NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Feature
Dynamics of Random Interfaces and Hydrodynamic Limits

Obituary
Reuben Hersh

Society
The Spanish Society of Statistics and Operations Research

European Mathematical Society

June 2020
Issue 116
ISSN 1027-488X

Marseille, venue of the EMS Executive Committee Meeting planned for March, 2020

8TH EUROPEAN CONGRESS OF MATHEMATICS

8ECM 2020 PORTOROŽ

20 - 26 JUNE 2021

PORTOROŽ SLOVENIA

Plenary Speakers

Peter Bühlmann
Xavier Cabré
Franc Forstnerič
Alice Guionnet
Gitta Kutyniok
Monika Ludwig
Janos Pach
Alfio Quarteroni
Karl-Theodor Sturm
Umberto Zannier

Public Speakers

Bojan Mohar
Sir Vaughan F. R. Jones
Andrei Okounkov
Stanislav Smirnov
Martin Hairer
Kathryn Hess
Robin Wilson

EMS Prize winners

Karim Adiprasito
Ana Caraiani
Alexander Efimov
Simion Filip
Aleksandr Logunov
Kaisa Matomäki
Phan Thành Nam
Joaquim Serra
Jack Thorne
Maryna Viazovska

Felix Klein Prize winner

Arnulf Jentzen

Invited Speakers

Andrej Bauer
Yves Benoist
Robert Berman
Martin Burger
Albert Cohen
Marius Crainic
Mirjam Dür
Alexander Efimov
Alison Etheridge
Rupert Frank
Aleksey Kostenko
Emmanuel Kowalski
Daniel Kressner
Daniela Kühn
Eugenia Malinnikova
Domenico Marinucci
Eva Miranda
Richard Nickl
Burak Özbağcı
Ilaria Perugia
Gabriel Peyré
Yuri Prokhorov
Alexander A. Razborov
Aner Shalev
László Székelyhidi
Špela Špenko
Anna-Karin Tornberg
Nick Trefethen, FRS
Maryna Viazovska
Stuart White

Otto Neugebauer Prize winner

Karine Chemla

WWW.8ECM.SI

Media Partner

videolectures.net
exchange ideas & share knowledge

Main Sponsor

Slovensko Zavarovalno Združenje
Slovenian Insurance Association

European Mathematical Society

Newsletter No. 116, June 2020

The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

ISSN 1027-488X

Published by the
EMS Publishing House
TU Berlin, Mathematikgebäude, Room MA266
Strasse des 17. Juni 137
10623 Berlin, Germany
homepage: https://ems.press

For advertisements and reprint permission requests contact: newsletter@ems.press

Scan the QR code to go to the Newsletter web page:
http://euro-math-soc.eu/newsletter

EMS Executive Committee

President

Prof. Volker Mehrmann
(2019–2022)
Technische Universität Berlin
Sekretariat MA 4-5
Straße des 17. Juni 136
10623 Berlin, Germany
e-mail: mehrmann@math.tu-berlin.de

Vice-Presidents

Prof. Armen Sergeev
(2017–2020)
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8
119991 Moscow
Russia
e-mail: sergeev@mi.ras.ru

Prof. Betül Tanbay
(2019–2022)
Department of Mathematics
Bogazici University
Bebek 34342 Istanbul
Turkey
e-mail: tanbay@boun.edu.t

Secretary

Prof. Sjoerd Verduyn Lunel
(2015–2022)
Department of Mathematics
Utrecht University
Budapestlaan 6
3584 CD Utrecht
The Netherlands
e-mail: s.m.verduynlunel@uu.nl

Treasurer

Prof. Mats Gyllenberg
(2015–2022)
Department of Mathematics
and Statistics
University of Helsinki
P.O. Box 68
00014 University of Helsinki
Finland
e-mail: mats.gyllenberg@helsinki.fi

Ordinary Members

Prof. Jorge Buescu
(2019–2022)
Department of Mathematics
Faculty of Science
University of Lisbon
Campo Grande
1749-006 Lisboa, Portugal
e-mail: jsbuescu@fc.ul.pt

Prof. Nicola Fusco
(2017–2020)
Dip. di Matematica e Applicazioni
Complesso Universitario di
Monte Sant' Angelo
Via Cintia
80126 Napoli
Italy
e-mail: n.fusco@unina.it

Prof. Stefan Jackowski
(2017–2020)
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa
Poland
e-mail: sjack@mimuw.edu.pl

Prof. Vicente Muñoz
(2017–2020)
Departamento de Algebra,
Geometría y Topología
Universidad de Málaga
Campus de Teatinos, s/n
29071 Málaga
Spain
e-mail: vicente.munoz@uma.es

Prof. Beatrice Pelloni
(2017–2020)
School of Mathematical &
Computer Sciences
Heriot-Watt University
Edinburgh EH14 4AS
UK
e-mail: b.pelloni@hw.ac.uk

EMS Secretariat

Ms Elvira Hyvönen
Department of Mathematics
and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
00014 University of Helsinki
Finland
Tel: (+358) 2941 51141
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dr. Richard H. Elwes
School of Mathematics
University of Leeds
Leeds, LS2 9JT
UK
e-mail: R.H.Elwes@leeds.ac.uk

EMS Agenda

2020

29 October
EMS 30 Years Anniversary Celebration
Edinburgh, UK

30 October – 1 November
EMS Executive Committee Meeting
Edinburgh, UK

EMS Scientific Events

2020

The events below may be subject to change due to the Covid-19 measures

6 – 10 July
28th International Conference on Operator Theory
Timisoara, Romania

23 – 30 August
Helsinki Summer School on Mathematical Ecology and Evolution
Turku, Finland

14 – 18 September
IAMP-EMS summer school "Quantum information in many-body physics: a mathematical invitation"
Technical University of Munich, Germany

22 – 24 September
"The Unity of Mathematics", Conference in Memory of Sir Michael Atiyah
Isaac Newton Institute, Cambridge, UK

2021

20 – 26 June
8th European Congress of Mathematics
Portorož, Slovenia

A Message from the President: EMS and COVID-19

Volker Mehrmann, President of the EMS

Dear EMS members,

In these unusual times of the COVID-19 crisis with so many tragic losses all over the world and, in particular, also all over Europe, with health systems coming to their limits, with lockdowns of whole countries, with the closing of schools and universities, with pure online-teaching, and many personal tragedies and inconveniences, also the EMS is challenged.

First of all the European Congress of Mathematics 8ECM in Portorož, Slovenia had to be postponed and is now planned for June 20–26, 2021 (see https://www.8ecm.si/). In view of this decision we have already announced the 2020 prize winners (https://www.8ecm.si/news/69).

Furthermore, the planned EMS council cannot take place in Bled in July but will be carried out in an online format on Saturday, July 4, 2020 to decide about future vice presidents, executive committee members, the budget for the years 2020/21 and the place of the next congress in 2024 (here Sevilla and Lisbon are the candidates).

If possible, we plan a restart of the non-virtual EMS activities and have an in-person council meeting in Edinburgh on October 30, 2020, where we will discuss the plans for the future of EMS that include the formation of activity groups, the organization of specialized meetings, and the start of a young academy. This meeting will be combined with the executive and presidents meetings that had to be cancelled in March.

We are also planning a celebration of the 30th anniversary of the EMS with a one-day meeting in Edinburgh on October 29, 2020. More details about these events will be announced in due time.

As we all know, a crisis always also offers a chance for positive developments. As we have noticed in the crisis, mathematics is entering every-day life much more than usual, with growth models, statistical analysis, infection prognostics, etc. Mathematical models are used for political decision making more than ever, but very little is reported about uncertainty in the data and the models. This development is a chance for mathematics but also bears the risk that we later go back to the times where scientific results are questioned because of this uncertainty. To my opinion it is the duty of the mathematical community to increase research in this direction and also to inform the public about the results. The EMS has already started initiatives in this direction, the Applied Mathematics Committee is forming a special research group on COVID-19 related mathematical research and the Committee for Raising Public Awareness is informing the public about the mathematics behind the pandemic crisis. But much more is needed and I appeal to the mathematical community to get more active in this respect.

With this appeal I wish you all the best, in particular good health, and hope to meet many of you soon again in person.

A Message from the Editor-in-Chief: EMS Newsletter – to a New Future

Valentin A. Zagrebnov, Editor-in-Chief of the EMS Newsletter

Dear Readers,
Dear Members of the European Mathematical Society,

Here I am with my farewell message as Editor-in-Chief of the EMS Newsletter. I prepared this article in April, whereas you will not read it until the June issue (No. 116). Therefore, this message resumes my and the Editorial Team's activities at the time when only the March issue (No. 115) was available.

To keep it short, I basically present below only my Report addressed to the EMS Executive Committee that was planned to be held at the *Centre Internationale de Rencontres Mathématiques* (CIRM) – Marseille Luminy, on 13–16 March 2020. Since on account of the COVID-19 epidemic the CIRM was closed at the beginning of March, all scientic events there had been annulled. The EMS Executive Committee meeting was also cancelled and switched to a video conference. The cover photograph with a view of the Vieux-Port de Marseille and Notre Dame de la Garde, prepared on the occasion of this meeting for the March issue, did not appear then either. Now, however, you can admire this sunny picture as the cover page of the current June issue (No. 116).

