM took place in Seoul in 2014. Of course, China is a superb example of amazing increase in the mathematical level due to a very strong support by the highest authorities, who understood that a strong mathematical level is a key for technological innovation and economic development of the country.

And Japan. Japan lifted itself by the hair in connection with the Meiji restoration and entirely on its own power caught up technologically with the West, but, I guess, they had a long intellectual tradition and a solid educational system in place, which only needed some prompting.
That is true. Mathematics has a long tradition in Japan. If you ask me about South East Asia I will give you particulars.

I do.
Asia is very diverse. One real success story is Vietnam. After the wars with the French and then the Americans, French mathematicians went there. I am, in particular, thinking of Laurent Schwartz in 1967…

… I recall him from a colloquium dinner at Columbia in the late 70s waxing about butterflies he collected and telling stories about tigers having acquired a taste for human flesh as a consequence of the war…
… there will not be so many tigers left, but please let me continue.

By all means.
And Grothendieck went too.

And Lê Dung Tráng, who was Vietnamese.
And do not forget the Fields medalist Ngô Báo Châu at Hyderabad. Vietnam has by now come on its own.

He was educated in Paris and did his work there.
This reminds me of what I was going to respond to one of your initial remarks. It is not good to educate people and have them leave permanently. Good maybe for the individual, as you pointed out, but bad for the country. On the other hand, having them return home is not always good either. Invariably they will be isolated and wither. One solution is that they should divide their time between the West for their own mathematical inspiration and their homes to make their countries benefit. In this way they will achieve the best of two worlds.

Seems an excellent idea.
I know quite a few very good mathematicians from India who have a position in the West and return to their home country on a regular basis and teach there. On the opposite, the mathematicians who left Iran without completing their military service, like Caucher Birkar (his name was Feridoun Derakhshani, he changed it for "immigrant mathematician" in Kurdish) cannot go back to their country of birth. As to other countries I have had much experience with Cambodia. Good and bad. The main public uni-

versity is really weak, the private ones and the Institut de Technologie du Cambodge are better. The people at the public ones are so badly paid that they all teach in some of the private institutions, which hence cannot be much better. However there are more than a dozen young PhD holders in Cambodia now (there was essentially only one 14 years ago when CIMPA started to be involved). The International Science Programme (ISP) at Uppsala University (Sweden) supports a network SEAMaN (South-East Asia Mathematical Network) involving Cambodia, Myanmar and Laos. The first five editions (2008–2012) of the International Conference on Mathematics and Mathematics Education in Developing Countries took place in Cambodia, the next five editions (2013–2017) were organized in Myanmar. Starting in November 2018 they will take place in Laos.

In Bangladesh there is an International Mathematics Conference (under the slogan Bridging South Asian Mathematicians) every second year. There are several universities there including a brand new German University of Bangladesh supported by German funds.

And Malaysia?
There is one ERCE in Kuala Lumpur and several universities with a good level in mathematics – but not at the level of their neighbour Singapore, which is outstanding, like Hong Kong. Thailand has a larger number of mathematicians and several quite good universities. In terms of number of publications, Malaysia, Thailand, Indonesia and the Philippines are among the most productive.

But they might have the requisite educational infrastructure.
Then we should not forget the Philippines which are doing fine, thanks to the international support which started more than twenty years ago. And Pakistan where we had a remarkable institute in Lahore once – this was the very first ERCE. But, after the previous director left, it is no more at an international level and has lost its label. Yet, thanks to it, there are many mathematicians in most universities of Pakistan with a good research record.

It is interesting that Pakistan has diverged so much from India after the partition. But it was funded as a specifically Islamic state, while India ostensibly was secular and multi-cultural, although this is changing as of late.
Still Pakistan is very similar to India, and I do not think that the religious character of the state has had much effect on science and technology. True, there is no Tata Institute in Pakistan, but parts of India, especially in the east, are intellectually backwards and we can perhaps make a difference. Burma (Myanmar) opened only recently. Besides, IMU and CIMPA support a very promising initiative called the Nepal Algebra Project: for six years a course on Galois theory has been taught in Tribhuvan University (Kathmandu) by five mathematicians for two weeks each. At the end of this program the same course will be taught by local staff of the university. We also care about more remote places like Mongolia, where a program of cooperation in mathematics is supported by CIMPA.

What about the Arabian peninsula? Saudi Arabia has a lot of money to throw around. Would that not be an excellent place for a research institution to serve not only it but also the surrounding Arab countries?
It is true that Saudi Arabia, by virtue of its financial muscle, is trying to establish themselves by importing not only goods but human resources. You know of course of the Faisal Prize, no doubt coming with a ridiculously high sum of money, and they hire foreigners, pay them very well, and for the duration of their appointed stay, relieve them of their passports.

But there is no tradition in Saudi Arabia.
None that I know of. In this part of the world Oman seems to be the best location in terms of research.

What about other Mid-Eastern countries?
Jordan was surprisingly good when I visited it. Much better than Lebanon, but the latter was racked by a nasty civil war, and people, including academics tend to fight each other for the most obscure reasons that make no sense to outsiders. In Syria, there were several active mathematics departments – I am afraid that this is no longer the case.

When was that?
It was in 2003, way before the civil war. And then there was Iraq.

When was that?
Early on. I actually visited twice: in 2000, 2002. It was during the time of Saddam Hussein. When I arrived at the ICM in Beijing thereafter I was severely upbraided by a compatriot who found it very inappropriate that while I was representing France in my capacity as the President of the French Mathematical Society, I went to Saddam Hussein's Iraq. Anyway, Iraq has an old tradition in mathematics, and there were several very active universities. We found four promising students, two of them women by the way, and asked to bring them to Paris. The Iraqi authorities insisted that the women travel either with their husbands or fathers. One husband tagged along and the other woman managed to get away somehow. She is now back in Iraq doing a very good job. She organizes workshops supported by CIMPA in the Kurdish part of Iraq. Caucher Birkar, who just received a Fields Medal in Rio, is one of the teachers of these workshops.

Egypt?
Egypt is exceptional for me as it is a country I went for purely touristic reasons. But I plan to participate in an ICTP research school there in two or three years.

We could go on for a long time.
Yes we could.

But I guess we will not. Thank you very much for your time.
No problem. I really want this to become better known.

Interview with Bernadette Faye

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

Bernadette Faye is one of the students in the Third World who has benefited from support from the CDC, and an interview with whom is meant to complement the previous interview with Michel Waldschmidt.

UP: Where are you from?
BF: I am from Senegal. I grew up in Dakar. A big city.

What is your background?
I come from a poor family. My mother took care of the home, while my father was driving a taxi.

So you went to school obviously.
It's compulsory. I did not like it at first, I hid under the bed in the morning, I did not want to leave my mother. But I got used to it and started to like it.

You were taught in your native language?
No, in Senegal we are taught in French from the very start. There are several native languages. However, we have one which is considered as a national language. It's called "Wolof". Nowadays there is a movement to replace French by Wolof from primary school.

I guess French is seen as a vestige of colonial times, but it is very convenient: it gives you a ticket to a much larger world.
That is true. We can travel all around Western Africa and communicate easily.

You were good at math?
I liked it and it came naturally to me, but I did not think much of it.

When did you discover it as a discipline on is own?
When I attended university, I had inspiring teachers.

How did you get to university? Is that common or are there financial restrictions? And maybe other restrictions?
High school is open to everyone, so were also universities. But then so many wanted to go, so they had to impose restrictions. In practice, there are no financial restrictions since the registration fees are not so high. However, most of those who are admitted to the universities don't finish undergrad since they are eager to find employment so they can contribute to the upkeep of their family. Family is very important in Africa.

I guess unlike the pampered West the family is essential to physical survival, and I do not speak about the immediate family, but also cousins and grandparents. As a result I suspect that there is less emphasis on the individual as such apart from the family, so the issue of individual careers is not so strong as in the West.
I agree. In my family my parents were very supportive of us children and they were willing to go to great sacrifices in order for us to get an education. It was never a question of me sacrificing myself in order to support the family.

What about the student population at the university?
Half of the high-school students continue to university, but only 5 percent or so are any good, meaning that they are thinking of it as an intellectual adventure. Most students only want to pass exams, find the shorter path to get a job and be done with it.

Just as in the West. So what did you do mathematically?
After obtaining my bachelor's degree in mathematics, I was selected to pursue a master's program in cryptography and information security in the Department of Mathematics and Computer Science at the University of Dakar (UCAD). It was considered applied and very useful, but I soon got more interested in the underlying number theory. In 2012, I joined the African Institute for Mathematical Science (AIMS) in Senegal, where I met a researcher from UNAM in Mexico, Florian Luca, who supervised me for my master's degree. Then, after obtaining a PhD fellowship from the TWAS-OWSD, I started my PhD studies in co-supervision between UCAD and the University of Wits in Johannesburg, where Florian had got a new position. I graduated in December 2017. This dual program enriched me a lot with professional and academic experience in both universities.

Going to South Africa must have been quite a move. It would have been a no-no until the 90s, and your French would not have been so helpful.
In the past, it would of course have been impossible for me to go. But South Africa is one of the leading countries in Africa in sciences and technologies. It has good infrastructure. I am speaking about libraries and internet, etc.

Is there still much racial tension in South Africa twenty years on? And if so would it affect your life as a university student?

There is still very much tension, Black students usually do not study advanced mathematics so my fellow students were mostly white or international students.

You wanted to eventually return to Senegal, you never considered going to the West?
In Africa you are very close to your family, as we have discussed before, and it would be unthinkable for me not to be close to my parents and siblings, and of course I very much identify with Senegal as a country as well. And besides Senegal has an AIMS institute in Mbour, one of the few in Africa, so from that point of view it is a regional mathematical centre in Africa.

So where are you now? Not in Dakar?
I am not in Dakar, instead in a very small town out in the countryside. At first I was not so comfortable, being used to big cities, like Dakar and Johannesburg, but I got used to it, and now I feel quite comfortable, and appreciate the quietness.

But from a mathematical point of view it is not ideal?
No, of course not. There is a heavy teaching load, there is not much emphasis on research, in fact almost none. But that is to be expected.

But how do you survive mathematically?
I have co-workers all around with whom I correspond and also am able to visit as it is possible to apply for grants. It is not so easy though, and as a student I would have wished that we had given more instruction on how to find grants and how to apply for them.

Which are the granting institutions?
There are of course national ones, but also international ones which are very important. In the past two years, I received grants from the European Mathematical Society (EMS-CDC) who provided a travel grant and living expenses for research visits to Europe. This was very useful for me since I am in an earlier stage of my career and I still need a network of co-workers.

So how do you envision your future ideally?
That I will be able to travel and discuss with other mathematicians, but also that I may be able to interest some students in what I am doing and gradually build up a research group. But it will take time.

But you are young.
Yes, so I guess I can manage it.

Thank you very much for consenting to the interview.
It was a pleasure.

ICM 2018 in Rio – A Personal Account Part I

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

Arrival

As few mathematicians can have failed to notice, the most recent International Congress of Mathematicans took place in Rio de Janeiro. There is always a first, and this time it was stressed that the congress in Rio was the first one to take place in the Southern Hemisphere, just as Kyoto was the first to take place in Asia; Beijing and Seoul were next, to be followed by Hyderabad and now Rio de Janeiro.

The congresses are by now considered to be monstrous affairs very different from the original intimate gatherings where group pictures could be taken. In the past you attended them as part of a national delegation, as exemplified by the Strasbourg congress in 1920 when the Germans were not invited, and not as an individual. This of course made attendance a privilege, not to say an honour, and of course the social mathematical landscape was very different back then. There were fewer mathematicians, and perhaps more to the point, many fewer conferences, so the congresses served very important functions when it came to spreading ideas, functions which are now more or less superfluous. Thus there is widespread criticism of them, and most mathematicians choose not to attend, which of course from a logistic point of view is a blessing: a meeting involving say fifty thousand people would present a nightmare to most organizers. Maybe only the Chinese could manage such challenges. Thus one may a bit cynically claim that there are two kinds of people attending an ICM: those who have to, i.e., invited speakers and prizewinners and a few others serving important functions due to their mathematical status, and those who volunteer to go and are considered tourists. Personally I did not have to, and I did not want to think of myself as a mere tourist, but luckily there is a third category, namely a handful of journalists and representatives for mathematical publishers.[1] My purpose was to interview the Fields medalists for the benefit of the readers of this Newsletter, as I did in Hyderabad and Seoul,[2] but I would be dishonest to dismiss the attraction of the touristic opportunities, this too was my first visit to South America.[3]

My trip to Rio was a long one, with a tight connection at Charles de Gaulle at midnight. Consequently I arrived early in the morning in Rio and as usual being impatient and with a quick step I overshot the welcoming committee strategically placed close to the arrival gate. Its purpose was to provide transportation to the venue. As a result I paced the arrival hall back and forth looking in vain for the ICM transport (was I too early?) attracting the attention of all the cab drivers, one of whom was kind enough to direct me to an ATM machine on the second floor.[4] Eventually I located the welcoming staff and was whisked to the venue in a bus. I then realized that the congress was really taking place far out of central touristic Rio and that I had been lucky indeed to choose a hotel on the premises, and not a more central one for touristic purposes, (but after all I had decided right away, that this is work and not play).

Arriving at the somewhat bleak venue (what else can you expect regardless of location?) there seemed to be some general confusion and large parts were cordoned off for some reasons. There was then rumors of a fire, the effects of which might jeopardize, if not the congress as such, but at least the opening ceremony. Inauspicious beginnings indeed.

Furthermore, the first thing I did at the desk of the hotel was to enquire about transport to central Rio. My enquiries did not meet with any enthusiasm and some rather complicated instructions were given to me, with the caveat that the local bus, necessary for the first step, did not run on any schedule but only intermittently. Then I was given definite warnings that as a tourist I was very vulnerable in central Rio, and it was not a matter of finding yourself unwittingly in a favela, that most likely I would be stripped of my camera, wallet, iPhone or whatever with marketable value. But they added to console me, that most likely I would not get killed, but would probably be left physically unscathed. Although initially somewhat taken aback, my instinct was to scoff at the advice. Was it not their duty perhaps to warn their hapless visitors, maybe to escape possible litigation? I consider myself a seasoned traveller, and during my stint at Columbia in the late seventies I lived on the border with Harlem, and often took the subway late at night. And nothing ever happened to me. So I contacted my Brazilian friend and colleague, whom I think of as a tough lady. Her view was even more dire than that expressed at the hotel desk. That put a stop to my plans for solitary forays and I realized that any touristy initiatives would have to be through the officially arranged tours. It is of course in the spirit of a congress to encourage informal contacts between the participants, and tours are part of a long ICM tradition. And for the benefit of the readers curious about those aspects I will add some comments at the end of the article.

Opening ceremony and the Fields Medalists

The opening ceremony took place as scheduled, although in a different building. There were rumours that due to the fire, things had to be improvised at the last moment. Unlike the cases of all previous Congresses I have attended, since as a young man I stopped after Helsinki in 1978 (with the exception of Kyoto in 1990), there was no security, which of course was very convenient: no long lines, no tedious checks, no humiliation being treated as a potential terrorist. On the other hand there was a simple explanation for this: unlike in the cases of China, Spain, India and South Korea there was no head of state at the ceremony. Is this a sign that Brazil does not take such a congress seriously? In Beijing we were bussed to the opening ceremony along a street closed off for other traffic as befits the VIPs we may in secret consider ourselves to be, but for most of us, that event will turn out to be the only occasion to be subject to such considerations. The opening ceremony also took place at the very centre of Chinese power – the Great Hall of the People – to which, in spite of its name, few people have access. Admittedly the ICM in Beijing was exceptional in this respect and it is unfair to make the comparison. And besides, what can a head of state really contribute to the event, except to signal that it is important? But perhaps Brazil does not really care about mathematics, and by implication other intellectual pursuits; instead it is a hedonistic country geared towards soccer and dancing samba on the beaches. Maybe the congress did not even appear on national news?

In the absence of constraining facts there are no obstacles to speculation which is thus allowed to expand unfettered in all directions. But the opening was in fact spectacular in other ways. Half-naked dancers, billed as Aborigines, with fancy headgear and elaborate tattoos took to the stage as well as to the aisles of the auditorium, performing to loud primordial music. This surely would never have been allowed in the stately Great Hall of the People. Many members of the audience could not contain themselves but stood up, arms outstretched, gazes fixed,

[1] There is also a fourth category, once you take a less cynical view, namely mathematicians from the Third World whom the congresses are making increased efforts to fund. The upcoming one in St. Petersburg has promised to fund a thousand. It stands to reason that this category will become more and more important and add significantly to the *raison d'être* of the congresses.

[2] I take this opportunity to thank the Swedish SveFUM headed by K.O. Widman, whose generous support made possible not only this excursion of mine, but also that of the previous congress in Seoul, as well as incidentally the recent interview with R. Piene which appeared in these pages.

[3] As a mathematician you want to be precise and truthful, traits which outsiders may find pedantic, but back in 1979 I visited the islands of the Netherlands Antilles in the Caribbean, which, I understand, are part of the South American continental socle.

[4] Later I found out that the ones at the airport are notorious for being regularly skimmed and sure enough a few months later the credit card company contacted me concerning a suspicious post. I had the card promptly canceled and suffered no economic damage.

catching the moment on their iPhones, for possible later delectation.

The initial formalities having been dispensed with and some anonymous governmental minister having played the role of the absent head of state, there was time for the climax of the opening ceremony, nay the climax of the whole congress, namely the announcement of the Fields medalists and the presentation of their medals from the hand of the minister. When I was a young man the Fields medalists were old established men whom I already knew of. Nowadays they are mere babes and in most cases carrying names on their badges I am ashamed to say I have never heard of. Is it that mathematics has grown so much in the last forty years, or simply that I have lost touch? But that will of course not stop me from doing my assignment.

In Hyderabad I had no problem getting in touch with the medalists after the ceremony. In Seoul it was very different: a typical medalist was swamped with over a thousand e-mails of congratulations or other attempts at getting their attention, and I was just part of this mob. Eventually it worked itself out. Maybe the situation would be different in Rio; after all, Brazilians do not care about Fields medalists, they are instead heading for the beaches or the soccer stadiums. Maybe I will be able to promptly catch their attention after all? In fact the situation in Rio was somewhat better, although not much, and they all got their share of unsolicited attention. As a result I only got in touch with two of them, namely Alessio Figalli and Akshay Venkatesh. Unfortunately Peter Scholze was already at the airport leaving early for Germany when he read my message,[5] and Caucher Birkar regretfully wrote to me that he only came across my e-mail after he had returned home, so they were counted out. Thus in the previous issue of the Newsletter there were only two interviews, but I hope to complement them with the remaining two in a later one.

The conclusion of the ceremony involved the return of the dancers, now including some on stilts towering high above the audience. But the real drama of the ceremony, and which later made the news and put the spotlight on one of the Fields medalists, went unnoticed at the time. It involved the theft of Birkar's Fields medal, whether by design or accident I am not sure; anyway his bag, which he left momentarily unattended on a table in connection with the aftermath of the ceremony, was stolen, containing among other things the magic medal. So this is what happens when you do not have strict security. I do not know whether the culprits were ever identified and apprehended, but I doubt it, and anyway the organizers took no risks but managed to fix a second medal in just a few days (for some reason they had an extra base for one, only needing to be properly engraved), and as a result Birkar got the distinction, surely never to be repeated, of getting awarded the Fields medal twice.

But the Fields medal is not the only prize given out: we all know of the Nevanlinna Medal to be given for work in computer science[6] and in later years there have been further additions such as the Gauss Prize, the Chern Medal, and the Leelavati Award, the latter for the communication of mathematics to a wider audience. Those distinctions were given this time respectively to Constantinos Daskalakis, David Donaho, Masakai Kashiwara and Ali Nesin, but nevertheless the Fields Medal is the one which gets the attention. While most prizes merely confirm greatness, the Fields medal actually bestows it, and with the increasing number of mathematicians and the concomitant anonymization of the same, the need for it has increased as well.

Part II of this personal account will be published in the next issue of the EMS Newsletter.

[5] Ironically he was the one I was sure I would catch as I could use the channel of his mentor and my old friend Michael Rapoport, who actually introduced us at an early stage of the congress, and I could have easily set up an appointment on the spot, but failed to do so, as I thought it would be unnecessary!

[6] It has been suggested that the name should be changed, because seventy years after its end it has been decided that he was after all politically unsavory, in short a fascist, during the Second World War. It was discussed at the General Assembly and it was decided that the Nevanlinna Prize should be discontinued and the IMU Executive Committee be saddled with the task of setting up a new prize with appropriate funding but with the same scope and purpose as the former. And of course with a new name. Politics aside, one may wonder what Nevanlinna had to do with computer science. The prize was set up and funded by the Finns in connection with the Helsinki Congress in 1978, and was awarded for the first time at the next one, and they naturally wanted to secure a Finnish connection to it by naming it after one of their very great mathematicians, who was incidentally still alive at the time.

The 100th Jubilee of Riesz Theory

Albrecht Pietsch (University of Jena, Germany)

The *Riesz spectral theory of compact operators* was created just 100 years ago. This jubilee is a very welcome occasion for writing a detailed appreciation. Surely, the Riesz paper [24] and Banach's monograph [2], which grew out of his thesis [3], are the most important publications of classical Banach space theory; see also [21]. The presentation by both these authors is so convincing that most of their proofs can be used in today's lectures.

The Riesz paper was finished in Győr (Hungary) on January 19, 1916, and printed on December 3–5 1916. However, in most bibliographies it is dated to the year 1918, when volume 41 of *Acta Mathematica* was completed; a consequence of World War I. By the way, there is also a Hungarian version from February 14, 1916 with the title "Lineáris függvényegyenletekről"; see [25].

We stress the fact that F. Riesz wrote his contribution at a time when the concept of an abstract Banach space did not exist. Indeed, claiming in [24, p. 71]

> *Der in den neueren Untersuchungen über diverse Funktionalräume bewanderte Leser wird die allgemeinere Verwendbarkeit der Methode sofort erkennen,*

he exclusively used the space $C[a,b]$ of continuous functions on an interval $[a,b]$, equipped with the sup-norm. In other words, he stated that almost all of his results remain true in abstract Banach spaces.

