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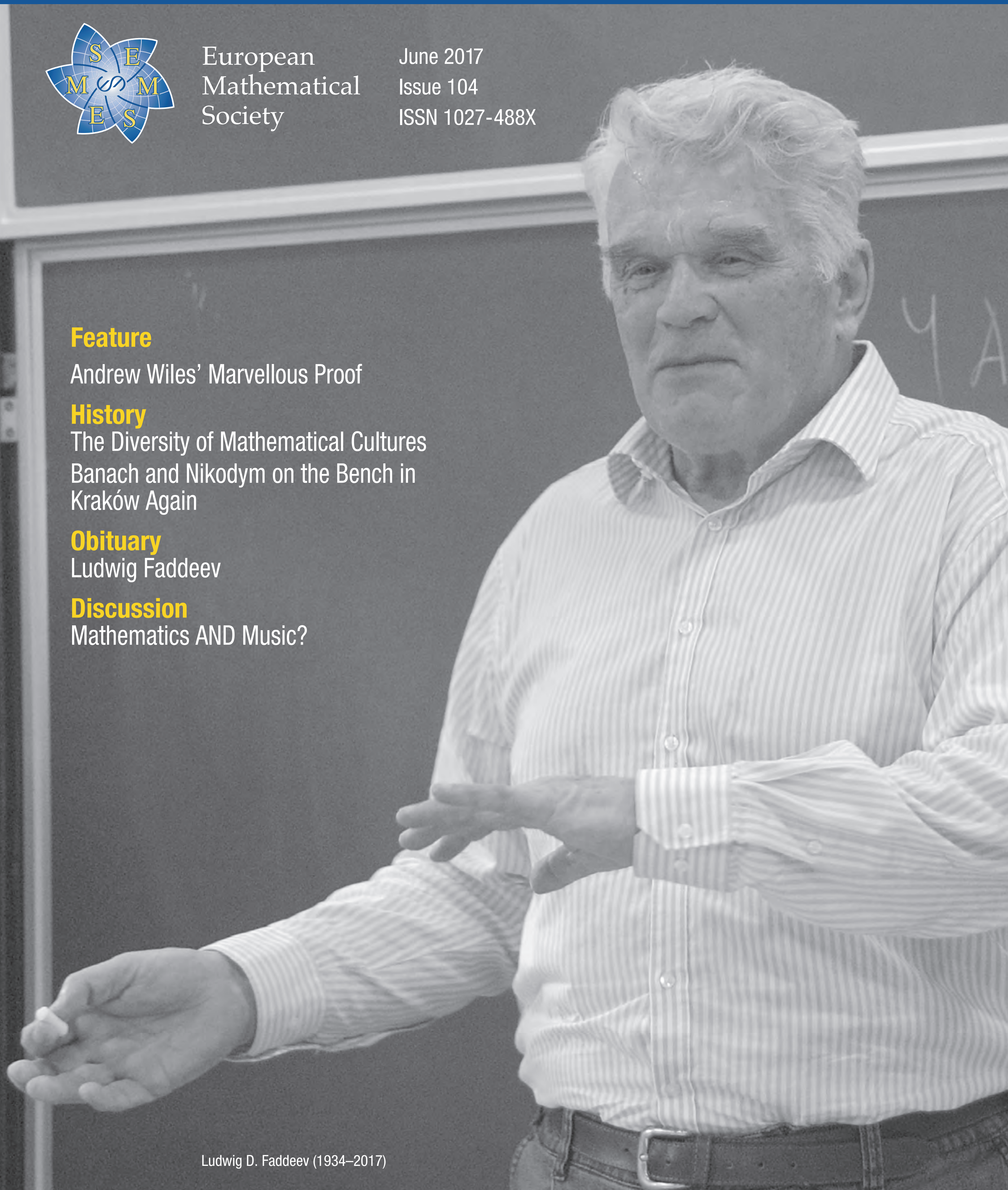

Ludwig D. Faddeev (1934–2017)

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http://euro-math-soc.eu/newsletter

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EMS Executive Committee

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11519 Prague 1
Czech Republic
e-mail: ems@ujf.cas.cz

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TU Berlin MA 4–5
Strasse des 17. Juni 136
10623 Berlin
Germany
e-mail: mehrmann@math.TU-Berlin.DE

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e-mail: s.m.verduynlunel@uu.nl

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UK
e-mail: b.pelloni@hw.ac.uk

Prof. Betül Tanbay
(2017–2020)
Department of Mathematics
Bogazici University
Bebek 34342 Istanbul
Turkey
e-mail: tanbay@boun.edu.tr

EMS Secretariat

Ms Elvira Hyvönen
Department of Mathematics
and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
00014 University of Helsinki
Finland
Tel: (+358) 2941 51141
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dr. Richard H. Elwes
School of Mathematics
University of Leeds
Leeds, LS2 9JT
UK
e-mail: R.H.Elwes@leeds.ac.uk

EMS Agenda

2017

20 November
Applied Mathematics Committee Meeting, Amsterdam

24–26 November
EMS Executive Committee Meeting, Portorož, Slovenia

EMS Scientific Events

2017

2–7 July
CNRS-PAN Mathematics Summer Institute, Cracow, Poland

10–19 July
Foundations of Computational Mathematics (FoCM'17), Barcelona, Spain
EMS Distinguished Speaker: Mireille Bousquet-Mélou
http://www.ub.edu/focm2017/

17–21 July
Summer School: Between Geometry and Relativity, ESI Vienna, Austria

23–29 July
7th PhD Summer School in Discrete Mathematics, Hotel Planja, Rogla, Slovenia

24–28 July
31st European Meeting of Statisticians, Helsinki, Finland
EMS-Bernoulli Society Joint Lecture: Alexander Holevo
http://ems2017.helsinki.fi

7–11 August
11th International Conference on Clifford Algebras and Their Applications, Ghent University, Belgium

14–19 August
Mathematics in the Modern World, Sobolev Institute of Mathematics, Novosibirsk, Russian Federation

27 August–1 September
23rd Czech and Slovak International Conference on Number Theory, Hotel Sepetná, Ostravice, Czech Republic

3–9 September
CISM-EMS School on Rationality, Stable Rationality and Birationally Rigidity of Complex Algebraic Varieties, Udine, Italy

24–29 September
5th Heidelberg Laureate Forum
http://www.heidelberg-laureate-forum.org/

15–19 November
ASTUCON – The 2nd Academic University Student Conference (Science, Technology Engineering, Mathematics), Larnaca, Cyprus

Editorial: If Mathematicians Unite…

Betül Tanbay (Bogazici University, Istanbul, Turkey), Member of the EMS Executive Committee

IMU is not quite the abbreviation of the title – but bringing mathematicians together is certainly the mission of institutions such as the IMU and the EMS. The first objective in the IMU statutes is "to promote international cooperation in mathematics" and the EMS similarly aims "to foster the interaction between mathematicians of different countries".

Globalisation may have brought a hope of general peace but today we see its failure in the sense of a world without borders. Yet there are two "success" stories to highlight at opposite ends of the spectrum, one fortunate and one extremely unfortunate: mathematics and terrorism. Thanks to technology, we are able to be present in Oslo and have a research partner in Cape Town. But also due to technology, we can be blown up on the metro in London or during a football game in Istanbul. How can we save mathematics and delete terrorism from the planet? Today we have to push the limits of our thinking processes and our imagination. We could not have carried out any mathematics had we not used our 'imagination'. So, aren't 'we' well-equipped?

In August 2017, the IMAGINARY exhibition (https://imaginary.org) will open in Van, one of the easternmost cities of the westernmost country of the East (Turkey) and the cradle of many civilisations, including some of the first Christians (Armenia), as well the Urartus, Byzance, Kurdish and Turkish cultures. The occasion is the second Caucasian Mathematics Conference (CMC, http://euro-math-soc.eu/cmc/), which has been organised under the auspices of the EMS by the mathematical societies of the Caucasian countries and their neighbours: Armenia, Azerbaijan, Georgia, Iran, Russia and Turkey. When the steering committee met during the very successful first CMC in Tbilisi in 2014, after a lot of "math and dance", it seemed very natural to all of us to have the second CMC in Turkey (in 2016). However, the unfortunate developments of that Summer made us postpone CMCII. Although the situation for locals was no worse than usual, several invited speakers changed their minds over the year and some even requested that the EMS boycott Turkey.

I fully respect an unwillingness to travel to a place where one does not feel secure. But let us think twice before we announce this reluctance as an academic boycott. It might also be good to remember that certain governments can be alike in their degree of authoritarianism, violence and funding of wars but very different in terms of their respect for academia and academicians or their support for mathematics.

A lot has been said about the controversy of academic boycotts but let us remember what Noam Chomsky, prominent supporter of the Palestinian cause, said about the calls for a boycott of Israel. He supported the "boycott and divestment of firms that are carrying out operations in the occupied territories" but he stated that a general boycott of Israel would be "a gift to Israeli hardliners and their American supporters".

So, mathematicians of the world … unite! We have a lot to lose!

Professor Betül Tanbay
Member of the EMS Executive Committee

International Prize "Tullio Levi-Civita" for the Mathematical and Mechanical Sciences

In honour of the famous Italian mathematical physicist Tullio Levi-Civita,* in 2010, the International Research Center for Mathematics & Mechanics of Complex Systems (M&MoCS) established the international prize "Tullio Levi-Civita". The prize recognises the high quality and undisputed originality of the scientific research of up to two distinguished Italian or foreign scientists in the field of mechanics and/or mathematics. Every winner must have contributed to the development of at least one young researcher and is expected to hold a series of lectures and join the research activities of the centre during a short stay. In 2016, the prize was awarded to Tudor Ratiu (École Polytechnique Fédérale de Lausanne EPFL, Switzerland) and Mauro Carfora (Università degli Studi di Pavia, Italy). Past winners of the prize include Lucio Russo (2010, Università degli Studi di Roma Tor Vergata), Pierre Seppecher (2010, Université de Toulon et du Var), Jean Jacques Marigo (2011, Ecole Polytechnique), Eric Carlen (2011, Rutgers University), Félix Darve (2012, Institut National Polytechnique de Grenoble), Alexander Seyranian (2012, Institute of Mechanics, Moscow State Lomonosov University), Kazou Aoki (2013, Kyoto University), David Steigmann (2013, University of California, Berkeley), Marcelo Epstein (2014, University of Calgary), Errico Presutti (2014, Gran Sasso Science Institute), Graeme W. Milton (2015, University of Utah) and Andrea Braides (2015, Università degli Studi di Roma Tor Vergata).

Moreover, every year, during the Levi Civita Lectures, the recipients of the International Levi Civita Prize and up to two young promising researchers are invited to give a talk. Finally, there is no need for applications since every scientist working in mechanics and mathematics will be considered for the prize by the scientific committee. Any further enquiries can be sent to memocs.cisterna@gmail.com and more information can be found on the website http://memocs.univaq.it.

M&MoCS
DICEAA
Università degli Studi dell'Aquila
Via Giovanni Gronchi 18
67100 L'Aquila, Italy
Tel. 06.90.28.67.84 Fax 0773.1871016
Website: http://memocs.univaq.it
E-mail: memocs.cisterna@gmail.com

* Tullio Levi Civita (1873–1941) was born into an Italian Jewish family and enrolled in 1890 at the University of Padua. He graduated in 1892 and his final dissertation, which was supervised by Ricci Curbastro, dealt with absolute invariants and tensor calculus. Levi-Civita was appointed to the Chair of Rational Mechanics at Padua in 1898. After the end of World War I, the University of Rome made strenuous efforts to attract many leading scientists and hence become an internationally recognised first-tier institution. Levi-Civita was always very international in his outlook and the ability of Rome to attract top quality students from abroad must have been a reason for him choosing to move there. In 1918, he was appointed to the Chair of Higher Analysis at Rome and, two years later, he was appointed to the Chair of Mechanics. Levi-Civita had a great command of pure mathematics, with a particularly strong geometric intuition, which he exploited in addressing a variety of problems in applied mathematics. He is best known for his work on absolute differential calculus, with its applications to the theory of Einstein's relativity, and on the calculus of tensors including covariant differentiation, continuing the work of Christoffel. Levi-Civita was also interested in the theory of stability and qualitative analysis of differential equations (because of his interest in geometry and geometric models) and classical and celestial mechanics. Indeed, he published many papers dealing with analytic dynamics. He examined special cases of the three-body problem and, near the end of his career, he became interested in the n-body problem. In the field of systems of partial differential equations, he extended the theory of Cauchy and Kovalevskaya. In addition, Levi-Civita made a major contribution to hydrodynamics, proving the existence of periodic finite-amplitude irrotational surface waves in a mono-dimensional fluid flow.

ICIAM Olga Taussky-Todd Lecture 2019 – Call for Nominations

The Olga Taussky-Todd Lecture is held every four years at the International Congress on Industrial and Applied Mathematics (ICIAM). This honour is conferred on a woman who has made outstanding contributions in applied mathematics and/or scientific computation.

The lecture is named in tribute to the memory of Olga Taussky-Todd, whose scientific legacy is in both theoretical and applied mathematics and whose work exemplifies the qualities to be recognised.

The Officers and Board of the ICIAM now call for nominations for the Olga Taussky-Todd Lecture, to be given at the ICIAM 2019 congress, which will take place in Valencia (Spain), 15–19 July 2019.

A nomination consists of:

- Full name and address of person nominated.
- Web homepage if applicable.
- Justification for nomination (on at most two pages, stating the nominator's reason for considering the candidate to be deserving of this honour, including an explanation of the scientific and practical influence of the candidate's work and publications).
- 2–3 letters of support from experts in the field (not mandatory), each of them on a maximum of two pages.
- CV of the nominee.
- Name and contact details of the proposer.

Nominations should be made electronically through the website https://iciamprizes.org/. The deadline for nominations is 30 September 2017.

Please contact president@iciam.org if you have any questions regarding the nomination procedure.

Olga Taussky-Todd Committee for 2019:
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Raymond Chan (The Chinese University of Hong Kong)
Ingrid Daubechies (Duke University)
Nick Higham (University of Manchester)
Sofia C. Ohlede (University College London)
Anna Karin Tornberg (KTH, Stockholm)

ICIAM, the International Council for Industrial and Applied Mathematics, is the world organisation for applied and industrial mathematics.

Its members are mathematical societies based in more than 30 countries.

For more information, see the council's webpage at http://www.iciam.org/.

Maria J. Esteban, ICIAM President

The Ferran Sunyer i Balaguer Prize 2017

The Ferran Sunyer i Balaguer Prize 2017 winners were:

Antoine Chambert-Loir (Université Paris-Diderot Paris 7), **Johannes Nicaise** (Imperial College London), and **Julien Sebag** (Université Rennes 1), for the work

Motivic Integration

Abstract: Over the last 12 years, since the introduction by Kontsevich in 1995, Motivic integration has been strongly developed by Denef and Loeser, to whom this monograph is dedicated (Nicaise and Sebaq were students of them). This theory evolved into a major branch of algebraic and arithmetic geometry. It has triggered developments in singularity theory, theory of motives, nonarchimedean and tropical geometry. The monograph under consideration is the first comprehensive exposition of this theory, going much beyond the two volumes of collected papers published by the LMS in 2011. In particular, most of the material in Sections 2 (Arc Schemes) and 3–4 (Greenberg Schemes) are new. Section 6 is a nice summary of applications of motivic integration.
This monograph will be published by Springer Basel in their Birkhäuser series *Progress in Mathematics.*

fundació FERRAN SUNYER I BALAGUER
Institut d'Estudis Catalans

Call for the

Ferran Sunyer i Balaguer Prize 2018

The prize will be awarded for a **mathematical monograph** of an expository nature presenting the latest developments in an active area of research in mathematics.

The prize consists of **15,000 Euros** and the winning monograph will be published in Springer Basel's **Birkhäuser** series "Progress in Mathematics".

DEADLINE FOR SUBMISSION:
1 December 2017
http://ffsb.iec.cat

Andrew Wiles' Marvellous Proof *

Henri Darmon (McGill University, Montreal, Canada)

Fermat famously claimed to have discovered "a truly marvellous proof" of his Last Theorem, which the margin of his copy of Diophantus' *Arithmetica* was too narrow to contain. While this proof (if it ever existed) is lost to posterity, Andrew Wiles' marvellous proof has been public for over two decades and has now earned him the Abel prize. According to the prize citation, Wiles merits this recognition "for his stunning proof of Fermat's Last Theorem by way of the modularity conjecture for semistable elliptic curves, opening a new era in number theory".

Few can remain insensitive to the allure of Fermat's Last Theorem, a riddle with roots in the mathematics of ancient Greece, simple enough to be understood and appreciated by a novice (like the 10-year-old Andrew Wiles browsing the shelves of his local public library), yet eluding the concerted efforts of the most brilliant minds for well over three centuries. It became, over its long history, the object of lucrative awards like the Wolfskehl prize and, more importantly, it motivated a cascade of fundamental discoveries: Fermat's method of infinite descent, Kummer's theory of ideals, the ABC conjecture, Frey's approach to ternary diophantine equations, Serre's conjecture on mod p Galois representations, …

Even without its seemingly serendipitous connection to Fermat's Last Theorem, Wiles' modularity theorem is a fundamental statement about elliptic curves (as evidenced, for instance, by the key role it plays in the proof of Theorem 2 of Karl Rubin's contribution to the issue of the Notices of the AMS mentioned above). It is also a centrepiece of the "Langlands programme", the imposing, ambitious edifice of results and conjectures that has come to dominate the number theorist's view of the world. This programme has been described as a "grand unified theory" of mathematics. Taking a Norwegian perspective, it connects the objects that occur in the works of Niels Hendrik Abel, such as elliptic curves and their associated abelian integrals and Galois representations, with (frequently infinite-dimensional) linear representations of the continuous transformation groups, the study of which was pioneered by Sophus Lie. This report focuses on the role of Wiles' Theorem and its "marvellous proof" in the Langlands programme, in order to justify the closing phrase in the prize citation: how Wiles' proof has opened "a new era in number theory" and continues to have a profound and lasting impact on mathematics.

Our "beginner's tour" of the Langlands programme will only give a partial and undoubtedly biased glimpse of the full panorama, reflecting the author's shortcomings as well as the inherent limitations of a treatment aimed at a general readership. We will motivate the Langlands programme by starting with a discussion of *diophantine equations*: for the purposes of this exposition, they are equations of the form

$$\mathcal{X}: \quad P(x_1, \ldots, x_{n+1}) = 0, \tag{1}$$

where P is a polynomial in the variables $x_1, \ldots, x_{n+1}$ with integer (or sometimes rational) coefficients. One can examine the set, denoted $\mathcal{X}(F)$, of solutions of (1) with coordinates in any ring F. As we shall see, the subject draws much of its fascination from the deep and subtle ways in which the behaviours of different solution sets can resonate with each other, even if the sets $\mathcal{X}(\mathbb{Z})$ or $\mathcal{X}(\mathbb{Q})$ of integer and rational solutions are foremost in our minds. Examples of diophantine equations include Fermat's equation $x^d + y^d = z^d$ and the Brahmagupta-Pell equation $x^2 - Dy^2 = 1$ with $D > 0$, as well as elliptic curve equations of the form $y^2 = x^3 + ax + b$, in which a and b are rational parameters, the solutions (x, y) with rational coordinates being the object of interest in the latter case.

It can be instructive to approach a diophantine equation by first studying its solutions over *simpler* rings, such as the complete fields of real or complex numbers. The set

$$\mathbb{Z}/n\mathbb{Z} := \{0, 1, \ldots, n-1\} \tag{2}$$

of remainders after division by an integer $n \geq 2$, equipped with its natural laws of addition, subtraction and multiplication, is another particularly simple collection of numbers, of *finite cardinality*. If $n = p$ is *prime*, this ring is even a *field*: it comes equipped with an operation of division by non-zero elements, just like the more familiar collections of rational, real and complex numbers. The fact that $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is a field is an algebraic characterisation of the primes that forms the basis for most known efficient primality tests and factorisation algorithms. One of the great contributions of Evariste Galois, in addition to the eponymous theory that plays such a crucial role in Wiles' work, is his discovery of a field of cardinality p^r for any prime power p^r. This field, denoted $\mathbb{F}_{p^r}$ and sometimes referred to as the Galois field with p^r elements, is even *unique* up to isomorphism.

For a diophantine equation $\mathcal{X}$ as in (1), the most basic invariant of the set

$$\mathcal{X}(\mathbb{F}_{p^r}) := \{(x_1, \ldots, x_{n+1}) \in \mathbb{F}_{p^r}^{n+1} \\ \text{such that} \quad P(x_1, \ldots, x_{n+1}) = 0\} \tag{3}$$

of solutions over $\mathbb{F}_{p^r}$ is of course its *cardinality*

$$N_{p^r} := \#\mathcal{X}(\mathbb{F}_{p^r}). \tag{4}$$

What patterns (if any) are satisfied by the sequence

$$N_p, N_{p^2}, N_{p^3}, \ldots, N_{p^r}, \ldots? \tag{5}$$

* This report is a very slightly expanded transcript of the Abel prize lecture delivered by the author on 25 May 2016 at the University of Oslo. It is published with the permission of the *Notices of the AMS*: reprinted from Volume 64, Issue 3, March 2017.

This sequence can be packaged into a generating series like

$$\sum_{r=1}^{\infty} N_{p^r} T^r \qquad \text{or} \qquad \sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r. \tag{6}$$

For technical reasons, it is best to consider the exponential of the latter:

$$\zeta_p(\mathcal{X}; T) := \exp\left(\sum_{r=1}^{\infty} \frac{N_{p^r}}{r} T^r\right). \tag{7}$$

This power series in T is known as the *zeta function* of $\mathcal{X}$ over $\mathbb{F}_p$. It has integer coefficients and enjoys the following remarkable properties:

1. It is a *rational function* in T:
$$\zeta_p(\mathcal{X}; T) = \frac{Q(T)}{R(T)}, \tag{8}$$
where $Q(T)$ and $R(T)$ are polynomials in T whose degrees (for all but finitely many p) are *independent of* p and determined by the shape – the complex topology – of the set $\mathcal{X}(\mathbb{C})$ of complex solutions.
2. The reciprocal roots of $Q(T)$ and $R(T)$ are complex numbers of absolute value $p^{i/2}$ with i an integer in the interval $0 \le i \le 2n$.

The first statement – the rationality of the zeta function, which was proven by Bernard Dwork in the early 1960s – is a key part of the Weil conjectures, whose formulation in the 1940s unleashed a revolution in arithmetic geometry, driving the development of étale cohomology by Grothendieck and his school. The second statement, which asserts that the complex function $\zeta_p(\mathcal{X}; p^{-s})$ has its roots on the real lines $\Re(s) = i/2$ with i as above, is known as the Riemann hypothesis for the zeta functions of diophantine equations over finite fields. It was proven by Pierre Deligne in 1974 and is one of the major achievements for which he was awarded the Abel prize in 2013.

That the asymptotic behaviour of N_p can lead to deep insights into the behaviour of the associated diophantine equations is one of the key ideas behind the Birch and Swinnerton-Dyer conjecture. Understanding the patterns satisfied by the functions

$$p \mapsto N_p \qquad \text{and} \qquad p \mapsto \zeta_p(\mathcal{X}; T) \tag{9}$$

as the prime p varies will also serve as our motivating question for the Langlands programme.

It turns out to be fruitful to package the zeta functions over all the finite fields into a single function of a complex variable s, by taking the infinite product

$$\zeta(\mathcal{X}; s) = \prod_p \zeta_p(\mathcal{X}; p^{-s}) \tag{10}$$

as p ranges over all the prime numbers. In the case of the simplest non-trivial diophantine equation $x = 0$, whose solution set over $\mathbb{F}_{p^r}$ consists of a single point, one has $N_{p^r} = 1$ for all p and therefore

$$\zeta_p(x = 0; T) = \exp\left(\sum_{r\ge1} \frac{T^r}{r}\right) = (1 - T)^{-1}. \tag{11}$$

It follows that

$$\zeta(x = 0; s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{12}$$

$$= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right) \tag{13}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \zeta(s). \tag{14}$$

The zeta function of even the humblest diophantine equation is thus a central object of mathematics: the celebrated Riemann zeta function, which is tied to some of the deepest questions concerning the distribution of prime numbers. In his great memoir of 1860, Riemann proved that, even though (13) and (14) only converge absolutely on the right half-plane $\Re(s) > 1$, the function $\zeta(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ (with a single pole at $s = 1$) and possesses an elegant functional equation relating its values at s and $1 - s$. The zeta functions of linear equations $\mathcal{X}$ in $n + 1$ variables are just shifts of the Riemann zeta function, since N_{p^r} is equal to p^{nr}, and therefore $\zeta(\mathcal{X}; s) = \zeta(s - n)$.

Moving on to equations of degree two, the general quadratic equation in one variable is of the form $ax^2 + bx + c = 0$ and its behaviour is governed by its *discriminant*

$$\Delta := b^2 - 4ac. \tag{15}$$

This purely algebraic fact remains true over the finite fields and, for primes $p \nmid 2a\Delta$, one has

$$N_p = \begin{cases} 0 & \text{if } \Delta \text{ is a non-square modulo } p, \\ 2 & \text{if } \Delta \text{ is a square modulo } p. \end{cases} \tag{16}$$

A priori, the criterion for whether $N_p = 2$ or 0 — whether the integer Δ is or is not a quadratic residue modulo p — seems like a subtle condition on the prime p. To get a better feeling for this condition, consider the example of the equation $x^2 - x - 1$, for which $\Delta = 5$. Calculating whether 5 is a square or not modulo p for the first few primes $p \le 101$ leads to the following list

$$N_p = \begin{cases} 2 & \text{for } p = 11, 19, 29, 31, 41, 59, 61, 71, 79, \\ & \qquad\qquad\qquad 89, 101, \ldots \\ 0 & \text{for } p = 2, 3, 7, 13, 17, 23, 37, 43, 47, 53, \\ & \qquad\qquad\qquad 67, 73, 83, \ldots \end{cases} \tag{17}$$

A clear pattern emerges from this experiment: whether $N_p = 0$ or 2 seems to depend only on the rightmost digit of p, i.e. on what the remainder of p is modulo 10. One is led to surmise that

$$N_p = \begin{cases} 2 & \text{if } p \equiv 1, 4 \pmod 5, \\ 0 & \text{if } p \equiv 2, 3 \pmod 5, \end{cases} \tag{18}$$

a formula that represents a dramatic improvement over (16), allowing a much more efficient calculation of N_p for example. The guess in (18) is in fact a consequence of Gauss' celebrated law of quadratic reciprocity:

Theorem (Quadratic reciprocity) *For any equation $ax^2 + bx + c$, with $\Delta := b^2 - 4ac$, the value of the function $p \mapsto N_p$*

(for $p \nmid a\Delta$) depends only on the residue class of p modulo 4Δ, and hence is periodic with period length dividing $4|\Delta|$.

The repeating pattern satisfied by the N_p's as p varies greatly facilitates the manipulation of the zeta functions of quadratic equations. For example, the zeta function of $\mathcal{X}$: $x^2 - x - 1 = 0$ is equal to

$$\zeta(\mathcal{X};s) = \zeta(s) \times \Big\{ \Big(1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s}\Big) + \Big(\frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s}\Big) + \Big(\frac{1}{11^s} - \frac{1}{12^s} - \frac{1}{13^s} + \frac{1}{14^s}\Big) + \cdots \Big\}. \quad (19)$$

The series that occurs on the right side is a prototypical example of a *Dirichlet L-series*. These L-series, which are the key actors in the proof of Dirichlet's theorem on the infinitude of primes in arithmetic progressions, enjoy many of the same analytic properties as the Riemann zeta function: an analytic continuation to the entire complex plane and a functional equation relating their values at s and $1 - s$. They are also expected to satisfy a Riemann hypothesis that generalises Riemann's original statement and is just as deep and elusive.

It is a (not completely trivial) fact that the zeta function of the general quadratic equation in n variables

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c = 0 \quad (20)$$

involves the same basic constituents – Dirichlet series – as in the one variable case. This means that quadratic diophantine equations in any number of variables are well understood, at least as far as their zeta functions are concerned.

The plot thickens when equations of higher degree are considered. Consider, for instance, the cubic equation $x^3 - x - 1$ of discriminant $\Delta = -23$. For all $p \neq 23$, this cubic equation has no multiple roots over $\mathbb{F}_{p^r}$ and therefore $N_p = 0$, 1 or 3. A simple expression for N_p in this case is given by the following theorem of Hecke:

Theorem (Hecke). *The following holds for all primes $p \neq 23$:*
1. *If p is not a square modulo 23 then $N_p = 1$.*
2. *If p is a square modulo 23 then*

$$N_p = \begin{cases} 0 & \text{if } p = 2a^2 + ab + 3b^2, \\ 3 & \text{if } p = a^2 + ab + 6b^2, \end{cases} \quad (21)$$

for some $a, b \in \mathbb{Z}$.

Hecke's theorem implies that

$$\zeta(x^3 - x - 1; s) \quad = \quad \zeta(s) \times \sum_{n=1}^{\infty} a_n n^{-s}, \quad (22)$$

where the generating series

$$F(q) := \sum_{n=1}^{\infty} a_n q^n = q - q^2 - q^3 + q^6 + q^8 - q^{13} - q^{16} + q^{23} + \cdots \quad (23)$$

is given by the explicit formula

$$F(q) = \frac{1}{2} \left(\sum_{a,b \in \mathbb{Z}} q^{a^2+ab+6b^2} - q^{2a^2+ab+3b^2} \right). \quad (24)$$

The function $f(z) = F(e^{2\pi i z})$ that arises by setting $q = e^{2\pi i z}$ in (24) is a prototypical example of a *modular form*: namely, it satisfies the transformation rule

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d) f(z), \quad \begin{cases} a,b,c,d \in \mathbb{Z}, & ad - bc = 1, \\ 23|c, & \left(\frac{a}{23}\right) = 1, \end{cases} \quad (25)$$

under so-called *modular substitutions* of the form $z \mapsto \frac{az+b}{cz+d}$. This property follows from the *Poisson summation formula* applied to the expression in (24). Thanks to (25), the zeta function of $\mathcal{X}$ can be manipulated with the same ease as the zeta functions of Riemann and Dirichlet. Indeed, Hecke showed that the L-series $\sum_{n=1}^{\infty} a_n n^{-s}$ attached to a modular form $\sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ possess very similar analytic properties, notably an analytic continuation and a Riemann-style functional equation.

The generating series $F(q)$ can also be expressed as an infinite product:

$$\frac{1}{2} \left(\sum_{a,b \in \mathbb{Z}} q^{a^2+ab+6b^2} - q^{2a^2+ab+3b^2} \right) = q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{23n}). \quad (26)$$

The first few terms of this power series identity can readily be verified numerically but its proof is highly non-obvious and indirect. It exploits the circumstance that the space of holomorphic functions of z satisfying the transformation rule (25) together with suitable growth properties is a one-dimensional complex vector space that also contains the infinite product above. Indeed, the latter is equal to $\eta(q)\eta(q^{23})$, where

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (27)$$

is the Dedekind eta function whose logarithmic derivative (after viewing η as a function of z through the change of variables $q = e^{2\pi i z}$) is given by

$$\frac{\eta'(z)}{\eta(z)} = -\pi i \left(\frac{-1}{12} + 2 \sum_{n=1}^{\infty} \Big(\sum_{d|n} d \Big) e^{2\pi i n z} \right) \quad (28)$$

$$= \frac{i}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2}, \quad (29)$$

where the term attached to $(m, n) = (0, 0)$ is excluded from the last sum. The Dedekind η-function is also connected to the generating series for the partition function $p(n)$ describing the number of ways in which n can be expressed as a sum of positive integers via the identity

$$\eta^{-1}(q) = q^{-1/24} \sum_{n=0}^{\infty} p(n) q^n, \quad (30)$$

which plays a starring role alongside Jeremy Irons and Dev Patel in a recent film about the life of Srinivasa Ramanujan.

Commenting on the "unreasonable effectiveness and ubiquity of modular forms", Martin Eichler once wrote: "There are five elementary arithmetical operations: addition, subtraction, multiplication, division, … and modular forms." Equations (26), (29) and (30) are just a few of the many wondrous identities that abound, like exotic strains of fragrant wild orchids, in what Roger Godement has called the "garden of modular delights".

The example above, and many others of a similar type, are described in Jean-Pierre Serre's delightful monograph [2], touching on themes that were also covered in Serre's lecture at the inaugural Abel Prize ceremony in 2003.

Hecke was able to establish that all cubic polynomials in one variable are *modular*, i.e., the coefficients of their zeta functions obey patterns just like those of (24) and (25). Wiles' achievement was to extend this result to a large class of cubic diophantine equations in two variables over the rational numbers: the *elliptic curve* equations, which can be brought into the form

$$y^2 = x^3 + ax + b \tag{31}$$

after a suitable change of variables and which are non-singular, a condition equivalent to the assertion that the *discriminant* $\Delta := -16(4a^3 + 27b^2)$ is non-zero.

To illustrate Wiles' theorem with a concrete example, consider the equation

$$E : y^2 = x^3 - x, \tag{32}$$

of discriminant $\Delta = 64$. After setting

$$\zeta(E; s) = \zeta(s-1) \times \left(a_1 + a_2 2^{-s} + a_3 3^{-s} + a_4 4^{-s} + \cdots\right)^{-1}, \tag{33}$$

the associated generating series satisfies the following identities reminiscent of (24) and (26),

$$\begin{aligned} F(q) &= \sum a_n q^n \\ &= q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} + \cdots && (34) \\ &= \sum_{a,b} a \cdot q^{(a^2+b^2)} && (35) \\ &= q \prod_{n=1}^{\infty} (1-q^{4n})^2 (1-q^{8n})^2, && (36) \end{aligned}$$

where the sum in (35) runs over the $(a, b) \in \mathbb{Z}^2$ for which the Gaussian integer $a + bi$ is congruent to 1 modulo $(1 + i)^3$. (This identity follows from Deuring's study of zeta functions of elliptic curves *with complex multiplication*, and may even have been known earlier.) Once again, the holomorphic function $f(z) := F(e^{2\pi iz})$ is a modular form, satisfying the slightly different transformation rule

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z), \quad \begin{cases} a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \\ 32|c. \end{cases} \tag{37}$$

Note the exponent 2 that appears in this formula. Because of it, the function $f(z)$ is said to be a *modular form of weight* 2 and level 32. The modular forms of (25) attached to cubic equations in one variable are of weight 1 but otherwise the parallel of (35) and (36) with (24) and (26) is striking. The original conjecture of Shimura-Taniyama and Weil asserts the same pattern for all elliptic curves:

Conjecture (Shimura, Taniyama, Weil). *Let E be any elliptic curve. Then,*

$$\zeta(E; s) = \zeta(s-1) \times \left(\sum_{n=1}^{\infty} a_n n^{-s}\right)^{-1}, \tag{38}$$

where $f_E(z) := \sum a_n e^{2\pi inz}$ *is a modular form of weight* 2.

The conjecture was actually more precise and predicted that the level of f_E – i.e., the integer that appears in the transformation property for f_E, as the integers 23 and 32 do in (25) and (37) respectively – is equal to the *arithmetic conductor* of E. This conductor, which is divisible only by primes for which the equation defining E becomes singular modulo p, is a measure of the arithmetic complexity of E and can be calculated explicitly from an equation for E by an algorithm of Tate. An elliptic curve is said to be *semistable* if its arithmetic conductor is squarefree. This class of elliptic curves includes those of the form

$$y^2 = x(x-a)(x-b), \tag{39}$$

with $\gcd(a, b) = 1$ and $16|b$. The most famous elliptic curves in this class are those that ultimately do not exist: the "Frey curves" $y^2 = x(x - a^p)(x + b^p)$ arising from putative solutions to Fermat's equation $a^p + b^p = c^p$, whose non-existence had previously been established in a landmark article of Kenneth Ribet, under the assumption of their modularity. It is the proof of the Shimura-Taniyama-Weil conjecture for semistable elliptic curves that earned Andrew Wiles the Abel prize:

Theorem (Wiles). *Let E be a semistable elliptic curve. Then E satisfies the Shimura-Taniyama-Weil conjecture.*

The semistability assumption in Wiles' theorem was later removed by Christophe Breuil, Brian Conrad, Fred Diamond and Richard Taylor around 1999. (See, for instance, the account that appeared in the Notices at the time [1].)

As a prelude to describing some of the important ideas in its proof, one must first try to explain why Wiles' theorem occupies such a central position in mathematics. The Langlands programme places it in a larger context by offering a vast generalisation of what it means for a diophantine equation to be "associated to a modular form". The key is to view modular forms attached to cubic equations or to elliptic curves, as in (24) or (34), as vectors in certain irreducible infinite-dimensional representations of the locally compact topological group

$$\mathbf{GL}_2(\mathbb{A}_\mathbb{Q}) = \prod_p{}' \, \mathbf{GL}_2(\mathbb{Q}_p) \times \mathbf{GL}_2(\mathbb{R}), \tag{40}$$

where $\prod_p'$ denotes a restricted direct product over all the prime numbers, consisting of elements $(\gamma_p)_p$ for which the p-th component γ_p belongs to the maximal compact subgroup $\mathbf{GL}_2(\mathbb{Z}_p)$ for all but finitely many p. The shift in emphasis from modular forms to the so-called *automorphic representations* that they span is decisive. Langlands showed how to attach an L-function to any irreducible automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ for an arbitrary reductive algebraic group G, of which the matrix groups $\mathbf{GL}_n$ and more general algebraic groups of Lie type are prototypical examples. This greatly enlarges the notion of what it means to be "modular": a diophantine equation is now said to have this property if its zeta function can be expressed in terms of the Langlands L-functions attached to automorphic representations. One of the fundamental goals in the Langlands programme is to establish further cases of the following conjecture:

Conjecture. *All diophantine equations are modular in the above sense.*

This conjecture can be viewed as a far-reaching generalisation of quadratic reciprocity and underlies the nonabelian reciprocity laws that are at the heart of Andrew Wiles' achievement.

Before Wiles' proof, the following general classes of diophantine equations were known to be modular:
- Quadratic equations, by Gauss' law of quadratic reciprocity.
- Cubic equations in one variable, by the work of Hecke and Maass.
- Quartic equations in one variable.

This last case deserves further comment, since it has not been discussed previously and plays a primordial role in Wiles' proof. The modularity of quartic equations follows from the seminal work of Langlands and Tunnell. While it is beyond the scope of this survey to describe their methods, it must be emphasised that Langlands and Tunnell make essential use of the *solvability by radicals* of the general quartic equation, whose underlying symmetry group is contained in the permutation group S_4 on 4 letters. Solvable extensions are obtained from a succession of abelian extensions, which fall within the purview of class field theory developed in the 19th and first half of the 20th centuries. On the other hand, the modularity of the general equation of degree > 4 in one variable, which cannot be solved by radicals, seemed to lie well beyond the scope of the techniques that were available in the "pre-Wiles era". The reader who perseveres to the end of this essay will be given a glimpse of how our knowledge of the modularity of the general quintic equation has progressed dramatically in the wake of Wiles' breakthrough.