Report to the EMS Executive Committee, 13–16 March, Marseille

General comments
The latest issue, i.e. No. 115, is available on line: https://www.ems-ph.org/journals/newsletter/pdf/2020-03-115.pdf. There are no special comments concerning the fifteenth issue published since the beginning of my term on the 1st of July 2016.

Editorial Team
The table below summarises the composition of the current EMS Newsletter Editorial Team:

Rotation of the Editorial Team
I strongly suggest that the EMS Executive Committee invite M.Th. Rassias and F.P. da Costa to continue their very efficient activity as members of the Editorial Team for the next 4-year term starting on 01 January 2021.

In turn, I am leaving the Editor-in-Chief position on the 30th of June 2020 after publishing the EMS Newsletter No. 116 (June 2020).

Reciprocity agreement and other projects
The reciprocity agreement with the SMF *La gazette des mathématiciens* and the contacts with the *Société Mathematique de France* are still efficient due to important activity by J.-B. Bru and F.P. da Costa. The new Editor-in-Chief of the SMF *La gazette des mathématiciens*, Professor Damien Gayet, has confirmed the continuity of this agreement.

Addendum
Since the 1st of July 2016, the EMS Newsletter has a reciprocity agreement with the *News Bulletin of the International Association of Mathematical Physics*. Editor-in-Chief of the *News Bulletin*, Professor Evans Harrell, has also confirmed the continuity of this agreement.

Translations for *Mathematical Advances in Translation* (PRC) and for the Chinese mathematical journal (Taipei) are now running at a regular level. This was confirmed during my visit in April 2019 to the Mathematical Institute of CAS in Peking by Professor Lingzhi Li, Editor-in-Chief of the *Mathematical Advances in Translation*. Recently, the interest in continuing this collaboration was confirmed by Ms. Xinli Pi, a new contact person responsible for *Mathematical Advances in Translation*.

A new era
It began in 2019, when the founder and Managing Director of the EMS Publishing House in Zürich, Thomas Hintermann, retired at the end of August. The last three issues of the EMS Newsletter, i.e., No. 114–No. 116 have already appeared within the new EMS Publishing House in Berlin under the leadership of André Gaul and his team.

Within this impetus, a number of projects planned by the EMS President Volker Mehrmann, together with the Executive Committee and the new EMS Publishing House, are already in the stage of realisation. One of these projects is preparing a new future for the EMS Newsletter in 2020. I wish great success to all persons involved in the construction of this promising future.

Acknowledgements:
My entering into the Editor-in-Chief activity for the EMS Newsletter was initiated in June 2016 by the former President of the European Mathematical Society, Pavel Exner. I am thankful to him for this opportunity to be useful to the EMS in this way and to all my colleagues not indifferent to the life and development of our Society.

The success of the EMS Newsletter has always been due to the hard work of the Editorial Team. I want to thank all the former and present members, and especially the active members amongst them, for a collective effort that produces a volume every three months with about sixty pages of interesting and attractive contributions.

My special thanks I would like to address to Sylvia Fellmann Lotrovsky for her very efficient assistance and prompt support, for her clever suggestions and patience in solving delicate problems. I very much appreciate her permanent aid from the issue No. 101 up until this one, No. 116.

Current composition of the Editorial Board of the EMS Newsletter.

Name	Begin	End	Notes	Terms
Valentin A.Zagrebnov	2016-07-01	2020-06-30	Editor-in-Chief	4
Michael Th. Rassias	2017-01-01	2020-12-31	Problem corner, Interviews	4
Fernando P. da Costa	2017-01-01	2020-12-31	Societies	4
Dierk Schleicher	2017-01-01	2020-12-31	Features, Discussions	4+4
Jean-Paul Allouche	2017-01-01	2020-12-31	Book reviews	4+4
Volker Remmert	2018-01-01	2021-12-31	History, Archives	4+4
Gemma Huguet	2018-01-01	2021-12-31	Research Centres	4
Octavio Paniagua Taboada	2018-01-01	2021-12-31	zbMATH	4
Jean-Bernard Bru	2018-01-01	2021-12-31	Contact with SMF, YMCo	4
Ulf Persson	2018-06-01	2022-06-01	Social Media	4
Jean-Luc Dorier	2015-01-01	2022-12-31	Education	4+4
Vladimir Popov	2015-01-01	2022-12-31	Features, Discussions	4+4
Vladimir Salnikov	2015-01-01	2022-12-31	YMCo	4+4

Dynamics of Random Interfaces and Hydrodynamic Limits

Fabio Lucio Toninelli (Technical University of Vienna, Austria)

1 Introduction

In this article, we will present a panorama of mathematical results and open problems concerning the large-scale properties, both in space and time, of the dynamics of random interfaces. This is a very broad field of research, whose motivation comes from physics [1], and which involves several branches of mathematics, notably probabilities (Markov processes) and analysis (deterministic and stochastic PDEs, calculus of variations), but also combinatorics (random tilings of the plane, Schur processes) and probabilistic algorithms (Markov "Monte Carlo" chains). We will only focus on a few selected aspects.

To introduce the topic, we would first like to take two examples from everyday life which illustrate the type of problems we are talking about. We will then move on to the mathematical modelling of such physical phenomena. As a first example, let us imagine the growth of the snow cover in a garden on a winter's day. Although the trajectory of individual flakes is an essentially random process, if we look at the landscape from a certain distance, we will get the impression that the snow height profile grows in a manner that is regular enough to form a smooth surface. However, if you look closer, you will notice a much more irregular and rough structure, on a larger scale than the typical size of the individual flakes, because of the fact that the flakes do not spread out uniformly on the surface. Note that there are growth phenomena that are of much greater practical importance to study, such as the growth of bacterial colonies in biology or the epitaxial growth of crystals by molecular jet deposition in solid-state physics [1]. Our second example concerns the coexistence of thermodynamic phases. For example, we imagine a container at a temperature of 0° C containing ice cubes floating in water. Although at this temperature both ice and water are in a thermodynamically stable state, we can observe a temporal evolution of the shape of the ice cubes (and thus of the water/ice interface). Once again, the evolution appears regular and deterministic on a large scale, and essentially random if we observe the interface under the microscope. Note that this second example is of a very different nature: if the former is a growth phenomenon (the snow cover increases with time), in the latter the water/ice interface is in a state of equilibrium because water and ice are both stable at the particular temperature of 0° C. This distinction between two very different physical situations will have repercussions on the mathematical models that we will introduce in the following.

The aim of this research area is to understand both the macroscopic (deterministic) evolution on a large scale and the fluctuations around this macroscopic behaviour. Note that the point of view of the mathematician working on these problems is to abstract from the microscopic details of physical or specific biological systems and try to extract "universal" types of behaviour.

Obviously, there is no way to study such physical phenomena on the basis of the fundamental laws governing the movement of water molecules or snow flakes. Following the usual spirit of statistical physics, the idea is rather to introduce strongly simplified mathematical models which, however, retain the essential qualitative aspects of real systems. First of all, the corresponding interface is usually modelled by a real-valued (or, often, integer-valued) *height function* $h(x)$, where the spatial coordinate x takes its values in a d-dimensional network $x \in \mathbb{Z}^d$. $h(x)$ should be interpreted as the vertical coordinate of the interface above point x. Note that this type of description is already a huge simplification, not only because physical space is not discrete, but also because any realistic interface will have "overhangs" that prevent us from identifying it with the graph of a function h. As to the dimension d, the physically most intuitive case is that of $d = 2$, in which case the height function describes a two-dimensional interface in the usual three-dimensional space. However, it is interesting to consider the case of any dimension, not only for the sake of mathematical generality, but also because, for example, the one-dimensional case $d = 1$ is very rich in physical applications. For example, in the case of a burning sheet of paper, the combustion front is a one-dimensional interface that propagates in a two-dimensional space (the sheet).

A second crucial simplification is to assume that the temporal evolution of the height function h is a stochastic process, and more particularly a time homogeneous Markovian process. The interface will therefore be described by a random function $h(x, t)$, with $t \geq 0$ being a continuous parameter that designates time: the Markov property implies that the probability of transition from a configuration $h_A = \{h_A(x)\}_{x\in\mathbb{Z}^d}$ at time t_A to a configuration $h_B = \{h_B(x)\}_{x\in\mathbb{Z}^d}$ at time $t_B > t_A$ depends only on h_A, h_B and $t_B - t_A$, and not on the whole history of the system between times 0 and t_A, nor on the individual values of t_A, t_B. Conceptually, it may seem very risky to replace the deterministic laws of physics by random laws of evolution, but this corresponds exactly to the usual approach of statistical physics to replace the Hamiltonian (deterministic) equations of a set of molecules by a stochastic process. Since the "true" interactions between elementary components (molecules, snow flakes...) of the physical system are essentially local (i.e., of short range) in space, we will choose a Markovian process whose elementary transitions are themselves local: for example, during an elementary transition $h \mapsto h'$, the height at a certain point x changes from the value $h(x)$ to the value $h'(x) = h(x) + n, n \in \mathbb{Z}$, while the other values remain unchanged. We denote by $c(h \to h') > 0$ the

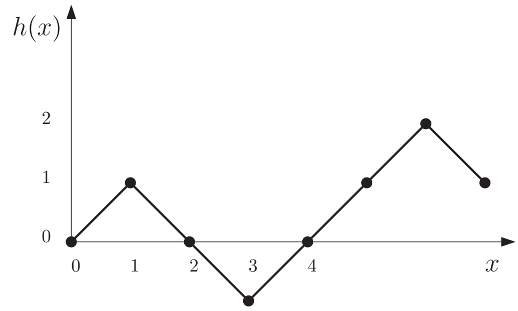


Figure 1. A portion of the height profile of the simple exclusion model

infinitesimal transition rate, i.e., the probability of transition between h et h' in an infinitesimal time interval $[0, \delta]$ will be given by $c(h \to h')\delta + O(\delta^2)$. To respect the local nature of the physical interactions, it will be assumed that the transition rate depends only on the configuration of h around point x, rather than on the function h as a whole.