Around the year 1905, D. Hilbert and his pupil E. Schmidt had developed a determinant-free approach to Fredholm's theory of integral equations of the second kind, which is based on ℓ_2 and $L_2[a,b]$ (in a hidden form); see [11]. Subsequently, F. Riesz introduced the spaces $L_p[a,b]$ and ℓ_p with $1 \le p < \infty$; see [26, 27]. Since orthogonality was no longer available for $p \ne 2$, he tried to overcome this trouble in the most simple case, namely, $C[a,b]$.

To understand the situation in which F. Riesz started his investigations and to see their most important applications, the reader may consult the beautiful survey on *'Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten'* [10] written by E. Hellinger and O. Toeplitz.

D. Hilbert treated completely continuous bilinear forms on $\ell_2 \times \ell_2$, and F. Riesz observed that the corresponding operators are characterised by the property that weakly convergent sequences are mapped to norm convergent sequences; see [27, p. 96]. On the other hand, in the paper under review, he refers to an operator T on $C[a,b]$ as *completely continuous* if every bounded sequence (f_n) contains a subsequence whose image (Tf_{n_i}) is norm convergent. For the moment, we will distinguish these concepts by saying that an operator is completely continuous 'in the sense of Hilbert' or 'in the sense of Riesz'. Naturally, F. Riesz knew that both kinds of complete continuity coincide for operators on ℓ_2. For general spaces, completely continuous operators in the sense of Riesz are also completely continuous in the sense of Hilbert, whereas the converse implication fails. However, the following counterexamples were available only later. J. Schur [30, § 4] showed that the identity map of ℓ_1 is completely continuous in the sense of Hilbert and, of course, it fails to be completely continuous in the sense of Riesz. Moreover, let $C(\mathbb{T})$ be the space of all continuous 2π-periodic functions. Then the rule

$$P : f(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{int} \mapsto (\gamma_{2^k})_{k=0}^{\infty}$$

defines a 2-summing operator from $C(\mathbb{T})$ onto ℓ_2, which is completely continuous in the sense of Hilbert but not in the sense of Riesz; see [33, Sec. III.F].

In Hille's monograph *'Functional analysis and semi-groups'* [13, p. 49] the term 'compact' was used instead of completely continuous in the sense of Riesz. Luckily, this proposal has prevailed and our temporary suffix 'in the sense of Hilbert' becomes unnecessary. From now on, we will employ the attributes 'compact' and 'completely continuous' in this way, which has become standard.

Let $\mathfrak{L}(X,Y)$ denote the Banach spaces of all (bounded, linear) operators from the Banach space X into the Banach space Y. If $X = Y$, then we simply write $\mathfrak{L}(X)$ instead of $\mathfrak{L}(X,X)$. The identity map of X is denoted by I or, more precisely, by I_X. Every $A \in \mathfrak{L}(X,Y)$ has the range $\mathcal{M}(A) := \{Ax : x \in X\}$ and the null space $\mathcal{N}(A) := \{x \in X : Ax = o\}$.

The main results of F. Riesz say that the following properties hold for any compact operator $T \in \mathfrak{L}(X)$:

If

$$\mathcal{M}((I-T)^m) = \mathcal{M}((I-T)^{m+1}),$$

then

$$\mathcal{M}((I-T)^{m+1}) = \mathcal{M}((I-T)^{m+2}).$$

The ranges $\mathcal{M}((I-T)^m)$ are closed and form a non-increasing sequence, which stabilizes for some index m_0.

If

$$\mathcal{N}((I-T)^n) = \mathcal{N}((I-T)^{n+1}),$$

then

$$\mathcal{N}((I-T)^{n+1}) = \mathcal{N}((I-T)^{n+2}).$$

The null spaces $\mathcal{N}((I-T)^n)$ are finite-dimensional and form a non-decreasing sequence, which stabilizes for some index n_0.

The indices m_0 and n_0 coincide when they are chosen as small as possible; their joint value is denoted by p. Then X is the direct sum of the T-invariant subspaces $\mathcal{M}((I-T)^p)$ and $\mathcal{N}((I-T)^p)$.

In the *regular* case $p = 0$, the operator $I - T$ is an isomorphism. In other words, the equation $x - Tx = a$ admits a unique solution $x \in X$ for every $a \in X$. In the *singular* case $p > 0$, the restriction of $I - T$ to $\mathcal{M}((I-T)^p)$ is an isomorphism and the restriction to the finite-dimensional space $\mathcal{N}((I-T)^p)$ is nilpotent. This means that solving the equation $x - Tx = a$ is reduced to a problem of classical linear algebra, at least in principle.

The general concept of a *dual* (adjoint, conjugate) operator, which is based on the Hahn–Banach extension theorem, was introduced only at the end of the 1920s; see [4, Théorème 1]. Hence F. Riesz had to restrict his considerations to the very special case of *transposed* integral equations, which are generated by continuous kernels $K(x, y)$ and $K(y, x)$. The missing keystone was laid by J. Schauder [29]. His main result says that the dual operator $T^* : Y^* \to X^*$ is compact if and only if so is the original operator $T : X \to Y$. Moreover, $\dim(\mathcal{N}(I^* - T^*)) = \dim(\mathcal{N}(I - T))$. In view of this important contribution, the joint outcome is often called the *Riesz–Schauder theory*.

An intermediate result is due to T. H. Hildebrandt [12], who used – in a hidden form – the codimension of $\mathcal{M}(I - T)$. Indeed, in view of the fact that $\mathcal{N}(I^* - T^*)$ and $[X/\mathcal{M}(I - T)]^*$ are isometric, we get $\dim[\mathcal{N}(I^* - T^*)] = \operatorname{cod}[\mathcal{M}(I - T)]$. Therefore $\dim[\mathcal{N}(I^* - T^*)] = \dim[\mathcal{N}(I - T)]$ is equivalent to the formula $\operatorname{cod}[\mathcal{M}(I - T)] = \dim[\mathcal{N}(I - T)]$, which does not require the knowledge of any dual operator.

Let $A \in \mathfrak{L}(X, Y)$. Following [9, pp. 307–308], the equation $Ax = b$ is said to be *normally solvable* provided that, for given $b \in Y$, there exists a solution $x \in X$ if and only if $\langle b, y^* \rangle = 0$ whenever $A^* y^* = \mathrm{o}$. Remarkably, this happens just in the case when the range $\mathcal{M}(A)$ is closed. Hence all operators $I - T$ with compact $T \in \mathfrak{L}(X)$ are normally solvable.

The preceding results remain true when $I - T$ is replaced by $I - \zeta T$ with any complex parameter ζ. If $I - \zeta T$ is singular, then we call ζ a *characteristic value*. F. Riesz proved that the characteristic values have no finite accumulation point. Note that he referred to those numbers as *eigenvalues*. This term is now commonly used for $\lambda \in \mathbb{C}$ when working with the scale $\lambda I - T$.

The Riesz paper has stimulated many remarkable developments. Some of them will be sketched in the rest of this review. For more detailed information the reader is referred to [19, Sect. 2.6, Subsect. 5.2.2, 5.2.3, and 8.3.1 (short biography)].

Already F. Riesz [24, p. 74] has observed that the class of compact operators is an ideal, now denoted by $\mathfrak{K}$. Further related ideals are $\mathfrak{F}$, the class of finite rank operators, and $\mathfrak{V}$, the class of completely continuous operators. Note that $\mathfrak{F} \subset \mathfrak{K} \subset \mathfrak{V}$. From Banach's monograph [2, Chap. VI, Théorème 2], we know that $\mathfrak{K}$ is closed in the norm topology of $\mathfrak{L}$. Therefore the closure $\overline{\mathfrak{F}}$, whose members are the *approximable* operators, is contained in $\mathfrak{K}$. A long-standing open problem asked whether even equality holds. The famous negative answer was finally given by P. Enflo [6] when he constructed a Banach space without the *approximation property*.

According to the Russian terminology, $A \in \mathfrak{L}(X, Y)$ is referred to as a Φ*-operator* (Φ stands for Fredholm) if there are operators $U, V \in \mathfrak{L}(Y, X)$, $S \in \mathfrak{K}(X)$, and $T \in \mathfrak{K}(Y)$ such that $UA = I_X - S$ and $AV = I_Y - T$; see [7, p. 195]. The preceding definition means that A is invertible modulo the ideal $\mathfrak{K}$, which can even be replaced by $\mathfrak{F}$. We know from F. V. Atkinson [1, Theorem 1] that Φ-operators are characterised by the property of having finite-dimensional null spaces and finite-codimensional closed ranges. By the way, a famous lemma of T. Kato [14, p. 275] says that $\operatorname{cod}[\mathcal{M}(A)] < \infty$ automatically implies that $\mathcal{M}(A)$ is closed.

F. Riesz has shown that, for compact T, all operators $I - \zeta T$ with $\zeta \in \mathbb{C}$ have very nice properties. Therefore the question arose whether his results hold for more general operators. In a first step, S. M. Nikolskij [16] confirmed this expectation for operators that admit a compact power. To formulate a complete answer, we refer to $T \in \mathfrak{L}(X)$ as a *Riesz operator* if every $I - \zeta T$ with $\zeta \in \mathbb{C}$ behaves in the desired way. To treat real operators, one must pass to their complexifications. Riesz operators can be characterised by various conditions of a quite different flavour.

(1) Every $I - \zeta T$ with $\zeta \in \mathbb{C}$ is a Φ-operator.
(2) According to A. F. Ruston [28, Theorem 3.1], T is *quasinilpotent* with respect to the quotient norm

$$|||T^n||| = \inf\{\|T^n - K\| : K \in \mathfrak{K}(X)\}.$$

This means that

$$\lim_{n\to\infty} |||T^n|||^{1/n} = 0.$$

(3) The resolvent $(I - \zeta T)^{-1}$ is a meromorphic $\mathfrak{L}(X)$-valued function on the complex plane such that the singular part of the Laurent expansion at every pole has finite rank coefficients; see [24, p. 90], [27, pp. 113–121], [29, Footnote 18 on p. 193], [5, p. 198], and [31, p. 660]. Note that the characteristic values coincide with the poles, whose order is just the index p at which the sequences $\{\mathcal{M}((I - \zeta T)^m)\}$ and $\{\mathcal{N}((I - \zeta T)^n)\}$ stabilize.
(4) For every $\varepsilon > 0$ there exists some n such that $T^n(B_X)$ can be covered by a finite number of balls $y + \varepsilon^n B_X$. Here B_X denotes the closed unit ball of X. A presentation of the Riesz theory based on a slightly modified geometric property is given in [18, Sec. 3.2 and 7.4.1].

T. T. West [32, Counterexamples] observed that the set of all Riesz operators on some Banach space may fail to be closed under addition, multiplication, and passing to the limit with respect to the operator norm. So it makes sense to look for closed ideals $\mathfrak{A}$ such that all components $\mathfrak{A}(X)$ consist of Riesz operators. The classical examples are $\overline{\mathfrak{F}}$ and $\mathfrak{K}$.

A much larger ideal $\mathfrak{S}$, introduced by T. Kato [14, pp. 284–288], consists of the *strictly singular* (semicompact) operators $T \in \mathfrak{L}(X, Y)$ defined by the following property: If there exists a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all x in a closed subspace M, then M is finite-dimensional. Dualisation yields the closed ideal of *stricly cosingular* (co-semicompact) operators. Both ideals are extensively treated in a monograph [22, pp. 252–263, 315–317] written by D. Przeworska-Rolewicz and S. Rolewicz. An increasing 1-parameter scale of closed ideals lying between $\mathfrak{K}$ and $\mathfrak{S}$ was constructed in [20].

The largest ideal of this kind, here abbreviated by $\mathfrak{N}$, was introduced in [17, p. 57]. Its components $\mathfrak{N}(X, Y)$ are formed by all $T \in \mathfrak{L}(X, Y)$ such that $I_X + AT$, or equivalently $I_Y + TA$, is a Φ-operator for every $A \in \mathfrak{A}(Y, X)$. Since this definition is based on earlier results of I. Ts. Gokhberg, A. S. Markus, and I. A. Feldman [8] as well as of D. Kleinecke [15], the members of $\mathfrak{N}$ were called *Gochberg operators* or *inessential* (which does not mean that *Gochberg* is *inessential*).

Bibliography

[1] F. V. Atkinson, Die normale Auflösbarkeit linearer Gleichungen in normierten Räumen, *Mat. Sb., N. Ser.* 28, 3–14, (1951; Zbl0042.12001).

[2] S. Banach, *Théorie des opérations linéaires*, PAN, Warszawa, (1932; Zbl0005.20901).

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.* 3, 133–181, (1922; JFM 48.0201.01).

[4] S. Banach, Sur les fonctionnelles linéaires. II, *Studia Math.* 1, 223–239, (1929; JFM 55.0240.01).

[5] N. Dunford, Spectral theory I, Convergence to projections, *Trans. Am. Math. Soc.* 54, 185–217, (1951; Zbl0063.01185).

[6] P. Enflo, A counterexample to the approximation problem in Banach spaces, *Acta Math.* 130, 309–317, (1973; Zbl0067.46012).

[7] I. C. Gohberg, M. G. Kreĭn, The basic propositions on defect numbers , root numbers and indices of linear operators, *Usp. Mat. Nauk* 12, 43–118, (1957; Zbl0088.32101), *Am. Math. Soc., Transl., II*, Ser. 13, 185–264, (1960; Zbl0089.32201).

[8] I. Ts. Gokhberg, A. S. Markus, I. A. Feldman, Normally solvable operators and ideals associated with them, *Am. Math. Soc., Transl., II*, Ser. 61, 63–84 (1967); translation from *Izv. Mold. Fil. Akad. Nauk SSSR*, 10 (76), 51–69; (1960; Zbl0081.40601).

[9] F. Hausdorff, Zur Theorie der linearen metrischen Räume, *J. Reine Angew. Math.* 167, 294–311, (1932; JFM 58.1113.05 and Zbl0003.33104).

[10] E. Hellinger; O. Toeplitz, *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten*, Leipzig, B. G. Teubner (Encyklopädie der mathematischen Wissenschaften, II C 13), (1927 JFM 53.0350.01).

[11] D. Hilbert; E. Schmidt, Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten, *Teubner-Archiv zur Mathematik*, 11, (1989; Zbl0689.01012).

[12] T. H. Hildebrandt, Über vollstetige lineare Transformationen, *Acta Math.* 51, 311–318, (1928; JFM 54.0427.03).

[13] E. Hille, *Functional analysis and semi-groups,* AMS Colloquium Publ. 31, New York, (1948; Zbl0033.06501).

[14] T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Analyse Math.* 6, 261–322, (1958; Zbl0090.09003).

[15] D. Kleinecke, Almost-finite, compact, and inessential operators, *Proc. Am. Math. Soc.* 14, 863–868, (1963; Zbl0117.34201).

[16] S. M. Nikolskij, Lineare Gleichungen in metrischen Räumen, *Doklady Akad. Nauk SSSR* 2, 315–319, (1936; JFM 62.0452.03).

[17] A. Pietsch, *Theorie der Operatorenideale* (Zusammenfassung), Eigenverlag der Friedrich-Schiller-Universität, Jena, 1972, (1972; Zbl0238.46067).

[18] A. Pietsch, *Eigenvalues and s-numbers*. Cambridge Univ. Press, (1987; Zbl0615.467019).

[19] A. Pietsch, *History of Banach spaces and linear operators*. Birkhäuser, Boston, (2007; Zbl1121.46002).

[20] A. Pietsch, A 1-parameter scale of closed ideals formed by strictly singular operators, The Bernd Silbermann Anniversary Volume, *Oper. Theory, Adv. Appl.* 135, 261–265, Birkhäuser, Basel, (2002; Zbl1036.47010).

[21] A. Pietsch, 'Looking back' review on the occasion of Banach's 125th birthday in 2017, (Zbl0005.20901).

[22] D. Przeworska-Rolewicz, S. Rolewicz, *Equations in linear spaces*, PAN, Warszawa, (1968; Zbl0181.40501).

[23] F. Riesz *Gesammelte Arbeiten*, Band I–II, Herausgegeben von Á. Császár, Verlag der Ungarischen Akademie der Wissenschaften, Budapest, (1960; Zbl0101.00201).

[24] F. Riesz, Über lineare Funktionalgleichungen, *Acta Math.* 41, 71–98, (1916; JFM 46.0635.01).

[25] F. Riesz, Lineáris függvényegyenletekről, see [23, Vol. II, pp. 1017-1052].

[26] F. Riesz, Untersuchungen über Systeme integrierbarer Funktionen, *Math. Ann.* 69, 449–497, (1910; JFM 41.0383.01).

[27] F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Paris, Gauthier-Villars, (1913; JFM 44.0401.01).

[28] A. F. Ruston, Operators with a Fredholm theory, *J. London Math. Soc.* 29 318–326, (1954 Zbl0055.10902).

[29] J. Schauder, Über lineare, vollstetige Funktionaloperationen, *Studia Math.* 2, 183–196, (1930; JFM 56.0354.01).

[30] J. Schur, Über lineare Transformationen in der Theorie der unendlichen Reihen, *J. für Math.* 151, 79–111, (1920; JFM 47.0197.01).

[31] A. E. Taylor, Analysis in complex Banach spaces, *Bull. Am. Math. Soc.* 49, 652–669, (1943; Zbl0063.07311).

[32] T. T. West, Riesz operators in Banach spaces, *Proc. London Math. Soc.*, III. Ser. 16, 131–140, (1966; Zbl0139.08401).

[33] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge Studies in Advanced Math. 25, Cambridge, (1991; Zbl0724.46012).

Albrecht Pietsch [a.pietsch@uni-jena.de], born 1934 in Zittau (Saxony), studied at the University of Leipzig from 1953 to 1958. He received the Dr. rer. nat. from TU Dresden in 1959 and the Dr. rer. nat. habil. from Humboldt University Berlin in 1963. Starting in 1965, Pietsch was a professor of analysis at the University of Jena for 35 years, where he supervised 27 PhD students. Besides his monographs Nuclear locally convex spaces, Operator ideals, Eigenvalues and *s*-numbers, Orthonormal systems and Banach space geometry *(co-author: J. Wenzel) and* History of Banach spaces and linear operators*, he published more than 100 research papers. From 1971 to 1990, 'Nationalpreisträger' Pietsch was a member of the* Akademie der Wissenschaften der DDR*, and since 1974 he has been a member of the* Leopoldina *(now* Nationale Akademie der Wissenschaften *Deutschlands). Honorary doctorates were awarded to him by the Universities of Paderborn (1998) and Pretoria (2008). Present status: Ostdeutscher Rentner.*

Up to a technical change concerning references, this article coincides with the 'Looking back' review ([21], 1916; JFM 46.0635.01).

Mathematics in the Historical Collections of École Polytechnique Part II

Frédéric Brechenmacher (École Polytechnique, Palaiseau, France)

Following the discussion developed in the previous issue of the EMS newsletter on the place of mathematics in the reserve of ancient books and in the museum collection of École polytechnique, the present paper is devoted to mathematics in the archives of École polytechnique.

The archives

When École polytechnique was created in 1794, its new library was given the responsibility of preserving some of the archives produced by the school, in addition to the rich collection of ancient books which had been gathered from revolutionary seizures, as described in the first part of this paper.[1] The historical value of the collection of ancient books benefitted from early recognition, which, in turn, helped preserve the archives.

The historical, administrative, and scientific archives of École polytechnique shed important light on various aspects of the evolution of mathematics since the creation of the school. The administrative archives especially contain the reports of the committees in charge of the various aspects of the school's academic programs, such as the structure and organisation of the competitive entrance exams and of the school's curriculum, and their relation to the expectations of both the specialised Écoles d'application, where the students complete their applied studies, and of future employers of École polytechnique graduates – businesses, administrative offices, research organisations, etc. The issue of the roles attributed to the teaching of mathematics has been a key topic in the debates of the school's committees for more than two centuries [Belhoste, 2001].

When the school was founded in 1794, its main features were the competitive entrance examination, the importance of mathematics, and the association of technical and mathematical education with military education.[2] The founding professors ("instituteurs") of analysis and mechanics were Joseph-Louis Lagrange and Gaspard Riche de Prony. Descriptive and differential geometry was in the hands of Gaspard Monge, who also served as director for two short periods. Each instituteur had an assisting adjoint, who were named "répétiteurs" after 1798. Among notable adjoints or répétiteurs in mathematics, one may cite Joseph Fourier, Auguste Comte, Edmond Laguerre, and Henri Poincaré. One major feature was the distinction made between teaching and examining so examiners were also appointed. For mechanics and analysis the initial examiners were Charles Bossut and Pierre-Simon Laplace.

Pierre-Simon Laplace. © Collections École polytechnique.

Mathematics served as the dominant criterion in the entrance examinations [Belhoste, 2002]. Even more important was the focal role of mathematics in the curriculum. The view that among all the sciences necessary to both civil and military engineers, mathematics has the most considerable rank was already common before the Revolution when mathematical education was considered an integral part of military education. More and more, mathematical culture was being identified with military culture in opposition to the juridical culture of the "noblesse de robe". The interest in mathematics arose not merely because of its direct usefulness: mathematics, and especially instruction in mathematics, was seen to have valuable moral uses. It sharpened powers of reasoning and inculcated an orderly manner of thinking. Furthermore, the learning process of mathematics was considered to foster habits of work, self-control, and discipline.

While the school had originally been conceived as the one and only institution to train engineers, the impracticality of the vision was soon recognised and the role of the school was thus changed in 1795 to that of a preparatory institution for the other schools, which were organised into a collection of "écoles d'application". This change would have important consequences for the roles attributed to the teaching of mathematics, especially through the influence of Laplace [Belhoste, 1994]. For six weeks in 1799 Laplace performed as Minster of the Interior. He proposed that the school have a governing council, the "Conseil de Perfectionnement", to supplement the "Conseil d'Instruction" on teaching details, and

[1] This responsibility was extended in the 19th century to the personal fonds donated by alumni and eventually in the 1930s to all the institutional archives produced by the school. On the archives of École polytechnique and their history, see [Billoux, 1985], [Billoux, 1989], [Billoux, 1999], [Azzola, 2013].

