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The Littlewood–Paley Theory

Interviews
Peter Scholze
Artur Avila

Research Center
Isaac Newton Institute for Mathematical Sciences

Artur Avila (photo courtesy of Daryan Dornelles/Divulgação IMPA)

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EMS Executive Committee

EMS Agenda

2019

11–13 October
EMS Executive Committee Meeting, Yerevan, Armenia

EMS Scientific Events

2019

8–12 July
EQUADIFF2019,
Leiden University, The Netherlands
https://www.universiteitleiden.nl/equadiff2019

9–13 July
SIAM Conference on Applied Algebraic Geometry
Bern, Switzerland

15–19 July
ICIAM 2019
Valencia, Spain

22–26 July
30th International Workshop on Operator Theory and its Applications – IWOTA 2019
Lisbon, Portugal

29 July–2 August
British Combinatorial Conference 2019
Glasgow, UK

26–29 August
Caucasian Mathematics Conference (CMC-III)
Rostov-on-Don, Russian Federation

2020

5–11 July
8th European Congress of Mathematics
Portorož, Slovenia

5–11 July
ICCOPT 2019 – 6th International Conference on Continuous Optimisation
Berlin, Germany

Report from the EMS Executive Committee Meeting in Barcelona, 9–11 November 2018

Richard Elwes, EMS Publicity Officer

Barcelona is a beautiful city at any time of year, and last November, unlike most of Europe, it was a luxuriantly warm place for the EMS's Executive Committee (EC) to gather. The meeting was generously hosted by the Catalan Mathematical Society (CMS) at the Institut d'Estudis Catalans, in the magnificent 17th century surroundings of the Casa de Convalescència de Sant Pau.

On Friday evening, the meeting was addressed by Xavier Jarque i Ribera, then President of the CMS, who told us about his society. The CMS comes under the umbrella of the Institute of Catalan Studies (founded in 1911), within its Science and Technology Section. It comprises around 800 members, both University and High School teachers (as well as a few professional members outside academia). It is an active society, awarding prizes for research, running several mathematical publications, and organising conferences, including the triennial Barcelona Mathematical Days. Yet its biggest activities are focussed on education, notably the annual *Proves Cangur*, a day of mathematical exercises in which over 100,000 school students participate. The CMS has long been an important member of the EMS family, including organising the 3rd European Congress in 2000, and providing the EMS President for the period 2011–2014, Marta Sanz-Solé.

Officers' Reports and Membership

The meeting was opened by the chair, Pavel Exner, for the final time as EMS President. After some initial formalities, he provided an update on his activities since the Council meeting in June 2018. EMS Treasurer Mats Gyllenberg followed with an update on the society's budget. The finances continue in a healthy state, and pleasingly the expenditure on scientific activities has risen accordingly. The Treasurer noted that donations to the Committee for Developing Countries (whose budget is handled largely separately) have recently increased too, following a specific call. However, there will be more to do to continue to raise funds for this important stream of activities. The Secretary and Vice-Presidents had nothing specific to report (with the Secretary Sjoerd Verduyn Lunel unable to attend, the Publicity Officer Richard Elwes took notes throughout the meeting, from which the official minutes were later constructed).

The EC was pleased to formally approve an application for institutional membership from the Department of Mathematics at the Technion (Israel Institute of Technology), along with 51 new individual members. Following the recent introduction of EMS individual life-membership, several applications have been received, and the EC discussed the implementation of this system.

Scientific Meetings

The EC considered the ongoing preparations for the 8th European Congress of Mathematics (ECM8, 5–11 July 2020, Portorož, Slovenia) including the calls for nominations for the EMS, Felix Klein, and Otto Neugebauer Prizes. Looking ahead, the call for organizing ECM9 in 2024 will appear in the December 2019 issue of this Newsletter.

The EC then discussed applications for financial support for other scientific activities. Regrettably, there will be no EMS distinguished speaker for 2019. However, the EC agreed to provide support the Diderot Mathematical forum on "Mathematics and Architecture" to be held on 8th June 2019 simultaneously in Helsinki, Porto, and Prague (Czech Republic), along with 14 mathematical Summer Schools over the course of 2019. (A further Summer School will be formally endorsed by the EMS, without financial assistance.)

Committees and Projects

The EC examined the make-up of the EMS's ten vitally important standing committees, several of which were due major personnel renewals. The EC was pleased to appoint new members to several committees, as well as naming Sophie Dabo as Chair of the Committee for Developing Countries with Francesco Pappalardi as Vice-Chair, David Abrahams as Chair of the ERCOM (Scientific Directors of European Research Centres in the Mathematical Sciences) committee, and to reappoint Roberto Natalini as Chair of the Committee for Raising Public Awareness of Mathematics.

The EC then considered reports from the committees, beginning with first report from Stéphane Cordier, new Chair of the Applied Mathematics Committee, in attendance as a guest. Likewise, Jürg Kramer Chair of the Education Committee, also present as a guest, reported on his committee's activities, including a draft survey on the problematic secondary-tertiary transition in mathematics education (this is now available at www.bit.ly/2DjyvvE – you are kindly requested to spend around 15 minutes completing the survey). The EC continued with discussion of the reports from the Committees on Developing Countries, Ethics, European Solidarity, Raising Public Awareness of Mathematics, and Women in Mathematics.

The EC discussed a number of ongoing projects in which the EMS is involved, including the European Digital

The Executive Committee and guests.

Mathematics Library (https://eudml.org), the online Encyclopaedia of Mathematics (www.encyclopediaofmath.org), EU-MATHS-IN (the European Service Network Of Mathematics For Industry And Innovation), and plans for a future Global Digital Mathematics Library.

Publicity and Publications

EMS Publicity Officer Richard Elwes presented his report, commenting that the day marked exactly five years since the EMS joined Twitter (@EMSNewsletter). The society currently has over 4000 followers there, and over 2000 on Facebook (@EMSNewsletter).

The Editor-in-Chief of the EMS Newsletter, Valentin Zagrebnov, presented his report. The EMS President Pavel Exner then updated the committee with details of the proposed official move of the EMS Publishing House from Zurich to Berlin. The EC endorsed this plan, including the formal establishment of a new foundation under German law. On behalf of the society, the members of the EC expressed their profound gratitude to the Publishing House's founding Managing Director Thomas Hintermann, who is due to retire in 2019.

The EC discussed other matters relating to publications, including the EMS's quarterly e-news, and Zentralblatt (https://zbmath.org).

Relation with Funding Organisations and Political Bodies

The President reported on recent developments regarding the EU funding framework Horizon 2020 and its successor programme Horizon Europe. The committee agreed on the importance of grants being evaluated by suitably qualified individuals, and recalled that EU-MATHS-IN encourages mathematicians to register as possible evaluators.

The committee discussed developments regarding the European Research Council, as well as the European Science Open Forum (ESOF) whose next meeting will be in 2020 in Trieste. It is hoped that the EMS committee for Raising Public Awareness will provide some mathematical content.

The President led a conversation about "Plan S", an initiative for Open Access, with a focus on its possible consequences for mathematics.

Relations with other Mathematical Bodies

The President provided an updated on the International Mathematical Union (of which the EMS is an adhering organisation) including details of the new leadership under President Carlos Kenig, starting in 2019, and the next ICM (to be held in 2022 in Saint Petersburg).

The committee discussed the EMS's joint work with other mathematical organisations, including ICIAM (The International Council for Industrial and Applied Mathematics), CIMPA (the International Centre of Pure and Applied Mathematics), the Abel Prize, the Gordin Prize, and the Bernoulli Society.

Close

The committee expressed its thanks to Xavier Jarque i Ribera and the Catalan Mathematical Society, for the warm hospitality and the event's flawless organisation.

The end of this meeting marked an important moment in the life of the EMS, being the final society meeting with Pavel Exner as Chair, his presidency ending in just a few weeks' time. On behalf of the whole European mathematical community, incoming EMS President Volker Mehrmann presented him with a card and a gift, communicating our sincere appreciation for all his hard work, and wise leadership, over the last 4 years. Pavel was touched by this gesture, and brought the meeting to a close with an expression of confidence that the future of the EMS is in safe hands.

Three EMS Presidents: Volker Mehrmann (2019–2023), Pavel Exner (2015–2018), Marta Sanz-Solé (2011–2014).

Discrete Maths Summer in Slovenia

Nino Bašić (University of Primorska, Koper, Slovenia), Ademir Hujdurović (University of Primorska, Koper, Slovenia), Matjaž Konvalinka (University of Ljubljana, Slovenia) and Klavdija Kutnar (University of Primorska, Koper, Slovenia)

This summer, mathematical activities in Slovenia will be marked by discrete maths. Some basic information regarding four events are presented in subsequent sections: Maps ∩ Configurations ∩ Polytopes ∩ Molecules ⊆ Graphs: The mathematics of Tomaž Pisanski on the occasion of his 70th birthday, Ljubljana, May 23–25; the 9th Slovenian International Conference on Graph Theory, Lake Bled, June 23–29; the 31st International Conference on Formal Power Series and Algebraic Combinatorics, Ljubljana, July 1–5; and the 9th PhD Summer School in Discrete Mathematics, Rogla, June 30–July 6.

Bled'19

Bled'19 is the 9th edition of the quadrennial Slovenian Graph Theory Conference. The conference has progressed a long way since its inaugural edition in 1985 (held in Dubrovnik, Croatia). The number of participants has grown from just 30 in 1985 to over 300 at the 8th edition held in Kranjska Gora in 2015, representing all six continents. It has become one of the largest – and for many areas of graph theory, the premier – graph theory conference series, and is attended by leading researchers in graph theory, as well as many postdocs and talented PhD students. For Bled'19, the number of participants is expected to be comparable to that of 2015, with over 220 talks. In addition to the keynote speakers named below, the conference will consist of minisymposia from specific fields, ranging from algebraic, algorithmic, geometric, topological and other aspects of graph theory to a general session and poster session.

The conference is organized by IMFM – Institute of Mathematics, Physics and Mechanics, Ljubljana, in collaboration with SDAMS (Slovenian Discrete and Applied Mathematics Society), UL FMF (University of Ljubljana, Faculty of Mathematics and Physics), UM FNM (University of Maribor, Faculty of Natural Sciences and Mathematics), UP FAMNIT (University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies), and UP IAM (University of Primorska, Andrej Marušič Institute).

Plenary and invited speakers together with organizers of the 8th Slovenian Conference on Graph Theory.

List of keynote speakers:

Noga Alon (Princeton University, USA, and Tel Aviv University, Israel)
Marco Buratti (University of Perugia, Italy)
Gareth Jones (University of Southampton, UK)
Gábor Korchmáros (University of Basilicata, Italy)
Daniel Král' (Warwick University, UK, and Masaryk University, Czech Republic)
Daniela Kühn (University of Birmingham, UK)
Sergei Lando (National Research University Higher School of Economics, Russia)
János Pach (EPFL, Lausanne and Rényi Institute, Hungary)
Cheryl E. Praeger (University of Western Australia, Australia)
Zsolt Tuza (Rényi Institute and University of Pannonia, Hungary)
Xuding Zhu (Zhejiang Normal University, China)

Scientific Committee:

Sandi Klavžar, Dragan Marušič, Bojan Mohar (chair), Tomaž Pisanski.

For further information please visit the conference web site: http://conferences.matheo.si/e/bled19.

FPSAC'19

The International Conference on Formal Power Series and Algebraic Combinatorics is a major annual combinatorial conference that is organised in a different city every year. The 31st edition of FPSAC will take place in Ljubljana, Slovenia. Topics include all aspects of combinatorics and their relations with other areas of mathematics, physics, computer science and biology. The conference will include invited lectures, contributed presentations, poster sessions and software demonstrations. There will be no parallel sessions. The number of attendants is usually between 200 and 250, and we expect a similar number this year.

The conference is organised by the University of Ljubljana, Faculty of Mathematics and Physics in collaboration with UM FNM (University of Maribor, Faculty of Natural Sciences and Mathematics), UP FAMNIT (University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies), IMFM (Institute of Mathematics, Physics and Mechanics, Ljubljana) and SDAMS (Slovenian Discrete and Applied Mathematics Society), and is sponsored by a variety of organizations and companies, including the National Science Foundation.

List of keynote speakers:
Andrej Bauer (University of Ljubljana and IMFM, Slovenia)
Alin Bostan (Inria, France)
Sandra Di Rocco (KTH, Royal Institute of Technology, Sweden)
Eric Katz (Ohio State University, USA)
Caroline Klivans (Brown University, USA)
Nataša Pržulj (University College London, UK)
Vic Reiner (University of Minnesota, USA)
Stephan Wagner (Stellenbosch University, South Africa)
Chuánmíng Zōng (Tianjin University, China)

Program Committee Chairs:
Sara Billey, Marko Petkovšek, Günter Ziegler.

Organizing Committee Chair:
Matjaž Konvalinka.

For further information please visit the conference web site: http://fpsac2019.fmf.uni-lj.si/.

9th PhD Summer School in Discrete Mathematics

The University of Primorska, together with the Slovenian Discrete and Applied Mathematics Society, is organising the 9th PhD Summer School in Discrete Mathematics, which will be held in Rogla, Slovenia, between June the 30th and July the 6th, 2019. This PhD Summer School began in 2011 and has since become an annual event. It is aimed at bringing together PhD students and postdocs with senior lecturers. The Summer School is truly international, with participants from all over the world. Each year there are around 60 participants, more than half of them students. Financial support is offered for several PhD students to attend this Summer School, which includes half board accommodation and exemption from payment of the conference fee. Students will have the opportunity to present their results in 15-minute talks, and a three-member committee will decide the winner of the *Best Student Talk Award*. The winner will receive a certificate and free participation at the 10th PhD Summer School in Discrete Mathematics.

The main part of the summer school consists of the following two mini-courses:

Minicourse 1: Combinatorial limits and their applications in extremal combinatorics, Daniel Král' (Masaryk University, Czech Republic, and University of Warwick, UK)
Minicourse 2: Coxeter groups, Alice Devillers (The University of Western Australia, Perth, Australia).

In addition to the two mini-courses, several invited talks will be given during the summer school.

Confirmed invited speakers are:
Vida Dujmović (University of Ottawa, Canada)
Miguel Angel Pizana (Universidad Autónoma Metropolitana-Iztapalapa, Mexico)
Jeroen Schillewaert (University of Auckland, New Zealand)
Klara Stokes (Maynooth University, Ireland)

For further information, please visit the conference web site: https://conferences.famnit.upr.si/event/12/.

Scientific Committee:
Ademir Hujdurović, Klavdija Kutnar, Aleksander Malnič, Dragan Marušič, Štefko Miklavič, Primož Šparl.

Supporters:
EMS - European Mathematical Society, ARRS - Slovenian Research Agency, MIZŠ - Ministry of Education, Science and Sport RS.

Group photo from the 8th PhD Summer School in Discrete Mathematics.

Maps ∩ Configurations ∩ Polytopes ∩ Molecules ⊆ Graphs: The mathematics of Tomaž Pisanski on the occasion of his 70th birthday

This conference, which was held in Ljubljana, Slovenia, between May the 23rd and 25th, 2019, was organised to celebrate the mathematics of Tomaž Pisanski on the occasion of his 70th birthday. Tomaž Pisanski, known as Tomo to his friends, works in several areas of discrete and computational mathematics. Combinatorial configurations, abstract polytopes, maps on surfaces and chemical graph theory are just a few areas of his broad research interests. Tomo is the author or co-author of over 160 original scientific papers. Together with Brigitte Servatius he authored the book *Configurations from a Graphical Viewpoint*, which was published in 2013 by Birkhäuser. Tomo's scientific work has been cited over 3400 times according to Google Scholar. In 2008, together with Dragan Marušič, he co-founded *Ars*

Mathematica Contemporanea, the first international mathematical journal to be published in Slovenia. Many Slovenian mathematicians refer to Tomo as "the father of Slovenian discrete mathematics". The Mathematics Genealogy Project lists 16 of Tomo's PhD students and 79 academic descendants.

The invited speakers named below are world-class mathematicians who work in areas of mathematics that are close to Tomo. In addition to the invited talks, participants had the opportunity to deliver short, 15-minute presentations.

The conference was organised by UL FMF (University of Ljubljana, Faculty of Mathematics and Physics), in collaboration with UP FAMNIT (University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies), UP IAM (University of Primorska, Andrej Marušič Institute), SDAMS (Slovenian Discrete and Applied Mathematics Society), IMFM (Institute of Mathematics, Physics and Mechanics, Ljubljana), and Abelium d.o.o.

List of invited speakers:
Vladimir Batagelj (University of Ljubljana and University of Primorska, Slovenia)
Gunnar Brinkmann (Ghent University, Belgium)
Patrick W. Fowler (The University of Sheffield, United Kingdom)
Gábor Gévay (University of Szeged, Hungary)
Wilfried Imrich (Montanuniversität Leoben, Austria)
Asia Ivić Weiss (York University, Canada)
Sandi Klavžar (University of Ljubljana and University of Maribor, Slovenia)
Dimitri Leemans (Université Libre de Bruxelles, Belgium)
Dragan Marušič (University of Primorska, Slovenia)
Alexander D. Mednykh (Sobolev Institute of Mathematics and Novosibirsk State University, Russian Federation)
Bojan Mohar (Simon Fraser University, Canada, and IMFM, Slovenia)
Daniel Pellicer Covarrubias (National Autonomous University of Mexico, Mexico)
Egon Schulte (Northeastern University, United States)
Brigitte Servatius (Worcester Polytechnic Institute, United States)
Martin Škoviera (Comenius University in Bratislava, Slovakia)
Thomas W. Tucker (Colgate University, United States)
Steve Wilson (Northern Arizona University, United States)

Scientific Committee:
Klavdija Kutnar, Primož Potočnik, Arjana Žitnik.

Organising Committee:
Nino Bašić, Marko Boben, Boris Horvat, Jurij Kovič, Alen Orbanić, Arjana Žitnik.

For further information please visit the conference web site: http://conferences.matheo.si/e/mcpm.

Nino Bašić [nino.basic@upr.si] is an Assistant Professor of Mathematics and Computer Science at the University of Primorska. He is an active member of the SDAMS and a member of the Organising Committee of the 8ECM. His research interests include various aspects of graph theory and its applications to natural sciences.

Ademir Hujdurović [ademir.hujdurovic@upr.si] obtained his PhD degree in mathematics from the University of Primorska, Koper, Slovenia, in 2013. He is currently an Assistant Professor at the University of Primorska and Vice-Dean for International Cooperation of UP FAMNIT. He is also a member of the Organising Committee of the 8ECM. His main research interest is graph theory, mainly in graphs that posses a certain degree of symmetry.

Matjaž Konvalinka [matjaz.konvalinka@fmf.uni-lj.si] is an Associate Professor of Mathematics at the Faculty of Mathematics and Physics, University of Ljubljana, and the chair of the Organising Committee of FPSAC 2019. His main research interests are algebraic and enumerative combinatorics.

The photo and CV of Klavdija Kutnar can be found in previous Newsletter issues.

Determinantal Point Processes

Adrien Hardy (Université de Lille, France) and Mylène Maïda (Université de Lille, France)

1 Introduction

If you ask a five-year-old child to *randomly* draw points within a disc, they will probably produce a picture like this one:

which looks more like the picture on the left than the one of the right:

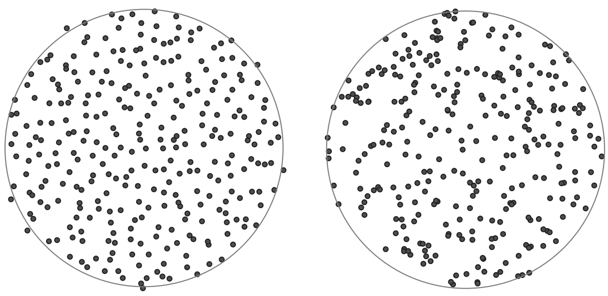


Figure 1. On the left, spectrum of a Ginibre matrix; on the right, sample of independent uniform points on the disc.

However, the figure on the right is a realisation of a point process[1] which naturally corresponds to the notion of randomness in mathematics: the well-known homogenous Poisson process, made of points that are chosen uniformly on the disc and *independently* of one to another. In particular, with the Poisson process, getting very close points is possible but the human brain tends to avoid such configurations for mysterious reasons.

The figure on the left-hand side shows the eigenvalues in the complex plane of a matrix with independent complex Gaussian entries[2], known as a *Ginibre matrix*. It seems that the intuition of randomness for a five-year-old child[3] fits better with these kind of structured configurations rather than realisations of a Poisson process. In this article, we want to discuss that underlying structure, which is a particular case of a class of point process with some intriguing properties: the *determinantal point processes*, hereafter abbreviated DPPs.

The following is neither going to be a formal introduction nor an exhaustive survey of this wide topic, but rather a personal choice of some elegant DPPs collected in various mathematical domains[4]. For a more extensive and classical presentation, see, e.g., [5, 6, 10, 11].

As for lovers of practical applications, let us imagine that we type the query "jaguar" into a search engine. Among the top results, we don't want to find 10 articles on sports cars, but also one on the animal with spotted fur, one on the film by Francis Veber and maybe one on an American football team. For a wide range of subjects, we need to introduce repulsion between similar items within the algorithms: if an item has been selected, the closely related items are less likely to be displayed. We are going to see, towards the end of this paper, that the DPP has already aroused curiosity among *machine learners* by providing implementable solutions to this kind of problem, and to many others.

My first DPP: The carries process.

Consider the following process, which might remind some of us of their primary school days: you have a column of digits which you add up one by one from top to bottom. At each step, you note the unit of the result on the right, note a dot if there is a number to carry, go on to the next line and so on.

	3		3	
+	6	=	9	
+	5	=	4	•
+	4	=	8	
+	4	=	2	•
+	3	=	5	
+	3	=	8	
+	7	=	5	•

If the first column contains random, independent and identically distributed (iid) digits following the uniform distribution on $[\![0, 9]\!] := \{0, \ldots, 9\}$, how will the distribution of the point process in relation to the carries be?

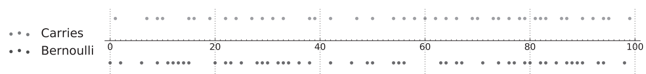


Figure 2. Above, the point process of carries; below, iid Bernoulli variables of the same parameter $p = 9/20$.

1 In the whole article, the notion of point process refers to a spatial process, namely random configurations (i.e. locally finite subsets) of points in space; indeed, the word *process* is commonly used in this context, although there is no relation to time.

2 A complex Gaussian is a complex random variable of which the real part and the imaginary part are two independent real Gaussian variables.

3 Study carried out on a sample of three participants, without quota sampling.

4 Let us mention some noteworthy DPPs which are not quoted in our article: for a given graph, the process of the edges of a uniform spanning tree is a DPP; non-intersecting random walks on a bipartite graph form, at fixed time, a DPP; the zeros of the hyperbolic Gaussian analytic function form a DPP; etc.

As a column of iid uniform digits on $[\![0,9]\!]$ leads, after successive additions, to a column of iid units that are also uniform on $[\![0,9]\!]$, it is equivalent to consider the following descent point process, treated in a more general framework by Borodin, Diaconis and Fulman [4]: consider a column of $n+1$ digits $S_0, S_1, \ldots, S_n$ that are iid uniform on $[\![0,9]\!]$ and, at each line, we note a dot on the right if the digit in this line is strictly inferior to the previous one. More precisely, if we put

$$X_i := \mathbf{1}_{\{S_i < S_{i-1}\}} \in \{0,1\},$$

the descent point process is given by the random configurations $D_n := \{i \in [\![1,n]\!] : X_i = 1\}$.

For instance, in the following example, we obtain $D_7 = \{2,4,7\}$:

i	S_i		X_i
0	3		
1	9		0
2	4	•	**1**
3	8		0
4	2	•	**1**
5	5		0
6	8		0
7	5	•	**1**

The simple computation

$$\begin{aligned}\mathbb{P}(\{i\} \subset D_n) &= \mathbb{P}(X_i = 1) = \mathbb{P}(S_i < S_{i-1}) \\ &= \frac{1}{10^2}\binom{10}{2} = \frac{9}{20} < \frac{1}{2}\end{aligned}$$

shows that, as one may expect, at every line the probabilty to have a descent/carry is a little bit less than 1/2. Thus, the sequence $X_1, \ldots, X_n$ is a sequence of random Bernoulli variables[5] of parameter 9/20 but they are not independent. Intuitively, if you have a carry in a given line, the digit in that line tends to be small and there are less chances to have one in the next line. On Figure 2, we compare a realisation of the process D_{100} with a random configuration $B_{100} = \{i \in [\![1,100]\!] : Y_i = 1\}$ where $Y_1, \ldots, Y_{100}$ are iid Bernoulli variables of the same parameter 9/20.

There is clearly a property of *negative association* or *repulsion* between the adjacent descents, a discrete and unidimensional analogue of what we observed in Figure 1 on the left. The following computation confirms that two adjacent descents are negatively correlated:

$$\begin{aligned}\mathbb{P}(\{i, i+1\} \subset D_n) &= \mathbb{P}(X_i = 1 \text{ and } X_{i+1} = 1) \\ &= \mathbb{P}(S_{i+1} < S_i < S_{i-1}) \\ &= \frac{1}{10^3}\binom{10}{3} \\ &= \frac{3}{25} < \left(\frac{9}{20}\right)^2 \\ &= \mathbb{P}(\{i\} \subset D_n)\,\mathbb{P}(\{i+1\} \subset D_n).\end{aligned}$$

5 That is to say, $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = 9/20$.

On the contrary, if $|i - j| > 1$, the independence implies

$$\begin{aligned}\mathbb{P}(\{i, j\} \subset D_n) &= \mathbb{P}(X_i = 1 \text{ and } X_j = 1) \\ &= \mathbb{P}(S_i < S_{i-1})\mathbb{P}(S_j < S_{j-1}) \\ &= \left(\frac{9}{20}\right)^2;\end{aligned}$$

we say that the process is of range 1.

Now, in order to completely describe the law of the descent point process, it is enough to determine $\mathbb{P}(A \subset D_n)$ for any subset A of $[\![1,n]\!]$. If A has cardinal k, it is called *k-point correlation function*, which we denote by $\rho_k(A)$. We have already determined the one-point correlation function $\rho_1(\{i\}) = 9/20$ for any i, as well as the two-point correlation function: for all $i \neq j$,

$$\rho_2(\{i,j\}) = \begin{cases} \left(\frac{9}{20}\right)^2 & \text{if } |i-j| > 1, \\ \frac{3}{25} & \text{if } |i-j| = 1. \end{cases}$$

When $k \geq 3$, if A is a sequence of k consecutive numbers of $[\![1,n]\!]$, one has $\rho_k(A) = \frac{1}{10^{k+1}}\binom{10}{k+1}$. Otherwise, we can write $A = A_1 \cup A_2$ with A_1 and A_2 at distance at least 2 and with respective cardinals k_1 and k_2, so that $\rho_k(A) = \rho_{k_1}(A_1)\rho_{k_2}(A_2)$, because the process is of range 1. Going a little further, we can verify that this collection of correlation functions is encoded by a function with two variables $K : [\![1,n]\!]^2 \to \mathbb{R}$ in the sense that, for any $k \in [\![1,n]\!]$ and $s_1, \ldots, s_k \in [\![1,n]\!]$ pairwise distinct, one has:

$$\rho_k(\{s_1, \ldots, s_k\}) = \det\Big[K(s_i, s_j)\Big]_{1 \leq i,j \leq k}.$$

We say that the point process D_n is *determinantal with kernel K*. Borodin, Diaconis and Fulman [4] obtain a rather explicit formula for the kernel:

$$K(i,j) := \kappa(j-i) \quad \text{where} \quad \sum_{m \in \mathbb{Z}} \kappa(m) z^m = \frac{1}{1-(1-z)^{10}}.$$

More generally, they show that any point process of range 1 on a segment of $\mathbb{Z}$ is determinantal. They also study the descents of a random permutation: a permutation σ of $[\![1,n]\!]$ has a descent at i if $\sigma(i-1) > \sigma(i)$. If σ is chosen uniformly in the symmetric group, then the descent process is determinantal with kernel

$$K_{\text{per}}(i,j) := \kappa_{\text{per}}(j-i) \quad \text{where} \quad \sum_{m \in \mathbb{Z}} \kappa_{\text{per}}(m) z^m = \frac{1}{1-e^z}.$$

Below, we are going to come across some other interesting DPP related to random permutations.

DPP and random matrices

While the mathematical study of DPPs began in the 1970s with Odile Macchi's thesis, inspired by the formalism of fermions in quantum mechanics, their appearance in random matrix theory has widely popularised them. In the following, we present some results that show that the set of eigenvalues of certain matrix models form a DPP. For all the results that are mentioned in this paragraph, we recommend for instance the monograph [1].

A first example is given by the eigenvalues of unitary matrices sampled "uniformly". More precisely, for $n \geq 1$, we equip the unitary group $U_n(\mathbb{C}) := \{U \in M_n(\mathbb{C}) : UU^* = I_n\}$

with its unique probability measure ν_n invariant by left and right multiplication, namely its normalised Haar measure. We hereby obtain a random variable random V with values in $U_n(\mathbb{C})$ by specifying that $\mathbb{P}(V \in A) = \nu_n(A)$ for any Borel set $A \subset U_n(\mathbb{C})$. To obtain the joint probability distribution of the n (random) eigenvalues of V, we perform a change of variables mapping a unitary matrix to the set of its eigenvalues and eigenvectors,[6] and then integrate on its eigenvectors. If we denote by $e^{i\theta_1}, \dots, e^{i\theta_n}$ the eigenvalues of the random matrix V, a classical Jacobian computation in Lie group theory (due to Weyl) shows that the phases $(\theta_1, \dots, \theta_n) \in [-\pi, \pi]^n$ follow the probability law:

$$d\mathbb{P}(\theta_1, \dots, \theta_n) = \frac{1}{n!} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}. \tag{1.1}$$

By noting that the interaction term between the eigenvalues is the square of a Vandermonde determinant, we obtain

$$d\mathbb{P}(\theta_1, \dots, \theta_n) = \frac{1}{n!} \det\left[K_{U_n(\mathbb{C})}(s_i, s_j)\right]_{1\le i,j\le n} \prod_{j=1}^{n} \frac{d\theta_j}{2\pi} \tag{1.2}$$

where we introduced the kernel

$$K_{U_n(\mathbb{C})}(x, y) := \sum_{k=0}^{n-1} \varphi_k(x)\overline{\varphi_k(y)}, \qquad \varphi_k(x) := \frac{e^{ikx}}{\sqrt{2\pi}}.$$

Observe that $K_{U_n(\mathbb{C})}(x, y)$ is the kernel of the projection operator onto the subspace of $L^2(-\pi, \pi)$ of the trigonometric polynomials of degree at most $n - 1$. Taking into account the continuous character of the eigenangles, we define the "infinitesimal" version of the k-point correlation function we introduced above for the carries process: for every $k \ge 1$ and $s_1, \dots, s_k \in [-\pi, \pi]$ pairwise distinct,

$$\rho_k(s_1, \dots, s_k) := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^k} \mathbb{P}\Big(\forall j \in [\![1, k]\!], \{\theta_1, \dots, \theta_n\} \cap [s_j, s_j + \varepsilon] \ne \emptyset\Big).$$

In other words, for any reasonable test function $\varphi : [-\pi, \pi]^k \to \mathbb{C}$, one has

$$\int_{[-\pi,\pi]^k} \varphi(s)\rho_k(s)ds = \int \sum_{i_1\ne\cdots\ne i_k} \varphi(\theta_{i_1}, \dots, \theta_{i_k})\, d\mathbb{P}(\theta_1, \dots, \theta_n). \tag{1.3}$$

Thanks to the invariance by permutation of $d\mathbb{P}(\theta_1, \dots, \theta_n)$, we see that

$$\begin{aligned}
&\rho_k(s_1, \dots, s_k) \\
&= \frac{n!}{(n-k)!} \int_{[-\pi,\pi]^{n-k}} d\mathbb{P}(s_1, \dots, s_k, s_{k+1}, \dots, s_n) \\
&= \frac{1}{(n-k)!} \int_{[-\pi,\pi]^{n-k}} \det\left[K_{U_n(\mathbb{C})}(s_i, s_j)\right]_{1\le i,j\le n} \prod_{j=k+1}^{n} \frac{ds_j}{2\pi} \\
&= \det\left[K_{U_n(\mathbb{C})}(s_i, s_j)\right]_{1\le i,j\le k},
\end{aligned} \tag{1.4}$$

6 This function is not a well-defined diffeomorphism. We fix it by reducing the source space to the matrices with simple eigenvalues (its complementary being of mesure zero for ν_n), and the target space by ordering the eigenvalues and by taking a quotient to get rid of the freedom for the eigenvectors.