In order to make this discussion more concrete, we will first describe the most canonical and best studied example of stochastic interface evolution [9]: the one-dimensional simple exclusion process. In this model, the configurations of allowed interfaces $h = \{h(x)\}_{x\in\mathbb{Z}}$ are the following: each height $h(x)$ takes its values in $\mathbb{Z}$ and its gradients $h(x) - h(x-1)$ take only the values $+1$ or -1. One can thus visualise such an interface as a one-dimensional zigzag path in two-dimensional space, as in Figure 1.

Now let us describe the associated Markov process. Transitions occur at real random times, and each is an elementary transition of one of two types:

- If $h(x-1) = h(x+1) = h(x)+1$, i.e., if the height has a local minimum in x, then we can have the transition $h \to h^{(x,+)}$, where $h^{(x,+)}(y) = h(y)$ for all $y \neq x$ while $h^{(x,+)}(x) = h(x)+2$ (the local minimum in x becomes a local maximum).
- Inversely, if $h(x-1) = h(x+1) = h(x) - 1$ (local maximum in x) then we can have the transition $h \to h^{(x,-)}$, where we let $h^{(x,-)}$ be the configuration where the height in x is decreased by 2, while it is unchanged everywhere else.

We still have to specify the rates associated with these two types of transitions. Given a real number $p \in [0, 1]$, we will assign the rate $c(h \to h^{(x,+)}) := p$ to transitions that increase the interface, and the rate $c(h \to h^{(x,-)}) := 1 - p$ to those that decrease it. The special case $p = 1/2$ is known as the *Symmetric Simple Exclusion Process* (SSEP), the case $p \neq 1/2$ is the *Asymmetric Simple Exclusion Process* (ASEP) and the special case $p \in \{0, 1\}$, where only transitions of one of the two types are allowed, is known as *Totally Asymmetric Simple Exclusion* (TASEP).

An equivalent description of the simple exclusion process is as follows: each x has an independent random clock which rings at random exponential time intervals of mean 1. If the clock in x rings at time t, a coin is tossed which gives $\wedge$ with probability p and $\vee$ with probability $1 - p$. If we obtain $\wedge$ and the height function at time t has a local minimum in x, we then perform the transition $h \mapsto h^{(x,+)}$, otherwise we do nothing; analogously, if we get $\vee$ and the height function at time t has a local maximum in x, we then perform the transition $h \mapsto h^{(x,-)}$, otherwise we do nothing. Note that, since the clocks at two points $x \neq x'$ are independent, we will never (except with zero probability) have two clocks ringing at the same instant of time, and thus each transition changes the height h at a single point x.

As we said at the very beginning of this note, we are interested in the large-scale behaviour of interface dynamics. There are several ways to understand the expression "large-scale behaviour". To understand this point, let us forget for a moment about interface dynamics and think about a much simpler probabilistic question: the asymptotic behaviour for large N of the sum

$$S_N = \sum_{i=1}^{N} X_i$$

of N independent and equally distributed random variables X_i which, for simplicity, are assumed to have a finite variance

$$\sigma^2 := \mathbb{E}(X_1^2) - (\mathbb{E}X_1)^2 < \infty,$$

where $\mathbb{E}(f)$ denotes the mean of a random variable f. The first question is, what is the asymptotic behaviour of $(1/N)S_N$, and the well-known answer is the *law of large numbers*

$$\lim_{N\to\infty} \frac{S_N}{N} = \mathbb{E}(X_1),$$

where the convergence of the random variable S_N/N towards the deterministic quantity $\mathbb{E}(X_1)$ is valid with probability 1. Next, there is the question of what the fluctuations are of S_N/N around its deterministic limit: we will obtain the *central limit theorem*

$$\sqrt{N}\left(\frac{S_N}{N} - \mathbb{E}(X_1)\right) \overset{N\to\infty}{\Rightarrow} \mathcal{N}(0, \sigma^2)$$

where $\mathcal{N}(0, \sigma^2)$ is a random normal (i.e., Gaussian) centered variable of variance σ^2 and "$\Rightarrow$" means that the law of the random variable on the left tends towards the law of the random variable on the right. If the central limit theorem describes typical or "normal" fluctuations of S_N/N at scale $1/\sqrt{N}$, we can also be interested in atypical deviations (say, deviations of order 1). We will then have statements of the "large deviations" type:

$$-\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(\left|\frac{S_N}{N} - \mathbb{E}(X_1)\right| \geq \delta\right) = I(\delta)$$

for a certain function $I(\cdot)$ which depends on the law of variables X_i. Here, $\mathbb{P}(A)$ denotes the probability of an event A. Although the evolution of a random interface is something much more complicated than a sum of independent variables, these three types of questions (law of large numbers, central limit theorem, large deviations) are also very natural in this context.

In this discussion, for interface dynamics, we will concentrate mainly on the first two questions ("law of large numbers" and "fluctuations"). In the context of interface dynamics, but more generally of out-of-equilibrium statistical physics, we will rather speak of "hydrodynamic limit" than "law of large numbers".[1] The question can be asked as follows. Suppose

1 The terminology of "hydrodynamic limit" comes from fluid dynamics: if the motion of molecules is governed at the microscopic level by Hamiltonian equations with chaotic and essentially random behaviour, at the laboratory scale we observe the deterministic evolution of a finite number of macroscopic parameters (pressure, density, ...), described by partial differential equations.

that the initial condition of the dynamics, i.e., the configuration $h(\cdot,0) = \{h(x,0)\}_{x\in\mathbb{Z}^d}$ of height at time $t = 0$, approximates a limit height profile $\phi_0 : \mathbb{R}^d \mapsto \mathbb{R}$. More precisely, let us assume that the initial profile $h(\cdot,0)$ satisfies, for all $u \in \mathbb{R}^d$,

$$\lim_{\epsilon\to 0} \epsilon h(\lfloor \epsilon^{-1} u \rfloor, 0) = \phi_0(u). \tag{1}$$

For example, for the simple exclusion process, given a function ϕ_0 of Lipschitz constant inferior to 1, we could choose $h(x,0)$ as being the nearest integer to

$$\frac{1}{\epsilon}\phi_0(\epsilon x)$$

and having the same parity as x (the latter condition ensures that $h(x,0) - h(x-1,0) \in \{-1,+1\}$). We say that a dynamic admits a hydrodynamic limit if there is a *deterministic* function $\phi = \phi(u,t), t \geq 0, u \in \mathbb{R}^d$ and an exponent $\gamma \geq 0$ such that

$$\lim_{\epsilon\to 0} \epsilon h(\lfloor \epsilon^{-1} u \rfloor, \epsilon^{-\gamma} t) = \phi(u,t). \tag{2}$$

Here, the convergence has to be interpreted as a convergence in probability: for all $\delta > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}\left(|\epsilon h(\lfloor \epsilon^{-1} u \rfloor, \epsilon^{-\gamma} t) - \phi(u,t)| \geq \delta\right) = 0.$$

As can easily be imagined, there will only be one choice for the exponent γ that allows us to find a well-defined and nottrivial hydrodynamic limit. For example, if γ is too small (e.g., $\gamma = 0$) we have $\phi(x,t) = \phi(x,0)$ for all t. In all the examples discussed in this note, this exponent is either $\gamma = 1$ or $\gamma = 2$. In the first case, we will say that we have a hydrodynamic limit in the Eulerian (or hyperbolic) scale, while in the second case we will speak of a diffusive scale. Basically, the interface growth models (such as the example of snowfall) correspond to the Eulerian scale, while the interface dynamics models describing the motion of boundaries between coexisting thermodynamic phases (such as the example of the water/ice interface at 0° C) correspond to the diffusive scale.