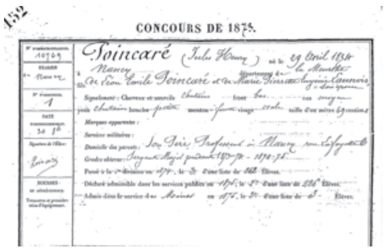
[2] For a short synthesis on the role played by mathematics in the educational purpose of the school from 1794 to 1850 see [Grattan-Guinness, 2005].

a “Conseil d’Administration” for management. Laplace was one of its founding members, and he exercised much influence there, in particular reducing the time given to Monge’s descriptive geometry and transferring much of it to mechanics and analysis. This opposition was led mainly by Laplace’s desire to confine the programmes at École polytechnique to teaching general theories, which would then be applied in the other more specialised schools. This kind of difference over curriculum policy in the school would continue for a long time: the archives of the reports of the school’s councils highlight that the issue of the roles attributed to mathematics fuelled a never-ending tension in the school curriculum, between the general and the special, and between the theoretical and the applied.[3]

École polytechnique’s archives do not only shed light on the school’s academic programs but also on the more general evolution of the French educational system. Because of the reorganisation and centralisation of the system of state technical schools as “écoles d’application” for École polytechnique in 1795, the standards and practices of the school were spread to other schools beyond the artillery and military engineering school. École polytechnique actually came to play a central role in French education in serving as the model for the French system of “grandes écoles”. The Polytechnique, coupled with the Revolutionary and Napoleonic attempts to create a centralised, uniform system of education, did much to impose national standards of mathematical instruction in France and abroad.

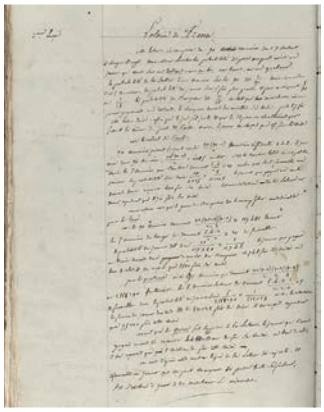
The administrative archives of École polytechnique also contain the registration files (“fiches matricules”) of all the students of the school. These files have been both digitalised and type-scripted in a database which is publicly available online for the time period 1794–1943.[4] In addition to administrative information about the schooling of each student, including their entrance and exit rankings, these files provide biographical information about their genealogy, physical descriptions (complemented by a photograph after 1861), information about the école d’application the student joined after the Polytechnique and, quite often, details about professional careers. These files therefore provide crucial information for historians of mathematics. They especially shed light on the diversity in the careers of polytechnicians and, thereby, on the diversity of practitioners of mathematics over time.

The scientific archives of École polytechnique provide a lot of information about the courses of mathematics since the creation of the school, as well as about other academic activities such as the publication of the *Journal de l’École polytechnique*, which was initially established to fulfil the requirement of publishing lecture courses, but soon evolved into an academic journal publishing genuine research, mostly (and eventually solely) in mathematics,[5] as well as the activity reports of the two mathematics laboratories of the school, the Laurent Schwartz Mathematics Centre (CMLS), created in 1965, and the Centre of Applied Mathematics (CMAP) created in 1974.


CONCOURS DE 1873.
Poincaré (Jules Henri)

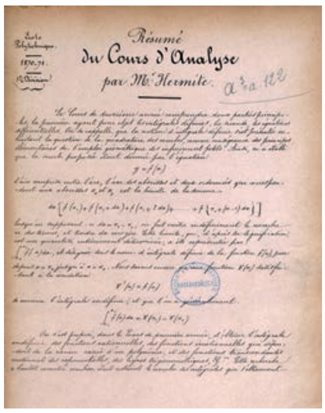
The registration file of Henri Poincaré in 1873. © Collections École polytechnique.

In the early years, several courses appeared at first in the school’s journal, and later as books edited by Paris publishers. The archives allow us to compare this published material to the courses that were actually taught to students, which are documented by handouts that were at first copied by hand and later reproduced by authography or other techniques. These hand-out copies allow

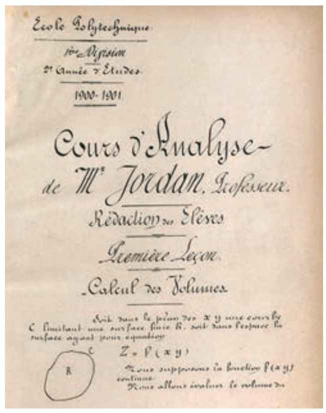


Manuscript of François Arago’s course on social arithmetics, 1815. © Collections École polytechnique.

Manuscript of Joseph Liouville’s course on differential calculus, 1847. © Collections École polytechnique.


Résumé
du Cours d’Analyse
par Mr Hermite

Manuscript of an abstract of Charles Hermite’s course on analysis, 1870–1871. © Collections École polytechnique.


École Polytechnique
1re Division
2e Année d’Études
1900-1901
Cours d’Analyse
de Mr Jordan, Professeur.
Rédaction des Élèves
Première Leçon
Calcul des Volumes

Manuscript of Camille Jordan’s course on analysis, 1900–1901. © Collections École polytechnique.

[3] See [Belhoste, 2001] and, for a case study at the time of World War I, [Gilain, 2014].

[4] According to French legislation on archival materials, only the registration files of students who graduated more than 75 years ago can be made public. To access the registration file database, see https://bibli-aleph.polytechnique.fr.

[5] The publication of the *Journal de l’École polytechnique* started in 1795 and stopped in 1939 until the journal was revived in 2013. See https://jep.math.cnrs.fr/index.php/JEP/.

us to investigate in minute detail the evolution of specific aspects in the teaching of mathematics. To give just one example, the yearly editions of the handouts of Jacques Hadamard's course on analysis between 1912 and 1937 illustrate the gradual incorporation of vector calculus in the mathematics curriculum.

A number of historical studies have been devoted to several aspects of the teaching of mathematics at École polytechnique, such as with the development of a curriculum in calculus centred on the concepts of functions and limits from Lagrange to Cauchy,[6] or with the role played by the three editions of Jordan's *Cours d'analyse* in providing new foundations to the theory of functions by incorporating the contemporary works of mathematicians such as Weierstrass or Peano.[7] In addition to the handouts of the courses, the archives contain many other types of documents, manuscripts of mathematical memoirs, such as of Louis Poinsot and Jean-Victor Poncelet [Belhoste, 1998], notebooks of professors, such as Georges Humbert who was in charge of the *Cours d'analyse* from 1895–1920, reading notes on various mathematical publications taken by professors, such as Joseph Liouville [Verdier, 2010], handwritten notes taken by students, such as August Comte [Gentil, 2002] or Benoit Mandelbrot, and manuscripts documenting the activities of the répétiteurs who were initially in charge of assisting the professors by organising individual oral examinations, and by supervising the works of the students.

Augustin-Louis Cauchy. © Collections École polytechnique.

Siméon Denis Poisson. © Collections École polytechnique.

Principal academic staff in mathematics at École polytechnique from 1794 to 1959

Analysis		Geometry		Mechanics[8]	
Prony	1794–1814	Monge	1794–1809	Lagrange	1794–1798
Ferry	1794–1798	Hachette	1794–1815	Fourier	1795–1797
Labey	1799–1806	Arago	1810–1815	Garnier	1798–1801
Ampère	1807–1808	Leroy	1816–1848	Lacroix	1799–1808
Poinsot	1809–1811	De La Gournerie	1849–1863	Poisson	1802–1814
Labey	1812–1814	Mannheim	1864–1901	Ampère	1809–1827
Cauchy	1815–1829	Haag	1901–1911	Mathieu	1828–1832
Arago	1816–1829	d'Ocagne	1912–1936	Navier	1833–1836
Savary	1830–1841	Julia	1936–1965	Duhamel	1837–1839
Navier	1831–1832			Sturm	1840–1850
Mathieu	1833–1838			Delaunay	1851–1871
Liouville	1839–1850			Belanger	1851–1860
Chasles	1842–1850			Bour	1861–1865
Duhamel	1851–1868			Philipps	1866–1879
Sturm	1851–1855			Resal	1872–1895
Bertrand	1856–1894			Bresse	1880–1883
Hermite	1869–1876			Sarrau	1884–1895
Jordan	1877–1911			Léauté	1894–1904
Humbert	1895–1920			Lecornu	1904–1927
Hadamard	1912–1937			Painlevé	1905–1933
Lévy	1920–1959			Jouguet	1927–1941
Chapelon	1937–1954			Platrier	1933–1952
Favard	1954–1964			Beghin	1941–1948
Schwartz	1959–1980			Roy	1948–1969
Mandel	1952–1973				

[6] See [Grabiner, 1981], [Belhoste, 1984], [Fraser, 1985], [Gilain, 1989], [Grattan-Guinness, 1990], [Bottazzini, 1992], [Bradley-Sandifer, 2009], [Phili, 2012].
[7] See [Gispert, 1982] and [Brechenmacher, 2016a].
[8] Mechanics was not distinguished from analysis before 1832.

A caricature of Hadamard in *Le petit crapal*. © Collections École polytechnique.

The historical archives of the school contain a lot of information about the social activities of the students, including richly illustrated student journals, such as *Le petit crapal* (1896–1932), and the *Séances des Ombres* (1893–1929), which consisted of a theatre of Chinese shadows made of cardboard caricatures, accompanied by songs written and performed by the students, and which were often dedicated to the professors and their teaching.

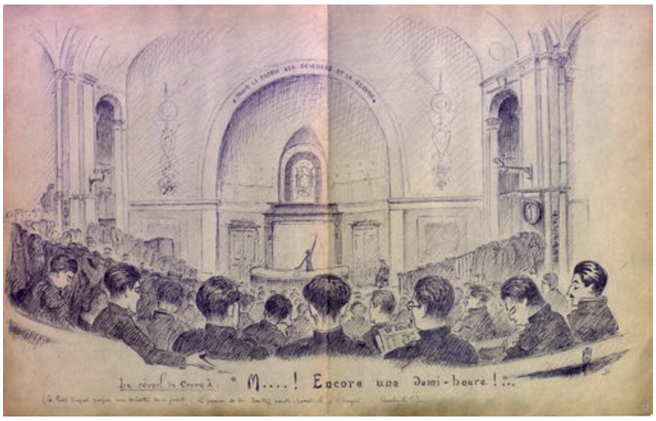
A caricature of Jordan's cours d'analyse for the Ombres in 1901. © Collections École polytechnique.

These archives provide rare and important sources for the history of mathematical education: while historians can usually make use of sources documenting the works of professors, and sometimes of students, the archives of the social activities of polytechnicians provide much rarer information, such as comments, caricatures and criticisms. This material especially sheds light on the reception by the students of the courses they were taught.

The archives of the students also highlight the central role played by mathematics in the construction of the persona of the polytechnicians. Often used to denote the École, its students, or its alumni, the letter X can be seen as an allusion both to the traditional symbol of the unknown in an algebraic equation and to the crossing cannon displayed on its coat of arms, a reminder of the privileged links the École polytechnique always had with artillery. The students' academic slang expressions also highlight the central place of mathematics in the identity of polytechnicians, whose swords are, for instance, designated as "tangents". In contrast, foreign students, who did not have the right to wear a sword in the 19th century, used to be designated as "constants" because their tangent was equal to 0. This central place of mathematics is especially made clear in the 19th century dictionary of student slang. At the entry "ana" (for analysis), a succession of anecdotes about the courses of Lagrange, Poisson, Ampère, Sturm and Hermite come to the following conclusion:

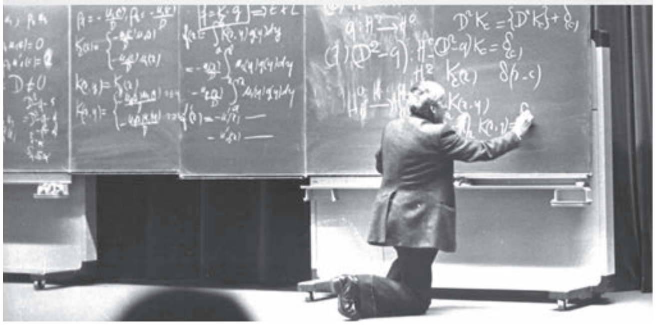
A caricature of the geometric theory of contact by a student of the promotion 1922. © Collections École polytechnique.

> It is of the very importance devoted to the teaching of *ana*, which language is made of x's and y's, that comes the universal use of the nickname X for designating the polytechnicians. Even though the polytechnicians do not always become mathematicians, all of them have enough knowledge in differential and integral calculus for the needs of the public services. At the most troubled times of our history, in 1830 and 1848, this knowledge has been particularly useful to distinguish the polytechnicians from all the individuals who costumed themselves in polytechnicians to pretend they aimed at maintaining order. When we met those, we asked them to differentiate $\sin x$ or $\log x$, and if they did not succeed to provide the right answer, we immediately sent them to jail [Albert-Lévy and Pinet, 1894, p. 25].



A drawing representing students sleeping during a plenary lecture at École polytechnique. © Collections École polytechnique.

École polytechnique has also maintained a collection of audio recordings, films, and videos. This collection dates back to the 1930s, even though most of the films were shot after the school was moved from Paris to Palaiseau in the 1970s. It documents a variety of activities, such as the students' military training, social activities, official events and scientific lectures. This audio-visual collection especially contains a great number of mathematics lectures after 1976, such as those of Laurent Schwartz, which have recently been digitalised. The collection also contains a number of films that were made by mathematicians at École polytechnique for other universities, as well as for private companies, in the 1980s and 1990s, and which especially document the development of mathematical models and computer simulations in those years.

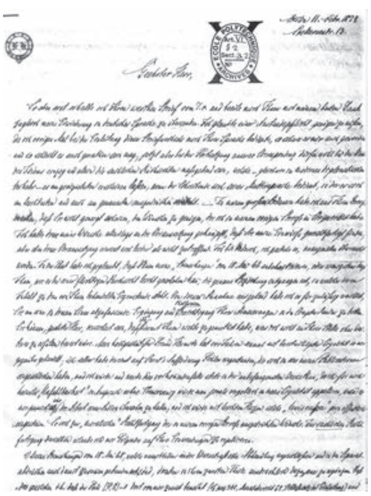


Screenshot of a lecture by Laurent Schwartz in 1976. © Collections École polytechnique.

Personal archive fonds

The historical collections maintain a series of personal archive fonds, either donated to the school by professors or alumni, or purchased by alumni associations such as the AX (the association of alumni of École polytechnique) or the SABIX (the association of the friends of the school's library).

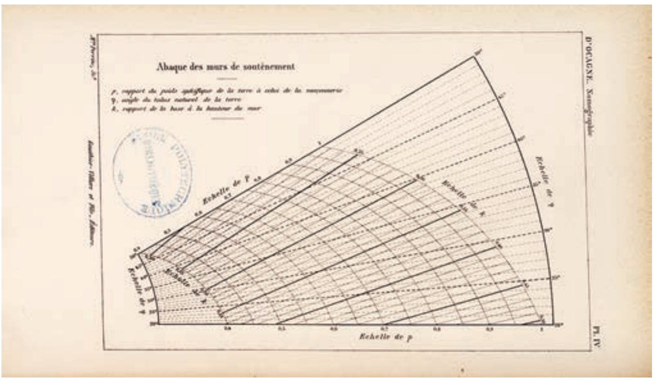
Among the archives donated by professors, we may cite three personal archive fonds especially relevant to the history mathematics: the ones of Camille Jordan, Maurice d'Ocagne, and Laurent Schwartz. The scientific correspondence of Camille Jordan contains about 200 letters sent to Jordan between 1867 and 1896, as well as a few drafts of letters written by Jordan himself [Billoux, 1985]. The epistolary exchanges with Brioschi, Cremona, Clebsch, Sylow, Kronecker, and Sylvester shed important light on Jordan's mathematical works on substitution groups and algebraic forms in the 1860s and 1870s, as well as on the reception of Jordan's 1870 *Traité des substitutions et des équations algébriques*.[9] The correspondence between Jordan and Peano in the 1880s and 1890s documents the evolution of the three successive editions of Jordan's *Cours d'analyse de l'École polytechnique*, especially on issues relative to the foundations of analysis, such as with the interplay between Rolle's theorem and the mean values theorem [Gispert, 1982]. An important part of the correspondence is related to Jordan's editorial responsibilities as the director of the *Journal de mathématiques pures et appliquées* from 1884 to 1922.[10] It includes discussions with the other directors of the main mathematical journals of the time, such as Klein in Göttingen, Gösta Mittag-Leffler in Stockholm, Paul Mansion in Ghent, and Giovanni Guccia in Palermo. Other letters shed light on various aspects of the organisation of mathematics in Paris in the late 19th century through the relationships between Jordan, Hermite, Jules Tannery, Halphen, and Humbert [Brechenmacher, 2016].



A letter from Leopold Kronecker to Jordan in 1874. © Collections École polytechnique. Fonds Camille Jordan.

The personal archive fond of Maurice d'Ocagne is nine times bigger than the one of Jordan. Its inventory, which has only been done partially (on about one-third of the archive fond), gives access to about 500 manuscript documents that contain material related to Ocagne's lectures on geometry at École polytechnique as well as hundreds of letters from contemporary mathematicians, mostly in connection with Ocagne's mathematical works on geometry and with his editorial responsibilities for the *Nouvelles annales de mathématiques*. This collection is complementary to the fonds Ocagne has deposited at other institutions: the Academy of science for his mathematical memoirs and notes, the École des ponts et chaussées for his mathematical works on nomography (including the part of his correspondence related to nomography and that Ocagne separated from the one on geometry he deposited at École polytechnique),[11] the Conservatoire des arts et métiers for his collection of calculating devices, and the Department of Military Geography for the archives of the bureau of nomography that Ocagne directed during World War I.

Ocagne in the picture album of his promotion. © Collections École polytechnique.



A nomogram. © Collections École polytechnique.

[9] See [Brechenmacher, 2007a], [Brechenmacher, 2016a], [Brechenmacher, 2016b], [Lê, 2015].

[10] On the creation of this journal by Liouville, see [Verdier, 2009]; on the direction of the journal by Jordan, see [Brechenmacher, 2009] and [Brechenmacher, 2016a].

[11] Nomography is a mathematical theory that formalises a set of practical engineering techniques that developed in connection with the major programs of public works of the 19th century. It was, for instance, crucial for the earthwork required for constructing a railway or a canal to estimate embankments and excavations: these 'cut and fill' issues required fast calculations and practical precision, for which several instruments and graphical calculating devices were invented in the second half of the 19th century and were later applied to other issues, such as firing tables for cannon artillery. See [Tournès, 2014].

This fragmentation of personal archives in a variety of locations is quite typical of the French mathematicians of the 19th century, who used to accumulate several academic positions and who were therefore likely to deposit their archives in various places, such as the Academy of science, the Collège de France, École polytechnique, etc.

In contrast, Laurent Schwartz chose to deposit the whole of his personal archives at École polytechnique in 2002 [Guichardet, 2005]. Parts of these archives are public while others are still private and will not be made public until 2032, for certain archives, and 2062 for the whole collection. Schwartz's personal archive fond has been entirely inventoried and partially digitalised.[12] It is organised in five sections :

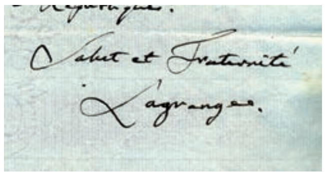
- Mathematics: research and supervision of research
- Teaching material and material related to École polytechnique
- Issues related to scientific teaching and research in France and abroad
- Human rights defence
- Personal archives

The first section contains thousands of manuscripts consisting of mathematical research, notes taken during conferences, drafts of lectures and conferences on various topics, from abstract algebra to partial differential equations or stochastic processes on manifolds, as well as an extensive correspondence between Schwartz and other mathematicians. The second section especially documents Schwartz's teaching at École polytechnique, as well the creation in 1965 of the first research laboratory of mathematics in France, which Schwartz directed until 1983, and which is nowadays designated as Centre de mathématiques Laurent Schwartz. The third and fourth sections shed light on Schwartz's political and humanitarian commitments in France, Vietnam, USSR, Eastern Europe, Africa, or South America, especially with Schwartz's participation in the creations of the Audin Comittee during the Algerian war in 1957 and of the International Committee of Mathematicians in 1973.[13] As a consequence of his opposition to the Algerian war, Schwartz was dismissed from École polytechnique in 1961, and reinstated in his position in 1963. New documents on this affair have been donated to the École polytechnique by Laurent Schwartz's daughter, Claudine Schwartz, on the occasion of an exposition organised in 2015 at the École polytechnique library for celebrating the centenary of the birth of Laurent Schwartz.

Several personal archive fonds have also been purchased by alumni associations. In 1974, the library acquired a collection of documents of Joseph-Louis Lagrange, one of the first professors of École polytechnique. This collection contains familial and biographical documents, a series of diplomas awarded to Lagrange by several academies, and a series of 37 letters sent by Lagrange to his father and his brother in Turin.[14] This donation has enriched Lagrange's personal archive fond at École polytechnique, which includes several engravings, lithographies and paintings, as well as administrative archives. This fond has especially been exhibited to the public on the occasion of the bicentenary of Lagrange's death in 2013,[15] and has provided the material for a documentary film produced by Institut Henri Poincaré.[16]

Laurent Schwartz in Vietnam. © Collections École polytechnique. Fonds Laurent Schwartz.

A letter from Lagrange to his father in 1777. © Collections École polytechnique.



Lagrange's signature. © Collections École polytechnique.

In 2002 the alumni association AX purchased the personal archives of Gaspard Monge, which had until then remained in the possession of Monge's family. This extensive collection contains biographical documents, iconographic documents, Monge's correspondence, scientific manuscripts, and a variety of archives which document Monge's teaching as well as his political responsibilities [Thooris, 2007]. After it was donated to École polytechnique, the fond was inventoried by Marie Dupont, who has recently devoted a thesis to a critical edition of Monge's correspondence [Dupont, 2014], and a selection of Monge's scientific manuscripts has been digitalised.[17] In 2018, Monge's archive fond was presented to the pub-

[12] For accessing Schwartz's archive fond, see the database at the following link: https://bibli-aleph.polytechnique.fr.

[13] See [Paumier, 2014] and [Barany, 2016].