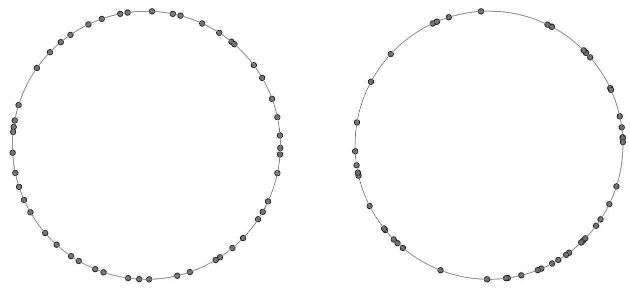


Figure 3. On the left, spectrum of a realisation of V; on the right, independent samples of uniform points on the unit circle.

where the last identity is obtained by developing the determinant under the integral and by using that $K_{U_n(\mathbb{C})}(x, y)$ is the kernel of a projection operator:

$$\int_{-\pi}^{\pi} K_{U_n(\mathbb{C})}(s_i, s)K_{U_n(\mathbb{C})}(s, s_j) \frac{ds}{2\pi} = K_{U_n(\mathbb{C})}(s_i, s_j).$$

Thus, the point process of the eigenangles $\{\theta_1, \dots, \theta_n\}$ is determinantal. Let us visually compare a realisation of this process to iid uniform variables on the unit circle: as in the case of the descents, we observe a more regular distribution than in the independent case. The eigenangles are well-distributed and there is little variability; while in the uniform case, we have clusters of points which can be noticeably different from one realisation to another.

This repulsion is also visible in the interaction term of (1.1). It induces behaviours that are surprising for an unprepared probabilist. For example, the variance[7] of a sum of n eigenangles is much smaller than the one of n iid angles. Indeed, if $\eta_1, \dots, \eta_n$ are iid uniformly distributed on $[-\pi, \pi]$, one has:

$$\mathrm{Var}\Big[\sum_{j=1}^{n} e^{i\eta_j}\Big] = \sum_{j=1}^{n} \mathrm{Var}\Big[e^{i\eta_j}\Big] = n.$$

On the other hand, for the eigenangles we compute, using (1.4) with $k = 1, 2$:

$$\begin{aligned}
\mathrm{Var}\Big[\sum_{j=1}^{n} e^{i\theta_j}\Big] &= \int_{-\pi}^{\pi} |e^{ix}|^2 K_{U_n(\mathbb{C})}(x, x)dx \\
&\quad - \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(x-y)}|K_{U_n(\mathbb{C})}(x, y)|^2 dx\, dy \\
&= n - \sum_{k,\ell=0}^{n-1} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(x-y)(1+k-\ell)} dx\, dy \\
&= 1.
\end{aligned}$$

For the sum of n random complex numbers of module 1 to have a unit variance, independently of n, many cancellations need to take place, and this forces the realisations of the eigenvalues not to differ too much from an equally-spaced configuration on the unit circle. This confirms the visual impression of Figure 3.

A natural question is to study asymptotically the spacing between these eigenangles when $n \to \infty$, after an appropriate scaling. Indeed, as there are n eigenangles well spread on $[-\pi, \pi]$, the point process $\{\frac{n}{2\pi}\theta_1, \dots, \frac{n}{2\pi}\theta_n\}$ is a subset of

7 The variance of a random complex variable Z is defined by $\mathrm{Var}(Z) := \mathbb{E}|Z|^2 - |\mathbb{E}(Z)|^2$.

$[-\frac{n}{2}, \frac{n}{2}]$, where the standard distance between two consecutive points is of order one. A change of variables yields a DPP with kernel:[8]

$$\widetilde{K}_{U_n(\mathbb{C})}(x,y) := \mathrm{e}^{-\mathrm{i}\frac{n-1}{2}(x-y)} \frac{2\pi}{n} K_{U_n(\mathbb{C})}\Big(\frac{2\pi x}{n}, \frac{2\pi y}{n}\Big) = \frac{\sin \pi(x-y)}{n \sin\left(\frac{\pi}{n}(x-y)\right)}.$$

One obtains the uniform local convergence

$$\widetilde{K}_{U_n(\mathbb{C})}(x,y) \xrightarrow[n\to\infty]{} K_{\sin}(x,y) := \frac{\sin \pi(x-y)}{\pi(x-y)}$$

where, by convention, $K_{\sin}(x,x) := 1$. In probabilistic terms, when $n \to \infty$, the DPP of the normalised eigenangles $\frac{n}{2\pi}\theta_1, \ldots, \frac{n}{2\pi}\theta_n$ of the random unitary matrices distributed according to the Haar measure on $U_n(\mathbb{C})$ converge, in the sense of the local uniform convergence of correlation functions, towards a limiting point process which is the DPP associated with the *sine kernel* $K_{\sin}(x,y)$: for all $k \geq 1$ and any reasonable function $\varphi : \mathbb{R}^k \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}\Bigg[\sum_{i_1 \neq \cdots \neq i_k} \varphi(\tfrac{n}{2\pi}\theta_{i_1}, \ldots, \tfrac{n}{2\pi}\theta_{i_k})\Bigg] = \int_{\mathbb{R}^k} \varphi(s) \det\Big[K_{\sin}(s_i, s_j)\Big]_{i,j=1}^{k} \mathrm{d}s. \quad (1.5)$$

Furthermore, $K_{\sin}$, seen as an operator acting on $L^2(\mathbb{R})$, is the projection on the space of functions whose Fourier transform has support in $[-\frac{1}{2}, \frac{1}{2}]$. This latter DPP generates almost surely infinite configurations on $\mathbb{R}$.

A paradigm in random matrix theory, already appearing in the pioneering works of Dyson and Wigner, is the idea that the local behaviour of the eigenvalues exhibits universality, unlike their global behaviour. To illustrate this phenomenon, let us first introduce another popular model of random matrices: the Gaussian Unitary Ensemble (GUE). This time we work on the space of Hermitian matrices $H_n(\mathbb{C}) := \{M \in \mathcal{M}_n(\mathbb{C}) : M^* = M\}$ equipped with a Gaussian measure. More precisely, the isomorphism $H_n(\mathbb{C}) \simeq \mathbb{R}^{n^2}$ induces a Lebesgue measure $\mathrm{d}M$ as well as an Euclidean norm $\|M\| = \mathrm{Tr}(\mathrm{M}^2)^{1/2}$ on $H_n(\mathbb{C})$, and we consider the Gaussian measure $g_n(M)$ with density proportional to $\mathrm{e}^{-\frac{n}{2}\|M\|^2}$. A change of variables and a computation of the Jacobian similar to the one for unitary matrices shows that the joint probability distribution of the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ of the random matrix of law $g_n(M)$ is given by:

$$\mathrm{d}\mathbb{P}(\lambda_1, \ldots, \lambda_n) := \frac{1}{Z_n} \prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_{j=1}^{n} \mathrm{e}^{-\frac{n}{2}\lambda_j^2} \,\mathrm{d}\lambda_j\,,$$

where $Z_n > 0$ is an explicit normalising constant[9]. After further manipulations one obtains that the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ is a DPP on $\mathbb{R}$ with kernel:

$$K_{H_n(\mathbb{C})}(x,y) := \sum_{k=0}^{n-1} \Psi_k^n(x)\Psi_k^n(y)$$

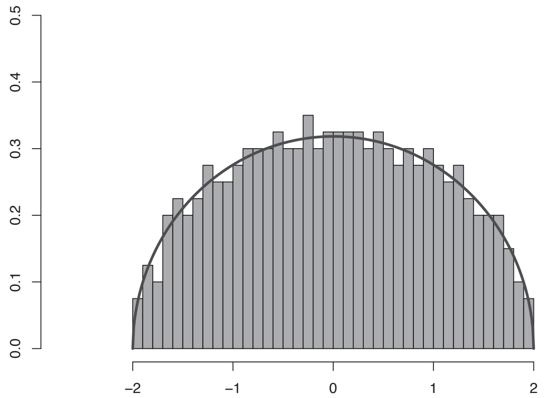


Figure 4. Histogramme des valeurs propres d'une réalisation du GUE de taille 300 et loi du demi-cercle

where, for all $n \geq 1$, $(\Psi_k^n)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$ given by renormalised Hermite functions.[10] Here again, we are dealing with a projection kernel on a subspace of $L^2(\mathbb{R})$ of finite dimension, the subspace of functions which can be written $P(x)\mathrm{e}^{-nx^2/4}$ where P is a polynomial of degree at the most $n-1$. When $n \to \infty$, as illustrated in Figure 4, the eigenvalues concentrate on the compact $[-2,2]$ with a density $\rho(x) := \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)$, known as the semicircle *law*.

If we now zoom in at a factor n, around a point x_0 from $(-2,2)$, it can be shown that the local behaviour of the eigenvalues around that point is again governed by the sine kernel $K_{\sin}$. More precisely, one has the uniform local convergence

$$\frac{1}{\rho(x_0)n} K_{H_n(\mathbb{C})}\left(x_0 + \frac{x}{\rho(x_0)n}, x_0 + \frac{y}{\rho(x_0)n}\right) \xrightarrow[n\to\infty]{} K_{\sin}(x,y).$$

Note that the limit does not depend on x_0. The proof of this convergence is a little more delicate than in the case of unitary matrices. It requires a close analysis of the asymptotic behavior of the Hermite functions.

On the other hand, if we have a closer look at the points at the edge of the semicircle, after some rescaling, another kernel appears, known as the *Airy kernel*,

$$K_{\mathrm{Airy}}(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

Here, $\mathrm{Ai}(x)$ is the Airy function, which satisfies $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and which plays a role in the study of the optical properties of rainbows. For example, at point 2 (the analysis at -2 is the same),

$$K_{H_n(\mathbb{C})}^{\mathrm{edge}}(x,y) := \frac{1}{n^{2/3}} K_{H_n(\mathbb{C})}\left(2 + \frac{x}{n^{2/3}}, 2 + \frac{y}{n^{2/3}}\right) \xrightarrow[n\to\infty]{} K_{\mathrm{Airy}}(x,y).$$

It is possible to show that this convergence takes place in the sense of the convergence of the trace-class operators on $L^2(s,\infty)$ for any $s > 0$. We can deduce the fluctuations of the largest eigenvalue around the right edge. Indeed, the distribution function of the largest particle of a DPP on $\mathbb{R}$ can be expressed in terms of a Fredholm determinant, namely

$$\mathbb{P}\Big(n^{2/3}(\max_{j=1}^{n} \lambda_j - 2) \leq s\Big) = \det(I - K_{H_n(\mathbb{C})}^{\mathrm{edge}})_{L^2(s,\infty)}\,,$$

8 We allowed ourselves to discretely add the term $\mathrm{e}^{-\mathrm{i}\frac{n-1}{2}(x-y)}$ because modifying a kernel $K(x,y)$ to $K(x,y)\frac{f(x)}{f(y)}$ where f does not vanish does not change the determinants associated with the correlation functions; thus, these two kernels generate the same DPP.

9 It is a ratio of products of Gamma functions, obtained as a limit case for Selberg's integral formula.

10 If $H_k(x) := (-1)^k \mathrm{e}^{x^2/2}(\frac{\mathrm{d}}{\mathrm{d}x})^k(\mathrm{e}^{-x^2/2})$ is the k-th Hermite polynomial, we use $\Psi_k^n(x) := c_k^n H_k(\sqrt{n}x)\mathrm{e}^{-nx^2/4}$ with $c_k^n := \int_{\mathbb{R}} H_k(\sqrt{n}x)^2\mathrm{e}^{-nx^2/2}\mathrm{d}x$.

which is continuous for this topology. Thus we obtain, for all $s \in \mathbb{R}$,

$$\mathbb{P}\Big(n^{2/3}\big(\max_{j=1}^{n}\lambda_j - 2\big) \leq s\Big) \xrightarrow[n\to\infty]{} F_2(s) := \det(I - K_{\mathrm{Airy}})_{L^2(s,\infty)}\,.$$

The probability law associated with the distribution function F_2 is known as the *Tracy–Widom law*. The latter has established an explicit formula for $F_2(s)$ in terms of the Hastings-McLeod solution of the Painlevé II equation. It is also the law of the largest particle of the DPP associated with the *Airy kernel*. We are going to meet this law again a little bit later, in a different context than random matrices.

After a series of works that cannot possibly all be quoted here, it could be observed that the sine and Airy kernels are ubiquitous in the description of the local behaviour of the eigenvalues in a huge number of random matrix models, linked with DPP or not. We speak of *universality phenomenon* in random matrices. It is surprising to observe that the two universal DPPs also appear outside the context of random matrices; in the following, we are going to present three particularly striking examples.

Orthogonal polynomials and the sine kernel

At first we would like to present the elegant approach of Lubinsky [9] for the DPPs related with a family of orthogonal polynomials. We consider a positive Borel measure μ supported in $[-1,1]$ such that $L^2(\mu)$ contains all polynomials. We can then define an orthonormal family of polynomials $(p_k)_{k\in\mathbb{N}}$ in $L^2(\mu)$ with positive dominant coefficients γ_k. We denote by $w(x)$ the Radon-Nikodym derivative of μ with respect to the Lebesgue measure, such that $\mu(x) = w(x)\mathrm{d}x + \mu_s$ with μ_s being singular with respect to the Lebesgue measure. We next introduce the Christoffel–Darboux kernel, well-known in approximation theory:

$$K_n^{(\mu)}(x,y) = \sqrt{w(x)w(y)}\sum_{k=0}^{n-1} p_k(x)p_k(y).$$

The DPP associated with this kernel almost surely generates configurations of n points in $[-1,1]$. We suppose that μ is a *regular* measure on $[-1,1]$, in the sense that $\gamma_k^{1/k}$ converges towards 2 when $k \to \infty$. This is the case for, e.g., the classic families like the Legendre, Tchebychev and, more generally, Jacobi polynomials. It is also the case once $w(x) > 0$ almost everywhere on $[-1,1]$. Lubinsky proves the following result: for any x_0 of $(-1,1)$ such that μ is absolutely continuous on a neighbourhood of x_0, and whose density w is continuous and strictly positive at x_0, one has the uniform local convergence:

$$\frac{1}{n\rho(x_0)}\,K_n^{(\mu)}\left(x_0 + \frac{x}{n\rho(x_0)}, x_0 + \frac{y}{n\rho(x_0)}\right) \xrightarrow[n\to\infty]{} K_{\sin}(x,y)$$

where $\rho(x) := 1/(\pi(\sqrt{1-x^2}))$. Note that, under the stronger assumption that $\mu(x) = w(x)\mathrm{d}x$, with a positive and continuous w on $[-1,1]$, one has a global convergence of the DPP points associated with $K_n^{(\mu)}$ towards the arcsine law with density $\rho(x)$, as illustrated by Figure 5. The proof of this result is surprisingly short, as compared to what is usually done in this field of mathematics. It uses elementary analysis in a very clever way to offer a robust method for the comparison between the kernels associated with different measures.

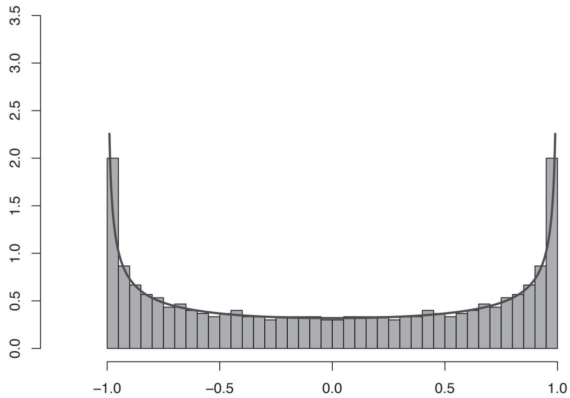


Figure 5. Histogram of a DPP of size 300 **associated to the Christoffel–Darboux kernel and the arcsine law**

Riemann zeta function and the sine kernel

Here we are going to discuss a somewhat unexpected occurrence of the sine process in analytic number theory. For a complex number s with real part $\Re\mathfrak{e}(s) > 1$, the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p\text{ prime}}\frac{1}{1-p^{-s}}.$$

This function admits a meromorphic extension to $\mathbb{C}$, which possesses "trivial" zeros at the negative even integers: $-2,-4,-6$, etc. The other zeros are localized in the region $0 \leq \Re\mathfrak{e}(s) \leq 1$. The celebrated Riemann conjecture claims that those zeros all have a real part equal to $1/2$. Under Riemann hypothesis, one can thus write those zeros into the form $1/2 \pm \mathrm{i}t_j$ with $0 < t_1 < t_2 < \ldots$ due to the symmetry $\zeta(\bar{s}) = \overline{\zeta(s)}$. Thus, a classical result is that, as $n \to \infty$,

$$N_n := \#\{j \geq 1 :\ t_j \leq n\} \sim \frac{n}{2\pi}\log\frac{n}{2\pi}\,,$$

which leads us to set

$$w_j := \frac{t_j}{2\pi}\log\frac{t_j}{2\pi}\,,$$

so that $\#\{j \geq 1 :\ w_j \leq n\} \sim n$ when $n \to \infty$; the typical distance between the consecutive w_j should be of order one. In order to better understand the distances between the renormalized zeros, at least asymptotically, we consider the analogue of correlation functions for point processes (1.3):

$$R_k^{(n)}(\varphi) := \frac{1}{n}\sum_{1\leq i_1\neq\cdots\neq i_k\leq n}\varphi(w_{i_1},\ldots,w_{i_k}),$$

for $k \geq 1$ and smooth functions $\varphi : \mathbb{R}^k \to \mathbb{R}$. Rudnick and Sarnak obtain, under rather strong assumptions on φ, that

$$\lim_{n\to\infty} R_k^{(n)}(\varphi) = \int_{s_1+\cdots+s_k=0}\varphi(s)\det\Big[K_{\sin}(s_i,s_j)\Big]_{i,j=1}^{k}\mathrm{d}s\,,$$

which is a result similar to (1.5). Thus, asymptotically, the distances between the renormalized zeros behave like a typical realisation of the sine process. The restriction of the integration domain to the hyperplane $s_1 + \cdots + s_k = 0$ is due to the fact that $R_k^{(n)}(\varphi)$ may depend only on the distances between the w_j's. In the case of pairwise correlations, one has, for instance, for $\varphi(x,y) = f(x-y)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{1\leq j\neq\ell\leq n} f(w_j - w_\ell) = \int_{-\infty}^{\infty} f(y)\left(1 - \left(\frac{\sin\pi y}{\pi y}\right)^2\right)\mathrm{d}y.$$

This particular case was proven by Montgomery in the 1970s, for the functions $f : \mathbb{R} \to \mathbb{R}$ whose Fourier transform is C^∞ and with support in $[-1, 1]$. The Montgomery conjecture, which is still open, states that the result must be true without any support restriction. It is said that, in 1972, Montgomery, who was then still a student but already author of the above result, met the physicist Dyson at an afternoon tea in Princeton. Dyson, an expert in random matrices, immediately recognized in Montgomery's result the famous sine kernel.

Random permutations and Airy kernel

We end with a surprising occurrence of the Tracy–Widom law in the study of random permutations. If $\sigma \in \mathfrak{S}_n$ is a permutation of $[\![1, n]\!]$, we say that $\sigma(i_1) < \sigma(i_2) < \ldots < \sigma(i_k)$ with $i_1 < i_2 < \ldots < i_k$ is an increasing subsequence of σ. We denote by $\ell(\sigma)$ the longest length of an increasing subsequence. We thus have $1 \leq \ell(\sigma) \leq n$. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 3 & 2 & 6 & 1 & 7 & 9 & 4 & 8 \end{pmatrix},$$

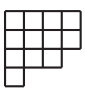
then $\ell(\sigma) = 4$ and it is reached at the increasing subsequence 5, 6, 7, 8 (but also at 2, 6, 7, 9). If now σ_n is a random permutation sampled uniformly on the symmetric group $\mathfrak{S}_n$, the study of the asymptotic behavior of $\ell(\sigma_n)$ for large n is known as the *Ulam problem*. After the works of Hammersley, Vershik, Kerov, Logan and Schepp, one obtains that

$$\mathbb{E}[\ell(\sigma_n)] \sim 2\sqrt{n}, \qquad n \to \infty.$$

An important breakthrough has been made by Baik, Deift and Johansson [2] who obtained the fluctuations of $\ell(\sigma_n)$ around its average value: for any $s \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(n^{-1/6}(\ell(\sigma_n) - 2\sqrt{n}\,) \leq s\right) = F_2(s).$$

In other words, the random variable $n^{-1/6}(\ell(\sigma_n) - 2\sqrt{n})$ converges (in the weak * topology) towards the Tracy–Widom distribution when $n \to \infty$. To understand how the problem is related to DPPs, we won't present the original proof by Baik, Deift and Johansson but rather the one by Borodin, Okounkov and Olshanski [3]. Their analysis is based on the Robinston–Schensted (RS) correspondence, well-known in representation theory of the symmetric group, which, to a permutation $\sigma \in \mathfrak{S}_n$ associates a pair of Young tableaux of the same shape. More precisely, a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of an integer n, that is, a decreasing sequence of positive integers with sum n, is encoded by a *Ferrers diagram* of n boxes with λ_i boxes on the i-th line. For example, the partition $(4, 4, 3, 1)$ of $n = 12$ is encoded by the diagram:



Given a Ferrers diagram λ with n boxes, a *Young tableau* of shape λ is a filling of the n boxes with the integers from 1 to n in a strictly increasing way along the lines and columns. The RS correspondence associates to a permutation σ two Young tableaux of the same shape. We successively place the integers $\sigma(1), \sigma(2), \ldots$ in the first tableau, starting in the first line and with the following rule: every new element $\sigma(j)$ is inserted into the first line. If $\sigma(j)$ is the largest integer of the line, we place it into a new box on the right. Otherwise, it takes the place of the smallest integer that is larger than itself. The latter is thus put down to the next line where the same rules apply. The second tableau keeps track of the order in which the boxes have been created. Thus for

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 3 & 2 & 6 & 1 & 7 & 9 & 4 & 8 \end{pmatrix},$$

the RS correspondence follows the successive steps:

5	3	2	26	16	167	1679	1479	1478
	5	3	3	2	2	2	26	269
		5	5	3	3	3	3	3
				5	5	5	5	5
1	1	1	14	14	146	1467	1467	1467
	2	2	2	2	2	2	28	289
		3	3	3	3	3	3	3
				5	5	5	5	5

Two key properties are used here. First, $\ell(\sigma) = \lambda_1$, the length of the first line of the tableaux obtained by the RS correspondence. Then, when we apply the RS correspondence to the uniform permutation σ_n, the probability that the shape of the resulting tableaux is the partition λ of n equals $\mathrm{PL}_n(\lambda) := \frac{1}{n!}\dim(\lambda)^2$. Here $\dim(\lambda)$ is the number of Young tableaux with shape λ; it is also the dimension of the irreducible representation of $\mathfrak{S}_n$ indexed by the partition λ. This probability distribution PL_n on the Ferrers diagrams with n boxes is called the *Plancherel measure*. Consider now the *poissonization* $\mathbb{P}^\theta$ of parameter $\theta > 0$ of the Plancherel measures, which generates Ferrers diagrams of random size: we sample an integer N with Poisson law of parameter θ and then sample a Ferrers diagram of N boxes following the Plancherel measure PL_N. Thus, if one denotes by $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$ the number of boxes of a Ferrers diagram λ, the probability under $\mathbb{P}^\theta$ to obtain a diagram λ is:

$$\mathbb{P}^\theta(\lambda) := \mathrm{e}^{-\theta}\theta^{|\lambda|}\left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2.$$

This poissonisation is motivated by the following fact: if $\lambda = (\lambda_1, \lambda_2, \ldots)$ has $\mathbb{P}^\theta$ as its distribution, then the random configuration $x_i := \lambda_i - i$ is a DPP on $\mathbb{Z}$. Its kernel $K^\theta(x, y)$ can be written as a double integral on complex contours; this rewriting is well-suited for asymptotic analysis (saddle-point method). From there, Borodin, Okounkov and Olshanski have proven that

$$\lim_{\theta\to\infty} \theta^{1/6} K^\theta(2\sqrt{\theta} + x\theta^{1/6}, 2\sqrt{\theta} + y\theta^{1/6}) = K_{\mathrm{Airy}}(x, y).$$

The convergence of $n^{-1/6}(\ell(\sigma_n) - 2\sqrt{n}\,) = n^{-1/6}(\lambda_1 - 2\sqrt{n}\,)$ towards the Tracy–Widom law cited above is a consequence of the latter, because, under $\mathbb{P}^\theta$, the number of boxes of a tableau concentrates around θ when $\theta \to \infty$ (depoissonisation procedure). In other words, for large integer θ, the distributions $\mathbb{P}^\theta$ and PL_θ generate diagrams which are similar in a sense that can be quantified.

DPP and machine learning

To conclude our little journey through the vast land of DPP, we would like to go back to the jaguar example from the introduction and outline some aspects of the use of DPP in the context of machine learning. The main message is that when it comes to modelling situations which involve repulsion, diversity and negative correlations between objects, DPP can provide efficient models that are easy to handle and easy to sample. To go deeper into the subject, we recommend for example [7].

In most of these applications, one tries to select a (random) subset of objects in a database, that is, in a discrete set $\mathcal{E}$ of cardinal n, possibly very large; typically images or texts. We then consider a class of DPP which generate configurations $\Xi \subset \mathcal{E}$ of random cardinal. In that context, we attribute to the i-th element of $\mathcal{E}$ a vector $B_i \in \mathbb{R}^d$, where the dimension d is fixed by the user. For instance, B_i encodes the pixels of the i-th image and d will depend on the chosen resolution. We then consider the semi-definite positive matrix $L := B^t B$ where B is the $d \times n$ matrix with columns B_i. We next take $K := L(I + L)^{-1}$ as a kernel of a DPP $\Xi \subset \mathcal{E}$. Thus, for all $A \subset \mathcal{E}$,

$$\mathbb{P}(A \subset \Xi) = \det(K_A) := \det[K_{ij}]_{i,j \in A}\,,$$

with the convention $\det(K_\emptyset) := 1$. In terms of the matrix L, one has

$$\mathbb{P}(\Xi = A) = \frac{\det(L_A)}{\det(I + L)}\,.$$

We see that $\mathbb{P}(\Xi = A)$ equals, up to a normalisation constant, the square volume of the polytope generated by the columns B_i of B with $i \in A$. Thus, if $B_i = q_i \varphi_i$ with $q_i \in \mathbb{R}^+$ and $\varphi_i \in \mathbb{R}^d$ with $\|\varphi_i\| = 1$, then q_i is interpreted as a measure of the importance of the i-th object of $\mathcal{E}$, while $S_{ij} := \varphi_i^t \varphi_j \in [-1, 1]$ represents a measure of the similarity between the i-th and the j-th object. More precisely, $\mathbb{P}(\Xi = A)$ is proportional to $\prod_{i \in A} q_i^2 \det(S_A)$. In practice, every object of $\mathcal{E}$ needs to be labelled with its attribute (q_i, φ_i), i.e., in the same way that PageRank assigns a level of importance to every web page, and then we compute the associated matrices L and K. A realisation of the DPP with kernel K will then provide a subset of $\mathcal{E}$ presenting diversity in the sense of the matrix S. For example, if $\mathcal{E}$ is the set of images associated with the key word "jaguar" on the internet, and we have previously labelled each of its elements with the attributes (q_i, φ_i), then a realisation of the DPP Ξ will provide subset of images exhibiting diversity.

It is also possible to restrict oneself to a parametrised family of kernels K_θ and estimate θ by means of the usual statistical methods from the training data. By this method, Kulesza and Taskar [7] develop, as an example, the extractive text summarisation where you have numerous texts on the same subject (i.e., newspaper articles on a time line) and you want to extract a summary, this is to say, a small number of sentences which contain as much information as possible.

The interest of these models is mainly due to the fact that they are easy to implement in an exact manner: one can reply to most of the questions on inference in polynomial time, essentially by multiplying, inverting, diagonalising or computing the determinants of matrices of size n, at a cost of $O(n^3)$ elementary operations. In particular, we have exact algorithms at our disposal for sampling a DPP at this cost.

The use of DPP in machine learning, as well as in spatial statistics ([8]) or in numerical integration, is still in its infancy but seems to spark the interest of more and more users of applied mathematics.

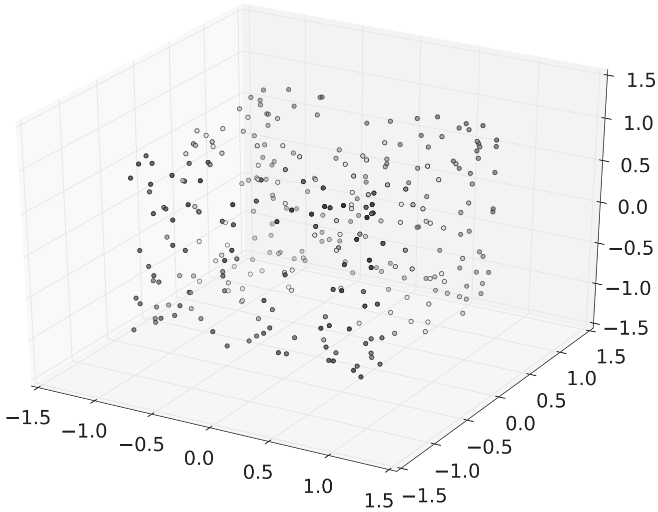


Figure 6. Sampling of a DPP associated to orthogonal polynomials in several variables on the cube

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The Littlewood–Paley Theory: A Common Thread of Many Works in Nonlinear Analysis

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In this article we present the Littlewood–Paley theory and illustrate the effectiveness of this microlocal analysis tool in the study of partial differential equations in a context which is the least technical possible. As we shall see below, the Littlewood–Paley theory provides a robust approach not only to the separate study of the various regimes of solutions to nonlinear partial differential equations, but also to the fine study of functional inequalities, and how to make them accurate.