Let us illustrate this fact by returning to the case of simple exclusion. Let us first take the case $p = 1$ (or more generally $p \neq 1/2$). In this case, the growth of the interface is intrinsically irreversible: only transitions where the height increases are allowed. If we exclude the pathological case where the macroscopic slope ρ of the interface is ± 1, we have a positive density $r = (1+\rho)/2$ of points x with $h(x) - h(x-1) = 1$, and a positive density $r = (1-\rho)/2$ of points x with $h(x) - h(x-1) = -1$. There will thus also be a positive density of points x where $h(\cdot)$ has a local maximum (resp. a local minimum). Since in a time of order 1 any local minimum has a probability of order 1 of becoming a local maximum, it is intuitive that after a time of order t, on average the height function increases by an amount of order t. One must therefore choose $\gamma = 1$ to hope to obtain a nontrivial hydrodynamic limit. The situation is very different for the symmetric exclusion where $p = 1/2$: in this case, since the positive and negative transitions tend to compensate each other (just as ice cubes have no natural tendency to grow or melt at 0° C), one must look at much longer time scales to see a non-trivial evolution. The fact that $\gamma = 2$ is the right choice requires a little more thought and we refer the reader to the discussion in Section 2.

Since we mentioned the word "irreversible", the following observation should be made. Suppose that the height profile $\{h(x)\}_{x\in\mathbb{Z}}$ at time zero is random and in particular that each gradient $h(x) - h(x-1)$ is independent and takes on the value $+1$ with probability r and the value -1 with probability $1-r$, for a certain $r \in (0,1)$. It is easy to verify that, both for the simple symmetric and for the asymmetric exclusion, such a distribution is invariant for the dynamics: the law of gradients $h(x,t) - h(x-1,t)$ will be exactly the same at time $t > 0$. It is thus a stationary state (or *invariant measure*: we will use these two terms as synonyms). What distinguishes the symmetric case $p = 1/2$ from the asymmetric case is therefore not the invariant measure, but the reversibility. Let us suppose that at initial time (and thus for all times) the gradients are distributed according to the stationary measure, and compare the law of the gradient process $\{h(x,t) - h(x-1,t)\}_{x\in\mathbb{Z},t\geq 0}$ and that of the process after time reversal, $\{h(x,-t) - h(x-1,-t)\}_{x\in\mathbb{Z},t\geq 0}$. It turns out that these two processes are governed by the same law (i.e., the process is reversible) if and only if $p = 1/2$.

It's natural to ask what kind of evolution will follow the "hydrodynamic" profile $\phi(u,t)$. If, as for the simple exclusion, we started from a Markovian evolution whose transition rates depend only on the local gradients of the configuration h, we can expect that $\partial_t\phi(u,t)$ depends only on the spatial derivatives of $\phi(u,t)$ at point u. In other words, we expect that ϕ follows a PDE. More precisely, it turns out that:

- in the case $\gamma = 2$ the hydrodynamic PDE will be of the parabolic type:

$$\partial_t\phi(u,t) = \mu(\nabla\phi(u,t)) \sum_{i,j=1}^{d} \sigma_{i,j}(\nabla\phi(u,t))\partial^2_{u_i u_j}\phi(u,t) \tag{3}$$

  with $\mu(\nabla\phi) > 0$ a function which is named "mobility" and $(\sigma_{i,j})_{i,j=1,\dots,d}$ a symmetric and positive definite matrix whose elements are the second derivatives of the surface tension σ of the model. For the reader willing to understand the physical meaning of μ and σ, we refer to the very instructive discussion of [14];
- in the case of growth models where $\gamma = 1$, the PDE will be of the "Hamilton–Jacobi" type

$$\partial_t\phi(u,t) = v(\nabla\phi(u,t)). \tag{4}$$

  The function $v(\cdot)$, generally non-linear, describes the interface growth rate as a function of the local slope $\nabla\phi$.

Note that, as a consequence of (2), the hydrodynamic PDE must be invariant by reparameterisation $\phi(u,t) \mapsto c\phi(c^{-1}u, c^{-\gamma}t)$ for all $c > 0$. Indeed, the two equations (3) and (4) satisfy this property.

Let us return to our analogy with the sum of independent random variables: after the law of large numbers, it is the turn of fluctuations. For diffusive models such as the simple symmetric exclusion or its analogues in dimension $d \geq 1$, let us define the "height fluctuation field" as

$$\hat{h}^{(\epsilon)}(u,t) := \epsilon^{1-d/2}\left\{h(\lfloor \epsilon^{-1} u \rfloor, \epsilon^{-2} t) - \mathbb{E}h(\lfloor \epsilon^{-1} u \rfloor, \epsilon^{-2} t)\right\}. \tag{5}$$

Although $\hat{h}^{(\epsilon)}$ is not a sum of independent variables, its randomness results from the cumulative effect of many random events (the transitions of the dynamics in a time interval of the order ϵ^{-2}); it is therefore natural to expect a central limit theorem, i.e., that $\hat{h}^{(\epsilon)}(u,t)$ tends towards a Gaussian process for $\epsilon \to 0$. More precisely, suppose for simplicity that the initial condition is such that the profile ϕ_0 (cf. (1)) is affine: $\phi_0(u) = \rho \cdot u, \rho \in \mathbb{R}^d$, in such a way that the solution of the hydrodynamic equation (3) is simply $\phi(u,t) = \phi_0(u) = \rho \cdot u$. Under fairly general assumptions, one would expect [13, 14] that the fluctuation field (5) converges, in the sense of distributions, towards the solution $\hat{\phi}$ of the following linear SPDE (stochastic PDE):

$$\partial_t \hat{\phi}(u,t) = \mu(\nabla\phi) \sum_{i,j=1}^{d} \sigma_{i,j}(\nabla\phi) \partial^2_{u_i u_j} \hat{\phi}(u,t) + \sqrt{2\mu(\nabla\phi)} \dot{W}(u,t). \quad (6)$$

This equation requires explanations. First of all, $\dot{W}(u,t)$ denotes the "space-time white noise", i.e., a Gaussian random function (more precisely, a random distribution) indexed by the space u and the time t, of mean zero ($\mathbb{E}\dot{W}(u,t) = 0$) and covariance

$$\mathbb{E}\dot{W}(u,t)\dot{W}(v,s) = \delta(t-s)\delta(u-v).$$

Note, furthermore, that the equation is linear in $\hat{\phi}$, since $\nabla\phi = \rho$ is a constant. For this reason, the solution $\hat{\phi}(u,t)$ is a Gaussian process. The origin of the equation (6) is quite easy to understand intuitively: the first term, proportional to the second derivatives, is obtained by writing the height function h as its hydrodynamic limit $\phi(u,t)$ plus fluctuations $\hat{h}$, and by linearising the hydrodynamic PDE (3) around $\phi(\cdot,\cdot)$. As for the noise, to understand the choice of the factor $\sqrt{\ldots}$, which guarantees the correct stationary measure of the process (6) when $t \to \infty$, we refer to the discussion in [13, Sec. II.3.5] done in a similar context.

The reader might expect that for growth models (such as TASEP, for example), the fluctuation field also tends towards the solution of a certain SPDE. However, this is not generally the case, and indeed the mathematical (and even heuristic) understanding of this question is still very incomplete. We are going to explain a little more about this in Section 3.

The remainder of this paper is organised as follows. In Section 2 we will discuss reversible interface dynamics, which correspond to $\gamma = 2$ and generalise our elementary example of symmetric simple exclusion. In Section 3 we discuss growth models. In particular, we will focus on the case of $d = 2$, where there is a non-trivial relationship between the interface fluctuations and the convexity properties of the function $v : \mathbb{R}^d \mapsto \mathbb{R}$ that appears in the PDE (4).

To conclude this introduction, it is important to make two remarks:

- As is often the case in statistical physics, on the basis of physical intuition, some phenomena are expected to be qualitatively "universal", i.e., independent of the models' details. In our context, this is the case, for example, for the convergence towards hydrodynamic PDEs of the type (3)–(4), or the convergence of fluctuations towards the SPDE (6) in the diffusive case. However, it is only in very specific cases that mathematical proofs can be obtained. For example, with respect to the convergence of the fluctuation field to the solution of (6) for interface dynamics in dimension $d > 1$, the reference [7] is essentially the only known result.
- Most of the known mathematical results in this area are specific to the case of the spatial dimension $d = 1$. The results in dimensions $d \geq 2$ are much rarer and we will mention some of the most recent ones.

2 Reversible interface dynamics

Until quite recently, the only example of reversible interface dynamics in dimension $d > 1$ for which a hydrodynamic limit of the type (2)–(3) was mathematically proven was the Ginzburg–Landau type gradient model with symmetric and convex potential [6]. This is a model of a very different nature from simple exclusion, since the height variables take on continuous real values and their evolution does not proceed by discrete jumps, but follows a "Langevin" type dynamic. This means that each height $h(x)$ is subject to independent Brownian noises, with a drift that depends on the differences $h(x) - h(y)$ for all y neighbouring x and that acts as a restoring force that tends to flatten the interface by penalising large gradients. The limit PDE obtained is of a very special type, since the mobility coefficient $\mu(\cdot)$ in (3) is constant. It is natural to wonder if there are other examples of non-trivial dynamics in dimension $d > 1$ where the mathematical proof of (2)–(3) can be obtained, and in particular if we can find examples – if possible somewhat closer in spirit to the simple exclusion – with non-constant mobility $\mu(\cdot)$.