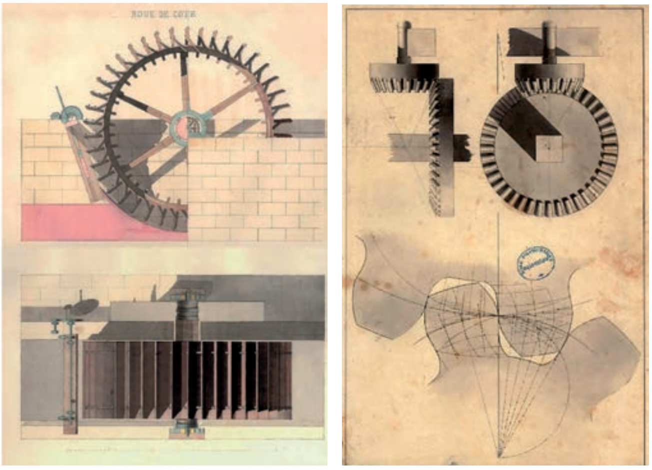
[14] For a description of these letters, see [Borgato and Luigi, 1989].

[15] A virtual edition of this exhibition can be visited at https://www.polytechnique.edu/bibliotheque/fr/lagrange-1736-1813-0.

[16] This film can be watched on the video channel of Institut Poincaré.

[17] See http://numix.sabix.org/monge.html.

lic in the first temporary exhibition of the mus'X, the new museum of École polytechnique: "Gaspard Monge, savant & républicain. Quand les sciences forment les citoyens".[18]



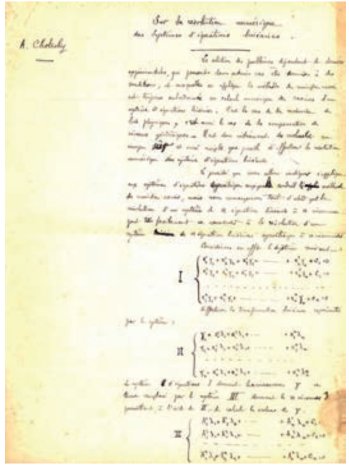
Drawings made by a student on applications of descriptive geometry to hydraulics and mechanics. © Collections École polytechnique.

Let us conclude this presentation by mentioning three personal archive fonds of mathematicians that have been donated to the École polytechnique library in the past decades: a collection of documents of Henri Poincaré, the personal archive fond of André Cholesky, and the correspondence of Gabriel Lamé. In 1977, François Poincaré donated to the library a copy of about 300 letters of his ancestor Henri Poincaré, along with the latter's tangent (i.e., polytechnician sword) and French Academy uniform.[19] The letters consist of the correspondence between Henri Poincaré and his family while the former was a student at Polytechnique. These letters have been inventoried and digitalised by the Archives Henri Poincaré in Nancy,[20] and have recently been published with a critical edition by Laurent Rollet [Rollet, 2017].

The young Henri Poincaré at École polytechnique. © Collections École polytechnique.

The personal archive fond of André Cholesky was donated to the library by his grandson in 2003. These archives have been inventoried by the Centre of Historical Resources with the help of the mathematician Claude Brezinski.[21] They especially contain a manuscript, dated December 2, 1910, which provides the first known description of the Cholesky decomposition of a Hermitian positive-definite matrix.[22]

Cholesky in the picture album of his promotion in 1895 at Polytechnique.



Cholesky's 1910 manuscript on the resolution of systems of linear equations. © Collections École polytechnique. Fonds André Cholesky.

In 2012, descendants of the mathematician Gabriel Lamé donated about 200 of his letters to the École library. This correspondence contains letters both from and to Lamé from 1815 to 1832. It documents several aspects of the life of the mathematician, such as his training at École polytechnique as well as his service in Russia as an engineer from 1820 to 1831 [Gouzevitch, 2009].

The historical collections: a resource for research, education, and popularisation of mathematics

École polytechnique's historical collections are famous on a world scale in the field of the history of science and technology. The Centre of Historical Resource therefore welcomes researchers on a regular basis. The collections are especially a key resource for the LinX, École polytechnique's research laboratory in the humanities and social sciences, which brings together a wide range of scholars in the human, social, and natural sciences to support interdisciplinary research, teaching, and outreach on the roles of science, technology, medicine, and public health in past and present societies.

Historical collections are also an important source for developing innovative teaching initiatives. Since 2012, the Department of Humanities and Social Sciences at École polytechnique has promoted a project-oriented approach based on a direct contact with its patrimonial collections. This project-oriented approach aims to initiate students in research activity in the history of science and technology. The investigation into novel historical sources, such as archives or instruments which have never been studied before, allows students to contribute to contemporary research through various types of outcomes: research papers, indexes and digitalisation of

[18] See the following link: https://www.polytechnique.edu/bibliotheque/fr/exposition-temporaire-du-musix-gaspard-monge.
[19] See [Moatti, 2012] and [Azzola and Thooris, 2012].
[20] See the numerical platform devoted to Henri Poincaré's paper :http://henripoincarepapers.univ-lorraine.fr/.
[21] See [Brezinski, 2005a], [Brezinski, 2005b], and [Brezinski, 2005c].
[22] See [Brezinski and Tournès, 2014].

archival sources, software for the representation of historical data, exhibitions, documentary films, numerical models of ancient instruments, and other types of applications of mathematics to the development of mediation devices for museum collections.

A selection of student research projects are published on the numerical platform of the École polytechnique library on a regular basis.[23]

Through these publications, the platform aims to shed light on various aspects of the historical collections as well as to enhance public understanding of science in relation to social concerns. The development of this numerical platform has recently given birth to the more ambitious project of giving access to all the historical collections of École polytechnique through a unified digital museum, which will moreover aim to provide e-learning resources on the history of science and technology, especially in the form of MOOCs targeting various types of audiences.

The recent creation of a physical museum, the mus'x, also works to enhance public understanding of science in relation to pressing social concerns.The development of both digital and physical museums makes available new historical sources and opens new perspectives for research in the fields of science, technology, and society. It calls for collective work from researchers both in science and technology and in the humanities and social sciences in order to construct platforms not only for research, but also for the diffusion of these historical sources to a much broader audience. Such diffusion of sources and research can inform contemporary research in the study of science in society, and, more generally, the general public.

References

[Albert-Lévy and Pinet, 1894] Lévy, A. and Pinet, G. (1894). *L'argot de l'X: illustré par les X*. E. Testard, Paris.

[Azzola, 2013] Azzola, O. (2013). Deux cents ans d'archives à l'École polytechnique: du comptable aux archivistes. *La Gazette des archives*, 231:63–76.

[Azzola and Thooris, 2012] Azzola, O. and Thooris, M.-C. (2012). L'École polytechnique honore la mémoire d'Henri Poincaré. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 51:41–44.

[Azzola and Verdier, 2010] Azzola, O. and Verdier, N. (2010). Extraits choisis du "fonds Drouineau / Liouville". *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 45:17–22.

[Barany, 2016] Barany, M. (2016). *Distributions in Postwar Mathematics.* PhD thesis, Princeton University.

[Belhoste, 1985] Belhoste, B. (1984–1985). Le Cours d'analyse de Cauchy à l'Ecole polytechnique en seconde année. *Sciences et techniques en perspective*, 9:101–178.

[Belhoste, 1994] Belhoste, B. (1994). Un modèle à l'épreuve. L'École polytechnique de 1794 au Second Empire. In Belhoste, B., Dahan Dalmedico, A. and Picon, A., editors: *Un modèle à l'épreuve. L'École polytechnique de 1794 au Second Empire*, pages 9–30. Dunod, Paris.

[Belhoste, 1998] Belhoste, B. (1998). De l'École polytechnique à Saratoff, les premiers travaux géométriques de Poncelet. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 19:9–29.

[Belhoste, 2001] Belhoste, B. (2001). The École polytechnique and Mathematics in Nineteenth-Century France. In Bottazzini, U. and Dahan, A., editors: *The École polytechnique and Mathematics in Nineteenth- Century France*. Routlege, London.

[Belhoste, 2002] Belhoste, B. (2002). Anatomie d'un concours. L'organisation de l'examen d'admission à l'École polytechnique de la Révolution à nos jours. *Histoire de l'éducation*, 94:141–175.

[Billoux, 1985] Billoux, C. (195). Le nouveau classement thématique des archives de l'École polytechnique. *Revue d'histoire des sciences*, 38(1):73–82.

[Billoux, 1985] Billoux, C. (1985). La correspondance mathématique de C. Jordan dans les archives de l'École polytechnique. *Historia Mathematica*, 12:80–88.

[Billoux, 1989] Billoux, C. (1989). Les archives de l'École polytechnique. *La Gazette des archives*, 145:125–135.

[Billoux, 1999] Billoux, C. (1999). La résurrection des archives. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 22:1–9.

[Borgato and Luigi, 1989] Borgato, M. T. and Luigi, P. (1989). The family letters of Joseph-Louis Lagrange. *Bollettino di Storia delle Scienze Matematiche*, 2:192– 318.

[Bottazzini, 1992] Bottazzini, U. (1992). Geometrical rigour and 'modern analysis', an introduction to Cauchy's Cours d'analyse. In Cauchy, A.-L., editor: *Cours d'analyse de l'École royale polytechnique, 1ère partie, Analyse algébrique*. CLUEB, Bologna.

[Bradley and Sandifer, 2009] Bradley, R. and Sandifer, C. (2009). *Cauchy's Cours d'analyse: An Annotated Translation*. Springer, New York.

[Brechenmacher, 2007] Brechenmacher, F. (2007). La controverse de 1874 entre Camille Jordan et Leopold Kronecker. *Revue d'histoire des mathématiques*, 13:187–257.

[Brechenmacher, 2009] Brechenmacher, F. (2009). Le 'journal de M. Liouville' sous la direction de Camille Jordan (1885–1922). *Bulletin de la Sabix*, 45:65–71.

[Brechenmacher, 2016a] Brechenmacher, F. (2016a). The 27 Italies of Camille Jordan. In Brechenmacher, F., Jouve, G., Mazliak, L. and Tazzioli, R., editors: *Images of Italian Mathematics in France*, pages 45–91. Springer.

[Brechenmacher, 2016b] Brechenmacher, F. (2016b). Algebraic generality vs arithmetic generality in the controversy between C. Jordan and L. Kronecker (1874). In Chemla, K., Chorlay, R. and Rabouin, D., editors: *The Oxford Handbook of Generality in Mathematics and the Sciences*, pages 433–467. Oxford University Press.

[Brezinski, 2005a] Brezinski, C. (2005a). Géodésie, topographie et cartographie. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 39:32–66.

[Brezinski, 2005b] Brezinski, C. (2005b). Restitution du carnet no 2 de cholesky. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 39:68–74.

[Brezinski, 2005c] Brezinski, C. (2005c). Restitution du carnet no 3 de cholesky. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 39:75–79.

[Brezinski and Tournès, 2014] Brezinski, C. and Tournès, D. (2014). *André-Louis Cholesky. Mathematician, Topographer and Army Officer*. Birkhäuser, Basel.

[Chabert and Gilain, 2004] Chabert, J.-L. and Gilain, C. (2004). Debating the Place of Mathematics at the École polytechnique around World War i. In Aubin, D. and Goldstein, C., editors: *The War of Guns and Mathematics: Mathematical Practices and Communities in France and Its Western Allies around World War I*, pages 275–306. American Mathematical Society, Providence.

[Dupont, 2014] Dupont, M. (2014). *Édition de la correspondance de Gaspard Monge (1795–1799): ce qu'elle révèle de l'engagement public d'un géomètre au cours de la Révolution française*. PhD thesis, University of Athens.

[23] See https://www.polytechnique.edu/bibliotheque/fr/portail-patrimoine-de-lecole-polytechnique.

[Fraser, 1985] Fraser, C. (1985). J. L. Lagrange's changing approach to the foundations of the calculus of variations. *Archive for History of Exact Sciences,* 32(2):151–191.

[Gentil, 2002] Gentil, B. (2002). Les notes de cours d'Auguste Comte, élève à l'École polytechnique (1814–1816). *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 30:51–56.

[Gierster, 1989] Gierster, J. (1989). Cauchy et le Cours d'Analyse de l'Ecole polytechnique. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 5:3–145.

[Gispert, 1982] Gispert, H. (1982). *Camille Jordan et les fondements de l'analyse*. Thèse de doctorat, Université Paris-Sud.

[Gouzévitch and Gouzévitch, 2009] Gouzévitch, I. and Gouzévitch, D. (2009). Gabriel Lamé à Saint-Pétersbourg (1820–1831). *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 44:20–43.

[Grabiner, 1981] Grabiner, J. V. (1981). *The Origins of Cauchy's Rigorous Calculus*. The MIT Press, Cambridge.

[Grattan-Guinness, 1990] Grattan-Guinness, I. (1990). *Convolutions in French Mathematics, 1800–1840: From the Calculus and Mechanics to Mathematical Analysis and Mathematical Physics*. Birkhäuser, Basel.

[Grattan-Guinness, 2005] Grattan-Guinness, I. (2005). The "Ecole Polytechnique", 1794–1850: Differences over Educational Purpose and Teaching Practice. *The American Mathematical Monthly*, 112(3):233–250.

[Guichardet, 2005] Guichardet, A. (2005). Laurent Schwartz et l'Ecole polytechnique. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 39.

[Lê, 2015] Lê, F. (2015). Geometrical equations: Forgotten premises of Felix Klein's Erlanger Programm. *Historia Mathematica*, 42(3):315–342.

[Moatti, 2012] Moatti, A. (2012). Les lettres personnelles de Poincaré à sa famille pendant sa scolarité à Polytechnique. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 51:32–36.

[Paumier, 2014] Paumier, A.-S. (2014). *Laurent Schwartz (1915–2002) et la vie collective des mathématiques*. Thèse de doctorat, Université Paris 6.

[Phili, 2012] Phili, C. (2012). Le manuscrit du cours de Lagrange à l'École polytechnique (1797–1798). In Féry, S., editor: *Aventures de l'analyse de Fermat à Borel. Mélanges en l'honneur de Christian Gilain*. Presses universitaires de Nancy, Nancy.

[Rollet, 2017] Rollet, L., editor (2017). *La correspondance de jeunesse d'Henri Poincaré. Les années de formation. De l'École polytechnique à l'École des Mines (1873–1878)*. Birkhäuser.

[Thooris, 2007] Thooris, M.-C. (2007). L'iconographie du fonds Monge: un inventaire singulier. *Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique*, 41.

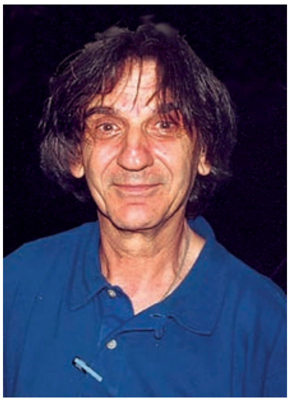
[Tournès, 2014] Tournès, D. (2014). Mathematics of engineers: Elements for a new history of numerical analysis. *Proceedings of the International Congress of Mathematicians*, 4:1255–1273.

[Verdier, 2009] Verdier, N. (2009). *Le journal de Liouville et la presse de son temps: une entreprise d'édition et de circulation des mathématiques au XIXe siècle (1824–1885)*. Thèse de doctorat, Université Paris-Sud.

Frédéric Brechenmacher [frederic.brechenmacher@polytechnique.edu] is a professor of the history of science at École Polytechnique. His research activities are devoted to the history of algebra from the 18th century to the 20th century. He recently supervised the creation of a science museum at École Polytechnique: the mus'x.

Michel Mendès France, 1936–2018[1]

Jean-Paul Allouche (CNRS, IMJ-PRG, Paris, France) and Jeffrey Shallit (University of Waterloo, Canada)



Michel Mendès France.

Our dear friend and colleague, Michel Mendès France, passed away on January 30, 2018. He was 82. He was the author of over 130 research papers in the mathematical and physics literature, and several books and book chapters, including *Les nombres premiers* (with W. J. Ellison) and *Les nombres premiers, entre l'ordre et le chaos* (with Gérald Tenenbaum).

Michel was a mathematician (even if he liked to say he was more of a physicist than a mathematician) of unusual creativity and breadth. His published work dealt with number theory (normal and transcendental numbers, continued fractions, Diophantine approximation), automata theory, theoretical physics (entropy, thermodynamics, the Ising model), the history of mathematics, and art, among other topics, and appeared over the period 1962 to 2018.

His most-cited paper, entitled "Suites algébriques, automates et substitutions", appeared in the *Bulletin de la Société Mathématique de France* in 1980. In this paper, Michel and his co-authors Christol, Kamae, and Rauzy, expanding on a result of Christol from 1979, established an unexpected connection between finite automata, formal power series over a finite field, and iterated morphisms over a finite alphabet, and illustrated this connection with several examples. It has been cited over 300 times in the scientific literature. His three-part paper "Folds!" (co-written with Dekking and van der Poorten) in the *Mathematical Intelligencer* was also highly cited and influential.

More recently, Michel worked on questions at the boundary between geometry and physics, developing a notion of entropy and temperature for planar curves. The definitions that he proposed, including pressure and volume, imply that at an infinite temperature, planar curves behave like a perfect gas!

Michel had the daily habit of checking the abstracts, and sometimes the full papers, of journals that had just arrived at the department (this was, of course, in the days when journals were still on paper) and – perhaps partly due to this – he was able to supply, at a moment's notice, the crucial reference needed on a wide variety of mathematical subjects. Beyond the classical dichotomy of those who pose interesting new problems and those who solve them (Michel being more a part of the former), there is another one: the dichotomy between those researchers who dig deeply into a subject (or add bricks to existing constructions), and those who prefer to find unexpected connections and build bridges between areas that, at first glance, appear entirely unrelated. Michel was part of this second group, and therefore had a strong influence on mathematicians who were pleased to join him in this quest.

Michel supervised several doctoral students. We found, in alphabetical order, these: Anne Bertrand, Emmanuel Cateland (co-directed with J.-P. Allouche), Michel Olivier, and Jia-yan Yao.

Michel was born on January 1, 1936, the son of (future Président du Conseil) Pierre Mendès France and Lily Cicurel. As a child, he was forced to leave France during World War II and spent part of his formative years in New York City, where he learned to speak English fluently.

Michel matriculated at the École Polytechnique in 1957 and graduated in the *Corps de l'armement*. His 1966 "Thèse d'état" (which corresponds to what is now called "Habilitation", i.e., a degree after the doctoral thesis which would itself correspond to a PhD) at the Faculté des sciences de Paris was entitled "Nombres normaux, fonctions pseudo-aléatoires" and his advisors were Charles Pisot and Jean Bass. The results of his thesis appeared in Volume 20 (1967) of the *Journal d'analyse mathématique*. He did a postdoctoral visit at the University of California, Berkeley. After a post at the Université de Paris, he became professor (and later, emeritus professor) at the Université de Bordeaux. He received the Prix Paul Doistau-Émile Blutet de l'information scientifique de l'Académie des sciences, shared in 1999 with Gérald Tenenbaum for their book *Les nombres premiers*.

In 2000 the Université de Bordeaux hosted a conference in honour of Michel's retirement. Among the attendees were the authors, Didier Nordon, Jacques Peyrière, Andrzej Schinzel, Hédi Daboussi, Gérald Tenenbaum, Martine Queffélec, Alan Baker, Vitaly Bergelson, Michel Dekking, Anne Bertrand-Mathis, Michael Keane, Zhi-Ying Wen, Bernard Derrida, Imre Ruzsa, Georges Rhin, Vera Sós, Alf van der Poorten, Wladyslaw Narkiewicz, Andrew Pollington, Étienne Fouvry, and Bahman Saffari. The list gives some idea of his wide collaborations and interests.

Both of us were privileged to be frequent collaborators with Michel. Working with him was always entertaining, consisting of mathematics, philosophy, a story from Michel's past (frequently hilarious and punctuated by the idiosyncratic interjection "Schlak!"), puns in two or more languages, and commentary on the issues of the day, often accompanied by espresso or red wine on

[1] This paper is a slightly modified translation of a paper published in French in *La Gazette des Mathématiciens* 158 (2018), 81–82. The EMS Newsletter thanks the Société Mathématique de France and *La Gazette des Mathématiciens* for authorising the publication of this translation.

Michel Mendès France with the authors, Jeffrey Shallit…

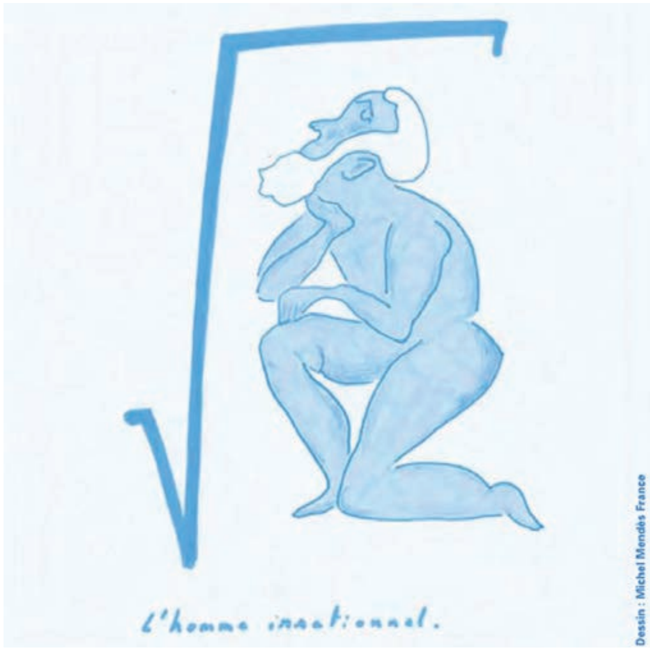
… and Jean-Paul Allouche.

the patio of the "Fac" at Bordeaux. To give some small taste of his wit, we repeat one of his most-quoted aphorisms: "A good research lecture in mathematics should be either beautiful, deep, surprising, … or short!"

His talks were often gently provocative, as (for example) when he claimed to prove a theorem by *descending* induction: first one proves that if the result is true for $n+1$, it must be true for n; it then only remains to prove it for n infinite. Since there are no infinite integers, the result follows! There always seemed to be a Bourbakian representative of right-thinking present who agitated on his chair, and finished by objecting that this did not seem to be a rigorous proof. This amused Michel, who knew well (as did the objector) that one could rearrange the argument to satisfy the guardians of the mathematical temple.