1 The Littlewood–Paley theory: A tool that has become indispensable

The Littlewood–Paley theory is a localization procedure in the frequency space that, since about three decades ago, has established itself as a very powerful tool in harmonic analysis. The first goal of this text is to present it in a way which is as simple as possible.[1] Its basic idea is contained in two fundamental inequalities, known as Bernstein's inequalities, that describe some properties of functions whose Fourier transform have compact support.

The first inequality says that, for a tempered distribution[2] in $\mathbb{R}^d$ whose Fourier transform is supported in an annulus of size λ, to differentiate first and then take the L^p norm is the same as to apply a homothety of ratio λ on the L^p norm. In the L^2 setting this remarkable property is an easy consequence of the action of the Fourier transform on derivatives and of the Fourier–Plancherel formula. The proof in the case of general L^p spaces uses Young's inequalities and the fact that the Fourier transform of a convolution is the product of the Fourier transforms.

On the other hand, the second inequality tells us that, for such a distribution, the change from the L^p norm to the L^q norm, with $q \geq p \geq 1$, costs $\lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)}$, which must be understood as a Sobolev embedding. It is proved, like the first inequality, using Young's inequalities and the relation between the Fourier transform and the convolution product.

Fourier analysis is at the heart of the Littlewood–Paley theory, which has inspired a large number of my works. It was in conducting experiments on the propagation of heat at the end of the 18th century that Joseph Fourier opened the door to that theory, which was hugely expanded in the 20th century and intervenes in the majority of branches of physics.

In this theory, which bears the name of its creator, one performs the frequency analyis of a function f of $L^1(\mathbb{R}^d)$ by the formula:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\xi} f(x)\, dx\,.$$

Under appropriate conditions, $\widehat{f}$ the Fourier transform of f (also denoted $\mathcal{F}f$ in the present text), allows the synthesis of f through the inversion formula:

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi\,.$$

1 For a more detailed presentation of this theory the reader can consult the monograph [3].

2 A tempered distribution is an element of the topological dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

As a consequence, we obtain the Fourier–Plancherel identity

$$\int_{\mathbb{R}^d} |f(x)|^2\,dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2\,d\xi\,.$$

In fact, for all functions f of $\mathcal{S}(\mathbb{R}^d)$, we have, due to Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\overline{f}(x)\,dx &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\widehat{f}(\xi)\,d\xi\Big)\overline{f}(x)\,dx\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\Big(\int_{\mathbb{R}^d}\mathrm{e}^{-ix\cdot\xi}f(x)\,dx\Big)}\,d\xi\,.\end{aligned}$$

This representation created a true revolution in the way we think about functions. To give $\widehat{f}$ is exactly equivalent to giving f, and this duality between analysis in amplitude (in the physical space described by x) and analysis in frequency (in the frequency space described by ξ) is of extraordinary importance in physics and in mathematics.

A fundamental fact from the theory of distributions is that the Fourier transform can be extended to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The crucial point is the fact that $\mathcal{F}$ is a well-known isomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (the space of smooth functions that, together with all their derivatives, decrease faster than every polynomial) and its extension to $\mathcal{S}'(\mathbb{R}^d)$ is defined by duality.[3]

Fourier transforms have a very large number of properties that we do not wish to list here. Let us just recall the two basic principles of these transforms that we cannot dissociate from the convolution product. The first principle of the Fourier transform is that regularity implies decreasing; the second one is that decreasing leads to regularity. The usefulness of these properties, that play a crucial role in the study of Fourier transforms in $\mathcal{S}(\mathbb{R}^d)$, will quickly become clear in what follows.

Fourier analysis allows us to explicitly solve linear equations with constant coefficients.[4] In particular, by combining the Fourier transform with the convolution product we can explicitly determine the solutions of the Schrödinger equation, a fundamental equation in quantum mechanics:

$$(\mathrm{S})\quad\begin{cases} i\,\partial_t v + \Delta v = 0\\ v_{|t=0} = v_0 \in \mathcal{S}(\mathbb{R}^d)\,.\end{cases}$$

In fact, taking the partial Fourier transform with respect to the variable x we obtain for every (t,ξ) in $\mathbb{R}\times\mathbb{R}^d$:

$$\begin{cases} i\,\partial_t\widehat{v}(t,\xi) - |\xi|^2\widehat{v}(t,\xi) = 0\\ \widehat{v}(0,\xi) = \widehat{v}_0(\xi)\,,\end{cases}$$

and integrating we get

$$\widehat{v}(t,\xi) = \mathrm{e}^{-it|\xi|^2}\widehat{v}_0(\xi).$$

Combining the inverse Fourier transform together with the properties of the Fourier transform and the convolution product, we deduce that the solution of (S) for $t \neq 0$ can be written as

$$v(t,\cdot) = \frac{\mathrm{e}^{i\frac{|x|^2}{4t}}}{(4\pi i t)^{\frac{d}{2}}} \star v_0\,.$$

By Young's inequality, it follows the fundamental dispersion property

$$\|v(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{|4\pi t|^{\frac{d}{2}}}\|v_0\|_{L^1(\mathbb{R}^d)}\,.$$

This technique of explicit representation of solutions can be adapted to all linear evolution equations with constant coefficients. However, it is not always straightforward to deduce the dispersion effects. In fact, for example, to establish dispersive estimations for the wave equation in $\mathbb{R}^d$ requires more elaborate techniques involving oscillating integrals, which necessitate a hypothesis of spectral localization in an annulus of Cauchy data.

The analysis of dispersion, a central problem in linear wave mechanics, provides a framework of formidable effectiveness for solving and analysing nonlinear dispersive partial differential equations. It is thanks to the remarkable work of Robert Strichartz [37] in the late 1970s that we have been able to transcribe dispersion phenomena, which correspond to a pointwise inequality, into robust inequalities. The idea of these estimates, known as Strichartz estimates, is to pass from a pointwise in time decay estimate to a spatial integrability gain after an appropriate time average. These Strichartz estimates, which have experienced a big boom these last few years, go along with the Littlewood–Paley theory: they can be expressed equally in Lebesgue spaces and in Besov spaces which we will define next.

The Littlewood–Paley theory was introduced by John Edensor Littlewood and Raymond Paley [29, 30] in the 1930s for the harmonic analysis of L^p spaces, but its systematic use in the analysis of partial differential equations is more recent. In fact, the main breakthrough of this theory was made after the seminal paper [12] by Jean-Michel Bony in 1981 on the paradifferential calculus that connects nonlinear functions and the Littlewood–Paley decomposition.

The main idea of this theory consists in sampling the frequencies by means of a decomposition of the frequency space in annulus of size 2^j, thus allowing the decomposition of a function into a sum of a countable number of regular functions whose Fourier transform is supported in an annulus of size 2^j:

$$f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f\,, \tag{1}$$

where the homogeneous dyadic blocks of f, $\dot{\Delta}_j f$, are defined by the filtering of f at frequencies of order 2^j. Observe that this so-called homogeneous Littlewood–Paley decomposition is valid modulo polynomials P. In fact, since the Fourier transform of every polynomial is supported at the origin, the identity (1) cannot be applied to polynomials. This restriction on the lower frequencies is overcome in the case of the inhomogeneous Littlewood–Paley decomposition:

$$f = \sum_{j\ge -1} \Delta_j f\,, \tag{2}$$

where $\Delta_j f := \dot{\Delta}_j f$ for j varying in $\mathbb{N}$ and $\Delta_{-1} f$ is an operator filtering the lower frequencies, that is: it only preserves the frequencies in a ball centered at the origin.

The Littlewood–Paley decompositions (1) and (2) introduced above are obtained by a decomposition in the space of frequencies arising from dyadic partitions of unity. More

3 For a complete presentation of the theory of distributions we can, for instance, see the fundamental references [34, 36].

4 Linear equations with variable coefficients and nonlinear equations require different methods.

precisely, if we are given a radial function χ belonging to $\mathcal{D}(B(0,4/3))$, identically equal to 1 in $B(0,3/4)$, we have the following identities

$$\chi + \sum_{j\geq 0} \varphi(2^{-j}\cdot) = 1 \ \text{ in } \mathbb{R}^d, \ \text{ and } \ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\cdot) = 1 \ \text{ in } \mathbb{R}^d \setminus \{0\},$$

where φ is the function defined by $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$.

With this normalization φ is a radial function of $\mathcal{D}(C)$ where C is the annulus centered at the origin with inner radius $3/4$ and outer radius $8/3$ and we define the homogeneous dyadic blocks $\dot{\Delta}_j$ by[5]

$$\dot{\Delta}_j f := \varphi(2^{-j}D)f := \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f) = 2^{jd}h(2^j\cdot) \star f \quad \text{with } h = \mathcal{F}^{-1}\varphi$$

and the inhomogeneous dyadic blocks Δ_j by $\Delta_j f := \dot{\Delta}_j f = 2^{jd}h(2^j\cdot) \star f$ if $j \geq 0$ and

$$\Delta_{-1} f := \chi(D)f := \mathcal{F}^{-1}(\chi\mathcal{F}f) = \widetilde{h} \star f, \ \text{ where } \widetilde{h} = \mathcal{F}^{-1}\chi.$$

In a similar way, we also introduce the low-frequency cut-off operators

$$\dot{S}_j f := \sum_{k\leq j-1} \dot{\Delta}_k f := \mathcal{F}^{-1}(\chi(2^{-j}\cdot)\mathcal{F}f) = 2^{jd}\widetilde{h}(2^j\cdot) \star f \quad \text{for } j \in \mathbb{Z}$$

and

$$S_j f := \sum_{k\leq j-1} \Delta_k f = 2^{jd}\widetilde{h}(2^j\cdot) \star f \ \text{ for } j \in \mathbb{N}.$$

It is worth noting that the dyadic blocks that are frequency cut-off operators are convolution operators. This property, which is a trivial consequence of the fact that the Fourier transform changes the convolution product to the pointwise product of functions, plays a central role in the techniques arising from Littlewood–Paley theory. In particular, all of these operators act in the spaces L^p in a uniform way with respect to p and j.

In what follows, it is also important to underline that the properties of the supports of the functions φ and χ give rise to quasi-orthogonal relations for the Littlewood–Paley decomposition, namely

$$\dot{\Delta}_j\dot{\Delta}_k = 0 \ \text{ and } \ \Delta_j\Delta_k = 0 \ \text{ if } |j-k| > 1,$$

which easily implies that

$$\forall \xi \in \mathbb{R}^d, \ \frac{1}{2} \leq \chi^2(\xi) + \sum_{j\geq 0} \varphi^2(2^{-j}\xi) \leq 1, \tag{3}$$

and

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \ \frac{1}{2} \leq \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{4}$$

Littlewood–Paley analysis allows the exact characterization of the regularity of a function f in terms of the decay properties of its dyadic blocks with respect the summation index j. We thus recover, in a more precise way, the idea already present in Fourier analysis: that space regularity is translated into frequency decay.

In particular, using the Fourier–Plancherel formula and the quasi-orthogonality properties (3)–(4), it is easy to observe that we can characterize a function f as an element of $L^2(\mathbb{R}^d)$ in terms of the sequence $(\|\dot{\Delta}_j f\|_{L^2(\mathbb{R}^d)})_{j\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, and the same for its inhomogeneous dyadic blocks. More precisely, thanks to an elementary Hilbertian analysis lemma, we can show the existence of a constant C such that we have

$$C^{-1}\sum_{j\in\mathbb{Z}} \|\dot{\Delta}_j f\|^2_{L^2(\mathbb{R}^d)} \leq \|f\|^2_{L^2(\mathbb{R}^d)} \leq C\sum_{j\in\mathbb{Z}} \|\dot{\Delta}_j f\|^2_{L^2(\mathbb{R}^d)},$$

and

$$C^{-1}\sum_{j\geq -1} \|\Delta_j f\|^2_{L^2(\mathbb{R}^d)} \leq \|f\|^2_{L^2(\mathbb{R}^d)} \leq C\sum_{j\geq -1} \|\Delta_j f\|^2_{L^2(\mathbb{R}^d)}.$$

Similarly, several classic norms can be written in terms of the Littlewood-Paley decomposition. This is, for example, the case of the Sobolev and Hölder norms. In particular, the fact that some function belongs to some Sobolev (resp. Hölder) space is related with properties of decay with respect to j of the L^2 (resp. L^∞) norm of $\dot{\Delta}_j u$ or $\Delta_j u$, according to whether they are homogeneous or nonhomogeneous spaces.

Let us recall that the nonhomogeneous Sobolev spaces $H^s(\mathbb{R}^d)$ that naturally show up in a large number of mathematical physics problems are, in the case when $s = m \in \mathbb{N}$, the subspaces of functions f of $L^2(\mathbb{R}^d)$ for which all derivatives (in the sense of distributions) of order smaller than or equal to m belong to $L^2(\mathbb{R}^d)$. It is then clear, given the quasi-orthogonality of the Littlewood–Paley decomposition and the action of the Fourier transform on the derivatives, that the fact that a function is in $H^m(\mathbb{R}^d)$ is characterized as follows:

$$\|f\|_{H^m(\mathbb{R}^d)} \sim \|(2^{jm}\|\Delta_j f\|_{L^2(\mathbb{R}^d)})\|_{\ell^2(j\geq -1)}.$$

A similar equivalence holds in the case of homogeneous Sobolev spaces $\dot{H}^m(\mathbb{R}^d)$, which are more appropriate to study scale invariant problems such as the incomprehensible Navier-Stokes system[6] and several variants of this system in meteorology and oceanography, or nonlinear wave equations that we have studied in [1, 2, 7], and many other equations such as those dealt with in [25, 26].

In general, to say that a function f belongs to $H^s(\mathbb{R}^d)$ means, roughly speaking, that f has s derivatives (fractional derivatives if s is noninteger) in $L^2(\mathbb{R}^d)$, and, as before, we can prove that there exists a constant C such that

$$C^{-1}\sum_{j\geq -1} 2^{2js}\|\Delta_j f\|^2_{L^2(\mathbb{R}^d)} \leq \|f\|^2_{H^s(\mathbb{R}^d)} \leq C\sum_{j\geq -1} 2^{2js}\|\Delta_j f\|^2_{L^2(\mathbb{R}^d)}.$$

This heuristic idea can also be applied to the homogeneous Sobolev norms, giving rise to the following correspondence in the setting of the Littlewood–Paley theory:

$$C^{-1}\sum_{j\in\mathbb{Z}} 2^{2js}\|\dot{\Delta}_j f\|^2_{L^2(\mathbb{R}^d)} \leq \|f\|^2_{\dot{H}^s(\mathbb{R}^d)} \leq C\sum_{j\in\mathbb{Z}} 2^{2js}\|\dot{\Delta}_j f\|^2_{L^2(\mathbb{R}^d)}.$$

5 By $\mathcal{F}^{-1}$ we denote the inverse Fourier transform in $\mathbb{R}^d$ and $\mathcal{F}(\varphi(2^{-j}D)f)(\xi) = \varphi(2^{-j}\xi)\widehat{f}(\xi)$ which shows that $\mathcal{F}(\dot{\Delta}_j f)$ is supported in the annulus 2^jC.

6 Recall that for the incomprehensible Navier-Stokes system the question of eventual creation of singularities after a finite time is one of the Millenium problems proposed by the Clay Mathematics Institute.

In examining these inequalities we observe that three parameters play a role: the regularity parameter s, the exponent of the Lebesgue norm used to measure the dyadic blocks $\dot\Delta_j f$ or $\Delta_j f$ and the type of sum preformed, either over $\mathbb{Z}$ or for $j \geq -1$. This observation allows, more generally, to efficiently characterise the norms of homogeneous or nonhomogeneous Besov spaces, respectively $\dot B^s_{p,r}(\mathbb{R}^d)$ and $B^s_{p,r}(\mathbb{R}^d)$. The norms of these spaces, which can be defined in terms of finite differences or using the heat kernel (as we can see, for example, in [3, 40]) can be expressed in terms of Littlewood–Paley decompositions[7]:

$$\|f\|_{B^s_{p,r}(\mathbb{R}^d)} \sim \Big(\sum_{j\geq -1} 2^{rjs}\,\|\Delta_j f\|^r_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{r}},$$

and

$$\|f\|_{\dot B^s_{p,r}(\mathbb{R}^d)} \sim \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\,\|\dot\Delta_j f\|^r_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{r}}.$$

Even if scale invariant, the homogeneous Sobolev spaces (and more generally the homogeneous Besov spaces) have to be manipulated with care, since, as was mentioned above, the homogeneous Littlewood–Paley decomposition (1) is only defined as modulo polynomials of arbitrary degree. There is no consensus about the definition of these spaces. In certain references, such as [11], they are defined as modulo polynomials of arbitrary degree. In others, such as [3], they are defined subject to a condition on the low frequencies. This condition requires limiting oneself to tempered distributions f satisfying (in the sense of distributions)

$$\|\dot S_j f\|_{L^\infty(\mathbb{R}^d)} \overset{j\to-\infty}{\longrightarrow} 0\,.$$

The dyadic decompositions provide not only the possibility of characterising a function as an element of almost all the classical spaces (Hölder, Sobolev, Besov, Lebesgue, Triebel–Lizorkin) by conditions concerning only its dyadic blocks, but they also allow us to define a plethora of functional spaces.

Littlewood–Paley decompositions and more simply the decomposition of functions into low and high frequency components are techniques that have proved their usefulness in the study of functional inequalities and in the analysis of nonlinear partial differential equations.

Sobolev embeddings are among the most celebrated of all functional inequalities. They provide key tools for the study of linear and nonlinear partial differential equations, in the elliptic, parabolic or hyperbolic framework. Sobolev inequalities express a strong integrability or regularity property for a function f in terms of integrability properties of some derivatives of f.

Among those inequalities, we can mention the Sobolev inequalities in Lebesgue spaces:

$$\dot H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)\,, \tag{5}$$

with $0 \leq s < d/2$ and $p = 2d/(d-2s)$.

Let us observe that the value $p = 2d/(d-2s)$ can easily be deduced using an homogeneity argument. In fact, if for every function v defined in $\mathbb{R}^d$ and all $\lambda > 0$ we define a function v_λ by $v_\lambda(x) = v(\lambda x)$, it is easy to verify that

$$\|v_\lambda\|_{L^p(\mathbb{R}^d)} = \lambda^{-\frac{d}{p}} \quad\text{and}\quad \|v_\lambda\|_{\dot H^s(\mathbb{R}^d)} = \lambda^{s-\frac{d}{2}}\|v_\lambda\|_{\dot H^s(\mathbb{R}^d)}\,.$$

Since both quantities $\|\cdot\|_{L^p(\mathbb{R}^d)}$ and $\|\cdot\|_{\dot H^s(\mathbb{R}^d)}$ have the same homogeneity degree when the Lebesgue index $p = 2d/(d-2s)$ (which means that they behave in the same way under a change of the unit of length), it is thus natural to compare them and we can assume in what follows that $\|f\|_{\dot H^s(\mathbb{R}^d)} = 1$.

We know that for all real number $p \geq 1$ and all measurable function f, we have, due to Fubini's theorem,

$$\|f\|^p_{L^p(\mathbb{R}^d)} = p\int_0^\infty \lambda^{p-1}\mu(|f|>\lambda)\,d\lambda\,.$$

To establish the Sobolev embedding (5), we decompose f into low and high frequency components in the following way:

$$f = f_{\ell,A} + f_{h,A} \quad\text{with}\quad f_{\ell,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\widehat{f})\,.$$

Since the support of the Fourier transform of $f_{\ell,A}$ is a compact set, the function $f_{\ell,A}$ is bounded and, more precisely, by using the inversion formula and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|f_{\ell,A}\|_{L^\infty(\mathbb{R}^d)} &\leq (2\pi)^{-d}\|\widehat{f_{\ell,A}}\|_{L^1(\mathbb{R}^d)}\\ &\leq (2\pi)^{-d}\int_{\mathbb{R}^d}|\xi|^s|\xi|^{-s}|\widehat{f_{\ell,A}}(\xi)|\,d\xi\\ &\leq C_s\,A^{\frac{d}{2}-s}\,\|f\|_{\dot H^s(\mathbb{R}^d)}\,.\end{aligned}$$

Now, the triangle inequality implies, for all $A > 0$,

$$(|f|>\lambda) \subset (|f_{\ell,A}|>\lambda/2)\cup(|f_{h,A}|>\lambda/2)\cdot$$

Consequently, by choosing

$$A = A_\lambda \overset{\text{def}}{=} \Big(\frac{\lambda}{4C_s}\Big)^{\frac{p}{d}},$$

we deduce that

$$\|f\|^p_{L^p(\mathbb{R}^d)} \leq p\int_0^\infty \lambda^{p-1}\mu(|f_{h,A_\lambda}|>\lambda/2)\,d\lambda\,.$$

Since, by the Bienaymé–Tchebychev inequality,

$$\mu(|f_{h,A_\lambda}|>\lambda/2) \leq 4\frac{\|f_{h,A_\lambda}\|^2_{L^2(\mathbb{R}^d)}}{\lambda^2}\,,$$

we obtain

$$\|f\|^p_{L^p(\mathbb{R}^d)} \leq 4p\int_0^\infty \lambda^{p-3}\|f_{h,A_\lambda}\|^2_{L^2(\mathbb{R}^d)}\,d\lambda\,.$$

Finally, by the Fourier–Plancherel identity,

$$\|f_{h,A_\lambda}\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^{-d}\int_{(|\xi|\geq A_\lambda)}|\widehat{f}(\xi)|^2\,d\xi\,,$$

which implies, due to Fubini's theorem, that for all $p > 2$

$$\begin{aligned}\|f\|^p_{L^p(\mathbb{R}^d)} &\leq 4p\,(2\pi)^{-d}\int_{\mathbb{R}^d}\Big(\int_0^{4C_s|\xi|^{\frac{d}{p}}}\lambda^{p-3}d\lambda\Big)|\widehat{f}(\xi)|^2\,d\xi\\ &\leq C_p\int_{\mathbb{R}^d}|\xi|^{\frac{d(p-2)}{p}}|\widehat{f}(\xi)|^2\,d\xi\,,\end{aligned}$$

where $C_p = (2\pi)^{-d}\frac{4p}{p-2}(4C_s)^{p-2}$. Since $s = d(\frac{1}{2}-\frac{1}{p})$, this concludes the proof of the Sobolev embedding.

The proof presented above is borrowed from [16]. We have other previous proofs of this estimate, namely one based on the Hardy-Littlewood-Sobolev inequality, which is for instance presented in [3]. We should note that the arguments of the above proof have inspired a number of other works,

7 Observe that the Besov spaces are independent of the dyadic blocks $\dot\Delta_j$ and Δ_j.

among which we refer to the paper [5] where the authors considered Sobolev embeddings in the Lorentz spaces $L^{p,q}$. Recall that the Lorentz spaces[8] were introduced in the 1950s by Lorentz so that $L^{p,\infty}$ are the weak spaces introduced by Marcinkiewicz in the 1930s, and $L^{p,p}$ are the usual Lebesgue spaces L^p.

This technique of decomposition into low and high frequencies was also relevant for the study of nonlinear partial differential equations, namely to establish that some Cauchy problems are globally well posed. Among these works we can refer to the article of Fujita–Kato [18] on the Navier-Stokes equations. In this type of approach, the idea is to decompose the Cauchy data (assumed here, for simplicity, in some Sobolev space $\dot{H}^s$) into low and high frequencies in such a way that the high frequency part has rather small norm in $\dot{H}^s$. If we have a global existence theorem for small initial data, then this high frequency part will give rise to a global solution to the problem, whereas the low frequency part (that will be regular) will satisfy a modified equation, and all we need to do is to prove that we can solve this perturbed equation.

The Sobolev embedding (5) is invariant by translation and scaling, but it is not invariant by oscillations, that is, by multiplication by oscillating functions, namely by those of the type $u_\epsilon(x) = e^{i\frac{(x|\omega)}{\epsilon}}\varphi(x)$, where ω is a unit vector of $\mathbb{R}^d$, and φ is a function in $\mathcal{S}(\mathbb{R}^d)$. Revisiting the proof of the Sobolev embedding presented above we can establish the following inequality due to Gérard-Meyer-Oru [20] :

$$\|u\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{(p-2)^{\frac{1}{p}}} \|u\|^{1-\frac{2}{p}}_{\dot{B}^{s-\frac{d}{2}}_{\infty,\infty}(\mathbb{R}^d)} \|u\|^{\frac{2}{p}}_{\dot{H}^s(\mathbb{R}^d)} . \qquad (6)$$

This Sobolev inequality is sharp, as the oscillatory example $u_\epsilon(x) = e^{i\frac{(x|\omega)}{\epsilon}}\varphi(x)$ shows. Many other examples show the optimality of the estimate (6), in particular a fractal example constructed in [4], supported in a Cantor type set, and the example of the chirp signal:

$$f(x) = x^{-\alpha}\sin\left(\frac{1}{x}\right), \quad \alpha > 0 ,$$

investigated in [5].

The refined estimate (6) is one of the key arguments in [19] where Patrick Gérard gave a characterisation of the defect of compactness of the critical Sobolev embedding (5) by means of profile decompositions.[9] We recall that the study of the defect of compactness of Sobolev embeddings of functional spaces, which goes back to the seminal works of Pierre-Louis Lions [27,28], provides a useful tool in the study of geometric problems and the understanding of the behaviour of solutions to nonlinear partial differential equations.

Nonlinear analysis has progressed substantially in the last decades due to profile decomposition techniques. This type of decomposition has been generalised, by different approaches, to other functional settings. In particular, we refer to the recent works [8,9] about the description of the defect of compactness of the critical Sobolev embedding of $H^1(\mathbb{R}^2)$ in $\mathcal{L}(\mathbb{R}^2)$, where $\mathcal{L}(\mathbb{R}^2)$, the so-called Orlicz space,[10] is the space of measurable functions $u : \mathbb{R}^2 \to \mathbb{C}$ for which there exists a real number $\lambda > 0$, such that

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1\right) \mathrm{d}x < \infty ,$$

as well as its generalisation to higher dimensions in [10]. This Sobolev embedding, which is based on the Trudinger-Moser inequalities, deals with the limiting case of the Sobolev embedding (5) and intervenes in numerous geometrical and physical problems, namely in the propagation of laser beams in different media. The study of this embedding is done in [10] by Fourier analysis arguments that highlight the fact that the elements responsible for the lack of compactness are, in this case and in contradistinction to the case of the Sobolev embedding (5), spread over the frequencies.

It is also noteworthy that an approach started by Stéphane Jaffard in [23] has allowed the extension of Patrick Gérard's result in [19] to the setting of the Triebel-Lizorkin spaces and has inspired the abstract analysis in [6]. This approach was based on the theory of wavelets, which, for its part, was inspired by the Littlewood–Paley, and will be discussed later.

As was referred to above, the second Bernstein inequality must be understood as a Sobolev embedding. In fact, it is easy to deduce from this second inequality that for all real numbers s, and for all $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$ we have

$$\dot{B}^s_{p_1,r_1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^d) , \qquad (7)$$

and analogously for the nonhomogeneous case.

Observe that these Sobolev embeddings are strict, as is shown, in the particular case of the Sobolev embedding $\dot{H}^s(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{2,\infty}(\mathbb{R}^d)$, by the following example based on the idea of lacunar series. Given a function χ of $\mathcal{S}(\mathbb{R}^d)$ whose Fourier transform is supported in a small ball centered at 0 with radius ϵ_0, and given a vector $\omega \in \mathbb{R}^d$ with Euclidean norm 3/2, we consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n(x) = \sqrt{n} \sum_{j\geq n} 2^{-js} \frac{1}{j+1} e^{i2^j(x|\omega)}\chi(x).$$

It is easy to observe that

$$\begin{cases} \dot{\Delta}_j f_n = 0 \quad \text{if} \quad j \leq n-1 \quad \text{and} \\ (\dot{\Delta}_j f_n)(x) = \dfrac{\sqrt{n}\, 2^{-js}}{j+1} e^{i2^j(x|\omega)}\chi(x) \quad \text{if} \quad j \geq n . \end{cases}$$

By an elementary computation, we conclude that

$$\|f_n\|^2_{\dot{H}^s(\mathbb{R}^d)} \sim n \sum_{j\geq n} \frac{1}{(j+1)^2} \sim 1 \;\text{ and }\; \|f_n\|_{\dot{B}^s_{2,\infty}(\mathbb{R}^d)} \lesssim \frac{1}{\sqrt{n}} ,$$

which clearly shows the strict inclusion of $\dot{H}^s(\mathbb{R}^d)$ into $\dot{B}^s_{2,\infty}(\mathbb{R}^d)$.

The techniques arising from the Littlewood–Paley theory allow also the analysis of the product of two tempered distributions (if it exists) by means of J.-M. Bony's paradifferential calculus. It does so in the following way: given two tempered distributions u and v, we write

$$u = \sum_p \Delta_p u \;\text{ and }\; v = \sum_q \Delta_q v .$$

Formally, if the product exists it is written as

$$uv = \sum_{p,q} \Delta_p u\, \Delta_q v .$$

8 For more details see [11,40].

9 Profile decompositions originate in the work of Brézis–Coron [15].

10 For an introduction to Orlicz spaces see [35,41].

The idea consists of decomposing the product uv into three parts: a first one with terms where the frequencies of u are large compared with those of v, a second one with terms where the frequencies of v are large compared with those of u and a third one for which the frequencies of u and v have comparable sizes. This leads to the following definition, first introduced by Jean-Michel Bony in [12]: we write

$$uv = T_u v + T_v u + R(u,v) \quad \text{with}$$
$$T_u v \stackrel{\text{def}}{=} \sum_{p \le q-2} \Delta_p u \Delta_q v = \sum_q S_{q-1} u \Delta_q v \qquad \text{and}$$
$$R(u,v) \stackrel{\text{def}}{=} \sum_{|q-p|\le 1} \Delta_q u \Delta_p v\,.$$

This so-called Jean-Michel Bony's decomposition is fundamental to the study of product laws as well as to the study of nonlinear partial differential equations. Clearly, it admits a homogeneous version. Let us recall that the bilinear operator $T_u v$ is called the paraproduct of v by u , whereas the symmetric bilinear operator $R(u, v)$ is called the remainder.

From the detailed study of the way the paraproduct and the remainder act on Sobolev, Hölder, and, more generally, Besov spaces, one can identify some principles:

– For two compactly supported distributions, the paraproduct is always defined, and the regularity of $T_u v$ is determined, mainly, by the regularity of v.
– On the other hand, the remainder is not always defined, but when it is the regularities of u and v add up to determine its regularity.