To do this, take for a moment our example of simple symmetric exclusion and suppose that the interface is defined not on all $\mathbb{Z}$ but on a one-dimensional torus of size $1/\epsilon$, which is supposed to be an even integer. In other words, the height function is $\{h(x)\}_{x=0,\ldots,1/\epsilon}$ with the periodicity constraint $h(0) \equiv h(\epsilon^{-1})$ and, as before, $h(x) - h(x-1) = \pm 1$. Given the height function $h(\cdot,t)$ at a time t, we calculate its average value at time $t + \delta$, with δ being infinitesimal. If the interface has a local minimum (respectively a local maximum) at point x, i.e., if $\Delta h(x,t) = +2$ (resp. $= -2$), then the height in x changes by $+2$ (resp. by -2) with probability $p \times (\delta + O(\delta^2)) = \delta/2 + O(\delta^2)$. It can be easily deduced that

$$\frac{d}{ds} \mathbb{E}[h(x,t+s)|h(\cdot,t)]\Big|_{s=0^+} = \frac{1}{2}\Delta h(x,t), \quad (7)$$

where $\mathbb{E}[\ldots|h(\cdot,t)]$ denotes the expectation conditioned to the height configuration at time t. Thus, we have that the time derivative of the height is proportional to the Laplacian of the height itself. Now, if we define $\tilde{h}(u,t) := \epsilon h(\epsilon^{-1}u, t)$ with $u = 0, \epsilon, 2\epsilon \ldots, 1$, the same computation leads to

$$\frac{d}{ds} \mathbb{E}[\tilde{h}(u,t+s)|\tilde{h}(\cdot,t)]\Big|_{s=0^+} = \frac{\epsilon^2}{2}\Delta \tilde{h}(u,t), \quad (8)$$

From there, one can be immediately convinced that the right time scale is diffusive ($\gamma = 2$) and that the hydrodynamic equation (3) of the simple symmetric exclusion will simply be the heat equation:

$$\partial_t \phi(u,t) = \frac{1}{2}\partial_u^2 \phi(u,t).$$

Indeed, it is not difficult to show convergence in the form (2), see e.g. [9, Ch. 4]. The example of the simple symmetric exclusion is instructive but very special: the time derivative of the height is given by a linear operator (the Laplacian) applied to the height itself, as shown in (7).

This type of miracle does not usually happen. Let us try to generalise the simple symmetric exclusion model to the dimension $d = 2$. If the height profile of the one-dimensional case is a zigzag path in the plane, the most natural two-dimensional analogue is a discrete interface like in the Figure 2.

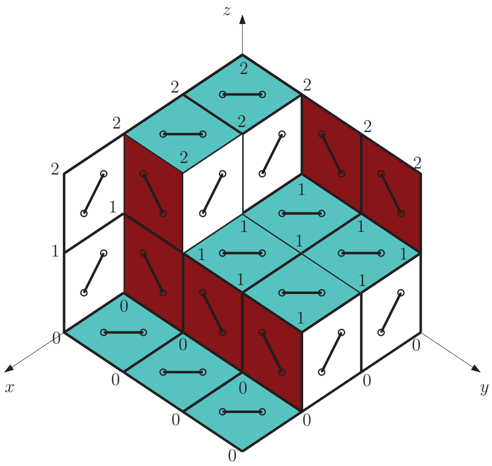


Figure 2. A monotone two-dimensional interface: the height function is (weakly) decreasing in x and y direction. Each level line is a zigzag path as in figure 1. The different contour lines do not intersect.

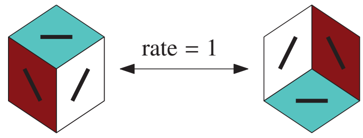


Figure 3. The elementary transitions of the dynamics correspond to adding or removing a unit cube.

There are several equivalent ways of interpreting this image. It can be seen as a tiling of the plane with three types of rhombus-shaped tiles, or as a monotone stack of cubes ("monotone" in the sense that the height of a column of cubes whose base coordinates are (x, y) is decreasing with respect to x and y; it is therefore a *plane partition*), or as a perfect matching of the hexagonal graph (the small segments designate the pairs of vertices that are matched). Note that any section of this interface through a plane perpendicular to one of the three axes gives a zigzag path, thus a height profile of the one-dimensional simple exclusion. However, these one-dimensional sections satisfy non-trivial non-intersection constraints and are therefore not independent.

A Markovian dynamic that naturally generalises the one-dimensional symmetric simple exclusion to the two-dimensional case is the one where the possible elementary transitions correspond to adding or removing an elementary cube, with transition rate 1: see Figure 3. Unfortunately, things are not nearly as easy as they are for the one-dimensional simple exclusion. Let us try to repeat the same calculation as (7). For that, for any vertex x of a rhombus, we note $h(x) \in \mathbb{Z}$ the height of the vertex of the corresponding cube with respect to the horizontal plane, as in Figure 2. We get

$$\frac{d}{ds}\mathbb{E}[h(x, t+s)|h(\cdot, t)]\Big|_{s=0^+} = A_x - B_x,$$

where A_x (resp. B_x) is 1 if the configuration around x at time t is as in the drawing on the right (resp. left) side of Figure 3, and 0 otherwise. There are two important differences as compared to Eq. (7): firstly, we have not obtained a closed equation for the height function; secondly, the result is not given by a discrete second-order operator applied to the height function, so it is not obvious a priori that the hydrodynamic equation (if it exists) is of the parabolic type.

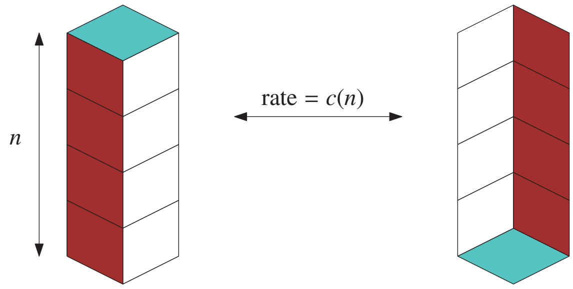


Figure 4

Indeed, it is an open problem (probably very difficult) to show that this dynamic admits a hydrodynamic limit: even from the heuristic point of view, we cannot guess an exact expression for the mobility μ which should appear in (3). We can be less ambitious and hope to show at least that the right time scale is diffusive. More precisely, consider the same tiling dynamics not in the whole plane, but in a two-dimensional domain of size $1/\epsilon$ (e.g., a two-dimensional torus). The only invariant (and reversible) probability measure of this dynamic is the uniform measure, which gives equal weight to any allowed tiling of the domain. Given a small constant δ (say $\delta = 1/100$), we define the *mixing time* T_{mix} of the dynamics as the smallest value of t such that, for any initial condition, the law of the process at time t is at a distance (in total variation) at most δ from the invariant measure. It is reasonable to ask how T_{mix} grows with the size $1/\epsilon$ of the system. If we interpret Markov dynamics as a probabilistic algorithm (of the "Markov Chain Monte Carlo" type) that takes a tiling η_0 at step zero and outputs the tiling η_T at time T, then having an upper bound $T_{mix} \leq N$ implies that the algorithm produces, after a simulation time N, an almost perfectly uniform tiling (up to some δ). This kind of question about the speed of convergence of probabilistic algorithms is very much studied in theoretical computer science. Coming back to our two-dimensional interface dynamics, an important conjecture in this domain is that T_{mix} is of order $\epsilon^{-2+o(1)}$. Under certain assumptions about the shape of the domain, this conjecture has been proven recently [4, 10]. The proof uses large-scale properties of uniform tilings of large planar domains, including the fact that their height fluctuations are only logarithmic with respect to the size of the domain. However, the issue is still far from being solved:

Open problem. For the dynamics restricted to a hexagonal domain of side $1/\epsilon$ (like in Figure 2 where $1/\epsilon = 3$) show that $T_{mix} = O(\epsilon^{-2+o(1)})$ for $\epsilon \to 0$. At present, the best rigorous upper bound is of order $\epsilon^{-4+o(1)}$.

Let us return to the problem of the hydrodynamic limit for the dynamics of monotone two-dimensional interfaces. Since this question is too difficult for the dynamics whose elementary updates are those of the Figure 3, we may ask whether a change in the transition rules for the Markov process can simplify things. For example, one could decide to allow more general transitions where n cubes are added/removed at a time, with a transition rate $c(n)$. See Figure 4. In the recent work [12], we showed, with some heuristic approximations, that if we choose $c(n)$ as being inversely proportional to n, the time derivative of the average height $h(x)$ is given by a discrete second-order operator acting on $h(\cdot)$. This suggested the possibility of obtaining a hydrodynamic limit in the diffusive time scale. In a later paper [11], we indeed succeeded in rigorously proving that for this dynamic there is a convergence as in (2), where ϕ satisfies a non-linear parabolic PDE of the type (3). The PDE is completely explicit and, in particular, the mobility coefficient $\mu(\cdot)$ is non-linear. A study of the hydrodynamic PDE and the specific form of the mobility $\mu(\cdot)$ shows that the equation has remarkable analytical properties: in particular it contracts both the L^1 and L^2 distances between its solutions (which is also true for the heat equation, but not for any equation of the form (3)). We refer the reader to [12] for a discussion of the relationship between these analytical properties and the fact that, for our particular choice of transition rate $c(n) \propto 1/n$, the stochastic process

$$t \mapsto V(t) := \sum_x (h^{(1)}(x,t) - h^{(2)}(x,t))$$

has the property of being decreasing on average (more precisely, it is a supermartingale). Here, $h^{(j)}(\cdot,t)$, $j = 1,2$ denote two height profiles who follow the Markovian dynamics with two different initial conditions $h^{(j)}(\cdot,0)$, for which we assume that $h^{(1)}(\cdot,0) \geq h^{(2)}(\cdot,0)$, while $V(t)$ denotes their volume difference.