At night we often had dinner at Michel's and Joan's house in Gradignan – a suburb of Bordeaux – where Michel delighted in playing host, showing us his superb collection of vintage posters from his time in Berkeley in the 1960s and finding eggs recently laid by the chickens pecking in the spacious grounds. His bookcases were crammed with books on all subjects and stretched to the top of the high ceilings. There were also sheep, and from time to time a sheep would escape and be returned by a local farmer who drove up with the sheep in the back seat. One day, one of us accidentally let a sheep into the kitchen and Michel had to be called to wrestle it out again. Michel sometimes cooked – the *poulet gros sel* being a particular favorite – and the conversation often continued late into the night. (One of us recalls inviting Michel and two friends to a restaurant on the rue Saint-Jacques in Paris. That day, the *poulet gros sel* suggested by the proprietor had been transformed into what we laughingly called an OESNI – objet extrêmement salé non identifié – but Michel only remarked diplomatically that there had been just a bit too much salt.)

Michel was also a talented artist. His drawings and collages, frequently with humorous or mathematical themes, came each year in the mail, and appeared, for example, in Didier Nordon's book *Les mathématiques pures n'existent pas!* (1981). An exhibition of his work entitled *Petits croquis en prose*} took place at the Musée des Beaux-Arts d'Angoulême in 1992.



L'homme irrationnel. Drawing by Michel Mendès France for the poster of the "Journée Annuelle 2012 de la SMF", first published for the cover of the book *Les mathématiques pures n'existent pas!* (D. Nordon 1981)

We were greatly privileged to be friends and collaborators with Michel for more than 35 years – and thanks to him, we first met when Michel invited one of us to Bordeaux in 1985.

An older brother, Bernard Mendès France, died in 1991. Michel is survived by his wife, Joan (née Horsley) and his children Tristan and Margot. We send them our deepest condolences.

The Spanish Society for Applied Mathematics and ICIAM 2019 – Valencia

Rosa Donat (Universitat de València), President of the Sociedad Española de Matemàtica Aplicada

I would like to thank the editors of the EMS Newsletter, and especially Fernando Pestana da Costa, for the opportunity to make this presentation on the Spanish Society for Applied Mathematics (or Sociedad Española de Matemática Aplicada in Spanish, with acronym SeMA) in this very special year in which we will be the host society of ICIAM 2019, https://iciam2019.com, the ninth edition in the series of ICIAM congresses.

SeMA was created in 1991, following the success and continuity of the Spanish Congresses of Differential Equations and their Applications (CEDYA) that began in 1978. It was legally founded on February 25, 1993, but there are documents reporting a meeting in 1988 of a set of about twenty mathematicians, representing most of the Spanish research groups in Applied Mathematics that, led by Professor Antonio Valle, decided that the XI edition of CEDYA, to be held in Málaga in 1989, would also be entitled "First Congress of Applied Mathematics" (I CMA). It was later, during the celebration of the XII CEDYA–II CMA at the Universidad de Oviedo in 1991, that more than a hundred mathematicians supported, by registering as members, the creation of SeMA as a society aiming to integrate researchers interested in applied mathematics living or working in Spain.

Professor Antonio Valle.

The mission envisioned for SeMA in its early years was to contribute in a coordinated manner to the development of mathematics in connection with its applications and the need to solve real-world problems in science and industry. The underlying motivation was the quantitative deep change that occurred in the preceding decade in the application of mathematical techniques and methodologies, due to the rapid development of scientific computing as a new emerging field.

During the last 25 years, SeMA has become a mature society, while contributing to the advancement of applied mathematics within the Spanish community. Nowadays the society has close to 500 individual members, some of them living or working outside of Spain, and around 50 institutional ones. Today the activity of SeMA is based upon the consideration that mathematical modelling, analysis, numerical simulation, and control techniques are essential tools to understand and solve many challenging problems appearing in physics, chemistry, engineering, biomedical sciences, geosciences, economics, or finance, among others.

Its Executive Council is composed of 11 members: the president, a vice-president, the secretary general, the treasurer and 7 members elected for a period of three years and renewable for a second term. The president is elected for a period of two years, renewable for a second term. The Executive Council decides on all important matters concerning the society and meets at least once a year. The General Assembly of the society is open to all members of SeMA and meets once every year to approve the budget and discuss other matters.

SeMA holds cooperation agreements and reciprocity membership with many mathematical societies. In Spain, RSME – Royal Spanish Society of Mathematics, SCM – Catalan Society of Mathematics, SEIO – Spanish Society of Statistics and Operational Research, as well as the AMS and SIAM in the USA, SIMAI in Italy, SMAI in France, and GAMM in Germany. In addition, SeMA is a corporate member of several international mathematical organizations, such as the European Community on Computational Methods in Applied Sciences (ECCOMAS), the International Council for Industrial and Applied Mathematics (ICIAM), the International Mathematical Union (IMU, through the Spanish CEMAT, that is, the Comité Español de Matemáticas), and the Centre International de Mathématiques Pures et Appliquées (CIMPA). SeMA is a full member of the European Mathematical Society (EMS).

There are two special scientific events in our society. The CEDYA/CMA congresses, held every two years, have consolidated themselves as a focal meeting point for applied mathematicians in Spain. The week-long meetings include plenary talks by international experts, special minisymposia, contributed lectures, and poster sessions. The latest edition, http://www.cedya2017.org, was held in Cartagena in 2017. In alternate years, and in collaboration with the French SMAI, the society sponsors the Spanish-French Jacques-Louis Lions schools for graduate students. These schools are addressed to young researchers, specially pre- and post-doctoral applied mathematicians and engineers, as well as to industrial technicians interested in learning state-of-the-art numerical simulation techniques that may be useful in their field. The latest school was held in Las Palmas de Gran Canaria (http://ehf2018.iusiani.ulpgc.es).

In addition, the society has strived to consolidate its scientific publications. The scientific part of the

Boletín de la Sociedad Española de Matemática Aplicada, has become a peer-reviewed international scientific journal: the *SeMA Journal*, published by Springer since 2010. It contains articles and review papers written in English on high-level achievements in applied mathematics, covering theoretical as well as numerical results and also their practical applications. In order to promote the publication of high-quality scientific papers and reviews in applied mathematics, SeMA sponsors yearly the SeMA Journal Best Paper Award.

The cooperation SEMA-SIMAI Springer Series of advanced textbooks and research monographs deserves special mention. Launched in 2013, the series enjoys a very successful acceptance in the international applied mathematics community, and its impact is well above average. There are 18 volumes already published, and a few more are in production or waiting for final approval. Modelling and simulation are very much appreciated in our series, as we understand that they must be at the core of the mission of mathematics reaching out to society.

Our society also seeks to recognise the scientific contributions of young researchers by sponsoring the SeMA Prize to Young Researchers, established in 1998 and awarded annually to a young researcher under 34. This award was renamed the *SeMA–Antonio Valle Prize to Young Researchers* in 2013, in recognition of the efforts dedicated by Professor Antonio Valle, first president of SeMA, to the promotion of young researchers in the field of applied mathematics.

A special mention also goes to the collaboration between SeMA, SIMAI, SMAI, and SBMAC (the Brazilian society for applied mathematics), who presently fund the Lagrange Prize, created on the initiative of the first three societies and first awarded at the ICIAM 1999 congress. The prize was established to provide international recognition to individual mathematicians who have made an exceptional contribution to applied mathematics throughout their careers. The recipient of the 2019 Lagrange Prize is G. Papanicolaou 'for his brilliant use of mathematics to solve important problems in science and engineering; in particular, problems involving inhomogeneity, wave propagation, random media, diffusion, scattering, focusing, imaging, and finance' (http://www.iciam.org/iciam-prizes-2019). The prize will be awarded at the opening ceremony of the ICIAM 2019 congress, on July 15, 2019.

As a full member of the International Council for Industrial and Applied Mathematics (ICIAM), SeMA was invited by ICIAM in 2011 to prepare a bid to host ICIAM 2019, the ninth ICIAM congress. After an internal competition between several Spanish cities, Valencia was chosen as the venue for the conference in the SeMA bid, which had the support of the entire mathematical community in Spain as well as most of the ICIAM member societies in the Mediterranean area and many societies in Latin America and northern and equatorial Africa. Dr. Alfio Quarteroni, agreed to be the Chairman of the SPC (Scientific Committee Panel) and His Majesty King Felipe VI of Spain agreed to chair the Honour Committee of the conference, a fact of enormous importance for the social relevance of the event in the media. SeMA's bid was finally selected by the ICIAM Board in its 2013 meeting in Beijing, in fair competition with the bids presented by Amsterdam and Brazil. Since then, the society has been actively involved in the preparation of this large congress that will take place in Valencia, from July 15 to 19, 2019.

Organising ICIAM 2019 is certainly generating a qualitative, and hopefully also a quantitative, leap in the relations between mathematics and industry in our territory, and it has undoubtedly helped to raise the national assessment of applied mathematics, and the international liaison role of Spain between Europe, Latin America, and Africa.

There are many features that make an ICIAM conference a special event. The 27 plenary lectures do, of course, stand out: the speakers appointed by the SPC are researchers of global excellence with a variety of gender, thematic, and geographical origins. In addition, ICIAM2019-Valencia will have five additional talks corresponding to the five ICIAM awards (http://www.iciam.org/iciam-prizes-2019), as well as two special lectures: the Olga Taussky-Todd and the Outreach Public lecture. Information on all the distinguished speakers at ICIAM 2019 can be found at https://iciam2019.com/index.php/scientific-program/.

As a new feature, ICIAM 2019 will hold a specific 'Industry Day', a mathematical and technology-transfer-oriented activity with a selection of speakers from industry, with a broad representation of the different sectors, whose target audience will be mainly the industrial attendees, but that is also addressed to the academic sector and open to the society in general.

ICIAM 2019 has, moreover, a special 'sub-venues' program. The Spanish cities of Bilbao, A Coruña, Málaga, Santiago de Compostela, Sevilla, and Zaragoza organise a dedicated program of satellite events, which reflect the commitment of the Spanish mathematical community with the success of the conference. The global Satellite Events Programme of ICIAM 2019 includes 30 events.

As of January 2019, ICIAM 2019 has already attracted more than 3500 talks. The call for minisymposia ended on December 5th (326 thematic minisymposia and 22 industrial minisymposia have been accepted). The contributed talks and posters submission periods end on January 21 and April 1, respectively.

SeMA and the organising committee of ICIAM 2019 have elaborated a Financial Aid Program for developing countries and young researchers (https://

iciam2019.org/index.php/infomation-for-delegates/financial-support) that will provide funds for around 250 scholarships. Banco Santander deserves special recognition for sponsoring 150 grants for young students from all over the world. As usual, the ICIAM council helps to provide support for around 20 scholarships for developing countries. SeMA has contributed to the Financial Aid Program of ICIAM 2019 by implementing a crowdfunding program (P2B) addressed to Spanish institutions, which will provide funding for around 50 scholarships.

An ICIAM conference is a truly special event for the host society, but also for its immediate neighbourhood. We invite all EMS members to join this thrilling journey across industrial and applied mathematics and to make ICIAM-2019-Valencia a great event of exceptional scientific quality.

Rosa Donat is a professor at the Departament de Matemàtiques of the Universitat de València. Her research interests cover the numerical analysis of hyperbolic partial differential equations and the theory of nonlinear approximation, with applications to nonlinear subdivision schemes.

ERME Column

Susana Carreira (University of Algarve and UIDEF, Institute of Education, University of Lisbon, Portugal), Berta Barquero (University of Barcelona, Spain), Gabriele Kaiser (University of Hamburg, Germany) and Jason Cooper (Weizmann Institute of Science, Israel)

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME) holds a bi-yearly conference (CERME), in which research is presented and discussed in Thematic Working Groups (TWG). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME's Thematic Working Group 6 – Applications and Modelling

There is a long tradition and worldwide consensus shared by researchers, practitioners, and policymakers on the important role of applications and modelling in mathematics education. In the last decades, especially in Europe, there has been a large movement of curriculum reforms including the integration of applications and modelling at different school levels. It has provided many achievements in the field (as surveyed by Blum, 2015), but has also opened up several questions related to the effects of the teaching and learning of mathematical modelling and to the promotion of new methods of assessment and evaluation, among other issues that are part of the current research agenda in the field.

CERME's Thematic Working Group 6 (TWG6) focuses on the research field of teaching and learning mathematical modelling at various educational levels. TWG6 started at the fourth ERME Congress (CERME4) in 2005 in Sant Feliu de Guíxols (Spain). It has since continued to be an active thematic working group in the biennial ERME conferences, receiving an increasing number of contributions. At the seven meetings from CERME4 to CERME10, the working group has already produced and presented 133 papers and posters. At CERME11, which took place recently (February 2019) in Utrecht, The Netherlands, 21 papers and 7 posters were presented. Those contributions came from 17 different countries, with most, but not all, being from European countries.

The overarching themes addressed in TWG6 show the diversity of research questions, the different school levels addressed – spanning primary to tertiary education – and the theoretical approaches endorsed (Kaiser & Sriraman, 2006). We summarise a selection of important topics that have been discussed recurrently at the CERME congresses:

The first topic focuses on the role of mathematical modelling and applications in connection to other school subjects and disciplines. There is a long tradition of discussing examples from engineering education in TWG6 at various ERME conferences; however, some papers discuss explicitly the role of mathematical modelling in interdisciplinary contexts, such as involving mathematics and history or archaeology, or mathematics and biology. Under this topic, a debate has also taken place on the differences between modelling at the practitioner level and at the school level. Industrial examples have played a powerful role in education, to analyse and replicate the authenticity and complexity of real tasks and their role in mathematical modelling from an educational point of view.

The second topic, developed under a cognitive perspective, focuses on students' cognitive processes when solving modelling problems and the existing cognitive barriers that students encounter when working on modelling problems. Cognitively oriented analyses have been a prominent approach of the researchers in the group at previous CERME conferences. Several theoretical and methodological tools, based on the conceptualisation of the modelling cycle and the underlying modelling competencies, have been developed and used for the analysis of the students' work.

The third topic refers to the study of constraints or barriers and favourable conditions for the inclusion of mathematical modelling in teaching and learning practices. The themes range from the analysis of the ways in which curricula and educational policies can favour the implementation of modelling and applications in school to the analysis of teachers' beliefs in relation to the teaching of mathematical modelling. Those questions have been examined at all previous conferences, thus revealing the significance and urgency of this topic.

The fourth topic, the instructional perspective, underlines the necessity of high-quality modelling education in order to promote an effective learning of mathematics. The question of how to implement effective modelling environments is a recurrent point in the work of TWG6. The discussion has included a variety of themes, such as the use of experimental materials and technology in modelling or new ways of assessing the learning of mathematics and modelling.

Finally, yet importantly, the topic of teacher education is a valued one within the group, based on the acknowledged need to prepare pre-service teachers and in-service teachers for the teaching of applications and modelling. Notwithstanding the importance of this topic, at previous CERME conferences it has been less prominent; however, at CERME10 and again at CERME11 there has been a significant increase in papers related to this topic. Teachers and their education are a key factor for the effective and efficient integration of mathematical modelling into mathematics education at various levels (for details see Barquero, Carreira & Kaiser, 2018).

The work of TWG6 was embedded in the discussions that took place at the biennial international conference series on the Teaching and Learning of Mathematical Modelling and Applications (ICTMA). At these conferences, empirical studies are strongly discussed, which contain examples for classroom activities and empirical evaluations of their implementation, including many non-European researchers, especially from East Asia. The recent literature review on the state of empirical research on the teaching and learning of mathematical modelling by Schukajlow, Kaiser, and Stillman (2018) provides in-depth insight into strengths and weaknesses of the current state of empirical research on modelling and applications. The survey pointed out that qualitative studies are strongly dominating quantitative studies, probably, amongst other reasons, due to the emphasis on implementation studies. Furthermore, it became clear that only a small number of papers have been published in journals in recent years, which should be changed in the future. TWG6 may support this goal by providing clear guidance on what it means to carry out high-quality research on the teaching and learning of mathematical modelling.

The relevance of mathematics and applied mathematics to the teaching and learning of modelling and applications is unequivocal. Pollak has been one of the influential authors in this discussion (Pollak, 2007) but research shows that only more recently has attention been given to the fact that mathematical modelling must find its way into school mathematics. Actually, there seems to be still a lack of dialogue between practitioners, industrial researchers, lab technicians, scientists, mathematicians, and mathematics education researchers in generating ideas, examples, tasks, and materials that value mathematical modelling for school levels. Generative inputs from the community of research mathematicians will certainly be useful from many perspectives, namely in offering interesting real-world problems in ways that are accessible to students and to school institutions as well as in collaborating in curricular reforms for the integration of modelling and applications. The design of modelling problems and challenges, such as the case of modelling competitions that are already taking place in different countries in the world (e.g., COMAP's Mathematical Contest in Modelling (MCM)), is also an opportunity to engage mathematicians in nurturing and promoting students' enjoyment of mathematics through mathematical modelling and applications.

References

Barquero, B., Carreira, S., & Kaiser, G. (2018). TWG06: Applications and modelling. Introduction to the papers of TWG06: Applications and modelling. CERME 10, Feb 2017, Dublin, Ireland. https://hal.archives-ouvertes.fr/hal-01933496/document

Blum, W. (2015). Quality teaching of mathematical modelling: What do we know, What can we do? In S. J. Cho (Ed.), The Proceedings of the 12th International Congress on Mathematical Education (pp. 73–96). Dordrecht: Springer.

Kaiser, G. & Sriraman, B. (2006). A global survey of international perspectives on modeling in mathematics education. ZDM Mathematics Education, 38(3), 302–310.

Pollak, H. (2007). Mathematical Modelling — a Conversation with Henry Pollak. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds), Modelling and Applications in Mathematics Education (pp. 109-120). Boston: Springer.

Schukajlow, S., Kaiser, G., & Stillman, G. (2018). Empirical research on teaching and learning of mathematical modelling: a survey on the current state-of-the-art. ZDM Mathematics Education, 50(1-2), 5–18.

Susana Carreira is associate professor at the University of Algarve and a member of the Research Unit of the Institute of Education, University of Lisbon. Her research is on mathematical modelling and applications, technology in mathematics teaching, and problem solving. She has been involved in CERME and in the International Conference on Teaching Modelling and Applications (ICTMA). In the former she has served as co-leader and as leader of TWG6 – Applications and Modelling.

Berta Barquero holds a doctoral degree in mathematics (2009) focusing on the design of teaching proposals to integrate modelling at university level. She is assistant professor at the Faculty of Education of the University of Barcelona. Her research focuses on modelling in school and university, interdisciplinary education, and primary- and secondary-school teacher education. She has been involved as co-leader and as leader in TWG6 – Applications and Modelling – in recent CERME conferences.

Gabriele Kaiser holds a master's degree as a teacher of mathematics and humanities for secondary level. For her doctorate in mathematics education (1986) she studied application, and modelling. Since 1998, she has been a full professor for mathematics education at the Faculty of Education of the University of Hamburg. Her areas of research are empirical studies on teacher education and teachers' professionalism, modelling and applications in school, international comparative studies, gender and cultural aspects in mathematics education.

Jason Cooper is a senior intern at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including the roles of advanced mathematical knowledge in teaching and contributions of research mathematicians to the professional development of teachers. He has been a member of the ERME board since 2015.

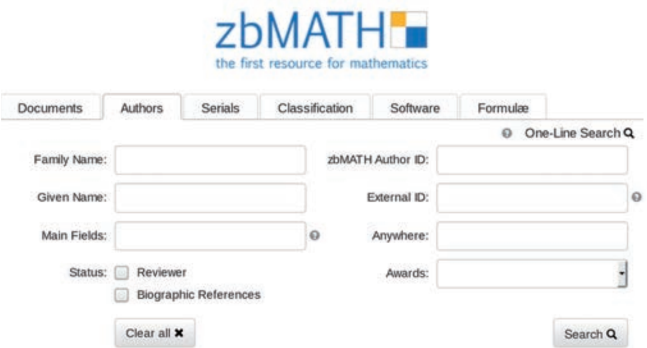
Current Developments on the zbMATH Interface

Octavio Paniagua Taboada (FIZ Karlsruhe, Berlin, Germany)

Like every year, zbMATH's staff of editors and developers has introduced new ideas and useful tools to enrich our database and provide the users with an easier and comfortable experience while using zbMATH. These developments have considered the great feedback of the mathematical community and other users of zbMATH, e.g., librarians.

New search options for authors and serials

Perhaps it has happened to you that looking for the articles of a certain author you find the work of other authors having the same name(s). Any additional information, like a middle name, could be really helpful in finding the author you are looking for. We have introduced some new fields (commands) that you can use in the author tab using one-line search. These fields are "**ln**" (last name) and "**fn**" (first name). Suppose that you are interested in an author called Wei Li. Then you can type the following commands in the author tab (one-line search): **ln:Li fn:Wei**. You will find 72 author profiles that correspond to that name[1] (in this case, Li and Wei could be both first name and surname). If we now invert the order **fn:Li ln:Wei** we find 14 author profiles. Compare this with the 122 results that we find if we just search for Wei Li in the author tab. In these cases or when the authors are exact homonyms, any additional information, such as mathematical area, or the profile in external databases like the Mathematics Genealogy Project, allows us to correctly distinguish them. If you already know exactly which author you are looking for, then it is really useful to use her/his personal author identifier "**ai**". You can also do a structured search in the author tab. This structured search looks like this:

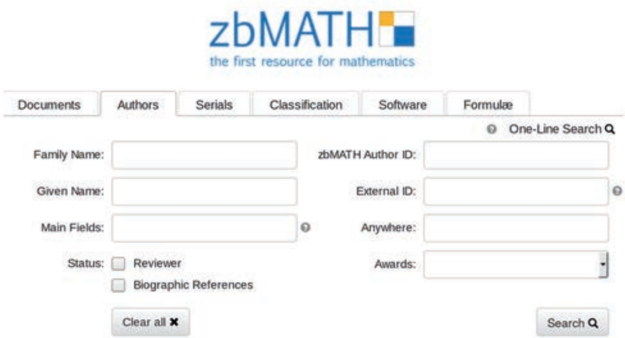


[1] Based on the current index on February 3rd 2019.