Jean-Michel Bony's paradifferential calculus has proven to be very effective in the study of evolution equations, which describe the behaviour of a physical phenomenon dependent of time. This method's relevance will be illustrated by presenting a method of microlocal decomposition we have introduced in [1, 2] in collaboration with Jean-Yves Chemin (see also [38, 39]) for the study of quasilinear wave equations of the type

$$(E)\begin{cases} \partial_t^2 u - \Delta u - \partial(G(u)\partial u) &= Q(\nabla u, \nabla u) \\ (u, \partial_t u)_{|t=0} &= (u_0, u_1) \end{cases}$$

with

$$\partial(G\partial u) = \sum_{1\le j,k\le d} \partial_j (G^{j,k}\partial_k u),$$

where Q is a quadratic form on $\mathbb{R}^{1+d}$; and G is a C^∞ function vanishing on 0, which, together with all its derivatives is bounded from $\mathbb{R}$ into the space of symmetric matrices on $\mathbb{R}^d$, and takes its values in a compact set K such that $Id + K$ is included in the cone of symmetric positive definite matrices.

By the classical theory of strictly hyperbolic equations,[11] we can solve such equations with Cauchy data (u_0, u_1) in the space $\dot H^s(\mathbb{R}^d) \times \dot H^{s-1}(\mathbb{R}^d)$ for $s > \frac{d}{2} + 1$. Notwithstanding, it is important to think about the scale invariance of such equations. It can be checked immediately that if u is a solution of equation (E), then the function u_λ defined by $u_\lambda(t, x) = u(\lambda t, \lambda x)$ is also a solution of (E). A large number of works have been concerned in solving nonlinear wave equations by trying to decrease as far as possible the index of minimal regularity of the initial data towards a space of initial data invariant by the above change of scale, for instance in the space $\dot H^{\frac{d}{2}}$.

The goal here is to solve equation (E) for less regular Cauchy data than what is required by energy methods. This approach fits in Christodoulou-Klainerman programme for general relativity, which also includes works by Klainerman, Bourgain, Tao and their schools. To get closer to scale invariant spaces for the initial data, it is obvious that we need to use the specific properties of the wave equation, namely the dispersion effects referred to above. This necessitates the proof of Strichartz type inequalities for that equation that we can interpret as a wave equation with variable and rough coefficients. It is the alliance of geometric optics and harmonic analysis through the paradifferential calculus of Jean-Michel Bony that allows us to establish these estimates, to improve the minimal regularity index and to give an answer to a longstanding open question.

As stated above, Strichartz estimates are obtained from dispersive phenomena coupled with an abstract functional argument known as TT^*-argument, developed by Ginibre and Velo in [21], and generalised by Keel and Tao in [24]. As also pointed out previously, dispersive phenomena are obtained for the wave equations with constant coefficients by applying a stationary phase argument to an explicit representation of the solution. The variable coefficients case requires more attention, since in this case we do not have an explicit representation, and we recur to geometric optics methods involving Hamilton–Jacobi and transport equations to approximate the solution. When the coefficients are rough, as, for example, in the quasilinear case, such an approach does not work, since the Hamilton–Jacobi equation produces singularities. It is the Littlewood–Paley theory that allow us to overcome this difficulty.

In fact, to perform such a method in this framework requires a regularisation of the coefficients. More precisely, using Bony's paradifferential calculus, we are left with the study of the part of the solution related to frequencies of size 2^j, which satisfies a wave equation with regular coefficients. By a classical method, we construct a microlocal approximation of the solution to this equation, that is valid in a time interval whose size depends on the frequency and that allows us to establish a microlocal Strichartz estimate. In fact, it seems impossible to construct a local approximation of the solution since the associated Hamilton–Jacobi equation generates singularities at a time related to the frequency: this is due to the fact that these regular coefficients keep memory of the original regularity of the solution. The local Strichartz estimate is obtained (with some loss) by decomposing the interval $[0, T]$ into intervals where the microlocal Strichartz estimate is satisfied.

The applications of the Littlewood–Paley theory, and particularly of the paradifferential calculus, are manifold and we cannot enumerate all of them here. For a wider range of perspectives, whether in the study of functional inequalities or the analysis of solutions to nonlinear partial differential equations arising in fluid mechanics or general relativity, we refer the reader to the monograph [3].

The Littlewood–Paley theory has inspired the wavelet theory, which is at the origin of numerous progresses in various

11 See, for instance, Chapter 4 of [3].

applied disciplines, such as signal and image processing techniques. We can illustrate wavelet theory in a simple setting by considering Haar's system introduced at the beginning of the 1920s by Alfred Haar in his PhD thesis. This system is defined by the functions

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad j, k \in \mathbb{Z},$$

where the generating wavelet

$$\psi = \chi_{[0,\frac{1}{2}[} - \chi_{[\frac{1}{2},1]}$$

is the piecewise constant function equal to 1 in $[0, \frac{1}{2}[$ and -1 in $[1/2, 1[$. This system constitutes an orthonormal basis of $L^2(\mathbb{R})$ and, thus, it is straightforward that all functions f of $L^2(\mathbb{R})$ can be decomposed as follows:

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}\,, \tag{8}$$

where $\langle f, \psi_{j,k}\rangle$ denotes the scalar product of f and $\psi_{j,k}$ in $L^2(\mathbb{R})$. In the wavelet decomposition (8), the homogeneous dyadic blocs $\dot\Delta_j f$ are replaced by the projections

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}\,,$$

where the index k provides an additional level of discretisation.

The main drawback of Haar's system is its lack of regularity, since the mother wavelet ψ is not continuous. Other more regular wavelet bases were constructed later on, allowing us to get decompositions in wavelets similar to (8), often taking into consideration the scaling of the space in question.

As in the Littlewood–Paley decompositions, we can characterise the belonging of a function to almost all classical functional spaces by conditions pertaining only to the absolute values of the coefficients of the function in a basis of unconditional normalised wavelets.[12]

For example, in the Besov space $\dot B^s_{p,p}(\mathbb{R}^d)$, $1 \le p < \infty$ and $s < \frac{d}{p}$, the wavelet decomposition of a function takes the form:

$$f = \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda, \tag{9}$$

where $\lambda = (j, k)$ includes the scale index $j = j(\lambda)$ and the space index $k = k(\lambda)$, and

$$\psi_\lambda = \psi_{j,k} = 2^{jr}\psi(2^j \cdot - k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^d,$$

where ψ is the mother wavelet, and $r = \frac{d}{p} - s\cdot$. The wavelet theory allows to characterise the belonging to $\dot B^s_{p,p}(\mathbb{R}^d)$ in terms of the coefficients in the above wavelet decomposition as follows:

$$\|f\|_{\dot B^s_{p,p}(\mathbb{R}^d)} \sim \|(d_\lambda)_{\lambda\in\nabla}\|_{\ell^p}\,. \tag{10}$$

The possibility of characterising the regularity of a function by the size of its wavelet coefficients is at the heart of the extensive applications of wavelet theory. In particular, we can translate the equivalence (10) by the decrease of the wavelet coefficients, with the exception of a small number of them. This property of concentration of information in a small number of coefficients, often called parsimony or sparsity, plays a crucial role in image processing. In this type of essentially nonlinear process, it is clear that the set of remaining coefficients depends on the function we are approaching. A general theory for the study of these phenomena, known as nonlinear approximation theory, was started by Ronald DeVore in the 1980s.

A first result in nonlinear approximation theory is the representation of a function by its N most significant coefficients. More precisely, given an element f of $\dot B^s_{p,p}(\mathbb{R}^d)$ admitting a decomposition given by (9) in the wavelet basis $(\psi_\lambda)_{\lambda\in\nabla}$, the goal is to keep only the nonlinear projection $Q_N f$ defined by

$$Q_N f = \sum_{\lambda\in E_N} d_\lambda \psi_\lambda,$$

where $E_N = E_N(f)$ is the subset of ∇ with cardinal N, which corresponds to the N largest wavelet coefficients $|d_\lambda|$.

Among the many applications of the nonlinear projection $Q_N f$, we can refer to the following estimate:

$$\sup_{\|f\|_{\dot B^s_{p,p}(\mathbb{R}^d)}\le 1} \|f - Q_N f\|_{\dot B^t_{q,q}(\mathbb{R}^d)} \le C N^{-\frac{s-t}{d}}\,, \tag{11}$$

that has played a key role in [6], in the study of the lack of compactness of the critical Sobolev embedding

$$\dot B^s_{p,p}(\mathbb{R}^d) \hookrightarrow \dot B^t_{q,q}(\mathbb{R}^d)\,,$$

with $0 < \frac{1}{p} - \frac{1}{q} = \frac{s-t}{d}\cdot$.

In fact, given a function f of $\dot B^s_{p,p}(\mathbb{R}^d)$ we obtain, from (10) and using $(d_m)_{m>0}$, the decreasing rearrangment of $|d_\lambda|$

$$\begin{aligned}
\|f - Q_N f\|_{\dot B^t_{q,q}(\mathbb{R}^d)} &\sim \Big(\sum_{\lambda\notin E_N} |d_\lambda|^q\Big)^{1/q} = \Big(\sum_{m>N} |d_m|^q\Big)^{1/q}\\
&\le |d_N|^{1-p/q}\Big(\sum_{m>N} |d_m|^p\Big)^{1/q}\\
&\le \Big(N^{-1}\sum_{m=1}^{N} |d_m|^p\Big)^{1/p-1/q}\Big(\sum_{m>N} |d_m|^p\Big)^{1/q}\\
&\le N^{-(1/p-1/q)}\Big(\sum_{m>0} |d_m|^p\Big)^{1/p}\\
&\le N^{-\frac{s-t}{d}}\|(d_\lambda)_{\lambda\in\nabla}\|_{\ell^p} \sim N^{-\frac{s-t}{d}}\|f\|_{\dot B^s_{p,p}(\mathbb{R}^d)}.
\end{aligned}$$

The success of wavelet theory either in signal and image processing, or in the field of numerical simulations of partial differential equations is now well established. For a general survey of applications of this theory, one can consult the monograph [31] and the references therein.

The Littlewood–Paley theory is considered the simplest tool of microlocal analysis. We can see microlocal analysis as the study of functions by the decomposition of the phase space, that is the space of (x, ξ). In a general way, this process consists of localising in physical space x then in the Fourier variable ξ, which corresponds to the localization in a ball for a metric of $T^\star\mathbb{R}^d$ (the cotangent space of $\mathbb{R}^d$): it is the Weyl–Hörmander calculus.[13] The interest in this type of process, introduced in the 1970s, is to allow for the analysis of fine properties of functions defined in the physical space by operating in the phase space, where the number of variables has doubled. This turned out to be particularly useful in the study of

12 For more details consult [17, 32, 33].

13 See, for example, [13, 14, 22].

nonlinear partial differential equations, namely, for instance, to take into consideration certain geometric specificities.

The whole issue of the Weyl–Hörmander calculus consists in the use of reasonable metrics (the so-called Hörmander metrics) in order to localise in phase space. As an example, the procedure of localising in the variable x in an Euclidean ball with size α, and afterwards in the Fourier variable in a ball of radius $\alpha(1+|\xi_0|^2)^{\frac{1}{2}}$ is equivalent to localize in a ball for the following metric, the so called $(1,0)$ metric:

$$g_{(x,\xi)}(dx^2, d\xi^2) = dx^2 + \frac{d\xi^2}{1+|\xi|^2}\,.$$

The so-called Weyl–Hörmander calculus, which achieved its present day formalism at the end of the 1970s in the works of L. Hörmander, generalises this metric. In fact, it consists of the description of reasonable ways to decompose the phase space. These decompositions are chosen according to the nature and the geometry of the problem under consideration. The admissible decompositions are those whose construction is based on Hörmander's metrics, which are functions g of $T^\star\mathbb{R}^d$ with its standard sympletic structure in the set of positive definite quadratic forms in $T^\star\mathbb{R}^d$ satisfying:

– a so-called slowness assumption stating that the metric does not change much on its own balls, and this in a uniform way;
– an uncertainty principle hypothesis that prevents too much localisation. In particular, the uncertainty principle imposes that the volume of a g_X ball of radius 1 is larger than or equal to the volume of the Euclidean ball of radius 1;
– and finally, a so-called temperance hypothesis that reflects the fact that we can estimate the ratio of metrics in arbitrary points by the dual metric.

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This is an English translation by Fernando Pestana da Costa of the French article *La théorie de Littlewood–Paley: fil conducteur de nombreux travaux en analyse non linéaire* by Hajer Bahouri, published in *La Gazette des Mathématiciens*, (No. 154, pp. 28–39, Octobre 2017). The EMS Newsletter thanks the author, *La Gazette des Mathématiciens*, and the *Société Mathématique de France* for authorisation to republish this article.

Interview with Fields Medalist Peter Scholze

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

UP: Let me start by asking you my standard, if somewhat silly, question I put to all new Fields medalists: were you surprised to get the medal?
PS: In view of all of the rumours I had been hearing for years beforehand that I was due to get it, it was in some sense not a surprise. But I actually felt pressurized by the rumours, so when I was finally informed that I was getting the medal, I also felt some relief.

So now you do not feel under any pressure to live up to the expectations that being a Fields medalist incurs?
I hope I can handle that.

So how was the experience of being the centre of attention at Rio, in particular at the prize ceremony? By the way, was this the first ICM you have attended?
No, I attended the one in Seoul as well. We had been warned about the media attention, and the number of e-mails we would receive.

Was it that bad? I thought that the media attention was much stronger in South Korea than in Brazil.
At the ceremony Figalli showed me that he had already got 200 e-mails on his phone.

That is nothing. In Seoul the medalists got into the thousands as I recall, so they were very hard to get in touch with. Figalli got in touch with me very promptly.
He is very efficient.

I certainly got that impression. By the way, did you know of the medalists already? When I was a young man, the names of the medalists were already familiar to me, but starting at the end of the last century most of them were unknown to me. I had heard of you before, but the other three names I had not heard of. Does it reflect that mathematics has become a much larger field and people are too caught up in their specialties to have an overview? Or is it just me being old and losing touch?
Of course I knew of Venkatesh, after all he is a number theorist, and Figalli I had also met before, but I must admit that I was not familiar with the work of Birkar. Yes, I think mathematics is becoming a larger field.

You left Rio early, was that because you wanted to get away from the attention?
I had already attended a very nice meeting in Rio before the actual ICM, during which I got a good opportunity to see Rio itself, and at some point I wanted to return home. I am sorry you missed me.

I am making up for it now. So, after this prelude let us be more systematic and start from the beginning. Your background. You grew up in Berlin?
That is true.

What do your parents do?
My father is a physicist and my mother started out in computer science.

So how and when did you discover mathematics? Did you have good teachers who were able to convey to you that there was such a thing as real mathematics, which is far more exciting than the standard mathematics offered at school?
I had good teachers but I was also very much drawn to mathematics.

Could you elaborate?
When I was around fifteen or sixteen, I realised that Fermat's last theorem had been proven, and I tried to figure out what the proof was about -- elliptic curves, modular forms, etc. I did not understand anything -- I did not know what a matrix was, really -- but it was extremely fascinating.

But how? Most students would never come across this, was it pointed out to you?
I don't exactly recall, but it certainly played a role that I had good teachers, and that I met many like-minded students at the mathematical olympiads.

I think this is natural for young burgeoning mathematicians, after all we are familiar with numbers from an early age, and may have played with them. But you were not discouraged by your lack of understanding?
No, on the contrary. Because it all was very exciting and intrigued me a lot and I was dying to learn what it all really meant...

…I can understand that. I had a similar experience, although at a much more elementary level, when I encountered Hardy's text book on Pure Mathematics. All those formulas with mystical symbols seemed like magic to me, and I thought that the way to happiness was through understanding and being familiar with them. Sorry, go on…
…So I actually started a programme of systematically unravelling the mysteries.

So one could say that you jumped into the middle of everything?
I guess so.

But other parts of mathematics, calculus for example. Did you not have that at your Gymnasium?
Not yet. At the time I got involved, I was too young for that. So calculus, complex analysis, even algebra and all that I picked up on the side as part of the process of educating myself.

When did you take your Abitur?
At nineteen.

That is the normal age. So you were not a prodigy in any formal sense. But you had participated in maths olympiads, I understand. When did you start?
I started in Year 7 (equivalent to seventh grade) in some regional olympiads. Later on I was able to reach the international olympiads.

You did very well, did you not, repeatedly winning gold medals?
That is true. I did not think that I was going to be able to do so, I thought that those who had won them before were much better than me, but to my surprise I was able to do as well.

So what is your opinion on those spectacles, are they a good thing or not?
I am not quite as negative as Rapoport is, but I can see the danger that people are diverted by obtuse and contrived combinatorial problems.

But did it give you a lot of self-confidence?
I guess so.

People who are good at olympiad problems usually are technically strong, or at least have the potential to become so, and as a research mathematician, especially if you want to be reasonably self-sufficient, such skills are invaluable, although not sufficient.
Yes. Mathematics is more than solving problems (let alone within time limits), and what excites me is getting a global understanding.

You mean, as actual designation implies, olympiads reduce mathematics to a form of athletics? What is your feeling towards competition in mathematics? As you have been very successful one may think that you would thrive on it.
First, as to the math olympiads, the major thing I got out of them was actually the social aspect. It was great meeting people all over the world with whom you could share interests...

…that was certainly my experience as well. Sorry, go on…
And as to the competitive aspect, as I told you already, what I get out of mathematics is the thrill of understanding, being the best is not necessarily important to me at all. On the other hand, I think that a certain element of competition is actually good. It pushes us to do our best.

To keep us on our toes, you mean?
Yes, if you prefer.

In chess they say the whole point of it is to see who is the best. Chess without competition would be meaningless. Hence, in chess you can rank people fairly accurately through an algorithm we need not be concerned with. If there were a similar system in mathematics, hiring would be greatly facilitated, but in mathematics there is no meaningful way you can rank people linearly; mathematics is too large and multifarious for that. However, in recent years there is much emphasis on various indices: citation indices, h-indices and whatever. The bureaucrats love it because it is of course objective and

can be computed with no regard whatsoever to the contents. All you need is to draw a graph; the nodes themselves mean nothing. It makes a travesty of mathematical competition.
I see what you are driving at. I know some very good people who, by the time they had established themselves as a leading mathematician, still had an h-index of maybe two.

In the past, people were judged by their work as such, now I fear there are just too many people, so it is no longer feasible. Is there too much mathematics being published? After all, most papers written are not published for their intrinsic interest, but because people need to get ahead and improve their lists of publications and citations. Although, most papers are lucky if they are at least read by the referees.
I have no opinion on that matter. In my own field I do not think this is a very serious problem.

Now let us get back. There you were at nineteen, quite advanced in mathematics, what did you do?
At the time I was wondering whether I should study in the US, but somehow I wanted to stay in Germany, yet I wanted to leave my home town Berlin. I was in contact with Altmann, who was an algebraic geometer and he suggested Bonn and mentioned Rapoport.

Rapoport was of course a bit sceptical about you, your performance at the olympiads did not cut any ice with him, and suspecting that you might be a mere charlatan he subjected you to a third-degree interrogation, from what I gather. How did you experience being on the receiving end, so to speak? Did it take long for you to convince him that you were not a charlatan but very much the real thing?
I have no particular memory of being subjected to very harsh treatment. But Rapoport is a very serious man you know.

I know.
Do you have any more questions?

I am loath to keep you, but let me just put a few more. I understand that you became enrolled as a graduate student right away.
No. Rapoport insisted that I go through the regular system, taking undergraduate maths courses. But he did give me a few problems to think about.

He was no doubt concerned about your general mathematical culture. Did you resent it?
No! The only thing that I didn't enjoy much at the time were the experiments in physics.

At what age did you finish your thesis?
At twenty-four.

That is not exceptional, but I suspect that you did some exceptional things before that.
I had found a simple way to compute the L-function of the modular curve, at bad places. A little later I realised that one could use these techniques to simplify a key step in the proof of the local Langlands correspondence for GL_n, for which I got a Clay scholarship.

And made your reputation outside Bonn, I surmise.
I suppose so.

Do you co-operate with people now, or is this really impossible to do so profitably with someone not at your level? It is hard to think of Gauss working with someone else in mathematics (with physics it was different as we know).
Of course I do. I like to share my ideas with colleagues.

Mathematics is very much a social thing.
It certainly is.

So, let me formulate a moral question for you. If you discovered that A implied the Riemann hypothesis ,say, would you keep quiet and wait until someone proved A and then step in because it is the one who puts in the last brick in an edifice who usually gets the major credit?
This sounds like some contrived speculative question that does not interest me.

Point well taken. What about your collaboration with your advisor Rapoport. Did it mainly concern suggestions on his part, or where you engaged in technicalities?
It was mostly one of general suggestions.

So you are self-sufficient in this respect. By the way, do you read a lot? And if so, do you read systematically, from cover to cover, or do you skim looking for the meat?
I read a lot. Some books I do read from cover to cover, especially when I am trying to learn the basics in a new field, but otherwise I often skim an article to find the information I care about.

You seem very focused on mathematics, do you have any other interests? When you were first pointed out to me proudly by Rapoport in a Bonn Cafeteria some years ago you were in a cast. Are you a skiier?
Oh no, this was just a stupid accident and had nothing to do with any athletic activity, and besides if I ski I do cross-country. I pretty much do all my thinking in mathematics.

Let us change tack. What you do in mathematics is mainly a matter of taste, and your taste was formed early on. Are there some parts of mathematics which you find distasteful?
[long silence]. Well, there are certainly parts of mathematics, and maybe also styles of proofs, that I like more than others.

Because even if logically impeccable it admits no global understanding but is simply thrown at you?

Proofs should be based on an idea, and the methods used compatible with the idea somehow.

It reminds me of Grothendieck who claimed that proofs should be natural and part of an overarching structure. He hated tricks and ad hoc intrusions. Proofs should be instructive and explaining, not just formal verifications.
That goes without saying.

You mentioned that your mother started out in computer science. Do you program? This is something which nowadays would come naturally to mathematically inclined children, but was not available when I was a child. And if so how would you compare programming to mathematics?
Before I encountered mathematics at fifteen I programmed. I even designed computer games. But it was all pretty childish and when I became caught up in mathematics I stopped doing it and have never taken it up again. I cannot make any comparisons, because my programming experience is that of a child, and that of mathematics of a mature professional adult.

I would say that programming is relaxation, akin to crossword puzzles, although I personally have no interest in the latter. You always know that you are going to succeed eventually, and you never get stuck in the same way as in mathematics, as you have a continual interaction with the computer and can engage in trial and error, in a way which is not available to you in mathematics. But you also discover how many mistakes you make, and would the same not be true of maths papers? Most are fixable of course but there may be serious mistakes which one will never discover, as the results are not interesting and no one really cares about them (maybe not even the authors!). But of course if someone claims to have proved an important hypothesis, the proof is subjected to relentless scrutiny and more often than not serious gaps are discovered: in some cases even leading to a total collapse of the approach. This also being the case with highly regarded mathematicians to boot.
Well, I guess that is somewhat true, but I am still very confident in the mathematical literature, certainly in the parts I know well. By the way, how much longer do you need? I do not want to miss the next lecture.

Do not worry, you will be sure to catch up quickly. And an interview like this is not based on a prepared list of questions to be ticked off; like all inquiries it is open-ended in the sense that every response suggests a new previously unthought of question.
That may be very well, but at this stage I prefer closed to open.

In that case I do not want to keep you any longer, thank you very much for your time.

Ulf Persson is on the Editorial Board of the EMS Newsletter. His photo and CV can be found in previous Newsletter issues.

A Discussion with Fields Medalist Artur Avila

Michael Th. Rassias (University of Zürich, Switzerland)

Artur Avila, born in 1979 in Rio de Janeiro, Brazil, is a world-renowned mathematician with outstanding contributions in a broad spectrum of topics, with his main work focusing on dynamical systems. His problem-solving mastery, technical power and ingenuity have led him to the proof of famous and long-standing open problems like the "Ten Martini Problem" and the "Zorich–Kontsevich conjecture", to name just a few. He was awarded the Fields Medal in 2014, being the first Latin American to win this award. He recently joined the University of Zürich, Switzerland, after spending years as a researcher in the CNRS (France), and remains associated with IMPA (Brazil) where he got his PhD in 2001.

Honoured to be working under the same *academic roof* in Zürich, I had the great privilege of meeting Artur a few months ago and spending some time with him discussing various topics. With this opportunity the following *discussion* arose, in which he talks about the path of his mathematical trajectory, as well as his interesting views on certain issues.

I know you participated in an IMO and won a gold medal. From which age did you get interested in mathematics?

My particular interest for mathematics started pretty early on, from the age of 4 or 5. I tried to learn beyond what was taught at school and I would ask for books in order to read by myself. Since my parents were not academics and would not know which books would be good to buy for me, there was some kind of a random process of buying several books, some good and some not, which were my original source of self-study. I was also very interested early on in the concept of science, like Physics, chemistry, etc., even in the primitive way I could understand it as a child. But at the time I certainly had no understanding that being a mathematician could be a career choice. It somehow seemed a bit more detached from the other sciences. By the nature of the field, what you are usually initially presented with is very old stuff. You don't see modern discoveries, whereas in physics for example, as a kid you hear about relativity and things that you can associate with recent discoveries and thus feel that the field is active. On the other hand, you might often get the impression that maths has been done by people in the past and there is nothing new to do. So the concept of becoming a mathematician did not initially exist.

Do you think that this might be related to how mathematics is taught in Brazil or is it more general?

I think that this is a more general difficulty that mathematics has with respect to other sciences. This happens because in mathematics we always build upon foundations. For example, what Euclid did is still valid and that is why we study it until today, whereas we don't try to learn physics, say, according to Aristotle, as that was completely wrong. In physics we are being taught more modern things. Maybe we start with Galileo, but we don't go back to Aristotle's time, while in mathematics we do start with Euclid. It is the nature of the subject to a certain extent. We go further back. So I don't think that it is mainly a matter of geography.

So how did you get more actively involved with mathematics?

At the age of 13, I heard about the existence of a national mathematical olympiad in Brazil without knowing exactly what that was. At the time I was reading calculus books and related things, but I was not familiar with this type of problem-solving. I got in touch with it and I thought that it was interesting. My first encounters were difficult but then I got really involved for a few years. That was my new impression of what mathematics could be. In Brazil olympiads are organised by IMPA to a large extent. Being from Rio De Janeiro, my fist contact with IMPA occured when I went there to collect a medal, as IMPA is also in Rio. So that is how I found out about its existence and the fact that research was done there. Also, IMPA has a very flexible system which even allows for high school students to study there and obtain their master's. So it can happen that a student starts a master's at the age of 16 and

Etienne Ghys (left) with Artur Avila (right) at the 2014 ICM in Seoul. (Photo by Joachim Heinze, Archives of the Mathematisches Forschungsinstitut Oberwolfach)

finishes it at 18 and then obtains his PhD at the age of 21. There are examples like that.

You are one such exceptional example. You started your PhD studies at the very young age of 19 at IMPA, correct?
So yes, when I found out about the existence of IMPA I thought it would be nice to get involved. In my mind, I imagined that it could probably help me with my olympiads training. Once I got a gold medal at the IMO I got an invitation from IMPA to take part in a pre-master's programme: a sort of scientific initiation program. It was a special course called "Introduction to Topology". I got a fellowship, I did well and then started the master's programme while in the last year of high school. Subsequently I commenced my PhD there as well.

Do you believe that the strategic way of thinking you had cultivated as an olympiad caliber problem-solver helped you later when approaching problems on a research level?
The olympiads gave me confidence to start with. So when I started the master's programme I felt that it was something I could really do and I worked intensively for it. But in research you learn how to think in a different manner. Of course tricks and little techniques inherited by the olympiads may be helpful not to waste time occasionally as you may recognise easily that something might fail to work, but in research you have to think about a problem whose solution is unknown. In competitions, the problems are designed to be solved. There is of course creativity involved, but the problems are such that they can be solved by the use of a specific toolbox. In research you deal with uncertainty. You also have to be able to recognise whether you should continue on a certain path or maybe start researching a little bit to the side, and know when to come back to the original problem again.

What is your advice to young talented students in Brazil and elsewhere?
I feel that the students have to take so many courses at the university that they have no time to engage at a deeper level with any of them. With so many required classes per week, they can only realistically strive for superficial understanding. It would be better to focus more on something more specific and understand it well. But of course, the question is where to focus. I advise students to try to identify, as soon as they can, what attracts them most and then put more energy into that, since you have to concentrate on something in order to understand it and you cannot achieve that by sharing time equally between several subjects. Unfortunately, the system is not designed to allow you to do this. Of course, it is easier to observe where things go wrong than to suggest how they could be done better. I would suggest to students to identify a professor and try to get into contact with him/her in order to get guidance.

You recently joined the University of Zürich (UZH), while prior to that you were in Paris. How do you find the new environment here?
The environment and the mathematical centre in Zürich is very active. There is a very good synergy between the UZH and ETH. An increasing number of good researchers stay here as well as just pass by, and thus the seminars are very active. Even though Paris, for example, has many more people, one can say that to some extent after a certain size of community you cannot really absorb everything that is going on. It is impossible to attend all of the seminars, meet with all of the people, etc. The nice thing about Zurich is that the size is such that you can absorb what happens and there is a good number of things happening. At the UZH we increased the dynamics group as well. Apart from me, two other researchers from Bristol have joined. We also established an "Ergodic theory and dynamical systems" seminar that is already becoming very active. We hope that it will be one of the central seminars in dynamical systems in Europe.

Was there a specific paper, book, lecture, or even a theorem you came across that had a lasting impact on you to the extent that it made you chose to become a mathematician?
In my case it was a long experience and a very gradual process. There were many things that grabbed my interest and several theorems that I liked while growing up. Additionally, my mental understanding of what mathematics is progressed over the years and I realise now that I was fortunate, since there were several stages in my development where the process might have not worked out so well. In general, a student might like olympiads and really enjoy problem-solving but could not be so attracted by the theory learning that happens at, say, in master's program. Or they may like learning all those wonderful theorems, but might not find it pleasurable to try and prove new theorems. Eventually ,what leads you to becoming a researcher is a path that comes from many different directions. But there are several little things in the process that might not work out for someone and lead them elsewhere.

So in my case there was no particular theorem or open problem that lead me to mathematics, such as Fermat's last theorem, Poincaré's conjecture, etc. It was a long and gradual process.

What is the first thing that comes to mind when you are thinking of the word "mathematics"?
Generally, my way of thinking is that of an analyst. So, when I think of "mathematics" the first things that come to mind are "analysis" and "inequalities". Inequalities are really central to my way of thinking.

Is there a mathematician who influenced you the most? Either through your mutual collaboration or interaction or even by studying his work?
There were several people who were important and influential. In research you have your own style, but it is very good to be exposed to several other views and possibilities. There is also a lot of advice you can get. You can of course choose to adopt some advice and some not. My PhD advisor Welington de Melo was influential to me. I was also influenced by Mikhail Lyubich, with whom I worked a lot. Later there was also Jean-Christophe Yoccoz, whose style was very different. These mathematicians had their own particular styles and I was exposed to that as I interacted with them directly. Those were close to me, but of course indirectly I was influenced by more distant people. For example, when I was shifting a bit between fields, I found Jean Bourgain's work to be an inspiration. His style and abilities sparked some admiration for the corresponding fields. His style of doing analysis is something I like very much and I have the highest respect for him.