3 Models of interface growth

The mathematical study of one-dimensional interface growth models, such as TASEP and its generalisations, has been extremely active for several years. A remarkable aspect is the emergence of universal *non-Gaussian* distributions for height fluctuations $h(x,t) - h(x,0)$ in the long time limit. Moreover, the asymptotic evolution of the fluctuation field is not governed by a SPDE, but a stochastic process ("KPZ fixed point") which is not yet fully understood. We are not going to develop this topic, nor its links with the one-dimensional Kardar–Parisi–Zhang (KPZ) equation (a highly singular non-linear SPDE), or with the eigenvalue statistics of large random matrices, among others, as these topics have already been the subject of several recent review articles, e.g., [5].

Instead, we will talk about growth models in dimension $d = 2$, a much less explored and known field, where new phenomena emerge. We will start with a few generalities. Given a Markovian growth model, such as TASEP for example, there are very natural quantities associated with it. (We stress, however, that in general it is a major mathematical challenge to prove that these quantities are well defined.) Suppose that the initial condition of the dynamics is a deterministic affine profile $h(x)$ of slope $\rho \in \mathbb{R}^d$, i.e., $h(x) = x \cdot \rho$ (one may have to take the integer part if the height function is discrete). It is intuitive (but it may be difficult to show in concrete examples!) that:

- the law of gradients $\{h(x,t) - h(x',t)\}$, with $x \in \mathbb{Z}^d$ and x' which takes its values among the $2d$ neighbors of x, tends for $t \to \infty$ towards a limit law π_ρ, stationary and invariant by translations;
- there is an asymptotic growth rate $v = v(\rho)$, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}(h(x,t) - h(x,0)) = v(\rho)$$

  and the function $v(\cdot)$ is the same as in the hydrodynamic PDE (4);
- the typical fluctuations of $h(x,t) - h(x,0)$ (measured for example by their standard deviation) behave for large t like t^β; $\beta \in \mathbb{R}$ is called the *growth exponent*;
- the typical fluctuations of $h(x,t) - h(y,t)$ for $t \to \infty$ and $|x-y|$ large behave like $|x-y|^\alpha$, with $\alpha \in \mathbb{R}$ being called the *roughness exponent* because the larger the exponent, the greater the fluctuations in the asymptotic height profile, as compared with an affine profile.

In the particular case of the dimension $d = 2$ we have the following conjecture, which links the convexity properties of $v(\cdot)$ to the exponents α, β:

Conjecture 1. *Let λ_1, λ_2 be the eigenvalues of the Hessian matrix D^2v of $v(\cdot)$. If the product $\lambda_1\lambda_2$ is negative or null (so if* $\det(D^2v) \leq 0$*), then $\alpha = \beta = 0$ and the growth of fluctuations for t or $|x-y|$ that tend to infinity is only logarithmic. If on the other hand $\lambda_1\lambda_2 > 0$, then α and β are strictly positive and universal exponents (i.e., independent of the particular values of λ_i).*

Note that λ_i are functions of the slope ρ and therefore the sign of $\det(D^2v)$ (and thus the values of the critical exponents) could in principle depend on ρ.

Conjecture 1 is a fairly typical universality statement in statistical physics: it is a fairly common fact that the microscopic models that describe a certain macroscopic phenomenon (here, the growth phenomenon) are divided into a small number (here, two) of "universality classes". The models in the same class are characterised by the same "critical exponents" (here, α and β) and the class a model belongs to is determined by certain qualitative symmetries (here, the sign of $\lambda_1\lambda_2$, thus the convexity properties of $v(\cdot)$). A small note on nomenclature: we will say that

- a growth model for which $\lambda_1\lambda_2 > 0$ belongs to the *KPZ universality class*
- a growth model for which $\lambda_1\lambda_2 \leq 0$ belongs to the *Anisotropic KPZ (or AKPZ) universality class*.

The above conjecture, which may seem arbitrary at first glance, is based on non-rigorous calculations by physicists [1] and is convincingly confirmed by numerical simulations of several growth models; we refer the reader to [15] for discussions and references. In the remainder of this section, our goal is to give an example of models for each of the two universality classes and to discuss one or two recent mathematical results that shed some light on the conjecture.

To make the discussion more concrete, we discuss two specific two-dimensional growth models. In order to make the link with the discussion of the previous section easier, suppose that the height profiles allowed for our interface are again monotone discrete functions corresponding to rhombus tilings of the plane, like in the Figure 2. In Section 2 our most natural candidate for a *reversible* dynamic for this interface was the Markovian process whose elementary transitions are those of the Figure 3, with symmetric rates. On the other hand, the natural candidate for a growth process is to allow only one of these two types of transition, for example one that adds an elementary cube, see Figure 5. This growth model is attractive from several points of view:

- it is very easy to be numerically defined, visualised and simulated;
- it is not difficult to show, by a super-additivity argument,

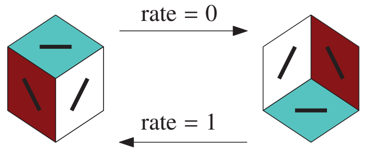


Figure 5. If we allow only those transitions that add a cube, we obtain a growth model that belongs to the KPZ universality class.

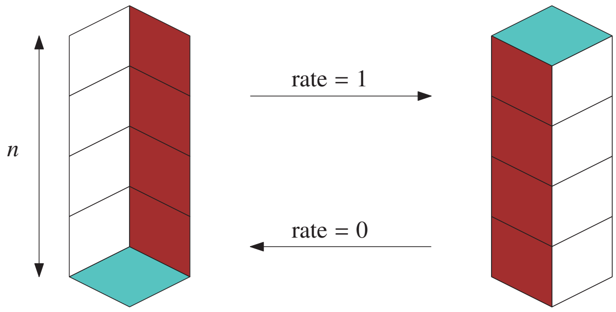


Figure 6. In the growth model studied in [2] we allow transitions that add *n* cubes at a time and the transition rate does not depend on *n*.

that the limiting velocity $v(\cdot)$ exists, *is convex*, and that the height profile satisfies a hydrodynamic limit as in (2) at Eulerian scale $\gamma = 1$, where the limiting PDE is given by (4).

Two remarks: the convexity of $v(\cdot)$ implies that its Hessian matrix will be positive definite and (assuming that its eigenvalues are strictly positive, which has not been rigorously shown because $v(\cdot)$ is not explicitly known) we have a candidate for the KPZ universality class. Secondly, it is well known that non-linear Hamilton–Jacobi equations of the type (4) develop singularities in finite time; however, when $v(\cdot)$ is convex, a natural way to obtain a (weak) solution defined globally in time is through the Hopf–Lax variational formula. Indeed, the super-additivity argument mentioned above gives the convergence of the height profile precisely towards this solution! Unfortunately, the mathematical study of this growth process stops here: we have no idea of how the translation invariant, stationary states π_ρ look like, there are no mathematical results on the critical exponents α, β (although they are known with great precision on the basis of numerical simulations!), nor an explicit expression for $v(\cdot)$. Indeed, we do *not* know *any mathematical result* on stationary states or on critical exponents for *any* two-dimensional growth model of the KPZ universality class.

As in the case of reversible dynamics, one may wonder whether things change qualitatively if we modify the transition rules of the Markov process a little. One possibility is generalising the growth process by allowing one, during each transition, to add $n \geq 1$ cubes at a time, as in the Figure 6. Note that the transition rate is chosen here independently of the value of n. This growth process was introduced in [2] and, quite surprisingly, it turns out that (see [15] for precise bibliographic references):

(i) the speed $v(\cdot)$ can be calculated explicitly and its Hessian has two eigenvalues of opposite sign;
(ii) the critical exponents α and β are zero;
(iii) the stationary states π_ρ can be explicitly determined;
(iv) under certain restrictions on the initial height profile, one can show convergence (in the sense of (2), with $\gamma = 1$) to the viscosity solution of the PDE (4). (The viscosity solution is a weak solution which simply reduces to the Hopf–Lax solution in the case of where $v(\cdot)$ is convex.)

This growth model is therefore a representative of the Anisotropic KPZ universality class. If the reader is confused by the fact that an apparently minor detail of the growth process (the possibility of adding only one cube at a time or several) may change the qualitative properties of the behaviour on a large scale, it can be reassuring that this is not intuitive even for experts. Indeed, it is not generally easy to guess the universality class of a growth model a priori, just by starting from its definition. On the other hand, it is important to mention that the "anisotropic" model we have just discussed is not an isolated example in the literature. Indeed, there is a whole class of growth models [2] that can be formulated in terms of Schur processes, a combinatorial object introduced by Okounkov and Reshetikhin in relation to plane partitions. The specific model described above is a special case of this class, and results (i)–(iv) have been rigorously demonstrated for several others.