Prior to Wiles' proof, modularity was also not known for any interesting general class of equations (of degree > 2, say) in more than one variable; in particular, it had only been verified for finitely many elliptic curves over $\mathbb{Q}$ up to isomorphism over $\bar{\mathbb{Q}}$ (including the elliptic curves over $\mathbb{Q}$ with complex multiplication, of which the elliptic curve of (31) is an example). Wiles' modularity theorem confirmed the Langlands conjectures in the important test case of elliptic curves, which may seem to be (and, in fact, are) very special diophantine equations but which have provided a fertile terrain for arithmetic investigations, both in theory and in applications (e.g., cryptography and coding theory).

Returning to the main theme of this report, Wiles' proof is also important for having introduced a revolutionary new approach, which has opened the floodgates for many further breakthroughs in the Langlands programme.

To expand on this point, we need to present another of the *dramatis personae* in Wiles' proof: *Galois representations*. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, namely, the automorphism group of the field of all algebraic numbers. It is a profinite group, endowed with a natural topology for which the subgroups $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$ with L ranging over the finite extensions of $\mathbb{Q}$ form a basis of open subgroups. Following the original point of view taken by Galois himself, the group $G_{\mathbb{Q}}$ acts naturally as permutations on the roots of polynomials with rational coefficients. Given a finite set S of primes, one may consider only the monic polynomials with integer coefficients whose discriminant is divisible only by primes $\ell \in S$ (eventually after a change of variables). The topological group $G_{\mathbb{Q}}$ operates on the roots of such polynomials through a quotient, denoted $G_{\mathbb{Q},S}$: the automorphism group of the maximal algebraic extension *unramified* outside of S, which can be regarded as the symmetry group of all the zero-dimensional varieties over $\mathbb{Q}$ having "non-singular reduction outside of S".

In addition to the permutation representations of $G_{\mathbb{Q}}$ that were so essential in Galois' original formulation of his theory, it has become important to study the (continuous) *linear* representations

$$\varrho : G_{\mathbb{Q},S} \longrightarrow GL_n(L) \tag{41}$$

of this Galois group, where L is a complete field, such as the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers, the finite field $\mathbb{F}_{\ell^r}$ equipped with the discrete topology, or a finite extension $L \subset \bar{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

Galois representations were an important theme in the work of Abel and remain central in modern times. Many illustrious mathematicians in the 20th century have contributed to their study, including three former Abel prize winners: Jean-Pierre Serre, John Tate and Pierre Deligne. Working on Galois representations might seem to be a prerequisite for an algebraic number theorist to receive the Abel prize!

Like diophantine equations, Galois representations also give rise to analogous zeta functions. More precisely, the group $G_{\mathbb{Q},S}$ contains, for each prime $p \notin S$, a distinguished element called the *Frobenius element* at p, denoted σ_p. Strictly speaking, this element is only well defined up to conjugacy in $G_{\mathbb{Q},S}$ but this is enough to make the arithmetic sequence

$$N_{p^r}(\varrho) := \mathrm{Trace}(\varrho(\sigma_p^r)) \tag{42}$$

well defined. The zeta function $\zeta(\varrho; s)$ packages the information from this sequence in exactly the same way as in the definition of $\zeta(\mathcal{X}; s)$.

For example, if $\mathcal{X}$ is attached to a polynomial P of degree d in one variable, the action of $G_{\mathbb{Q},S}$ on the roots of P gives rise to a d-dimensional permutation representation

$$\varrho_{\mathcal{X}} : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_d(\mathbb{Q}) \tag{43}$$

and $\zeta(\mathcal{X}, s) = \zeta(\varrho_{\mathcal{X}}, s)$. This connection goes far deeper, extending to diophantine equations in $n+1$ variables for general $n \geq 0$. The glorious insight at the origin of the Weil conjectures is that $\zeta(\mathcal{X}; s)$ can be expressed in terms of the zeta functions of Galois representations arising in the *étale cohomology* of $\mathcal{X}$, a cohomology theory with ℓ-adic coefficients that associates to $\mathcal{X}$ a collection

$$\left\{H^i_{\mathrm{et}}(\mathcal{X}_{/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)\right\}_{0\leq i\leq 2n}$$

of finite-dimensional $\mathbb{Q}_\ell$-vector spaces endowed with a continuous linear action of $G_{\mathbb{Q},S}$. (Here, S is the set of primes q consisting of ℓ and the primes for which the equation of $\mathcal{X}$ becomes singular after being reduced modulo q.) These representations generalise the representation $\varrho_{\mathcal{X}}$ of (43), insofar as the latter is realised by the action of $G_{\mathbb{Q},S}$ on $H^0_{\mathrm{et}}(\mathcal{X}_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ after extending the coefficients from $\mathbb{Q}$ to $\mathbb{Q}_\ell$.

Theorem (Weil, Grothendieck, ...). *If $\mathcal{X}$ is a diophantine equation having good reduction outside of S, there exist Galois representations ϱ_1 and ϱ_2 of $G_{\mathbb{Q},S}$ for which*

$$\zeta(\mathcal{X}; s) = \zeta(\varrho_1; s)/\zeta(\varrho_2; s). \tag{44}$$

The representations ϱ_1 and ϱ_2 occur in $\oplus H^i_{\mathrm{et}}(\mathcal{X}_{/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$, where the direct sum ranges over the odd and even values of $0 \le i \le 2n$ for ϱ_1 and ϱ_2 respectively. More canonically, there are always *irreducible* representations $\varrho_1, \ldots, \varrho_t$ of $G_{\mathbb{Q},s}$ and integers $d_1, \ldots d_t$ such that

$$\zeta(\mathcal{X}; s) = \prod_{i=1}^{t} \zeta(\varrho_i; s)^{d_i}, \tag{45}$$

arising from the decompositions of the (semisimplification of) the $H^i_{\mathrm{et}}(\mathcal{X}_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ into a sum of irreducible representations. The $\zeta(\varrho_i, s)$ can be viewed as the "atomic constituents" of $\zeta(\mathcal{X}, s)$, and reveal much of the "hidden structure" in the underlying equation. The decomposition of $\zeta(\mathcal{X}; s)$ into a product of different $\zeta(\varrho_i; s)$ is not unlike the decomposition of a wave function into its simple harmonics.

A Galois representation is said to be *modular* if its zeta function can be expressed in terms of generating series attached to modular forms and automorphic representations, and is said to be *geometric* if it can be realised in an étale cohomology group of a diophantine equation as above. The "main conjecture of the Langlands programme" can now be amended as follows:

Conjecture. *All geometric Galois representations of $G_{\mathbb{Q},S}$ are modular.*

Given a Galois representation

$$\varrho : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_n(\mathbb{Z}_\ell) \tag{46}$$

with ℓ-adic coefficients, one may consider the resulting mod ℓ representation

$$\bar{\varrho} : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_n(\mathbb{F}_\ell). \tag{47}$$

The passage from ϱ to $\bar{\varrho}$ amounts to replacing the quantities $N_{p^r}(\varrho) \in \mathbb{Z}_\ell$ as p^r ranges over all the prime powers with their mod ℓ reduction. Such a passage would seem rather contrived for the sequences $N_{p^r}(\mathcal{X})$ – why study the solution counts of a diophantine equation over different finite fields, taken modulo ℓ? – if one did not know *a priori* that these counts arise from ℓ-adic Galois representations with coefficients in $\mathbb{Z}_\ell$. There is a corresponding notion of what it means for $\bar{\varrho}$ to be modular, namely, that the data of $N_{p^r}(\bar{\varrho})$ agrees, very loosely speaking, with the mod ℓ reduction of similar data arising from an automorphic representation. We can now state Wiles' celebrated *modularity lifting theorem*, which lies at the heart of his strategy:

Wiles' modularity lifting theorem. *Let*

$$\varrho : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_2(\mathbb{Z}_\ell) \tag{48}$$

be an irreducible geometric Galois representation satisfying a few technical conditions (involving, for the most part, the restriction of ϱ to the subgroup $G_{\mathbb{Q}_\ell} = Gal(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ of $G_{\mathbb{Q},S}$). If $\bar{\varrho}$ is modular and irreducible then so is ϱ.

This stunning result was completely new at the time: nothing remotely like it had ever been proved before! Since then, "modularity lifting theorems" have proliferated and their study, in ever more general and delicate settings, has spawned an industry and led to a plethora of fundamental advances in the Langlands programme.

Let us first explain how Wiles himself parlays his original modularity lifting theorem into a proof of the Shimura-Taniyama-Weil conjecture for semistable elliptic curves. Given such an elliptic curve E, consider the groups

$$E[3^n] := \left\{P \in E(\bar{\mathbb{Q}}) \ : \ 3^n P = 0\right\}, \qquad T_3(E) := \varprojlim E[3^n], \tag{49}$$

the inverse limit being taken relative to the multiplication-by-3 maps. The groups $E[3^n]$ and $T_3(E)$ are free modules of rank 2 over $(\mathbb{Z}/3^n\mathbb{Z})$ and $\mathbb{Z}_3$ respectively and are endowed with continuous linear actions of $G_{\mathbb{Q},S}$, where S is a set of primes containing 3 and the primes that divide the conductor of E. One obtains the associated Galois representations:

$$\begin{aligned} \bar{\varrho}_{E,3} &: G_{\mathbb{Q},S} \longrightarrow \mathrm{Aut}(E[3]) \simeq \mathbf{GL}_2(\mathbb{F}_3), \\ \varrho_{E,3} &: G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_2(\mathbb{Z}_3). \end{aligned} \tag{50}$$

The theorem of Langlands and Tunnell about the modularity of the general quartic equation leads to the conclusion that $\bar{\varrho}_{E,3}$ is modular. This rests on the happy circumstance that

$$\mathbf{GL}_2(\mathbb{F}_3)/\langle\pm 1\rangle \simeq S_4 \tag{51}$$

and, hence, that $E[3]$ has essentially the same symmetry group as the general quartic equation! The isomorphism in (51) can be realised by considering the action of $\mathbf{GL}_2(\mathbb{F}_3)$ on the set $\{0, 1, 2, \infty\}$ of points on the projective line over $\mathbb{F}_3$.

If E is semistable, Wiles is able to check that both $\varrho_{E,3}$ and $\bar{\varrho}_{E,3}$ satisfy the conditions necessary to apply the modularity lifting theorem, at least when $\bar{\varrho}_{E,3}$ is *irreducible*. It then follows that $\varrho_{E,3}$ is modular and therefore so is E, since $\zeta(E; s)$ and $\zeta(\varrho_{E,3}; s)$ are the same.

Note the key role played by the result of Langlands-Tunnell in the above strategy. It is a dramatic illustration of the unity and historical continuity of mathematics that the solution in radicals of the general quartic equation, one of the great feats of the algebraists of the Italian renaissance, is precisely what allowed Langlands, Tunnell and Wiles to prove their modularity results more than five centuries later.

Having established the modularity of all semistable elliptic curves E for which $\bar{\varrho}_{E,3}$ is irreducible, Wiles disposes of the others by applying his lifting theorem to the prime $\ell = 5$ instead of $\ell = 3$. The Galois representation $\bar{\varrho}_{E,5}$ is always irreducible in this setting because no elliptic curve over $\mathbb{Q}$ can have a rational subgroup of order 15. Nonetheless, the approach of exploiting $\ell = 5$ seems hopeless at first glance because the Galois representation $E[5]$ is not known to be modular a priori, for much the same reason that the general quintic equation cannot be solved by radicals. (Indeed, the symmetry group $\mathbf{SL}_2(\mathbb{F}_5)$ is a double cover of the alternating group A_5 on 5 letters and thus closely related to the symmetry group underlying the general quintic.) To establish the modularity of $E[5]$, Wiles constructs an auxiliary semistable elliptic curve E' satisfying

$$\bar{\varrho}_{E',5} = \bar{\varrho}_{E,5}, \qquad \bar{\varrho}_{E',3} \text{ is irreducible.} \tag{52}$$

It then follows from the argument in the previous paragraph that E' is modular, hence that $E'[5] = E[5]$ is modular as well, putting E within striking range of the modularity lifting theorem with $\ell = 5$. This lovely epilogue of Wiles' proof, which came to be known as the "3-5 switch", may have been viewed as an expedient trick at the time. But, since then, the prime switching argument has become firmly embedded in the subject and many variants of it have been exploited to spectacular effect in deriving new modularity results.

Wiles' modularity lifting theorem reveals that "modularity is contagious" and can often be passed on to an ℓ-adic Galois representation from its mod ℓ reduction. It is this simple principle that accounts for the tremendous impact that the modularity lifting theorem and the many variants proven since then continue to have on the subject. Indeed, the modularity of elliptic curves was only the first in a series of spectacular applications of the ideas introduced by Wiles and, since 1994, the subject has witnessed a real golden age, in which open problems that previously seemed completely out of reach have succumbed one by one.

Among these developments, let us mention a few below:

- The two-dimensional Artin conjecture, first formulated in 1923, concerns the modularity of all odd, two-dimensional Galois representations

  $$\varrho : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_2(\mathbb{C}). \tag{53}$$

  The image of such a ϱ modulo the scalar matrices is isomorphic either to a dihedral group, to A_4, to S_4 or to A_5. Thanks to the earlier work of Hecke, Langlands and Tunnell, only the case of the projective image A_5 remained to be disposed of. Many new cases of the two-dimensional Artin conjecture were proven in this setting by Kevin Buzzard, Mark Dickinson, Nick Shepherd-Barron and Richard Taylor around 2003, using the modularity of all mod 5 Galois representations arising from elliptic curves as a starting point.
- Serre's Conjecture, which was formulated in 1987, asserts the modularity of all odd, two-dimensional Galois representations

  $$\varrho : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_2(\mathbb{F}_{p^r}), \tag{54}$$

  with coefficients in a finite field. This result was proven by Chandrasekhar Khare and Jean-Pierre Wintenberger in 2008 using a glorious extension of the "3 – 5 switching technique" in which essentially all the primes are used. (See Khare's report in the Notices of the AMS mentioned above.) This result also implies the two-dimensional Artin conjecture in the general case.
- The two-dimensional Fontaine–Mazur conjecture concerning the modularity of odd, two-dimensional p-adic Galois representations

  $$\varrho : G_{\mathbb{Q},S} \longrightarrow \mathbf{GL}_2(\bar{\mathbb{Q}}_p) \tag{55}$$

  satisfying certain technical conditions with respect to their restrictions to the Galois group of $\mathbb{Q}_p$ was proven in many cases as a consequence of work of Pierre Colmez, Matthew Emerton and Mark Kisin.
- The Sato–Tate conjecture concerning the distribution of the numbers $N_p(E)$ for an elliptic curve E as the prime p varies, whose proof was known to follow from the modularity of all the symmetric power Galois representations attached to E, was proven in large part by Laurent Clozel, Michael Harris, Nick Shepherd-Barron and Richard Taylor around 2006.
- One can also make sense of what it should mean for diophantine equations over more general number fields to be modular. The modularity of elliptic curves over all real quadratic fields has been proven very recently by Nuno Freitas, Bao Le Hung and Samir Siksek by combining the ever more general and powerful modularity lifting theorems currently available with a careful diophantine study of the elliptic curves that could a priori fall outside the scope of these lifting theorems.
- Among the spectacular recent developments building on Wiles' ideas is the proof, by Laurent Clozel and Jack Thorne, of the modularity of certain symmetric powers of the Galois representations attached to holomorphic modular forms, which is described in Thorne's contribution to the Notices of the AMS mentioned above.

These results are just a sample of the transformative impact of modularity lifting theorems. The Langlands programme remains a lively area, with many alluring mysteries yet to be explored. It is hard to predict where the next breakthroughs will come but surely they will continue to capitalise on the rich legacy of Andrew Wiles' marvellous proof.

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Henri Darmon [darmon@math.mcgill.ca] is a James McGill Professor of Mathematics at McGill University and a member of CICMA (Centre Interuniversitaire en Calcul Mathématique Algébrique) and CRM (Centre de Recherches Mathématiques). He received the 2017 AMS Cole Prize in Number Theory and the 2017 CRM-Fields-PIMS Prize for his contributions to the arithmetic of elliptic curves and modular forms.

The Diversity of Mathematical Cultures: One Past and Some Possible Futures

Karine Chemla (Laboratory SPHERE, CNRS, Université Paris Diderot, France and Project ERC SAW[1])

Introduction

For a long time now, as a historian as well as an observer of contemporary mathematical practices, I have been struck by the diversity of ways of doing mathematics. I am not speaking here of the variety of individual styles, which has already been the subject of many works, but rather the diversity of collectively shared ways of practising mathematics. I feel this phenomenon deserves more attention than it has received to date. In this article, I would like to explain some of the reasons that convince me of the value that could be represented by the description of these social and cultural realities – realities that it seems to me appropriate to understand as different "mathematical cultures". I have had the opportunity to clarify, with the help of an example developed at length, what I mean by this last expression [1] and I will return to this question below. However, before I go any further, I must rule out one possible source of misunderstanding. While I propose speaking about 'mathematical cultures', this is totally unrelated to another, all-too-common interpretation of the same expression that seems to me meaningless and, what is more, dangerous.

Indeed, since the 19th century, a certain way of thinking about the diversity of mathematical practices has become dominant: it is the antithesis of the thesis for which I argue here. To remain brief, I will forego nuance and illustrate this alternative concept with a statement made by the physicist Jean-Baptiste Biot, which has the merit of, in only a few lines, revealing many facets of the representation that I reject. In an 1841 review of Jean-Jacques Sédillot's translation of a work in Arabic entitled *Traité des instruments astronomiques des Arabes*, Biot published the following verdict (the italics are mine, except for the final sentence in Latin) [2]:

> …one finds [in this book] renewed evidence for this peculiar *habit of mind*, whereby the Arabs, like the Chinese and Hindus, *limited* their scientific writings to the statement of a *series of rules*, that, once given, were only to be *verified by their very application, without requiring any logical demonstration or connection* between them, which gives these *Oriental nations a remarkable character of dissimilarity*, I would even add of *intellectual inferiority, compared to the Greeks*, with whom *all propositions are established by reasoning, and generate logically-deduced consequences.* This *fixed* writing of scientific methods, *in the form of precepts*, must have represented a significant *hindrance for the development of new ideas* for the peoples for which it was in use, and it is in sharp *contrast* with our *European* maxim: *nullius in verba*.[2]

When Biot concludes by quoting the motto of the Royal Society: "take no one's word for it", which enjoined its members to reject all forms of authority, it is to draw a contrast. The maxim calls for a form of "freedom" in thinking, which, for Biot, characterises Europe – elsewhere he says the Occident – and which has been extolled regularly ever since as the specific intellectual attitude that allowed the emergence of "modern science". According to Biot, the "Orientals", however, contented themselves with stating sequences of "rules" (in modern terms: algorithms) and then proceeded with prescriptions ("precepts"), which, in his view, supposed, by contrast, obedience from their users, meaning that it was therefore impossible for them to bring about a scientific revolution. This is one of the key elements of a broader opposition between mathematical practices of different peoples that Biot shapes along these lines. Thus, on another level, the "Orientals" would not feel the need to demonstrate, making do with simple "verifications". It is, in Biot's eyes, the function of the mathematical problems contained in their texts and that he designates as "applications". However, he insists, by contrast, that "the Greeks" demonstrate everything. As a consequence, in "Oriental" mathematical practice, the rules presented no interrelationships, while the "Occidentals", conversely, created deductive edifices.

All in all, as the above statement shows, Biot believed in a fundamental difference in nature between peoples, the presentation of which required only two categories: in one camp, the "Oriental nations" and, in the other, "The Greeks" and the "Europeans", among whom he

[1] I present here some of the results of research carried out in the context of the SAW ("Mathematical Sciences in the Ancient Worlds") project that has been financed by the European Research Council, in the context of the 7th programme framework (FP7/2007–2013, ERC Grant agreement n. 269804). This article is a translation, by Richard Kennedy, of "La diversité des cultures mathématiques: un passé et quelques futurs possibles", *Gazette des mathématiciens*, 150, 2016, p. 16–30 (online at http://www.smf.emath.fr/files/150-bd.pdf). It derives from the plenary lecture that I gave at the European Congress of Mathematics (Berlin, July 2016). A more complete version of this text is to be published in the proceedings of this conference; I will also make it available on HAL-SHS. I am grateful to Bruno Belhoste and Nad Fachard for their invaluable help throughout the preparation of this article.

[2] This document was first published by F. Charette [3].

positioned himself. For him, this difference was reflected in the contrast between their mathematical practices – a contrast to which he assigns long-term consequences (one camp experiences progress while the other advances with great difficulty). If we examine more closely how Biot articulates the difference between peoples and the contrast between mathematics, we notice that, to his eyes, the way in which "Orientals" carry out mathematics derives from a "particular habit of mind" common to these peoples: mathematics here only illustrates a more general fact. It is as much his belief in the truth of the general fact as the confirmation that he believes to have found in the description of their mathematical activities that leads Biot to express a hierarchy between the peoples. However, conversely, the declaration gives mathematics and modern science as the proof of the superiority of the Greeks and of Europe. History of science served for a long time, in fact, as a laboratory for developing conceptions with which some have believed it possible to consider the "characteristics" of peoples and establish the theory of an irreducible disparity between them. In the context of the SAW Project, we started an historical study of these forms of history of science and their uses but pursuit of this here would lead us too far. Rather, let us return to our subject.

Biot wrote these lines in 1841. I can testify that many elements of the representation of the diversity of mathematical practice to which he subscribed still persist today, in various forms, and are even very widespread, if not within the mathematical community, at least more widely in our society. In the context of today's world, the effects are potentially as destructive as they have been in the past. It is interesting to examine the documentary base from which Biot established his verdict. This is quite straightforward for China, as Biot's son, Edouard (1803–1850), was the first specialist of China to publish in Europe on the history of mathematics, and the four articles he wrote on the subject between 1835 and 1841 were all discussed with his father. Like a good number of sinologists of the time, Edouard never travelled to China and his investigations had to be limited to documents available in Europe. The Bibliothèque Royale's collections in Paris gave him access to a book on mathematics, written in Chinese and published in China in 1593, to which he devoted his first two articles. In 1839, he published a study on a second work, which he was able to consult thanks to the fact that his mentor in sinology, Stanislas Julien, loaned it to him. Isolated in his work on the mathematics in Ancient China, Edouard incorrectly dated this book, completed in 1259, to the 8th century and appears not to have understood the algebraic symbolism that was central to the author's project. Finally, once again in the Bibliothèque Royale, he found a work dating from the start of the Common Era and addressing mathematical knowledge necessary for astronomy and cosmography, a translation of which he published in June 1841. It is essentially from these data that, in the same year, Jean-Baptist Biot would formulate his definitive opinion of the mathematics of the Chinese "people" from antiquity up to his time.

The fact that today we can read several dozen mathematical books written in China between the last centuries before the Common Era and the 19th century does not mean that it makes any more sense to talk about "Chinese mathematics". In any case, it is not "mathematical cultures" conceived in terms of this type that I am thinking of when I propose to argue in favour of the interest there would be in considering the diversity of collectively shared ways of doing mathematics. Entities such as "nations" or "peoples" seem far too vast for what I have in mind. Wanting, at all costs, to say something about mathematics in a context of this magnitude, we would find ourselves condemned, like Biot, to generalising unduly. Or else the search for a common denominator for the mathematics of a "nation" or a "people" would lead us to stand much too far from those whom we are observing (and whom I will, as anthropologists do, call "actors"). At such a distance we would only grasp some commonalities of little significance, frequently minimising everything that contradicts the overall conclusion, and it would be by decree that we give these common points as characteristics of the entity observed. In both cases, it is by postulate that "nation" or "people" are posited as relevant frameworks and therefore we shouldn't be surprised to find the postulate in the conclusions.

Another approach to mathematical cultures

Like the majority of historians, I prefer to work from documents. And what has struck me, in considering the writings produced in a variety of contexts, is that these documents form clusters, which attest shared but different ways of doing mathematics. What types of human collectives do these clusters of writings bear witness to? We cannot give a general answer to this question and it would be necessary to examine them case by case. Below, I will outline some ways of addressing it. My main objective here will be, however, to illustrate, with examples, the phenomena which interest me and that I propose to approach in terms of different "cultures". Along the way, these examples will allow me to explain why I am convinced of the importance of taking these phenomena into account to interpret our documents in a more thorough and rigorous way and, through this exploration, I will also bring out some new general questions that they seem to raise.

The first illustration of what I mean by a "mathematical culture" comes from a field with which I am familiar. This is not by chance: an approach of this type requires an intimate knowledge of the sources. I chose this example from ancient history, as the problems of the interpretation of documents are often more acute when the writings were produced in the distant past. I hope, therefore, that the help in interpretation that can be afforded by an approach in terms of culture will be all the more obvious. I will consider, then, a cluster of Ancient Chinese mathematical works presented to the throne in 656 by Li Chunfeng and the scholars working under his direction: *The Ten Canons of Mathematics*.

By order of the Emperor, Li and his colleagues set about the preparation of this anthology, selecting classics from the past as well as commentaries that had been written about them and then preparing a critical edition and their own commentaries for all these documents. From 656, with the work having just been completed, these books were used in the School of Mathematics that had been established within the Imperial University, where students could follow a specialised curriculum in mathematics in order to gain access to a career in the bureaucracy. These ten canons, in a different chronological order to their composition, formed the content of two curricula [4]. I am interested here in only the most elementary of these curricula and I will, in fact, mention only two of the books studied in this context: the first canon worked on, *The Mathematical Classic of Master Sun*, which was devised around the year 400 (even if the text that we can read shows marks of modifications dating from the 8th century), and the canon that formed the centrepiece of this course, *The Nine Chapters on Mathematical Procedures*, the completion of which I date to the 1st century CE. This book and the commentaries written about it by Liu Hui in the 3rd century and by the team working with Li Chunfeng in the 7th century required, in fact, a far greater number of years of study in comparison to the other books.

These historical elements allow us to formulate two important points of method. If these canons and their commentaries were taught in the same curricula, it means that 7th century actors considered them de facto as associated with the same mathematical culture. In addition, the first six canons of the first curriculum are essentially composed of mathematical problems and algorithms allowing them to be solved; they are therefore difficult to interpret. By contrast, however, the commentaries chosen or written by Li Chunfeng's team comprise discussions on mathematics and explicit references to the practice of mathematics. These commentators are, in fact, the earliest readers of the canons that we are able to observe and they provide us with essential clues to describe the mathematical culture that makes up my first example. I stress this point: the description of a mathematical culture must not derive from impressions or intuition but instead rely on historical demonstrations based on documents. The assertions that I will formulate below are, as far as possible, based on long arguments but I will not elaborate on them here, instead referring the reader to previous publications.

The key question at present is to understand how the mathematical activity testified by these documents was practised. A typical page from canons like *The Nine Chapters* (this is how I will abbreviate the title henceforth) is composed of problems and algorithms, while the commentaries, which appear in smaller characters and often between the sentences making up the algorithms, systematically establish the correctness of these algorithms, interspersing these developments with all sorts of remarks and discussions.

As the oldest editions show, all these writings only contain characters, without any graphical representations of any sort. However, the canons, like the commentaries, make reference to rods, with which the numbers were represented on a surface on which the calculations were carried out. Without representations in the texts of the use made of the rods or the calculating surface, everything that took place on that surface has had to be reconstructed from clues gleaned from the writings and on the basis of historical arguments. Our situation is probably comparable with that of future historians who will concern themselves with understanding the part of the activity of mathematics that takes place today on our blackboards.

The rods constitute the first physical object mentioned in the texts and we will see that they played a key role in the mathematical culture testified by the canons. Furthermore, canons like *The Mathematical Classic of Master Sun* and *The Nine Chapters* neither contain nor mention any figures, nor even any visual aids. However, in the context of certain demonstrations, the commentaries do evoke figures and blocks, opting for one or the other according to whether they are dealing with plane geometry or space geometry. With the blocks, which evoke the plaster and string models used by certain mathematical milieus in the second half of the 19th and the beginning of the 20th centuries, we thus encounter a second type of physical object that mathematics activity had recourse to. The early editions of these classics do not contain the figures that the commentators refer to and the examination of clues that we could gather about them has led me to conclude that they, too, were physical objects at the time. I will refer to them with the term "diagram", to remind us that they are visual aids different from those we usually associate with the term "figure".

In summary, and in contrast to what later documents attest, the mathematical activity evidenced by our first cluster of writings is based on books containing only text and also on three types of object: rods, blocks and diagrams [5].

In the course of a series of articles, I have shown how the description of what the actors did with the elements contained in the writings, as well as with the objects we have just identified (which only partly covers what I mean by the expression "way of doing mathematics"), is essential to interpreting the writings and obtaining a more complete grasp of the mathematical knowledge they had. Here, I will illustrate this thesis with the aid of only one of these aspects, concentrating on the way in which the actors worked with the calculating surface, according to what can be reconstructed, and showing how this approach allows us to understand the knowledge that they had developed around arithmetical operations.

My reasoning starts with the first pages of *The Mathematical Classic of Master Sun*, that is to say, the start of the elementary curriculum. Here, the work describes, among other things, the use of rods to represent numbers on the calculating surface: without entering into detail, in this description, we can recognise a decimal place-value system, in the sense that writing the symbols 123 in these positions implies that 1 means a hundred, 2 twenty and 3 three.

Then, based on this system of numeration (which, therefore, was purely physical and did not appear at the time in the writings), the book offers two algorithms, one for multiplying and the other for dividing – the division here is called *chu*. The text of this second algorithm does not open with a prescription but with an assertion: "this algorithm is exactly opposed to that for multiplication". The meaning of this statement is not obvious solely on the basis of the text of these two algorithms. However, the calculations for the execution of these two operations that we can reconstruct on the basis of the texts, and for which I give an example in Figure 1, suggest an interpretation. They will be essential to my argument and hence I enter into detail here.

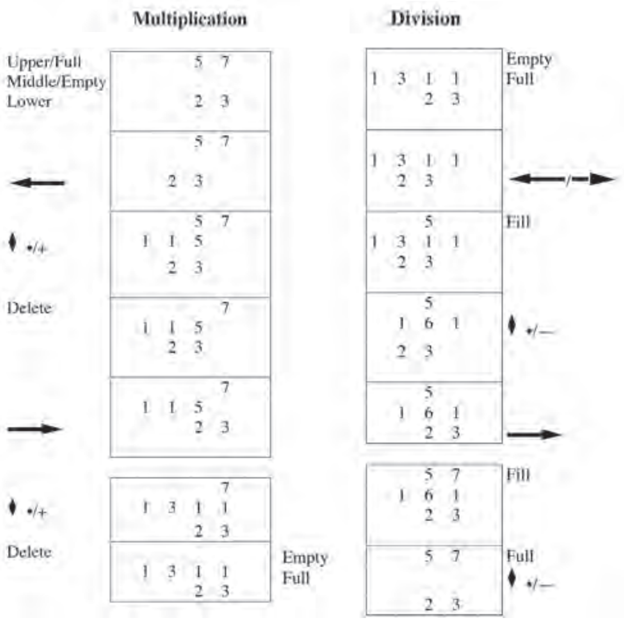

Figure 1.

These algorithms are based on two types of "positions", both designated by the same Chinese term (*wei*). Firstly, the numbers are written horizontally, as a series of decimal positions. This place-value notation echoes the property of the algorithms to iterate the same series of elementary operations along this sequence of digits. Moreover, both algorithms make use of three vertical positions, one above the other (upper, middle and lower). The multiplication starts by placing the multiplier (23 in my example) and the multiplicand (57) in the upper and lower positions respectively, leaving the middle position empty. The initial layout for the division of 1311 by 23 is completely opposed with respect to the middle and upper positions: contrary to the multiplication, it is the upper position that is empty at the start of the calculation, while the middle position is full, as it contains the dividend. For both operations, the calculation proceeds in the same way, by filling the line of these two that is empty while emptying the line that is full. Operating on initial configurations that are opposed, the processes that follow are themselves opposed to each other. The "result" of the multiplication is produced in the middle, while that of the division is produced at the top. Here, we thus see that a relationship of opposition between the two operations is shaped, through the precise fashioning of the processes of execution on the calculating surface. It is the first property of interest for us in these flows of operations that execute multiplication and division. We will return to it.

In contrast to the middle and upper positions, which are opposed to each other between multiplication and division, the lower position similarly receives operators that are the multiplier and the divisor. Both act in the same way during the execution of their respective operations: their significant digits are not modified, but their decimal positions are, being displaced at each iteration. The layout of the two operations and the algorithms have the effect that the execution of the multiplication ends at the starting point of the division and vice versa: if you run multiplication and division one after the other, the operations cancel each other out. This is a second property of interest to us in these flows of operations.

These arrangements, partly opposed and partly identical, of the algorithms on the calculating surface correspond to flows of calculations that allow practitioners to see the relationship of opposition between multiplication and division. Thus, once the multiplication operands are positioned, the multiplier 23 is moved to the left until its units digit is vertically under the digit with the highest magnitude in the multiplicand (5). The multiplier is thus multiplied by the power of 10 corresponding to this latter digit. The products of the digits of 23 by 5 can then be added progressively to the middle position, immediately above the corresponding digits in the multiplier. Once this sub-procedure is completed, 5 is deleted from the upper line, 23 is shifted one position to the right and the same sub-procedure is repeated with 7, the following digit, which in turn will be deleted at the end of the execution. Thus, it is in this way that "that which the multiplication produces" finds itself "in the middle" while the multiplicand is, for its part, deleted. The execution of a division will "produce", in an opposed way, the result "in the upper position", while the number in the middle position will be progressively deleted. By contrast, the digits in the quotient are, in effect, progressively added to the upper position (5 then 7), while, in the appropriate corresponding position, the products of the digits in the divisor and first 5, and then 7, are progressively subtracted from (and not added to) the dividend. Incidentally, if we had divided not 1311 but 1312 by 23, the quotient would be given as 57 + 1/23. The fact that the results of divisions are always exact plays a critical role, but that necessitates another development that I am not able to give here.

In the context of this way of doing mathematics, inculcated from the beginning of the first curriculum in the School of Mathematics, the algorithms for multiplication and division have therefore been shaped to allow a global vision, position by position, of a network of oppositions and similarities in the very dynamic of the executions on the calculating surface. It is, I think, to this and not the fact that multiplication and division cancel the effect of each other, that the declaration in *The Mathematical Classic of Master Sun* (placed at the beginning of the text on the algorithm for division) refers when it states "this algorithm is exactly opposed to that for multiplication". This conclusion deserves further examination.

Firstly, it implies that the physical practices that mathematical activity brought into play in this context must be reconstituted so that we can fully interpret the writings. This assertion has a de facto general validity. Moreover, the interpretations I suggest for both the declaration in the text and the procedures for calculation imply that the processes for carrying out the operations on the calculating surface do not only have the aim of producing results but also of expressing properties – here a form of relation between multiplication and division.

This basic example suffices to illustrate what I mean by different "mathematical cultures" and it also provides a glimpse of the interest their description takes on. The ideas brought into play in the algorithms represented in Figure 1 are identical to those that inspire the way we ourselves have learned to carry out multiplication and division. Yet, in the eyes of the actors who employed one or the other, the *meaning* of the two sets of algorithms differs in part and we will see that this difference has important consequences. By contrast to this other way of working, our practices for calculation do not invite us to interpret as meaningful the relations between flows of operations executing multiplication and division, or to work with these flows. This is one of the features that confers its uniqueness to the practice of calculation provided in the first curriculum of the 7th century in China, and the declaration in *The Mathematical Classic of Master Sun* allows us to grasp what is at stake. Let us now analyse what is brought to us by the knowledge of this specific element of such a "way of doing mathematics".

Work on the relations between the operations

The statement in *The Mathematical Classic of Master Sun*, combined with the flows of calculations we can reconstitute on the basis of the texts of algorithms, allows us to establish the existence of a practice of calculation unique to a certain context: the use of "position" to explore and express an interpretation of the relation between operations. In doing so, it reveals the existence of mathematical interest in such relations. Understanding this practice will, more generally, allow us to grasp mathematical knowledge on the relation between operations as it was produced in this context. This is all the more important because historians had not really perceived this knowledge before. Only by reading what the texts and the physical inscriptions express in a specific way do we uncover part of the actors' mathematical knowledge and also a fundamental question that inspired their research.

Moreover, the fact of having uncovered such a practice also provides us with tools for interpreting other texts in the same corpus and for going further into the reconstitution of the actors' practices on the calculating surface. Thus, we can better understand the theoretical work that the actors carried out on the operations and also comprehend the history of this work. The operations of multiplication, and especially of division, as well as their execution on the calculating surface described above, will then prove to have played a key role in this history.

To establish this point, we will return firstly to *The Nine Chapters*, whose text attests the same practice of calculation on the calculating surface, as well as the same interest for the relations between operations. Let us examine, for example – without, for the moment, trying to interpret them – the texts of the algorithms provided for the extraction of square and cube roots (I only quote the beginnings here, which are sufficient to bring out the phenomena that interest me):[3]

> "Procedure for the extraction of the square root: One places the number-product as **dividend**. Borrowing one rod, one **moves it forward**, jumping *one* column. Once the **quotient** is obtained, one multiplies *once* the borrowed rod by it, which makes the **divisor**, then one **divides by this**. After having **divided**, one *doubles* the **divisor**, which makes the determined **divisor**. If one **divides again**, one reduces the **divisor by moving it backward**. One again places a borrowed rod, and moves it forward like at the beginning. One **multiplies this once** by **the new quotient**. (…)"
>
> "Procedure for the extraction of the cube root: One places the number-product as **dividend**. Borrowing one rod, one **moves it forward**, jumping *two* columns. Once the **quotient** is obtained, one multiplies *twice* the borrowed rod by it, which makes the **divisor**, then **one divides by this**. After having **divided**, one *triples* this, which makes the determined **divisor**. **If one divides again**, one **reduces (the divisor) by moving it backward**. One multiplies the quantity obtained by three, and one places this in the middle row. Once more borrowing a rod, one places it in the row underneath. One moves them forward, that which is in the middle jumping one column, that which is underneath jumping *two* columns. **One again places a quotient** and multiplies by it that which is in the middle once, and that which is underneath, *twice*. (…)"

If we consider these algorithm texts independently of any context, they are difficult to interpret with certainty. In particular, the layout of the calculations to which they refer seem unfathomable. However, two key points are evident.