You started your PhD studies at the very young age of 19 at IMPA. Your doctoral advisor there was Welington de Melo who was working on dynamical systems and particularly on one-dimensional dynamics, which is a very demanding field. Did he allure you somehow towards that dynamical systems or had you already chosen your path?
When I initially went to IMPA I had a very weak background and I was just finding out about the possibility of doing research. I wanted to learn the basics at the time and I was concentrated on trying to do well at my Master's programme. During that period I attended a PhD course in differential topology by de Melo. I was not very keen on attending that course at first, as de Melo had the reputation of being very tough with students and it was considered difficult to get a good mark in his courses. But it went well and he liked me. After the completion of the course he approached me and he proposed that we discuss some topics in complex analysis, quasiconformal mappings, etc. Through that process he naturally became my advisor. At IMPA, dynamics was a top field and they were strong at it. De Melo was a bit isolated, though. He was working on one-dimensional dynamics: a field that had already become very important but was and remains very hard. So one had to put all ones energy into this specific field and it made it hard to interact with others working on different things. The techniques and main problems are very difficult. An example of the complications involved is the following: we often have to work with fractal sets with very complicated combinatorics, and just to get the correct language to describe their structure is very hard. In analysis in some cases you might

Artur Avila (left) with Ashkan Nikeghbali (right) at the University of Zürich.

just take a dyadic division, break everything into squares and so on. But here you will need to do this in fractal sets, and just the description of the pieces is complicated. You have to describe the set and the topology is very intricate. For this reason, until today, whenever I have to supervise PhD students I tell them that if you do one-dimensional dynamics you will have to put all your effort and energy into that particular field, which is very specialised. Otherwise, we can cover almost everything else in dynamics. So I often propose to them to follow the second option, as it might not be a very good idea to get too specialised so early on. Of course, when I started I did not know any of this, and I just said OK and worked on one-dimensional dynamics. It turned out to be very interesting and very hard as well. Later I learned more things and expanded in other directions too. I still work on one-dimensional dynamics, but now most of my work is on other things.

One may state that mathematics has witnessed great expansion during the last, say, one hundred years, with many different areas emerging and various methods discovered, bridging seemingly different fields. How do you see the future of mathematics in that respect? Do you think that interdisciplinarity might be the theme of the future for example?
Things tend to become more and more technical and thus people get further apart as they don't speak the same language. It has become clear that for an analyst it is more difficult to understand what they are now doing in algebra. My view on this is that people should just do what they want. If someone wants to think about prime numbers he should do that, if he wants to think about fractals then that is fine as well. Some researchers will make progress working on isolated topics and others might want to bridge together areas and examine the connections between discoveries. Not everyone has to do that. Some people might just want to play on their own. You cannot anticipate what is going to work out. It is also not necessary to decide which fields of mathematics have to be stimulated, since we don't know whether the next discovery in geometry will depend on a discovery in algebra and so on.

This is also my point of view regarding focusing on pure or applied mathematics. What might be crucial in the solution of some very specific problem (with industrial applications or whatever) often happens to be just some beautiful object that someone studied for its own sake. The motivation of the latter (often similar to the motivation a child has for playing a game) is irrelevant to the fact that what has been discovered turns out to be useful to the former. And of course it works both ways: much of my work deals with theories that first arose in connection to physics, but for me what matters is that these theories are rich mathematically, and I study them from this perspective. So "pure" mathematicians have to recognise that many of the nice objects they have at their disposal with rich mathematical theorems that one works on just because they are beautiful, without caring about the motivation or the application, might have not been discovered if it was not for some physical model that someone was examining because they cared about the physics of it. Thus, by admitting this and letting people work on what they want, the whole community benefits from the eventual (unpredictable) interactions.

It has happened many times in the past that great mathematicians moved from area to area, making contributions to various different mathematical domains. Stephen Smale is one such example, part of whose work is on dynamical systems like yours, but who has also worked in topology and mathematical economics with the latter being fairly distant from the other two fields. Do you see yourself exploring completely new research terrains in the future?

With Smale there was a natural evolution from topology to dynamical systems, since part of his work towards Poincare's conjecture was already leading him to understand certain types of dynamical systems. Sometimes of course there is an explicit desire for one to do something different. So I believe that Smale actually forced himself to turn to another direction. In my case, things happened to evolve in directions that I had not anticipated. It has happened that I have turned my attention to something which I found attractive and later it turned out to be related to things I had been doing. The connection proved to be deeper than I would initially have expected. I assume that this will tend to continue to happen. Personally, I do not put in a great effort to work on different areas, except if I find something attractive and there is a natural progression that leads me there.

On a humorous note, if – like in the case of Ramanujan – there were a supreme being that could hand you solutions to important problems in your sleep, and say that you could have only one such dream, which problem would you like to see the solution for?

Mm… The first thing that comes to mind is the local connectivity of the Mandelbrot set. I don't know whether I would change my mind if I put more thought to into it, but this is the first problem that I thought of when you asked me this question.

I don't actively work on it. I could possibly work on it in the future if the corresponding techniques have evolved, but it is certainly a beautiful problem that is captivating.

Many scientists from a broad spectrum of areas have expressed various views and opinions about the possible future consequences of the advancement of artificial intelligence (AI) in connection to the so-called "AI-control problem". Being a young scientist, during whose life it is very likely that great advances in AI might be achieved, affecting our lives, do you have any views on this?

Being a dynamicist, my training has taught me to a large extent that there are a lot of limitations when we make predictions. To my understanding, when it comes to AI peoples' hopes might be too high at the moment, as one tends to get too optimistic following a few unexpected successes. It is of course correct to continue the effort, as probably there is more success to come, but it is reasonable to imagine that obstacles will be identified so that things will not advance as fast as it was hoped. Even though it has happened many times in science that several things that we didn't think could be done in a specific timeframe were actually achieved and we were very surprised by the outcome, we should not get too carried away. So, I am not particularly scared about the consequences of the advancement of AI, since expecting big advancements in AI is already large assumption. Expecting that AI will become something applicable to almost all situations involves too much wishful thinking, even in the long run.

When we learn more about a corresponding domain we are also able to better understand its limitations. The point is to try to discover directions where the techniques allow you to go. It is very important and at the same time difficult to know what you cannot achieve. A lot of good quality research is conducted to see exactly what is impossible to be done, in order to know what the barriers of the techniques are. Barriers that will not be overcome, since they are theoretical ones. That is, it is important to know that it is not a matter of putting more effort towards a goal if that goal is really intractable. This is what happened in mathematics when we learned that there are unsolvable problems and all kind of limitations to formalism. Our understanding changed completely from Hilbert to Gödel, could this have been predicted? In technology, while there was very fast progress with the clock speed of chips, we could anticipate that this needed to stop at some level as there were known physical limitations. Maybe, with respect to AI, there are still unknown theoretical limitations hiding in information theory or even dynamical systems.

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ICM 2018 in Rio – A Personal Account Part II

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

This article is the second part of the personal account of the author who attended the ICM2018 in Rio. Since Part I of the report triggered a number of reactions I suggest that readers consult the section Letters to Editor (p. 59) for the comments by M. Viana, Chairman of the IMC2018 organising committee, V. Mehrmann, President of EMS, and a reply by U. Persson.

The EMS Newsletter is open for other remarks focused on the value for our community of the IMU congresses of mathematicians.

Note that the views expressed in the Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

The Editor-in-Chief

Daily life at the congress

The days at a congress tend to be very similar to each other, as they are from one Congress to another. There are of course the talks, which serve the same purpose as food at a social gathering. It is not food which is the main thing, but getting together; however, without the former, the latter would be a bit harder. Mathematical talks are either of the show variety or seriously meant to convey both conceptual ideas and technical information, although one should of course be wary of drawing too precise a line between them: all talks have at least a little of both. A serious talk has a very narrow audience and is typically given at specialised conferences. As noted, one complaint about an ICM I have often heard over the years is that it does not have much to offer the professional mathematicians, who instead give priority to their specialised meetings in their own fields. Not much can be done about that obviously, as an ICM has a different ambition, namely to create a feeling of community including all mathematicians, thus the show talks are the order of the day, trying to address all mathematicians rather than just experts. There are of course the prestigious plenary talks taking place in the main hall, then there are invited speakers who will have to make do with smaller halls, while the "self-invited" are reduced to short communications or poster sessions. This is typical for meetings at big science.

The plenary talks get most attention and are reviewed by mathematicians as if they were movies. As movies most of them fall short of expectations. There are real showmen among mathematicians who love to perform and never give a boring talk. Atiyah is the outstanding example. At 89 he is still going strong[1] and he carried off by virtue of charm and authority a talk you and I would have been booed off of the stage for had we attempted it. But most mathematicians have no such talents, nor ambitions for that matter. Nevertheless, in my opinion there were some very good talks, and, unlike in many other disciplines, mathematicians tend to be intellectually very honest and there is seldom a talk without at least some palpable content. However, the organisers had decided not to print out any programmes, instead providing apps. At least for older people like me, this was a definite disadvantage, and made me miss many talks.

Another source of entertainment is to browse among the various bookstalls, but with the advent of laptops and smartphones there are no longer computer rooms set up, which I recall from Beijing and Madrid. Nevertheless, the organisers had thoughtfully decided to provide a few laptops on a couple of tables in the hall of the publishers. I actually found that very helpful, especially as they were not in high demand so you could easily access them.

Then, more excitingly, there is a life of parties and receptions, to which I fear most participants are not privy. I myself only got a glimpse of it through my invitations by the Norwegians, who have made a mathematical claim with the Abel Prize, arranging an Abel lecture at each Congress. This time I was also invited to the reception of the London Mathematical Society. Both of these receptions took place at the same location (but of course at different times), as well as many other similar events, I surmise; namely at the so-called i-bar room on the top floor of the Mercure hotel, just next to the venue, and where all of the VIPs had been offered accommodation. The room provided splendid views of the Atlantic Ocean and one could also imagine the presence of the big city just behind the horizon.[2] I suspect I only went to the lower circles of 'Paradiso'.

Meeting Marcelo Viana, the Chairman of the Organising Committee

I have been speculating above[3] about the congress, especially the fire. How much did it really affect it? Maybe the emptiness of the customary bags you always receive at the registration could be due to the fire, their contents having been devoured by it. Incidentally, I wonder how far this tradition of giving out bags goes back? Did Hilbert carry one with the logo of the ICM Paris in 1900? More generally, could all of the perceived shortcomings of the present Congress be due to the fire? Presenting

[1] As of the time of writing. Sadly we all know that he passed away on January 11 this year.

[2] How far away? Could it have been on the tenth floor, say 30 meters above ground, a quick estimate gives 20 km, somewhat short of the distance to the centre.

[3] See Part I of my personal account (Newsletter 111).

myself as an editor of the EMS Newsletter (which is true) with an assignment to cover the Congress (which may be stretching truth a little, but can be retroactively confirmed with this article), I did get prompt access to the Chairman of the Organising Committee by the name of Marcelo Viana. He happily received me with a very warm smile, commiserating as I limped in, having run to the appointment, fearful of keeping him waiting, and as a consequence stretching a muscle. No need to have hurried, he assured me, being in a good mood, which he later revealed, on the promise of temporary secrecy, was due to the fact that the replacement for the stolen medal was underway.

'Don't blame the fire, blame me!' he disarmingly exclaimed when I brought up the case of the fire and its possibly devastating effects on the congress. In fact, the effects of the fire had been greatly exaggerated, I was told, it was actually not much of an interruption. It started on the evening of July the 29th (therefore just before my arrival the following morning) but was quickly put out. It was due to a hot air balloon touching the building in which the congress kept some equipment. The fire never went beyond licking the outside of the same; everything inside was unscathed. However, flying a balloon requires special permission in Brazil and is otherwise illegal, so a police investigation was necessary. Consequently, the building was cordoned off and became off limits during the investigation. Inside they had stored their video equipment to be used for the ceremony and the plenary talks, and they were forced to rent new gadgets and set it up: a process which had initially taken a week and now had to be performed in 48 hours. It certainly was hard work, but it worked out in the end (indeed, so perfectly that no one noticed, I reflected, but that is typical of competent management, great things being done silently with no visible effects). And otherwise the running of the congress was not effected in any way. The bags were empty not because their contents had perished in some putative fire, but because they had never existed. (If paperless congresses will be the case in the future, maybe the tradition to hand them out will be discontinued?)

A decision had been taken to dispense with paper for ecological reasons. So much paper was simply being discarded by the participants, who now more and more have come to rely on digital means. Maybe a wrong decision, he admitted modestly, the next congress may decide to reintroduce paper in the form of programmes, daily newsletters and such things, but it was ecologically a sound one, he maintained. Maybe, but paper, which by the way is biodegradable, makes up for but a tiny fraction of the ecological footprint caused by a congress; one need only think about what it takes to transport the bulk of the participants halfway across the globe. Instead, they had provided apps to be downloaded, and if your phones did not operate in Brazil, or at a prohibitive cost, there would of course always be the home page (it is notoriously difficult to navigate home pages in my experience, but that could be due to my age) and if you did not lug around your laptop there were TV-monitors everywhere, I was told. But the problems with the latter, I thought, was that they showed all kinds of footage and you had to stand and wait until the daily programme showed up (if ever). I never saw one displayed on a screen, but maybe I lacked the necessary patience? And then a TV monitor cannot be put into your pocket (unless it is in the form of a phone). But of course, digital programmes and newsletters can be continually updated; the advantages of the digital approach are obvious and easy to formulate, while the disadvantages are less so, and harder to put in words. But then in the end, it may simply boil down to being of an older generation formed by outdated habits. Yet, he conceded, he also prefers physical books, but he doubts that someone like Avila ever checked out a book in the library, although of course mathematicians in general tend to be conservative and like books, maybe, I reflect, as they have the pretensions of writing for posterity, sometimes even for eternity, not just for the immediate future as people in big science.

As to the President not being present, that was just a coincidence into which I should not read too much, he admonished me (in particular not suspect that Brazilian society does not care about mathematics). The presence of the President was needed at a meeting of the World Bank, he explained, as Brazil is getting a new loan on very favorable terms, and this is good for mathematics as well, he added, concluding that he was very happy to have him there instead of here, where he could not have done much good.

As to security in Rio, he himself was scared when he visited New York in the 90s (it was much worse in the 70s when I lived there, I pointed out) and he was also afraid in Rio when he first moved here. But with care and avoiding the bad areas you are basically OK, and besides, what can you expect? it is a third world country after all with lots of poverty (this goes for India as well and it is comparatively safe, or am I naive?). But during the carnival, crime rates go down; people are out to enjoy themselves, on the other hand traffic accidents go up, he added. But as this is the most common way of coming to grief, I remarked, we have a high tolerance for it. And there are always reasons to worry, he continued, and used the example of the yellow fever warnings. Should participants be warned and vaccinations recommended? But that warning was issued in February, thus during summer. In winter, as it is now, there is no yellow fever, and besides you are only at risk in the countryside. He did not post it. Later on he relented due to some pressure from the health authorities. Better safe than sorry. And then he had to hurry to his next meeting. And I thought to myself that I never saw the yellow fever warning, shrugged my shoulders and limped out of the building.

Tourism, or Rio by day

Why fly a quarter of the way round the globe to Rio and not go there at all? Tourism, regardless of the purpose of your visit to an ICM, is an important component of it, if not necessarily the only one. If you cannot go by yourself you can join an organised tour. You may ordinarily scoff at such ventures if you fancy yourself to be an independent and adventurous traveller, but of course now you are

travelling with your colleagues: extending your attendance, if by other means. Three things come to mind to the tourist when thinking of Rio; namely, the Jesus statue, Sugar Loaf mountain and Copacabana beach. Pictures of the Jesus statute show it overlooking and dwarfing the city below, and thus one expects that it will present a dominating view wherever one is in the city. But big as it is, it is only 38 metres from the bottom of its pedestal to the crown of its head. Situated on the top of the mountain Corcovado, which rises 710 metres above the beach, it is about five kilometres from the centre itself, thus its apparent size is only about thirty minutes of arc, the same as that of the Sun or the Moon, and hence dwarfed in the general background. However, the view from above the mountain is spectacular and worth a trip, not just a detour. The Sugar Mountain, to which you can get access by two cable cars, is much closer to the centre, and offers a nice view of it, if not quite as spectacular. And we were told by the guide, the cable cars played an important role in a Bond movie from the early 70s. Both of these sights were offered on the standard Rio tour, but not Copacabana beach, immortalized, at least from the perspective of a mathematician, by Stephen Smale and Richard Feynman (most people would have other associations).

However, what I found more interesting was the excursion to Petropolis, which I took on the last day of the Congress. Petropolis is a city close to Rio but further inland and hence up in the mountains. I first encountered the name of the city in connection with the Austrian writer Stefan Zweig, who exiled himself there after the outbreak of the Second World War. His sojourn did not last more than a year and a half before he committed suicide in February 1942, together with his young wife and former secretary, out of depression concerning the war and the future of mankind (it is not clear whether his wife shared his existential misgivings, but she was apparently very loyal). In the 20s and 30s he was one of the most extreme bestselling writers in the world at a time when people read books much more than today, but after the war he descended into relative obscurity. However, in recent years his writing has had a modest revival due to sentimentality for the age in which he lived. In particular as he brings it to life in his autobiography *Die Welt von Gestern* (The World of Yesterday), completed the day before he died. Unfortunately, 'Casa Zweig' was closed the day we visited.

More interesting to most people may be the Imperial Palace in the city. Brazil had two emperors, Pedro I and his son Pedro II, something I have to admit I only learned during my visit. The first was originally a Portuguese Prince, who together with his family was in Brazilian exile during the Napoleonic Wars (Portugal and Sweden, along with Britain, were irreconcilable foes of Napoleonic France). He liked it so much that he refused to return to Portugal after Napoleon's downfall, and declared Brazil independent in 1822 and himself as its emperor. However, he abdicated in 1831, leaving the crown with his young son, not much more than a toddler. The son, however, turned out to be what we ideally expect of an exemplary monarch. A patron of the arts

Casa Stefan Zweig, Petropolis.

Crystal Palace, Petropolis.

and sciences, a champion of liberal causes (in his case the abolition of slavery), he enjoyed widespread popularity. He was deposed in 1889, exiled himself to France and died two years later. The imperial connection during the 19th century left the city with a lot of interesting buildings, one particular and unexpected example being the replica of the Crystal Palace erected in London for the fabled 1851 exhibition, thus allowing oneself, be it in semi-tropical surroundings, temporary time travel back to mid 19th century London.

A day or so later I left Rio with an enhanced view of the country. The problem is the ridiculously low airfares of today, which not only enable but encourage frivolities such as weekend trips to New York. Ideally, a visit to Brazil would have started with a two week Atlantic crossing and then continued for at least a month or two. Air travel takes the romanticism out of travel, but the romantic scenario I am sketching is no longer an option in our age, except for individual cases (which of course in the past also held for exotic travel).

And as to ICMs. In spite of my grumblings I look forward, God willing, to visiting the next, adding St. Petersburg to the unbroken list of Beijing, Madrid, Hyderabad, Seoul and now Rio. In spite of the obvious disadvantages of a Congress they form a venerable tradition, which, as pointed out by Guillermo Curbera (a noted chronicler of the ICMs), is the envy of every other scientific discipline.

Ulf Persson is on the Editorial Board of the EMS Newsletter. His photo and CV can be found in previous Newsletter issues.

The Secondary-Tertiary Transition in Mathematics

What are our current challenges and what can we do about them?

Boris Koichu and Alon Pinto (Weizmann Institute of Science, Israel) on behalf of the EMS Education Committee

Introduction

Student transition from school-level mathematics to university-level mathematics, often referred to as the *secondary-tertiary transition* (hereafter STT) is an enduring, complicated and multi-faceted process. STT is a long-standing issue of concern, which has merited significant attention in mathematics education research and practice. In particular, STT was discussed on the pages of this Newsletter several years ago (Gueudet on behalf of the Education Committee of the EMS, 2013).

At its 2018 meeting in Cyprus, the EMS Education Committee recognised that our knowledge about successful ways of dealing with STT is still insufficient and that moving forward requires a large-scale effort on the part of all parties involved, including mathematics lecturers, school teachers, education researchers, policymakers and students in transition. As part of this effort, the Committee is conducting a survey among mathematicians. *The goal* of the survey is to collect and report to the mathematics community information needed in order to devise national and international actions that can essentially improve the state of the art with respect to STT.

Invitation

Thank you very much for devoting about 15 minutes to completing the survey!

You can find the survey by googling "EMS Committees" => Education => Reports => Survey (at the bottom of the page).

Additional thanks for sending this invitation to your colleagues who might be interested in taking part in the survey and thus contributing to the EMS collaborative effort to make substantial progress in relation to the STT. The survey is open until 15th September, 2019.

Background information on STT

Many university mathematics lecturers feel that teaching first-year university mathematics courses, such as real-analysis and linear algebra, is often a more difficult, frustrating and disappointing experience than they would had expected or would have liked them to be. In a recent survey of the state of the art with respect to the teaching and learning of proof, Stylianides, Stylianides, and Weber (2017) conclude that students at all levels struggle with proof writing, have difficulty translating informal reasoning into valid arguments, are often unable to validate proofs, and generally lack many of the competencies needed for proving. It is thus not surprising that many mathematicians characterize their first-year students as unprepared, unable or unwilling to cope with challenges of university-level mathematics (Nardi, 2008).

From the student side, the STT experience looks differently. For example, in a recent study on mathematics freshmen in Pisa University, Di Martino and Gregorio (2018) describe the emotional crisis experienced even by those students who had been successful mathematics learners in high school and who eventually succeeded at the university. Student testimonies show that the crisis is induced by unexpected failures in first year mathematics courses and intensified by the realisation that the strategies they developed at high school for learning mathematics, which had served them well, were failing them at the university level. As a result, many students feel helpless, ashamed and, in some cases, left alone. In brief, encounters with university mathematics require from students a deep reconstruction of their understanding of what mathematics is, of their attitude towards the subject, and of their perceived competences (Winsløw & Grønbæk, 2014).

Recent research tends to attribute students' difficulties during the STT to discontinuities between mathematics as experienced at school and as practiced at university rather than to inherent inability to do mathematics at a high level. The discontinuities concern for example: modes of thinking (e.g., formalisation and abstraction), modes of mathematical communication (e.g., proof writing), student agency over learning (e.g., the increased requirement for independent study), teacher-student interaction, assessment and grading and curriculum misalignment (Gueudet, 2008, Jablonka, Ashjari & Bergsten, 2017). In addition, the students' first encounters with university-level mathematics are considered from the perspective of characteristics of communities the students and mathematics lecturers belong to (Biza, Jaworski & Hemmi, 2014) and by accounting for differences between the university and school as educational institutions (Winsløw & Grønbæk, 2014).

Implications of mathematics education research with regard to STT include suggestions for reducing these discontinuities at the university level and at the school level. At the university level, suggestions include "bridging" mathematics courses, resources and courses for promoting lecturers' pedagogical awareness and enriching their arsenal of teaching and assessment strategies; pedagogical methods making tacit aspects of mathematics accessible for students during lectures and tutorials; and curriculum design principles for producing better textbooks (Gueudet et al., 2016). At the school level, education researchers and math educators have called for, sought and proposed curriculum reforms and teacher development programs that would bring the mathematics that is practiced and experienced in school closer to the mathematics within the discipline (Gueudet et al., 2016). As mentioned, a small portion of the proposed changes has been realised in practice.

Are hard feelings and experiences an inherent part of learning during STT, or can they be reduced? How do lecturers and students experience STT in different universities and countries? How did these experiences change (if at all) during the last decade in light of increasing attention and efforts to addressing STT? How and to what extent have different institutions and lecturers been addressing STT? Has technology been helpful in this regard? If so, how was technology integrated in university teaching, and what have been the results of this integration? How can successful efforts to address these challenges of STT be further disseminated? These and such questions are still open and serve as a motivation for the suggested survey and further action.

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Isaac Newton Institute for Mathematical Sciences

David Abrahams (Isaac Newton Institute for Mathematical Sciences, Cambridge, UK)

The INI building seen from the south-west.

The apple tree in the garden to the rear of the INI building – planted by Lady Atiyah, wife of the first Director of the Institute, as part of the opening ceremony in July 1992 – is said to be a descendent from the tree that inspired Isaac Newton.

In July 2017 the Isaac Newton Institute for mathematical sciences celebrated the 25th anniversary of its opening, a quarter of a century that had seen a total of 133 research programmes undertaken within its walls, seminar rooms, offices and collaborative work spaces. That figure is now at 145. Based on the University of Cambridge's mathematics campus, the Institute is privileged to be a part of one of the world's leading centres for the advancement of the science as a whole – a position in which it continues to thrive.

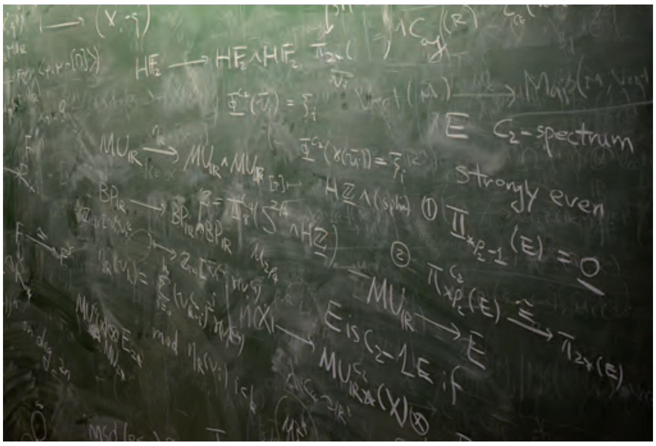
A place for collaboration

INI's distinctive architecture characterises it as a collaborative space, embracing both open, shared spaces ideal for group discussion and smaller, more private offices for quieter work in smaller groups. The building is typically occupied by up to 70 mathematical scientists at any one time – with three times that number during workshop weeks. Many of these participants will not have met before and others may not realise the relevance of other research to their own work. Whether early-career researchers or senior academics, or between those two groups, collaborations and networks are routinely formed within this space that last a lifetime. The Institute runs scientific programmes varying in length from four weeks to six months. At any particular time there are typically two or three programmes running in parallel, each with between 20–30 participants working at the Institute. Week-long workshops are run at regular intervals throughout these programmes to allow outside speakers or other researchers to contribute to the programme in shorter, more intense periods of interaction. Selection of these programmes is made by a panel of leading mathematicians. The criteria are that a programme should have great scientific merit and be at the forefront of current developments where a significant scientific breakthrough can be expected, be intra or interdisciplinary by nature, and to have the highest quality leadership and participants. By identifying subjects that have both substantive mathematical significance and clear common ground for collaborative study, the Institute is able to transcend the boundaries of departmental structures.

The Institute in numbers

Throughout a typical year the Isaac Newton Institute will host over 500 such programme participants and 1,500 workshop participants, amongst which regularly feature winners of the Abel Prize and the Fields Medal, as well as other prestigious awards. Together these groups will account for around 25,000 participant days of activity, allowing for both a broad scope of subjects to be covered and profound, meaningful progress to be made across them. Within the past two years, for example, programme themes have included, but not been limited to, a variety of pure and applied areas such as: symplectic geometry, computer vision, uncertainty quantification, homotopy theory, the mathematics of sea ice phenomena, the design of new materials, energy systems, and (inspired by the work of D'Arcy Thompson) the nature of form and self-organisation in living systems. The geographical spread of these researchers' home institutions is evenly split between the UK, the rest of Europe, and the remainder of the world. This makes INI a truly international centre and inspires the kind of cross-pollination of ideas which has the potential to truly move disciplines. A perhaps more quantifiable outcome can be highlighted by the number of research papers produced within the four-year span of

A typical workshop underway in INI's main Seminar Room.



INI boasts over 130 square metres of blackboards.

the Institute's most recently completed grant cycle. The number of research papers produced since 2014 stands at around 350, excluding any for which the authors have not acknowledged the Institute.

An increasing output

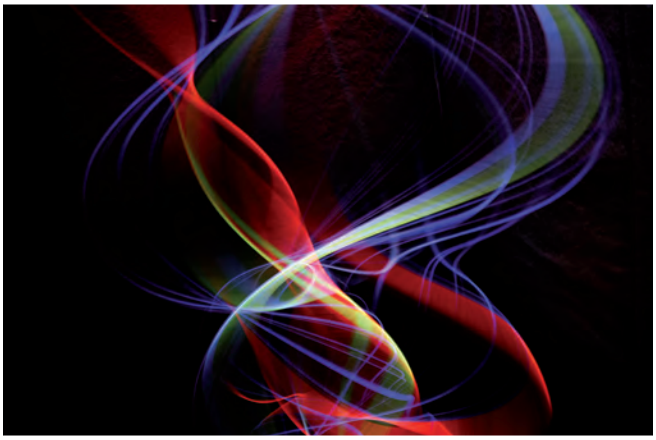
A further point worth highlighting is that as the programme frequency at INI has grown in recent years, so has the output detailed above. The decision, taken in 2015, to regularly increase the number of programmes run in parallel from two to three, has resulted in as many as nine separate programmes occurring within a calendar year. Once the selection of follow-on workshops from past programmes and other one-off events are added to this, the Institute is now achieving within 12 months what it previously may have done across 18. This growth, of course, necessitates upgrades to the building's infrastructure as well as an augmentation of the administrative structures needed to support such large numbers of visitors. Consequently, in July 2018 work began on the long-anticipated refurbishment of INI's Benians Court properties – themselves just one of the options available for participant housing. This exciting new redevelopment involves the comprehensive update of 24 apartments, all of which are regularly used to house INI visitors and participants. Of those flats, 18 are being altered to contain two double rooms with en-suite facilities, with improved kitchen areas and standardised quality throughout the furnishings. Of the remaining six, two will be fully renovated to ensure complete disabled access whilst four will keep a configuration better suited to housing families. With work on the final phase of the project underway at the time of going to press, work is set to finish on schedule in the summer of this year.

Equality within the science

As detailed above, throughout its 27 years of operation INI has welcomed pre-eminent mathematicians from across the world – and this has been in part thanks to its Rothschild Visiting Distinguished Fellowship scheme, set up to facilitate the participation of a keynote participant for each programme. In addition, as of 2019, a generous £250,000 donation from the Turner-Kirk Charitable Trust has allowed the establishment of the new Kirk Distinguished Visiting Fellowship Scheme. This fellowship also provides funding for one senior mathematician per programme; however, these field-leading figures are chosen from under-represented groups within higher mathematical research. As a consequence, they will primarily be senior women mathematicians – as demonstrated by the five scientists so far selected for 2019. It is the Institute's hope that this prestigious fellowship scheme, promoting influential role models within their field, will give confidence and self-belief to early career researchers and hence address the historical gender imbalance that persists within the science.

Collaborative working across myriad disciplines lies at the core of INI's philosophy.

Shared spaces are present throughout the building, making discussions lively and regular.



An interactive artwork displayed as part of the "Growth form and self-organisation" programme.