Notice that there is more information that you can provide about the author you are looking for, e.g., awards, main fields, zbMATH author ID, or other external IDs (such as MGP and ORCID).

Likewise, we have implemented similar improvements for serials. In the serials tab (formerly called "journals") you can use the one-line search and an enhanced structured search. In the one-line search you can use the following fields: "**jt**" (meaning journal or serial title), "**pu**" (publisher), "**sn**" (ISSN code), "**tp**" (type of publication), and the field "**st**" allows you to find, for instance, open access journals (**st: vo**). There is also a new structured search at the serials tab, where you have additional options like language, time period, country, main fields, etc.

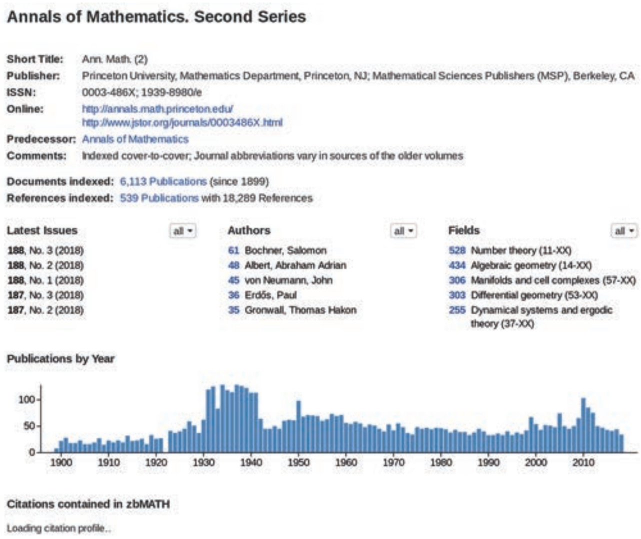
Additionally, you will find some small question marks (this is also the case for the author structured search), which explain the possibilities that you have or provide you with some examples. The structured search for series has this appearance:



As a final remark, you will notice that the documents structured search also has a few changes.

New filters and facets at the zbMATH interface

zbMATH now has some new filters that are displayed after you make a query. Among these filters you will find *Document Type* (serial articles, collections articles, books), *Reviewing State* (reviewed, title only, pending, scanned review), *Biographic Reference*, *Software*, and *Volume* (this filter appears when all results of your search belong to the same serials). Furthermore, when you load a series profile, you find the facet "*Latest Issues*", where it is possible to check the content of the latest volumes. In the figure below you can see a current series profile:



In addition, you will notice the new look of the results page when you perform a serials query.

zbMATH on Twitter

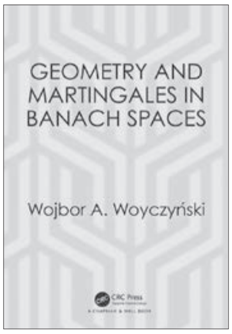
Social networks have proved capable of reaching worldwide a great number of users. Recently, zbMATH has created an account on Twitter. Here you can check on news about zbMATH, important mathematical news, information about mathematicians, history, and some anecdotes. You can find us at @zbMATH, or you can click at the twitter icon at the page footer of zbMATH.

MSC2020: work in progress

After the feedback of the mathematical community, the common effort between zbMATH and Mathematical Reviews is going forward to revise and improve the current Mathematics Subject Classification (MSC) schema. Several concrete proposals and modifications are already being considered by both staffs. The results of this work will be announced at the end of this year.

Photo and CV of the author can be found in previous Newsletter issues.

Book Reviews



Woyczyński, Wojbor A.

Geometry and Martingales in Banach Spaces

CRC Press, 2019
xiii, 315 p.
ISBN 978-1-138-61637-0

Reviewer: Dirk Werner

The Newsletter thanks zbMATH and Dirk Werner for permission to republish this review, which originally appeared as Zbl 1403.46003.

In the 1970s there was frenetic activity in the field of probability in Banach spaces, propelled by mathematicians like, to name but a few, A. Araujo, P. Assouad, E. Giné, J. Hoffmann-Jørgensen, G. Pisier, L. Schwartz, N.N. Vakhaniya, J. Zinn and of course the present author, W. Woyczyński. The monograph under review sums up many of the results obtained in this decade, highlighting the interplay between probabilistic ideas and properties of Banach spaces, specifically the Radon–Nikodým property (RNP) and local properties; indeed, the local theory of Banach spaces emerged as a result of these activities.

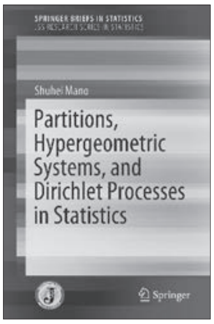
The book starts with a chapter on preliminaries from probability and Banach space theory. Chapter 2 presents the characterisation of the RNP by dentability; in essence, the point is that both properties are equivalent to a martingale convergence property. Chapter 3 discusses martingales with values in spaces having convexity or smoothness properties and, in particular, martingales valued in superreflexive spaces.

The remaining chapters deal with sums of independent random vectors in Banach spaces and hence with martingales with independent increments. The role of c_0 is worked out in Chapter 4, and Chapter 5 introduces spaces with cotype q and investigates their structure, including the Maurey–Pisier(–Krivine) theorem, q-summing operators into such spaces and probabilistic consequences in this setting (law of large numbers, central limit theorem, law of the iterated logarithm). Chapters 6 and 7 discuss the counterpart of (Rademacher) type p (including infratype and stable type) with Chapter 7 concentrating on type 2.

B-convex spaces (i.e., spaces of type >1) and their probabilistic significance are the subject of Chapter 8, and Chapter 9 discusses a Marcinkiewicz–Zygmund type law of large numbers in certain Banach spaces.

The author's proofs are detailed, and he gives precise references to the original sources. The text could, however, have benefitted from thorough proof-reading (for example, the name Assouad is never spelt correctly). Also, as mentioned above, this monograph reflects the discoveries made in the 1970s, and there are basically no post-1980 references or comments about more recent developments; e.g., the name Talagrand is not even mentioned. Readers might therefore wish to complement this otherwise fine monograph by other presentations like [M. Ledoux and M. Talagrand, *Probability in Banach* spaces. Berlin etc.: Springer (1991; Zbl 0748.60004)] (by now a classic itself and reprinted as such (2011; Zbl 1226.60003)), [G. Pisier, *Martingales in Banach spaces*. Cambridge: Cambridge University Press (2016; Zbl 1382.46002)], or [T. Hytönen et al., *Analysis in Banach spaces. Volume I. Martingales and Littlewood–Paley theory*. Cham: Springer (2016; Zbl 1366.46001); *Volume II. Probabilistic methods and operator theory*. Cham: Springer (2017; Zbl 1402.46002)].

Dirk Werner is a professor of mathematics at Free University Berlin. His main areas of research are functional analysis and operator theory with a special emphasis on the theory of Banach spaces.



Shuhei Mano

Partitions, Hypergeometric Systems, and Dirichlet Processes in Statistics

Springer, 2018
viii, 135 p.
ISBN 978-4-431-55886-6

Reviewer: Fraser Daly

The Newsletter thanks zbMATH and Fraser Daly for permission to republish this review, which originally appeared as Zbl 06921105.

This short monograph presents key ideas in the study of random partitions, hypergeometric systems and Dirichlet processes, and in particular the links between them, in the context of statistical inference, taking advantage of the underlying exchangeability present in these models.

The first chapter gives a short (and very readable) introduction to the key ideas that will be needed through the rest of the book: partitions and exchangeability. Two simple motivating examples are then given, to help the reader understand the goal of the work before starting the more technical material presented in Chapters 2–5. The first example is a two-cluster Bayesian mixture model, and the second a problem of testing goodness-of-fit in a Poisson regression model.

Chapter 2 discusses probability measures on combinatorial structures. The author begins by introducing measures on partitions, and in particular the multiplicative measures and combinatorial structures associated with binomial, negative binomial and Poisson coefficients: powersets, multisets and assemblies. Some examples are given (for example, integer partitions with or without the constraint that the elements in the partition be distinct) with clear references to lead the reader to other sources of examples. Exponential structures are then discussed, and examples are again provided (such as set partitions, where the numbers of possible instances are given by the Bell numbers). There is some brief discussion of statistical applications, including to the first of the motivating examples of Chapter 1. Tilting of probability measures and partial Bell polynomials are introduced (again, accompanied by examples), before the chapter concludes with asymptotic results on the size of extreme (both large and small) parts of Gibbs partitions, in particular for the Pitman partition and Ewens sampling formula.

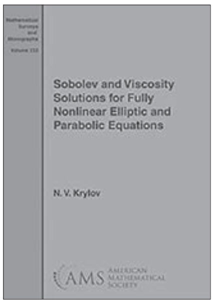
The main object of Chapter 3 is to introduce the A-hypergeometric system and distribution for a non-negative, integer-valued matrix A. The hypergeometric system is a system of linear partial differential equations; the author makes clear the relationship of this system to the integer partitions and partial Bell polynomials (as introduced in the previous chapter) in the case where A has two rows. The A-hypergeometric distribution is a discrete probability measure appearing, for example, in multinomial sampling from log-affine models, whose normalising constants are the A-hypergeometric polynomials. An explicit algorithm (using holonomic gradient methods) is given for the evaluation of these normalising constants, as they will be needed in algorithms for sampling from these distributions which are discussed in Chapter 5. Statistical properties (for example, nonexistence of the MLE in the case of a two-row matrix A) are also discussed, as well as links to the second motivating example from Chapter 1.

The first part of Chapter 4 introduces concepts from probability theory (such as conditional probability and martingale limit theorems) with which the reader is not assumed to be familiar, and also gives a proof of de Finetti's representation theorem. The main part of the chapter is then spent introducing and exploring properties of the Dirichlet process. This includes two constructions of the Dirichlet process (one based on the gamma process, and the other on stick-breaking), its use as a conjugate prior in Bayesian statistics, sampling from the Dirichlet process, and its relationship with exchangeability and the Ewens sampling formula. The chapter concludes with some discussion of other prior processes in Bayesian statistics related to the problems discussed in Chapter 2.

Finally, Chapter 5 explores aspects of sampling and statistical inference. Both direct and MCMC samplers are given (and illustrated with examples) for A-hypergeometric distributions and for random partitions. In the case of random partitions, direct sampling is achieved either by making use of the direct sampling from the A-hypergeometric distribution, or by simulating random graphs generated by processes on partitions. The final section considers maximum likelihood estimation of curved exponential families (using tools from information geometry), which are closely linked with A-hypergeometric distributions.

The book concludes with two appendices, giving background material on symmetric functions and on stochastic processes on partitions. Overall, this monograph should prove to be a useful asset for researchers in various aspects of statistical inference and combinatorial probability. The author assumes some familiarity with discrete probability and with some concepts from algebra (ideals and Gröbner bases, for example), but more advanced topics are introduced from scratch, and references are given throughout to further resources that the reader may consult, either for background or more advanced material.

Fraser Daly has been an assistant professor at Heriot-Watt University since 2013. He previously held postdoctoral positions in Bristol and Zürich, and obtained his PhD from the University of Nottingham. His research is in applied probability, with particular emphasis on approximations for random systems and processes with dependence.



N.V. Krylov

Sobolev and Viscosity Solutions for Fully Nonlinear Elliptic and Parabolic Equations

AMS, 2018
xiv, 441 p.
ISBN 978-1-4704-4740-3

Reviewer: Vincenzo Vespri

The Newsletter thanks zbMATH and Vincenzo Vespri for the permission to republish this review, originally appeared as Zbl 1401.35001.

The arguments considered in this interesting book concern first boundary-value problems for fully nonlinear second-order uniformly elliptic and parabolic equations with discontinuous coefficients. Until 60 years ago, there was not a general theory about fully nonlinear equations except the ones in dimension 2. With no general theory at hand, only specific classes of fully nonlinear equations were investigated. The first class of such equations is related to the famous Monge–Ampère equation which was studied for quite some time by using the theory of convex surfaces. The first breakthrough was given by Aleksandrov in1958 with the introduction of the notion of generalized solutions. Its smoothness, before 1971, was proved by Bakel'man only in dimension two. Then, Pogorelov proved interior smoothness for any dimension. In the multidimensional case, the smoothness up to the boundary was proved only after an approach to general nonlinear equations was developed in the early 1980s.

Another class of nonlinear elliptic and parabolic equations, so-called Bellman's equations, arose from the probabilistic theory of controlled stochastic diffusion processes. When the diffusion is not controlled, Bellman's equations become quasilinear. Krylov, by using probabilistic methods, proved the solvability of general degenerate nonlinear elliptic Bellman's equations in the whole space in the class of functions with bounded second-order spacial derivatives. This theory was all presented in probabilistic terms. Up until 1979 no other approaches were found. The probabilistic approach was developed by the author, Nisio, Pragarauskas, Safonov, Lions, Menaldi, and many others. In 1979 Brézis and Evans proved the existence of a $C^{2+\alpha}$ solution for Bellman's equation with two elliptic operators. The results of Evans, and Lions were obtained by the PDE methods, but the solutions there had only bounded derivatives, and the only novelty in comparison with earlier results was that purely analytical methods were used.

A real breakthrough came in 1982 when Evans and Krylov proved the solvability in a $C^{2+\alpha}$ of a wide class of fully nonlinear equations. The starting point for these proofs was the fact that Krylov and Safonov proved that the solutions of linear equations with measurable coefficients are Hölder continuous. Another step forward were Fang-Hua Lin's estimates for elliptic and parabolic equations. These estimates play a crucial role in the approach to the Sobolev space theory. Actually, these results opened the way to build a solvability theory in classes a $C^{2+\alpha}$ with the operator convex or concave with respect to D^2u.

The next breakthrough came when Safonov introduced a new technique that allows to consider operators that are only Hölder continuous with respect to the independent variables. Safonov's approach is also remarkable in the respect that it works equally well for linear and fully nonlinear equations. Then, naturally the interest in equations arose when the operator is neither convex nor concave with respect to D^2u. These equations appear as the Isaacs equations in the theory of stochastic differential games. Today, it is known their solutions are in $C^{1+\alpha}$. In this case, the notion of the so-called viscosity solutions introduced by Lions turned out to be quite effective because this notion does not require the solutions to have any derivatives.

The next dramatic turn of the theory thanks Caffarelli when in 1989 proved a priori estimates in Sobolev $W^{2,p}$ classes. This book presents in a clear way all these results. The auxiliary results, such as Aleksandrov's elliptic and parabolic estimates, the Krylov-Safonov and the Evans-Krylov theorems, are taken from old sources, and are the topic of the first three chapters of this book. The following results are based on a generalization of the Fefferman-Stein theorem, FangHua Lin's type estimates, and the so-called "ersatz" existence theorems, saying that one can slightly modify "any" equation and get a "cut-off" equation that will have solutions with bounded derivatives. These theorems allow to prove the solvability in Sobolev classes for equations that are quite far from the ones which are convex or concave with respect to the Hessians of the unknown functions. In studying viscosity solutions, these theorems also allow to deal with classical approximating solutions, thus avoiding sometimes heavy constructions from the usual theory of viscosity solutions.

The exposition is self-contained and extremely clear. This makes this book perfect for an advanced PhD class.

Vincenzo Vespri is full professor of mathematics at the University of Florence and the author of more than 100 scientific publications concerning the regularity of weak solutions to evolution equations. More precisely, he worked on the regularity of the solutions to the p-Laplacian and to the porous medium equations. In the last year, he worked also in more applied topics. He is an evaluator of projects in pure, applied, and industrial mathematics for the Italian government and the European Community.

Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

As for me, all the various journeys on which
one by one I found myself engaged,
were leading me to Analysis Situs.

Henri Poincaré (1854–1912)

The present column is devoted to topology. The proposed problems range from tractable to fairly demanding, so that a wide range of our readers could try to tackle them. As always, there is also a proposed open research problem. The open problem in this column, along with the relevant discussion, is contributed by Simon Donaldson.

The word *topology* is derived from the two Greek words τόπος meaning *place* and λόγος meaning *study*. In mathematics, topology is considered to be the study of those properties of geometrical objects that remain invariant under topological transformations. But what is a topological transformation? A transformation of a geometric figure is called *topological* if under this transformation the relations of adjacency of various parts of the figure are not destroyed and also no new ones appear. That is, in such a transformation, the parts of a geometric figure that were in contact will remain in contact, and the parts that were not in contact cannot come into contact. Therefore, under a topological transformation we can stretch, twist, crumple, and bend, but we can neither tear nor glue.

In the above, the notion of continuity plays an integral role and for this reason topology progressed being influenced by the rigorous construction of mathematical analysis. Among the mathematicians who have been involved in the development of topology, the ones who are generally considered to have played the most profound roles are G. Leibniz, L. Euler, F. Gauss, B. Riemann, E. Betti, and most importantly H. Poincaré.

Leibniz coined the term *geometria situs* to describe the field of mathematics that is known today as topology, but it wasn't until Euler that an important topological concept arose with his proof of the now famous Euler polyhedron formula. In this study Euler introduced the concept of what is now called Euler's characteristic. Later, Gauss also made essential contributions to the field. Subsequently, Riemann's work had a profound impact in the development of topology when he introduced the concept of a Riemann surface. He introduced the concept of connectivity of a surface, which helped classify topologically compact orientable surfaces. Inspired by Riemann's concept of connectivity, Betti introduced connectivity numbers of surfaces, now known as Betti numbers. In this manner Betti established the concept of boundary and generalised Riemann's concept of connectivity. Later, based also on Riemann and Betti, Poincaré made monumental contributions to the development of topology and this is the reason why he is generally acknowledged as the father of this field.

In 1895, in his famous memoir *Analysis situs*, Poincaré established the difference between curves deformable to one another and curves bounding a larger space, respectively leading to the concepts of homotopy, fundamental groups, and homology. Poincaré was the first to discover that topological arguments could be applied to prove the existence of periodic solutions in the three-body problem of celestial mechanics.

Topology constitutes one of the most central fields of mathematics. Notwithstanding its very abstract nature, there is a staggering number of applications of topology to various other fields of Science such as astronomy, physics, biology, computer science and robotics.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

204. Note that in any topological space with an isolated point, any two dense sets must intersect. Show that there is a 0-dimensional, Hausdorff topological space X with no isolated points so that still, there are no disjoint dense sets in X.

(Daniel Soukup, Kurt Gödel Research Center, University of Vienna, Austria)

205. For $X = \{\{x, y\} : x, y \in \mathbb{Q}\}$, find a function $b : X \to \mathbb{N}$ such that

$$\{b(\{x, y\}) : x, y \in B\} = \mathbb{N},$$

whenever $B \subseteq \mathbb{Q}$ is homeomorphic to $\mathbb{Q}$.

(Boriša Kuzeljević, University of Novi sad, Department of Mathematics and Informatics, Serbia)

206. Suppose that $(G, \cdot)$ is a group, with identity element e and (G, τ) is a compact metrisable topological space. Suppose also that $L_g : (G, \tau) \to (G, \tau)$ and $R_g : (G, \tau) \to (G, \tau)$ defined by, $L_g(x) := g \cdot x$ and $R_g(x) := x \cdot g$ for all $x \in G$, are continuous functions. Show that $(G, \cdot, \tau)$ is in fact a topological group.

(Warren B. Moors, Department of Mathematics, The University of Auckland, New Zealand)

207. We will say that a nonempty subset A of a normed linear space $(X, \|\cdot\|)$ is a *uniquely remotal set* if for each $x \in X$,

$$\{y \in A : \|y - x\| = \sup\{\|a - x\| : a \in A\}\}$$

is a singleton. Clearly, nonempty uniquely remotal sets are bounded. Show that if $(X, \|\cdot\|)$ is a finite-dimensional normed linear space and A is a nonempty closed and convex uniquely remotal subset of X, then A is a singleton set.

(Warren B. Moors, Department of Mathematics, The University of Auckland, New Zealand)

208. Let X be any set. A family $\mathcal{F}$ of functions from X to $\{0, 1\}$ is said to separate *countable sets and points* if for every countable set $B \subseteq X$ and every $x \in X \setminus B$, there is a function $f \in \mathcal{F}$ so that $f(x) = 1$ and $f[B] = \{0\}$.
Let κ and λ be infinite cardinals with $\lambda \leq 2^\kappa$. Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\lambda$ the usual product topology. Show that the following are equivalent:

1. there is a family $\mathcal{F}$ of λ many functions from κ to $\{0, 1\}$ such that $\mathcal{F}$ separates countable sets and points;
2. there is a subspace $X \subseteq \{0, 1\}^\lambda$ of size κ such that every countable subset of X is closed in X.

(Dilip Raghavan, Department of Mathematics, National University of Singapore, Singapore)

209. A subset X of a partial order $(P, \leq)$ is *cofinal* in P if for each $p \in P$ there is an $x \in X$ satisfying $p \leq x$. Let $\beta\omega$ denote the Stone–Čech compactification of the natural numbers, and let ω^* denote the Stone–Čech remainder, $\beta\omega \setminus \omega$. A neighbourhood base $\mathcal{N}_x$ at a point x forms a directed partial order under reverse inclusion. A neighbourhood base $(\mathcal{N}_x, \supseteq)$ is said to be *cofinal* in another neighborhood base $(\mathcal{N}_y, \supseteq)$ if there is a map $f : \mathcal{N}_x \to \mathcal{N}_y$ such that f maps each neighbourhood base at x to a neighborhood base at y. Assume the continuum hypothesis. Show that there are at least two points x, y in ω^* with neighbourhood bases $(\mathcal{N}_x, \supseteq)$ and $(\mathcal{N}_y, \supseteq)$ which are cofinally incomparable; that is, neither is cofinal in the other.