I have marked in bold type the terms that these texts take from the algorithm for division. They clearly show that the formulations of the algorithms for extraction – tacitly, i.e. without any other form of commentary – shape these calculation procedures as types of divisions. Based on what we have seen above, we can advance the hypothesis that the texts, like the executions, state a form of relation between extraction and division. We thus again find the interest that the actors manifested for this very question and its exploration with the help of the same working tools, as well as, now, also the algorithm texts.

Furthermore, in the translations of the two texts, I have underlined the terms and expressions that indicate how the root extractions are not real divisions. They show the modifications to the division algorithm

[3] [6] contains a complete, annotated translation of these texts.

through which the extractions have been cast in the divisions mould. These terms and expressions do, however, also demonstrate an interest for the relations between operations since they match each other from one text to another. This correspondence reveals how correlated modifications of the division algorithm lead to the extraction of square and cube roots respectively. The words in italics highlight how even the differences between these modifications are correlated from one algorithm to the other. The use, in the square root text, of an expression like "multiply once" instead of simply "multiply", which accentuates the parallel with the expression "multiply twice" in the corresponding statement of the cube root text, brings out the authors' wish to write the texts in relation to each other.

All these properties confirm what I have advanced above: alongside the work on the flows of calculation executing the operations on the calculating surface, we see emerging, through the formulations of the algorithm texts, a second facet in the modalities of exploring the relations between operations. Above, we encountered a specific practice using physical objects (rods and positions on the calculating surface). We discover now a specific way of working – and of expressing mathematical meanings – with certain elements that make up the texts themselves. These remarks provide us with tools to rigorously reconstitute the flows of calculation to which the extraction procedures for square and cube roots refer. The key hypothesis that the previous argument allows us to advance, a hypothesis that plays a key role in this reconstitution, states that the processes of execution highlight, or "write", the similarity between extractions and division on the calculating surface in the same way that they allowed the reading above of the opposition between multiplication and division. Therefore, we know that the first digit of the root (or "quotient") $a.10^n$ and then those that followed were placed successively in the upper position, while the number A whose root was being sought was positioned as "dividend". In the lower position, a number acting as "divisor" distinguished itself from the homonymous position of the division by the fact that its value had to be adjusted. The interpretation gives the flow of calculations reconstituted in Figure 2.

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8	(Step)
			2	2	2	2	2	Upper
55225	55225	55225	55225	55225	15225	15225	15225	Middle
1	1	1	1	2	2	4	4	Lower

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	Step 7	Step 8
			$a.10^n$	$a.10^n$	$a.10^n$	$a.10^n$	$a.10^n$
A	A	A	A	A	$A - a.a.10^{2n}$	$A - (a.10^n)^2$	$A - (a.10^n)^2$
1	10^2	10^{2n}	10^{2n}	$a.10^{2n}$	$a.10^{2n}$	$2a.10^{2n}$	$2a.10^{2n-1}$

Figure 2.

If we had read the texts with the sole intention of knowing how roots were extracted – as most historians have actually read them – we would have missed the *work* carried out to shape a set of relations between these operations as well as the *ways of working* developed in order to carry out this research (use of positions and the dynamic of the calculations, and formulation of the algorithm texts). Certainly, we would have convinced ourselves, once again, that the ideas applied are essentially identical to those used in the algorithms that some of us learned in our youth for extracting roots. But we would have missed out on what makes the difference between this latter algorithm and the one in *The Nine Chapters*. The reconstitution of the practice of writing algorithm texts, like the practice of calculation on the calculating surface (two facets of this specific way of doing mathematics that illustrate what I mean by "mathematical culture"), invites a different reading of the texts as flows of calculation and consequently allows us to grasp another facet of the actors' mathematical work that no other discourses express. I think this point clearly illustrates the link I stated between, on one hand, the description of the actors' "mathematical culture" and, on the other, a better understanding of their mathematical knowledge, as well as the questions they were pursuing.

Another clue confirms the conclusions that one can draw from this form of interpretation, which derives from attention being paid to the practices: it comes from the way in which these operations were prescribed in the algorithm texts. Indeed, the texts refer to the operand of a root extraction by the term 'dividend" and prescribe the operation, as appropriate, by the expressions: "one divides this by extraction of the square root" or "one divides this by extraction of the cube root". In other words, the prescription states, again without further ado, the same structure for all the operations, signalling that *chu* division was their foundation.

This is not all and, for us to go further, it will be useful to evoke the demonstrations that the commentator Liu Hui developed to establish the correctness of root extraction algorithms and, in particular, the diagram on which the proof is based in the case of the square root extraction. These demonstrations are the opportunity for Liu Hui to correlate the elementary steps of the extractions with those of *chu* division. Moreover, in order to develop the meaning of the steps in the extraction, the commentator introduces a diagram for the square root and blocks for the cube root. While the text of the commentary refers to these, there are no illustrations in the text and, here again, it is down to historians to reconstitute them. Figure 3 illustrates the reconstitution that

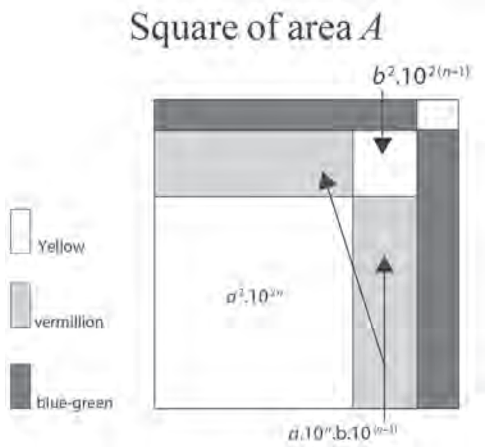

Figure 3.

historians all agree to propose for the visual aid relating to the square root upon which Liu Hui based the explanation of the meaning of the calculations. I reproduce the colours that the text of the commentary indicates – this is a common feature of the diagrams in the context of this culture. Furthermore, I add marks allowing the execution of the extraction to be linked to the figure. It is possible that the diagram used by the commentator contained characters performing the same function but we have no evidence of this. We will return later to this diagram, insofar as we will see that it plays a role in the structuring of an altogether broader set of operations.

Chu division as the foundation for a set of operations

To summarise the conclusions we have obtained thus far: we have encountered several characteristic features of a mathematical culture by concentrating on the practice of computation. Among these features, we have identified the use of positions on the calculating surface to establish links between the operations through the flows of calculation. The decimal positions of the place-value notation for numbers are a part of this landscape, insomuch as they constitute one of the types of position that the practice of computation brings into play. Their utilisation meshes with the use of algorithms operating uniformly on sequences of digits of the operands and producing, with regard to the operations of the division family, the results digit by digit. Furthermore, *chu* division has been shown to play a central role in this context. The combination of all of these features is found in two other subjects dealt with in *The Nine Chapters*. We will analyse them one after the other.

The first concerns the resolution of systems of linear equations, which are the subject of Chapter 8 in the book. The central algorithm describes, firstly, an initial layout of the data (i.e. the coefficients of the equation) on the calculating surface and, thus, easily allows the reconstitution as follows (in modern terms, the system given on the left corresponds to the inscription on the surface reconstituted on the right):

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\cdot \\
\cdot \\
\cdot \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \\

\end{array}
\qquad
\begin{array}{cccc}
a_{n1} & \cdots & a_{21} & a_{11} \\
a_{n2} & \cdots & a_{22} & a_{12} \\
\cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot \\
a_{nn} & \cdots & a_{2n} & a_{1n} \\
b_n & \cdots & b_2 & b_1
\end{array}
$$

Thus, in the layout described in *The Nine Chapters*, each linear equation corresponds to a column and the coefficients attached to the same unknown are all placed in the same line. Here, again, the actors have developed a place-value notation for the system of equations. The algorithm itself corresponds to the Gauss elimination method. It operates as follows. Assuming that the upper terms in the two right-most columns are non-zero, the upper term in the right column multiplies the column immediately to its left, whereupon the upper term in this second column is eliminated by operating on these two columns. This sub-procedure is repeated until the following triangular system is obtained:

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
0 \quad\;\; + c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\
\cdot \\
\cdot \\
\cdot \\
0 + 0 + \ldots + 0 + c_{nn}x_n = d_n \\

\end{array}
\qquad
\begin{array}{cccc}
0 & \ldots & 0 & a_{11} \\
0 & \ldots & c_{22} & a_{12} \\
 & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot \\
c_{nn} & \ldots & c_{2n} & a_{1n} \\
d_n & \ldots & d_2 & b_1
\end{array}
$$

The algorithm is concluded by determining x_n by a simple division then successively calculating the other unknowns in a similar manner. Note in passing that the division that produces x_n presents the dividend under the divisor. Here, too, a uniform algorithm meshes with a place-value notation of the system, since it determines the sequence of the unknowns by means of an iteration of the same sub-procedures, which deal with the positions in a uniform way. Positive and negative marks are introduced during the chapter to allow the operations to be completed in all cases and then to extend the set of systems that the algorithm can handle. Finally, let us note that the way the data are structured on the calculating surface is central to the operations the algorithm uses.

In the same way as before, the interpretation of the algorithm as identical to the Gauss elimination method is relevant but it only partially captures the mathematical knowledge developed. Indeed, the observation of the same elements as above (the terms employed to designate the operands, the algorithm texts and the calculation flows) highlights something else quite unexpected here. It appears that the constant terms in the equations are given the name "dividends", while the coefficients of the unknowns are described as forming "divisors in square" – this is, in my view, the meaning of the name of the algorithm (in Chinese *fang cheng*, "measures in square"). Finally, the central operation of eliminating the upper non-zero terms from the columns is prescribed as a "vertical *chu* division". It appears, once again, that the actors' work was not limited to determining an algorithm to produce the results. In addition, they further carried out a conceptual reflection on the relations between the operations, which led to conceiving the resolution of systems of linear operations as a generalised division, opposing a sequence of dividends to a square of divisors (here the dividends are also under the divisors), and articulating the forms of horizontal and vertical division [7]. Again, we find, on one hand, an interest for the structuring of a set of operations and, on the other, *chu* division as the foundation of this enlarged set. In this context, the positions seem, once more, to have served as the work tool for carrying out the exploration.

We have brought out, by means of the observation of aspects of the mathematical culture, a reflection by the actors on the operations and the relations between them. We notice that bringing this work programme to light, which no text appears to formulate explicitly, allows us to give meaning to a growing set of clues contained in the texts and to grasp a facet of the mathematical knowledge specific to the actors that have, up to this point, remained invisible. That the linear equation is conceived in this context as the opposition between a dividend and a set of divisors is actually only one aspect of a more general fact, as we will now see by turning to the second subject dealt with in *The Nine Chapters*, where positions and *chu* division also play a key role.

I will introduce this subject by showing how the description of facets of the mathematical culture in the context in which the text was written, with the restitution of the flows of calculation on the surface and the diagrams that can be deduced from clues in the text, provide essential tools for interpretation. The algorithm to understand is formulated following the problem that I represent in Figure 4 (by respecting the representation of the cardinal directions usual at that time).

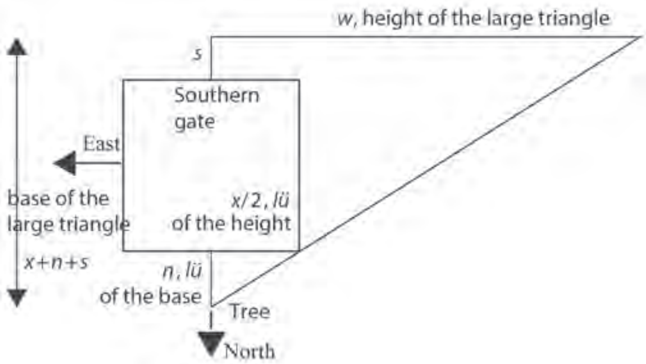

Figure 4.

It is a question of determining the (length of the) side of a square town, knowing that a person walking s *bu* (unit of distance) outside the southern gate then westwards w *bu* sees a tree situated n *bu* from the northern gate. The algorithm *The Nine Chapters* proposes is formulated as follows (I emphasise certain words in bold text):

> "One multiplies the quantity of *bu* walked to the west by the quantity of *bu* outside the northern gate, and one doubles this which gives the **dividend**. Adding together the quantities of *bu* outside the southern gate and the northern gate makes the **joined divisor**. And one **divides this by the extraction of the square root**, which gives the side of the square town."

The algorithm thus calculates two operands ("dividend" and "joined divisor") and prescribes the operation as a "square root extraction". What is the meaning of this operation? Actually, it can neither be a square root extraction, as this operation should only have one operand, nor can it be a division. Here, Liu Hui's commentary provides valuable clues for dealing with the conundrum. The commentator describes a graphical process that does not correspond to any illustration in the text and that I translate as a sequence of figures. In Figure 4, I have marked the height and the base of a large triangle. The term *lü* attached to the height and the base of a second triangle in the figure indicates the similarity of these two triangles. From this observation, Liu Hui draws the equality of the areas of the horizontal rectangle, with sides w and n, and the vertical rectangle, with sides $x/2$ and $n + x + s$ (see Figure 5).

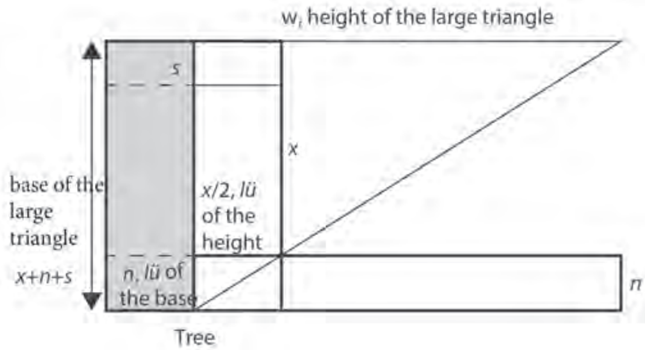

Figure 5.

Twice this area corresponds to what the algorithm calls the "dividend": Liu Hui interprets this as the area of the vertical rectangle to which one adds the grey rectangle. It corresponds to the rectangle in Figure 6a. Liu Hui finally interprets the calculation of the "joined divisor" by the prescription of joining the upper and lower rectangles, which produces the shaded rectangle in Figure 6b. Establishing this figure concludes his commentary.

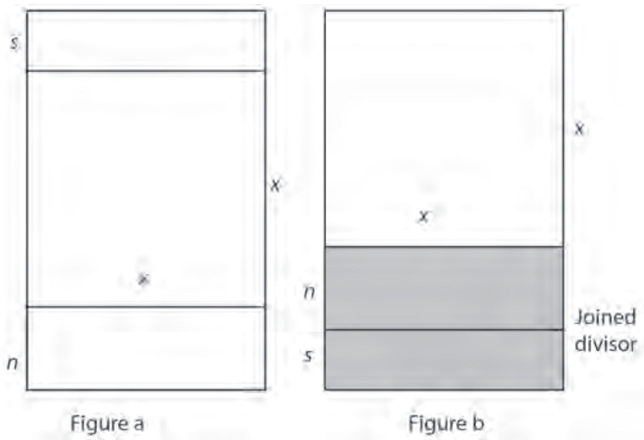

Figure 6.

The commentator thus interprets the operation used in the algorithm as the equation represented in modern terms as:

$$x^2 + (s + n)\, x = 2\, nw.$$

Why is its execution prescribed as a root extraction? The answer to this question is obtained by considering the demonstration Liu Hui formulated for the algorithm executing this last operation and, in particular, the diagram that he introduced to state the meaning of the operations in the algorithm, whose reconstitution is provided in Figure 3. The commentator interprets steps 1 to 6 of the algorithm (see Figure 2) as having the aim of subtracting the area of the square of side $a.10^n$ from A. If this square is removed from the figure, we are left with a gnomon, shown in Figure 7.

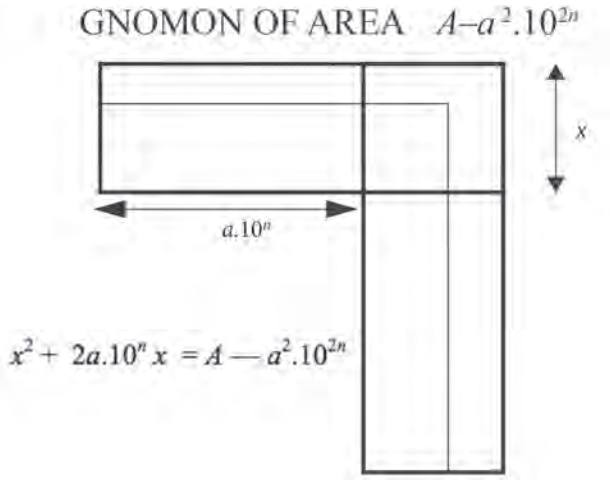

Figure 7.

By unfolding this gnomon, one obtains a figure comparable to that with which Liu Hui established the quadratic equation. Indeed, by omitting the first part of the root extraction algorithm (thus by removing from the result the digit of the root calculated to this point) and by commencing the algorithm at step 8, one solves the quadratic equation that writes this gnomon (or rectangle). Now, if we observe the configuration of the calculating surface at step 8, we notice that there are, at this point in the calculation, two terms that correspond exactly to the operands of the quadratic equation described by the algorithm, the interpretation of which is under consideration. Thus, the quadratic equation is an operation that derives from root extraction, in that the procedure that executes it is a subprocedure of the execution of the extraction: we understand at one and the same time how it is introduced and how it is executed in this context. Several consequences follow from this.

Firstly, we note that a large number of ingredients enter into the development of our interpretation: the demonstrations by which the commentator established the correctness of both the algorithm solving the problem given and the root extraction algorithm; and the reconstitutions of the diagrams and the flows of calculation on the calculating surface, based on knowledge of the practices at work in this context. We see that, in the context of a given way of "doing mathematics", the elementary practices (practices of diagrams, practices of computations, etc.) mesh with one another in a specific way.

Furthermore, we observe here, once again, that the operations for square root extraction and quadratic equations are linked by processes of calculation on the calculating surface, and especially by the way of managing the positions. Both in terms of the processes of calculation and the role played by diagrams, the relation is established in a different way from that which we have described for multiplication and division. Nevertheless, as above, the link between the operations is also expressed by the terminology chosen to designate the operands and the operations. All this explains, at one and the same time, the graphical means used to establish the equation and the fact that only two operands are identified for the quadratic equation (in the context of traditions that developed on this basis, the term in x^2 seems to have been identified only in the 11th century).

This last remark raises a crucial and particularly interesting question. The fact that the quadratic equation is only associated with two operands highlights a *correlation* between the ways of doing mathematics (here, in particular, the practice of computations on the surface linked to the establishment of relations between operations) and the concepts or, more broadly, the mathematical knowledge produced. This fact is, I think, wholly general and only a careful examination of "mathematical cultures" will allow us to explore it further. For me, this issue provides a fundamental reason to justify the interest in the diversity of ways of doing mathematics. What is at stake is understanding how mathematical knowledge is correlated to shared, collective ways of working. This is one of the new questions to which we are led and one which I hope historians of mathematics and mathematicians will consider jointly.

But there is more. If we return to the quadratic equation, we realise that the text of *The Nine Chapters* only contains the names of the operands and the formulation of the prescription. By methodically reconstituting the ways of working, we have been able to reveal a representation of the equation on the calculating surface and a process of execution, as well as a graphical representation essential to its establishment. In fact, all the quadratic equations established in Ancient Chinese sources correspond to the reading of gnomons or of rectangles in geometrical configurations in the same way. In other words, if we had not paid attention to concrete practices with physical objects, we would have missed key aspects of the ways in which the actors worked in this context with this mathematical object and the tools they forged for this purpose.

More important for our purposes, we would also have missed out on the work and the resources that the actors deployed to structure a series of operations. Yet we are now discovering the extent of the knowledge developed on this subject. We see that, in this context, both linear and quadratic equations were conceptualised as forms of division. In fact, many traditions that gained momentum by relying on the canons at the centre of this mathematical culture, be they in China, Korea or Japan, would develop knowledge about algebraic equations in this conceptual framework. And I show, in the complete version of the article to appear in the proceedings of the EMC, that it is again only one aspect of a much more general phenomenon.

Another mathematical culture in Ancient China and some issues at stake

Let us recapitulate what the observation of certain facets of a mathematical culture (in the main, the use of positions and the processes for calculation) has allowed us to do so far. We have relied on it for a more rigorous interpretation of the texts. We have also reconstituted ways of working with mathematical entities. Finally, we have grasped a body of knowledge that the actors had developed on the subject of the relations that link certain operations and the systematic study, which had been, until now, overlooked by historians. In this context, the operation of *chu* division has emerged as pivotal. I have approached all these aspects from the basis of a cluster of documents originating from Ancient China: the canons published with certain commentaries in the 7th century and used as textbooks in the official mathematics curriculum.

Recently, two other clusters of mathematical documents also originating from Ancient China have resurfaced and a quick observation of the way of doing mathematics they bear witness to allows us to raise some very interesting questions, both specific and general. I will only refer here to the first cluster of documents, directing the reader to the article published in the proceedings for the operation of the second.

I will speak, therefore, only about documents newly provided by archaeology. Since the 1970s, a growing number of tombs sealed in China in the last centuries before the Common Era have been excavated and archaeologists soon discovered that, in some of them, libraries had been interred among the funerary objects supposed to accompany the dead person in the afterlife. These documents provide fresh perspectives on the final centuries before the Common Era and have shaken up our knowledge of this period. During the Winter of 1983–1984, a first mathematical document, the size of a book, came to light among a series of writings of this type. Since then, excavations and the antiquities market have produced several other similar documents and we can expect new finds, all profoundly altering our understanding of the history of mathematics in China at the time. For the time being, only two of the mathematical texts discovered have been published completely (the first in 2001); for the others, we can only consult some extracts pending their full publication. The conclusions I propose are thus fragile and could be contradicted by new discoveries.

Of these documents, those we can study all seem to reflect the same way of practising mathematics: in the terms that I introduced above, they form the same cluster. Moreover, as far as we can see, the writings have several features in common with the canons and their commentaries. We can suppose then that all the documents had close historical ties, without, for the moment, being able to specify their exact nature. What is important for us is that these two clusters of documents also present significant differences to each other, which leads me to advance the hypothesis that these two clusters bear witness to different ways of doing mathematics, even if both present similarities. For my purposes, I will concentrate here on a set of similarities and key differences.

Firstly, like the canons and their commentaries, these documents contain no illustrations, being made up solely of Chinese characters and punctuation marks. They do, however, also refer to counting rods to represent numbers and to the practice of laying out numerical values away from the text. However, no traces have been detected of the use of a decimal place-value system for writing numbers. On the contrary, a certain number of clues gleaned from the operations suggest that the numbers were represented using a different number system.

This is the first of a series of facts that appear to indicate that the surface on which the calculations were carried out was the subject of a different practice. In fact, more generally, no reference is made to the use of a system of positions in the execution of the algorithms and nor do any of these writings use terms like "line", "column" or "position".

So much for the physical aspects of the practice of calculation. If we now turn to the operations, an initial fact is immediately striking: division seems to have been seen as a specific operation, different from all the others. A first clue for this is the fact that, while the other operations can all be prescribed by simple verbs, division is always, at least in this context, prescribed by complex expressions. In particular, the term *chu* alone cannot prescribe a division, contrary to what we have seen earlier for the other cluster of texts. And when it is encountered in isolation, it refers in fact to a subtraction. It seems then that one can perceive a change in the meaning of the verb *chu* and a change in the practice of division.

These recently discovered documents contain algorithms for square root extraction. But these procedures do not determine the roots decimal position by decimal position and do not seem to iterate sub-procedures on numbers written in a place-value form. Neither do they appear to present a relation to a process like that of division as we saw in the canons. More generally, no trace appears to reveal an interest for the relations between the operations.

Finally, none of the algorithms in which we have seen the close relation with division and the use of positions, such as the resolution of systems of linear or quadratic equations, appear, for the moment, among the subjects dealt with in these documents.

In conclusion, whether from the perspective of ways of working with the processes of calculation or from the perspective of knowledge or of the projects that actors formed, in these documents we have none of the elements from the constellation of facts described earlier. This suggests another issue of interest which is, in my opinion, wholly general. In fact, these new documents invite the thought that *The Nine Chapters* and the other canons bear witness to the emergence, no later than the 1st century of the Common Era, of two closely linked things: on one hand, a way of doing mathematics (more precisely a way of working with the processes of calculation and an interest in uniform algorithms) and, on the other, new knowledge, among which I include new ways of carrying out known operations, several new operations, a way of understanding the relations between these operations and a decimal place-value numbering system. Thus, at the same time, a way of working and a body of knowledge appear in concert.

When historians of mathematics have become interested in the activity of mathematics as such, they have, in general, studied, with a few exceptions I cannot develop here, the history of mathematical knowledge. Yet, the phenomena that I have mentioned above suggest that a history of ways of doing mathematics is also meaningful. What is more, these two dimensions (mathematical knowledge and mathematical practice) appear to constitute inseparable facets of the same reality. This is what we have seen for Ancient China, and I think it is the same everywhere and at all times.

I pose the conjecture that these two facets transformed themselves jointly. It is, without doubt, one of the fundamental reasons why the description of ways of doing could help in the interpretation of writings and allow a better understanding of the knowledge to which they attest. This close articulation between these two types of facts constitutes another reason why the history of mathematics should be interested in the description of mathematical cultures. After all, ways of doing mathematics do not appear from nowhere. They have been shaped and transformed by the actors during the process of exploring the problems that they sought to solve and the questions they pondered. Ways of doing mathematics represent one of the results of actors' research: mathematical work thus produces both knowledge and practices. This is, in any case, one of the principal motivations of my plea for the history of mathematics to take as a subject of study not only the knowledge but also the practices and ultimately the relations that exist between one and the other.

I have presented arguments on the value of studying the ways of doing mathematics by illustrating my arguments with examples taken from Ancient Chinese documents. Many other clusters of texts produced closer to us, even today, seem to me to call for the same analysis. I conclude this article with the wish that the general issues that I have formulated inspire discussion and research into the mathematical cultures in other periods and in other fields, and even cooperation between mathematicians and historians to address contemporary mathematical practices in this light. I am convinced that such cooperation would be fruitful for the historians and, who knows, could provide some interesting insights to today's mathematics.

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Karine Chemla studied mathematics at the Ecole Normale Supérieure de Jeunes Filles (1976–1982) and the history of mathematics at the Institute for the History of Natural Sciences (Beijing, China, 1981). She is now a senior researcher at the French National Centre for Scientific Research (CNRS) in the laboratory SPHERE (CNRS & University Paris Diderot) and, from 2011 to 2016, she was a principal investigator on the ERC Advanced Research Grant "Mathematical Sciences in the Ancient Worlds" (SAW, https://sawerc.hypotheses.org). She focuses, from a historical anthropology viewpoint, on the relationship between mathematics and the various cultures in the context of which it is practised. Chemla published, with Guo Shuchun, Les neuf chapitres *(Dunod, 2004). She edited* The History of Mathematical Proof in Ancient Traditions *(Cambridge University Press, 2012),* Texts, Textual Acts and the History of Science *(with J. Virbel, Springer, Book series Archimedes, 2015),* The Oxford Handbook of Generality in Mathematics and the Sciences *(with R. Chorlay and D. Rabouin, Oxford University Press, 2016) and* Cultures Without Culturalism: The Making of Scientific Knowledge *(with Evelyn Fox Keller, Duke University Press, 2017).*

Banach and Nikodym on the Bench in Kraków Again

Danuta Ciesielska (Institute for the History of Science, Polish Academy of Sciences, Warsaw, Poland) and Krzysztof Ciesielski (Jagiellonian University, Kraków, Poland)

Sometimes accidental events have a huge impact on history. One such event took place 100 years ago. An evening walk of a 29-year-old mathematician resulted in the "discovery" of an extremely talented mathematician and, consequently, several outstanding results in mathematics. This meeting has recently been commemorated in Kraków.

Steinhaus

Let us start with an introduction of Hugo Steinhaus (1887–1972). He studied in Göttingen, where, in 1911, he obtained his PhD under the supervision of David Hilbert. He was an exceptional person with a broad knowledge of different branches of mathematics. A significant part of his scientific work involves practical, sometimes very surprising, applications of mathematics. He was remarkably multi-faceted and was a man of great culture and deep knowledge of literature. His aphorisms, remarks and thoughts are famous to this day. Unfortunately, most of them, including the best, are not translatable. One of his thoughts was: "It is easy to go from the house of reality to the forest of mathematics, but only few know how to go back." Once, when somebody was decorated with a medal, Steinhaus said: "Now I know what to do in order to be awarded a medal. Nothing, but for a very long time." He used to say that "a computer is an extremely efficient idiot". He was an accomplished populariser of mathematics. His book "Mathematical Snapshots", first published in 1938, was translated into many languages. However, in the second decade of the 20th century, he was just a young, well-educated mathematician without an occupation. In the Summer of 1916, in the middle of World War I, Hugo Steinhaus was spending some months in Kraków. This Polish city was then a safe fortress in the Austro-Hungarian empire.

A famous meeting

Once, during his evening walk in the Planty Gardens in the centre of Kraków, Steinhaus heard the words "Lebesgue integral". At that time, this was a recent idea known almost exclusively to specialists. Steinhaus was intrigued. He joined the conversation between two young men, who turned out to be Stefan Banach and Otton Nikodym. They told Steinhaus that they had a habit of evening walks and discussing mathematics. In fact, they were usually talking about mathematics with their friend Witold Wilkosz but, this evening, Wilkosz was not with them. During the conversation, Steinhaus presented a problem he was currently working on. The problem concerned the convergence in the first moment of partial sums of the Fourier series of an integrable function. A few days later, Banach visited Steinhaus and presented to him a correct solution of the problem. Then, Steinhaus realised that Banach had an incredible mathematical talent. He started taking care of Banach. The solution of this problem was published in the *Bulletin International de l'Académie Sciences de Cracovie* [3], presented by Stanisław Zaremba (1863–1942), the most well known Polish mathematician of that period. Later, Steinhaus, an author of many important papers, used to say that his best mathematical discovery was the "discovery" of Stefan Banach.

Banach, Nikodym and Wilkosz before 1916

Who were the three young men who had a habit of evening discussions about mathematics? Let us start with the most famous of them, Stefan Banach.

Figure 1. Stefan Banach in 1919. (Courtesy of the Banach family collection [11]).

Banach was born in Kraków in 1892. He was born out of wedlock and took his surname from his mother, Katarzyna Banach. His father, Stefan Greczek, was a soldier serving in the Austrian army. Entrusted into the care of a laundry owner Franciszka Płowa a few months after birth, Banach was brought up by her and by her daughter Maria. In 1902, he entered Gymnasium (secondary school) No.4 in Kraków. On completion of his secondary education in 1910, Banach decided to study engineering in Lvov (Lemberg). He was interested in mathematics but considered it to be a nearly complete science to which very little was left to be added. He decided to study in Lvov, as there was no technical university in Kraków. At the outbreak of World War I, Banach returned to Kraków. He attended some mathematical lectures at the Jagiellonian University and enriched his mathematical knowledge with independent studies.

Wilkosz, born in 1891 in Kraków, was Banach's schoolmate. Endowed with outstanding mathematical and linguistic abilities, he initially linked his future with the latter. However, after two years of philological studies in Kraków, he changed direction to mathematics and studied in Turin. In Italy, he prepared his doctoral dissertation under the direction of Giuseppe Peano but, due to the outbreak of war, the final exams did not take place and Wilkosz had to return to Kraków. He continued mathematical studies at the Jagiellonian University.

Nikodym was older than his friends. He was born in 1887 in Demycze, a suburb of the small Galician city Zabłotów, to a family with Polish, Czech, Italian and French roots. He graduated in 1911 from the university in Lvov, obtaining the right to teach mathematics and physics in secondary schools. He moved to Kraków and taught in Gymnasium No.4, the same one that Banach and Wilkosz had previously attended. In those days, it was usual that, after graduating university, mathematicians would start off teaching at schools and only after some years would join the university faculty.

Banach after 1916

Who knows how the future of Banach would have evolved if Steinhaus had not heard the words "Lebesgue integral"? The result of the accidental meeting in the Planty Gardens was many other meetings; Steinhaus had found excellent interlocutors and brilliant mathematicians. In the Autumn of 1917, Steinhaus left Kraków but he did not forget about Banach. Banach and his colleagues continued to deal with mathematics. After Poland regained its independence (the country lost its independence in 1795), the Polish Mathematical Society was formed in the Spring of 1919 in the building at No. 12 St. Anne Street that housed the Mathematics Seminar of the Jagiellonian University. Banach and Nikodym were among the founders (see Figure 2). Stanisław Zaremba became the first president of the society. For more details about the initial period of activity of the Polish Mathematical Society, see [14].

In 1920, Steinhaus was offered a Chair in Mathematics at the university in Lvov, which, in 1919, was given the name Jan Kazimierz University (Jan Kazimierz was the Polish king who founded this university in 1661). Through Steinhaus' intercession, Banach was appointed to an assistantship at Lvov Technical University. In December 1920, Banach passed his PhD exam at the Jan Kazimierz University. In his PhD dissertation, the notion of the space known later as Banach space was introduced (the name "Banach space" was probably used for the first time by Maurice Fréchet in 1928). The paper was published in 1922 ([1]). In the same paper, the famous Banach Fixed Point Theorem was proved. In 1922, Banach was appointed as a professor at the Jan Kazimierz University.

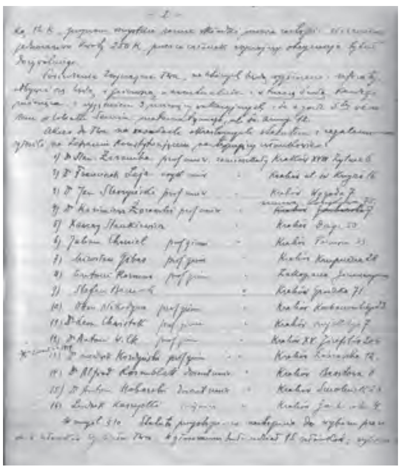

Figure 2. A fragment of the second page of the minutes, with the list of the founders of the Polish Mathematical Society; Banach and Nikodym are listed in positions 9 and 10. (Archives of the Polish Mathematical Society)

Banach continued research, starting from the work in his PhD dissertation. This resulted in the great development of functional analysis (the term "functional analysis" only came into use in the 1940s). In 1928, Banach and Steinhaus founded the journal *Studia Mathematica*. At the end of the 1920s, Banach proved some theorems that are regarded by many authorities as the three basic principles of functional analysis: the Hahn–Banach Theorem, the Banach Closed Graph Theorem and (obtained in collaboration with Steinhaus) the Banach–Steinhaus Theorem. In 1931, Banach's fundamental monograph on functional analysis *Teoria operacyj. Operacje linowe* (Theory of operations. Linear operations) was published and, one year later, its French translation ([2]) appeared. For a few decades, this monograph was the most fundamental book on functional analysis.

Banach did not only work on functional analysis. For example, he obtained some results in measure theory and the foundations of mathematics. In particular, Banach and Alfred Tarski proved that it is possible to break up a 3D ball into a finite number of pieces that can be recombined to form two balls, each of them congruent to the initial one. The proof relies on the properties of the group $SO(3)$ and the axiom of choice. The theorem is nowadays called the Banach–Tarski Theorem on paradoxical decomposition of the ball.

For more information on Banach, see [7, 9, 11, 12]. An excellent description of mathematical results obtained by Banach and his collaborators in Lvov can be found in [10].

In 1939, Lvov was captured by the Soviet Union and, in 1941, Nazi Germans soldiers took Lvov for four years. After World War II, Lvov was joined to the Soviet Union and Banach planned to go to Kraków, where he was offered a Chair at the Jagiellonian University. He died just a few days before the planned move. He is buried in Lychakov Cemetery in Lvov.

Banach published about 80 papers and some monographs and textbooks. It is very interesting that, in the Zentralblatt für Mathematik database, Banach is the mathematician who is most frequently listed in the titles of papers (on 15 April 2017, the score was 23846; the main role played here, of course, is Banach spaces, with a result of 17821).

Nikodym and Wilkosz after 1916

Wilkosz obtained his PhD from the Jagiellonian University in 1918 and he was later appointed to a chair there. He had a very broad range of mathematical interests. Apart from important scientific results from various branches of mathematics (mathematical analysis, geometry, topology and set theory), he was very active in popularisation and teaching. He wrote about 50 scientific papers and several monographs, textbooks and popular books. In particular, he was an author of the first monograph on topology in Poland ([16]) and the first monograph on topology by a Polish author published abroad ([17]). He was also a pioneer in radio engineering and broadcasting

Figure 3. Otton Nikodym in about 1930. (Courtesy of the Mathematical Institute of the Polish Academy of Science)

Figure 4. Witold Wilkosz. (Courtesy of the Jagiellonian University).

in Poland. He constructed a new radio receiver, known later as "Wilkosz's radio". In the 1920s, he reformed the system of mathematical studies at the Jagiellonian University. He died in 1941 in Kraków.

Nikodym continued teaching in the gymnasium for a few years. He did not endeavour to obtain a PhD. He used to ask: "Will I be wiser because of that?" Nevertheless, he obtained a PhD in 1925 from Warsaw University. Up to World War II, he spent some time in Kraków and some time in Warsaw. In addition to his intensive research, he was writing textbooks and scientific monographs. Manuscripts of two monographs prepared for printing just before the war were lost after the 1944 Warsaw Uprising. He commented: "So I will not have to make corrections" and did not write them again. After World War II, he moved to the USA and continued his research in a wide range of areas, including measure theory and differential equations.

He is also an author of mathematical results of great importance. Before World War II, he published more than 30 scientific papers and a total of about 100. One of his most famous results is the Radon-Nikodym Theorem about the existence of a certain measurable function (nowadays often known as the Radon–Nikodym derivative). The result obtained by Nikodym was published in a paper concerning Radon's integral and the version of the theorem obtained by Johann Radon in a special case in 1913 [13]. The theorem is also known as the Lebesgue-Radon–Nikodym Theorem or the Lebesgue–Nikodym Theorem. Another famous result connected with the name of Nikodym is the Nikodym–Grothendieck Boundedness Theorem. It says that if a family of scalar bounded, finitely additive measures defined on σ-algebra A is simply bounded then it is uniformly bounded on A. The result of Nikodym from the 1930s was generalised about 30 years later by Alexandre Grothendieck.

Nikodym died in 1974 in Utica. In his tomb, there is a mosaic designed by his wife Stanisława, who was a mathematician and an artist. For more information about Nikodym and his wife, see [5, 15].

Nikodym's name also appears frequently in the titles of mathematical papers. According to the Zentralblatt für Mathematik database, his score in the titles of papers is 783; he is one of the leading Polish mathematicians in this "competition", with a result similar to those of Sierpiński, Schauder and Marcinkiewicz.