Nilanjana Datta and Andreas Winter of the "Beyond I.I.D. in information theory" follow-on workshop (2018).

Outreach and public-engagement

Since 2002 INI has maintained a list of correspondents in UK Universities, other academic institutions and research and development groups in industry and commerce. These individuals act as a channel of communication between the Institute and the mathematical sciences community in the UK. Correspondents are regularly informed about activities of the Institute, and it is their responsibility to ensure that the information is disseminated to the relevant departments in their institution, as well as to provide regular feedback to the Institute. Following a refresh and re-launch in January 2018, which took the number of active members to 84 and established a new Chair to better coordinate the group, the Correspondents Network was branded jointly with the International Centre for Mathematical Sciences (ICMS). This rebranded network launched with the INI-ICMS Correspondents Day meeting in Cambridge on 24 January 2018, and has continued most recently with the 2019 meeting on 30 January 2019 at ICMS' new facilities at The Bayes Centre in central Edinburgh. Within the past two years INI has also taken steps to further its outreach activities and to broaden its communication channels. Examples of this include: hosting the 25th anniversary celebrations of 2017, which welcomed esteemed and popular figures ranging from Sir Andrew Wiles and the late Sir Michael Atiyah to Dr Simon Singh and Dr Hannah Fry; contributing to annual popular science events such as New Scientist Live at London's Excel centre and the Cambridge Science Festival (the most recent INI speaker being Amazon Alexa's Head of Applied Science Dr Craig Saunders); the publication of a video interview series focused on the programme organisers; the recording of a regular podcast interview series, and more. The Institute is also proud to announce that, as of January 2019, INI's knowledge exchange arm has been renamed the Newton Gateway to Mathematics (formerly known as the Turing Gateway to Mathematics) – a move which clarifies its key role within INI and highlights its essential task of fostering long-lasting relationships between mathematical scientists and other academic disciplines, industry, commerce and the public sector.

Three of John Robinson's sculptures mark the entrance to the INI building, all based on the interlocking forms of the Borromean Rings.

A portrait of Newton displayed at INI. This copy of the original by Sir Godfrey Kneller (1689) was painted by Barrington Bramley and donated to the Institute in 1992.

David Abrahams has devoted research effort to the broad area of applied mathematics, and specifically to the theoretical understanding of wave processes, since graduating in 1982 with a PhD in applied mathematics from Imperial College London. Prior to taking up the post of Director of the Isaac Newton Institute, he held the Beyer Chair of Applied Mathematics at the University of Manchester, and between 2014–16 was Scientific Director of the International Centre for Mathematical Sciences (ICMS) in Edinburgh.

The Norwegian Statistical Association

Ørnulf Borgan (University of Oslo, Norway) and Marie Lilleborge (Cancer Registry of Norway, Oslo, Norway)

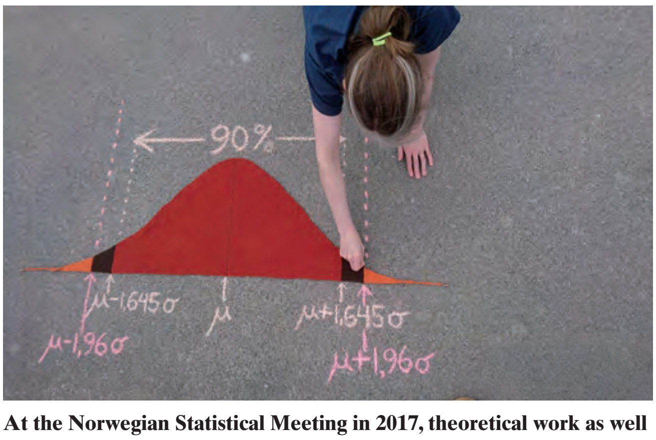
Norsk statistisk forening

The Norwegian Statistical Association was founded 83 years ago, on April the 28th, 1936. The association actually has roots going back 100 years; it is the continuation of a club for statisticians founded on January the 7th, 1919. The first president of the Norwegian Statistical Association was Professor Ingvar Wedervang, a Norwegian economist and statistician. Professor Wedervang, together with the later Nobel laureate Ragnar Frisch, was a founder of the Department of Economics at the University of Oslo in 1932. He was also a co-founder, and the first rector, of the Norwegian School of Economics, which was also established in 1936 in Bergen as Norway's first business school.

Until 1985, The Norwegian Statistical Association was mainly an association for statisticians, economists and others working at The Central Bureau of Statistics in Oslo (now Statistics Norway). Then the association was reorganised as an association for all Norwegian statisticians, and today the majority of its about 300 members have a university degree in statistics.

The main goal of the Norwegian Statistical Association is to connect Norwegian statisticians, work for their professional interests and promote research in statistics and statistics in schools and society. The association also aims to strengthen the relationship between statisticians from the Nordic countries and increase contact with the international statistical community. The association is a member of the European Mathematical Society (EMS), the International Statistical Institute (ISI) and the Federation of European National Statistical Societies (FenStatS).

Together with its Nordic sister associations – the Danish Society for Theoretical Statistics, the Finnish Statistical Society, and the Swedish Statistical Society – the Norwegian Statistical Association is responsible for publication of *The Scandinavian Journal of Statistics* (SJS), and the editors in chief for the journal circulate among the four Nordic countries. The *Scandinavian Journal of Statistics* was founded in 1974, and it is internationally recognized today as one of the world's leading statistical journals. At the start, the journal was published by the Swedish publisher Almqvist and Wiksell, but since the early 1990s it has been published by international publishers; first Blackwell and then Wiley. The Scandinavian Journal of Statistics has four issues per year, and as of today, more than 80% of its submissions are from outside of Scandinavia.



At the Norwegian Statistical Meeting in 2017, theoretical work as well as knitted normal distributions were presented. Kathrine Frey Frøslie is the author of the popular science knitting blog Statistrikk. (Photo: Ingeborg Frey Frøslie)

The four Nordic statistical associations also collaborate on the organisation of the Nordic Conferences in Mathematical Statistics (NORDSTAT). The first of these biennial conferences took place in Aarhus in Denmark in 1965, and the site of the conference circulates among the four Nordic countries. Since 2008, the Baltic countries also take part in the organisation of the NORDSTAT conferences, and NORDSTAT2008 and NORDSTAT2018 took place in Lithuania and Estonia, respectively.

The Norwegian Statistical Association has three local chapters, located in Oslo, Bergen and Trondheim; corresponding to the three largest universities in Norway. The chapters have their own boards, and organize seminars and social events for their members. The chapters also take turns in organising the biennial Norwegian Statistical Meetings.

The Norwegian Statistical Meetings usually last for three days, with invited speakers and scientific contributions by PhD and master students and researchers. The meetings serve as a meeting point for statisticians from all over Norway. Most presentations are held in Norwegian, with slides in English. This is contrary to university seminars in Norway, where all presentations and discussions are currently held in English. The Norwegian Statistical Meetings are also the arena of the general assembly of the Norwegian Statistical Association, and the board of directors of the association is elected at these meetings. The next Norwegian Statistical Meeting will be held in Sola outside Stavanger this summer, June the 17th–20th 2019.

The Norwegian Statistical Association issues two awards. They are named after the Norwegian statistician Erling Sverdrup, who played an instrumental role in building up and modernising the fields of mathematical statistics and actuarial science in Norway. The creation of the two prizes was announced in 2007. The two Sverdrup Prizes are given to "An eminent representative of the statistics profession" and to "A young (<40 years) stat-

Four winners of the Sverdrup Prize for an eminent representative of the statistics profession, from left to right: Tore Schweder, Dag Tjøstheim, Nils Lid Hjort and Odd Aalen. (Photo: Celine Marie Løken Cunen)

istician with best journal article". The prizes have been awarded every second year since 2009. The winners of the prize for an eminent representative of the statistics profession have been Dag Bjarne Tjøstheim (2009), Tore Schweder (2011), Nils Lid Hjort (2013), Odd Olai Aalen (2015) and Ørnulf Borgan (2017). The winners of the young statisticians prize have been Sara Martino (2009); Ida Scheel (2011), Ingrid Hobæk Haff and Kjetil Røysland (2013); Tore Selland Kleppe (2015) and Geir-Arne Fuglstad (2017).

The Norwegian Statistical Association publishes its own member magazine in Norwegian called *Tilfeldig Gang* (Random Walk). Since the beginning in 1984, it has served as a newsletter for members of the association, with 2–4 issues per year. All current and historical issues are available online for free on the webpage of the association.

Ørnulf Borgan has been a professor of statistics at the University of Oslo since 1993. His main research interest is survival and event history analysis, and he is co-author of the monographs "Statistical models based on counting processes" (Springer, 1993) and "Survival and event history analysis: a process point of view" (Springer, 2008). Borgan has been editor-in-chief of Scandinavian Journal of Statistics (2007–2009) and associate editor of Annals of Statistics (1998–2003). In 2017, he was awarded the Sverdrup prize for an eminent representative of the statistics profession. Borgan is an elected member of the Norwegian Academy of Science and Letters and a fellow of the American Statistical Association.

Marie Lilleborge is the second female president of the Norwegian Statistical Association, and currently a PostDoc at the Cancer Registry of Norway. Her research interests include probabilistic graphical models, algorithms and data analysis. She received her MSc degree in Industrial Mathematics at the Norwegian University of Science and Technology in 2012, and her PhD in Statistics at the University of Oslo in 2017.

The Program in Interdisciplinary Studies of the Institute for Advanced Study, Princeton

Michael Th. Rassias (University of Zürich, Switzerland) and Olaf Witkowski (Tokyo Institute of Technology, Japan)

The Institute for Advanced Study (IAS), Princeton, is one of the very few places in the world where everyone is totally absorbed in pure research. At the IAS there is no student body, just chalk, blackboards, books and scholars offered complete freedom to think and explore. It was founded in 1930 by Abraham Flexner, Louis Bamberger and Caroline Bamberger Fuld, and since then it has served as the academic home of emblematic figures of science like Albert Einstein, Kurt Gödel, John von Neumann and J. Robert Oppenheimer, to name just a few.

This small academic utopia is also the home of the so-called Program in Interdisciplinary Studies (PIDS), which provides a meeting place for members from the IAS, Princeton, and beyond. These scholars engage in conversations and research projects that address the wider questions that rarely get asked within individual disciplines; questions like, what is the nature of knowledge in different academic areas, in the humanities, the social sciences, the natural sciences and mathematics? How may these fields stimulate each other? And how do such types of knowledge compare with those found in the arts, in the business world, in ancient traditions and in daily life?

The Program started in 2002, as the response to the perceived need to have more communication between

Susan Clark (left) and the Head of the PIDS, Professor Piet Hut (right).

Professor Freeman Dyson's talk in the "After Hours Conversations" of the PIDS, on October 18, 2018.

the four main Schools at IAS (Mathematics, History, Natural Science, Social Science), and possibly even collaborations between the Schools. Through PIDS, one can be exposed to various different scientific disciplines and, by engaging in discussions with scientists hailing from a broad spectrum of scientific areas, one can possibly discover unexpected applications and interconnections of one's own field to something completely different. For example, number theorists may interact with nuclear physicists, philosophers with astronomers, musicologists with computer scientists etc., in a very intellectually stimulating environment. The nature of PIDS – attracting members from all Schools within the IAS and beyond – can allure even the purest of researchers, like pure mathematicians whose work may seem distant from other fields, to engage in conversations which often lead to intriguing new interdisciplinary questions. Commencing from concrete problems, discussions often stretch to explorations of a more philosophical flavour, such as for example the emergence of life, the origin of mathematics, or the nature of knowledge.

The output of such collaborative projects from PIDS members ranges from the very pure, like works on Goldbach's conjecture (cf. [5, 11]) and Riemann's hypothesis (cf. [6, 7]); or abstract, such as the philosophy of "Math Matter and Mind" (cf. [3]); to more applied, like works on astrophysics (cf. [4]), artificial life simulations (cf. [1, 2, 12, 13]), and biology (cf. [8]). In a recent volume dedicated to the Riemann zeta function, co-edited by current PIDS visitor Michael Th. Rassias in collaboration with Hugh L. Montgomery and Ashkan Nikeghbali [9], F. Dyson of the IAS writes about a fascinating interdisciplinary approach to Riemann's hypothesis. He argues that one could combine physics and pure mathematics by studying one dimensional quasi-crystals and associating them with the zeros of the Riemann zeta function. In a more playful yet challenging spirit, prompted by conversations with Monica Manolescu and Siobhan Roberts as a PIDS visitor in 2012, Philip Ording published a book, [9] challenging the conception of mathematics as having a single style strictly characterized by symbolic notation and abstraction.

Apart from the individual as well as collaborative research that members of PIDS conduct, there are several outreach and dissemination activities initiated by this Program, taking place at the IAS, Princeton, but also elsewhere, such as in New York, Tokyo, etc. One central facet of these activities are the so-called After Hours Conversations, which constitute informal short talks held in the bar area of the IAS twice a week during the first two months of each semester. These talks are open to faculty, members, visitors, staff, spouses and partners in an effort to encourage cross-discipline communication at the IAS. Professors Piet Hut (head of PIDS), Didier Fassin, Helmut Hofer and Myles Jackson moderate these sessions. In these talks, someone gives an informal presentation of no more than 10 minutes, intended for a general audience. The topic of the talk consists of a brief description of a major open problem in the speaker's field of expertise, together with suggestions for possible future progress with respect to that problem. The talk is then followed by 20 minutes of discussion. Afterwards, the remaining participants are free to mingle in more general discussions in smaller groups, preferably with others not from their own School. The lively and intriguing nature of the After Hours Conversations has been very successful, with prominent figures of science having delivered such brief talks, like famed physicist Freeman Dyson. Inspired by these meetings, the School of Mathematics recently introduced the so-called Mathematical Conversations.

Following in this interdisciplinary spirit, the head of PIDS was also invited to contribute to the experience of PIDS and be one of the founding members for a new research centre within the Tokyo Institute of Technology, working on elucidating the rise of initial life on Earth and its subsequent evolution to complex life. This centre, called the Earth-Life Science Institute (ELSI for short), in order to address both the origin and evolution of life, required putting together and co-ordinating a strongly interdisciplinary hub of researchers from very different backgrounds. Following the experience of PIDS, from astrophysicists, planetary scientists, to geochemists, paleontologists, astrobiologists and computer scientists, all researchers had to learn to form a common language to communicate across fields, and design bridging structures to collaborate efficiently as a diverse research institution.

In another instance of such an initiative spreading further, the head as well as members of PIDS in col-

Co-founder of YHouse Professor Caleb Scharf, at YHouse's event "Learning About the Brain and Brain-Inspired Learning", on May 3, 2017.

laboration with scientists from other institutions have also founded a pilot experiment in the form of a virtual research institute called YHouse, which was very active for a couple of years starting in 2016. This design constitutes a unique gathering space ("House") based in Manhattan, New York, dedicated to accelerating our understanding of "why?" questions ("Y") on the nature of awareness through multiple areas of science, technology, and humanities. YHouse's aspiration is to create a culture of open-minded rigour and a spirit of transdisciplinary inquiry that breaks free of the institutional barriers in academia and society.

The presence of a centre as PIDS is essential to the life of an institute, as it not only catalyses work across fields, it also challenges the researcher's area of comfort and fosters very creative work through interdisciplinary exchanges (academic, but also outreach and citizen science). In a sense, it provides a comfortable place to work out of one's comfort zone. This way, PIDS represents an avant-garde initiative in a top institution, which deserves to be considered seriously as a model for future universities and research centres.

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Michael Th. Rassias is on the Editorial Board of the EMS Newsletter. His photo and CV can be found in previous Newsletter issues.

Olaf Witkowski is a Visiting Member at the Institute for Advanced Study, Princeton, and a Research Scientist at the Earth-Life Science Institute. He is also a Chief Scientist at the Cross Research Institute as well as Founding Member of YHouse. His research is in machine intelligence and artificial life.

Practices for Identifying, Supporting and Developing Mathematical Giftedness in School Children: The Scene of Hungary (short version)

Katalin Fried (ELTE, Budapest, Hungary) and Csaba Szabó (ELTE, Budapest, Hungary) on behalf of the EMS Committee of Education*

An extended version of this paper is published on the Committee's website http://euro-math-soc.eu/sites/default/files/giftedness-long-HUN.pdf.

Traditions

The traditions of modern Hungarian mathematics education go back to the 19th century. Around that time several scientific forums were founded: in 1781 a society called the Hungarian Scientific Society, in 1824 the Science Popularising Society (which still exists) and in 1825 The Hungarian Academy of Sciences.

The predecessor of the Bolyai Mathematical Society was founded in 1891 under the name Mathematical and Physical Society. By the end of the 19th century several well-known mathematicians took part in public education, like Rátz László, the teacher of Neumann János (known as John von Neumann) and Wigner Jenő (known as EugeneWigner) or Beke Manó, and Mikola Sándor. In addition, in 1893 Arany Dániel founded the *Mathematical Journal for Secondary School Students* (age 14–18), *KöMaL* for short, including a problem solving contest. Many of our famous mathematicians can think of this journal as the start of their careers. Just to mention a few of them Fejér Lipót; Erdős Pál; Szele Tibor; Surányi János; Révész Pál; Szegő Gábor; Hajós György; Hódi Endre; Fuchs László; Császár Ákos; Károlyházy Frigyes.

In 1894 the Hungarian Mathematical and Physical Society initiated a competition for the age group of 18, the first in the world. Its problems were published worldwide in the so-called Hungarian problem book. Apart from a few months during the war years, *KöMal* has been published ever since.

As the result of the rise in mathematical education a new generation of young and talented mathematicians emerged, for whom new competitions needed to be organised. In 1923 a new competition was initiated, and is still organised today every year for the age group 16–18. Mathematicians like Szekeres György; Klein Eszter; Grünwald Géza; Svéd György; Wachsberger Márta; Erdős Pál; Turán Pál; Gallai Tibor etc. were part of this new generation.

After WWII, in the Socialist era, it was considered to be important to promote the children of the industrial and agricultural workers in order to find more talented children. The school system was subdivided into an 8-year primary school and a 4-year secondary school education. Mass education, however, required a more sophisticated policy and curriculum. In the 1950s a new textbook was written for Year 9 schoolchildren by Péter Rózsa and Gallai Tibor, both excellent theoretical mathematicians. In 1962 a special type of mathematics programme started for grades 9–12. Mass education also required new didactic strategies. According to the ideas of Varga Tamás, if you do not start teaching mathematics from the very beginning, you might have problems later when you want to teach more abstract thoughts. In 1978 the reform was introduced in all schools in Hungary.

Education

In Hungarian schools, the minimum number of lessons in mathematics in a class is 3 lessons per week, and the maximum is 10. In today's school system in Hungary it is possible to change the status of a child every 2–4 years, however, this does not happen very frequently in sparsely populated areas. All children in Years 1 to 4 learn approximately the same content. In Years 11 and 12 pupils have to specialise in taking more or fewer lessons in mathematics and it is very rare that pupils change class or school.

At the beginning of the 1960s teacher training was separated from theoretical and applied mathematician education at university level. The training for prospective teachers for Years 1 to 4 and Years 9 to 12 had almost had the same structure for the previous one hundred years. The training of prospective teachers for Years 5 to 8 had always been in a contradictory position. Today, education of prospective teachers is split into two significantly different trainings to fit the needs of these levels of education: teachers for children in Years 1 to 4 are prepared for teaching all subjects on a basic level with a didactic and psychological background; teachers for Years 5 to 12 specialise in teaching two subjects and are taught their subjects to a high level with the required didactic background. Their education includes, among others, a course called 'Elementary Mathematics' for 6 semesters, which

* http://euro-math-soc.eu/sites/default/files/giftedness-short-HUN.pdf

aims at developing their problem-solving skills; and a course called 'Teaching Mathematics' for 4 semesters which discusses how to teach certain mathematical topics or manage talented pupils.

Competitions and off-school programmes

Apart from education at school, many children take part in after-school activities. In summer some children are invited to such camps according to the results they have achieved at competitions, but for most of the children the camp has to be a playful pastime with a lot of physical activity. There is a huge amount of interest in such camps, so the number of mathematics camps is increasing.

There are numerous competitions in mathematics. Some competitions can be entered directly, that is, without taking part in school, district or county-level competitions, but most competitions have rounds.

- The 'Kürschák József' competition is the most famous competition, organised by the Bolyai Mathematical Society since 1894, for those students who are about to finish or have finished their secondary school studies within a year.
- The National Competition for Years 11 to 12 is organised by the State of Hungary in all major subjects. The competition in mathematics started in 1923.
- The 'Arany Dániel' competition, organised by the Bolyai Mathematical Society, is for Years 9 to 10. The first such competition was organized in 1947.
- The 'Varga Tamás' competition, organized by MATEGYE on behalf of the Bolyai Mathematical Society, is for Years 7–8. The competition started in 1988.
- The 'Zrínyi Ilona' competition, organised by MATEGYE since 1990, is for Years 2 to 12, with a system of multiple choice questions.
- The 'Kalmár László' competition has been organised by the Science Popularising Society for Years 3 to 8 since 1971.

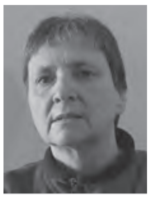
There are more nationwide competitions but the ones mentioned above are the most popular ones.

At the beginning of the 1970s a new initiative to aid gifted students started. Nowadays, the programme runs under the name of 'The Joy of Thinking' (MaMut (MAtheMatical amUsemenT)) and is organised by those youngsters who took part in these programmes when they were students. They invite the best pupils to their weekend and summer camps to be a part of their famous talent-care programme. Prospective teachers are also invited to observe and learn their teaching methods. It is an honour to take part in the camp, both as a pupil and as a prospective teacher. Back in the the 1970s the Ministry of Education organised a camp for the best students from mathematics competitions as part of their prize. These camps were popular among children and teachers as well.

'BEAR events'

Lately, a new style of after-school maths activity has come from a group of young people who, as they say, ``feel passionate about mathematics education''.
Their activities include outdoor maths challenges, maths camps, experience days and treasure hunts. In 2017, about 15,000 people were involved in their programmes.

This wide selection of competitions and the different levels of mathematics education makes it possible for every pupil to achieve their best. In this way, talented pupils are supported within the system. The idea of learning mathematics through problem-solving seems to be very successful. These many competitions and camps make it possible to keep track of the work of the most talented children, keep up their interest and encourage them to develop their skills by preparing for these competitions.



Katalin Fried has been engaged with teacher training at Eötvös Loránd University in Budapest since 1983. She has earned her PhD in mathematics. As an associate professor she is teaching preservice mathematics teachers to several subjects related to mathematical didactics.

Csaba Szabó is a professor at the Department of Algebra and Number Theory at the Faculty of Sciences of Eötvös Loránd University in Budapest. He has widened his research interest ten years ago from the theory of general algebraic systems to the methodology of teaching mathematics. He is a co-founder and leader of the "Theory and Psychology of Learning Mathematics Research Group" in Hungary.

ICMI Column

Hilda Borko (Stanford University) and Despina Potari (Kapodostrian University of Athens)

ICMI Study 25 – Teachers of Mathematics Working and Learning in Collaborative Groups

As announced in the EMS Newsletter 110, ICMI has launched its 25th ICMI study with co-chairs Hilda Borko (USA, hildab@stanford.edu) and Despina Potari (Greece, dpotari@math.uoa.gr) (the full IPC was given in the article in the EMS Newsletter 110).

The first meeting of the IPC took place in Berlin in February, and it produced the discussion document, of which a short abstract is given below.

The full-length version and all of the details in order to participate can be found online: https://www.mathunion.org/fileadmin/ICMI/ICMI%20studies/ICMI%20Study%2025/190218%20ICMI-25_To%20Distribute_190304_edit.pdf

1. The need for the study

Collaboration implies careful negotiation, joint decision-making, effective communication and learning in a venture that focuses on the promotion of professional dialogue. Across education systems, and at all educational levels, mathematics teachers work and learn through various forms of collaboration. Such collaborative work of teachers has a long tradition in mathematics education as it is critical as a way of bringing educational innovation into the everyday practice of teaching.

This attention to teachers learning through collaboration is especially relevant as countries around the world strive to improve educational experiences for all children and to see these improvements reflected on international assessments such as PISA and TIMSS.

Efforts to understand what teachers do as they work in collaborative groups, and how these experiences lead to improvement in their expertise and teaching practice, has led to increasing interest in examining the different activities, processes, contexts and outcomes for teacher collaboration around the world.

However, the ICME-13 Survey also identified several gaps and limitations, not only in the existing research base but also in the coverage of relevant topics related to teacher collaboration.

These gaps and limitations highlight the need for the ICMI Study 25. We hope that this Study will help us to better understand and address these challenges in the study of the processes and outcomes of mathematics teacher collaboration.

2. Aims and rationale

The Study's theme of teachers working and learning in collaborative groups implies a focus on teachers as they work within teams, communities, schools and other educational institutions, teacher education classes, professional development courses, local or national networks – that is, in any formal or informal groupings. Teachers' collaborative work might also include people who support their learning and development such as teacher educators, coaches, mentors or university academics. Collaboration can extend over different periods of time, and take place in face-to-face settings or at a distance. The role of online platforms and technology-enabled social networks is an additional focus in supporting "virtual" collaboration.

We encourage reporting on promising forms of collaborative work among different groups of participants (e.g., teachers/researchers, teachers/curriculum designers, teachers from different disciplines) and collaboration that addresses different goals (e.g., design of tasks, lessons and curriculum materials; improvement of teaching; development of mathematical and pedagogical understanding). The Study will acknowledge that learning is mutual; that is, those who work collaboratively with teachers to develop their practice are also learning from these interactions.

The primary aims of the study are to report the state of the art in the area of mathematics teacher collaboration with respect to theory, research, practice and policy; and to suggest new directions of research that take into account contextual, cultural, national and political dimensions. Because there are different ways of understanding teacher collaboration and its characteristics, enablers and consequences, the Study will include multiple theoretical perspectives and methodological approaches. We encourage contributions that report research using a variety of methodological approaches including large-scale experimental and descriptive studies, case studies and research approaches characterised by iterative or cyclical processes such as design research and action research. We also solicit contributions from teachers as well as researchers, to ensure that teachers' voices are given prominence in accounts of their learning.

3. Themes and questions

The areas and questions that the Study will investigate are outlined below, organized into four themes. These areas are not independent, and some questions can reasonably be placed in more than one area.

A. Theoretical perspectives on studying mathematics teacher collaboration

A number of theoretical and methodological perspectives have been used to study teacher collaboration, illuminating the dynamics of teachers' collaborative working and the communities in which they work. Some of these perspectives are listed in the full discussion document.

These theoretical and methodological perspectives suggest several questions to be explored in this ICMI Study:

- How do the different theoretical perspectives or networks of theories enhance understanding of the processes of teacher collaboration?
- How do they enhance understanding of the outcomes of teacher collaboration?
- What is illuminated by the different perspectives and methodologies and what needs further investigation?
- What are promising research designs and data collection and analysis methods to study teacher collaboration?

B. Contexts, forms and outcomes of mathematics teacher collaboration

The assumption underlying this Study is that teachers *learn* through collaboration; however, it can be challenging to investigate and explain the *processes* through which this learning occurs and to gather evidence of *what* teachers learn. The goals of teacher collaboration are multi-faceted and might be related to the mathematics content, to the learning experience of students, to the development of mathematics teaching that promotes students' learning (e.g., to implement new curriculum materials), to the design of resources such as classroom and assessment tasks, to the creation of a community in which ongoing professional learning is supported, or even to day-to-day teaching (e.g. lesson preparation, team teaching). Similarly, the outcomes of the collaboration also vary. Outcomes related to teachers' and teacher educators' interactions are addressed in Theme C and those related to instructional materials are addressed in Theme D.

The Study will address the various forms of teacher collaboration, their outcomes related to teaching and learning, and the contexts in which they are offered.

- What models of teacher collaboration have been developed? What are the design features, goals, and outcomes of the different models?
- How effective are various models for promoting different outcomes?
- Which forms of collaboration are appropriate in different contexts?
- What are the affordances and limitations of each form of teacher collaboration?
- What are the benefits and the challenges that online teacher collaboration poses to the teachers?

C. Roles, identities and interactions of various participants in mathematics teacher collaboration (e.g., lead teachers, facilitators, mathematicians, researchers, policy makers)

Collaborative groups can include different "actors", such as teachers, facilitators, mathematicians, researchers, administrators, policy makers or other professionals, in various combinations. These participants can assume a variety of roles in collaborative activities, including learners, leaders, designers, researchers and more. The literature indicates that different roles can support productive interactions.

In collaborative interactions, the learning of *all* participants is important. The nature of roles that people play can vary in different countries and cultural contexts. A variety of research-informed approaches for supporting teachers to work collaboratively and also for developing teachers as leaders have emerged around the world. Challenges faced by those taking on the role of facilitating teacher collaborations can include, on the one hand, supporting teachers to develop their teaching and, on the other hand, valuing and promoting their own goals and perspectives.

We invite contributions focusing on these issues, as reflected in the following questions:

- What is the role of lead teachers, facilitators, mentors and teacher educators in supporting teacher collaboration?
- How are different roles and identities shaped and developed among various "actors" (teachers, leaders, mathematicians, researchers, etc.) within a collaborative group? How do lead teachers negotiate their dual roles and identities as both teachers and facilitators of peer-collaboration?
- What are characteristics of a good facilitator of teacher collaboration? How can these facilitators be prepared and supported?
- How can different stakeholders impact teacher collaboration?
- What types of learning environments enhance or hinder mutual learning of teachers and other participants in collaborative interactions?

D. Tools and resources used/designed for teacher collaboration and resulting from teacher collaboration

This theme focuses on the role of tools and resources in facilitating and supporting teacher collaboration. Tools, as well as resources, are understood in a broad sense "that goes beyond the material objects, to include human and cultural resources" . Taking into account their diversity, we are interested here in tools and resources with respect to teachers' collaboration: tools and resources *for* teacher collaboration and tools and resources *from* teacher collaboration

Resources *for* and *from* teacher collaboration can be considered as two ingredients of continuous processes: *adopting* a resource always leads to *adapting* it, and that is more the case in the context of teacher collaboration. Using and designing are then to be considered as two intertwined processes. Taking into account this dialectical point of view, the Study will investigate the roles of resources in facilitating teachers' collaboration, and how those roles differ in different contexts. It will focus on the following questions:

- What resources are available to support teacher collaboration? With what effects, both on the collaboration and on the resources themselves?
- What resources are missing for supporting teacher collaboration? How and to what extent can teachers overcome these missing resources?
- To what extent and under which conditions do digital environments (e.g., mobile devices, platforms, applications) constitute opportunities for teacher collaboration? How have these resources been used to support teacher collaboration?
- Which resources can be used (and how) to sustain and scale up collaboration over time?
- How are teachers engaged in the design of resources in collaboration? What are the outcomes of these collaborations?

4. The Study Conference

The Study Conference will take place in the Institute of Education of the University of Lisbon from the 3rd to the 7th of February 2020, with a reception on the evening of Monday the 3rd of February. Participation in the Study Conference will be by invitation only, for one author of each submitted contribution that is accepted.