(Natasha Dobrinen, Department of Mathematics, University of Denver, USA)

II An Open Problem, by Simon Donaldson

(Department of Mathematics, Imperial College, London, UK)

210* A problem in 4-manifold topology. This is not a new problem, it has been well known to 4-manifold specialists for the 20 years since the paper [1] of Fintushel and Stern, which is our basic reference. (Other good background references include [2] and [4].) The question involves a simple topological construction, *knot surgery*, introduced by Fintushel and Stern, involving a compact 4-manifold M and a knot K (i.e., an embedded circle in the 3-sphere S^3). We assume that there is an embedded 2-dimensional torus T in M with trivial normal bundle. We fix an identification of a neighbourhood N of T in M with a product $D^2 \times T$, where D^2 is the 2-dimensional disc . Thus the boundary of N is identified with the 3-dimensional torus $T^3 = S^1 \times T = S^1 \times S^1 \times S^1$. Likewise, a tubular neighbourhood ν of the knot K in S^3 can be identified with $D^2 \times K$, with boundary $S^1 \times K = S^1 \times S^1$. Thus the product $Y_K = (S^3 \setminus \nu) \times S^1$ has the same boundary, a 3-torus, as the complement $M \setminus N$ and we define a new compact 4-manifold

$$M_{K,\phi} = (M \setminus N) \cup_\phi Y_K,$$

where the notation means that the two spaces are glued along their common boundary using a diffeomorphism $\phi : \partial N \to \partial Y_K$. This map ϕ is chosen to take the circle ∂D^2 in the boundary of N, which bounds a disc in N, to the "longitude" in the boundary of ν, which is distinguished by the fact that it bounds a surface in the complement $S^3 \setminus \nu$. This condition does not completely fix ϕ but for the case of main interest here it is known that the resulting manifold is independent of the choice of ϕ, so we just write M_K. For the trivial knot K_0 the complement $S^3 \setminus \nu$ is diffeomorphic to $S^1 \times D^2$, so Y_{K_0} is the same as N and M_{K_0} is the same as M – the construction just cuts out N and then puts it back again.

The general problem is this: *For two knots K_1, K_2, when is the 4-manifold M_{K_1} diffeomorphic to M_{K_2}?* But there is no need to be so ambitious so we can ask the following: *Can we find interesting examples of M, K_1, K_2 such that M_{K_1} and M_{K_2} either are, or are not, diffeomorphic?*

The simplest way in which one might detect the effect of this knot surgery is through the fundamental group. For a non-trivial knot K, the fundamental group of the complement $S^3 \setminus \nu$ is a complicated nonabelian group, but it has the property that it is normally generated by the loops in the boundary 2-torus. That is, the only normal subgroup of $\pi_1(S^3 \setminus \nu)$ which contains $\pi_1(\partial\nu)$ is the whole group. It follows that if the complement $M \setminus T$ is simply connected then the same is true of M_K. In particular, this will be true if M is simply connected and there is a 2-sphere Σ in M which meets T transversely in a single point. From now on we restrict attention to the case when the 4-manifold M is the 4-manifold underlying a complex *K3 surface* X and $T \subset K$ is a complex curve. Regarded as complex manifolds there is a huge moduli space of K3 surfaces (only some of which contain complex curves) but it is known that all such pairs (X, T) are equivalent up to diffeomorphism. For one explicit model we could take X to be the quartic surface in $\mathbf{CP}^3$ defined by the equation

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.$$

If $\kappa \in \mathbf{C}$ is a fourth root of -1 then the line L defined by the equations $z_1 = \kappa z_0, z_3 = \kappa z_2$ lies in X and for a generic plane Π through L the intersection of X with Π is the union of L and a smooth plane curve of degree 3. It is well known that smooth plane cubics are (as differentiable manifolds) 2-dimensional tori, so this gives our torus $T \subset X$, which one can check has trivial normal bundle. Using the manifest symmetries of X we can find another line L' in X which is skew to L and then L' meets T in just one point. A standard general result in complex algebraic geometry (the Lefschetz hyperplane theorem) shows that X is simply connected and since L' is a 2-sphere (as a differentiable manifold) we see that $X \setminus T$ is simply connected. There are many other possible models for (X, T) that one can take, for example using the "Kummer construction" via the quotient of a 4-torus by an involution.

To set our problem in context we recall that, in 1982, Freedman obtained a complete classification of simply connected 4-manifolds up to *homeomorphism*: everything is determined by the homology. At the level of homology all knot complements look the same and it follows that all the manifolds X_K are homeomorphic to the K3 surface X. By contrast the classification up to *diffeomorphism*, which is the setting for our problem, is a complete mystery. The only tools available come from the Seiberg–Witten equations which yield the *Seiberg–Witten invariants*. Ignoring some significant technicalities, these invariants of a smooth 4-manifold M take the form of a finite number of distinguished classes ("basic classes") in the homology $H_2(M)$, with for each basic class β a non-zero integer $SW(\beta)$. So there is a way to show that 4-manifolds are not diffeomorphic, by showing that their Seiberg–Witten invariants are different, but if the Seiberg–Witten invariants are the same one has no technique to decide if the manifolds are in fact diffeomorphic, except for constructing a diffeomorphism by hand, if such exists. The special importance of the K3 surface X appears here in the fact that it has the simplest possible non-trivial Seiberg–Witten invariant: there is just one basic class $0 \in H_2(X)$ and $SW(0) = 1$.

The main result of Fintushel and Stern in [1] is a calculation of the Seiberg–Witten invariants of the knot-surgered manifolds X_K. To explain their result we need to recall the *Alexander polynomial* of a knot K. While the knotting is invisible in the homology of the complement $S^3 \setminus \nu$ we get something interesting by passing to the infinite cyclic cover. The action of the covering transformations makes the 1-dimensional homology of this covering space a module over the group ring of $\mathbf{Z}$, which is the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$ of Laurent series with integer coefficients. One finds that this is a torsion module Λ/I, for a principal ideal $I \subset \Lambda$ and the generator of this ideal I gives the Alexander polynomial $p_K \in \Lambda$. From this point of view p_K is defined up to multiplication by a unit in Λ but there is a way to normalise so that

$$p_K(t) = a_0 + \sum_{i=1}^{g} a_i(t^i + t^{-i}),$$

for integers a_i with $a_0 + 2\sum_{i=1}^{g} a_i = 1$.

Fintushel and Stern show that X_K has basic classes $\pm 2i[T]$, where $[T]$ is the homology class of a "parallel" copy of T in the complement $X \setminus N$ (which is contained in all X_K) and $SW(2i[T]) = a_i$. In other words, the Seiberg–Witten invariants capture exactly the Alexander polynomial of K. It is easy to construct distinct knots with same Alexander polynomial, so our question becomes the following: *if K_1, K_2 are knots with the same Alexander polynomial, are the 4-manifolds X_{K_1}, X_{K_2} diffeomorphic?*

As we have outlined, this question is a prototype – in an explicit and elementary setting – for the fundamental mystery of four-dimensional differential topology. There are also important connections with symplectic topology. A knot is called "fibred" if there is a fibration $\pi : S^3 \setminus \nu \to S^1$, extending the standard fibration on the 2-torus boundary. The fibre S is the complement of a disc in a compact surface of genus g and in this case the Alexander polynomial is just t^{-g} times the characteristic polynomial of the action of the monodromy on $H_1(S)$. In particular the polynomial is "monic", with leading coefficent a_g equal to ± 1. On the other hand there are knots K with monic Alexander polynomial which are not fibred and distinct fibred knots may have the same Alexander polynomial. If K is fibred then one can construct a symplectic structure ω_K on X_K. Conversely if X_K has a symplectic structure then results of Taubes on Seiberg–Witten invariants, combined with the calculation of Fintushel and Stern, show that p_K must be monic. So we have further questions such as

1. *If p_K is monic but K is not fibred, does X_K admit a symplectic structure?*
2. *If K_1, K_2 are fibred knots and (X_{K_1}, ω_{K_1}) is symplectomorphic to (X_{K_2}, ω_{K_2}) are K_1, K_2 equivalent?*

Another question in the same vein as (1) is whether a 4-manifold $S^1 \times Z^3$ admits a symplectic structure if and only if the 3-manifold Z^3 fibres over the circle. This was proved by Friedl and Vidussi [3] and by Kutluhan and Taubes [5] (with an extra technical assumption).

If we take the product $X_K \times S^2$ we move into the realm of high-dimensional geometric topology: the subtleties of 4 dimensions disappear and all the manifolds are diffeomorphic. But in the symplectic theory there are still interesting questions:

For which fibred knots K_1, K_2 are $(X_{K_i} \times S^2, \omega_{K_i} + \omega_{S^2})$ symplectomorphic?

It seems likely that the Alexander polynomials must be the same, using Taubes' result relating the Seiberg–Witten and Gromov–Witten invariants.

References

[1] R. Fintushel and R. Stern, *Knots, links and 4-manifolds*. *Inventiones Math.* **134** (1998), 363–400.

[2] R, Fintushel and R. Stern, *Six lectures on 4-manifolds* in *Low dimensional topology*. IAS/Park City Math. Series, Vol. 15 Amer. Math. Soc. 2009.

[3] S. Friedl and S. Vidussi, Twisted Alexander polynomials detect fibered 3-manifolds. *Annals of Math.* **173** (2011), 1587–1643.

[4] R. Gompf and A. Stipsicz, *Four-manifolds and Kirby calculus*. Grad. Studies in Math. Amer. Math. Soc. (1999)

[5] C. Kutluhan and C. Taubes, *Seiberg–Witten Floer homology and symplectic forms on $S^1 \times M^3$*. *Geometry and Topology* **13** (2009), 493–525.

III Solutions

197. In a game, a player moves a counter on the integers according to the following rules. During each round, a fair die is thrown. If the die shows "5" or "6", the counter is moved up one position and if it shows "1" or "2", it is moved down one position. If the die shows "3" or "4", the counter is moved up one position if the current position is positive, down one position if the current position is negative and stays at the same position if the current position is 0. Let X_n denote the position of the player after n rounds when starting at $X_0 = 1$. Find the probability p that $\lim X_n = +\infty$ and show that $X_n/n \to 1/3$ with probability p and $X_n/n \to -1/3$ with probability $1 - p$.

(Andreas Eberle, Institute for Applied Mathematics, Probability Theory, Bonn, Germany)

Solution by the proposer. For $x \in \mathbb{Z}$, we denote by $p(x)$ the probability that $\lim X_n = +\infty$ if the counter starts at $X_0 = x$. If $x > 0$ then during the first round, the player moves to $x + 1$ with probability $2/3$ and to $x - 1$ with probability $1/3$. By conditioning on the first step, we see that for $x > 0$,

$$\begin{aligned} p(x) &= \frac{2}{3}\,\mathbb{P}[\lim X_n = \infty \,|\, X_1 = x + 1] \\ &\qquad\qquad + \frac{1}{3}\,\mathbb{P}[\lim X_n = \infty \,|\, X_1 = x - 1] \\ &= \frac{2}{3}\,p(x + 1) + \frac{1}{3}\,p(x - 1). \end{aligned}$$

This intuitive argument is made mathematically rigorous by applying the Markov property for the process $(X_n)_{n \in \mathbb{Z}_+}$. Similarly, we can consider the cases $x < 0$ and $x = 0$. We obtain the linear system

$$p(x) = \frac{2}{3}\,p(x + 1) + \frac{1}{3}\,p(x - 1) \qquad \text{for all } x > 0, \tag{1}$$

$$p(x) = \frac{1}{3}\,p(x + 1) + \frac{2}{3}\,p(x - 1) \qquad \text{for all } x < 0, \tag{2}$$

$$p(0) = \frac{1}{3}\,p(1) + \frac{1}{3}\,p(0) + \frac{1}{3}\,p(-1). \tag{3}$$

The equation (1) can be rewritten as the difference equation

$$\frac{2}{3}\,(p(x + 1) - p(x)) = \frac{1}{3}\,(p(x) - p(x - 1)) \qquad \text{for all } x > 0.$$

Thus the general solution of (1) is given by

$$p(x) = a + b \cdot (1 - 2^{-x}) \qquad \text{for } x \geq 0, \tag{4}$$

where a and b are real constants. Similarly, the general solution of (2) is

$$p(x) = c + d \cdot (1 - 2^{x}) \text{ for } x \leq 0.$$

Matching coefficients at $x = 0$ shows that $c = a$, and taking into account (3) implies $d = -b$. Hence

$$p(x) = a - b \cdot (1 - 2^{x}) \qquad \text{for } x \leq 0. \tag{5}$$

Finally, we observe that

$$\lim_{x \to +\infty} p(x) = 1 \qquad \text{and} \qquad \lim_{x \to -\infty} p(x) = 0. \tag{6}$$

To see this let $S_n = \sum_{i=1}^{n} Y_i$ where $Y_i = +1$ if the dice shows "3", "4", "5", or "6" in the ith round, and $Y_i = -1$ otherwise. Then by the law of large numbers, with probability 1, $\lim S_n/n = 1/3$, and thus $\lim S_n = \infty$. Moreover, if $\inf S_n > -X_0$ then $X_0 + S_n$ is always positive, and hence $X_n = X_0 + S_n$ for all n. Thus

$$\liminf_{x \to \infty} p(x) \geq \lim_{x \to \infty} \mathbb{P}[\inf S_n > -x] = \mathbb{P}[\inf S_n > -\infty] = 1.$$

This shows that $\lim_{x\to\infty} p(x) = 1$, and, by a similar argument, $\lim_{x\to-\infty} p(x) = 0$. By (4), (5), and (6), $a + b = 1$ and $a - b = 0$, i.e., $a = b = 1/2$. Hence

$$p(x) = \begin{cases} 1 - 2^{-x-1} & \text{for } x \geq 0, \\ 2^{x-1} & \text{for } x \leq 0. \end{cases}$$

In particular, for $X_0 = 1$ we obtain

$$p = \mathbb{P}[\lim X_n = \infty] = p(1) = 3/4.$$

Moreover, by symmetry,

$$\mathbb{P}[\lim X_n = -\infty] = p(-1) = 1/4 = 1 - p.$$

Hence with probability 1, we have either $\lim X_n = +\infty$ or $\lim X_n = -\infty$. In the first case, $X_n - X_{n-1} = Y_n$ for sufficiently large n, and hence

$$\lim(X_n/n) = \lim(S_n/n) = 1/3.$$

Similarly, in the second case,

$$\lim(X_n/n) = -1/3. \qquad \square$$

Also solved by Mihaly Bencze (Romania), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

198. Let $B := (B_t)_{t\geq 0}$ be Brownian motion in the complex plane. Suppose that $B_0 = 1$.

(a) Let T_1 be the first time that B hits the imaginary axis, T_2 be the first time after T_1 that B hits the real axis, T_3 be the first time after T_2 that B hits the imaginary axis, etc. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(b) More generally, let ℓ_n be lines through 0 for $n \geq 1$ such that $1 \notin \ell_1$. Let $T_1 := \inf\{t \geq 0\,;\ B_t \in \ell_1\}$ and recursively define $T_{n+1} := \inf\{t > T_n\,;\ B_t \in \ell_{n+1}\}$ for $n \geq 1$. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(c) In the context of part (b), let α_n be the smaller of the two angles between ℓ_n and ℓ_{n+1}. Show that $\sum_{n=1}^{\infty} \alpha_n = \infty$ iff, for all $\epsilon > 0$, the probability that $\epsilon \leq |B_{T_n}| \leq 1/\epsilon$ tends to 0 as $n \to \infty$.

(d) In the context of part (a), show that

$$\lim_{n\to\infty} \mathbf{P}\Big[\exp(-\delta_n \sqrt{n}) \leq |B_{T_n}| \leq \exp(\delta_n \sqrt{n})\Big] = \int_{-2\delta/\pi}^{2\delta/\pi} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$$

if $\delta_n \geq 0$ tend to $\delta \in [0, \infty]$.

(Russell Lyons, Department of Mathematics, Indiana University, USA [Partially supported by the National Science Foundation under grant DMS-1612363])

Solution by the proposer. We skip (a) and pass directly to (b). Denote inversion in the unit circle by $\phi(z) := 1/\bar{z}$. It is well known that $W := (\phi(B_t))_{t\geq 0}$ is a time-change of Brownian motion. Since ϕ maps each line ℓ_n to itself, $T_1 = \inf\{t \geq 0\,;\ W_t \in \ell_1\}$ and $T_{n+1} = \inf\{t > T_n\,;\ W_t \in \ell_{n+1}\}$ for $n \geq 1$. Thus, B_{T_n} and W_{T_n} have the same distribution. However, $|B_t| \leq 1$ iff $|W_t| \geq 1$. Since the chance that $|B_{T_n}| = 1$ is 0 for each n, we obtain (b).

In light of (b), the conclusion of (c) is equivalent to $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] = 1/2$ for all $\epsilon > 0$.

Let $T := \lim_{n\to\infty} T_n$. If $T < \infty$ a.s., then $\lim_{n\to\infty} B_{T_n} = B_T$, which implies that for some $\epsilon > 0$, $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] \neq 1/2$. Now suppose that $T = \infty$ a.s. Neighbourhood recurrence of B shows that for each $\epsilon > 0$, there is some $t < \infty$ such that $|B_t| \leq \epsilon$. Let S_ϵ be the first such time t. The strong Markov property, scaling, rotational symmetry, and part (b) shows that $\mathbf{P}\big[|B_{T_n}| < \epsilon \;\big|\; T_n > S_\epsilon\big] = 1/2$. Because $\lim_{n\to\infty} \mathbf{P}[T_n > S_\epsilon] = 1$, this shows that $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] = 1/2$.

It remains to show that if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $T < \infty$ a.s., whereas if $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $T = \infty$ a.s. If $\limsup_{n\to\infty} \alpha_n > 0$, as in (a), then this is clear from the fact that then a.s. there is no limiting argument of B_{T_n}. In general, we use the skew-product representation of B as $B_t = \exp(X_{H_t} + iY_{H_t})$, where X and Y are independent real Brownian motions started at 0 and $H_t := \int_0^t ds/|B_s|^2$ (the form of H will not matter to us). Note that H_{T_n} are functions of Y and thus independent of X. Also, $H_\infty = \infty$ a.s. It is well known that the expected time for Y to visit either $\alpha > 0$ or $-\beta < 0$ is $\alpha\beta$, whence the expectation of $H_{T_{n+1}} - H_{T_n}$ equals $(\pi - \alpha_n)\alpha_n$. Note that $H_{T_{n+1}} - H_{T_n}$ are independent (and nonnegative). Therefore, if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\sum_n (H_{T_{n+1}} - H_{T_n}) < \infty$ a.s., and otherwise (by Kolmogorov's three-series theorem) $\sum_n (H_{T_{n+1}} - H_{T_n}) = \infty$ a.s. This is the same as $H_T < \infty$ a.s. or $H_T = \infty$ a.s. respectively, which in turn is equivalent to $T < \infty$ a.s. or $T = \infty$ a.s.

For part (d), note that $H_{T_n}/n \to \pi^2/4$ in probability by the weak large of large numbers. With $\epsilon_n := \exp(-\delta_n \sqrt{n})$, we have

$$\begin{aligned} \mathbf{P}[\epsilon_n \leq |B_{T_n}| \leq 1/\epsilon_n] &= \mathbf{P}[-\delta_n \sqrt{n} \leq X_{H_{T_n}} \leq \delta_n \sqrt{n}\,] \\ &= \mathbf{P}\Big[-\frac{2\delta_n}{\pi} \leq X_{\frac{4H_{T_n}}{\pi^2 n}} \leq \frac{2\delta_n}{\pi}\Big] \end{aligned}$$

by Brownian scaling. Now let $n \to \infty$. $\square$

Remark. The proof of (c) could be shortened by using the skew-product representation throughout, but the proof given is more elementary in the context of (a). Part (b) could also be proved with the skew-product representation.

Also solved by Mihaly Bencze (Romania), Sotirios E. Louridas (Greece), and Socratis Varelogiannis (France).

199. Suppose that each carioca (native of Rio de Janeiro) likes at least half of the other 2^{23} cariocas. Prove that there exists a set A of 1000 cariocas with the following property: for each pair of cariocas in A, there exists a *distinct* carioca who likes both of them.

(Rob Morris, IMPA, Rio de Janeiro, Brazil)

Solution by the proposer. Choose 10 random cariocas (possibly with repetition), and consider the set X of cariocas that they all like. Observe that, writing V for the set of all n cariocas, we have

$$\mathbb{E}[|X|] \geq \sum_{v\in V} \left(\frac{d(v)}{n}\right)^{10} \geq 2^{-11} n \geq 2000,$$

where $d(v)$ is the number of cariocas that like carioca v. Indeed, the first inequality follows by linearity of expectation, together with the fact that $v \in X$ if and only if the 10 random cariocas all like v, the second inequality uses the convexity of the function $f(x) = x^{10}$ and the fact that $\sum_{v\in V} d(v) \geq \binom{n}{2}$, and the third holds since $n \geq 2^{23}$.

Now, let Y be the set of pairs of cariocas in X such that fewer than $\binom{1000}{2}$ cariocas like both of them. Observe that

$$\mathbb{E}[|Y|] \leq \binom{n}{2} \left(\frac{\binom{1000}{2}}{n}\right)^{10} \leq \frac{2^{189}}{n^8} \leq 1000,$$

again using the fact that $n \geq 2^{23}$. It follows that $\mathbb{E}[|X| - |Y|] \geq 1000$, and hence there exists a choice of 10 cariocas such that $|X| - |Y| \geq$

1000. Removing one carioca from each pair in X such that fewer than $\binom{1000}{2}$ cariocas like both of them, we obtain a set A of 1000 cariocas, so that no pair in A has this property. But now we can greedily (i.e., one by one) find a distinct carioca w for each $u, v \in A$ such that w likes both u and v, as required.

This proof is due to N. Alon, M. Krivelevich, and B. Sudakov [1, 5], and is based on an earlier idea of W. T. Gowers [3] and (independently) A. V. Kostochka and V. Rödl [4]. It is a simple example of a powerful technique known as *dependent random choice*; see [2]. □

References

[1] N. Alon, M. Krivelevich and B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, *Combin. Probab. Comput-ing*, **12** (2003), 477–494.

[2] J. Fox and B. Sudakov, Dependent Random Choice, *Random Structures Algorithms*, **38** (2011), 1–32.

[3] W.T. Gowers, A new proof of Szemerédi's theorem for arithmetic progressions of length four, *Geom. Funct. Anal.*, **8** (1998), 529–551

[4] A.V. Kostochka and V. Rödl, On graphs with small Ramsey numbers, *J. Graph Theory*, **37** (2001), 198–204.

[5] B. Sudakov, Few remarks on the Ramsey–Turán-type problems, *J. Combin. Theory, Ser. B*, **88** (2003), 99–106.

Also solved by Mihaly Bencze (Romania) and Socratis Varelogiannis (France).