Figure 5. The grave of Otton Nikodym.
(Photo by Janusz Łysko)

Banach, Nikodym and Wilkosz were invited speakers at the International Congresses of Mathematicians: Nikodym in Bologna in 1928, Wilkosz in Zurich in 1932 and Banach as a plenary speaker at the ICM 1936 in Oslo and an invited speaker in 1928. It is interesting that, although Banach, Nikodym and Wilkosz were good friends and they were discussing mathematics a lot (Banach and Nikodym even worked later on similar areas), any two of them have never written a joint paper.

A monument

In many cities, statues of people can be seen sitting on benches. Such monuments are now quite popular but is there a better justification for a commemorative bench than the event described here, which had such an effect on science? The concept of placing a bench with a statue of Banach has been considered several times before but the idea has always ended at the concept stage. There are many difficulties, as there are four basic problems to solve:

- Finance. Making such an object would cost a lot and there is no chance of getting any financial support from official sources.
- Permission. Kraków is a historical city, the Planty Gardens are in the city centre and it is extremely difficult to get permission to place a memorial plaque there, let alone a monument. For example, presenting mathematics to a broad audience in the city centre during the 6ECM immediately caused a reaction from the city guard (see [4]) – fortunately, permission for "maths busking" was then provided (although the formalities took a couple of months).
- Design of the sculpture. Such a monument must be beautiful and representative of the 1916 reality.
- Management of the event. People are needed to manage the project (from the concept until the unveiling of the monument). This is a very troublesome and time-consuming task.

Many mathematicians have wondered about the precise location of the bench where Steinhaus encountered Banach? The Planty Gardens is a large park of about 21 hectares surrounding the historical centre of Kraków. The meeting could have happened in many different places. In his memoirs and articles about Banach, Steinhaus does not indicate a precise place. However, after careful analysis of the problem, we came to a conclusion. Of course, one cannot be sure but it is almost certain that this meeting happened at the extreme end of the Planty Gardens, close to Wawel Castle and the house where Banach lived (for details, see [6]). Thus, there was a renewed motivation for making a bench memorialising this significant event. Moreover, the 100th anniversary of the event was approaching.

In 2014, the Dean of the Faculty of Mathematics and Computer Science of the Jagiellonian University appointed a special committee to act upon the creation of the bench. It consisted of seven people: Artur Birczyński, Danuta Ciesielska, Krzysztof Ciesielski (chair), Małgorzata Jantos, Jerzy Ombach, Piotr Tworzewski and Karol Życzkowski. Our idea was to present not only Banach but Nikodym as well, sitting on a bench as it was in 1916. And now, everyone can come to them and join them like Steinhaus did.

Stefan Dousa, an outstanding Polish sculptor, agreed to design the monument. Dousa is a professor at the Kraków University of Technology and a creator of many of the magnificent monuments, plaques and medals in Poland, as well as many European countries and the USA. Moreover, Dousa likes mathematicians. He had already made a memorable medal for the 6ECM (see [8]). We got all the required permissions from the city authorities (with great help from M. Jantos, who is a member of the City Council). One of the good reasons that permission was granted was that, in this case, a monument in the form of figures on a bench was perfectly justified. Finally, a sponsor was found. ASTOR, a company that provides modern technologies in the fields of industrial robotics, IT solutions and technical knowledge through training and consulting, agreed to finance the monument. The name ASTOR is an acronym for *Automatyka, Sterowanie, Transmisja, Oprogramowanie, Robotyka* (Automation, Control, Transmission, Software, Robotics).

Unveiling

It was known in what area of the Planty Gardens the bench should be placed and it was Stefan Dousa who picked the final location of the bronze monument. It is on the way from Wawel Castle to the Main Square, the route most frequently used by tourists. The unveiling was

Figure 6. From left to right: Piotr Idzi (Dousa's assistant), D. Ciesielska and S. Dousa in Dousa's workplace and a plaster model of a monument.

planned not for the Summer (the exact date of the historic meeting is unknown but it is known that it was in the Summer) but in October, after the start of the academic year. The celebration took place on 14 October 2016. More than 200 people attended, including many high profile guests. Several of them came from abroad. Some members of the Council of the European Mathematical Society and authorities of the Polish Mathematical Society were also present. The monument was unveiled by Stanisław Kistryn (Vice-Rector of the Jagiellonian University), Stefan Życzkowski (President of ASTOR), Monika Waksmundzka-Hajnos from Lublin (a niece of Banach) and Banach's nephew, John Greczek from the USA. The film of the ceremony can be seen at https://www.youtube.com/watch?v=813R1905hUc. After the ceremony, the guests visited the ASTOR Innovation Room and were invited for dinner in the *CK Browar Restaurant*. The choice of this location was because the restaurant is in the basement of the building where Gymnasium No.4 was located 100 years ago.

Figure 7. The unveiling of the bench. From left to right: S. Dousa, M. Waksmundzka-Hajnos, J. Greczek, S. Kistryn and S. Życzkowski. (Courtesy of the Jagiellonian University)

In addition to the figures sitting on the bench, the backrest of the bench is enscribed with the logo of ASTOR and the inscription: "On the 100th anniversary of the most famous mathematical discussion at the Planty Gardens." On the seat of the bench, next to the figures, mathematical symbols are carved. We decided that they would not be symbols of Banach's and Nikodym's best-known results but, instead, a formula from the paper in which the solution of the problem communicated by Steinhaus to Banach at their first meeting was published [3].

Figure 8. The plaque close to the bench.

Moreover, close to the bench, there is a special plaque (see Fig. 8) with information (in Polish and English) about the event in 1916 and the figures.

Now, visiting the Planty Gardens in this area, one can see that the bench is of

Figure 9. The bench.

great interest. People often sit on it, take photographs with Banach and Nikodym, study the inscriptions, etc. It is really a great promotion of mathematics. Moreover, the bench is really marvellous and perfectly made by Dousa. The faces of Banach and Nikodym are very similar to their photographs dated 100 years ago. As one approaches the bench, it almost looks as if there were two real human beings talking to each other.

One mathematician, after a first look at the bench, commented: "It is obvious that they talk about mathematics."

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Danuta Ciesielska [smciesie@cyfronet.krakow.pl] is a mathematician and an historian of science. She is a Section Editor of the journal "Antiquitates Mathematicae" of the Polish Mathematical Society, which is devoted to the history of mathematics. She is a member of the Commission on the History of Science of the Polish Academy of Arts and Sciences.

Krzysztof Ciesielski [krzysztof.ciesielski@im.uj.edu.pl] was an Associate Editor of the EMS Newsletter, 1999–2012, and a member of the Raising Public Awareness of Mathematics Committee of the EMS, 2009–2016. He is currently an Associate Editor of the journal "Wiadomości Matematyczne" of the Polish Mathematical Society.

The Institut Mittag-Leffler and its Archives: A Mathematician and his Legacy

Eva Kaufholz-Soldat (Johannes Gutenberg-Universität, Mainz, Germany)

It is quite impossible to write about the history of the Institut Mittag-Leffler (*IML*) without, at the same time, discussing the mathematician who conceived it and after whom it is named.[1]

In writing his obituary, G.H. Hardy noted that "[t]here have been greater mathematicians during the last fifty years, but no one who has done in his way more for mathematics" [10, p. 160]. And, indeed, even though he produced quite deep results during his active years, Gösta Mittag-Leffler is mostly remembered for his vital role in the mathematical community at the turn of the century and as the founder of *Acta Mathematica*, which appeared in 1882 for the first time. André Weil aptly called Mittag-Leffler's new journal his deed of true genius [24, p. 12].

Its success as one of the first truly international journals was partly due to Sweden's neutrality, making it possible for French and German mathematicians to publish in the same journal roughly a decade after the Franco-Prussian War [2, 5]. But, even more importantly, *Acta* was a testimony to Mittag-Leffler's outstanding organisational talent, including his ability to find a number of prominent sponsors, the Swedish King Oscar II among them. And when he married Signe Lindfors the very same year, their romantic honeymoon trip through Europe was surely dampened by the fact that he was constantly meeting with French and German mathematicians in order to secure publications for *Acta* [2, pp. 6, 8].

Already a decade before, Mittag-Leffler had made important contacts in Berlin and Paris, the two foremost mathematical centres at the time. It was in May 1872, right after he finished his dissertation, that he was awarded the *Byzantine Grant*, which allowed him to travel to Central Europe, a trip that proved decisive for the rest of his mathematical career [22, pp. 143ff].

In Paris, he got to meet Charles Hermite, from which a friendly relationship developed that lasted until the latter's death in 1901. An extensive correspondence bears witness to that friendship, which was especially important for Mittag-Leffler in helping him make connections with the French mathematical community and for securing publications for *Acta*. While the letters from Hermite to Mittag-Leffler were edited and published by Pierre Dugac [6–8], the other side of the exchange still exists in the *IML* in the form of drafts, waiting for anyone willing to battle with Mittag-Leffler's difficult handwriting.

It was in 1881 that Mittag-Leffler first wrote to Henri Poincaré, soon after Hermite had called his attention to this talented young mathematician, who had just finished his doctoral thesis under his direction. Their subsequent letters along with other archival materials relating to Poincaré at the *IML* are without doubt among the better known holdings there. In 1999, Philippe Nabonnand (Poincaré Archives, Nancy) edited and published this correspondence. Unlike many others, Mittag-Leffler kept his letters to Poincaré on a strictly professional level, whereas for Poincaré it is the most regular and extensive scientific correspondence that has been preserved [18]. As a result, their symbiotic relationship is very well documented. From the very beginning, Mittag-Leffler recognised Poincaré's potential and he would continue to promote him throughout his career.

He also hoped that Poincaré would play a role for *Acta* similar to the one Niels Henrik Abel had so importantly played for Crelle's *Journal für die reine und angewandte Mathematik*. In the late 1820s, Abel's publications had helped launch this fledgling journal, which quickly became one of the most renowned mathematical publications of its time [14, p. 199]. Mittag-Leffler was not to be disappointed. Indeed, many of Poincaré's most important papers would appear under his editorship, such as his publications on Fuchsian functions and, of course, his now famous contribution on the *n-body-problem*, which won the prize established in honour of King Oscar II of Sweden. At first, however, it was overlooked that Poincaré's original paper, published in *Acta* in 1889, contained such a serious error that the issue had to be reprinted. Poincaré reworked that article and, in the new version, he presented what could be considered a first description of chaotic behaviour, even though this concept would only be formulated much later, in 1960, when Edward Lorenz introduced modern chaos theory [1]. Scans of some of the documents concerning this dramatic episode in the history of modern mathematics can be found on the *IML*'s homepage, together with a short overview of the events.[2] Unfortunately, no other documents are available online as yet, nor is there an index of all the material that can be found on-site (this would be very helpful).

According to Mittag Leffler, it was partly due to Hermite's influence that he met Weierstraß for the first time. Unlikely as it may seem, the French mathematician supposedly advised him to go to Berlin to attend his lectures, calling him the "master of us all". Not long after his arrival

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[1] My grateful thanks to both Mikael Rågstedt and David E. Rowe for their valuable help and suggestions for this article.

[2] http://www.mittag-leffler.se/library/prize-competition.

there in 1875, Mittag-Leffler wrote to his former teacher Hjalmar Holmgren that he had nowhere "found so much to learn as here". This was mainly due to Weierstraß, who, together with Leopold Kronecker and Eduard Kummer, had led the Prussian capital into what would later be called the "golden era" of Berlin mathematics. Still, his account was not all praise, as he described Weierstraß's manner of lecturing as lying "beneath all criticism, and even the least important French mathematician, were he to deliver such lectures, would be considered completely incompetent as a teacher". Nevertheless, he was deeply impressed by the clarity of his thought and the systematic approach he took [19, pp. 62f].

It was under Weierstraß's influence that Mittag-Leffler produced his most noteworthy mathematical result: the theorem that bears his name was, in fact, closely linked to the master's programme on the foundations of complex analysis [23]. Afterwards, he would dedicate a large part of his life to spreading and defending the principles that Weierstraß had established as standards for mathematical research. It is therefore not surprising that he based his own lecture course on Weierstraß's ideals when he became the first professor of mathematics at the newly founded *högskola* in Stockholm.

Weierstraß seldom published his latest results, preferring to present them in his lecture courses. Recognising this, from the time he came to Berlin in the 1870s, Mittag-Leffler made diligent efforts to take notes himself but also collect lecture notes prepared by other students. Others certainly did so as well; Weierstraß's lectures were lauded for their innovative character and studied by aspiring mathematicians who longed to get their hands on good written versions of them. But no one would manage to assemble more of these lecture notes than Mittag-Leffler, who eventually owned a collection of around 50 of them, some prepared by different transcribers who attended the same course[20, p. 7].

In fact, Mittag-Leffler's passion as a lifelong collector of mathematical literature and memorabilia is one of the reasons that the *IML* is so unique today. Not only did he purchase almost all the important mathematical publications and mathematical journals printed during his lifetime – which can still be found in the *IML* library, usually in complete sets – he also acquired the correspondence between some of the most eminent mathematicians in history, for example, letters written by and to Carl Gustav Jacob Jacobi, most of the latter written by Adrien-Marie Legendre. Occasionally, this passion for owning something from the hand of a personal hero resulted in rather odd purchases, such as a poem by Louis Cauchy or his book of household accounts. Over time, he also acquired an impressive collection of roughly 1500 books printed before 1800, including several first editions, such as Archimedes' *Opera* (1544) and Galilei's *Dialogo* (1632) and *Discorsi* (1638), as well as Newton's *Principia* (1687) and *Opticks* (1704). At some point, the famous explorer Adolf Erik Nordenskiöld presented Mittag-Leffler with a very special gift, the first incunable of Euclid's *Elements* from 1482, which also happens to be the first printed book to include pictures. Since Mittag-Leffler already owned it, there are now two copies available to the researcher. One might, in fact, compare this rare version with any of the various other early editions and translations of the *Elements* that can be found on-site.

Another unusual part of his library is the so-called *Boncampagni Collection*, named after its creator. Much like Mittag-Leffler, Baldassarre Boncompagni Ludovisi, Prince of Piombino (1821–94), began his career as a mathematician but is remembered today mainly for his role as a bibliophile and promoter of the discipline. He, too, relied on his contacts with other influential men of the time, most notably Pope Pius IX. After the 1840s, he began to take a deep interest in the history of mathematics, founding the first journal devoted to this field, *Bullettino di bibliografia e di storia delle scienze matematiche e fisiche*, which appeared from 1868 to 1887. He also assembled a large library on the history of the exact sciences, much of it consisting of medieval *abbaci* and early treatises of arithmetic. To do so, he relied on a large network of contacts who kept him informed about such works, which he then purchased whenever possible. When he could not, he often employed artists who were responsible for producing duplicates [15, pp. 257–260]. These exquisite copies were designed so as to be as true to the originals as possible, even to the point of having imitation creases, cracks or missing pieces painted into them. Four years after Prince Boncampagni's death, Mittag-Leffler bought this unique collection at an auction in 1898. While this short description gives a vague idea of this collection,[3] the books have been sitting on shelves, untouched and awaiting future research.

Mittag-Leffler's ability to collect such expensive books reflects the fact that he proved to be a rather successful entrepreneur [22, passim]. Additionally, his wife Signe contributed a considerable dowry to the marriage, as she was heir to the significant fortune her grandfather had made in the tobacco business [9, pp. 363f]. Together, the Mittag-Lefflers could thus dispose of assets that far exceeded those of a typical professor. Such means also enabled them in 1890 to begin building their impressive mansion, nestled in the rolling hills of Djursholm, just north of Stockholm [22, p. 391]. This served as their home for the rest of their lives, even though two of its three floors were designed to be libraries from the very beginning. Moreover, these were never conceived as being for private use alone, as becomes clear in the ex-libris carefully glued into every book: *Sibi et amicis* – for me and my friends. In this respect, the villa's future life as a research institute was foretold.

Mittag-Leffler had long been driven by a desire not only to make a name for himself in the mathematical community but to fashion an enduring legacy, a longing that only grew the older he became. Shortly before his 60th birthday in 1906, he wrote about his future plans in his diary, including arrangements for "financing for the legacy that I want to leave behind" [22, p. 504]. Thus, the establishment of a mathematical institute that would

[3] Items 502-1,223 on pp. 95-141 in the *Catalogo della biblioteca Boncompagni*, Vol. I, Rome, 1898.

bear his name after his death fitted perfectly with his mindset.

Probably inspired by the wish to realise this dream but also to ensure that his own ideas would not be misconstrued, he set these down a decade later in his last will and testament. Therein, he designated that on his 70th birthday, he would bequeath his villa and fortune to the *Royal Swedish Academy of Sciences*, under the condition that they use both to found a mathematical institute.[4]

That date fell in the middle of the Great War so the official opening was postponed until 1919. Nevertheless, it remained a pure formality [9, p. 364] and it was only long after Mittag-Leffler's death in 1927 that his dream would become a reality. After the *IML* had lain more or less dormant for several decades, discussions about closing it for good began to surface. But, luckily enough, Lennart Carleson had different ideas and he was in a position to act on them. In 1966, he had proven a conjecture by Nikolai Luzin about the convergence of certain types of Fourier series, a result so impressive that he received numerous job offers from universities in the United States. To discourage him from relocating, the Swedish government awarded him a special professorship, which allowed him to work wherever he wanted. And so he chose to go to the *IML*, taking up residence there a mere year later. From there, he began efforts to secure the necessary financial means and he was able to play the role of Prince Charming, whose kiss reawakened the Sleeping Beauty of the old villa, which could now begin to bloom as an international research institute.

In doing so, Mittag-Leffler mysteriously came to assist him from beyond the grave [11, p. 1053f]. As it happened, the cellars of his villa were filled with countless complete sets of *Acta Mathematica*, some of them probably intended for a special purpose. For Mittag-Leffler had not only bequeathed his fortune to establish the *IML*, he had also designated how this was to be spent. Part of the funding was to be earmarked for stipends to support young mathematicians from the Nordic countries and, if their work proved to be of exceptional quality, to reward them with gold medals and a complete set of the journal [22, p. 584]. Now, however, at a time when universities were expanding worldwide, the Institut had the opportunity, instead, to sell several hundred of these, thereby contributing to its solid financial basis as an added bonus for Carleson's endeavours [11, p. 1053].

Ever since that time, the villa has served as an international research institute, hosting numerous longer mathematical programmes, now supplemented with conferences during the Summer. Mathematical topics have regularly been set by the directors for up to one year, during which leading experts and young research scholars working in the chosen field are invited to attend. But, under Carleson's directorship, the *IML* has also become an important repository of historical documents. While it would take until the 1990s before Reinhard Bölling systematically ordered and catalogued the archival materials on-site [20, p. 9], 20 years earlier Ivor Grattan-Guinness had taken a first look at the treasures hidden in various boxes and cabinets. He was delighted to find that he could now determine – at least partly – "the whereabouts, or fate, of Weierstraß's *Nachlass*" [9, p. 365]. Later, Gert Schubring discovered a small number of Weierstraß's letters in Berlin [21].

That most of Weierstraß's personal papers are now located in a country that played no role in his career is, of course, due to Mittag-Leffler, who procured them after the *Nachlass* had passed through many hands. After Karl Weierstraß died in 1897, his estate was initially handled by his former student and confident Johannes Knoblauch, at the request of Weierstraß's sister Elise, but was later given to their brother Peter, who passed away in 1904. None of the Weierstraß siblings had (legitimate) children,[5] so there was no official heir. Thereafter, Mittag-Leffler contacted H. Schulz, who was apparently the husband of Peter's housekeeper during the last months of his life and acted as a custodian for his estate. Mittag-Leffler persuaded him to sell him "a small mountain" of correspondence, some original writings, diplomas, official documents and paintings of the Berlin mathematician in his name. And, thus, a significant part of Weierstraß's *Nachlass* was moved to his home in Djursholm. In later years, it even grew larger as other mathematicians – Eugen Netto, Georg Cantor, Paul du Bois-Reymond and Leo Königsberger – also agreed to hand over their respective correspondence with Weierstraß to Mittag-Leffler [20].

That Mittag-Leffler tried to save as much as possible of Weierstraß's papers can only partly be explained as a wish to ensure these documents would not get lost over time. As has already been mentioned, he modelled his own lecture courses on those given by the Berlin mathematician. He did so, however, not only to spread the principles of his idol but also in order "to appropriate a part of the legacy of the great Weierstrass for himself" [14, p. 118]. So his motivation, once again, had much to do with fashioning his own place in the history of mathematics. Already in the early 1880s, he had been strengthening his bonds with Weierstraß, particularly when, through much personal effort, he managed in 1883 to appoint the latter's "favourite pupil", Sofia Kovalevskaya, at the *högskola* in Stockholm.

Ever since, Kovalevskaya has attracted a great deal of attention: not only was she one of the first women to hold an official teaching position at a university, she was also a writer with connections to current radical political movements. Whereas her lecture notes and other math-

[4] While Mittag-Leffler suffered serious financial setbacks during the latter stages of his life, in part because his financial advisors took advantage of him, there are no sources confirming that this was motivated by an imminent loss of his possessions due to any depreciation of investments in German bonds during World War I, as claimed earlier by Ivor Grattan-Guinness [9, p. 364].

[5] While none of the Weierstraß siblings officially had any children, there is a certain possibility that Karl fathered a son out of wedlock with the widow of his best friend Carl Borchhardt [3].

ematical papers, as well as various manuscripts from her literary ventures, have remained largely untouched, much of her vast *Nachlass* at the *IML* has been used in biographical studies, most notably by Ann Hibner Koblitz and Pelageya Kochina [12, 13]. Another important publication stemming from her personal papers were the letters Weierstraß wrote to her, which were published by Reinhard Bölling in 1993 [4].

Still, Bölling was not the first to publish from this correspondence, as Mittag-Leffler himself had already cited letters from it in recounting the life of Weierstrass at the *ICM* held in Paris in 1900. It was only toward the end of his life, however, that Mittag-Leffler actually published several of the letters that would later find their way into Bölling's collection.

One reason for the delay was a promise he had made to Weierstraß. Deeply struck by the death of his very dear friend, the Berlin mathematician had burned all her letters to him but he allowed Mittag-Leffler to keep the ones he wrote to her under the condition that the Swede would not publish these before he died. Mittag-Leffler kept true to his word: not until 1923, more than 25 years after Weierstraß's death in 1897, did he publish excerpts from them. These appeared in an article in volume 39 of *Acta* that was based on Mittag-Leffler's lecture from 1900 [17].

That volume, like the preceding one, had been dedicated to the lives and scientific achievements of Karl Weierstraß, Henri Poincaré and Sofia Kovalevskaya, all of whom were, by then, long since deceased. As Mikael Rågstedt, the current librarian of the IML, has noted: "Mittag-Leffler throughout his life showed a special fascination with an exclusive group of mathematicians, [often behaving like] a knight who stood up for them and tried to restore them to their rightful scientific position." Certainly, Mittag-Leffler proved his loyalty to these three famous figures all his life but, as Rågstedt also noted, "chivalry is not the only qualification of a knight, and Mittag-Leffler also knew how to make use of his idols as armour and lances for his own ambitions" [20, p. 2]. In this spirit, his efforts to promote the careers of Poincaré and Kovalevskaya were not just a little self-serving. More striking still were his efforts to make himself the legitimate heir to Weierstraß's legacy by way of the biographical genre, an aim he pursued for decades with increasing zeal [14, p. 118]. This culminated in a long essay on his idol's life contained in [16], for which he could rely on those personal papers out of the *Nachlass* he had acquired almost 20 years earlier.

While Mittag-Leffler never finished the extensive biography of the Berlin mathematician he had hoped to write, it would certainly be wrong to claim that Weierstraß's *Nachlass* has been completely ignored since his time. Quite a few items from it have been cited in various publications since its rediscovery by Grattan-Guinness. Nevertheless, it still awaits a proper assessment as a whole. This would certainly shed more light on the life and work of one of the most important mathematicians of the 19th century, including the school he founded in Berlin, which is still lacking a comprehensive study.

Much the same could be said about another major resource found at the *IML*, one which had already impressed André Weil when, as a fellow of the Rockefeller Foundation, he visited Mittag-Leffler in 1927, just months before the latter's death. What he discovered was Mittag-Leffler's correspondence with "the great ones of the past half-century; […] all there to keep me company while everyone was asleep, opening up for me the secret recesses of their minds" [24, p. 11]. Here, one can find letters from the likes of Georg Cantor, Marie Curie, Gottlob Frege, G.H. Hardy, David Hilbert, Camille Jordan, Felix Klein, Sophus Lie and Paul Painlevé. As would be expected, much of this deals with matters concerning *Acta*, as Mittag-Leffler was always on the lookout for possible publications, very often with a remarkable instinct for groundbreaking advances.

As Mittag-Leffler had a tendency to keep records of everything, drafts of his own letters are also available in most cases, making it possible to reconstruct a fairly complete picture of his correspondence in many instances. While quite a few of these documents have been cited in various publications, most recently in Arild Stubhaug's biography of Mittag-Leffler [22], a full study of this correspondence would certainly be a most rewarding undertaking. For, as Weil rightly noted, Mittag-Leffler had a talent for turning colleagues into friends and the "rare quality of sympathy in the bonds of friendship which Mittag-Leffler had succeeded in establishing with men so diversely gifted, inducing them to confide their innermost thoughts to him with such abandon" is certainly striking [24, p. 11]. Thus, the letters bear witness to important advances in the discipline in a technical sense but also provide insights into what was going on behind the scenes, allowing for a rich and multifaceted picture of a larger part of the mathematical community at the time.

For security reasons, the originals of these letters have recently been moved to the *Royal Swedish Academy of Sciences* in Stockholm. Earlier, Mittag-Leffler's private materials (about 80 shelf metres), most notably including his diaries, were transferred to the *National Library*. Still, the *IML* has kept high quality copies of the scientific letters, which are easily accessible to anyone who feels inspired to follow in Weil's footsteps. As the *IML* also provides accommodation for researchers, the grand mansion in serene Djursholm offers ideal conditions either to study these or to tackle other historical papers, such as those in Weierstraß's *Nachlass*. And this can be done in virtually the same atmosphere that Mittag-Leffler enjoyed, breathing the spirit of the 19th and early 20th centuries in his spacious but charming rooms, furnished mostly with the original chairs, lamps and shelves of the day. But one can do that while taking advantage of the perks of the digital age, provided in most discrete ways so as not to disturb the enchanting atmosphere of a place that seems to have fallen out of time.

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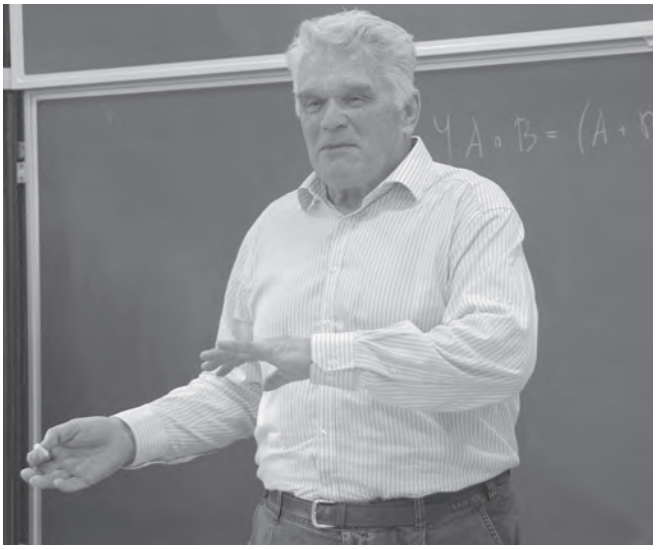
Eva Kaufholz-Soldat [kaufholz@uni-mainz.de] is a PhD student in the history of mathematics at Johannes Gutenberg-Universität, Mainz. She is currently writing her thesis on the reception of Sofia Kovalevskaya during the long 19th century.

Ludwig Faddeev (1934–2017) – His Work and Legacy

Irina Aref'eva (Steklov Mathematical Institute, Moscow), Michael Semenov-Tian-Shansky (St. Petersburg Branch of the Steklov Mathematical Institute and Université de Bourgogne, Dijon, France), and Leon Takhtajan (University of Stony Brook, USA)



Ludwig Faddeev. Aarhus University, August 2010.

The mathematical community has suffered a heavy loss with the death of one of the world's greatest mathematicians and theoretical physicists Ludwig Faddeev, who passed away on 26 February after a heavy illness. Despite his ailing health, Faddeev remained active until the last months of his life. In August 2016, he attended a special meeting, the 23th European Conference on Few-Body Problems in Physics, held in Aarhus (Denmark), where a new award to recognise distinguished achievements in few-body physics, the Faddeev Medal, was inaugurated. This proved to be the last honour, among the many others, that he received in his lifetime.

Professor Ludwig Faddeev is widely known for his contributions to mathematics and theoretical physics, which have largely reshaped modern mathematical physics. His work on quantum field theory prepared the ground for the gauge fields theory revolution of the 1970s. His contributions to the many-body problem in quantum mechanics and to the inverse scattering problem belong to the deepest achievements in these areas. His pioneering work on the quantum inverse scattering method started a wide new field of research, ranging from solvable models in quantum field theory to quantum groups.

For more than 60 years, Professor Faddeev was associated with the Steklov Mathematical Institute. In 1976–2000, he was serving as Director of the Leningrad (later, St. Petersburg) branch of the Institute and Head of the Laboratory of Mathematical Problems in Physics, where he brought together a score of his pupils and colleagues. Although they are now dispersed over several countries and continents, the Faddeev school is still highly united and plays a prominent role in modern mathematical physics.

Early years

Professor Faddeev was born in 1934 in Leningrad (now St. Petersburg) into a family of prominent Soviet mathematicians. His mother Professor V.N. Faddeeva was among the pioneers of computational methods and, for many years, headed the Laboratory of Computational Methods of the Steklov Institute in Leningrad. His father Professor D.K. Faddeev was one of the best Soviet algebraists; he is particularly remembered for his contributions to homological algebra, Galois theory and representation theory. His university teaching has shaped several generations of Soviet algebraists. He was also a distinguished musician and a brilliant pianist. The choice of the rather rare name Ludwig for his elder son reflected the parents' hope that he would one day become a professional musician. This hope would not materialise because of the hardships of wartime during his childhood but Ludwig still had a profound knowledge of classical music. Together with his father, he played Bruckner's and Mahler's symphonies (in a four-hands transcription) and his favourite composers were Berlioz and Richard Strauss.

The younger years of Faddeev were a time when the country was recovering from the ravages of war and also a time of great hope after the death of Stalin. Among his generation, there is a remarkable number of first-rate mathematicians: Arnold, Berezin, Maslov, Novikov and Sinai, to mention just a few. This spectacular explosion of talent was, to a great extent, due to a very simple circumstance: in this time of still tough ideological controls, mathematics was a domain of freedom that naturally attracted bright young people. Another crucial point was the solid scientific tradition that survived the turmoil of revolution and the repressions of the 1930s.

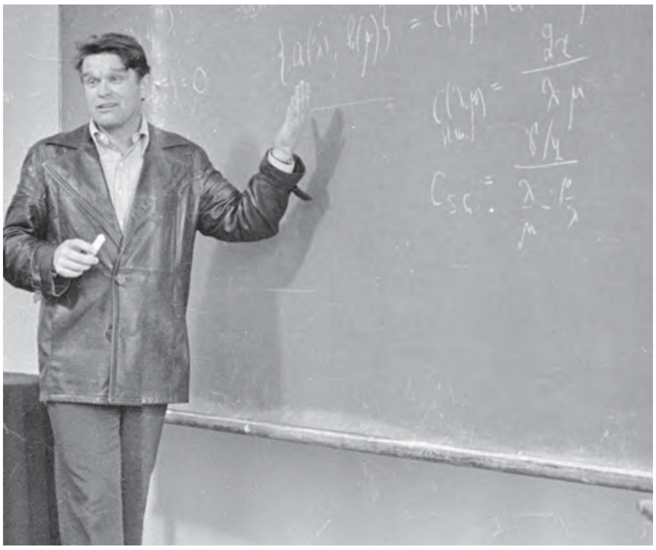
One important decision Faddeev took as a 17-year-old youth was the choice of the Physics Department of Leningrad University. At the time, his father was Dean of the Mathematics Department and Ludwig wanted to make his own way. The Physics Department had a brilliant tradition in theoretical physics and especially in general relativity and in quantum mechanics (then only 25 years old!), marked by such names as A.A. Friedmann and V.A. Fock. The higher mathematics course, which was supervised by Academician V.I. Smirnov, was directly oriented towards the needs of quantum theory, with an emphasis on operator theory, spectral theory of differential operators, etc. Starting from 1954, the mathematical education of the young theorists was mainly entrusted to Professor O.A. Ladyzhenskaya, then the youngest and most brilliant professor of the Chair of Mathematics. Before that, the Chair of Mathematics at the Physics Department, created by Academician Smirnov in the 1930s, was mainly considered an auxiliary one. Now, for the first time, it was given an independent status and was allowed to have its own students, give special courses and supervise diploma work. Faddeev was thus in the very first group of students who defended their university theses in mathematical physics. Already in 1954, Professor Ladyzhenskaya had organised a special learning seminar in quantum field theory, where Faddeev was one the main speakers. This seminar shaped his early interest in mathematical problems of quantum field theory. Professor Ladyzhenskaya also became Faddeev's thesis adviser during his PhD studentship at the Leningrad Branch of the Steklov Mathematical Institute. Her portrait remained on Faddeev's writing desk in his study at the Steklov Institute up to the last day of his life.

First papers: quantum scattering and the inverse problem

The first published papers of Faddeev dealt with potential scattering and spectral decomposition for Schrödinger operators with continuous spectrum. His concise proof of the dispersion relations for the scattering amplitude was included in the famous Landau and Lifshitz textbook on quantum mechanics. In his PhD thesis, he gave a complete solution of the inverse scattering problem for the Schrödinger operator on the line. This work was written in the aftermath of the fundamental results on inverse scattering due to I.M. Gelfand, B.M. Levitan and V.A. Marchenko, who had solved the inverse scattering problem for the radial Schrödinger equation (which arises from the three-dimensional Schrödinger equation after a separation of variables). The case of the Schrödinger equation on the line is slightly more difficult because of the multiple continuous spectrum. Over a decade later, this paper proved to be of crucial importance as it contained all the background of the future inverse scattering method in the theory of integrable systems. In the course of this study, Faddeev also prepared a comprehensive review of quantum inverse scattering. At the invitation of Academician N.N. Bogolyubov, it was submitted at the inauguration meeting of the Laboratory of Theoretical Physics in Dubna, in the presence of Gelfand, Levitan, Krein, Marchenko and other big names; its written version, published in 1959 in Uspekhi, became a standard reference in the field.

Quantum three-body problem

The next big subject Faddeev chose was the quantum three-body problem. At the time, he was already heavily attracted to the intricate and complicated problems of quantum field theory but believed that, before launching into the insecure waters of QFT, it was important to resolve a really difficult technical problem. While the difficulties of the quantum three-body problem are of an



First Conference on Quantum Solitons. Leningrad, October 1978.

entirely different nature than those of its famous classical counterpart, it represents a real challenge because of the complicated structure of the continuous spectrum. Before Faddeev's work, only some partial results had been obtained by physicists (under very restrictive and not quite self-consistent assumptions on the interaction potentials). Faddeev's original approach to this problem was based on experience he had gained in his work with the so-called Friedrichs model in perturbation theory and also in the study of an instructive example from QFT, the Thirring model. The key idea consists of a clever rearrangement of the integral equations associated with the multi-body scattering problem (which basically result from the Hilbert identity for the resolvent of the Schrödinger operator) into a much more manageable and symmetric system for the so-called T-operators (generalising the pairwise scattering amplitudes for different particles). This new system of integral equations, called Faddeev equations, is already Fredholm, in contrast to the initial equation for the resolvent. It has become the basis of efficient numerical computations in various applications (ranging from quantum chemistry to nuclear physics). Faddeev's work on the quantum three-body problem triggered tremendous activity in the area (pursued up to the present day); his own decision, however, was definitely to move to other subjects. After the publication of his now famous monograph on three-body scattering (1963, English translation 1965), while some of his pupils continued working in this direction for another decade or more, he decided that it was now time to attack QFT.

Quantum gauge theory

The bid was, in fact, a very difficult one since QFT was positively out of grace in the Soviet Union at the time. The great success of quantum electrodynamics in the late 1940s and early 1950s was followed by a decade of fruitless attempts to apply QFT to strong interactions. Still more importantly, QFT was plagued by the so-called "zero charge paradox", discovered by Landau and Pomeranchuk and believed to point out the logical inconsistency of quantum electrodynamics and QFT in general. In his short note dedicated to the memory of Wolfgang Pauli, written shortly before the tragic car accident that put an end to his scientific career, Landau insisted, with a reference to this paradox, that the Hamiltonian method in field theory was now totally dead and needed to be buried ("with all the honours it deserves"). Due to the brevity of life, he concluded, we cannot allow ourselves the luxury of spending our time on problems that do not lead to new results. Landau's words were considered by his pupils in the 1960s as the Teacher's Testament and when, in 1966, Faddeev, together with his pupil V.N. Popov, obtained a breakthrough in quantum Yang–Mills theory, their paper could not be published in any of the Soviet scientific journals nor sent abroad (for which a positive opinion of the Nuclear Physics Department of the Academy of Sciences was necessary). A short note by Faddeev and Popov was finally published in Physics Letters (with a year delay), while the full text was made available only as a preprint of the Kiev Institute of Theoretical Physics (with hand-written formulae); its English version only appeared in 1973 at the time of the big boost triggered by the gauge fields revolution of the early 1970s.

The choice of Yang–Mills theory reflected Faddeev's characteristic non-conformism but also his fundamental belief that a good physical theory should have mathematical beauty. His original idea was to understand the quantisation of general relativity, a theory of incontestable great beauty but also of notorious difficulty. Yang–Mills theory seemed, at the time, just a kind of useful model example. We know now that this example proved to be an exceptionally successful one: it allowed the generalisation of quantum electrodynamics by unifying electromagnetic and weak interactions, and the building of a consistent theory of strong interactions. Geometrically, Yang–Mills theory is, in fact, very close to general relativity (while the latter deals with the tangent bundle of the spacetime, Yang–Mills theory brings into play abstract vector bundles). All these exciting developments had already taken place in the 1970s; the key discoveries, due to G. 't Hooft, D. Gross, F. Wilczek and D. Politzer, were that Yang–Mills theory is renormalisable and free of the zero charge paradox. The culmination of this "gauge fields revolution" was the creation of the "standard model" in high energy physics. The earlier results of Faddeev and Popov provided both the technical base and the conceptual base for these developments, marked by several Nobel prizes.