The accepted papers will be published in an electronic volume of conference and finally an ICMI Study 25 volume will be developed on the basis of the papers and the discussion in the working groups. This volume will be published by Springer as part of the new ICMI Study Series.

5. Call for contributions

The IPC for ICMI Study 25 invites submissions of several types including: reports of research studies, syntheses and meta-analyses of empirical studies, discussions of theoretical and methodological issues and examinations of the ways that teacher collaboration has taken place in local or national contexts. Studies from different cultural, political, and educational contexts and submissions by researchers, teachers and policy makers are encouraged so that mathematics teacher collaboration can be addressed in its complexity.

The papers should be clearly related to the themes that are discussed in Section 3 and address the questions associated with the themes. Authors must select one of the themes to which their paper will be submitted.

The papers should be submitted through the ICMI Study 25 online system. A template for submission of papers is available on the Study website (see below).

Papers must be a maximum of 8 pages and not have been submitted or published elsewhere. The working title of the paper must contain the author(s) name(s) and the theme letter to which it is submitted, for example: JamesThemeB.

Deadlines

30th of June, 2019: Submissions must be made online through the ICMI Study website no later than the 30th of June but earlier if possible

30th of September, 2019: Decisions from the reviewing process will be sent to the corresponding author by the 30th of September.

Information about registration, costs and details of accommodation may be found on the ICMI Study 25 website: icmistudy25.ie.ulisboa.pt

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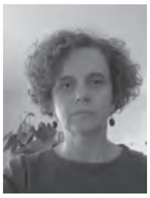

Despina Potari [dpotari@math.uoa.gr] is a professor of mathematics education at the Mathematics Department of the National and Kapodostrian University of Athens. She has been visiting professor in different universities in Europe and US and is currently at Linnaeus University in Sweden. Her research interest is mainly on the development of mathematics teaching and learning and teacher development and in particular on the role of different contexts and tools in the classroom setting as well on mathematics teacher collaboration. She has published in international research journals, conference proceedings and book chapters. She is an editor-in-chief of the Journal of Mathematics Teacher Education, a member of editorial boards and teams and a reviewer for international journals and conferences.

ERME Column

Maria Chimoni (University of Cyprus), Reinhard Oldenburg (University of Augsburg, Germany), Heidi Strømskag (Norwegian University of Science and Technology) and Jason Cooper (Weizmann Institute of Science, Israel)

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME), holds a bi-annual conference (CERME), in which research is presented and discussed in Thematic Working Groups (TWG). We will continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME's Thematic Working Group 3 – Algebraic thinking

Thematic Working Group 3 (TWG3) focuses on a prominent area of research in mathematics education: algebraic thinking. This working group featured at all (bi-annual) CERME conferences except CERME 2. Typically, 20-30 papers are presented at the conferences with authors from all over Europe, and to a minor extent from the Americas and Africa. These numbers reflect the timeless interest of researchers into the teaching and learning of algebra and their consensus on the important role of algebraic thinking for improving students' mathematical knowledge and performance from primary through to university level.

As a basis for algebraic thinking, most of the studies reported can be seen to draw on a conceptualisation of school algebra as consisting of three interrelated principal activities: generational activity (i.e. the creation of algebraic expressions and equations); transformational activity (i.e. syntactically guided manipulations of formalisms); and global/meta-level activity (i.e. activities for which algebra is used as a tool) [1]. This view is in contrast to the "traditional" image of algebra as a set of procedures that are taught and learned in the middle school and are disconnected from other mathematical domains, students' earlier experiences with mathematics and college/university-level content. Like the discipline as a whole, the papers presented in TWG3 are characterised by diversity regarding their specific algebraic topics and their theoretical and methodological approaches. One of the central issues addressed in TWG3 has been the nature and characteristics of algebraic thinking, indicating the groups' interest in understanding students' ways of using algebraic tools, rather than in algebra as an epistemic body of knowledge. Participants have examined the notion of algebraic thinking from a multitude of theoretical, historical, and epistemological perspectives. Specifically, a number of papers used conceptual frameworks (developed with algebraic thinking as their momentum) as models for describing the structure of algebra and algebraic thinking, as well as clarifying particular algebra learning goals and instructional activities. Other papers have used general theories of teaching and learning as lenses to analyse algebra (e.g. semiotic theory, genetic epistemology, theory of sense and reference, theory of mediating tools, cognitive theory of instrument use). These theories assist in further explaining how individuals acquire and deploy algebraic abilities. Further, some papers have used holistic theories that have guided the instructional design of the studies reported (e.g. the theory of didactical situation in mathematics, the anthropological theory of the didactic, variation theory). This multitude of theories and research perspectives reveal that many questions remain open, such as what kind of research problems in the teaching and learning of algebra are related to which theoretical frameworks, and to what extent are these various perspectives complementary or contradictory.

A recurrent theme in TWG3 has been the process of generalisation, which is considered as central to algebraic thinking and at the very heart of mathematical activity. Virtually all papers have touched upon it to some extent, while some papers have addressed generalisation more explicitly. A considerable number of papers have examined individuals' abilities to detect figural and numerical patterns, as well as generalising those patterns and representing them symbolically.

An important breakthrough in the last 1–2 decades has been the success of early algebra. In the past, researchers tended to assume that a certain level of cognitive development was needed before symbolic algebra could be understood, yet recent research has shown that primary school students are able to use symbolic algebraic thinking tools in a sensible manner. The importance of generalisation already explained above has been supported by these works. Furthermore, many papers on early algebra have viewed the establishment of generalisations as a key characteristic of early algebraic thinking and as a foundation for developing understanding of the more formal algebra in later years and at university level. Moreover, these papers have presented curricular and instructional approaches that were found to facilitate students' moving from intuitive ways of thinking about mathematical relationships and structure to more formalised ways.

A considerable number of papers in TWG3 focus on students' understanding and different thinking levels about fundamental algebraic concepts, such as functions, equations, equivalence and the equals sign. Especially, students' difficulties, errors and misconceptions have been a central theme. For example, students tend to interpret the equal sign as an operational sign that says: calculate what's on the left. As a consequence, young students may be baffled by problems with unknowns such as $3+\_=8$ (many think the answer is 11), and even students in higher grades, where $3+5=8$ is recognised as a valid equation, may consider $8=3+5$ to be invalid. Discussions within the TWG have highlighted the possible impact of such difficulties on students' transition to university.

Interestingly, even some basic epistemic issues remain unclear. For example, can two equations be considered equivalent if one is over a field (i.e. the variable represents a number) and the other is over a polynomial ring, and thus have different semantics?

Another interest of researchers in TWG3 has been the nature and objective of activities that promote the development of algebraic thinking. A number of papers have presented teaching experiments and design research studies that suggest ways for approaching important algebraic concepts. These studies have provided evidence of ways in which students' algebraic thinking evolves within appropriate classroom environments. Some of these studies discuss task design and the use of technological tools, while others focus on teachers' roles and actions for prompting students to develop and apply algebraic processes and reasoning.

A limited number of studies draw on methodological and theoretical approaches from complementary fields such as cognitive science. For example, the notion of embodied cognition has been examined through the electroencephalographic brain activity of university students while performing algebraic, geometric and numerical reasoning tasks. Preliminary results suggest that bodily movements have a positive effect on the cognitive processing of demanding mathematical tasks. We suggest that this line of studies might further be extended and developed, as recent results in brain science show that people with amazing accomplishments in mathematics have more communication between different areas of the brain [2–3]. What encourages brain connections is when we see mathematics in different ways: e.g. as numbers, visuals, words, algebraic expressions, algorithms, gestures [2]. The three principal activities involved in algebraic thinking mentioned above (generational, transformational, global/meta-level activity) presuppose creating, manipulating and transforming between different representations of mathematical objects. In this way, these activities encourage brain connections, a matter we believe should be of interest to teachers in school as well as to university mathematicians.

Overall, the corpus of work on algebraic thinking at CERME discloses the rich and varied perspectives according to which the participants have been examining its nature, its learning and its teaching. The chapter on algebraic thinking in the book *Developing Research in Mathematics Education* gives more details on these issues [4]. The spirit of communication and collaboration, as well as the structure of TWG discussions at CERME, have proven to be effective for spotting critical issues and inspiring further research. TWG3 remains an active group that contributes to our understanding of how the development of algebraic thinking can be supported so that more students around the world can gain access to algebra as a valuable tool in their everyday and academic lives.

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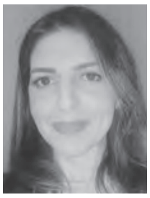


Maria Chimoni is a research fellow at the Department of Education, University of Cyprus. Her research interests concern the development of algebraic thinking through cognitive and pedagogical perspectives, curriculum design for mathematics, and development of mathematics textbooks.

Reinhard Oldenburg holds the chair of Mathematics Education in the Department of Mathematics at Augsburg University, Germany. His research is devoted to the learning and teaching calculus and algebra, especially using technology. At CERME10 (2017) and CERME11 (2019) he led TWG3 on Algebraic Thinking.

Heidi Strømskag is an associate professor at the Department of Mathematical Sciences at the Norwegian University of Science and Technology in Trondheim. Her research focuses on didactical design in mathematics teacher education for secondary school – using didactical engineering – with special interest in mathematical modelling and algebraic thinking. She has been Norway's representative to ICMI since 2018.

Jason Cooper is a senior intern at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including roles of advanced mathematical knowledge in teaching and contributions of research mathematicians to the professional development of teachers.

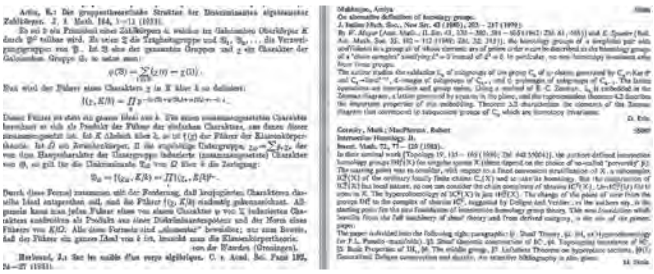
Four Decades of TeX at zbMATH

Moritz Schubotz (FIZ Karlsruhe, Berlin, Germany) and Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

In April 2019, zbMATH was completely transformed from TeX to LaTeX sources. On this occasion, we give a brief history of typesetting Zentralblatt volumes, and describe the challenges, methods and benefits of the transition.

A very short overview of typesetting Zentralblatt

TeX and LaTeX have been the standard tools for creating documents for at least two generations of mathematicians. Today it is almost inconceivable that mathematical content was typeset before. Indeed, mathematical typesetting has a much longer history at zbMATH. zbMATH has existed for about 150 years if one includes its printed predecessors, *Jahrbuch über die Fortschritte der Mathematik* and *Zentralblatt für Mathematik*. Although it would be somewhat tempting to retell the history of mathematical typesetting in these periods based on the appearance of the *Jahrbuch* and *Zentralblatt* volumes in a similar manner as, e.g., [5], this would go beyond the scope of the present article. Instead, the figure below provides impressions of volumes based on various lead types, linotypes, or phototypesetting using IBM golfball typewriters.



Formulæ from Zentralblatt volumes 1 and 529.

In general, mathematics was always very expensive to typeset.[1] The various developments until the 1970s aimed to make this process more efficient. Famously, the quality of the more cost-efficient phototypesetting technique decreased in comparison to classical lead techniques [5]. Indeed, in the *Zentralblatt* volumes from the years of phototypesetting one comes across several formulae which needed to be added by hand, before one could start to create the phototypesetting master by cutting and glueing.

However, at *Zentralblatt*, the shift to phototypesetting was inevitable due to the growing amount of content, which resulted in the production of immense register volumes. This created the most obvious and urgent need for further digitisation.

From the beginnings of TeX to its adaptation

Donald Knuth developed the TeX system to overcome the limitations of phototypesetting. As a welcome side effect, the fact that TeX is an open system established the autonomy of mathematical writing. With hindsight, this appears to be an inevitable development, though in the transition period it was not. For instance, a subject of detailed debate was whether it would be more efficient to employ scientists as their typesetters, instead of having specialised staff [9].

A major milestone in this process was the TeX82 version with both its improvements and stability. The 1982 meeting of the TeX Users Group at Stanford was the first TUG meeting lasting for about a week. Incidentally, a *Zentralblatt* editor – who was involved at that time in the editors' exchange program with Mathematical Reviews[2] – took part, and advocated later in Berlin the adaptation of TeX for the production of the *Zentralblatt* volumes. At Mathematical Reviews TeX became fully productive as of 1985, after several years of preparation [7]. In contrast, it took until 1992 before the advent of TeX at *Zentralblatt*. In the meantime, the phototypesetting technique became no longer sustainable. In 1984, a proprietary, internal Springer system was employed that resembled many of the typesetting functions of TeX. The commands of the Springer system were designed for specialised technical staff instead of self-use. A key argument for the Springer system in contrast to TeX was the resulting lower number keystrokes for volume production.

From today's viewpoint, the greatest advantage of moving away from phototypesetting was that at least the texts and formulae were available in a digital form for the volumes 531–734. However, the disadvantage was that after the Springer system became outdated in the early 1990s, the introduction of TeX required conversion: simultaneously to the Springer-based production plan. Since the transition to TeX also included a migration from Springer servers to FIZ Karlsruhe, the expected delay occurred.[3] Fortunately, the TeX expertise acquired in the meantime allowed for a successful transition. It turned out that even most formulae could be translated automatically, though some constructions

1 Typically, there were only a few highly specialised typesetters involved. One anecdote that has passed through generations was that one single typesetter was responsible for producing Zentralblatt volumes for many years. Eventually, he was able to spot errors in mathematical formulae without any semantical knowledge.

2 That was also the time when a merger of both services was actively pursued.

3 The 3-month hiatus between Vol. 734 and 735 is by far the largest post-war gap.

had inherent problems; e.g., matrices needed conversion from column-based encoding to row-based encoding.[4]

From TeX to LaTeX

For some years after the switch to TeX, the production of printed volumes had been the main objective. The main objective was the appropriate presentation. A standardised encoding, which is desirable from an information retrieval viewpoint, was less relevant. This changed in the second half of the 1990s after the online database became the primary objective. In 2004, adapting to these needs, the database production switched to a PostgreSQL system. This stored all available information in ASCII-coded TeX texts. This framework hadn't changed significantly until recently.

Of course, despite the impressive robustness and stability of TeX, technical development didn't stop in the 80s. Probably already in the mid-90s LaTeX was the preferred dialect for many users. Today LaTeX accounts for more than 90% of zbMATH review submissions. Since the database routines previously interacted with TeX, reviews in LaTeX required a re-standardisation. Furthermore, an increasing amount of data is provided in the UTF-8 format. Its conversion to TeX encoding for internal storage and later reconversion for online presentation may cause errors and a loss of information. This pertains especially to bibliographic reference data which may include the need to encode native script such as Arabic, Chinese, Farsi, Hebrew, Japanese, Korean, or Russian. This makes it desirable to have the XeTeX expansion available.

How does one convert ~ 20 million formulae?

Therefore, a switch of the production system underlying zbMATH to XeTeX/LaTeX was in preparation for several years. The initial work-intensive step was made when MathML was introduced in 2010 [1]. Standard tools for MathML conversion employed by zbMATH, such as Tralics [8], require LaTeX source. Thus, it was necessary to convert different TeX commands to LaTeX – at least, those commands which could be processed by MathML converters. This part of the conversion could be amply addressed by regular expressions. However, it turned out to be necessary to build the full expression tree for TeX formulae. This step, which was finally taken in April of this year, was preceded by an upgrade of PostgreSQL, which allows for UTF-8 handling and a 14-hour routine that converted approximately 18 million standard inline mathematical expressions and 600,000 displaystyle formulae. These included more sophisticated environments, like \alignat or \gathered. Fortunately, only a few environments, e.g., commutative diagrams, require additional manual transition. Overall, the introduction of LaTeX resulted in a pause of zbMATH updates of about a month. In contrast to the introduction of TeX, which took three months, this is a significant improvement. The most visible difference for zbMATH users looking at the review sources is the replacement of \$…\$ by \(…\). Standardisation will allow for much easier integrity checks, and for a much more seamless integration of the submitted reviews. Additionally, considerably better presentation is possible by new functions available via LaTeX packages. MathJax (needed for maths presentation in browsers not capable of MathML) works better on widely used LaTeX commands instead of less frequent TeX commands. Another advantage pertains to the MathML generation: while Tralics works well for presentation purposes, it hasn't been developed further for some years, so it is reasonable to look for alternative options like LaTeXML [6]. The availability of LaTeX code makes such alternative implementations much more feasible.

[4] By looking at the sources of these, a reader can easily indicate that they were not genuine TeX-coded; e.g., single variables were not set in formula italics in the old system, so they lack conversion until now.

Paving ground for future developments

Even more importantly, further developments in mathematical information retrieval and processing will most likely be based on LaTeX. LaTeX became the de facto standard in mathematics, and most working mathematicians use LaTeX in their publication workflows. Moreover, most websites use LaTeX as an input language for mathematical formulae. To make mathematical content better discoverable, multiple approaches exist for enhancing semantics in mathematical formulae. For example, the NIST Digital Library of Mathematical Functions developed a set of semantic LaTeX macros for mathematics. These macros are easy to use for mathematicians fluent with LaTeX. By using these macros, with minimal overhead, information retrieval systems would be able to better disambiguate the syntax and semantics for mathematical expressions. Eventually, this will provide better search and recommendation functionality for users of mathematical digital libraries [2]. To some extent, such approaches have already been applied to realise the zbMATH formula search [4, 3], but having standardised LaTeX sources at hand will certainly make further developments in this direction much more feasible.

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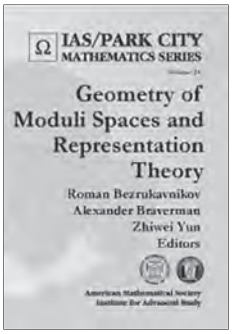
Moritz Schubotz [moritz.schubotz@fiz-karlsruhe.de] is a senior researcher for mathematical information retrieval and open science. He maintains the support for mathematical formulæ in Wikipedia and is off-site collaborator at NIST.

Olaf Teschke's photo and CV of the author can be found in previous Newsletter issues.

Book Reviews



Okounkov, Andrei
Lectures on *K*-theoretic computations in enumerative geometry, in:
Bezrukavnikov, Roman et al. (Eds.), Geometry of Moduli Spaces and Representation Theory
AMS and IAS, 2017
ISBN 978-1-4704-3574-5

Reviewer: Letterio Gatto

The Newsletter thanks zbMATH and Letterio Gatto for permission to republish this review, which originally appeared as Zbl 1402.19001.

The major concern of the book under review, to whom the author modestly refers to as "Lecture Notes", is the algebraic enumerative geometry understood in the most modern of the possible modern senses. This, however, without neglecting a constant and respectful eye to the classical roots of the subject, never left aside but enhanced by that deeper understanding gained from new visions and perspectives. Self-defined as "Lecture Notes", this collection of ten dense chapters form, to say the truth, an impressive monumental book, 375 pages long, whose title "*K*-theoretic computations in enumerative geometry", that seems very focused, looks more quite a pretest for an amazing interdisciplinary carousel of mathematical fireworks than a way to confine the subject between artificial walls.

The treatise, that's the way we think more appropriate to refer to the "Lecture Notes", is written with an excited and exciting mood. It is a vehicle through which the author transmits enthusiasm and arouses the interests of many many readers, regardless of the harsh reality, namely, that only a few elected and very dedicated people will be really able to grasp the deepness and the subtleties of most of the contents, which is what one needs to feel breathtaking emotions.

In the very first lines of his work, the author declares that the lectures are intended "for graduate students who want to learn how to do the computations from the title". That gave the reviewer, mostly unaware of the material developed in the notes, the pleasant sensation to feel younger, a little bit like a graduate student, although certainly not among the best.

The author clearly emphasizes that the lectures are focused on computations. As he puts it, it is very important to keep a "connection between abstract notions of algebraic geometry, which can be very abstract indeed, and something we can see, feel, or test with code". If the journey is promising, however, it certainly is not effortless: everyone is warned that the exciting but tiring reading is demanding, that everyone is asked to work himself and is exhorted to read with an active, and not passive, attitude.

Who wishes to be initiated to the mysteries of the enumerative *K*-theory through these notes, is invited, nearly every half a page, to think her/himself to work out an exercise, to propose creative examples. To say it with the same words of the author, in fact, while "it is a challenge to adequately illustrate a modern algebraic geometry narrative, one can become familiar with main characters of the notes by working out examples" and his hope is "that these notes will be placed alongside a pad of paper, a pencil, an eraser, and a symbolic computation interface."

One difficulty in reviewing this book is that it is not just an expository work, although it is also that, and not just a research book, although it is also that. In fact it covers a wide range of interrelated subjects that even now, while the reviewer is writing the present report, or the reader is reading the inadequate reviewer's words, is in rapidly expansion. This treatise forms the body not of a static but of a dynamic book. It sounds like the foundation of a new literary genre, that of mathematical snapshots taken by experienced photographers from the window of a running train, while the landscape rapidly changes before the eyes. The titanic flavor of the work, in spite of the prudent warning of the author ("These notes are meant to be a partial sample of basic techniques and results, and this is not an attempt to write an A to Z technical manual on the subject, nor to present a panorama of geometric applications that these techniques have"), comes from the (successful) attempt to draw the state of art of the subject as photographed in the exactly the same instant the notes were going to be written.

Stopping by listing too easy and obvious praises, both to the text and the fresh style it is written with, the best way to end this review is probably that of giving the reader the idea of how its content is organized. The book is divided into ten chapter, that the author calls Sections: hard work already begins in Section 2, despite its title "Before we begin". This is a kind of warning, like the game level 0 where one must gains points to access to the next one. The first section "Aim and Scopes" is dramatically intriguing and transmits a lot of excitement to the potential reader who will tend to forget, due to the beauty of the exposition, how much the book is demanding.

The third section is dedicated to the study of the still mysterious land of Hilbert schemes of points in threefolds: by contrast with the case of points on a surface, even its dimension is not known. This topic is used as a pretest to introduce the traveler into the realm of the Donaldson–Thomas (DT) invariants. A central result of this section is the Nekrasov formula related with the DT theory, originally born in the context of string theory. A remarkable family of equivariant symplectic resolutions are provided by Nakajima quiver varieties, that have found many geometric and representation-theoretic applications and whose construction is recalled in Section 4.

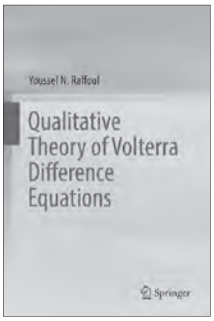
Next level: Section 5. Devoted to the DT theory on symmetric powers of curves, instead of working with stable maps, whose first appearance dates back to the milestone papers by Kontsevich and Manin ["Gromov–Witten classes, quantum cohomology, and enumerative geometry", Mirror symmetry, II, 607–653, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997], the author deals with the more general *quasimaps*, extensively treated in Section 4.3. These are not the same as the stable ones, but the author implicitly suggests a research direction to check his guess that the methods used by A. Givental and V. Tonita [Math. Sci. Res. Inst. Publ. 62, 43–91 (2014; Zbl 1335.19002)] may be possibly adapted to the situation considered in Section 6, where quasimaps and related material are extensively treated.

Vertices and edges (certain kind of tensors associated to fixed points and invariant curves under the torus action of a toric threefold) have direct analog in K-theory, and are the "Nuts and bolts" of the subject, widely treated in Section 7, through which the mathematical performer. according to the Master, may measure his own degree of professional skill.

Section 8 is concerned with equivariant difference equations: key words for this chapter are *difference equations for vertices*. The last two chapters are entitled respectively "Stable envelops" and "Quantum Knizhnik–Zamolodchikov equations". These are rather technical sections, and one will not even attempt to popularize them within the few lines left in this already too long review. The best recommendation is to read over and over the very first section, "Aim and Scope", to arouse once more in the beginner the desire to deeply penetrate, and become intimate with, the content of the notes.

The notes, the book, the paper, or whatever this mathematical exposition be, concludes itself with a huge list of related literature to whom the reader is really invited to refer to. As if the reviewed masterpiece were, more than a usual report, an Arianna thread, a sort of compass necessary to avoid getting lost in the labyrinth of a unusually spectacular mathematics that the young brave reader might like to venture out.

Letterio Gatto serves as Associate Professor at the Dipartimento di Scienze di Matematiche of Politecnico di Torino, where he teaches linear algebra and geometry for engineering students. He has reviewed nearly 100 papers and books. His main research areas are algebraic geometry and algebra. With Parham Salehyan he shares the authorship of the Springer IMPA Lecture Notes "Hasse-Schmidt derivations on Grassmann algebras", a subject he has been studying within the last fifteen years. He is fond of Brazil, where he periodically spends research periods. In the last four years, he has collaborated with the Italian Embassy in Brasilia on the promotion of scientific meetings of mathematical interest in Brazil.



Youssef N. Raffoul

Qualitative Theory of Volterra Difference Equations

Springer, 2018
xiv, 324 p.
ISBN 978-3-319-97189-6

Reviewer: Rodica Luca

The Newsletter thanks zbMATH and Rodica Luca for the permission to republish this review, originally appeared as Zbl 1402.39001.

The monograph is concerned with a systematic study of the qualitative theory of boundedness, stability and periodicity of Volterra difference equations. The book is organized in six chapters, an extensive bibliography and an index. Chapter 1 contains a brief introduction to functional difference equations including Volterra difference equations. The author provides the z-transform and known theorems for the stability of the zero solution of Volterra difference equations on convolution type. The Lyapunov functions for autononomous difference equations are presented and some results for the stability and boundedness are also given. The concept of total stability and its connection with the uniform asymptotic stability for perturbed Volterra difference equations are introduced and utilized. The total stability relies on the notion of the resolvent equation which is used to express the solutions of Volterra difference equations, and to obtain theorems for the stability of the zero solution. Necessary and sufficient conditions for the uniform asymptotic stability of the zero solution via resolvent applied to perturbed Volterra difference equations are also addressed.

Chapter 2 deals with functional difference equations for which there exists a Lyapunov functional satisfying certain inequalities in terms of wedges. For these equations, the author proves general theorems for the boundedness of solutions and stability of the zero solution. By construction of suitable Lyapunov functionals, he obtains results regarding the boundedness, uniform ultimate boundedness and stability of solutions for finite, scalar and vector Volterra difference equations. Theorems concerning functional difference equations with finite or infinite delays with many applications are also presented. Some theorems which guide us in the construction of suitable Lyapunov functionals for specific nonlinear Volterra difference equations are finally given in this chapter.

In Chapter 3, the author provides a brief introduction for metric spaces, Banach spaces, Cauchy sequences, compactness and contraction mapping principle. Then he presents a detailed study of stability and boundedness of ordinary and functional difference equations (such as: highly nonlinear delay equations, difference equations with multiple and functional delays, neutral Volterra equations, almost-linear Volterra equations and delay functional difference equations) by using the fixed point theory. For the Volterra summation equations, the author gives an exposition on the method of Lyapunov functionals and the fixed point theorems. At the end of Chapter 3, he uses the concept of Large Contraction instead of the regular contraction, and the reformulated classical contraction mapping principle and the Krasnoselskii fixed point theorem to investigate the boundedness and the periodicity of solutions to some difference equations.

Chapter 4 is devoted to the study of periodic solutions and asymptotically periodic solutions for functional difference equations and systems with finite and infinite delays. The author applies the Schaefer fixed point theorem, the Krasnoselskii fixed point theorem and the Schauder fixed point theorem, and obtains various results concerning difference equations with finite and infinite delays. Applications of the given results to infinite delay Volterra difference equations are presented by constructing suitable Lyapunov functionals to obtain a priori bound for all possible solutions. The existence of periodic and asymptotically periodic solutions for coupled infinite delay Volterra systems is also investigated. Functional difference systems that have constants as their solutions are finally considered and, by using the contraction mapping principle, the unique constants to which each solution converges are determined.

Chapter 5 is focused on the applications of functional difference equations and Volterra difference equations to population dynamics and epidemics. First, the author introduces different types of population models, and then he presents the cone theory and uses it to prove the existence of positive periodic solutions for functional difference equations. An infinite delay population model which governs the growth of population of a single species whose members compete among themselves for the limited amount of food available for sustaining the population is investigated. The existence of a positive periodic solution is obtained by using the cone theory. The permanence phenomenon for an $(l+m)$-species Lotka–Volterra competition-predation system with several delays is finally studied.

Chapter 6 is dedicated to the exponential stability, l_p-stability and instability of the zero solution for Volterra difference equations by using the Lyapunov functional. Scalar Volterra difference equations are firstly investigated and exponential stability results are obtained. Then, by combining the method of Lyapunov functionals of convolution type, the Laplace transform and the z-transform, the author presents boundedness and stability properties of solutions. A nonstandard discretization scheme is applied to continuous Volterra integro-differential equations and he proves that the stability of the zero solution and the boundedness of solutions for the continuous dynamical system are preserved. The concept of semigroup and its application to Volterra difference equations are finally presented in this chapter.

We remark that each chapter is finalized by a section entitled "Open problems", where the author presents for the interested readers some new and interesting research problems related to the subjects of the chapter.

The book is well-written and the presentation is rigorous and very clear. This monograph is a great source for graduate students in mathematics and science and for all researchers interested in the qualitative theory of Volterra difference equations and functional difference equations.

Rodica Luca (rluca@math.tuiasi.ro) is a professor at the Department of Mathematics of the "Gheorghe Asachi" Technical University of Iasi, Romania. She received her PhD in mathematics at the "Alexandru Ioan Cuza" University of Iasi in 1996. She received the "Spiru Haret" Prize of the Romanian Academy in 2018 for the monograph "Boundary Value Problems for Systems of Differential, Difference and Fractional Equations. Positive Solutions" published by Elsevier in 2016, and written in collaboration with professor Johnny Henderson. Her research interests are boundary value problems for nonlinear systems of ordinary differential equations, finite difference equations, and fractional differential equations, and initial-boundary value problems for nonlinear hyperbolic systems.

Letters to the Editor

ICM 2018 – What Really Happened in Rio de Janeiro

Marcelo Viana (Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil)

I have attended every International Congress of Mathematicians since Kyoto 1990. Twice I was an invited or plenary speaker. Last year, in Rio de Janeiro, I chaired the organising committee. On all of these occasions, I was there because I enjoy the lectures, the discussions with fellow mathematicians from around the world and the unique role played by the Congress. Over the years, I have read many good accounts of the ICM activities. And then there was the unbelievable piece that the EMS Newsletter just published: *ICM 2018 in Rio – A Personal Account Part I*, by Ulf Persson.

The author, an editor for the Newsletter, gets basic facts wrong: Hyderabad 2010 preceded Seoul 2014, not the other way around. He insults Congress participants: ICMs are attended by several thousand people but, to believe his article, apart from a few hundred who have to be there, including journalists, the rest are merely there as tourists. He seems poorly informed; only upon arrival does he find where the venue is located. His goal is to interview the new Fields medalists, and yet he fails to meet half of them, despite their being available for several days at the Congress, and being interviewed by everyone else. Bizarrely, he complains that Fields medalists used to be "old established men" and now are "mere babes (sic)". Maybe this just proves that youth and old age are relative. I cannot disagree with that.