200. Let X, Y, Z be independent and uniformly distributed in $[0, 1]$. What is the probability that three sticks of length X, Y and Z can be assembled together to form a triangle?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

Solution by the proposers. When x, y, z are fixed and $z = \max(x, y, z)$, three sticks of length x, y, and z can be assembled together to form a triangle if and only if $x + y \leq z$. Let A be the event that three sticks of length X, Y, and Z can be assembled together to form a triangle. We want to know the probability of A. Let $M = \max(X, Y, Z)$ We first compute $P(A|Z = M)$. This is

$$P(X + Y \leq Z|Z = M)\ .$$

We condition on Z: for a fixed $z \in [0, 1]$, $P(X + Y \leq z|Z = z = M)$ is

$$P(X + Y \leq z|X \leq z, Y \leq z) = \frac{P(X + Y \leq z)}{P(X \leq z, Y \leq z)}\ .$$

The denominator is z^2. Since the density is uniform, the nominator is the area of the triangle bounded by the x and y axes and the line $y = z - x$. This area is $\frac{z^2}{2}$, and hence we obtain that $P(A|Z = z = M) = \frac{1}{2}$. De-conditioning, we obtain

$$P(A|Z = M) = \int_0^1 \frac{1}{2} \cdot 1dz = \frac{1}{2}\ .$$

Similarly, $P(A|X = M) = P(A|Y = M) = \frac{1}{2}$. By symmetry, X, Y, and Z are equally likely to be maximal, hence $P(X = M) = P(Y = M) = P(Z = M) = \frac{1}{3}$. We conclude that $P(A) = \frac{1}{2}$. □

Also solved by Mihaly Bencze (Romania), Jim Kelesis (Greece), Panagiotis Krasopoulos (Greece), Peter Marioni (USA), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

201. Suppose that each hour, one of the following four events may happen to a certain type of cell: it may die, it may split into two cells, it may split into three cells, or it may remain a single cell. Suppose these four events are equally likely. Start with a population consisting of a single cell. What is the probability that the population eventually goes extinct?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

Solution by the proposer. Let Z_n be the number of cells after n hours (the sequence $Z_0, Z_1, \ldots$ is called a branching process). We have that $Z_0 = 1$ and the mass function f_{Z_1} of Z_1 is $f_{Z_1}(k) = \frac{1}{4}$ for $k = 0, 1, 2, 3$. Thus its generating function is

$$G_{Z_1}(s) = \sum_{k=0}^{\infty} f_{Z_1}(k)s^k = \frac{1}{4}\left(1 + s + s^2 + s^3\right).$$

Write $G := G_{Z_1}$, and let $G_n := G_{Z_n}$. We claim that for $n \geq 2$, $G_n = G_{n-1} \circ G$. Indeed, for $1 \leq i \leq Z_{n-1}$, let X_i be the number of cells that the ith cell reproduced into. Then $Z_n = X_1 + X_2 + \cdots + X_{Z_{n-1}}$ and $G_{X_i}(s) = G(s)$. Thus,

$$\begin{aligned} G_n(s) &= \mathbf{E}(s^{Z_n}) = \mathbf{E}\left(\mathbf{E}(s^{Z_n}|Z_{n-1})\right) \\ &= \sum_{m=0}^{\infty} (G(s))^m P(Z_{n-1} = m) = G_{n-1}(G(s)). \end{aligned}$$

We now claim that the probability η of extinction is the least nonnegative solution to the equation $G(s) = s$. Indeed, let $\eta_n := P(Z_n = 0)$. Note that $(\eta_n)_{n\in\mathbb{N}}$ is an increasing sequence with limit η. Moreover, $G_n(0) = \eta_n$, and so in particular $G(0) = \eta_1$, $\eta_2 = G(G(0)) = G(\eta_1)$, and in general $G(\eta_n) = \eta_{n+1}$. G is continuous, so taking the limit on both sides, $G(\eta) = \eta$. Clearly, η is nonnegative, and if η' is another nonegative solution, then $0 \leq \eta'$, so (using that $G(s)$ is nondecreasing for $s \geq 0$) $\eta_1 = G(0) \leq G(\eta') = \eta'$, and so $\eta_2 = G(\eta_1) \leq G(\eta') = \eta'$, and so on. Thus $\eta_n \leq \eta'$ for all n, and hence $\eta \leq \eta'$.

We have shown that the desired probability of extinction is the least nonnegative solution of $\frac{1}{4}(1 + s + s^2 + s^3) = s$, i.e., of

$$p(s) := \frac{s^3}{4} + \frac{s^2}{4} - \frac{3s}{4} + \frac{1}{4} = 0.$$

To find the roots of p, note that $G(1) = 1$ (probabilities must sum to 1), so after a polynomial division,

$$p(s) = (s - 1)\left(\frac{s^2}{4} + \frac{s}{2} - \frac{1}{4}\right).$$

The second factor has roots $-1 \pm \sqrt{2}$. Discarding the negative solution (and noting that $\sqrt{2} - 1 < 1$), we obtain that the probability of extinction is $\eta = \sqrt{2} - 1$. □

Also solved by Mihaly Bencze (Romania), Jim Kelesis (Greece), Panagiotis Krasopoulos (Greece), and Peter Marioni (USA).

202. We are flipping a fair coin repeatedly and recording the outcomes.

(1) How many coin flips do we need on average to see three tails in a row?

(2) Suppose that we stop when we first see heads, heads, tails (H, H, T) or tails, heads, tails (T, H, T) come up in this order on three consecutive flips. What is the probability that we stop at H, H, T?

(Benedek Valkó, Department of Mathematics, University of Wisconsin Madison, Madison, Wisconsin, USA)

Solution by the proposer. Let $X_k \in \{H, T\}$ denote the outcome of the kth coin flip. By assumption

$$P(X_k = H) = P(X_k = T) = \frac{1}{2}$$

and the random variables $\{X_k, k \geq 1\}$ are independent. For a finite sequence

$$A = (a_1, \ldots, a_n),\ a_i \in \{H, T\}$$

we set

$$\tau_A = \inf\{k \geq n : X_{k-n+1} = a_1, X_{k-n+2} = a_2, \ldots, X_k = a_n\}$$

to be the first time we see the sequence A appearing as the result of consecutive outcomes in our sequence of coin flips.

The first observation is that for any finite A we have $P(\tau_A < \infty) = 1$ and $E[\tau_A] < \infty$. To prove this, we divide up the infinite coin flip sequence into consecutive blocks of n (the length of A), and set $\widetilde{\tau}_A$ to be the first time we see A appearing in one of these blocks:

$$\widetilde{\tau}_A = \inf\{k \geq 1 : X_{(k-1)n+1} = a_1, X_{(k-1)n+2} = a_2, \ldots, X_{kn} = a_n\}.$$

Since the outcomes in non-overlapping blocks are independent, and the probability that we see A in a given block of length n is 2^{-n}, we have

$$P(\widetilde{\tau}_A = j) = 2^{-n}(1 - 2^{-n})^{j-1} \ \text{ for } j \geq 1.$$

From this we get

$$P(\widetilde{\tau} < \infty) = \sum_{j=1}^{\infty} P(\widetilde{\tau}_A = j) = 1$$

and

$$E[\widetilde{\tau}_A] = \sum_{j=1}^{\infty} j \cdot P(\widetilde{\tau}_A = j) = 2^n < \infty.$$

By definition $\tau_A \leq n \cdot \widetilde{\tau}_A$, which implies $P(\tau_A < \infty) = 1$ and $E[\tau_A] < \infty$.

Now we turn to the actual questions.

(a) We have to compute $E[\tau_A]$ where $A = (T, T, T)$. Consider the following events:

$$B_1 = \{X_1 = H\}, \qquad B_2 = \{X_1 = T, X_2 = H\},$$
$$B_3 = \{X_1 = T, X_2 = T, X_3 = H\}, \quad B_4 = \{X_1 = X_2 = X_3 = T\}.$$

These are disjoint, and one of them will always happen, i.e., they form a partition of our sample space. Hence we can compute $E[\tau_A]$ by averaging the conditional expectations:

$$E[\tau_A] = \sum_{i=1}^{4} E[\tau_A | B_i] P(B_i).$$

The probabilities $P(B_i)$ can be computed using the independence of the different coin flips: $P(B_1) = 1/2$, $P(B_2) = 1/4$, $P(B_3) = P(B_4) = 1/8$. If $X_1 = X_2 = X_3 = T$ (i.e., B_4 occurs) then $\tau_A = 3$ which means that $E[\tau_A|B_4] = 3$. If the first coin flip is heads (i.e., B_1 occurs) then the first T, T, T sequence will have to start at least at the second coin flip. By the independence of the coin flips this means that τ_A conditioned on B_1 behaves the same way as $\tau_A + 1$, which implies

$$E[\tau_A|B_1] = E[\tau_A + 1] = E[\tau_A] + 1\,.$$

We can show $E[\tau_A|B_2] = E[\tau_A] + 2$ and $E[\tau_A|B_3] = E[\tau_A] + 3$ the same way. This gives

$$E[\tau_A] = \frac{1}{2}(E[\tau_A] + 1) + \frac{1}{4}(E[\tau_A] + 2) + \frac{1}{8}(E[\tau_A] + 3) + \frac{1}{8} \cdot 3,$$

and solving this we get $E[\tau_A] = 14$. (Note that we need $E[\tau_A] < \infty$ for the last step.)

(b) Let $A_1 = (H, H, T)$ and $A_2 = (T, H, T)$. We need to compute $P(\tau_{A_1} < \tau_{A_2})$. Introduce the events

$$C_1 = \{X_1 = X_2 = H\},\ C_2 = \{X_1 = T, X_2 = H\},$$
$$C_3 = \{X_1 = X_2 = T\},\ C_4 = \{X_1 = H, X_2 = T\}.$$

These events form a partition of our sample space, hence we can compute $P(\tau_{A_1} < \tau_{A_2})$ by averaging the corresponding conditional probabilities:

$$\begin{aligned} P(\tau_{A_1} < \tau_{A_2}) &= \sum_{i=1}^{4} P(\tau_{A_1} < \tau_{A_2}|C_i) P(C_i) \\ &= \frac{1}{4}\sum_{i=1}^{4} P(\tau_{A_1} < \tau_{A_2}|C_i). \end{aligned}$$

Given that the first two flips are heads we will have $\tau_{A_1} < \tau_{A_2}$, as the first appearing tails will form a sequence of H, H, T (and T, H, T cannot happen before that). Thus $P(\tau_{A_1} < \tau_{A_2}|C_1) = 1$. Suppose now that $X_1 = T$ and $X_2 = H$. If $X_3 = T$ (which has a conditional probability of $1/2$) then $\tau_{A_2} = 3 < \tau_{A_1}$. If $X_3 = H$ then we are in a similar situation as before: H, H, T will come up before T, H, T. Thus $P(\tau_{A_1} < \tau_{A_2}|C_2) = \frac{1}{2}$. A similar argument shows that $P(\tau_{A_1} < \tau_{A_2}|C_3) = P(\tau_{A_1} < \tau_{A_2}|C_4) = \frac{1}{2}$ as well (in both cases one has to consider the first heads appearing after the second flip). This gives

$$P(\tau_{A_1} < \tau_{A_2}) = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{5}{8}.$$

□

Remarks.

- One can always compute the expectation $E[\tau_A]$ for a given finite sequence of length n by setting up a system of linear equations for the conditional expectations of τ_A with respect to the first $n-1$ possible coin flips. These equations are the consequence of the fact that we only need to 'remember' the last $n-1$ coin flips to check whether we complete the sequence at a given coin flip. This idea can also be used to compute $P(\tau_{A_1} < \tau_{A_2})$ for any two given sequences A_1, A_2.
- It might be surprising to note that there are sequences A_1, A_2 so that $E[\tau_{A_1}] > E[\tau_{A_2}]$ but $P(\tau_{A_1} < \tau_{A_2}) > 1/2$. Moreover, there are sequences A_1, A_2, A_3 so that A_1 is more likely to come up before A_2, A_2 is more likely to come up before A_3, and A_3 is more likely to come up before A_1.
- Using a bit more sophisticated methods (martingales and optional stopping, see, e.g., [1]) one can prove an explicit formula for $E[\tau_A]$. If $A = (a_1, \ldots, a_n)$ then

$$E[\tau_A] = \sum_{k=1}^{n} 2^k \cdot \mathbb{1}(a_{n-k+1} = a_1, a_{n-k+2} = a_2, \ldots, a_n = a_k).$$

Thus the more ways the sequence A can 'overlap' with itself the larger the expected wait time for its first appearance. A similar formula can be derived for $P(\tau_{A_1} < \tau_{A_2})$.

References

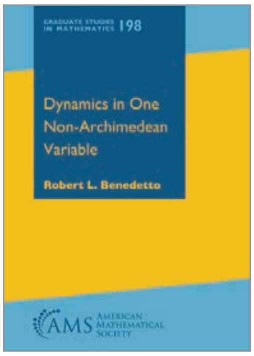
[1] David Williams *Probability with Martingales*, Cambridge University Press, 2001.

Also solved by Marcello Galeotti (Italy), Jim Kelesis (Greece), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

We encourage you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *differential equations*.

AMS AMERICAN MATHEMATICAL SOCIETY



DYNAMICS IN ONE NON-ARCHIMEDEAN VARIABLE

Robert L. Benedetto, Amherst College

Presents the fundamentals of non-archimedean dynamics, including a unified exposition of Rivera-Letelier's classification theorem, as well as results on wandering domains, repelling periodic points, and equilibrium measures. The Berkovich projective line is developed from the ground up, as are relevant results in non-archimedean analysis.

Graduate Studies in Mathematics, Vol. 198

Mar 2019 418pp 9781470446888 Hardback €103.00

LINEAR ALGEBRA

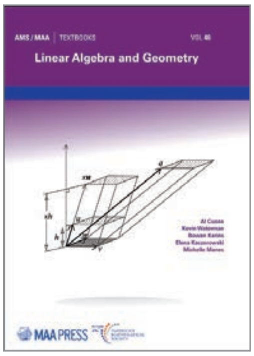
Concepts and Applications

Przemyslaw Bogacki, Old Dominion University

Provides a complete coverage of core linear algebra topics, including vectors and matrices, systems of linear equations, general vector spaces, linear transformations, eigenvalues, and eigenvectors. All results are carefully, clearly, and rigorously proven. The exposition is very accessible.

MAA Textbooks, Vol. 47
MAA Press

Mar 2019 383pp 9781470443849 Hardback €107.00



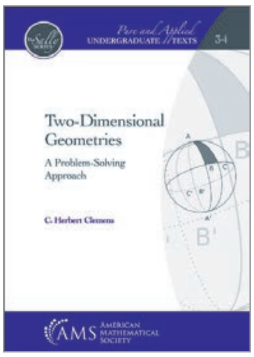
LINEAR ALGEBRA AND GEOMETRY

Al Cuoco, Education Development Center, Inc. et al

Organised around carefully sequenced problems, *Linear Algebra and Geometry* will help students build both the tools and the habits that provide a solid basis for further study in mathematics. This volume uses elementary geometry to build the beautiful edifice of results and methods that make linear algebra such an important field.

MAA Textbooks, Vol. 46
MAA Press

Apr 2019 597pp 9781470443504 Hardback €107.00



TWO-DIMENSIONAL GEOMETRIES

A Problem-Solving Approach

C. Herbert Clemens, Ohio State University

Uses a problem-solving approach to actively engage students in the learning process. The book bridges the study of plane geometry and the study of curves and surfaces of non-constant curvature in three-dimensional Euclidean space.

Pure and Applied Undergraduate Texts, Vol. 34

Apr 2019 142pp 9781470447601 Hardback €96.00

Free delivery at **eurospanbookstore.com/ams**

AMS is distributed by EUROSPAN

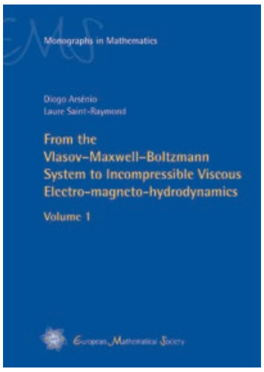
CUSTOMER SERVICES:
Tel: +44 (0)1767 604972
Fax: +44 (0)1767 601640
Email: eurospan@turpin-distribution.com

FURTHER INFORMATION:
Tel: +44 (0)20 7240 0856
Fax: +44 (0)20 7379 0609
Email: info@eurospan.co.uk

New books published by the

European Mathematical Society

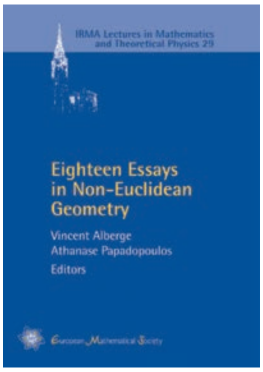
Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.



Diogo Arsénio (Université Paris Diderot, France) and Laure Saint-Raymond (École Normale Supérieure, Lyon, France)
From the Vlasov–Maxwell–Boltzmann System to Incompressible Viscous Electro-magneto-hydrodynamics. Volume 1 (EMS Monographs in Mathematics)

ISBN 978-3-03719-193-4. 2019. Approx. 414 pages. Hardcover. 16.5 x 23.5 cm. 78.00 Euro

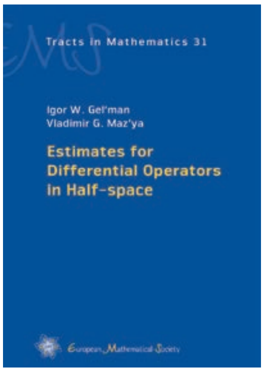
The Vlasov–Maxwell–Boltzmann system is a microscopic model to describe the dynamics of charged particles subject to self-induced electromagnetic forces. At the macroscopic scale, in the incompressible viscous fluid limit, the evolution of the plasma is governed by equations of Navier–Stokes–Fourier type, with some electromagnetic forcing that may take on various forms depending on the number of species and on the strength of the interactions. From the mathematical point of view, these models have very different behaviors. Their analysis therefore requires various mathematical methods which this book aims at presenting in a systematic, painstaking and exhaustive way.
The first part of this work is devoted to the systematic formal analysis of viscous hydrodynamic limits of the Vlasov–Maxwell–Boltzmann system leading to a precise classification of physically relevant models for viscous incompressible plasmas, some of which have not been previously described in the literature. In the second part, the convergence results are made precise and rigorous, assuming the existence of renormalized solutions for the Vlasov–Maxwell–Boltzmann system. The analysis is based essentially on the scaled entropy inequality. The third and fourth parts will be published in a second volume.



Eighteen Essays in Non-Euclidean Geometry (IRMA Lectures in Mathematics and Theoretical Physics, Vol. 29)
Vincent Alberge (Fordham University, Bronx, USA) and Athanase Papadopoulos (Université de Strasbourg, France), Editors

ISBN 978-3-03719-196-5. 2019. 475 pages. Hardcover. 17 x 24 cm. 78.00 Euro

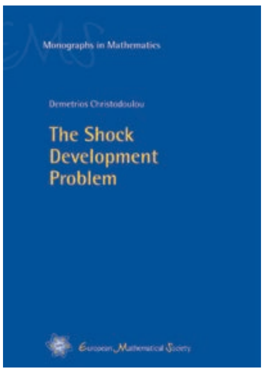
This book consists of a series of self-contained essays in non-Euclidean geometry in a broad sense, including the classical geometries of constant curvature (spherical and hyperbolic), de Sitter, anti-de Sitter, co-Euclidean, co-Minkowski, Hermitian geometries, and some axiomatically defined geometries. Some of these essays deal with very classical questions and others address problems that are at the heart of present day research, but all of them are concerned with fundamental topics.
All the essays are self-contained and most of them can be understood by the general educated mathematician. They should be useful to researchers and to students of non-Euclidean geometry, and they are intended to be references for the various topics they present.



Igor W. Gel'man (Israel) and Vladimir G. Maz'ya (Linköping University, Sweden, and University of Liverpool, UK)
Estimates for Differential Operators in Half-space (EMS Tracts in Mathematics)

ISBN 978-3-03719-191-0. 2019. 264 pages. Hardcover. 17 x 24 cm. 48.00 Euro

Inequalities for differential operators play a fundamental role in the modern theory of partial differential equations. Among the numerous applications of such inequalities are existence and uniqueness theorems, error estimates for numerical approximations of solutions and for residual terms in asymptotic formulas, as well as results on the structure of the spectrum. The inequalities cover a wide range of differential operators, boundary conditions and norms of the corresponding function spaces.
The book focuses on estimates up to the boundary of a domain. It contains a great variety of inequalities for differential and pseudodifferential operators with constant coefficients. Results of final character are obtained, without any restrictions on the type of differential operators. Algebraic necessary and sufficient conditions for the validity of the corresponding a priori estimates are presented. General criteria are systematically applied to particular types of operators found in classical equations and systems of mathematical physics. The book will be interesting and useful to a wide audience, including graduate students and specialists in the theory of differential equations.



Demetrios Christodoulou (ETH Zürich, Switzerland)
The Shock Development Problem (EMS Monographs in Mathematics)

ISBN 978-3-03719-192-7. 2019. 932 pages. Hardcover. 16.5 x 23.5 cm. 128.00 Euro

This monograph addresses the problem of the development of shocks in the context of the Eulerian equations of the mechanics of compressible fluids. The mathematical problem is that of an initial-boundary value problem for a nonlinear hyperbolic system of partial differential equations with a free boundary and singular initial conditions.
The free boundary is the shock hypersurface and the boundary conditions are jump conditions relative to a prior solution, conditions following from the integral form of the mass, momentum and energy conservation laws. The prior solution is provided by the author's previous work which studies the maximal classical development of smooth initial data. New geometric and analytic methods are introduced to solve the problem. Geometry enters as the acoustical structure, a Lorentzian metric structure defined on the spacetime manifold by the fluid. This acoustical structure interacts with the background spacetime structure. Reformulating the equations as two coupled first order systems, the characteristic system, which is fully nonlinear, and the wave system, which is quasilinear, a complete regularization of the problem is achieved.

European Mathematical Society Publishing House
Seminar for Applied Mathematics, ETH-Zentrum SEW A21
Scheuchzerstrasse 70
CH-8092 Zürich, Switzerland
orders@ems-ph.org
www.ems-ph.org