Turning back to the Faddeev–Popov paper, it is worth stressing the conciseness and clarity of their approach, which was to become the basic language of the new theory. The new QFT formalism they proposed was, for the first time, entirely based on the ample use of functional integrals. Functional integrals had already been introduced into quantum mechanics by R. Feynmann in the 1940s but, for some strange reason, he never used them in quantum field theory, even though, as we understand now, they provide the easiest and most straightforward way to deduce his famous diagram expansion. In the early 1960s, Feynmann also examined the quantisation of Yang–Mills theory (he, too, regarded it as a model example before addressing quantum gravity). Feynmann discovered the inconsistency of the naive perturbative expansion for Yang–Mills theory but did not manage to resolve this problem. The use of functional integrals makes all computations completely transparent. The main point is to determine the correct symplectic measure on the quotient phase space of the theory (passing to the symplectic quotient accounts for the gauge symmetry of the theory, which is, in fact, its key property). This brings into play a specific regularised determinant of a differential operator, itself represented as an auxiliary Berezin functional integral over anti-commuting variables. The associated extra "non-physical" particles are the famous Faddeev–Popov ghosts that soon became sort of a mascot of the new method.

As the ideas of quantum field theory were spreading over new areas of mathematics (in particular, representation theory and topology), the force and flexibility of the Faddeev–Popov approach were fully confirmed once again. The refined "method of ghosts" developed into a convenient cohomology technique directly connected to supersymmetry concepts (the BRST method).

Automorphic functions and three-dimensional inverse scattering

The work on quantum Yang–Mills theory is probably the best known of Faddeev's results of the late 1960s, although it is by no means the only one. In the aftermath of his fundamental works on perturbation theory for operators with continuous spectrum, Faddeev addressed the spectral theory of the automorphic Laplace operator on the Poincaré upper half-plane (the standard model of the non-Euclidean plane). The key problem that attracted much attention at the time was the famous trace formula found by A. Selberg, which is particularly non-trivial for discrete subgroups with a non-compact fundamental domain. At I.M. Gelfand's initiative, Faddeev applied to this problem the methods he had developed in his study of scattering theory and perturbation theory for operators with continuous spectrum. This resulted in a non-arithmetic proof of the spectral theorem for the automorphic Laplacian, followed by a proof of the Selberg trace formula (in joint work with his PhD students A. Venkov and V. Kalinin). In another development, Faddeev (together with B. Pavlov) explored the non-stationary scattering problem for the automorphic wave equation, which allows the interpretation of the zeros of Riemann's zeta function as quantum mechanical resonances.

Simultaneously, Faddeev obtained a crucial advancement in the three-dimensional inverse problem for the Schrödinger operator. The key difficulty here was to find an adequate substitute for the so-called Volterra transformation operators, which play a prominent role in the treatment of the one-dimensional inverse problem. This was done in a 1965 paper but a complete exposition had to wait for about a decade because of intensive work on other subjects. Although these results are less widely known, Faddeev considered them as his best analytic results.

During the conference "Mathematical Physics. Its Past, Present and Future". Euler Mathematical Institute, St. Petersburg, March 2014.

Classical integrability

Another exciting area was the theory of integrable systems, which was started by the famous paper of Gardner, Greene, Kruskal and Miura (1967) on the Korteweg–de Vries equation. Faddeev learned about these developments a few years later in 1971, during a conference on the inverse scattering problem. As it turned out, the technique he had developed in his PhD thesis had now become directly relevant for the new method. His first major contribution to the new theory was his joint paper with V. Zakharov, which established that the KdV equation is completely integrable in a technical sense.

The ideological importance of this paper was immense: it provided a first ever non-trivial example of an infinite-dimensional, completely integrable system and triggered a complete change of the paradigm in the study of non-linear evolution equations. From the very beginning, Faddeev's interest in the study of these equations was fuelled not by their role as useful models in mechanics or hydrodynamics but rather by their possible application to quantum field theory. The original KdV equation is not quite appropriate in this respect because of its non-relativistic kinematics but very quickly Faddeev came upon a truly exciting example, the now famous Sine–Gordon equation, which he studied together with his young student L. Takhtajan. Soliton solutions for this model may be interpreted as genuine relativistic particles and hence the particle content of the corresponding QFT model is much richer than suggested by naive perturbation theory. The Sine-Gordon equation was the first indication of the important role of classical quasi-particle solutions in quantum field theory and, more generally, of how rich correctly chosen non-linear QFT models could be.

The way to a full justification of these bold predictions proved to be quite long and difficult. By that time, Faddeev had created, at the Leningrad Branch of the Steklov Institute, an independent Laboratory of Mathematical Problems in Physics, which gathered his young students around him. With some pride, Faddeev defined his own role in this small team as that of a playing coach. Many results and key developments of the next two decades were largely due to their collective work. Faddeev's weekly seminar at the Steklov Institute became a focus of research activity in various aspects of integrability, in QFT and in infinite dimensional Lie groups and Lie algebras. This versatile activity and lecture courses that Faddeev delivered in the 1970s, notably at the Summer School in Les Houches, have contributed substantially to the fundamental reshaping of mathematical physics in general, with its new emphasis on interdisciplinary research and the increased role of geometric and algebraic ideas.

The quantum inverse scattering method

Besides the study of various examples of integrable systems, the mid-1970s were also marked by the first attempts to understand quantisation of integrable models in QFT, at first at the semiclassical level. This demanded a good deal of heavy technical work, which was needed to confirm the stability of solitons, contrary to the initial scepticism of theoretical physicists. This work prepared the way for a major breakthrough at the end of the decade, when a new systematic method for solving quantum counterparts of classical integrable systems was created. This was a truly fundamental discovery that united ideas from the classical inverse scattering method, the recent developments in quantum statistical physics (due mainly to R. Baxter) and the old technical insights of quantum mechanics (the Bethe ansatz). The keystone of the new method was the beautiful algebra based on the notion of the "quantum R-matrix". One of the important examples of a quantum R-matrix was extracted from an old paper of C.N. Yang and hence the main algebraic identity satisfied by quantum R-matrices was given the name of quantum Yang–Baxter identity (a name by which it became universally known). Faddeev's programme talk with a sketch of the quantum inverse scattering method was delivered in May 1978; within a year, all his major conjectures were confirmed, with key contributions from Faddeev's pupils and collaborators: E. Sklyanin, L. Takhtajan, P. Kulish and others. One of the highlights of the new method was the solution of the quantum Sine–Gordon model.

The new algebra, focused on the quantum Yang–Baxter identity, soon led to the discovery of new algebraic objects that have subsequently been baptised quantum groups. A first example of a quantised universal enveloping algebra is due to P. Kulish and N. Reshetikhin; further examples and appropriate axiomatics are due to V. Drinfeld. Quantum groups started a new chapter in non-commutative algebra, with numerous applications ranging from knot theory and low-dimensional topology to combinatorics and representation theory. A few years later, Faddeev, together with Reshetikhin and Takhtajan, developed an original approach to the quantisation of Lie groups and Lie algebras based entirely on the use of quantum R-matrices. While the notion of quantum groups has won tremendous popularity, it should be noted that it only formalised the 'easy part' of the quantum inverse scattering method, its true core certainly being

An excursion to the French Alps. Aiguille du Midi, March 2015.

the algebraic and analytic technique used to solve the spectral problem for quantum Hamiltonians. In its simplest version, this was the algebraic Bethe ansatz invented by Faddeev, together with Sklyanin and Takhtajan; it was followed by the more elaborate techniques of the functional Bethe ansatz and quantum separation of variables, introduced by Sklyanin a few years later. Active work in this direction is still going on up to the present day.

Quantum anomalies and the search of knotted solitons

The rapid development of the quantum inverse scattering method had, to a certain extent, pushed aside the four-dimensional physics in the work of Faddeev's laboratory. Still, there are quite a few important results that were obtained in this direction as well. In the 1980s, there was the joint work of Faddeev and S. Shatashvili on quantum anomalies (in particular, the Gauss law anomaly in Yang–Mills theory), which resulted in the discovery of a new, interesting cohomology class and an associated abelian extension of the three-dimensional current group. Faddeev was particularly fond of these results, since they brought to bear, rather unexpectedly, the discoveries in homological algebra of his father Professor D.K. Faddeev from the 1940s.

One more research direction was the search of nontrivial, soliton-like solutions of non-linear equations in three and four dimensions, based on the use of the Hopf invariant. In the 1990s, Faddeev's collaborator A. Niemi confirmed numerically the existence of stable "knotted" solutions of the modified Skyrme model proposed by Faddeev. These solutions play a key role in the description of the hypothetical "glueball" solutions of the Yang–Mills equations related to one of the possible scenarios of quark confinement.

Later years

The decay of the Soviet Union and the deep crisis of the country brought about profound changes in the composition of the Faddeev group. Many of his former students and collaborators were dispersed over various laboratories and universities all over the world. There were also several early losses to deplore, provoked by the stresses of the situation in the 1990s. Those who stayed at the Steklov Institute were spending a good share of their time abroad as well. There were still quite a few gifted students but they too could only find decent jobs abroad. In the early 1990s, the support provided by the Soros Foundation was of great help but gradually it became clear that fundamental research and science in general are by no means a priority of the new Russian authorities.

During these years, Faddeev travelled a lot but his fundamental desire was to stay at home. He declined, in particular, an invitation to head the Institute of Theoretical Physics at Stony Brook after the retirement of C.N. Yang. His constant preoccupation was to save mathematics in Russia, keeping afloat both the Steklov Institute and the Mathematics Division of the Russian Academy of Sciences. Over the years, this task was getting more and more painful, causing much distress and disillusionment. He largely returned to a more solitary style of work characteristic of his younger years, in contrast to the team style of the 1980s. Still, some new fruitful collaborations emerged during this period, along with quite a few old ones. Among his important discoveries of this period, one should mention the concept of modular duality for quantum groups. This concept, which emerged from the study of integrable quantum models in discrete space-time, opened a totally new and very promising chapter in representation theory of quantum groups. While early work focused mainly on the highest weight representations of quantum groups, Faddeev's work started the study of principal series representations, which proved to be extremely rich in various interdisciplinary connections, with links to non-commutative geometry, finite difference operators, new classes of special functions, etc. Research in this area is now actively pursued by Faddeev's pupils.

Faddeev's work in mathematics and physics won him wide international recognition. He has been awarded many prestigious awards, among them the Dirac Medal (1995), the Max Planck Medal (1996), the Euler Medal (2002), the Henri Poincaré Prize (2006), the Shaw Prize (jointly with V. Arnold, 2008) and the Lomonosov Medal (2014). He was elected to leading academies including the Royal Academy of Sweden (1989), the National Academy of the USA (1990), the French Academy of Sciences (2000) and the Royal Society (2010). Since 1976, he has been a full member of the Soviet (now, Russian) Academy of Sciences. In 1986–1990, he served as President of the International Mathematical Union.

Faddeev's legacy retains all its importance for current research as well as for the future of mathematical physics. One striking example of this is given by the recent discovery of unexpected links between Yang–Mills theory and the quantum inverse scattering method. In the early years of gauge fields theory, there existed a somewhat romantic hope that Yang–Mills theory itself was

Steklov Institute, St. Petersburg, September 2010. From left to right: I. Aref'eva, S. Shatashvili, P. Kulish, L. Faddeev, L. Takhtajan, A. Volkov.

At a workshop in Chamonix, March 2015. From left to right: L. Faddeev, A. Volkov, L. Takhtajan, A. Alexeev, M. Semenov-Tian-Shansky.

integrable. This proved to be false but one of its versions, supersymmetric Yang–Mills theory, is indeed close to integrability or exact solvability. As discovered recently by Shatashvili and Nekrasov, the description of the vacuum sector in supersymmetric Yang–Mills theory (in dimension 4) directly leads to quantum integrable systems (both of standard and of new types). All the main ingredients of the quantum inverse scattering method are naturally incorporated into this new approach. This fascinating link between the seemingly very remote aspects of Faddeev's legacy is a spectacular confirmation of its depth and vitality. Our feelings now may be expressed by the line of an old Roman poet: *letum non omnia finit*. Faddeev's works and ideas remain a source of inspiration for all of us and are destined for a long and fruitful life in posterity.

Irina Aref'eva has worked in the Laboratory of Mathematical Physics of the Leningrad Branch of Steklov Mathematical Institute since 1970. Since 1977 she is a scientific researcher and later Professor at the Departement of Theoretical Physics of the Steklov Mathematical Institute in Moscow. Her research areas are non-perturbative approaches in Quantum Field Theory and Superstring theories.

Michael Semenov-Tian-Shansky has worked in the Laboratory of Mathematical Problems of Physics of the Leningrad (now, St.Petersburg) Branch of Steklov Institute since 1972. He heads this Laboratory since 2016. Since 1993 he was Professor at the Université de Bourgogne (since 2016, Professor Emeritus). His research area is Representation Theory of Lie Groups and the Theory of Classical and Quantum Integrable Systems.

Leon Takhtajan has worked in the Laboratory of Mathematical Problems of Physics of the Steklov Institute since 1973. Since 1992 he is Professor of Mathematics at Stony Brook University, Stony Brook, NY, USA. He is also working at the Euler International Mathematical Institute. His research area is the Theory of Classical and Quantum Integrable Systems with application to complex and algebraic analysis.

Mathematics AND Music?[1,2]

Personal Views on a Difficult Relationship

Christian Krattenthaler (Universität Wien, Austria)[3]

Preamble

⟨*Robert Schumann (1810–1856): "Aveu" from Carnaval op. 9*⟩ [4]

Mathematics and music – stress on "AND" – question mark, this is our topic today. In order to enter the subject: when I am involved in a conversation, and the person with whom I am talking discovers that, on the one hand, I am professor of mathematics at the University of Vienna and, on the other hand, have been a concert pianist in a previous life, then it happens frequently that this person spontaneously exclaims:

"Mathematics and Music – they are so close to each other!"

To which I reply:

"Is that really so?"

What do I want to say? Frankly, I have always had big troubles with the topic "Mathematics AND Music", namely when mathematics and music are brought together, are set in relation to each other, or when one merely searches for connections between them. Yes, it is correct, tones and intervals obey strict mathematical rules, due to laws of physics; but does this establish a connection between mathematics and music? Yes, it is also true that Johann Sebastian Bach frequently wove numbers into his compositions.[5] But is this mathematics? It is also correct that compositions are often built in rather complex ways, that they have complicated forms. But is this mathematics in music? Conversely, if mathematics – here I mean structure – becomes too dominant in music, as for example in serialism, where all parameters – pitch, rhythm, volume, etc. – are subject to strict rules, is the result still music?

Without further ado, I confess: I cannot see any direct, substantial connections between mathematics and music. In particular, I never have understood what mathematics has to do with, say, that touching confession, declaration of love[6] from Robert Schumann – I suppose dedicated to his beloved Clara –, which I played at the beginning. If you had come to hear my answer to the question of the title of my talk: here it is! You could then safely go home. Of course, that would be too cheap, and, moreover, we would not have addressed a further question.

Let me rst take a step back. Not very long ago, a prominent visitor said to the Dean of the Fakultät für Mathematik of the University of Vienna:

"I hear that you are chairing a department of pianists!"

What did this visitor want to say? If you go through the list of members of the Fakultät für Mathematik – myself among them[7] –, then it is indeed remarkable how many of them are enthusiastic pianists. (The dean I refer to is one of them, by the way.) Moreover, there are others who play other instruments, there are those who are passionate choir singers, and there are others who do not play an instrument or sing but instead are devoted opera- and

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1 This is the English translation of the (slightly extended) script of a talk-performance that the author gave on May 16, 2013 in the math.space in the museums' quarter in Vienna. Since the author's performances of the piano pieces cannot be reproduced on printed paper, for each piece he provides a hint for an excellent performance.

2 The German original appeared in *Int. Math. Nachr.* 224 (2013), 29–60. The English translation appears here with the kind permission of the Austrian Mathematical Society.

3 I am deeply indebted to Theresia Eisenkölbl, who designed the computer presentation for this talk, parts of which have been incorporated into this article. I also thank Reinhard Winkler, for a careful reading of a rst version of the manuscript, and for numerous corrections and insightful comments. Last, but not least, I am extremely grateful to Tomack Gilmore for signicant and essential help with the English translation of the German original.

4 I did not find anything on YouTube which really convinces me. Tal-Haim Samnon's performance (http://www.youtube.com/watch?v=EN2gUDaHqvo) matches the character, but drawls sometimes too much.

5 It is well documented, for instance (see, for example, Ludwig Prautzsch, *Die verborgene Symbolsprache Johann Sebastian Bachs*, Band 1: Zeichen- und Zahlenalphabet der kirchenmusikalischen Werke. Merseburger, Kassel 2004), that Bach put numbers of psalm verses into his passions, at the places where these are cited. However, this remains concealed from a listener since this cannot be "extracted" by just listening; it can only be discovered and substantiated through a careful study of the score. This was an extraneous task that Bach chose to take upon himself.
The number that plays the biggest role in Bach's work is the number 14. In a sense, it is Bach's signature mark (in the same way as painters sign their paintings by putting their signature marks on them). In order to understand this, one has to observe that the number 14 is the sum of the positions of the letters B, A, C, and H in our alphabet (to be precise, 2+1+3+8=14). To mention an example, the number of pieces in the "Musicalisches Opfer" ("Musical Offering") is exactly 14 (if one counts correctly, of course, as one of the canons can be performed in two different ways).

6 "Confession", "declaration of love" – this is the meaning of the French word "aveu".

7 I was not dean then…

Figure 1. Gustav Mahler, Dmitry Shostakovich, Arnold Schönberg.

Figure 2. Andrew Wiles, Grigori Perelman.

concert goers. In other words, the proportion of members of the Fakultät who have a great affinity for music is much higher than average. The same holds if one looks at other mathematics departments.

On the other hand, it is also surprising to see how many musicians also have an affinity for mathematics. A prominent example is the young pianist and composer Kit Armstrong, who, as is well known, studied with Alfred Brendel in London, but, on the side, also completed a mathematics degree at the Université "Pierre et Marie Curie" in Paris.

Hence, the question that presents itself at this point is:

> *"Why are there so many mathematicians who also have a strong affinity for music, and why are there so many musicians who also have a strong affinity for mathematics?"*

On a superficial level, we could phrase this question as follows:

> *"How do we imagine the typical mathematician – I mean, the typical sharp thinker, the typical intellectual?"*

I would say that the portraits in Figure 1 match this image perfectly. You agree, don't you? We can cross-check:

> *"How do we imagine the typical musician – I mean, the typical sensitive artist?"*

Exactly like the portraits in Figure 2, right?

For those who are not so familiar with the names "Wiles" and "Perelman", I should perhaps explain: Andrew Wiles, a British mathematician, is famous for having solved a 300 year old problem that goes by the name of "Fermat's Last Theorem". We shall hear more about this later. On the other hand, Grigori Perelman, a Russian – very eccentric – mathematician, is famous for the proof of a 100 year old conjecture of Henri Poincaré on four-dimensional geometry.

Before we attempt to answer the above question, we should perhaps first make precise what we are talking about. I am a mathematician, and in mathematics all objects must first be precisely defined before one can talk about them. So, what is the definition of mathematics, what is the definition of music?

> *Music is … arises … comes about, when tones are produced … when tones and noise are produced (I must not forget noise!). So, if these tones and noise sound, together …*

I am sorry, I see that this does not really work. Let us try something easier! Mathematics – this is very simple:

> *Mathematics is … art of calculation. Mathematics deals with numbers, … geometric objects, … more abstract objects – such as for instance algebraic structures and such – and …*

No, no, this makes no sense!

Actually, what I am doing here is completely stupid. Today one no longer racks one's brains, today one has Wikipedia! So, what does Wikipedia say about music?[8]

> *Music is an art form and cultural activity whose medium is sound and silence, which exist in time. The common elements of music are pitch (which governs melody and harmony), rhythm (and its associated concepts tempo, meter, and articulation), dynamics (loudness and softness), and the sonic qualities of timbre and texture (which are sometimes termed the "color" of a musical sound).*

Well … I would say: not completely wrong … But I don't think that this is convincing. What does Wikipedia say about mathematics?[9]

> *Mathematics is the study of topics such as quantity (numbers), structure, space, and change.*

Is *this* really mathematics?

What do I want to prove via this somewhat clumsy exercise? Of course, it is impossible to precisely say, to precisely define what music is, and it is equally impossible to precisely define what mathematics is (even if this may seem a little strange to the mathematical layman). Very good!

Nevertheless, I can precisely say what I mean when I talk here of music, when I talk here of mathematics. When I talk here of music, then I mean the *art form* music; art wants to express something, music wants to convey something to the listener with the help of tones and noise, it wants to give something to the audience to

[8] http://en.wikipedia.org, as of 12 November 2016.
[9] http://en.wikipedia.org, as of 12 November 2016.

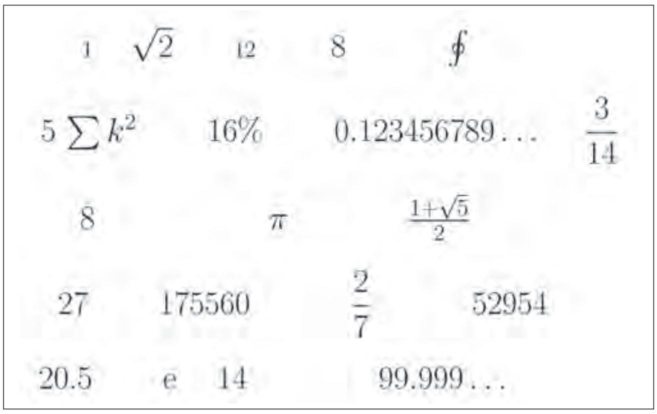

Figure 3.

take home. In order to make this absolutely clear: when I randomly press a few keys of a piano and then maybe bang the lid, then these were a few tones and one noise. This was *not* music; it did not say anything, and it did not want to say anything.

When I talk here of mathematics, I mean the *science* of mathematics; that is, we are talking of discovering new grounds, of solving mathematical problems, of investigating and studying mathematical phenomena, and of revealing the structure and connections lurking behind. In order to completely clarify this point: when I randomly type mathematical numbers and symbols on the page (like in Figure 3), this is *not* mathematics.

I can now precisely explain my difficulties with the topic "Mathematics AND Music". When Bach weaves numbers into his compositions, then these are numbers, this is not mathematics. Moreover, these numbers do nothing for the message of the work as it is transmitted to the audience. When compositions take on complex forms, then, from the point of view of the science of mathematics, this is either trivial or completely without interest. When mathematics – structure – starts to dominate music – when, in the extreme case, we program a computer to produce ("compose") tones and then eagerly await the result, then out will come *tones* but *no music*. This will *convey nothing*. What music shall do for mathematics, is entirely unclear anyway.[10] Thus for me the interesting question is not that of the connections between mathematics and music, but rather:

> *"Why are there so many mathematicians who also have a strong affinity for music, and why are there so many musicians who also have a strong affinity for mathematics?"*

To give it away, the thesis which I shall defend here is:

> *Both Mathematics AND Music are food for the soul AND the brain.*

Maybe there is a region in our brain which resonates – responds to – particularly when emotion and intellect come together, form a symbiosis. Maybe this provides the explanation for the phenomenon which is touched upon in the above question. In the following, I shall attempt to substantiate this thesis.

Soul in music

You will say: "This is like carrying coals to Newcastle! Of course, emotion plays an enormously important role in music." You are obviously completely right. Nevertheless, I want to say a few words about this, because not only can it have many different facets, but also it gives me the opportunity to play the piano a little…

You remember: music wants to express something, wants to transmit something to the audience. This may be many different things. For example, music may simply spread good cheer…

> ⟨*Scott Joplin (1867/1868?–1917): Maple Leaf Rag (beginning)*⟩[11]

or bad…

> ⟨*Robert Schumann (1810–1856): Pantalon et Colombine (beginning) from Carnaval op. 9*⟩[12]

Music can be heartbreakingly sad…

> ⟨*Franz Schubert (1797–1828): Andantino (beginning) from the Sonata in A major, D 959*⟩[13]

or transcendentally joyful…

[10] If one ignores that the reconstruction and analysis of sound documents pose very interesting and challenging mathematical problems; see for example: A. Boggess and F. Narcowich, *A first course in wavelets with Fourier analysis*, second edition, John Wiley & Sons, Inc., 2009. However, also here we are not talking of a true substantial relation or connection between mathematics and music: the substance lies entirely on the side of mathematics, music *as an art form* is not affected here.
In this context, one could also think of some colleagues who have apparently better ideas if they have music playing on the side. I do not belong to these: bad music is annoying, and good music – it enthrals me, I have to listen to it, I can't think about mathematics at the same time. In any case, this leads us somewhat off-course…
Closest to a true connection between mathematics and music is research as done, for example, by Gerhard Widmer (even if his work rather belongs to Artificial Intelligence; see http://www.cp.jku.at/people/widmer/), who, with the help of mathematical models, investigates specialities and peculiarities of interpretations of pianists, or, on the other hand, tries to "teach" computers how to "interpret" musical scores agogically – varying the tempo as the piece moves along – "correctly" – on a piano. He is however well aware of the limits of such studies and experiments, even if it is not clear where these lie exactly.

[11] Absolutely worth listening to is the pianola roll recording played by Scott Joplin himself: http://www.youtube.com/watch?v=pMAtL7n_-rc.

[12] Arturo Benedetti Michelangeli knows how to perfectly portray a squabbling couple (who then reconcile, only to begin squabbling again, etc.): http: //www.youtube.com/watch?v=LgpDYQcmZB4.

[13] The "measure of all things" concerning Franz Schubert's work for piano is, without any doubt, Alfred Brendel: http://www.youtube.com/watch?v=Il6-lZYDpqY.

⟨*Franz Schubert (1797–1828): Impromptu A flat major, D 899, Nr. 4 (end)*⟩[14]

Music may radiate elegance, and what better accomplishes this than a waltz by Chopin?

⟨*Frédéric Chopin (1810–1849): Grande Valse Brillante E flat major, op. 18 (beginning)*⟩[15]

We come to *humour in music*. This is an entire topic in itself. The grand master of humour in music was without any doubt Joseph Haydn. All of you know his most famous joke: that sudden fortissimo chord from the entire orchestra in his "Surprise Symphony". There, as you all will recall, the second movement begins with the most trivial theme that one can imagine, and, as if that were not enough, this theme is repeated! It is understandable that one nods off a bit at this point, before, suddenly, the orchestra roars off completely without warning. Today, we are used to quite a bit, however at the time the effect was certainly enormous … I want to draw your attention to a little detail that is not that obvious at first sight. Joseph Haydn grew up in deepest Lower Austria, later living in Vienna and in Eisenstadt in Burgenland.[16] This joke, however, represents[17] typical *British* humour: it is told "with a straight face". After that fortissimo chord, one constantly – and nervously – awaits further consequences as the movement continues (for instance, in the form of further shock effects…). But, no: nothing happens at all, the music continues as if nothing had occurred…

Normally, however, humour in music is of a finer nature. Usually, the expectations of the listener are led astray, and it is in this manner that humourous effects are produced. A nice example for this is the first of the Humoresken by Max Reger. This piece has quite a graceful main theme, which however cannot develop as it would like. This main theme dominates two short sections at the beginning and at the end that frame a middle section, which considers itself as slightly too important, and thereby also creates an amusing impression.

⟨*Max Reger (1873–1916): Humoreske D major, op. 20/1*⟩[18]

I have a final point to offer: *Tour de Force*! I think you know: the thunderous hammering of keys in the Liszt Sonata, for example…

⟨*Franz Liszt (1811–1886): Sonata b minor (excerpt)* ⟩[19]

If you listened attentively then you will have observed that I studiously avoided one word in particular: the word "*nice*" (as well as the word "*beautiful*"). On this point, I shall digress a little.

Not long ago, I attended a performance of the opera "Mathis der Maler" by Paul Hindemith. The opera is finished, the applause has ceased and then I hear one person saying to her neighbour: "Very nice!" I was quite taken aback. What was that? One must know that "Mathis der Maler" is set during the peasants' wars in Germany. This was a very dark epoch. The peasants revolted against the abuses that were visited upon them by their landlords, and the latter crushed this revolt mercilessly. During the opera, one of the leaders of the peasants' movement is cruelly slaughtered openly on-stage. At the heart of the story lies the conflict of conscience of the artist Mathis over how to behave during these times. Should he continue to work on his canvases and sculptures, or should he "engage himself in politics"? Finally, he joins the peasants' revolt and, of course, achieves nothing. At the end of the opera, a voice announces that the artist should stick to his art, but this is not really convincing. Clearly, Paul Hindemith projects his own personal conflict over how to behave as an artist in the face of the Nazi regime into this opera. The music reflects all of this. It is disturbing, very intelligent, but one cannot characterise it as "nice." Bluntly: there are very few pure major chords in this opera…

I want to drive this particular point home:

*Music **does not want** to be nice!*

What I mean is: music wants to say something, wants to express something to a listener. This *may* be accompanied by beauty, but then beauty *is not an end in itself*, it is always a *means to an end*. But it *need not* be accompanied by beauty. "Sacre de Printemps" by Igor Stravinsky is eruptive, explosive, but it is not "nice" or "beautiful". The last movement of the "Great Sonata for the Hammerklavier" in B flat major, op. 106, by Ludwig van Beethoven, the movement containing the fugue, is many things – grand, bold, unprecedented –, but it is certainly not "nice" or "beautiful". In fact, one has to wait for a hundred years until again a piece is written which contains similar harmonic abrasiveness. Even in the work of Johann Sebastian Bach one cannot call many of his compositions "nice", since frequently consistent progression of voices is more important than "nicely sounding" harmonies. When saying this, I have in mind some of the canons in the "Goldberg Variations", each of which has its distinctive character, but which are not always "nice".

Hence, when, after a performance of "Mathis der Maler", I hear: "Very nice!", then I am strongly reminded of the celebrated standard phrase of the "alte Kaiser"

[14] Alfred Brendel: http://www.youtube.com/watch?v=V0z7mUV5rSc

[15] Inimitable in his elegant, natural style of playing is Arthur Rubinstein: http://www.youtube.com/watch?v=laSh3D_77ZM, even if he does not take "brillante" too strictly…

[16] A rural region of Austria to the south-east of Vienna.

[17] It is fitting that that symphony is one that Haydn wrote for London.

[18] Marc-André Hamelin does quite well in http://www.youtube.com/watch?v=ba5js057WGM.

[19] I like a recording from the Salzburg Festival, of which I possess a CD and in which Emil Gilels plays extraordinarily. On YouTube there exists a recording in three parts that is not quite as good: http://www.youtube.com/watch?v=7yhGSrn3idI, http://www.youtube.com/watch?v=gyQ-MnjRvsE, http://www.youtube.com/watch?v=EKUAFRosm48.

Franz Joseph,[20] who used to apply it whenever he was confronted with cultural intrusions:

> *"Es war sehr schön, es hat mich sehr gefreut!"*[21]

For somebody, who apparently did not have any affinity for culture, this was seemingly the best he could say about it…

Let us return to the actual subject of this essay.

Soul in mathematics

For non-mathematicians, this will look like a pretty difficult topic. After all, we have all learned in high school that mathematics is a dust-dry, abstract matter, which is about applying recipes that have been known for centuries to more or less intelligent exercises, and hoping that one has selected the correct recipe… (For the vindication of my mathematics teacher, I must say that I did not learn *this* in high school.) Anyhow, I believe that on the topic of "soul in mathematics" we should hand over to the earlier mentioned Andrew Wiles.

As I have already said, Wiles is famous for having proved "Fermat's Last Theorem." The statement of this theorem can be understood by any high school student, and I shall therefore present it here.

Theorem (Wiles, Taylor 1995). (*Fermat's Last Theorem*) *Let n be a natural number which is at least 3. Then there are no natural numbers*[22] *x; y; z such that*

$$x^n + y^n = z^n.$$

Pierre de Fermat scribbled this assertion over 300 years ago into the margin of a page of an exemplary of Diophantos' book "Arithmetica".[23] In order to increase the suspense, he also added that he has found a "truly wonderful" proof of this, but that the page margin was not wide enough to hold this proof. Since then, many very clever people racked their brains about this problem. As a matter of fact, much of number theory ignited itself on exactly this problem. However, for over 300 years nobody could find a proof of Fermat's assertion. We may therefore safely assume that Fermat did not really have a proof, in any case not something that we would accept as a proof nowadays. It was a big sensation when Andrew Wiles announced at the end of a series of lectures that he gave at the Isaac Newton Institute in Cambridge in 1993 that he had found a proof. Now, in mathematics it is not sufficient to just announce that one has found the proof of a theorem (as Fermat did). The proof must be written down in order to let others read and check it – Wiles did that; the result was an article of 200 pages, which itself was based on previous work by numerous other authors –, and the writeup must be submitted to a scientific journal for publication – Wiles also did that –, after which referees carefully verify this proof. During this process, it was discovered after a short while that Wiles' proof contained a gap that he was unable to fill. It needed another two years until Wiles, in joint work with his former student Richard Taylor, succeeded in repairing this hole. In a BBC documentary,[24] Andrew Wiles says the following about the moment when he realised that now all difficulties are overcome:

> [Wiles is visibly deeply moved and speaks haltingly]
> *When I was sitting here, at this desk – it was a Monday morning, September 19 – and I was trying convincing myself that it did not work, seeing exactly what the problem was, when suddenly, totally unexpectedly, I had this incredible revelation. I realised [that] what was holding me up was exactly what would resolve the problem that I had in my Iwasawa theory attempt three years earlier.*
> *It was–. it was the most – the most important moment of my working life …* [At this point, Wiles is finally no longer able to continue; the scene is faded out.]
> *It was so indescribably beautiful, it was so simple and so elegant … – and I just stared in disbelief for 20 minutes …. – then during the day I walked to our department, I keep coming back to my desk, looking to see, it was still there, it was still there…*

Impressive, isn't it? Contrary to widespread perception, mathematics seems to be a highly emotional activity. I noticed various emotions, including everything from "heartbreakingly sad" – at the point when the construction of the proof was in danger of collapsing – up to "transcendently joyful" – at the point when Wiles realised that he has now mastered all difficulties. You may argue that Wiles is so moved because it was him who first solved this famous problem. This is certainly a component. However, it falls short of the full truth. Wiles also says: "This was so indescribably beautiful, so elegant!". Mathematics must have other qualities than just being "dust-dry" and "abstract". We should hence discuss some of these qualities in greater depth.

As I have already said, once a mathematician has proved a fantastic theorem, then this proof must be written down and be submitted for publication, whereupon the corresponding article is examined. The referees do

[20] Franz Joseph I. (1830–1916), Emperor of Austria 1848–1916.

[21] It was very nice! I enjoyed it very much!

[22] In order to avoid any misunderstanding, when I speak of "natural" numbers, I mean the numbers 1, 2, 3,…, which corresponds to the original meaning of the word "natural". Nowadays, unfortunately, one learns in school that the "natural numbers" consist of the numbers 0, 1, 2,… This may indeed be handy in some situations but it is simply a perversion of the word "natural", since 0 is without any doubt not a *natural* number.

[23] The background/context of this assertion is the sharp contrast to the situation for $n = 2$: in that case, there are infinitely many solutions to the equation $x^2 + y^2 = z^2$ in the natural numbers x, y, z, which can be precisely characterised and which are known as "Pythagorean triples". Two of these we know from high school: $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$.

[24] The complete documentary can be seen at http://www.youtube.com/watch?v=7FnXgprKgSE. The cited passage appears roughly 5 minutes before the end. The very beginning of the documentary is also remarkable…

not only judge correctness of proofs but also the other qualities of the article. A standard phrase that a referee might use to show that they like the article is:

"This is a very nice paper!"

In view of the previous digression on "beauty" of music: funny, isn't it? Mathematicians also don't know anything better than saying "nice"… However, if the referee provides a sound opinion then they would also tell more specifically what they like about the article. Then we may sometimes read:

"This is a very elegant proof!"

What is an "elegant proof"? In other words, what is a "mathematical waltz by Chopin"? Usually, we are talking about the situation where – in a proof – the mathematician is facing a seemingly insurmountable obstacle. With the help of a relatively simple, but not at all obvious, idea, the mathematician succeeds however to – elegantly – circumnavigate this obstacle. I shall try to give an example, the theorem, known to everybody, that there are infinitely many prime numbers.

Theorem. *There are infinitely many prime numbers.*

Proof. If one looks at this assertion, what would we have to do in order to prove it? It seems that we would have to construct infinitely many primes. We would do even better if we could find a formula which gives us all prime numbers (or at least infinitely many). This is pretty hopeless.[25]

However, there is an – elegant – way around this. Let us suppose that there are only finitely many prime numbers. If, under this assumption, we succeed in deriving a contradiction, then our original assumption must have been wrong. Thus, we would have shown that there are indeed infinitely many prime numbers.

So, let us suppose that there are only finitely many prime numbers; say, $2, 3, 5, 7, 11, 13, \dots, 1031$.

We now consider

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \dots \cdot 1031 + 1:$$

This (huge) number can be decomposed into a product of prime factors. Each of these prime factors must, on the one hand, divide this number and, on the other hand, must appear among the prime numbers $2, 3, 5, 7, 11, 13, \dots$, 1031. (Remember that we assumed that these are *all* the prime numbers!) Let p be such a prime factor. p cannot equal 2 since the above number is visibly an odd number. But p can also not equal 3 since 3 does not divide a number of the form $3X + 1$, of which the above number is an example. For an analogous reason, the prime factor cannot equal 5,…, and it cannot equal 1031. Hence, this *cannot* have been all the prime numbers. □

Now you will object: "This is all fine, however this is not a rigorous – valid in generality – mathematical proof." After all, 1031 is just one *special* prime number. You are right, but the rigorous proof looks exactly the same. The only thing that needs to be done is to replace 2, 3, 5, …, 1031 by symbols: $p_1, p_2, p_3, \dots, p_n$.

Proof. Let us suppose that there are only finitely many prime numbers, say, $p_1, p_2, p_3, p_4, p_5, p_6, \dots, p_n$.

We now consider

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6 \cdot \dots \cdot p_n + 1:$$

This (huge) number can be decomposed into a product of prime factors. Each of these prime factors must, on the one hand, divide this number and, on the other hand, must appear among the prime numbers p_1, p_2, … , p_n. (Remember that we assumed that these are all the prime numbers!) Let p be such a prime factor. p cannot equal p_1 since p_1 does not divide the number above. But p can also not equal p_2 since p_2 does not divide the above number. For the same reason, the prime factor cannot equal p_3, …, and p_n. Hence, this *cannot* have been all the prime numbers. □

We come to *Humour in Mathematics.* Can there really be humour, or indeed jokes, in mathematics? Well, this must be the case, since sometimes we may read in a reviewer's report:

"This is a funny construction!"

How does humour appear in mathematics? Humour in mathematics is – as in music – normally of a finer nature. Also here, the expectations of the reader of a proof are led astray before, suddenly, a little detail surfaces, which we had not noticed earlier, but it is exactly this little detail which is the last (but decisive!) little brick that is needed to complete the argument. At this point, a mathematician must smile (how could (s)he have overlooked this?), and it delights her/his soul.