Without going into technical details, it should be pointed out here that a peculiarity of the growth models of [2] (such as the one in Figure s6) which makes them amenable to mathematical analysis is that their stationary measures π_ρ have the following determinantal property. Given n points $x_1, \ldots, x_n$ of the plane and n colors $c_1, \ldots, c_n$, where every c_i is one of the three colors of the tiles of a rhombus tiling, let $A = A(x_1, c_1; \ldots; x_n, c_n)$ be the event: "there is a rhombus of colour c_i at point x_i, for all $i \leq n$". Then, the probability of A is given by the determinant of an $n \times n$ matrix, whose elements are Fourier coefficients of a certain explicit function, defined on the two-dimensional torus $[0, 1]^2$. This remarkable determinantal property also extends to spatio-temporal correlations in the sense that, under certain restrictions, the probability of events of the type: "there is a rhombus of colour c_i at point x_i and at time t_i for all $i \leq n$" is also given by a determinant.

In conclusion, it can be said that the mathematical understanding of the growth models of the Anisotropic KPZ class is much more satisfactory than that of the models of the KPZ class as a whole. However, there are two aspects that have remained rather mysterious until very recently:

1. Firstly, for all models of the AKPZ class for which a rigorous study is possible, the proof that $\lambda_1\lambda_2 \leq 0$ requires an explicit and very unenlightening calculation of the second derivatives of $v(\cdot)$, which must be done on a case-by-case basis, and which is based on the explicit formula of $v(\cdot)$. However, according to conjecture 1 formulated above, the negativity of this product should be linked to qualitative properties of the model.
2. In addition, it often happens that although the function $v(\rho)$ has a very complicated expression in terms of the slope ρ, it can be rewritten as a very simple harmonic function in terms of a natural complex variable $z(\rho)$, introduced by Kenyon and Okounkov [8] in the context of dimer models.

In a very recent work [3] we shed some light on these two points and we understood that, for all the growth models of the Anisotropic KPZ class known so far, the fact that $\lambda_1\lambda_2 \leq 0$ and the harmonicity property of $v(\cdot)$ have a rather simple geometrical origin. We give here the main steps of our reasoning:

(1) First of all, one naturally associates a continuous family $\mathcal{F}$ of probability measures π with these growth models (this family includes the steady states π_ρ but also measures that are not invariant by translation). In general, $\pi \in \mathcal{F}$ *is not* a stationary state of the Markov process. However, if the initial condition is distributed according to $\pi^{(0)} \in \mathcal{F}$, then at any time $t > 0$, the law $\pi^{(t)}$ of the interface still belongs to $\mathcal{F}$.

(2) The second observation is that a typical height profile $\{h(x)\}_{x\in\mathbb{Z}^2}$, which is randomly drawn according to $\pi \in \mathcal{F}$, is close on a large scale to a minimizer ϕ_0 of a functional of the type

$$\int_{\mathbb{R}^2} \sigma(\nabla\phi)dx \tag{9}$$

where $\sigma(\cdot)$ is a convex function (surface tension). More explicitly, the fact that $h(\cdot)$ and $\phi_0(\cdot)$ are "close on a large scale" means that, with probability 1,

$$\lim_{\epsilon\to 0} \epsilon h(\lfloor \epsilon^{-1}x \rfloor) = \phi_0(x).$$

The minimizers ϕ_0 of (9) satisfy the associated Euler–Lagrange equations:

$$\mathrm{div}(\nabla\sigma \circ \nabla\phi_0(x)) = 0,$$

where div denotes the divergence with respect to spatial variables $x \in \mathbb{R}^2$, whereas $\nabla\sigma \circ \nabla\phi_0$ denotes the gradient of $\sigma(\rho)$ with respect to its argument ρ, computed for $\rho = \nabla\phi_0(x)$.

(3) From these two facts it follows that, if the initial datum ϕ_0 of the hydrodynamic PDE (4) is a solution to the Euler–Lagrange equation, this is also the case for the solution $\phi(\cdot, t)$ at any time $t > 0$.

(4) This link between the Euler–Lagrange equation (involving σ) and the hydrodynamic equation (involving v) implies a non-linear relationship between the Hessian $D^2 v$ of $v(\cdot)$ and the (positive definite) Hessian of σ. This relation implies in turn that the determinant of $D^2 v$ is not strictly positive (this last step requires a few lines of computations that we do not develop here).

This is therefore a rigorous first step, no longer based on explicit calculations for a specific model but on general analytical arguments, which sheds some light on conjecture 1. However, the road to its full justification remains very long: one reason is that we barely know anything about the models of the KPZ class, where $v(\cdot)$ is strictly convex or strictly concave, and about their critical exponents α, β, which are supposed to be universal.

Bibliography

[1] A.L. Barabási and H.E. Stanley. *Fractal Concepts in Surface Growth*. Cambridge University Press, Cambridge, 1995.
[2] A. Borodin and P.L. Ferrari. Anisotropic Growth of Random Surfaces in 2 + 1 Dimensions. *Comm. Math. Phys.*, 325:603–684, 2014.
[3] A. Borodin and F. L. Toninelli. Two-dimensional anisotropic KPZ growth and limit shapes. *J. Stat. Mech. Theory Exp.*, 2018(8):083205, 2018.
[4] P. Caputo, F. Martinelli, and F. L. Toninelli. Mixing times of monotone surfaces and sos interfaces: A mean curvature approach. *Comm. Math. Phys.*, 311(1):157–189, 2012.
[5] I. Corwin. The Kardar–Parisi–Zhang equation and universality class. *Random Matrices: Theory Appl.*, 01:1130001, 2012.
[6] T. Funaki and H. Spohn. Motion by mean curvature from the Ginzburg–Landau $\nabla\phi$ interface model. *Comm. Math. Phys.*, 185:1–36, 1997.
[7] G. Giacomin, S. Olla, and H. Spohn. Equilibrium fluctuations for $\nabla\varphi$ interface model. *Ann. Probab.*, 29(3): 1138–1172, 2001.
[8] R. Kenyon and A. Okounkov. Limit shapes and the complex Burgers equation. *Acta Math.*, 199:263–302, 2007.
[9] C. Kipnis and C. Landim. *Scaling Limits of Interacting Particle Systems*. Springer Verlag, Berlin, 1999.
[10] B. Laslier and F. L. Toninelli. Lozenge tilings, Glauber dynamics and macroscopic shape. *Comm. Math. Phys.*, 338(3):1287–1326, 2015.
[11] B. Laslier and F. L. Toninelli. Lozenge tiling dynamics and convergence to the hydrodynamic equation. *Comm. Math. Phys.*, 358(3):1117–1149, 2018.
[12] B. Laslier and F.L. Toninelli. Hydrodynamic limit equation for a lozenge tiling glauber dynamics. *Ann. Henri Poincaré*, 18(6): 2007–2043, 2017.
[13] H. Spohn. *Large Scale Dynamics of Interacting Particles*. Texts and Monographs in Physics. Springer Verlag, Heidelberg, 1991.
[14] H. Spohn. Interface motion in models with stochastic dynamics. *J. Stat. Phys.*, 71(5–6):1081–1132, 1993.
[15] Fabio Toninelli. (2 + 1)-dimensional interface dynamics: mixing time, hydrodynamic limit and anisotropic KPZ growth. In *Proceedings of the International Congress of Mathematicians – Rio de Janeiro 2018. Vol. III. Invited lectures*, pages 2733–2758. World Sci. Publ., Hackensack, NJ, 2018.

Fabio Toninelli [fabio.toninelli@tuwien.ac.at] is a Professor at TU Wien from February 2020 (on leave from CNRS, Universite Claude Bernard Lyon 1). He obtained his PhD thesis in physics at Scuola normale superiore in Pisa. His main research domains are probability and classical statistical physics.

Acknowledgments. The *EMS Newsletter* thanks *La Gazette des Mathématiciens* for authorisation to republish this text, which is an English translation of the paper [*La Gazette des Mathématiciens*, Number 160, pp. 15–26, April 2019].

Yuri L. Daletskii and the Development of Infinite Dimensional Analysis

Yana Belopolskaya (St Petersburg University of Architecture and Civil Engineering, Russia) and Alexei Daletskii (The University of York, UK)

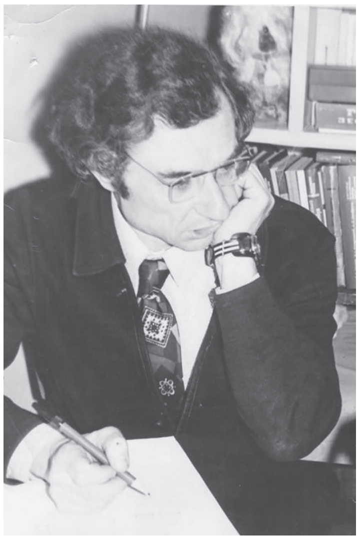
Yuri L. Daletskii after demobilisation from the army, 1946.