But truth and fairness are not relative concepts, and the article contains little of either. Instead, prejudice and smear are all over the place. Many of the falsehoods were refuted in an interview I granted the author, but he chose to ignore it and print them anyway.

At the heart of the text lies the claim that "Brazil does not really care about mathematics, and by implication other intellectual pursuits; instead it is a hedonistic country geared towards soccer and dancing samba on the beaches," which is insulting and plainly wrong. How does anyone dare to reduce a diverse, culturally rich nation of 210 million to such stereotypes?

That such a piece of bigotry was published by the EMS Newsletter defies rational explanation. Even if "the views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team", that does not exempt the EMS and, more directly, the Editorial Board from responsibility.[1]

I cannot address the whole article here – there are just too many wrongs to be righted – so I have to settle with setting the record straight on a few main issues. Hopefully, this will help prevent such an atrocity from ever happening again: the EMS Newsletter deserves better.

Brazilians do not care about maths

At the ICMs 2002 to 2014, the opening ceremony was chaired by the host country's head of state. It was not so in Rio de Janeiro, and one may ask whether that might reflect a lack of interest for science and mathematics on the part of our government. The answer is no.

To begin with, even under adverse economical conditions, the government provided the bulk of the (6 million euros) budget of the Congress, besides funding the International Math Olympiad 2017. Surely that counts for more than a speech or two. Besides, the reason the President could not come was that at that time he was signing a US$ 1.5 billion IDB loan to science in Brazil. Not something we would complain about. Last but not least, the government was officially represented at the ceremony by the Minister of Education, the ICM's main sponsor.

All of this was explained to the author in our interview; he just chose to omit it. It is clear from the text that purporting to believe our government does not value mathematics is a pretext for claiming that is part of who we are as a nation: "Brazilians do not care about Fields medalists, they are instead heading for the beaches or the soccer stadiums". This is absolutely astonishing.[2]

[1] President Volker Mehrmann of the EMS immediately denounced the article, offering to express his opinion as an addendum to the present text. The Editorial Board acknowledged they have a moral responsibility, and accepted my request to publish this rebuttal, but have yet to comment publicly on the matter. I am told that my reaction "will be a subject of our analysis of the attitude and the quality of this author, who is a member of the Editorial Board". Hopefully, that will result in something good and meaningful.

[2] In Zürich 1994 the President of the Confederation did not attend the opening ceremony. Would the EMS Newsletter publish a claim that " the Swiss do not care about Fields medalists, instead they are geared towards eating cheese fondue"? The Swiss government was represented by the Minister of Home Affairs, who delivered what must have been the most intelligent speech about mathematics ever uttered by a politician. Check pages xiv–xvii of the proceedings.

After all, we are talking about a country where 2014 Fields medalist Artur Avila has household celebrity status, and which celebrated its promotion to IMU group 5 as a matter of national pride. A country that holds a national mathematical Olympiad attended by 20 million kids (yes, 10% of our population!) every year, and engages in a very successful nationwide popularisation initiative,[3] the Biennium of Mathematics 2017–2018.

Besides, over five thousand Brazilian schoolkids and teachers visited the ICM venue to attend public lectures by the likes of Ingrid Daubechies and Cédric Villani, and stood in line to get the autographs of Fields medalists. That is, of course, not mentioned in the article. Had the author just cared to be at one of those occasions (yes, they were advertised), he would have been able to get his interviews…

Cédric Villani addresses over 1,200 Brazilian school teachers and students at one of the five ICM 2018 public lectures.

The opening ceremony

We invited the 576 gold medalists of the Brazilian Mathematical Olympiad to participate in the opening ceremony, including the Fields medal awards. As I mentioned in my speech, their presence also symbolised that this ICM in Brazil has always been about the future.[4] They came back the next day to receive their own medals in that same hall, in a ceremony that was attended by three Fields medalists. The author does not care to mention any of that; perhaps it would ruin a pre-established narrative.

Instead, he explains that: "half-naked dancers, billed as Aborigines, with fancy headgear and elaborate tattoos took to the stage as well as to the aisles of the auditorium, performing to loud primordial music". Seriously? In 2019, is this how you want to portray indigenous peoples of the Americas performing their ancestral rites in a ceremony that celebrated (to general applause: ask your colleagues and check tinyurl.com/yyo47c8p) the diversity of sources of the host country's culture? Were we talking of European traditions, would such language be used in the EMS Newsletter?


The gold medalists for the Brazilian Math Olympiad stand to be awarded at the venue of the ICM 2018 opening ceremony.

Security in Rio

Another subject the author seems to think he knows a lot about, despite this being his first visit to South America. A cab driver directed him to the ATMs on the second floor of the Rio airport, which, he later found out, "are notorious for being regularly skimmed". We do not know who said so, nor how reliable that person is. I am myself a regular user of ATMs at the airport, and have never heard of that theory, nor ever had any problems with them. But the author has 'proof': "sure enough, a few months later the credit card company contacted me concerning a suspicious post". Again, we are not told whether the company actually linked the post to those ATMs.[5] And, come to think of it, "a few months" is an awfully long time for such things…

Rio is a city of 6.7 million people in a developing country; it does have serious security issues,[6] and visitors are well advised to exert caution. But enriching a story with unsupported allegations, possibly just because they may sound plausible,[7] is unethical.

And the fact remains that, for all of the scary stories, not one single security incident involving any of the 3,000 plus ICM participants was reported to the organisers during the whole Congress in Rio. Except, of course, for the infamous Fields medal episode.

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3 Formally proclaimed by the Brazilian parliament, and supported by the government.

4 Pierre Pansu wrote at Images des Mathématiques (images.math.cnrs.fr): "Je trouve que la présence des lycéens a donné un sens à l'ensemble de la cérémonie. […] faire rêver les jeunes, leur donner envie de se lancer dans des études de mathématiques et de se frotter à la recherche, cela mérite un peu de décorum, et cela justifie le travail de toutes les personnes impliquées".

5 Shortly after the ICM, a massive breach of credit card data was reported that affected some 500 million customers of the hotel chain Starwood Guest worldwide. A couple of our partner hotels were affiliated with Starwood, but the breach had nothing to do with the Congress.

6 Still its reputation is overblown. Some major cities in developed countries have worse crime rates, but that is seldom mentioned. Exposure from being a major tourist attraction does Rio no favors in this regard.

7 Swimmer Ryan Lochte, a gold medalist at the Rio 2016 Olympics, falsely reported he had been robbed, rather than confessing to his girlfriend that he had been partying. Most people believed him – in Rio, what else, right? – for a while. It did not end well for him.

Theft of the medal

"It involved the theft of Birkar's Fields medal, whether by design or accident I am not sure". Is this an attempt at humour, or the birth of a conspiracy theory? Here are the facts:

Right after the end of the closing ceremony, I was informed that a briefcase containing Caucher Birkar's Fields medal had disappeared. We were horrified, of course, and we immediately took all measures within our reach to find it.

We reported the theft to the authorities, obviously, and a senior member of our staff was assigned to assist Caucher in all things related to the incident. Our team searched the venue thoroughly, and the briefcase was soon found, although the medal was gone. We also screened the security videos and, within the hour, we had found footage of the theft (check tinyurl.com/y5owfgun): a man, his face clearly visible, is seen picking up the briefcase from the chair behind Caucher as he stood up to be congratulated. We gave the video to the police, and it was widely publicised by the major news organisations.

The author's claim that there was no security is nonsense. The ICM venue was patrolled by 50 security agents, 18 of whom were assigned to pavilion 6, where the opening ceremony took place. Metal detectors were used at the entrance, and only registered participants carrying their ICM badges were let in.[8] The premises were surveyed by security cameras. And the access to pavilion 6 was controlled by a team of 9, who checked the badges once more. The fact is that in such large events,[9] involving thousands of people, it is nearly impossible to ensure that such an incident does not happen.

The second medal

We also had concerns of a more human nature. Caucher confided to our liaison person that his little son Zanko does not like it when he travels. To be "allowed" to come to Rio, Caucher had promised the 4-year-old he would give him the medal upon returning home. We had to act quickly.

Fields medals are minted in Canada, under the authority of the University of Toronto. As the chair of the organising committee, it was my responsibility to keep the medals (as well as those for the Nevanlinna, Gauss and Chern prizes) until they were handed out. I proposed to the IMU president and secretary general that we give an available spare copy of the Fields medal to Caucher (the organising committee paid for it), and they agreed.

In the interview that followed the "second Fields medal ceremony", a good-humoured Birkar joked that the theft had made him a lot more famous. The same is true for the Fields medal and the Congress itself, which got an unprecedented level of media exposure, so that some good came out of this incident after all.

[8] My wife and children were held at the entrance until I had their badges brought to them.

[9] A few months before, a similar incident happened at the Academy Awards ceremony, when the Oscar trophy of actress Frances McDormand was stolen. Fortunately, the trophy was later recovered.

ICM in the news

"Maybe the congress did not even appear on national news?" If the author actually cared for the answer, he just had to google it. Or pay attention.

No less than 154 journalists, from 9 different countries, covered ICM 2018. ICM TV, an independent operation, continuously produced and broadcast material about the Congress, that was also relayed to local commercial TV networks. Our own press team, consisting of 20 professional reporters (3 photographers) and 3 YouTubers, facilitated the work of their fellow journalists, in addition to producing ICM material themselves, including speakers' profiles and interviews with various participants.

As a result, 200 (overwhelmingly positive) reports about the Congress were featured on television channels and in newspapers around the world, 2/3 of them in Brazil. The ICM 2018 social networks reached 2.36 million users, and the website www.icm2018.org had 416,000 page views during the event alone.

Had he just cared to know, the author would have learned that wide coverage of mathematics is not unusual in my country: the Congress, the Fields medal and all things mathematical were regularly on the Brazilian news throughout the Biennium of Mathematics years, often on prime time.[10]

A few hard figures: 2,697 articles about IMPA or the Brazilian Math Olympiad in the written press or television channels; 2.44 million accesses to IMPA's website; 7 million accesses to our social networks. And that was in 2018 alone!

We also created a prize for scientific journalism sponsored by IMPA and the Brazilian Math Society. The awards were a highlight of the ICM's closing ceremony but, needless to say, they are not mentioned in the EMS Newsletter article.

What about the fire?

"There seemed to be some general confusion […] There were rumours that due to the fire, things had to be improvised at the last moment". Confusion? Rumours? Even if one discounts lazy information and sloppy statements, this account remains inaccurate and unfair. Let us get the facts:

On the evening of July 29, less than 60 hours before the start of the Congress, a fire broke out on the roof of the convention centre's pavilion 3, which was planned to host the opening ceremony and the plenary lectures. Our own fire brigade, soon joined by the state's firefighters, was able to contain the fire on the outside, but the whole setting (stage, equipment, chairs, carpets etc) was nevertheless ruined. This was a serious blow that put the whole Congress at risk. Our reaction was swift.

I flew back from São Paulo, where the IMU General Assembly was taking place, to take control of the situa-

[10] Jornal Nacional, the main evening news, with more than 20 million viewers every day, featured a one-week series with myself as the host: over 26 minutes of television prime time devoted to mathematics. How often does that happen, in any country?

tion. We arranged with the convention center to move the activities to nearby pavilion 6. We hired extra teams and all the necessary infrastructure (including another 28-metre-wide LED screen!). And we set a task force that worked around the clock for two whole days.

Thus, the opening ceremony was held as scheduled, only a few metres away from the original location. That was a remarkable feat of hard work and ingenuity, one for which I will always be grateful to our fantastic team. But there was nothing 'improvised' about the ceremony. The programme went on as planned and all rehearsals took place in the new pavilion, which also proved to have great audio and video facilities.

What do I take from it?

As I wrote in the IMU-Net: "Feedback from the participants has been invariably very positive, often outright enthusiastic, and that is most gratifying. We worked hard to make this ICM a memorable event for everyone, and it feels good to be told we achieved our goal." I stand by those words. It would take a lot more than one individual abusing his position to advance prejudice to taint our pride in what we have achieved.

Still, harm has been done that cannot be undone. Mathematicians in developing countries face many great challenges. Respect from our colleagues abroad is critical, not least because it reflects on the domestic standing of mathematics research in our countries. We can only hope that this was not in vain, and that good things will come out of it.

Marcelo Viana is a professor of mathematics and the current director of IMPA – Instituto de Matemática Pura e Aplicada, in Rio de Janeiro. His main research fields are ergodic theory and dynamical systems. He has supervised 39 Ph.D. students from 11 different countries. He was an invited speaker at ICM 1994 and a plenary speaker at ICMP 1994 (math physics) and ICM 1998. Viana is the recipient of several academic distinctions, including the membership of four academies of sciences. He is a former vice-president of the IMU and president of the Brazilian Math Society, and he chaired the ICM 2018 organizing committee. He writes weekly about mathematics in Folha de São Paulo, Brazil's most prominent newspaper.

Comments on my article "ICM 2018 in Rio – A Personal Account"

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

My article on the Rio Congress has caused some furore. This, as I understand it, is due to a misunderstanding of its nature, a misunderstanding further exacerbated by it being published in two parts. It is simply the report of a hapless and clueless 'tourist' at an ICM written in a picaresque manner. To those who unfortunately have misunderstood, I offer my sincere apologies, and also to those who may have understood, but thought that such a report is inappropriate in a journal such as the EMS Newsletter.

I am of course tempted to address (and rebut?) every single criticism as formulated in the letter by Marcelo Viana, but *pace* Oscar Wilde not every temptation should be indulged. Would that really be of interest to the general reader of the Newsletter? [Maybe not even to Viana himself?] Better to follow the example of a lecturer who explains the technical points afterwards in private to a few concerned. Anyone who is curious is encouraged to contact me.

Statement on the two articles ICM 2018 in Rio – A Personal Account Part I, by Ulf Persson ICM 2018 – What Really Happened in Rio de Janeiro, by Marcelo Viana

Volker Mehrmann (Technical University Berlin, Germany), President of the European Mathematical Society

Writing a personal account about a congress is very useful information for readers of the Newsletter and I think that they generally really appreciate such articles. Constructive criticism of the way congresses like the ICM are organised is certainly important and welcome, and the IMU has taken this issue up recently in a discussion blog.

Nevertheless, when writing such a personal account it should be clear that the facts are correct and the article should be respectful and not patronising towards the country and the community of the organisers. I did not attend the the ICM in Rio and only began my presidency in January 2019, but actually, increasing mutual respect for differences in people and cultures, and the differences in the various kinds of mathematics we are doing is a very important item on my agenda as president of the EMS. This includes decreasing the gap between rich and poor countries, filling in the trenches between pure and applied mathematics, and closing the gender gap in our community. In this respect, the Newsletter article by Ulf Persson was very counterproductive.

Despite our joint belief in the great importance and beauty of mathematics, this is not the general opinion in society or even other sciences, and we have to face the fact that we are a small community. Maybe this unpleasant incident can be seen as a catalytic wake-up-call in our community that arrogance and prejudice is not tolerable, and that we should treat each other with respect. Unfortunately, as we have seen on many occasions (not only this one), we still have a long way to go and I really hope that all of us in the mathematical community can follow this path together.

Erratum

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

In the first part of my article on the Rio Congress I mentioned in a footnote that the Nevanlinna prize was to be discontinued. This was correct, but my assumption that the Finns initiated the prize, natural as it might appear, was not. The true initiator of the Prize was actually Lennart Carleson, who during his tenure as a President of the IMU 1979–1982 strongly pressed for a prize in computer science, having at the time begun to appreciate the great potential computers could have for mathematics. There are many of us who agree with him that it is important for the status of mathematics to award such a prize. He had hoped to find financial backing for such a prize in Sweden, but this had not worked out.

Then the Finns stepped in, and got the privilege of proposing a name, and they suggested naming it after Rolf Nevanlinna who had recently passed away and who had served as IMU President. The IMU Executive Committee has discussed the name for the new prize. It was decided that it should definitely not be named after some commercial enterprise, and preferably not after any person, unless he or she has a totally unblemished reputation, neither should it be a long technical name, but something which is both 'snappy' and not to be confused with anything already existing. The solution turned out to be the *IMU Abacus Medal*. If anything represents mankind's involvment with computing, it is the abacus. (If I may include a piece of personal speculation, I find all this concern about, say, the cumbersome Roman numerals irrelevant. Clearly where computations were concerned an abacus was used: the paper and pen method being just an interlude in mankind's computational history.).

Everything is not yet settled, the IMU is looking for a bidder with a deadline of 1st October 2019. It is assumed that this will be successful, and the first medal will be awarded in St. Petersburg in 2022.

More details are to be found on the homepage of IMU, https://www.mathunion.org.

Personal Column

Please send information on mathematical awards and deaths to newsletter@ems-ph.org.

Awards

The Norwegian Academy of Science and Letters has decided to award the **Abel Prize for 2019** to **Karen Keskulla Uhlenbeck** of the University of Texas at Austin, USA "for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics."

The DMV (Deutsche Mathematiker-Vereinigung) awards its highest prize for outstanding research, the **Cantor Medal 2019**, to **Hélène Esnault** of Freie Universität Berlin (Algebra and Number Theory).

The **Richard von Mises Prize 2019** was awarded by the International Association of Applied Mathematics and Mechanics (GAMM) to **Dietmar Gallistl** (University of Twente) and **Philipp Junker** (Ruhr-Universität Bochum) in acknowledgement of their exceptional scientific achievements in the field of applied mathematics and mechanics.

The **2019 Dr.-Klaus-Körper Prizes** in appreciation for an excellent dissertation in applied mathematics and mechanics have been awarded by the International Association of Applied Mathematics and Mechanics (GAMM) to **Friederike Hellwig** (HU Berlin), **Barbara Verfürth** (Universität Augsburg), **Tim Brepols** (RWTH Aachen) and **Pawan Goyal** (MPI Magdeburg).

The **2018 State Prize of Ukraine in Science and Technology** was awarded to **Hennadii Feldmann** and **Mariya Shcherbina** (B. Verkin Institute for Low Temperature Physics and Engineering of NASU), **Anatoliy Kochubei** and **Alexei Rebenko** (Institute of Mathematics of NASU) as well as **Ihor Mykytyuk** (Ya. Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NASU) for a cycle of works "Qualitative Methods of Investigation of Models of Mathematical Physics".

The **Marc Yor Prize 2019** was jointly awarded to **Rémi Rhodes** (Université Aix-Marseille) and **Vincent Vargas** (CNRS, ENS Paris) for their works on multiplicative chaos and Liouville quantum gravity fields.

The **2019 Prix de thèse SMAI-GAMNI** went to **Camilla Fiorini** for her PhD at LMV, UVSQ, Versailles, France and ACUMES team, INREA, Sophia Antipolis, France.

The **2018 Prix de thèse Jacques Neveu** went to **Elsa Cazelles**, Bordeaux.

The Czech Mathematical Society has awarded the **Honorary Medal for Mathematics 2019** to **Jiří Neustupa** (Institute of Mathematics of the CAS, Prague) in recognition of his lifelong contribution to the development of Czech mathematics, international collaboration in research, and education of the new generation of mathematicians.

The Czech Academy of Sciences has awarded the **Honorary Bernard Bolzano Medal 2019** for Merits in Mathematical Sciences to **Miloslav Feistauer** (Charles University, Prague) and **Antonín Novotný** (Université de Toulon).

The **2018 SeMA Journal Best Paper Prize** has been awarded to **Gabriel R. Barrenechea**, **Volker John**, **Petr Knobloch** and **Richard Rankin**, researchers of the University of Strathclyde of Glasgow, WIAS Berlin, the Charles University of Prague and the University of Nottingham Ningbo, respectively, for their contribution "A unified analysis of algebraic flux correction schemes for convection–diffusion equations".

Saharon Shelah is the recipient of the **Rolf Schock Prize 2018** of The Royal Swedish Academy of Sciences.

Alex Lubotzky is the recipient of the **2018 Israel Prize** in Mathematics and Computer Science, handed out by the State of Israel.

Mike Hochman is the recipient of the **2018 Michael Brin Prize** for outstanding work in the theory of dynamical systems.

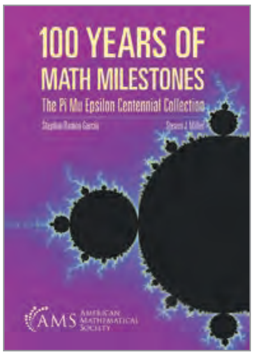
The **Shaw Prize in Mathematical Sciences 2019** is awarded to **Michel Talagrand**, Sorbonne University, France, for his work on concentration inequalities, on suprema of stochastic processes and on rigorous results for spin glasses.

Deaths

We regret to announce the deaths of:

Josef Peter Tschupik (28 March 2018, Innsbruck, Austria)
Rudolf Fritsch (12 June 2018, Munich, Germany)
David Simms (24 June 2018, Dublin, Ireland)
Beloslav Riecan (13 August 2018, Banska Bystrica, Slovakia)
Walter Knödel (19 October 2018, Stuttgart, Germany)
Jordan G. Brankov (1 December 2018, Sofia, Bulgaria)
Pavel Brunovsky (14 December 2018, Bratislava, Slovakia)
Jean Bourgain (22 December 2018, Bonheiden, Belgium)
Bogdan Bojarski (22 December 2018, Warsaw, Poland)
Elias Stein (23 December 2018, Somerset, England)
Peter Swinnerton-Dyer (26 December 2018, Cambridge, England)
Richard Timoney (1 January 2019, Dublin, Ireland)
Michael Atiyah (11 January 2019, Edinburgh, UK)
Gilbert Helmberg (18 February 2019, Innsbruck, Austria)
Petr Zabreiko (21 March 2019, Minsk, Belarus)
Hagen Neidhardt (23 March 2019, Berlin, Germany)
Francisco-Javier Sayas (2 April 2019, Newark, USA)
Vladimir Kirichenko (20 April 2019, Kiev, Ukraine)
Bezalel Peleg (9 May, 2019, Jerusalem, Israel)

AMS American Mathematical Society



100 YEARS OF MATH MILESTONES

The Pi Mu Epsilon Centennial Collection

Stephan Ramon Garcia, Pomona College & Steven J. Miller, Williams College

This book is an outgrowth of a collection of 100 problems chosen to celebrate the 100th anniversary of the undergraduate maths honor society Pi Mu Epsilon. Each chapter describes a problem or event, the progress made, and connections to entries from other years or other parts of mathematics. In places, some knowledge of analysis or algebra, number theory or probability will be helpful. Put together, these problems will be appealing and accessible to energetic and enthusiastic maths majors and aficionados of all stripes.

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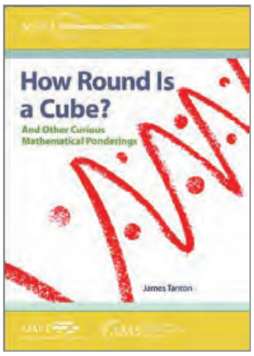

HOW ROUND IS A CUBE?

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James Tanton, Mathematical Association of America

A collection of 34 curiosities, each a quirky and delightful gem of mathematics and each a shining example of the joy and surprise that mathematics can bring. Intended for the general maths enthusiast, each essay begins with an intriguing puzzle, which either springboards into or unravels to become a wondrous piece of thinking. The essays are self-contained and rely only on tools from high-school mathematics (with only a few pieces that ever-so-briefly brush up against high-school calculus).

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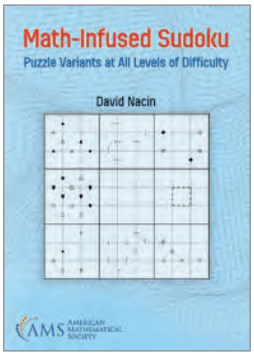

MATH-INFUSED SUDOKU

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David Nacin, William Paterson University

Presents nine variations on a classic and beloved puzzle type. Building upon the rules of Sudoku, the puzzles in this volume introduce new challenges by adding clues involving sums, differences, means, divisibility, and more. Each of the first eight chapters presents a rule system followed by a series of puzzles that progress in difficulty from easy to hard, allowing readers to develop and hone skills in logical problem solving, pattern discernment, and strategy building. In the final chapter, the eight puzzle variants are combined into a single complex puzzle type that puts all of the reader's new skills into play.

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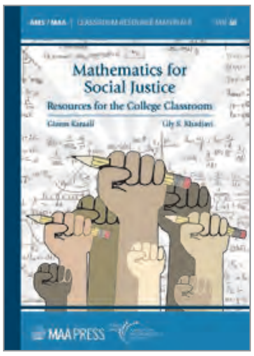

MATHEMATICS FOR SOCIAL JUSTICE

Resources for the College Classroom

Edited by Gizem Karaali, Pomona College & Lily S. Khadjavi, Loyola Marymount University

A collection of resources for mathematics faculty interested in incorporating questions of social justice into their classrooms. The book begins with a series of essays from instructors experienced in integrating social justice themes into their pedagogy; these essays contain political and pedagogical motivations as well as nuts-and-bolts teaching advice. The heart of the book is a collection of fourteen classroom-tested modules featuring ready-to-use activities and investigations for the college mathematics classroom.

Classroom Resource Materials, Vol. 60

MAA Press

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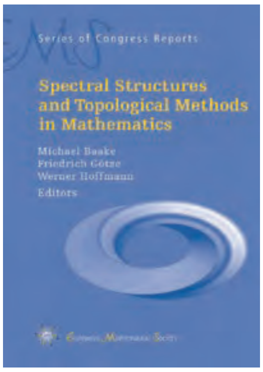
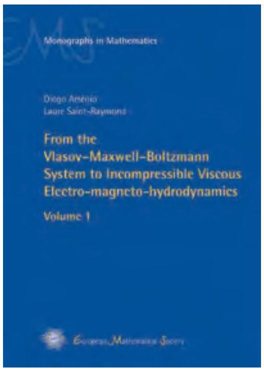


Spectral Structures and Topological Methods in Mathematics (EMS Series of Congress Reports)
Michael Baake, Friedrich Götze and Werner Hoffmann (all University of Bielefeld, Germany), Editors

ISBN 978-3-03719-197-2. 2019. Approx. 433 pages. Hardcover. 17 x 24 cm. 88.00 Euro

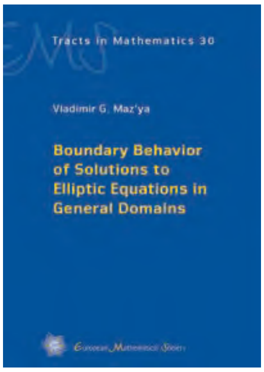
This book is a collection of survey articles about spectral structures and the application of topological methods bridging different mathematical disciplines, from pure to applied. The topics are based on work done in the Collaborative Research Centre (SFB) 701. Notable examples are non-crossing partitions, which connect representation theory, braid groups, non-commutative probability as well as spectral distributions of random matrices. The local distributions of such spectra are universal, also representing the local distribution of zeros of *L*-functions in number theory.
An overarching method is the use of zeta functions in the asymptotic counting of sublattices, group representations etc. Further examples connecting probability, analysis, dynamical systems and geometry are generating operators of deterministic or stochastic processes, stochastic differential equations, and fractals, relating them to the local geometry of such spaces and the convergence to stable and semi-stable states.



Diogo Arsénio (Université Paris Diderot, France) and Laure Saint-Raymond (École Normale Supérieure, Lyon, France)
From the Vlasov–Maxwell–Boltzmann System to Incompressible Viscous Electro-magneto-hydrodynamics. Volume 1 (EMS Monographs in Mathematics)

ISBN 978-3-03719-193-4. 2019. 418 pages. Hardcover. 16.5 x 23.5 cm. 78.00 Euro

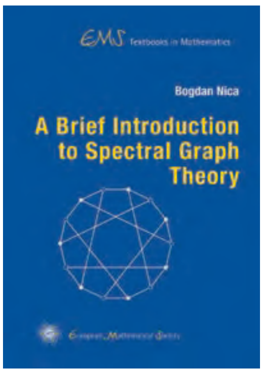
The Vlasov–Maxwell–Boltzmann system is a microscopic model to describe the dynamics of charged particles subject to self-induced electromagnetic forces. At the macroscopic scale, in the incompressible viscous fluid limit, the evolution of the plasma is governed by equations of Navier–Stokes–Fourier type, with some electromagnetic forcing that may take on various forms depending on the number of species and on the strength of the interactions. From the mathematical point of view, these models have very different behaviors. Their analysis therefore requires various mathematical methods which this book aims at presenting in a systematic, painstaking and exhaustive way.
The first part of this work is devoted to the systematic formal analysis of viscous hydrodynamic limits of the Vlasov–Maxwell–Boltzmann system leading to a precise classification of physically relevant models for viscous incompressible plasmas, some of which have not been previously described in the literature. In the second part, the convergence results are made precise and rigorous, assuming the existence of renormalized solutions for the Vlasov–Maxwell–Boltzmann system. The analysis is based essentially on the scaled entropy inequality. The third and fourth parts will be published in a second volume.



Vladimir Maz'ya (Linköping University, Sweden and University of Liverpool, UK)
Boundary Behavior of Solutions to Elliptic Equations in General Domains (EMS Tracts in Mathematics)

ISBN 978-3-03719-190-3. 2018. 441 pages. Hardcover. 17 x 24 cm. 78.00 Euro

The present book is a detailed exposition of the author and his collaborators' work on boundedness, continuity, and differentiability properties of solutions to elliptic equations in general domains, that is, in domains that are not a priori restricted by assumptions such as "piecewise smoothness" or being a "Lipschitz graph". The description of the boundary behavior of such solutions is one of the most difficult problems in the theory of partial differential equations. After the famous Wiener test, the main contributions to this area were made by the author. In particular, necessary and sufficient conditions for the validity of imbedding theorems are given, which provide criteria for the unique solvability of boundary value problems of second and higher order elliptic equations. Another striking result is a test for the regularity of a boundary point for polyharmonic equations.
The book will be interesting and useful for a wide audience. It is intended for specialists and graduate students working in the theory of partial differential equations.



Bogdan Nica (McGill University, Montreal, Canada)
A Brief Introduction to Spectral Graph Theory (EMS Textbooks in Mathematics)

ISBN 978-3-03719-188-0. 2018. 168 pages. Hardcover. 16.5 x 23.5 cm. 38.00 Euro

Spectral graph theory starts by associating matrices to graphs – notably, the adjacency matrix and the Laplacian matrix. The general theme is then, firstly, to compute or estimate the eigenvalues of such matrices, and secondly, to relate the eigenvalues to structural properties of graphs. As it turns out, the spectral perspective is a powerful tool. Some of its loveliest applications concern facts that are, in principle, purely graph theoretic or combinatorial.
This text is an introduction to spectral graph theory, but it could also be seen as an invitation to algebraic graph theory. The first half is devoted to graphs, finite fields, and how they come together. This part provides an appealing motivation and context of the second, spectral, half. The text is enriched by many exercises and their solutions.
The target audience are students from the upper undergraduate level onwards. We assume only a familiarity with linear algebra and basic group theory. Graph theory, finite fields, and character theory for abelian groups receive a concise overview and render the text essentially self-contained.