I shall try once more to give an example, this time extracted from the work of the celebrated Indian mathematician Srinivasa Ramanujan (see Figure 4). Born in 1887 in the vicinity of Madras (today Chennai), Ramanujan had a very modest upbringing. He had only a basic school education, but had always been interested in mathematics and mathematical problems. After finishing school, he worked as a clerk in the Madras Port trust, but in his leisure time

[25] Mathematics offers plenty of incredibly fascinating, respectively absurd, facts – depending on which point of view one is willing to take… the Russian mathematician Yuri Matiyasevich proved that there exist polynomials in several variables whose *positive* values – if the variables are specialised to concrete natural numbers – run through all prime numbers; see *Dokl. Akad. Nauk SSSR* 196 (1971), 770–773; *Soviet Math.\ Dokl*. 12 (1971), 249–254. Such polynomials have indeed been constructed explicitly. Not only do they have the "annoying" property that they attain (some) *negative* values, but in particular this property is satisfied *most of the time…* Hence, they are today just a curiosity, since, aside from their existence, they do not seem to be good for anything.

Figure 4. Srinivasa Ramanujan, Godfrey Harold Hardy.

he constantly worked on mathematical problems. At the age of 25, he sent his mathematical results to eminent mathematicians of the time. One of them, Godfrey Harold Hardy, Professor at the University of Cambridge, indeed read Ramanujan's letter and recognised the genius of the unknown author. He invited Ramanujan to come to Cambridge, and to study and work with him. Benefactors in India succeeded in collecting the money necessary to finance the journey to England, thus Ramanujan spent some years at the University of Cambridge. During this time, he wrote several very famous articles, often in collaboration with Hardy. Unfortunately, Ramanujan could not bear the British climate (as well as British nutrition…) and was frequently ill; within a year of returning to India he passed away at the age of only 32 years.

One of the objects that were very dear to Ramanujan in his mathematical work was (integer) partitions. A *partition* of a number n is the representation of this number as a sum of other natural numbers, where the summands are arranged in (weakly) increasing order. For $n = 1$, there is exactly one such representation, namely,

$$1$$

For $n = 2$, there are two, namely,

$$2, \quad 1+1,$$

For $n = 3$, there are three partitions,

$$3, \quad 1+2, \quad 1+1+1,$$

For $n = 4$, we already have five,

$$4, \quad 1+3, \quad 2+2, \quad 1+1+2, \quad 1+1+1+1,$$

and, for $n = 5$, we have

$$5, \quad 1+4, \quad 2+3, \quad 1+1+3, \quad 1+2+2, \quad 1+1+1+2, \\ 1+1+1+1+1.$$

Let $p(n)$ denote the number of partitions of n. Percy Alexander MacMahon, Major of the British army and at the same time an eminent mathematician then, had calculated the numbers $p(n)$ up to $n = 200$.[26] The first few numbers $p(n)$ are shown here:

$p(1) = 1$	$p(2) = 2$,	$p(3) = 3$,	$\mathbf{p(4) = 5}$,	$p(5) = 7$
$p(6) = 11$	$p(7) = 15$,	$p(8) = 22$,	$\mathbf{p(9) = 30}$,	$p(10) = 42$
$p(11) = 56$	$p(12) = 77$,	$p(13) = 101$,	$\mathbf{p(14) = 135}$,	$p(15) = 176$
$p(16) = 231$	$p(17) = 297$,	$p(18) = 385$,	$\mathbf{p(19) = 490}$,	$p(20) = 627$

Ramanujan studied MacMahon's tables intensively, and made remarkable observations. One of these is made explicit in the theorem below. It says that every fifth partition number is divisible by 5; see the bold face entries in the above table.[27]

Theorem ("*Ramanujan's most beautiful theorem*" 1919). *$p(5n + 4)$ is always divisible by 5.*

This theorem acquired the byname "Ramanujan's most beautiful theorem",[28] since it is so simple and elegant to formulate, and at the same time it is so unexpected. Moreover, Ramanujan himself found a proof for it. Here, I want to discuss a proof taken from a paper of Hirschhorn and Hunt,[29] which is very much in the spirit of Ramanujan. I am fully aware that the following is (mathematically) more demanding than everything else we have discussed so far. If you should not understand everything (or understand almost nothing…), this is fine. My point here is to indicate what "humour" may mean in mathematics.

The proof is based on an old result of Leonhard Euler. It says that the power series, in which the numbers $p(n)$ appear as coefficients, can be written in terms of an infinite product.

Theorem (Euler). *We have*

$$1 + p(1)q + p(2)q^2 + p(3)q^3 + p(4)q^4 + \cdots \\ = \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}.$$

Proof. It is quite possible that you feel uneasy when looking at these infinite sums and products. You may ask: "Does this really converge?"[30] But this is the wrong question! The above expressions should be regarded as *formal* expressions, which are added, multiplied, etc., naively.[31]

Let us adopt this formal point of view. Then, Euler's formula can be proved in the following way. The product on the right side consists throughout of factors of the

[26] And he did this without making a single mistake! Even if he did not accomplish this by listing all partitions of numbers up to 200, but rather by using a recurrence relation due to Euler, this constitutes – at a time that knew of no "computing machines" except paper and pencil – an extraordinary achievement!

[27] Ramanujan made similar observations for the prime numbers 7 and 11. Together with the theorem discussed in the text, these founded the research area of "partition congruences", which has witnessed important breakthroughs during the past few years; see page 1525 in the survey article "Srinivasa Ramanujan: Going Strong at 125, Part I", that appeared in the *Notices of the American Mathematical Society*, vol. 59, Nr. 11, 2012, edited by Krishnaswami Alladi, and is available at http://www.ams.org/notices/201211/rtx121101522p.pdf.

[28] Strictly speaking, it is the identity (*) below, which Hardy selected as "Ramanujan's most beautiful identity."

[29] *J. reine angew. Math.* 326 (1981), 1–17.

[30] It does for $|q| < 1$.

[31] All this can be made rigorous by the theory of so-called *formal power series.*

form $\frac{1}{1-q^k}$. In high school, we learned that the infinite geometric series can be summed:[32]

$$1 + Q + Q^2 + Q^3 + Q^4 + \cdots = \frac{1}{1-Q}.$$

We may apply this summation formula to each of the factors (so-to-speak: reading it backwards):

$$\begin{aligned}&\frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}\\ &\quad= \frac{1}{1-q}\cdot\frac{1}{1-q^2}\cdot\frac{1}{1-q^3}\cdot\frac{1}{1-q^4}\cdots\\ &\quad= (1+q^1+q^{1+1}+q^{1+1+1}+\cdots)\\ &\qquad\cdot(1+q^2+q^{2+2}+q^{2+2+2}+\cdots)\\ &\qquad\quad\cdot(1+q^3+q^{3+3}+q^{3+3+3}+\cdots)\cdot\cdots.\end{aligned}$$

Now we must imagine what happens if we expand this last product. Each term in the result arises by selecting one term from each factor, and by multiplying these terms. For example, if we select the term q^{1+1} from the first factor, the term q^{2+2+2} from the second, the term q^3 from the third factor and the term 1 from all remaining factors, then we obtain

$$q^{1+1+2+2+2+3}$$

upon multiplication of these terms. Now it costs just a few moments to convince oneself that the exponents of the expressions one obtains in this manner run through all partitions. Thus, the above product is indeed equal to the left-hand side of Euler's theorem.. □

We are now in the position to embark on the proof of Ramanujan's "most beautiful theorem".

Proof of Ramanujan's most beautiful theorem. In order to have a compact notation,[33] we abbreviate the product $(1-q)(1-q^2)(1-q^3)(1-q^4)$ by $(q;q)_\infty$. More generally, we write

$$(\alpha;q)_\infty = (1-\alpha)(1-\alpha q)(1-\alpha q^2)(1-\alpha q^3)\cdots.$$

The proof is based on several auxiliary results. These auxiliary results can be derived by means of elementary (but tricky) manipulations of power series and by the use of *Jacobi's triple product formula*

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}x^n = (q;q)_\infty\,(x;q)_\infty\,(q/x;q)_\infty.$$

It would however go definitely beyond the scope of this discussion to explain this in detail here.

Lemma. Let $\omega^5 = 1$, $\omega \neq 1$. *Then*

$$(q;q)_\infty\,(\omega q;\omega q)_\infty\,(\omega^2 q;\omega^2 q)_\infty\,(\omega^3 q;\omega^3 q)_\infty\,(\omega^4 q;\omega^4 q)_\infty = \frac{(q^5;q^5)^6_\infty}{(q^{25};q^{25})_\infty}.$$

This lemma entails two further lemmas.

Lemma. *We have*

$$\frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty} = q^{-1}R - 1 - qR^{-1},$$

where R is a power series in q^5.[34]

Lemma. *We have*

$$q^{-5}R^5 - 11 - q^5R^{-5} = \frac{(q^5;q^5)^6_\infty}{q^5(q^{25};q^{25})^6_\infty}.$$

Now, we may combine these lemmas[35] in order to find the following expression for the so-called "generating function" for the partition numbers:

$$\begin{aligned}&1 + p(1)q + p(2)q^2 + p(3)q^3 + \boldsymbol{p(4)q^4} + p(5)q^5\\ &+p(6)q^6 + p(7)q^7 + p(8)q^8 + \boldsymbol{p(9)q^9} + p(10)q^{10}\\ &\quad+p(11)q^{11} + p(12)q^{12} + p(13)q^{13} + \boldsymbol{p(14)q^{14}} + \cdots\\ &= \boldsymbol{q^4\frac{(q^{25};q^{25})^5_\infty}{(q^5;q^5)^6_\infty}}\cdot(q^{-4}R^4 + q^{-3}R^3 + 2q^{-2}R^2 + 3q^{-1}R + \mathbf{5}\\ &\qquad -3qR^{-1} + 2q^2R^{-2} - q^3R^{-3} + q^4R^{-4}). \quad (*)\end{aligned}$$

By this time, we have certainly lost sight of our overall goal. Why do we write such a complicated expression for the generating function of the partition numbers? What did we actually want to prove? It is at this point that the *punch line* reveals itself! We are actually only interested in the partition numbers $p(4), p(9), p(14), p(19)$, etc., that is, in

$$p(4)q^4 + p(9)q^9 + p(14)q^{14} + p(19)q^{19} + \cdots.$$

Let us look at the right-hand side of the above complicated expression: there we see the series R, which according to the lemma contains only powers of q^5. Also, the products $(q^5;q^5)_\infty$ and $(q^{25};q^{25})_\infty$ consist only of powers of q^5. At the front of this expression, there is the factor q^4. So, inside the big parentheses, the only terms that are of interest for us are powers of q^5; everything else can be neglected. However, if one actually looks inside carefully (the reader should recall that the series R only contains powers of q^5!) then the only term that is relevant is the lonely 5! In other words, from the horrendous formula above (the reader should concentrate on the terms in bold face), one can immediately extract that:

$$p(4)q^4 + p(9)q^9 + p(14)q^{14} + \cdots = q^4\frac{(q^{25};q^{25})^5_\infty}{(q^5;q^5)^6_\infty}\times\mathbf{5}.$$

The point here is: on the right-hand side everything gets multiplied by 5! Consequently, all coefficients on the left-hand side – that is, $p(4)$, $p(9)$, $p(14)$, $p(19)$, etc.

[32] The formula is also valid in the theory of formal power series.

[33] Ramanujan did not know this notation, and did not use any other short notation. Consequently, to read notes of Ramanujan constitutes a certain challenge.

[34] There also exists an explicit formula for the series R.

[35] The identity from the last lemma is "divided" by the one from the previous lemma, and then one substitutes Euler's theorem.

– are divisible by 5. This is exactly the desired assertion that we wanted to prove. □

I do not know how you were doing while going through this proof. Every time, I present it during a lecture course, there are always a few students who cannot help but smile when the punch line is revealed.

We come to the *Tour de Force*! Of course, what Andrew Wiles has accomplished is an incredible tour de force. Since this requires however large chunks of modern number theory and algebra, in a few minutes I can say exactly nothing about it. Therefore, I have chosen a different example for illustration – from my own research area –, namely Doron Zeilberger's (see Figure 5) *theorem on alternating sign matrices*. First of all, we need to know what an alternating sign matrix is. An alternating sign matrix is a square arrangement of 0's, 1's and (–1)'s which satisfies the following rule: if one reads along rows or columns and ignores the 0's then one reads alternatingly 1, –1, 1, … , 1. In order to avoid any misunderstanding: one starts and ends with a 1. Here is an example of an alternating sign matrix:

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$

You may well ask why mathematicians are interested in alternating sign matrices. I cannot say too much here for the sake of brevity. Alternating sign matrices arose originally in a natural fashion around 1980 in work of David Robbins and Howard Rumsey on a generalisation of determinants. Later, it was discovered that the same objects also appear in Theoretical Physics, albeit in different guise, namely as configurations in an – admittedly somewhat simplistic – model for the formation of ice. William Mills, David Robbins and Howard Rumsey asked themselves how many alternating sign matrices there are. More precisely:

> *How many alternating sign matrices with exactly n rows are there?*

Apparently there exists exactly one alternating sign matrix consisting of one row, namely

$$1.$$

There are two alternating sign matrices with two rows:

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

And there are 7 alternating sign matrices with three rows:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

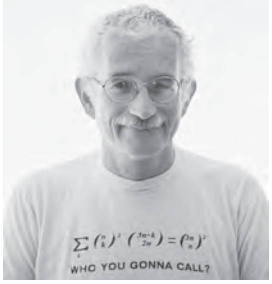

Figure 5. Doron Zeilberger.

If we denote the number of all alternating sign matrices consisting of exactly n rows by $A(n)$ then the following table

n	1	2	3	4	5	6
$A(n)$	1	2	7	42	429	7436

shows the first values of the sequence. Mills, Robbins and Rumsey studied these numbers carefully and made a remarkable discovery.

Conjecture (Mills, Robbins, Rumsey ~ 1980). *We have*

$$A(n) = \frac{1! \cdot 4! \cdot 7! \cdot \cdots \cdot (3n-2)!}{n! \cdot (n+1)! \cdot (n+2)! \cdot \cdots \cdot (2n-1)!},$$

where $m! = m \cdot (m-1) \cdot (m-2) \cdot \cdots \cdot 2 \cdot 1$.

This is extremely surprising. If a mathematician learns about the above question, then the immediate reaction would be that there cannot be any reasonable formula for the number of all alternating sign matrices consisting of exactly n rows. But, no! It seems that there is even an elegant, compact product formula!

But how to prove this? For more than 10 years, mathematicians did not even know how to attack this conjecture. Everybody was therefore very surprised when Doron Zeilberger announced in 1993 that he had found a proof. Along with the announcement, he distributed a 25-page article which contained that proof.

As you know, it is not sufficient to announce that one has proved something. The proof must be written down and submitted for publication, after which the corresponding article is refereed. Zeilberger submitted his article for publication and – you guess it – the referee found gaps in the proof. So, the article went back to Doron Zeilberger with the request to fill the gaps. Zeilberger did some repair work and resubmitted the article, and the referee found new gaps. The article went again back to Zeilberger, he did more modifications, resubmitted, and the article went back and forth in this manner several times, until the referee lost patience. He probably told Zeilberger roughly the following: "Dear Doron! Before you resubmit the article, please do something. Read your proof carefully from the very beginning to the end. If you should not be able to do that, then give the article to a student to check the proof; but, please, do something!"

Doron Zeilberger did do something. First of all, he read and checked his article carefully. Furthermore, he structured the proof completely hierarchically, so that the article could be read "locally"; in the sense that each part

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

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Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquet-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

Dedicated to my Friend, Mentor, and Guru, Dominique Foata.

Two stones build two houses. Three build six houses. Four build four and twenty houses. Five build hundred and twenty houses. Six build Seven hundreds and twenty houses. Seven build five thousands and forty houses. From now on, [exit and] ponder what the mouth cannot speak and the ear cannot hear.

(Sepher Yetsira IV,12)

Abstract: The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row- and column- sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\dots(3n-2)!]/[n!(n+1)!\dots(2n-1)!]$, as conjectured by Mills, Robbins, and Rumsey.

original version written December 1992. The Maple package ROBBINS accompanying this paper, can be downloaded from the www address in footnote 2 below.
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA.
E-mail:`zeilberg@math.temple.edu`. WWW:`http://www.math.temple.edu/~ zeilberg`. Supported in part by the NSF.
See the Exodion for affiliations, attribution, and short bios.

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Figure 6.

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[In order to view all of them type 'GOG(3,5):' in ROBBINS.]

On the TSSCPP side, it was shown in [MRR3] that TSSCPPs whose 3D Ferrers graphs lie in the cube $[0,2n]^3$ are in trivial bijection with triangular arrays $c_{i,j}$, $1 \le i \le n$, $1 \le j \le n-i+1$, of integers such that: (i) $1 \le c_{i,j} \le j$, (ii) $c_{i,j} \ge c_{i+1,j}$, and (iii) $c_{i,j} \le c_{i,j+1}$. We will call such triangles *n-Magog* triangles, and the corresponding chopped variety, with exactly the same conditions as above, but $c_{i,j}$ is only defined for $1 \le i \le k$ rather than for $1 \le i \le n$, *$n \times k$-Magog trapezoids.* For example the following is one of the 429 5−Magog triangles:

$$\begin{matrix} 1 & 2 & 3 & 3 & 5 \\ 1 & 2 & 2 & 3 & \\ 1 & 2 & 2 & & \\ 1 & 2 & & & \\ 1 & & & & \end{matrix} \quad .$$

[In order to view all of them type 'MAGOG(5,5):' in ROBBINS.] Retaining only the first three rows of the above Magog-triangle, yields one of the 387 5×3-Magog trapezoids:

$$\begin{matrix} 1 & 2 & 3 & 3 & 5 \\ 1 & 2 & 2 & 3 & \\ 1 & 2 & 2 & & \end{matrix} \quad .$$

[In order to view all of them type 'MAGOG(3,5):' in ROBBINS.]

Our goal is to prove the following statement, conjectured in [MRR3], and proved there for $k = 2$.

Lemma 1: For $n \ge k \ge 1$, the number of $n \times k$-Gog trapezoids equals the number of $n \times k$-Magog trapezoids.

[The number of n by k Magog trapezoids, for specific n and k, is obtained by typing `b(k,n);` while the number of n by k Gog trapezoids is given by `m(k,n);`. To verify lemma 1, type `S1(k,n):`.]

This would imply, by setting $n = k$, that,

Corollary 1': For $n \ge 1$, the number of n-Gog triangles equals the number of n-Magog triangles.

Since n-Gog triangles are equi-numerous with $n \times n$ alternating sign matrices, and n-Magog triangles are equi-numerous with TSSCPPs bounded in $[0,2n]^3$, this would imply, together with Andrews's[A2] affirmative resolution of the TSCCPP conjecture, the following result, that was conjectured in [MRR1].

The Alternating Sign Matrix Theorem: The number of $n \times n$ alternating sign matrices, for $n \ge 1$, is:

$$\frac{1!4!\dots(3n-2)!}{n!(n+1)!\dots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \quad .$$

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Figure 7.

could be read independently of the rest if one assumed that everything which appeared lower in the hierarchy was correct. Subsequently, he asked about 80 colleagues to check the article. He assigned to each of them 2 to 3 pages, and the task was to check these pages under the assumption that everything which appeared lower in the proof hierarchy was correct. So it happened. A few minor deficiencies were discovered in that way, which could be easily repaired, but nothing dramatic surfaced anymore, and the article was eventually published in 1995.[36] In Figure 6, we see the first page of the article. After the title, the aforementioned colleagues (the "checkers", totalling around 80) are listed. The article is no longer 25 pages long, but rather 85. As I said, the proof is structured completely hierarchically. The actual main theorem of the article is called Lemma 1 (see Figure 7). This is based on Sublemma 1.1 and Sublemma 1.2. The latter in turn are based on Subsublemma 1.1.1, Subsublemma 1.1.2, … , Subsublemma 1.2.1, Subsublemma 1.2.2, … , which in turn are based on Subsubsublemma 1.1.1.1, … , and so forth, up to Sub^6, that is, up to Subsubsubsubsubsublemma, one of which we see in Figure 8.

You get the impression: we are talking about a real tour de force. There is one thing, however, that cannot be said about it. One cannot claim that this is a "nice" proof, an elegant proof. In order to defend this, the same Doron Zeilberger – in a different context – went as far as to exclaim:[37]

"Extreme UGLINESS is new BEAUTY!"

I think we let this stand as it is. The sarcasts among you will say: "Yes, I always had the impression that this is exactly the idea of many modern composers." I would counter that at all times there existed better and worse composers. Once time passes, the worse ones tend to be forgotten, and only the outstanding composers remain. One can verify the latter phenomenon very well if one asks how many composers there were when Beethoven was a celebrity. Answer: innumerably many! If, however, one asks which of those are still known today, which ones are still performed today, then Franz Schubert comes immediately to one's mind (who "ironically" was largely unknown at the time), also Carl Maria von Weber and the Italian opera composers Gioachino Rossini and Gaetano Donizetti. This is it! The same thing will apply for us in 100 or 200 years from now. Most composers will be completely forgotten, and only the outstanding ones will survive. If I may make a personal comment on this

[36] *Electron. J. Combin.* 3 (no. 2, "The Foata Festschrift") (1996), #R13, 84 pp.

[37] Excerpt from a talk at the Third International Conference on "Formal Power Series and Algebraic Combinatorics", Bordeaux, 4 May 1991.

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We now need the following $(sub)^6$ lemma:

Subsubsubsubsubsublemma 1.2.1.2.1.1.1: Let U_j, $j = 1, \ldots, l$, be quantities in an associative algebra, then:

$$1 - \prod_{j=1}^{l} U_j = \sum_{j=1}^{l} \left\{ \prod_{h=1}^{j-1} U_h \right\} (1 - U_j) \quad .$$

Proof: The series on the right telescopes to the expression on the left. Alternatively, use increasing induction on l, starting with the tautologous ground case $l = 0$. $\square$

Using $(sub)^6$lemma 1.2.1.2.1.1.1 with

$$U_j = \prod_{i=r_{j-1}+2}^{r_j} (\bar{x}_{i-1} x_i) \quad ,$$

we get that $(Marvin)$ implies:

$$Jamie(x_1, \ldots, x_k) = \left\{ \prod_{m=1}^{l} \bar{x}_{r_m} \right\} \cdot \sum_{j=1}^{l} \left\{ \prod_{h=1}^{j-1} \prod_{i=r_{h-1}+2}^{r_h} (\bar{x}_{i-1} x_i) \right\} \cdot \left(1 - \prod_{i=r_{j-1}+2}^{r_j} (\bar{x}_{i-1} x_i) \right) \quad . \tag{$Marvin'$}$$

We can split $(Marvin')$ yet further apart, with the aid of the following $(sub)^6$lemma:

Subsubsubsubsubsublemma 1.2.1.2.1.1.2: Let U_j, $(j = K, \ldots, L)$, be quantities in an associative algebra, then:

$$1 - \prod_{i=K}^{L} U_i = \sum_{p=K}^{L} (1 - U_p) \left\{ \prod_{h=p+1}^{L} U_h \right\} \quad .$$

Proof: The sum on the right telescopes to the expression on the left. (Note that it is in the opposite direction to the way in which it happened in 1.2.1.2.1.1.1.) Alternatively, the identity is tautologous when $K = L + 1$, and follows by decreasing induction on K. This completes the proof of $(sub)^6$ lemma 1.2.1.2.1.1.2. $\square$.

Going back to $(Marvin')$, we use the last $(sub)^6$lemma (1.2.1.2.1.1.2), with $K = r_{j-1} + 2$, $L = r_j$, and $U_i := (\bar{x}_{i-1} x_i)$, to rewrite:

$$\begin{aligned} Jamie(x_1, \ldots, x_k) &= \left\{ \prod_{j=1}^{l} \bar{x}_{r_j} \right\} \cdot \sum_{j=1}^{l} \left\{ \prod_{h=1}^{j-1} \prod_{i=r_{h-1}+2}^{r_h} (\bar{x}_{i-1} x_i) \right\} \cdot \sum_{p=r_{j-1}+2}^{r_j} (1 - \bar{x}_{p-1} x_p) \prod_{i=p+1}^{r_j} (\bar{x}_{i-1} x_i) \\ &= \sum_{j=1}^{l} \sum_{p=r_{j-1}+2}^{r_j} \left\{ \prod_{m=1}^{l} \bar{x}_{r_m} \right\} \left\{ \prod_{h=1}^{j-1} \prod_{i=r_{h-1}+2}^{r_h} (\bar{x}_{i-1} x_i) \right\} \cdot (1 - \bar{x}_{p-1} x_p) \left\{ \prod_{i=p+1}^{r_j} (\bar{x}_{i-1} x_i) \right\} \quad . \end{aligned} \tag{$Marvin''$}$$

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Figure 8.

matter from a local patriotic view: I am convinced that Friedrich Cerha will be one of those composers whose music will still be performed in 100 or 200 years. His powerful, expressive musical language is impressive and also clearly present in pieces which I like less.

Figuratively – not literally – the above statement is essentially what Arnold Schönberg and the composers around him have done. The romantic sound idiom was, after it had also moved into expressionism, exhausted, at its end. No further development was possible. What Arnold Schönberg did then, when he turned to the twelve-tone technique, was radically rupture all common habits and rules. He based his music on a completely new foundation, with completely new rules. He believed –hoped – that in this way a new musical aesthetic would emerge. I, personally, regard this experiment as a failure. As I already said at a different occasion: I understand that a genius such as Arnold Schönberg tried this path, but I do not understand why he did not find an escape from this – as I see it – dead end of musical history. (That Schönberg was a musical genius is single-handedly proved by his string sextet "Verklärte Nacht".[38] This is such an incredibly touching and moving, and at the same time complex piece as only a genius can write. To me, it belongs to the greatest compositions ever.)

[38] "Transfigured Night".

Brains in mathematics

You may argue that there is little to say on this topic. Obviously, reason and thought are the essentials in mathematics. You are right, of course. Hence, we may consider this topic as checked…

Brains in Music

This is again an entire topic in itself. There is the widespread naive idea, concerning pianists, that a pianist must practise diligently, and in the evening of the concert he storms onto the stage, sits down at the piano, and cuts loose. Yes, this is a possibility, but this is not how it works. The audience will notice[39] that not much thought went into that interpretation. It will not really make sense, it will remain inconclusive. Indeed, if one looks at the great pianists, one will notice that emotion and thought always go together – form a symbiosis – certainly with different weighting in each individual case. The prototypical example is Alfred Brendel, where it is amply established by his books just how much thought went into his interpretations, and where simply watching him play was sufficient to understand what a sensitive and emotional artist he was.

Concerning composers, there is a similar widespread conception that it is most important to have good melodic ideas. Everything else just works by itself. In response to this, I can only say that at all times there are and have been many more composers with good melodic ideas than good (or even outstanding) composers. The great art is in bringing to bear the melodic ideas, the themes, and in building, forming, and developing the pieces. Here too the following applies: if one looks at the great composers, then emotion and thought always go hand in hand. For composers such as Bach, Beethoven, or Brahms, this is obvious anyway. However, it also applies to composers who are not really under suspicion of having approached composition in a particularly intellectual manner. In this latter category, I would see Franz Schubert, Anton Bruckner, or also Modest Mussorgsky. One will be surprised how much thought went into the compositions of even these composers. For Mussorgsky, it suffices to consider his "Pictures at an Exhibition," how the promenades keep the work artfully together, how the theme of the last picture, the "Great Gate of Kiev" is extracted from the theme of the promenade, which is itself formed in a self-referential way. Bruckner's scores are highly complex anyway. Even in the work of Schubert the role of reason and thought is much bigger than one would commonly believe. I want to give a glimpse of an idea here. The example I have chosen is the Great Sonata in A major, D 959, from Schubert's last year of life. This sonata has four movements. A broad first movement, whose proud opening theme is the following:

[39] Singular exception is presumably Martha Argerich, whose interpretations do not seem to be very reflective. Instead, she proceeds rather spontaneously when playing. I have the greatest respect for Martha Argerich. Her musicality is marvellous. I did however also listen to pieces played by her which, due to her spontaneous approach, – so-to-speak – disintegrated under her fingers…

We already know the heartbreakingly sad theme of the second movement:

There follows a playful Scherzo, which also contains Ländler[40] elements:

The concluding melodious Rondo begins as follows:

You will not have noticed, but maybe you sensed it: these four themes, so different in character, are bound together by a hidden brace. This is what I now want to expose.

If one looks at the opening theme of the first movement more closely, then one recognises that (in the upper voice) the note *a* is at first repeated several times, before it is "resolved" to a *g sharp* in the end, which is also ornamented by an *f sharp*. Thus, if one reduces the theme to its nucleus then it becomes clear that we are talking of a largely blown up suspension *a–g sharp*:

How does the second movement begin? The answer is: *a–g sharp*. How does the Scherzo fit in? This is more hidden. Here, one must look at the lower voice in order to discover *a–g sharp* again! The theme of the last movement even contains the suspension *a–g sharp* twice (namely in the second and in the fourth bar, both times in the upper voice).

Sure, these fine points are not *consciously* noticed by the listener, nonetheless, they do have an *unconscious* effect. In our concrete case, they contribute to the great unity of the sonata. It is, among other things, these details that make the difference between a masterpiece and compositions of average quality.

[40] A "Ländler" is a rural dance in Austria. The German translation of "rural" is "ländlich".

Figure 9.

Differences between mathematics and music

So far, I have talked a lot about parallels between mathematics and music. I should perhaps also address the differences between them. In short, there are many. Here, I only want to work out the most significant difference.

This begins with another parallel. When a composer has the great inspiration and a composition materialises in his/her head, then it must now be written down in order to be performed. This may then look as in Figure 9.

When a mathematician has a brilliant idea and proves a great new theorem, then it must now be written down in order for others to be able to study it. This may then look as in Figure 10.

If somebody cannot read scores and also does not understand anything of mathematics: I would say, there is no discernible difference between the two; each is as incomprehensible as the other…

Let us return to the score. It must now be brought to life. In the case of the "Appassionata", we need a pianist. This pianist must carefully study and practise the piece, and then perform it. And this performance – this is it! This is the *complete* composition! Nothing was added, nothing was omitted (if we neglect that the pianist stumbles possibly here and there…). And everybody can sit down and listen to it. No prior education is required for that. If one has an affinity with the musical language of Beethoven, then one will be captivated by the dark, tense atmosphere of the Appassionata.

Now you may object: but at the university, in the mathematics courses, there mathematics is "performed." Somehow, yes. However, this is actually different. You cannot simply sit down in a course and delight in the various qualities of the "performed" mathematics. Depending on how advanced a course is, it requires more or less prior knowledge from the listener in order to understand at all what is being discussed. (Even the courses in the first semester require certain prior knowledge, without which it is not advisable to attend such a course. Unfortunately, every year there arrive more freshmen than we would like for whom this is apparently not so clear…). In the courses, it is common practise to build on this prior knowledge, and to not repeat what is (should be) already

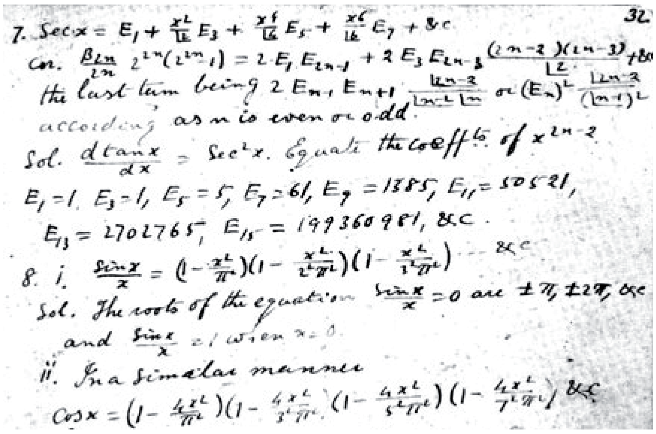

Figure 10.

known. Moreover, for conclusions which are somehow "obvious", it is left to the listener to fill in the details. In this sense, also in courses at the university there is no (complete) performance of mathematics.

You will insist: okay, but at mathematics conferences, there mathematicians present their latest results in front of their colleagues, *there* mathematics is "performed"! The former is certainly correct, but here as well there is no "performance" of mathematics which takes place in the same sense as music is performed. At a conference, you may have maybe 30 minutes, maybe an hour to present your latest result. What is presented the lecturer has thought about for weeks, months, sometimes years. This cannot be presented in all detail within 30 minutes or an hour. What one therefore does is explain the assertion of the newest theorem, and subsequently indicate which ideas go into the proof. If a listener wants to completely work through the proof, respectively wants to completely check the proof in all detail, then (s)he must study the article in which this proof is written down. So, also here, there is no "performance" of mathematics.

This has a very lamentable consequence for mathematicians: I would say that – very roughly estimated – 90 percent of the population are amenable to music. If one subtracts popular music, then there remain still – conservatively estimated – 10 percent who are addressable by – let me say – *expressive* music. [41]

How are matters with mathematics? I would formulate it as follows. You may perhaps remember from high school that mathematicians have a special symbol for infinitely small quantities: the ε! So, I would say that – roughly – ε percent of the population are amenable to the manifold qualities of the science of mathematics.

This is very grievous for mathematicians. Frequently, mathematicians are reproached for not stepping out of their ivory tower and not explaining a wide audience what they are doing. As a matter of fact, mathematicians would love to do exactly that: with great excitement, they would present and explain their newest theorem – their newest composition, so to speak – in front of a wide audience. However, because of the earlier described difficulties, it is impossible! In order to avoid any misunderstanding: I am not saying that one should not talk *about* mathematics. On the contrary! What I am doing here is, in a sense, also to talk *about* mathematics. However, if it comes to current research, then one will have to take recourse to metaphors, then one will only be able to vaguely indicate what is really going on. As I said: the performance of mathematics does not exist, and thus a mathematician will never be able to convey to a wide audience what (s)he experiences when (s)he deals with mathematical problems and their solutions. Here, mathematicians are always at a disadvantage when compared to musicians – and to researchers in other disciplines; music speaks directly to the listener, no "translation", no further explanation is necessary, and this is in sharp contrast to mathematics.[42]

Personal notes

What do mathematics and music mean to me? A lot, obviously. First of all, there is the inexplicable, magical component. If I am asked why I went for music and mathematics: I do not know. I remember very well that, as a 6–7-year old, I used to sing along with great passion. Why? I do not know. I also remember very well that, as a 13–14-year old, I was burningly interested in how big the probability is that, by throwing a given number of dice, one scores a certain total; so, for example, how likely it is to score 36 by throwing 10 dice. I computed (by hand) long tables, and I studied the numbers in these tables. After work of several years, I was indeed able to find a formula for this probability. Obviously, at the time, I did not have the slightest idea how to prove it.[43] Why was I so fascinated by this? I do not know.

What fascinates me today in mathematics and music? When it comes to mathematics, there is for one the challenge to "crack" open problems as they constantly arise in physics, in computer science, and also in mathematics itself. Interestingly, in my research work, in order to solve a problem, I frequently study long tables (nowadays computed by using a computer, of course), subsequently I try to guess a mathematical formula for the numbers in these tables (also partially with the help of a computer), and then – if successful – try to prove this newly discov-

[41] I never knew what to do with the labels "classical music" and "light music".

[42] Consequently, Cédric Villani, in his remarkable and controversial book "*Théorème vivant*" (in the English translation: "*Birth of a Theorem*") – in which he describes how the proof of the theorem emerged that significantly contributed to the award of the Fields Medal to him in 2010 – does not even try to explain the mathematics behind, but on the contrary intentionally remains often incomprehensible even to mathematicians who are not experts in the field of partial differential equations, in order to entirely concentrate on the emotional side of the involvement with mathematics. Villani is highly successful doing this, but – seen plainly – he does not *talk about* mathematics.

[43] Today I know that this formula can be easily proved with the help of generating functions or with the help of the principle of inclusion-exclusion.

ered conjecture. Moreover, I am of course fascinated by searching and discovering hidden structures and connections behind the problems and their solutions. Clearly, the aesthetic component in mathematics plays a big role for me as well.

Also in music, I am fascinated by fathoming new territory. It is extremely interesting to take a new[44] piece, and now start working on it. As we already discussed: a score must be brought to life. When one starts studying a piece, one often does not know which are the important points within it, how to understand the structure of the piece, and how the piece should develop when played. I remember very well the situation when I, together with my trio partners, started to prepare the third movement of Mozart's piano trio in C major, KV 548, for an encore of a concert evening. Each of us had – individually –prepared and practised one's own part, but the first "reading" of the movement ended up in a complete disaster: nothing made any sense at all. The violinist immediately pled that we choose a different piece as an encore … I insisted to give the piece a chance. So, we started to work on it, and, lo and behold, this "ugly duckling" slowly transformed itself into a lively, witty piece of music, which was a great pleasure to play for all of us.

Another important point is that, once one has worked out an interpretation of a piece, to present this – own – view of the composition in front of an audience. Each time, this is a tremendously interesting and exciting experience. One never knows in advance how this is going to develop, but the more this is tantalising, and the more this can be enriching.

In any case, for me, mathematics and music always have been two very different things that complement each other. And it is exactly this complementary aspect that I have always found so interesting and appealing. It is perhaps unhealthy to become obsessed with just one matter. When I am trying to solve a mathematical problem and I arrive at a dead end where I do not know how to proceed, then I may sit down at the piano and concentrate on something completely different, and in this way clear my mind. Maybe upon returning to the mathematical problem, I will have a new, fresh view of things, which allows me to progress again.

Conclusion

Thus, I arrive at the end of my discourse on "Mathematics AND Music?". To tell you a secret: it is absolutely allowed to remain largely incomprehensible during a mathematical talk; there is but one condition (in the words of the influential Italian/American mathematician Gian-Carlo Rota as a postulation of the audience to the speaker[45]):

> *"Give us something to take home!"*

In this sense, I hope that I was not too incomprehensible, and that there was something for you to take home. On this point, I have one thing further to offer, a piece of music at the end. Obviously, it must suit our motto "Soul AND Brains". Clearly, one could find many natural candidates, for example, in the work of Johann Sebastian Bach, or of Ludwig van Beethoven. However, this would be too simple, too conventional. Instead, I chose the Sonata Opus 1 by Alban Berg. He wrote this sonata at 23 years of age. It is, in a sense, the final "paper" of his music studies, which he mainly undertook under Arnold Schönberg. If you wish, it is Alban Berg's musical "thesis," in order to stress another analogue with mathematics. It fits excellently with our motto "Soul AND Brains". I would say that the musical language of this sonata can be classified as expressionistic. It is thus highly emotional. On the other hand, it is an incredibly dense musical construction, in which the *complete* piece of approximately 10 minutes is extracted from *one* nucleus – namely the theme at the beginning.

Enough of explanations. I will now play the Sonata Opus 1 by Alban Berg. I shall directly adjoin a prayer by Johannes Brahms. "Intermezzo" is the original title by Brahms, from the last piano pieces that he wrote. I have always liked to do this, since, first of all, the two pieces fit so well together, and, second, if one listens, then one understands where the musical language of Berg comes from.

⟨*Alban Berg (1885–1935): Sonata op. 1*⟩[46]

⟨*Johannes Brahms (1833–1897): Intermezzo in b minor, op. 119/1*⟩[47]

Christian Krattenthaler studied mathematics at the University of Vienna and piano at the Vienna University of Music and Performing Arts. After finishing his studies (mathematics 1984, piano 1986), he pursued both careers (lecturer at the University of Vienna and concert pianist) for a while. He terminated his activities as a concert pianist in 1991 because of an incurable ailment of both hands. After holding a position as a professor at the Université "Claude Bernard" Lyon-I from 2002 to 2005, he was appointed as a professor of discrete mathematics at the University of Vienna. For his scientific achievements, he was awarded the Wittgenstein-Prize in 2007.

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44 Meaning: "not yet studied".

45 The citation is from the talk "Ten Lessons I wish I had been Taught", which Rota delivered on 20 April 1996 at the occasion of a birthday conference in his honour at the Massachusetts Institute of Technology in Boston. It can be read in the *Notices of the American Mathematical Society*, vol. 44, Nr. 1, 1997, pp. 22–25 (see http://www.ams.org/notices/199701/comm-rota.pdf).

46 In danger of exhibiting a certain bias, Alfred Brendel's wonderfully balanced view can be enjoyed on YouTube in two parts: http://www.youtube.com/watch?v=PlV-ksfS7F8, http://www.youtube.com/watch?v=QxBGG74ztVo.

47 An old concert recording of this piece with the author himself at the piano can be found at http://www.mat.univie.ac.at/~kratt/klavier/brahms119-1.html.

Caucasus Mathematical Olympiad

Daud Mamiy (Adyghe State University, Maykop, Russia)

On 13–18 March 2017, Maykop (Republic of Adygea, Russian Federation) hosted the Second Caucasus Mathematical Olympiad (an International Olympiad for secondary school students).

The first Olympiad was held in the 2015–16 academic year at the "Sirius" Educational Centre (Sochi, Russia), with students participating from 15 regions of Southern Russia.

It was only in 2017 that the Caucasus Mathematical Olympiad accomplished its original intention, bringing together school students and representatives of various regions of Southern Russia and becoming international. From 15 southern regions of Russia and countries of the South Caucasus: Armenia, Abkhazia, and South Ossetia, 110 students took part in the Olympiad.

The creators of the Olympiad set the humanistic objective of contributing to the formation of a unified cultural and educational space, which would unite regions of Southern Russia, the Caucasus and countries from the Black Sea region, and strengthening ties between school students keen on mathematics from these regions. One of the main goals of the Olympiad is to prepare students for the final stages of the National Olympiads in the participating countries.

The Coordination Council oversaw the development of the Olympiad and its preparation. The Coordination Council of the Olympiad included Daud Mamiy, initiator of the Olympiad, Nazar Agahanov, member of the Coordination Council of the International Mathematical Olympiad, and Nikolai Andreev, Head of the Laboratory of Popularization and Promotion of Mathematics of the Steklov Mathematical Institute of the Russian Academy of Sciences.

In 2017, the members of the central committee of the Russian Mathematical Olympiad responsible for subject-specific methods of teaching formed the Problem Selection Committee of the Olympiad. Pavel Kozhevnikov served as the Chairman of the Problem Selection Committee. Kozhevnikov himself was a Gold Medallist of the International Mathematical Olympiad in 1992 (currently he is at the Moscow Institute of Physics and Technology, Laboratory of Popularization and Promotion of Mathematics of the Steklov Mathematical Institute of the Russian Academy of Sciences).

The Jury of the Caucasus Mathematical Olympiad consisted of members of the Jury of the Russian Mathematical Olympiad, who were previous winners of Russian and International Olympiads.

The Olympiad was held in two age groups: Junior League for students of Grades 8–9 and Senior League for students of Grades 10–11 (last two years of high school). The Coordination Council invited the winners of the regional and final rounds of the national Mathematical Olympiads of the participating countries to enter the competition. The Olympiad was held over two rounds in which the participants were asked to solve four problems.

The organisers tried to create an atmosphere that realised the intent of the Olympiad; the entire programme aimed to create a friendly and creative atmosphere. The organisers provided consultants who discussed and reviewed the problems of the previous rounds with the delegations from any regions where the level of mathematical training is not high enough.

In the evenings, volunteers from Maykop schools and students of Adyghe State University organised guitar and music concerts. Without doubt, the Olympiad participants and the members of the jury enjoyed the informal gatherings, where they talked about the problems, their preparation for Olympiads and anything else that was of interest.

The programme of the Olympiad was also designed to introduce the guests of Adygea to the local history, culture and traditions. The contestants visited the National Museum of the Republic of Adygea and the Northern Caucasus branch of the State Museum of Oriental Art, travelled to the foothills of the Western Caucasus, attended lectures on the history and culture of a local ethnic group called Circassians and became familiar with Circassian cuisine. Every night, dancers from the Dance Ensemble "Nart" of Adyghe State University taught everybody who was interested how to dance Caucasian dances.

The members of the Coordination Council were very involved during the Olympiad, attending numerous meetings with teachers of mathematics, university students and schoolchildren from almost all districts of the Republic of Adygea. Schoolteachers gathered at roundtables to discuss problems of student training for Olympiads. Teachers also attended lectures on how to engage students in mathematics, while young mathematics enthusiasts had the chance to attend lectures on popular mathematics that were taking place in all districts of the Republic of Adygea for three days.

The Olympiad events were updated on its website (http://cmo.adygmath.ru/en) and on the social network "VKontakte" (https://vk.com/cmo.adygmath). Volunteer students of Adyghe State University formed the press centre for the Olympiad and uploaded more than 1000 photos (http://cmo.adygmath.ru/node/31) of the most interesting moments of the Olympiad, as well as short videos and dozens of posts.

The opening and closing ceremonies of the Olympiad were very impressive and reflected the diversity of the cultures of people living in the Caucasus. It is not a coincidence that the closing ceremony ended with a song "Our Caucasus", beautifully performed by the soloists of the Adyghe State University musical theatre "Art Rhyton". The Olympiad closing ceremony was held in the Main Hall of the State Philharmonic of Adygea.

It should be added that the emblem of the Caucasus Mathematical Olympiad is the Caucasian tree of mathematical knowledge, designed by a talented Maykop painter Tatiana Vaganova. At the opening ceremony of the Olympiad, the delegates lit their symbolic signs on the branches of this tree and later, during the closing ceremony, they put the lights out. Most of the audience could not hold back their tears because they had made great friends and did not want to part from each other.

The jury awarded various diplomas and gifts to 72 participants from the regions and participating countries for their achievements. Forty-one of these participants were Olympiad champions and prize winners. The overall winners of the Olympiad were three students from Armenia and six students from regions of Russia (Adygea, Dagestan and the Krasnodar region).

Apart from the diplomas and prizes, each of the winners of the Olympiad received a pendant in the shape of a blade. The pendants were created by the famous Circassian jeweller and gunsmith, Asiya Eutykh, who made them from a unique alloy used by Circassians in ancient times.

Four winners of the Olympiad who succeeded in solving all eight problems got special prizes from Murat Kumpilov, the Head of the Republic of Adygea.

The second and third place winners were participants from the Republic of Armenia and from eight regions of Southern Russia: the Astrakhan region, the Republic of Adygea, the Volgograd region, the Republic of Dagestan, the Rostov region, the Krasnodar region, the Republic of Crimea and the Republic of Kalmykia. All winners received memorable prizes and gifts.

The organisers of this Olympiad were the Ministry of Education and Science of the Republic of Adygea and Adyghe State University. The idea of hosting the Olympiad in Adygea was actively supported by the authorities of the Republic, who helped cover the major cost of the Olympiad. The sponsor of the Olympiad was an ITV company, a Russian market leader in video-editing.

The success of the Olympiad is largely linked to the activity of the Adygea Regional School of Mathematics and Natural Sciences, which has extensive experience in hosting mathematical competitions and conferences at the highest level.

The Caucasus Mathematical Olympiad of 2017 achieved its objectives and goals. Apart from the mathematical contest experience, schoolchildren from different regions of the Caucasus got to know each other and made new friends. The results of the Second Caucasus Mathematical Olympiad highlight the necessity of carrying it forward into subsequent years. To maintain the current format, the Olympiad of 2018 will take place in Adygea again. We hope that the participants from other countries of the Caucasus and the Black Sea region will attend this event in the future. We will be happy to welcome students of Azerbaijan, Bulgaria, Georgia, Romania, Turkey, Ukraine and other countries. The Olympiad is further expected to be held under the direction of the Coordination Council in other countries of the Caucasus, the Black Sea region and Southern Russia.

Daud Mamiy (dmami@yandex.ru) is the Initiator of the Caucasus Mathematical Olympiad and Head of its Coordination Council. He received his degree in mathematics at Lomonosov Moscow State University in 1987 and his PhD in mathematics in 1991 also from Lomonosov Moscow State University. Daud Mamiy is the Dean of the Faculty of Mathematics and Computer Science of Adyghe State University. He is the Director of the ASU School for mathematically gifted students.

IMPA, Instituto de Matematica Pura e Aplicada

Henrique Bursztyn and Roberto Imbuzeiro Oliveira (both Instituto de Matematica Pura e Aplicada, Rio de Janeiro, Brasil)

IMPA's entrance.

IMPA's location is in the right bottom corner of the image.

The "Instituto de Matemática Pura e Aplicada" (IMPA) is a research centre for mathematics situated in Rio de Janeiro, Brazil. It was created 65 years ago with the mission of pursuing high-level research in pure and applied mathematics, training new researchers and disseminating mathematics at all levels. IMPA has been pivotal in the development of mathematics in Latin America, particularly through the training of leading professionals working in the region.

The institute has recently gained wider international recognition due to the 2014 Fields Medal awarded to Artur Ávila, a former PhD student and current faculty member of the institute.

IMPA's location in the hills of the Tijuca Forest in Rio de Janeiro provides scenic and tranquil surroundings, which contrast with the vibrant and stimulating scientific environment of the institute. Every year, IMPA gathers prominent mathematicians from around the world for short- and long-term visits, offering ideal conditions for scientific collaboration.

IMPA is also a leading centre for graduate studies in Latin America. It runs a PhD programme in mathematics and offers a number of options for Master's degrees. The institute hosts around 150 graduate students and nearly half of them come from abroad, contributing to the cultural diversity of IMPA's atmosphere. Postdoctoral programmes have also helped attract young international talent to the institute.

IMPA plays an active role in disseminating mathematics. A significant proportion of existing Portuguese mathematical literature (used in universities throughout the Portuguese speaking world) comes from IMPA's publishing house. IMPA runs Summer programmes that have up to 700 participants and organises a dozen international conferences every year. It hosts training programmes for high school teachers and is responsible for the Mathematics Olympiad for Public Schools (OBMEP), a national competition that involves 18 million pre-college children every year, discovering many new talents.

Brief history

IMPA was founded in 1952 as the first research centre affiliated with CNPq, the Brazilian National Council for Scientific and Technological Development. The institute started with only two researchers, Leopoldo Nachbin and Mauricio Peixoto, both mathematicians of international renown at the time. A few years later, topologist Elon Lima and number-theorist Paulo Ribemboim joined IMPA's research staff.

IMPA started its activities in the premises of CBPF (Brazilian Center for Research in Physics) in Urca, Rio de Janeiro, and moved to two other locations before settling, in 1981, in its current headquarters in the forest hills above Rio de Janeiro's botanical garden.

Master's and PhD programmes at IMPA started in 1962, through a cooperation agreement with the Federal University of Rio de Janeiro. IMPA underwent a major expansion in its research activities after 1970, upon the return of a new generation of Brazilian mathematicians who had completed their PhD degrees in distinguished foreign universities. This allowed IMPA to make a qualitative leap forward and widen its research and training activities. At that time, some of the most traditional research groups at IMPA were established, such as the groups on dynamical systems and differential geometry. The growth of IMPA's scientific staff gave new impetus to its Master's and PhD programmes, which became internationally recognised and responsible for the training of leading mathematicians in the region.

Over the last three decades, IMPA's scientific staff have increased in number to nearly 50 researchers, covering a wide array of areas in pure and applied mathematics.

Governance and funding

For nearly 50 years, IMPA was a public research institution under the auspices of the Brazilian Ministry of Science and Technology. This changed in 2001 when it transitioned to a new legal status as an "organização social" (OS). IMPA is now a private non-profit organisation of a special kind. Its aims and goals are specified by a contract with Brazil's Federal Government that is renewed every five years, contingent on good performance in the previous period.

The OS model has made the institute more dynamic and flexible. A streamlined hiring process has helped IMPA attract and retain talent. Institutional funding now combines private as well as public sources. Many new programmes and activities have been made feasible by the OS system. Of course, all of this has been possible only because IMPA has consistently surpassed its contractual goals.

IMPA's OS structure also specifies a governance structure with two main administrative bodies: the Board of Trustees and the Scientific Committee. The Board of Trustees supervises and regulates all of the institute's activities. It includes members of IMPA's faculty, external mathematicians, representatives from scientific societies and members from Government and society. The Scientific Committee, for its part, focuses on academic matters, such as hirings and scientific activities. It is composed of seven members from IMPA and five external members from the scientific community.

IMPA's previous facilities in downtown Rio de Janeiro.

Research at IMPA

IMPA's *raison d'étre* was to produce and promote mathematical research in Brazil and abroad. Founding members Leopoldo Nachbin and Maurício Peixoto were Brazil's first speakers at the International Congresses of Mathematicians (in 1962 and 1975 respectively). Faculty such as Jacob Palis, Welington de Melo and Ricardo Mañé in dynamical systems, César Camacho in complex geometry and foliations and Manfredo do Carmo in differential geometry helped propel their respective areas in South America and beyond.

IMPA's current faculty continues to play a leading role both regionally and globally. The institute has 40 regular professors (tenured and tenure-track), two special visiting professors (Etienne Ghys and Bruce Reed), two extraordinary professors (Artur Ávila and Harold Rosenberg) and five emeriti. Current research areas include: analysis and partial differential equations; algebraic geometry and representation theory; combinatorics; complex geometry and foliations; computer graphics; differential and symplectic geometry; dynamical systems; fluid mechanics; mathematical economics; mathematical finance; optimisation; and probability and statistics. IMPA's high profile in worldwide research is attested by the many recent distinctions received by its professors, including Ávila's Fields Medal and four invited talks at the 2014 International Congress of Mathematicians.

IMPA's current location.

IMPA's visitors and postdoctoral researchers are another major factor in making IMPA an exciting research environment. In 2016, we hosted the equivalent of 324 months of visits and postdocs. Researchers are attracted by the opportunity to collaborate with locals and other visitors in friendly, informal surroundings, with the stunning scenery of Tijuca Forest and the many attractions of Rio de Janeiro. A large number of visits to IMPA happen through cooperation agreements with international institutions, including CNRS (of which IMPA is a "Unité Mixte Internationelle"); École Normale Superieure, École Polytechnique, Brown University, Université Pierre et Marie Curie and Montréal's Centre de Recherche Mathématique.

Laboratories for applied research are another important aspect of IMPA's research life. The Fluid Dynamics Lab is devoted to fluid mechanics and its applications in the oil industry, LAMCA (the Laboratory for Mathematical Analysis in the Applied Sciences) pursues a range of projects in applied and industrial mathematics and VISGRAF is dedicated to research in computer graphics and vision.

Graduate studies

IMPA is one of Latin America's leading centres for graduate education in mathematics.

After an initial period of cooperation with the Federal University of Rio de Janeiro, IMPA's graduate programme was accredited by Brazil's Federal Council of Education in 1971 and it has been running regularly ever since. The programme has been instrumental in the development of mathematics in the region. Its alumni work in a large number of universities throughout Brazil and abroad. Through these alumni, many new centres for graduate studies have been created and consolidated around Brazil in recent years.

IMPA has granted over 450 PhD degrees in mathematics to date, with an average of 17 new doctors per

IMPA's library.

year over the past 10 years. Besides its PhD degree in mathematics, IMPA offers four streams of academic Master's degrees: pure mathematics, mathematical economics, computational mathematics and modelling, and computer graphics. As of 2008, IMPA started a professional Master's programme on mathematical methods in finance, with a focus on quantitative finance, financial instruments and risk analysis, that is geared towards qualified professionals holding non-academic positions.

Besides receiving students from all over Brazil, IMPA's international projection has had a direct impact on the diversity of its graduate students. Currently, nearly half of the institute's students come from abroad, mostly from other countries in Latin America but also from various places in Europe, Africa and Asia. Marcelo Viana, IMPA's current director, came to the institute as a PhD student in the mid 1980s after completing his undergraduate degree at the University of Porto in Portugal: "I was attracted by IMPA's international reputation, especially in dynamical systems. Here I found a lively and stimulating environment, ideal for the development of my research. In time, I realised there were plenty of opportunities for extending even more the scope of my work as a mathematician. This is an institution like no other." Viana was a plenary speaker at the 1998 ICM in Berlin and is the recipient of many international prizes, most recently the Grand Prix Scientifique Louis D., one of France's most prestigious scientific awards.

Other distinguished alumni of IMPA's PhD programme include 2014 Field's Medallist Artur Ávila, who first came to the institute as a high school student, after outstanding performances in mathematics Olympiads. Avila, who completed his PhD degree at the age of 19, was one of many students to profit from IMPA's flexibility in absorbing and training young talents. Avila's advisor Welington de Melo was himself a PhD student at IMPA in the 1970s under Jacob Palis.

Incoming classes for IMPA's PhD programme typically number around 20 students. A similar number of students start the Master's programmes every year. All students have financial support through scholarships provided by the national funding agencies CNPq and Capes, as well as Rio de Janeiro's agency Faperj. Further Information on how to apply can be found at http://www.impa.br/opencms/en/ensino/index_geral/index.html.

Programmes and events

IMPA's many events and programmes are a key aspect of its contribution to mathematics at the regional and global levels. In fact, it could be said that IMPA acts as a hub for mathematics in South America and (increasingly) worldwide.

VISGRAF Lab.

The institute regularly organises 14 conferences each year. These events range from small workshops to area schools and the 1000-plus-participant Brazilian Mathematical Colloquium, which is held at IMPA every two years.

Longer thematic programmes, each lasting between two and four months, are a recent addition to the institute's roster of events. Upcoming programmes are on parameter identification in mathematical models (October–November 2017) and graph theory (January–March 2018).

Lastly, IMPA has a well known Summer Programme, during the Southern Hemisphere Summer months of January and February. The importance of this programme for South American mathematics can hardly be overstated. Each year, the programme supports between 120 and 180 undergraduates and postgraduates for intensive classes. This has had a major impact on regional mathematical culture and many of the region's best mathematicians are alumni of this school. The programme also hosts a large number of visitors at the postdoctoral and senior levels, who come to take part in seminars, watch mini-courses on cutting edge research and collaborate with local and visiting colleagues.

Outreach and mathematical education

IMPA goes well beyond its role as a premier research centre in its efforts to disseminate mathematics. Brazil's best known books for college mathematics have mainly come from IMPA's publishing house. Lecture notes from the Brazilian Mathematical Colloquium have helped promote novel research directions throughout the country. IMPA has also promoted high school mathematics through the Brazilian Mathematical Olympiad (OBM).

In recent years, IMPA has become more involved with efforts to improve pre-college mathematical education in Brazil. Early initiatives include a series of books for high school mathematics as well as the PAPMEM teacher recycling programme. The PROFMAT Master's programme, which trains over 2,500 teachers around Brazil, was also started by the IMPA faculty.

IMPA's best known and by far its largest activity in pre-college mathematics is the Brazilian Mathematics Olympiad for Public Schools (OBMEP). Around 18 million children register for the OBMEP competition each year. Medal awardees are eligible for special training programmes and scientific initiation scholarships when they reach college. These have provided life-changing opportunities for many of the students involved, who often come from underprivileged backgrounds.

OBMEP has also had an impact beyond the competition winners. In fact, the competitive aspect is but one facet of a multipronged initiative that includes special training programmes for teachers, freely available teaching and study materials and a series of videos on YouTube. Detailed studies have shown OBMEP's measurable impact on schools nationwide.

In 2017, OBMEP and the older Olympiad OBM will be combined into a single competition, open to private as well as public school students. This will further broaden the impact of these initiatives.

The Mathematics Biennium: the ICM, the IMO and the Mathematics Festival

The years 2017 and 2018 mark a very special time for IMPA. In 2017, the International Mathematical Olympiad will be organised in Brazil for the first time. In 2018, the International Congress of Mathematicians will be held in Rio de Janeiro, the first such event to be held in either Latin America or the Southern Hemisphere.

IMPA is proud to be deeply involved with these two activities, which are landmarks for the Brazilian mathematical community. At the same time, these landmarks should not go unnoticed by society at large. On the contrary, the two events provide an opportunity to further promote mathematics in our country.

With this in mind, IMPA, the Brazilian Mathematical Society (SBM) and the Brazilian Academy of Sciences (ABC) approached Brazil's National Congress to make 2017 and 2018 Brazil's Mathematical Biennium. The "Joaquim Gomes de Souza" Biennium, named after Brazil's first research mathematician, was introduced into law in 2016.

The Biennium provides a boost to IMPA's dissemination efforts. This includes giving higher visibility to the ICM and the IMO and also promoting further activities. One important event is the "Festival da Matemática" in late April 2017. Hosted in Rio de Janeiro but with events nationwide, this festival features general-audience talks, movies, plays, exhibits and workshops geared toward young students and teachers. More information about the festival is available from its website (in Portuguese): http://www.festivaldamatematica.org.br/.

Henrique Bursztyn is a full professor at IMPA and the current head of graduate studies. His research lies in the areas of differential geometry and mathematical physics, including symplectic geometry, Poisson structures and Lie theory.

Roberto I. Oliveira is an associate professor at IMPA. He is also the current head for projects and planning at the institute. His research interests are in probability and related disciplines, such as statistics, quantum information and discrete mathematics, especially as they pertain to systems that have many degrees of freedom.

A Survey of Articles in the *Newsletter of the EMS* about the History and Activities of Full Member Societies of the EMS (Issue 1, September 1991 – Issue 104, June 2017)

Fernando Pestana da Costa (Universidade Aberta, Lisboa, Portugal), Editor of the EMS Newsletter

The architecture of the European Mathematical Society allows for individual and corporate members; in the latter category, there is a sub-category of "full members" that is made up of mathematical societies of the various countries of the European continent (plus Israel).

Full member societies have had a presence in the *Newsletter of the EMS* since its very first issue, with information articles about their history and activities. These are not just short announcements of their current initiatives (which are also regularly provided) but also short papers (typically 1–3 pages long) describing something of their history as well as their most important past and current activities.

Unfortunately, in spite of the existence of open access to all *Newsletter* issues on the webpage of the EMS Publishing House (http://www.ems-ph.org/journals/all_issues.php?issn=1027-488X), it is not exactly easy to check for published articles about EMS full members. In order to have a clearer idea of what has been published and so to be able to plan future actions, the Editorial Board of the *Newsletter* has recently compiled a survey of that information. We believe this survey may also be of interest to all EMS members and to the general readership of the *Newsletter*, as it provides a direct pointer to reliable published information about mathematical societies in Europe that can be difficult to obtain otherwise. In Table 1, we present the results of the survey. As part of Editorial Board duties, in due time and according to the planned Editorial Board policy, societies that have not yet been covered in the Newsletter will be asked to contribute.

Survey[1,2] of articles about EMS full member[3] societies published in the *Newsletter* up to issue 104

Name of Society	Country	Founding Date	Published in	
			Issue	Date
Armenian Mathematical Union	Armenia	1901		
Association for Mathematics Applied to Economics and Social Sciences	Italy	1976	96	06 / 2015
Association of Mathematicians of Republic of Macedonia	Macedonia			
Austrian Mathematical Society	Austria	1903		
Belarusian Mathematical Society	Belarus			
Belgian Mathematical Society	Belgium	1921	29	09 / 1998
			80	06 / 2011
Bosnian Mathematical Society	Bosnia			
Catalan Mathematical Society	Spain	1931	93	09 / 2014
Croatian Mathematical Society	Croatia			
Cyprus Mathematical Society	Cyprus	1983	66	12 / 2007

1 Information about the founding date of some societies could not be obtained from the data in the EMS webpage and archives. We thank Elvira Hyvönen, secretary of the EMS, her help in procuring this information.
2 We apologize for any omission or error and thank in advance all readers that care to send us corrections.
3 To the best of our knowledge all Societies in the table were full members at the time of the published article. Some may have ceased to be so in the meantime.

Czech Mathematical Society (Union of Czechs Mathematicians & Physicists)	Czech Republic	1862	1	09 / 1991
			43	03 / 2002
			85	09 / 2012
Danish Mathematical Society	Denmark	1873	35	03 / 2000
			88	06 / 2013
Edinburgh Mathematical Society	UK	1883	33	09 / 1999
			74	12 / 2009
Estonian Mathematical Society	Estonia	1926	41	09 / 2001
Finnish Mathematical Society	Finland	1868	9	09 / 1993
			32	06 / 1999
French Statistical Society	France	1997		
German Mathematical Society	Germany	1890	29	09 / 1998
			82	12 / 2011
Georgian Mathematical Union	Georgia	1923		
Hellenic Mathematical Society	Greece	1918	45	09 / 2002
Icelandic Mathematical Society	Iceland	1947		
Institute of Mathematics and its Applications	UK	1964	91	03 / 2014
Irish Mathematical Society	Ireland	1976	2	12 / 1991
			94	12 / 2014
Israel Mathematical Union	Israel		47	03 / 2003
Italian Mathematical Union	Italy	1922	38	12 / 2000
Janos Bolyai Mathematical Society	Hungary	1891	59	03 / 2006
Kharkov Mathematical Society	Ukraine	1879	34	12 / 1999
Kosovar Mathematical Society	Kosovo	2008		
Latvian Mathematical Society	Latvia	1993	48	06 / 2003
Lithuanian Mathematical Society	Lithuania	1962		
London Mathematical Society	UK	1865	31	03 / 1999
			37	09 / 2000
			99	03 / 2016
			103	03 / 2017
Luxembourg Mathematical Society	Luxembourg	1988	34	12 / 1999
Malta Mathematical Society	Malta			
Mathematical Society of France	France	1872	40	06 / 2001
Mathematical Society of Serbia	Serbia	1948		
Mathematical Society of the Republic of Moldova	Moldova	1950s		
Moscow Mathematical Society	Russia	1864	50	12 / 2003
			51	03 / 2004
Norwegian Mathematical Society	Norway	1918	41	09 / 2001
Norwegian Statistical Association	Norway	1919		
Polish Mathematical Society	Poland	1919	32	06 / 1999
			54	12 / 2004
Portuguese Mathematical Society	Portugal	1940	46	12 / 2002
			98	12 / 2015
Romanian Mathematical Society	Romania	1910	40	06 / 2001
			77	09 / 2010
Royal Dutch Mathematical Society	Netherlands	1778	35	03 / 2000
Royal Spanish Mathematical Society	Spain	1911		
Slovak Mathematical Society	Slovakia			
Società Italiana di Matematica Applicata e Industriale	Italy	1989		
Société de Mathématiques Appliquées et Industrielles	France	1983	6	12 / 1992
			40	06 / 2001
Society of Applied Mathematics and Mechanics	Germany	1922		
Society of Mathematicians and Physicists of Montenegro	Montenegro	1947		

Society of Mathematicians, Physicists and Astronomers of Slovenia	Slovenia	1949	47	03 / 2003
Spanish Society of Applied Mathematics	Spain	1991		
Spanish Society of Statistics and Operations Research	Spain	1962		
St. Petersburg Mathematical Society	Russia	1890	10	12 / 1993
Swedish Mathematical Society	Sweden	1950		
Swiss Mathematical Society	Switzerland	1910	33	09 / 1999
Turkish Mathematical Society	Turkey	1948		
Ukrainian Mathematical Society	Ukraine			
Union of Bulgarian Mathematicians	Bulgaria	1898	6	12 / 1992
			31	03 / 1999

ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

In the most recent issue of the ICMI Newsletter (March 2017), a double editorial was published from the new president Jill Adler and the former president Ferdinando Arzarello. With their authorisation, we re-publish this double editorial here today.

From the desk of the ICMI President Jill Adler

Four years ago, the immediate past president Ferdinando Arzarello wrote from the desk of the ICMI president about the multiple needs mathematics education must serve, where all should have opportunity not only for mathematical literacy but also the adventures of mathematics itself. He drew on the 2012 UNESCO booklet on basic mathematics education written a few years ago mainly by ICMI past president Michele Artigue, assisted by other colleagues in the ICMI. We are pleased to announce this has recently been translated into Portuguese, thanks to the work of ICMI executive committee member at large Yuriko Baldin (http://www.sbm.org.br/?s=desafios+do+ensino). The description of basic mathematics in the booklet highlights the multiple demands on mathematics education and so the need for reform in mathematics in schools, across all levels. In a globalising world, we understand our cultural diversity and the dangers of domination and alienation up ahead for any consideration of common curricula or what is quality teaching and learning of mathematics.

The ICMI launched two studies in response to these challenges. Study 23 on primary mathematics, co-chaired by Maria Bartonini Bussi (Italy) and Xuhua Sun (China), is the first study devoted to early learning and its study volume is almost ready for printing and dissemination. ICMI Study 24, entitled 'School Mathematics Curriculum Reforms: Challenges and Changes', has just been launched, with Renuka Vithal (South Africa) and Yoshi Shimizu (Japan) as the two co-chairs. The International Programme Committee has been finalised and will meet later this year to develop the discussion document that will frame its study conference in 2018. This is a strategic moment for the ICMI to collect, analyse, synthesise and then communicate our collective research and practical wisdom in school mathematics curriculum reform. As the new ICMI executive committee prepares for its first meeting in June this year, a key agenda item will be the ongoing support for Study 24, followed by an initial discussion of what might be our next study, and so some considered directions in which to further grow the organisation and its influence.

With these introductory comments, I hope I have illustrated the continuity and growth that now define the organisation, as one executive committee and its new president take over from another. Thus, accompanying this introductory message from the new president is a farewell letter from Ferdinando Arzarello. In this, he describes the developments and progress in the ICMI during his presidency and I take this opportunity, on our collective behalf, to thank Ferdinando and his executive committee for all their work. I will not refer again here to the ongoing continuous work of the ICMI. I will only say that being elected as the president of the ICMI is an immense honour, especially at a time when the organisation is so strong and where a critical component of our strength is our collaborative and increasingly productive relationship with the IMU and the global community of mathematicians.

I will use this opportunity for my first communication with our ICMI community to convey my greetings to you all, each and every participant in all ICMI activities, and to invite you all to participate with me and the new executive committee in ongoing communication about our work – past, present and future.

In particular, I wish to welcome all national representatives into your roles if you are new this year and to thank those who are continuing in their roles for their ongoing work. We have 94 member countries and each has a representative through whom there is a formal line of communication to and from the executive committee. If you were at ICME13 in Hamburg in July, you would know from Ferdinando's presentation during the opening ceremony that we have numerous member countries and therefore national representatives across some continents but not others. We hope in our term of office over the next few years to enable greater representation and participation from countries in Africa, South and Central America, Eastern Europe and parts of Asia. We are, of course, only too aware of the challenges facing us all with this, as our world seems to have become increasingly unequal and fragmented. You have been introduced to the nine members of the executive committee (http://www.mathunion.org/icmi/icmi/icmi/executive-committee/ec-2017-2020/) and you can see that we are spread across continents and so hopefully in a good position for this task.

The ICMI executive committee is currently preparing the agenda for our first meeting in June and so, right here, right now, we invite you to communicate with us, with ideas that you would like us to take forward and, of course, concerns with the organisation or its work. Formally, such communication should happen through the national representative in your country. If you are not aware who this is, the list is available at http://www.mathunion.org/icmi/icmi/members/icmi-representatives/. We hope, over the next four years, to strengthen these lines of communication. Of course, you could write directly to me (icmi_president@mathunion.com), to our secretary general Abraham Arcarvi, (Abraham.Arcavi@weizmann.ac.il), to our administrator Lena Koch (icmi.cdc.administrator@mathunion.org) or to any of the executive committee members.

Of course, we have been thinking about our growth. You will all know, and Ferdinando's letter provides detail on this, that a key direction for growth for some time now has been what can be described as a development agenda. Expanding the ICMI's reach into new communities has been a key concern. The Capacity and Networking Project (CANP), and substantial solidarity grants to support participation at the ICME, demonstrate our growing success. In my recent work in South Africa, I have been challenged by how reform ideas in mathematics education are taken up (or not), particularly in developing country contexts, and thus contexts of educational disadvantage. The 2015 millennium goals for universal primary education are becoming a reality. Coupled with this, however, is a concern that while most might now have access to school, in many areas this has not come with quality education. In 2012, the Conference of Commonwealth Education Ministers described this situation as "education for all, learning for some" (www.cedol.org). Just as we confront diversity as we study curriculum reform, so there are different orientations to what is quality mathematics teaching and quality mathematics learning.

I have been stimulated by recent literature and research related to educational development and comparative studies. There has been provocative debate in leading journals (the International Journal of Education Development and COMPARE) on pedagogy promoted in development projects and interventions, particularly in contexts where socio-economic conditions deny quality education to the majority of students in school. The current UNESCO goal is for sustainable development and while this is not specific to mathematics education, it is a programme we in the ICMI can think about, stimulate interest in and contribute to. The goal of sustainable development is shared with the IMU and the CDC (Commission for Developing Countries) and also carried out through its work with the ICSU, the International Council of Scientific Unions. At this moment, we are collaborating with the project Mathematics for Planet Earth, with a new project inviting modules that specifically speak to the African context.

We in the executive committee have also been inspired by the talk entitled Mathematics for Human Flourishing given to the Mathematics Association of America (MAA) by its outgoing president Francis Su in January this year. He framed his eloquent and passionate talk with the question: "How can the deeply human themes that drive us to do mathematics be channelled to build a more beautiful and just world in which all can truly flourish?" He suggests these themes are fun, beauty, truth, justice and love. These are sustainable development goals crafted in a different form for mathematics and so too for mathematics education (https://mathyawp.wordpress.com/2017/01/08/mathematics-for-human-flourishing/).

Being elected as president of the ICMI is thus not only an honour but also a huge responsibility, to ensure the continuing strength of the institution. This is a turbulent period in the world and there are increasing threats to collaboration and social justice and thus flourishing for all. However, as a South African, I have learned over and again how turbulence brings opportunities. During my mandate, and with a wonderful executive committee elected to support the ICMI's work over the next four years, we will work to maximise these opportunities.

Jill Adler, President of the ICMI
(ICMI_President at mathunion.org)

Farewell letter from ICMI Past President Ferdinando Arzarello

Dear Friends,

At the end of my term as ICMI president, I wish to thank you so much for the strongly collaborative spirit with which we have been able to work together in order to pursue the aims of our joint endeavour. For me, it has been an exciting period: ICMI programmes have allowed me to engage in wonderful challenges, whose results I do hope have been useful for supporting and improving mathematics education in many parts of the world.

In this mission, I was strongly sustained by the whole executive committee, whom I thank so much; I am particularly grateful to the secretary general and to the two vice-presidents for their precious advice in many circumstances. In the case of Abraham, the continual interactions have also developed a deep friendship: many times I realised that we were sharing a common interpretation of circumstances even before discussing them. I think that these peculiar interactions have also been useful for ICMI policy.

A special thank you must also go to Lena Koch, whose help, collaboration and suggestions incessantly supported and encouraged me in my day-to-day work as president. I think that the whole ICMI family, not only our executive committee, owes her a lot. Thank you Lena!

Usually, custom dictates that at the end of their term people should carry out an analysis of their activity. I do not wish to break this tradition but I also do not like to bore people with long inventories, so I will limit myself to listing some of the ICMI courses of actions that, as far as I can see, are worthwhile underlining.

I consider first the issues that pertain to the ICMI "by default". I will only sketch some of them since the secretary general has given detailed reports on all such activities on many occasions where all of us have been present (e.g. in his report at the last general assembly or in his speech at the ICME closing ceremony: https://lecture2go.uni-hamburg.de/l2go/-/get/v/19779). Because of this, I will limit myself to recalling the huge efforts that many of us have made in ICME events, through our participation in the scientific work of the IPC and the important decisions about support to participants from developing countries, and through the careful choice of the country that will host the next ICME. The competition among three top-level contenders like Australia, China and the USA (Hawaii) for hosting ICME 14 shows the increasing relevance that our organisation has for people working in mathematics education.

Another significant activity has concerned the organisation of the general assembly, where the main items of ICMI policy and institutional life are presented, discussed and approved: the minutes of its Hamburg meeting show the relevant contributions of the ICMI Affiliated Organisations (http://www.mathunion.org/icmi/icmi/icmi-as-an-organisation/general-assembly/) and the high participation of the country representatives. We have taken care, as far as possible, of the links with them and this systematic task has been fruitful. Also, the wide participation of members of the executive committee in the activities of the ICMI regional conferences and other relevant regional events all over the world shows the vitality of our institution.

Another important issue I am happy to mention here concerns the relationship between the ICMI and the IMU: it has improved greatly in recent years, mainly thanks to the intelligent policy of former presidents in that direction, which has continued in these last few years. I wish to thank the current and past presidents and secretaries of the IMU, Ingrid Daubechies and Martin Grötschel, and Shigefumi Mori and Helge Holden, for their support and help. The collaboration with them in everyday activities, as well as in specific programmes, has been wonderful and productive. Moreover, the support of the IMU secretariat, with its distinguished head Alexander Mielke and excellent team (Sylwia Markwardt, Lena Koch, Anita Orlowsky, Birgit Seeliger, Gerhard Telschow and Ramona Keuchel) has always been a precious concrete help for all our activities.

I will finish my farewell with some more "political" thoughts that I have progressively elaborated during my work in the ICMI, thanks to crucial interactions with the members of the executive committee and with many people of the wider ICMI family.

I think that, while designing our programmes, we have reflected extensively on the meaning of mathematics teaching/learning in the era of globalisation: curricula, teachers, classroom practices and cultural, political and social issues. The world frame in this matter (and not only in this) is full of contradictions, which have constituted a challenge for us and I should think a challenge for all, mathematicians, mathematics educators, policymakers and mathematical education societies alike. As pointed out in an important UNESCO document, on the one hand the universality of technological development and related needs for manpower skills are playing the role of strong historical motivation for reform that should lead to unified standards for mathematics in school. But, on the other hand, for real success in mathematics education it is crucial to avoid both the cultural distance of some proposed curricular reforms from the mathematical culture of the different countries, as well as students' alienation from their cultural environment, which can inhibit them from engaging in learning in a productive way.

Based on the inspiring experiences of previous ICMI executive committees, we have devoted many resources and much energy to some projects that we think are crucial for featuring our own policy; among them I recall the four I like the most:

- CANP activities, and how these have improved through the publication of their volumes at an international level, and a scientific survey of CANPs conducted by Lena Koch (her detailed and informative review will be uploaded to the ICMI website in the near future; in the meantime, a long summary is available at http://www.mathunion.org/fileadmin/ICMI/files/CANP/PP_CANP_ICMI_ICME_CANP_WORKSHOP.pdf).
- The ICMI Study 24 on 'School Mathematics Curriculum Reforms: Challenges and Changes', whose launching document, because of its complexity, required a lot of discussion within the executive committee.
- The new Emma Castelnuovo Award, which underlines the relevance of practices in addition to research in mathematics education (according to the ICMI spirit).
- The Klein project, which aims to bridge the gap between the mathematics traditionally taught in second-

ary school and its most recent results through vignettes that can inspire teachers in their daily activities.

From these experiences, some particularly significant aspects have emerged as major challenges for mathematics education at the beginning of the new millennium and, as is sometimes the custom of people who finish their term, I dare to leave them as a small legacy.

I think that the main fresh challenges concern:

- The education of younger pupils.
- Teacher training.
- The gender gap in mathematics learning.
- The use of internet technologies n.0 ($n > 1$) for mathematics teaching and learning.

For some of these, the ICMI has already done interesting things but I think the ICMI could do even more. For example, with Study 23, the ICMI has started to extend its concerns to primary education. This study focuses on a segment of students that were traditionally not a core concern. I think this most worthwhile new trend will continue in some way. Moreover, the publication of the Study 22 volume on Task Design certainly constitutes an important tool for researchers and practitioners.

Other challenges at the moment are at the stage of promising beta-projects: e.g. the organisation of a MOOC for researchers and teachers as a resource of high-level lectures given by ICME awardees. Others concern the work of specific IMU commissions (where the ICMI has its representing member), which are carrying out projects where the ICMI can provide a relevant contribution.

A last word on what I call the "sleep of reason", rephrasing the title of a well known F. Goya etching (Capricho 43: "El sueño de la razón produce monstruos"): 'The ongoing tremendous events in many parts of the world seem produced in fact by monsters that such a sleep generates. I think that we can react against this sleep, trying to realise another meaning of the Spanish word "sueño", or "dream": hence, let us struggle for a "dream of reason". Our contribution to this dream can be pursued by supporting and strengthening the diffusion of a solid mathematical education, rooted in the cultural contexts of the different countries but universal in its final content. It will contribute by helping people think for themselves and understand one another. It will only be a drop in the ocean but not a useless one and it is important that all of us do our best in this.'

This is the legacy I leave to the next ICMI executive committee, which will start its task on 1 January. I wish all its members and particularly Jill Adler (the new president) and Abraham Arcavi (the re-elected secretary general) all my best wishes for continuing and enhancing the ICMI mission over the next four years. As the past president, I will be an ex-officio member of the executive committee and I will have the privilege of continuing to work for the ICMI and collaborate with them in this exciting enterprise.

Dear Friends,
Our common work over these four years has been a really exciting human and cultural experience: apart from our realised programmes, of which all of us are justly proud, our mutual knowledge and friendship is one of the most solid results we achieved. Thank you again!

Torino, 30 Dec 2016
Ferdinando Arzarello, Past President of the ICMI

Results of the 2016 EMS User Survey for zbMATH

Ingo Brüggemann (Max Planck Institute for for Mathematics in the Sciences, Leipzig, Germany), Klaus Hulek (Gottfried Wilhelm Leibniz Universität Hannover, Germany) and Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

In 2005, the European Mathematical Society took the initiative to appoint a Scientific User Committee (SCUC) of zbMATH (then Zentralblatt MATH). The main intent was to actively involve the scientific community in a number of developments that were felt to be both necessary and important. Jean-Pierre Bourguignon, who had been a driving force for the creation of the SCUC, also assumed the duties of the first chair. The most urgent issues at the time were the lack of appropriate author identification, the question of use and misuse of reference data and citation measures, and interface functionality. Further developments since then have been the interlinking with open sources like EuDML and arXiv, the database of mathematical software swMATH and formula search.

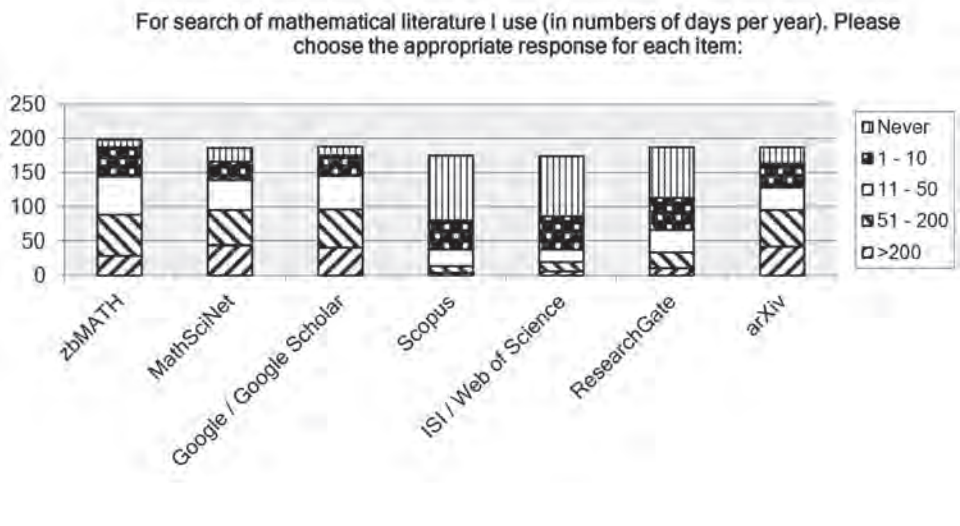
In 2012, the SCUC (most notably by the efforts of its then chair, Stephan Klaus, from the Oberwolfach Research Institute for Mathematics) prepared a user survey to gather information on the priorities for zbMATH developments from a user perspective. The survey was distributed at the 6th ECM at Kraków on both paper and online forms. Four years later, during the 7ECM at Berlin, a renewed survey was conducted with the main aim of evaluating recent developments but also identifying future directions. At the same time, the results also served as a report on the perception of zbMATH developments under more than a decade of guidance by the SCUC.

While the 7ECM was one of the main dissemination channels for the survey, others (like the zbMATH entry page and reviewer and EMS member mailings) ensured survey participation that reflected diverse usage quite well. Though 66% of the 209 respondents were from Europe (and 22% from Germany), indicating a slight conference participation bias, there was also participation of over 13% from the U.S. and 10% from South America; Asia was the only large region relatively underrepresented. While more than 60% of the participants held professorships, there was also broad participation from postdoctoral and doctoral researchers, students and mathematicians holding other positions. Also, the subject areas (according to the MSC) turned out to be as broad as one could reasonably expect.

The first questions were directed to search customs. On a general level, arguably the most significant result is that a majority of mathematicians rely on maths-specific services (arXiv, MathSciNet, zbMATH) in comparison to generic providers like ISI, Scopus and Google Scholar. From a zbMATH perspective, the most positive results are that this service is today used more frequently by 48% of users compared to five years ago (38% use it at about the same level and 14% use it less).

The preferred search topics are quite diverse and often specific: while author is by a slight margin the most preferred aspect, title comes in a close second, while more specific facets like MSC or even formulae are considered relevant by a majority. Even more surprising was the extensive use of free-text feedback for this question. Almost half of all respondents indicated the use of sometimes quite sophisticated combinations of search aspects. The option of extensive logical combinations, which has been additionally supported by filters for some years, is obviously heavily used. Likewise, basically all aspects of the search results (metadata, full text links, reviews, citations, profiles) are considered almost equally important.

Further questions were specifically concerned with new developments. Concerning the zbMATH author database, 52% of the answers confirm that it has



Survey responses concerning search customs of mathematicians.

improved significantly (41% of the participants agree with this statement in principle). On this matter, the quality of the information is, in general, much more highly valued than the quantity – the most important aspects are correct author disambiguation, comprehensive information and clean attribution. More specifically, for example, precise author disambiguation was considered much more important (more than 85%) than author citation statistics (30%). The SCUC also included here a question directly comparing to the MathSciNet author database, which was preferred by 25%, while 22% found zbMATH more reliable. The majority (53%) voted that both services have their strengths and weaknesses.

A second large development has been the gradual inclusion of reference data and profiles, which is currently available for a third of recent publications. A difficult aspect here is that this is an ongoing process, and data and profiles are therefore necessarily incomplete. An overwhelming majority (80%) understood this and agree with the inclusion of this feature at an early stage, while less than 2% were against this and would have preferred to omit potentially incomplete citation information. A general fear connected to the integration of reference data into documents and profiles was that this may give quantitative measures an unjustified prevalence over individual quality assessment (as provided by, for example, reviews). The survey does not yet support this – a large majority (about 75%) of users use citations to discover other interesting work rather than using them as a tool to assess impact. The general experience that mathematicians are quite aware of the fallacies connected to superficial use of statistics seems to prevail.

Several further functions have been implemented in the service, like filters, profiles, a software database and formula search. The functionality is generally appreciated (with average marks ranging from 3.5/5 for bibtex to >4/5 for the search function) and 86% of the respondents think that the service has improved decisively since 2011.

Completeness of entries and quality of reviews are issues for a service like zbMATH that require continued efforts and are naturally always a subject of discussion. While the completeness is generally viewed favourably (4.2/5, with some limitations for very recent articles) and reviews are usually considered as correct (4/5), there is room for improvement regarding the frequency (3.6/5), timeliness (3.7/5) and quality (3.8/5) of reviews, as well as for the suitability of reviewers (3.8/5). Since the decisive factor here is the availability of reviewers, we take the opportunity to encourage the reader to join the reviewer community to facilitate further improvements here.[1]

The question of possible future developments was naturally one of the most interesting ones for us. The diverse answers of fields that were considered relevant left no doubt that there will be an ample amount of work ahead of us in the years to come! User priorities ranged from aspects of historical importance like the digitisation of scans (considered highly desirable with 4/5), gradual improvements like further integration of full text links (3.9/5), research data information like swMATH (3.2/5) or institution codes (3.5/5) to areas like full-text formula search (3.4/5), where technology is still under development.

Finally, zbMATH offers many freely accessible features. Recently, free author and journal profiles have been added to the traditional three free hits, as well as free EMS member accounts, the swMATH facet and formula search. It is perhaps not surprising that all of these have been quite well received (with marks from 3.6/5 to 4.2/5) and several comments suggest that further steps would be welcomed in this direction. While such efforts are still limited by the need to maintain the resources for zbMATH production and development, we can promise to pursue all feasible solutions. This may also be illustrated by a free referencing tool recently made available to MathOverflow users (with hopefully more to come).

Overall, the survey supports the statement that the SCUC has accompanied a decade of exciting developments for zbMATH. It was decided at the last CC meeting that the duties of the SCUC will, in future, be transferred to the newly formed EMS Committee for Publications and Electronic Disseminations. We would like to take this occasion to thank all the SCUC members for their valuable contributions over the years!

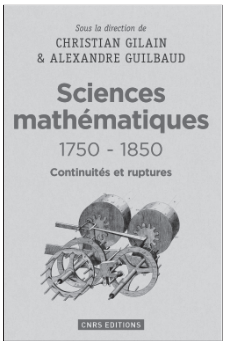
Ingo Brüggemann (Ingo.Brueggemann@mis.mpg.de) is head librarian of the Max Planck Institute for Mathematics in the Sciences, and served in this function during two terms in the SCUC of zbMATH, bringing into it a certain complementary viewpoint of an in-between. In his role of a Vice-Chair, he coordinated the EMS survey on behalf of the SCUC. His favourite zbMath feature, by the way, is the formula search: math texts are the most formalized of the sciences and this kind of search hints at a way to get a grip on really making the texts' contents machine readable.

Klaus Hulek studied mathematics at the University of Munich and the University of Oxford. He completed his PhD in Erlangen in 1979 and spent the academic year 1982/83 at Brown University, Providence, RI, USA. He received his habilitation in Erlangen in 1984. Klaus Hulek was a professor of mathematics in Bayreuth from 1985 to 1990, when he moved to Hanover. He held the position of Vice-President of Research of Leibniz Universität Hannover from 2005 to 2014, and from 2007 to 2014 he represented the German Rectors' Conference (HRK) at the Research Policy Working Group of the European University Association (EUA). Klaus Hulek was a member of the Institute for Advanced Study (IAS) in Princeton in 2015. His field of research is algebraic geometry.

Olaf Teschke [olaf.teschke@fiz-karlsruhe.de] is a member of the Editorial Board of the EMS Newsletter, responsible for the zbMATH Column.

[1] https://zbmath.org/become-a-reviewer/

Book Reviews



Christian Gilain &
Alexandre Guilbaud, eds.

Sciences mathématiques
1750–1850
Continuités et ruptures

CNRS Editions, 2015
560 p.
ISBN 978-2-271-08295-4

Reviewer: Catherine Goldstein

In the last few decades, the internet has provided us with almost unlimited access to thousands of the mathematical texts of the past, which can be used in a pedagogical environment or for popularising mathematics or even as a direct source for research. Meanwhile, though perhaps less conspicuously, the history of mathematics has flourished as an autonomous discipline, with its own key problems. Its objective is not to simply look back, searching in old texts for traces of current results, but to provide compelling answers to specific questions about the nature and the dynamics of the development of mathematics. The book under review is a good example of this. Despite its somewhat deceptive title, it is not an overview of simply what happened in mathematics between 1750 and 1850 but a state of the art example of historical research around a particular question: was there, as traditional historiography tends to state, a radical and global rupture in the way of doing mathematics around 1800?

The first part of the book, written by the two editors C. Gilain and A. Guilbaud, summarises this traditional point of view, tracing it back to Felix Klein's early 20th century *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (the first part of which was translated into English in 1979). In this still influential, although historiographically now dated, synthesis, Klein describes the early 1800s as the moment of a deep rupture in mathematics (implicitly restricted to what occurred in Western Europe) and this point of view has more or less prevailed since then. In this view, 18th century mathematics is seen as being non-rigorous and computational, primarily directed toward uses and applications and linked to and supported by monarchical institutions, in particular royal academies of science. Mathematics of the 19th century, on the other hand, is seen as becoming rigorous and increasingly specialised, with a keen interest in the foundations of mathematics and an emphasis on pure mathematics; it would develop in a democratic or meritocratic system, in universities or in new types of schools such as the French Polytechnique. Klein, and those who wrote after him, linked this change to the French Revolution and its new values; some authors have even incarnated the two sides of this break in individual mathematicians, opposing, for instance, Euler as a utilitarian man of the Enlightenment to "Romantic" mathematicians like Abel or Galois.

By thoroughly analysing and putting together numerous writings that, after Klein, aimed to document one aspect or another of this supposed rupture, Gilain and Guilbaud are able to display the contradictions of what now appears as much too simple a vision. For instance, they show that different authors associate the same mathematician to one or other side of the break (key examples are Lagrange, Fourier and Gauss); or that essential aspects of mathematical development, in particular those connected to the mathematisation of natural phenomena, are twisted or even ignored to fit this rough picture; or that theoretical discussions on the foundation of certain concepts reveal strong continuities from the mid 18th century up to the 1850s, an example being the concept of a function; or finally that institutional change, the creation of the Polytechnique being one example, is sometimes presented as the cause of the wider change, sometimes as its consequence and sometimes only as one of its components. Changes, then, often appear as local and specific to a certain aspect, each with its own chronology; they do not mesh globally into a supposed explosive birth of modernity.

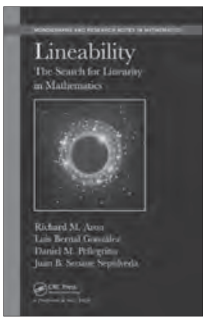
The second part of the book then enlarges the timeline to a full century (centred on 1800) in order to capture, through detailed examination of original sources, exactly what sort of continuities and discontinuities can be traced through the period, and where and how they operate. This part comprises 14 chapters by 18 authors; it addresses four key issues raised in the first part: the shaping of mathematics through institutional change, the relations between mathematics and its applications, foundational questions in mathematics, and the balance between computational and formal aspects of the field. By focusing in each chapter on specific examples, the authors are able to precisely locate the effect on mathematics of certain changes: some come from the subdisciplines involved, some from what counts as an acceptable answer and some simply from the changes in the nature of publishing, etc. One will find, for example, in-depth studies on: the so-called secular equation (that is, for us, the characteristic equation of a system of linear differential equations with constant coefficients), migrating from mechanics to astronomy to matrix algebra; a certain dice problem (the probability of obtaining a given sum when one throws n f-faced dice) using various tools (combinatorial, analytical and so on); the multiple 18th century origins of the early 19th century foundation of projective geometry or of a Diophantine problem; and the effects on mathematics of various educational reforms in German-speaking countries. All the chapters display a much richer, more complex picture of mathematics around 1800 and its diffusion, as well as a convincing refutation of the thesis of a global rupture at this time.

Besides this main axis, there are several interesting features in these contributions. For instance, they show that what counts as a mathematician – and even as mathematics – has no straightforward answer. Coulomb is usually considered more of an engineer-physicist than a mathematician and thus not taken into account in most histories of mathematics. However, his work on the mathematisation of electricity in the 1780s displayed new relations between theoretical laws, numerical approximations and the search for an agreement with precise, experimental measurements, which would subsequently inspire the 19th century's well known representatives of mathematical physics like Poisson and Fourier. Besides the usual stars of the period – Laplace, Cauchy and Gauss – a profusion of not so well known figures whose contributions are shown to be decisive appear in the pages of this book (as witnessed by a 12-page index of people!). The same is true for mathematical topics: finite difference equations and astronomical techniques provide illuminating insights into the development of the concept of computation or that of a "rigorous" solution of an equation.

What counts as social or cultural factors is also revisited in some of the contributions. One chapter, for instance, carefully links the crushing defeat of the French navy in 1763 to a deep reform of the curriculum of the officers and then to the writing of new textbooks and finally to the development of innovative algebraic tools and results. Two other chapters present very interesting surveys of the history of mathematical publishing during this period, including the typographical problems raised by formulas. Matching this systematic methodology in the use of sources, the book also offers useful tables and graphs, for instance on journals and textbook authors.

As explained in the final remarks by the two editors, extending both the period and the mathematics taken into account show, on the one hand, different, asynchronous breaks and, on the other, many localised continuities and traditions. In order to be able to reach a more balanced, synthetic and accurate picture of the period, the editors therefore plead for more detailed studies on 18th century mathematics, a subject they consider as still the poor sibling in the historiography of Western mathematics. The various chapters, however, point to a variety of phenomena contributing to the diffusion and evolution of mathematics; not only continuity and ruptures are at work but also oblivion, rediscovery, parallel developments, etc. One might perhaps regret that the book does not address the terms in which a new synthesis on the studied period could be written. Still, it certainly provides a variety of new material and key lines of thought, which will have to be integrated into any such synthesis.

Catherine Goldstein is Directrice de recherche at the CNRS (Institut de Mathématiques de Jussieu-Paris Gauche, Paris). Her research interests lie in the history of number theory and in the history of mathematical practices, combining close readings and a network analysis of texts. She has recently co-edited A War of Guns and Mathematics *(AMS, 2014) and* Les travaux combinatoires en France (1870–1914) et leur actualité *(PULIM, 2017).*



Richard M. Aron, Luis Bernal-González, Daniel M. Pellegrino and Juan B. Seoane Sepúlveda

Lineability. The Search for Linearity in Mathematics

CRC Press, 2016
xix, 308 p.
ISBN 978-1-4822-9909-0

Reviewer: Olav Nygaard (Kristiansand)

The Newsletter thanks zbMATH and Olav Nygaard for the permission to republish this review, originally appeared as Zbl 1348.46001.

At long last, the main results and techniques in the field of lineability have now become available in a book. Even more, it has turned out an extremely nice book, which is really written as an invitation for more researchers to enter the field.

Lineability, spaceability or algebrability of a set M in a Banach space X means that $M \cup \{0\}$ contains a linear space, a closed linear space or even an algebra. Naturally, these concepts have to be quantified in some measure of bigness, like dimension or cardinality.

When we study some set $M \subset X$, we of course check if it has an algebraic structure. And if not, what is a more natural question than looking for an algebraic structure inside M? However, as we know in general, asking is easier than answering, and reading the book under review teaches us that deciding whether M contains an algebraic substructure is really a non-trivial task. Practically every example of the book serves as a proof of that statement.

Let me describe two, to me very appetizing, examples before turning to the organization of the book:

Example 1: Let $X = C[0, 1]$ and recall Banach's beautiful result that the subset M consisting of nowhere differentiable functions is residual. So M is huge. We note that $k_f \in M$ if $f \in M$ and $k \neq 0$. But $0 \notin M$, so M is not linear. Now, if we add the origin to M, can we find an algebraic structure inside? In 1966, Vladimir Gurariy proved that

$M \cup \{0\}$ contains an infinite-dimensional linear space [*Sov. Math., Dokl.* 7, 500–502 (1966); translation from *Dokl. Akad. Nauk SSSR* 167, 971–973 (1966; Zbl 0185.20203)]. In the 1990s Gurariy, together with Vladimir Fonf and Mikhail Kadets, showed that $M \cup \{0\}$ contains an infinite-dimensional Banach space Z [*C. R. Acad. Bulg. Sci.* 52, No. 11–12, 13–16 (1999; Zbl 0945.26010)]. But there is not only one such Z; L. Rodríguez-Piazza [*Proc. Am. Math. Soc.* 123, No. 12, 3649–3654 (1995; Zbl 0844.46007)] showed that Z can be chosen to be any separable Banach space (here recall that $C[0,1]$ contains all separable Banach spaces as subspaces)! Have a look at pp.24–25 to learn even more.

Example 2: Let X be the dual of some Banach space Y and let M consist of the functionals in $X = Y^*$ that attain their norm, i.e., $f \in M$ if and only if there exists some norm-one element $x_f \in Y$ such that $f(x_f) = \|f\|$. A celebrated theorem of R.C. James says that M is not all of X unless Y is reflexive (and then, of course, $M = X$). The Bishop–Phelps theorem tells us that M is, however, always norm-dense in X. So, somehow M is big. This time 0 belongs to M. Sometimes M is linear, as in the case $Y = c_0$ (then M is simply the ℓ_1-elements with finally many non-zero coordinates). But, in general, M is not linear, as for $Y = \ell_1$ (add $(1, 1/2, 2/3, 3/4, \ldots)$ and $(-1, 0, 0, \ldots)$). We see that when $Y = c_0$, M is lineable, but not spaceable. Today we know, courtesy of a subtle example due to M. Rmoutil ["Norm-attaining functionals and proximinal subspaces" Preprint (2015), arXiv:1503.06112] that M need not even contain a two-dimensional linear space.

Let me now say a little about the content of each chapter, just to indicate the structure of the book.

A 20 page introductory chapter serves to prepare us with basic knowledge of cardinality, algebra and Baire category before it turns to formal definitions of quantified lineability, spaceability and algebrability. Its short and enormously informative style makes this chapter very valuable on its own, as this is knowledge needed for many directions of (functional) analysis.

The following chapters starts with a section called "What one needs to know" and ends with a "Notes and remarks" section after an "Exercises" section. I will not comment further on these sections except remarking that starting by a "What one needs to know" section is something many authors could learn from!

Chapter 1 is of 60 pages length and consists of seven core sections. In this chapter M is always a subset of a space of real-valued functions, mostly also defined on a subset of $\mathbb{R}$.

In Chapter 2, M is a subset of a space of complex functions, mostly holomorphic and defined on some subset of $\mathbb{C}$. This chapter counts around 40 pages and contains 6 core sections.

The "universe" of the set M in Chapter 3 is some sequence space or space of integrable functions. Here we find six core sections and the whole chapter has approximately 30 pages.

Chapter 4 is called "Universality, hypercyclicity and chaos". It has almost 50 pages and consists of eight core sections devoted to lineability, spaceability and algebrability in various classes of hypercyclic vectors.

In Chapter 5, M is the set of zeros of some polynomial defined on some Banach space. This is a 20 pages long chapter with only one core section.

Chapter 6 contains eight short core sections (typically 1–2 pages long) with results that do not fit naturally elsewhere. As an example, one will find an elaboration of Example 2 above as Section 6.4.

There are indeed some techniques that are used again and again throughout the text in Chapters 1–6. There are also some general principles, like Theorem 7.4.1 due to N. Kalton: If Y is a closed linear subspace of a Fréchet space X, then $X \setminus Y$ is spaceable if and only if Y has infinite codimension. The objective of the final Chapter 7 is to present general techniques and results, in general settings, concerning lineability, spaceability and algebrability.

A reference list of 387 items surely indicates how big this field has grown, and thus the value of collecting the main results of the field in a book. Being in addition so tastefully written, and as a bonus containing the very valuable general introductory chapter, I recommend it for the bookshelf of any researcher and supervisor in mathematical analysis as well as for the graduate student interested in lineability.

Olav Nygaard is a professor of mathematics at the University of Agder, Norway. He was born in 1967 in the village of Vegusdal, 60 km north of Kristiansand. After receiving his Master's degree in applied mathematics at the University of Bergen in 1991 he went into teacher education. He obtained his PhD in functional analysis in 2001, entitled "Approximation, Boundedness, Surjectivity" under the supervision of Åsvald Lima and Arne Stray. His main interests are geometry of Banach spaces, measure theory and mathematics teaching.

Personal Column

Please send information on mathematical awards and deaths to newsletter@ems-ph.org.

Awards

Michele Stecconi is the winner of the **2016 Tricerri Prize**.

Alberto Cogliati is the winner of the **UMI-SISM Prize 2016** and **Erika Luciano** received honourable mention.

The **2016 Federigo Enriques Prize** has been awarded to **Roberto Svaldi** and the **2016 Stefania Cotoneschi Prize** has been awarded to **Stefania Castellini**.

Marta Macho Stadler (Universidad del País Vasco, Spain) is the winner of the **Emakunde Prize 2016** from the Basque Institute for Women.

Martino Lupini has been awarded the **2017 Franco Montagna Prize** for his Ph.D. thesis "Operator Algebras and Abstract Classification".

Timothy Gowers (Cambridge University) and **Eva Miranda** (UPC Barcelona) have been awarded with a **Chaire d'Excellence de la Fondation Sciences Mathématiques de Paris**.

The **2017 Breakthrough Prize in Mathematics** has been awarded to **Jean Bourgain** (Institute for Advanced Study) for "combining deep theoretical insights with ingenious problem-solving ability" and for "an enormous impact on mathematics over the past forty years".

The **2017 New Horizons in Mathematics Prize** was awarded to **Mohammed Abouzaid** (Columbia University), **Hugo Duminil-Copin** (University of Geneva), and jointly to **Benjamin Elias** (University of Oregon) and **Geordie Williamson** (Kyoto University). Duminil-Copin and Williamson were among the winners of the EMS prizes earlier this year.

The **Abel Prize 2017** was awarded to Yves Meyer (École normale supérieure Paris-Saclay, France) "for his pivotal role in the development of the mathematical theory of wavelets".

The Royal Dutch Mathematical Society (KWG) has awarded the **2017 Brouwer Medal** to **Kenneth A. Ribet** (University of California, Berkeley) for "his contributions to number theory, in particular for the groundbreaking work in which he applies methods of algebraic geometry to number theoretical problems".

The Fields Institute announced that **Henri Darmon** (McGill University) is the winner of the **2017 CRM-Fields-PIMS Prize**.

Christophe Breuil (CNRS and Université Paris-Sud Orsay) has been awarded the **Médaille d'argent 2017**, and **Béatrice de Tilière** (CNRS, Université Paris-Est Créteil, Université Paris-Est Marne-la-Vallée) is the winner of the **Médaille de bronze 2017**.

Martin Hairer (FRS, University of Warwick) has been awarded an **Honorary Knight Commander** of the Order of the British Empire for "contributions to the arts and sciences, work with charitable and welfare organisations, and public service outside the Civil Service".

Gerd Faltings will be awarded the **Cantor Medal** of the German Mathematical Society (DMV).

Eva Miranda (UPC Barcelona) has been awarded with an **ICREA Academia Prize in Experimental Sciences and Mathematics**.

Roger Casals (Department of Mathematics, MIT) has received the **RET** (Spanish Network in Topology) **Award 2017** for the best thesis in topology.

The **Richard-von-Mises-Prize** in acknowledgment of their scientific achievements in the area of applied mathematics and mechanics was awarded to **Benjamin Klusemann** (Leuphana) and to **Christian Kuehn** (TU München).

Dr.-Klaus-Körper Awardees 2017 in appreciation for an excellent dissertation in Applied Mathematics and Mechanics are **Christoph Anton Meier** (MIT), **Philipp Christian Petersen** (TU Berlin), **Ronny Behnke** (TU Dresden) and **Patrick Kürschner** (MPI Magdeburg).

The **CNRS Innovation Medal 2017** is awarded to **Raphaèle Herbin**, **Jean-Pierre Nozières**, **Jean-Marie Tarascon** and **Jamal Tazi**. The CNRS Innovation Medal is awarded for exceptional research in applied mathematics leading to a technological, therapeutic or social breakthrough, thereby enhancing French scientific research.

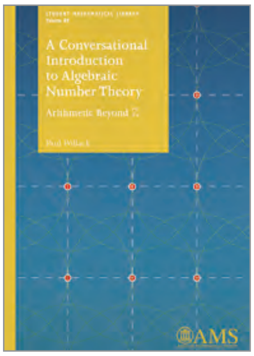
Maurice Duits (KTH, Stockholm) is the winner of the **Wallenberg Prize 2017**, awarded by the Swedish Mathematical Society, for important contributions to the analysis of random matrices and related stochastic models.

The **Show Prize in Mathematical Sciences** 2017 is awarded to **Claire Voisin** (Collège de France).

Deaths

We regret to announce the deaths of:

Javier Peralta Coronado (6 December 2016, Madrid, Spain)
Petr Hájek (26 December 2016, Prague, Czech Republic)
Bohuslav Balcar (17 February 2017, Prague, Czech Republic)
Igor R. Shafarevich (19 February 2017, Moscow, Russia)
Jiří Kopáček (22 February 2017, Prague, Czech Republic)
Ludwig D. Faddeev (26 February, 2017, St. Petersburg, Russia)
Peter Gruber (7 March 2017, Vienna, Austria)
Komaravolu Chandrasekharan (13 April, 2017, Zürich, Switzerland)
Vicente R. Varea Agudo (8 May 2017, Zaragoza, Spain)
Hans-Otto Georgii (16 May 2017, Munich, Germany)



A CONVERSATIONAL INTRODUCTION TO ALGEBRAIC NUMBER THEORY

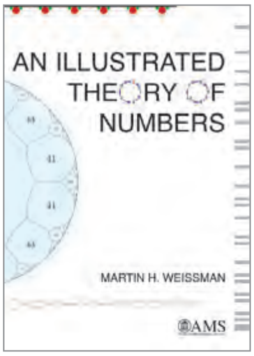
Arithmetic Beyond $\mathbb{Z}$

Paul Pollack, University of Georgia

An introduction to algebraic number theory, meaning the study of arithmetic in finite extensions of the rational number field $\mathbb{Q}$. Originating in the work of Gauss, the foundations of modern algebraic number theory are due to Dirichlet, Dedekind, Kronecker, Kummer, and others. This book lays out basic results, including the three "fundamental theorems": unique factorization of ideals, finiteness of the class number, and Dirichlet's units theorem.

Student Mathematical Library, Vol. 84

Sep 2017 311pp 9781470436537 Paperback €59.00



AN ILLUSTRATED THEORY OF NUMBERS

Martin H. Weissman, University of California

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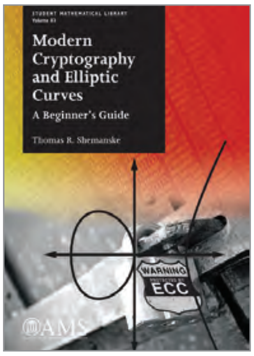

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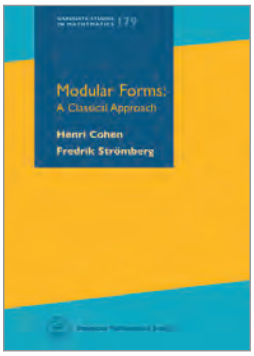

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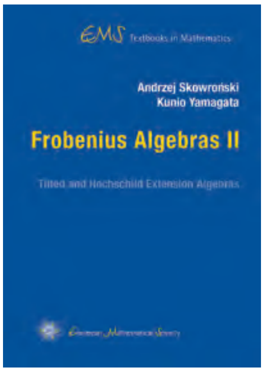
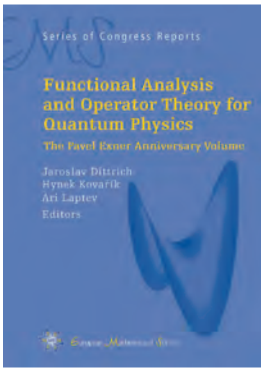


Andrzej Skowroński (Nicolaus Copernicus University, Toruń, Poland) and Kunio Yamagata (Tokyo University of Agriculture and Technology, Japan)
Frobenius Algebras II. Tilted and Hochschild Extension Algebras (EMS Textbooks in Mathematics)

ISBN 978-3-03719-174-3. 2017. 629 pages. Hardcover. 17 x 24 cm. 58.00 Euro

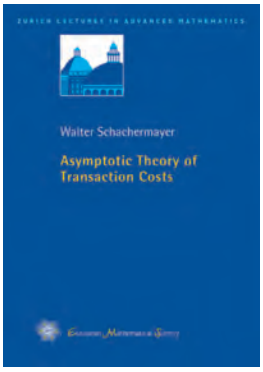
This is the second of three volumes which will provide a comprehensive introduction to the modern representation theory of Frobenius algebras. The first part of the book is devoted to fundamental results of the representation theory of finite dimensional hereditary algebras and their tilted algebras, which allow to describe the representation theory of prominent classes of Frobenius algebras.
The second part is devoted to basic classical and recent results concerning the Hochschild extensions of finite dimensional algebras by duality bimodules and their module categories. Moreover, the shapes of connected components of the stable Auslander-Reiten quivers of Frobenius algebras are described.
The only prerequisite in this volume is a basic knowledge of linear algebra and some results of the first volume. It includes complete proofs of all results presented and provides a rich supply of examples and exercises.
The text is primarily addressed to graduate students starting research in the representation theory of algebras as well mathematicians working in other fields.The book is accessible to advanced students and researchers of complex analysis and differential geometry.
The first volume (ISBN 978-3-03719-102-6) has appeared under the title *Frobenius Algebras I. Basic Representation Theory.*



Functional Analysis and Operator Theory for Quantum Physics. The Pavel Exner Anniversary Volume
Jaroslav Dittrich (Czech Academy of Sciences, Rez-Prague, Czech Republic), Hynek Kovařík (Università degli Studi di Brescia, Italy) and Ari Laptev (Imperial College London, UK), Editors

ISBN 978-3-03719-175-0. 2017. 597 pages. Hardcover. 17 x 24 cm. 98.00 Euro

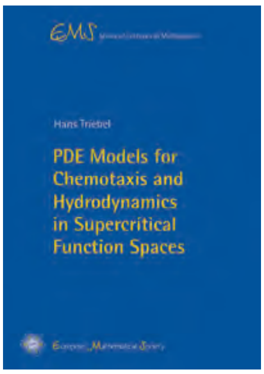
This volume is dedicated to Pavel Exner on the occasion of his 70th anniversary. It collects contributions by numerous scientists with expertise in mathematical physics and in particular in problems arising from quantum mechanics. The questions addressed in the contributions cover a large range of topics. A lot of attention was paid to differential operators with zero range interactions, which are often used as models in quantum mechanics. Several authors considered problems related to systems with mixed-dimensions such as quantum waveguides, quantum layers and quantum graphs. Eigenvalues and eigenfunctions of Laplace and Schrödinger operators are discussed too, as well as systems with adiabatic time evolution. Although most of the problems treated in the book have a quantum mechanical background, some contributions deal with issues which go well beyond this framework; for example the Cayley–Hamilton theorem, approximation formulae for contraction semigroups or factorization of analytic operator-valued Fredholm functions. As for the mathematical tools involved, the book provides a wide variety of techniques from functional analysis and operator theory.
Altogether the volume presents a collection of research papers which will be of interest to any active scientist working in one of the above mentioned fields.



Walter Schachermayer (Universität Wien, Austria)
Asymptotic Theory of Transaction Costs (Zürich Lectures in Advanced Mathematics)

ISBN 978-3-03719-173-6. 2017. 160 pages. Hardcover. 17 x 24 cm. 34.00 Euro

A classical topic in Mathematical Finance is the theory of portfolio optimization. Robert Merton's work from the early seventies had enormous impact on academic research as well as on the paradigms guiding practitioners.
One of the ramifications of this topic is the analysis of (small) proportional transaction costs, such as a Tobin tax. The lecture notes present some striking recent results of the asymptotic dependence of the relevant quantities when transaction costs tend to zero.
An appealing feature of the consideration of transaction costs is that it allows for the first time to reconcile the no arbitrage paradigm with the use of non-semimartingale models, such as fractional Brownian motion. This leads to the culminating theorem of the present lectures which roughly reads as follows: for a fractional Brownian motion stock price model we always find a shadow price process for given transaction costs. This process is a semimartingale and can therefore be dealt with using the usual machinery of mathematical finance.



Hans Triebel (University of Jena, Germany)
PDE Models for Chemotaxis and Hydrodynamics in Supercritical Function Spaces (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-171-2. 2017. 138 pages. Hardcover. 17 x 24 cm. 32.00 Euro

This book deals with PDE models for chemotaxis (the movement of biological cells or organisms in response of chemical gradients) and hydrodynamics (viscous, homogeneous, and incompressible fluid filling the entire space). The underlying Keller–Segel equations (chemotaxis), Navier–Stokes equations (hydrodynamics), and their numerous modifications and combinations are treated in the context of inhomogeneous spaces of Besov–Sobolev type paying special attention to mapping properties of related nonlinearities. Further models are considered, including (deterministic) Fokker–Planck equations and chemotaxis Navier–Stokes equations.
These notes are addressed to graduate students and mathematicians having a working knowledge of basic elements of the theory of function spaces, especially of Besov–Sobolev type and interested in mathematical biology and physics.

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