Yuri L'vovich Daletskii was born on 16 December 1926 in Chernigov, a small town near Kiev in Ukraine, at that time a republic of the Soviet Union, to a family of mixed Jewish-Polish descent. The English spelling of Yuri's surname is ambiguous, other versions (according to MathSciNet) being Dalecky, Daleckii and Daletsky. Yuri was raised by his mother and stepfather. At the beginning of the 1930s the family moved to Kiev, where Yuri went to school. The Second World War did not allow him to finish it. Fleeing from advancing German Nazi troops, Yuri's family headed to Eastern Ukraine and then the Caucasus. Yuri was then sent further east to Kemerovo, a city in Western Siberia, where he lived with his uncle.

At an early age, Yuri fell in love with mathematics and music. He had inherited a good voice from his parents and sang in a children's choir (Yuri's cousin became a professional opera singer). At the age of thirteen, Yuri survived paralytic poliomyelitis. Although the illness destroyed his singing voice and any chance of a professional musical career, his love and deep understanding of music became an important part of his personality. In order to overcome the illness' residual effects, Yuri took up gymnastics and became quite proficient at it, staying physically strong and fit throughout his life.

In his teens, Yuri spent a lot of time reading mathematical books, and his mathematical knowledge far exceeded school level. In 1944 he was admitted as a student to the Mining Institute in Kemerovo. Soon after, he volunteered for the Soviet army and participated in the war against Japan.

Yuri was demobilised in 1946 and returned to Kiev, where he started his studies at the Mechanics and Mathematics Faculty of Kiev University. Alongside the study and research work, Yuri actively participated in students' organisations and the public life of the university. Just weeks before the final examinations, he was expelled for criticising the rector (although the formal reason given was the absence of a secondary school certificate). During the next year, Yuri passed all school exams and was subsequently allowed to complete his university programme. He graduated in 1951, having written several research papers by that time. These works became the foundation of his PhD thesis completed under the supervision of Selim Krein, who played an important role in the development of the functional analysis school in Kiev. Soon after, Selim moved to Voronezh University. Yuri was very close to his teacher, always acknowledged his influence and remained in contact with him for his whole life.



At work (1971).

In 1951, Yuri took up a position at the Kiev Polytechnic Institute (The National Technical University of Ukraine at present), where he would remain for the entirety of his career, first as an assistant and eventually as a full professor and member of the Ukrainian Academy of Sciences. Yuri played a major role in forming the mathematical curriculum of the Institute. In the 70s and 80s he developed the mathematical programmes of the departments of Applied Mathematics, Mathematical Methods of System Analysis and the (new at that time) Faculty of Physics and Technology. Additionally, Yuri was one of the leaders of the successful independent post-graduate programme "Mathematics for Engineers", which was taught in Kiev for nearly two decades. Later, in the 90s, he also led mathematical programmes at the newly-founded Soros University.

Very soon Yuri became a significant figure on the Kiev mathematical scene. At that time, mathematical life in Soviet research centres was concentrated around big inter-institutional seminars, famous examples being Gelfand's and Dobrushin's seminars in Moscow. Yuri supervised major Kiev seminars "Random processes and distributions in functional spaces" (together with A. Skorokhod) and "Algebraic Structures in Mathematical Physics". He was also an important contributor to the seminar "Group methods in solid-state physics". Due to his friendly and energetic personality and vast knowledge of a variety of mathematical fields, Yuri played a

Symposium on soliton theory, Kiev, 1979. From left to right: H.P. McKean, Yu.L. Daletskii, I.M. Gelfand, Ya.I. Belopolskaya.

Warwick, 1991. From left to right: K.D. Elworthy, Yu.L. Daletskii, X.-M. Li.

connecting role between different mathematical groups in Kiev. He shared Israel Gelfand's philosophical view of mathematics as a single interconnected field, and was always interested in invariant algebraic structures "hiding" in various mathematical theories. To a large extent that was the philosophy and motivation of his seminar "Algebraic Structures in Mathematical Physics", which had a truly interdisciplinary flavour on the interface of algebra, geometry, stochastic analysis and mathematical physics.

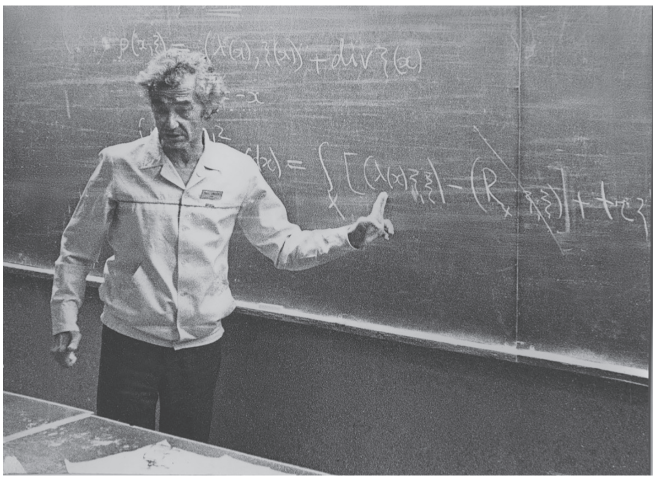
Yuri became well known within the mathematical world of the Soviet Union, which was, however, rather isolated from the rest of the world. He was an active participant and contributor to many conferences, among them the famous Voronezh Winter Mathematical School organised by Selim Krein, which year after year attracted the mathematical elite of the country.

Yuri's development as a scientist was very much influenced by his collaboration with two mathematical giants of the 20th Century – Mark Krein and Israel Gelfand. Mark Krein lived in Odessa, and Yuri liked to rent a holiday house by the sea, staying there with his family and combining work with long offshore swims. This collaboration resulted in the joint book *Stability of solutions of differential equations in Banach spaces* (1974), where the infinite-dimensional version of the Lyapunov stability theory was developed.

Yuri had been a frequent participant of Gelfand's seminar at the Moscow State University. More direct collaboration started, however, in the early 80s, when Gelfand was spending summers at his wife's home in Kiev. This collaboration lead to a number of works on non-commutative differential geometry and topology (joint with B. Tsygan, L. Takhtajan and others).

Yuri was a true pioneer of global infinite-dimensional stochastic analysis. His research in this field, rooted in Gelfand's ideas, formed the contents of two books: *Measures and Differential Equations in Infinite-Dimensional Spaces* (jointly with S. Fomin and actually finalised by Yuri after Fomin's death in 1983; the English translation appeared in 1991), and *Stochastic Equations and Differential Geometry* (jointly with Ya. Belopolskaya, 1989; English translation 1990). This research was acknowledged around the world, and soon after 1991 Yuri became a frequent traveller and visited most of the world's important mathematical research centres.

In the 1950s, a strong motivation for the development of the analysis of measures and random processes in infinitely many dimensions was the need for a rigorous theory of Feynman integrals. After Mark Kac's results on parabolic equations and his derivation of the celebrated Feynman–Kac formula for the Wiener measure, it looked possible to implement a similar approach in the study of the Schrödinger equation and represent its solutions via integrals with respect to probability measures on path spaces. Yuri showed (independently of R. Cameron) that no such measures could be constructed, that is, they appeared to be quasi-measures of unbounded variation. He developed a rigorous construction of the integrals with respect to such quasi-measures, including abstract



Fifth International Vilnius Conference on Probability Theory and Mathematical Statistics, Vilnius 1989.

Gaussian quasi-measures with complex variance parameters.

As a technical tool for these studies, Yuri proved a chronological product formula for evolution families of operators. In the simplest case, for a generator A of a strongly continuous semigroup e^{tA} and a bounded operator B in a Banach space, the semigroup generated by the sum $A+B$ has the following form:

$$e^{t(A+B)} = \lim_{\Delta t_k \to 0} \prod_k e^{\Delta t_k A} e^{\Delta t_k B}, \quad \text{where} \quad \textstyle\sum_k \Delta t_k = t.$$

This result, usually called the Trotter product formula in the literature, was obtained by Yuri independently and extended to the case of evolution families generated by non-autonomous equations $\dot{\phi}(t) = A(t)\phi(t)$ and $\dot{\psi}(t) = B(t)\psi(t)$. It lead to the notion of operator-valued multiplicative integrals, and was in turn used by Yuri to prove Feynman–Kac type formulae for abstract evolution equations, including various equations of parabolic, hyperbolic and Schrödinger type. This, in particular, allowed him to give a rigorous derivation of the Feynman–Kac formula for Schrödinger type equations with bounded smooth potentials; a generalisation for singular potentials was later obtained by Nelson.

Furthermore, Yuri went on to consider abstract evolution equations in infinite-dimensional (Hilbert or Banach) spaces. Simultaneously with Leonard Gross, he defined the infinite-dimensional Laplacian for functions with trace-class second derivatives. L. Gross (*J. Funct. Anal.* 1967) refers to Yuri's paper "Differential equations with functional derivatives and stochastic equations for generalised random processes" (*Dokl. Akad. Nauk USSR* 1966, with yet another spelling of Yuri's surname as *Daletzkii*), which had previously been unknown to him. These ideas led Yuri to the development of the general theory of stochastic differential equations in Banach spaces and corresponding probabilistic representations of solutions of parabolic equations, in the framework of Gelfand triples, building upon the classical finite-dimensional results of Itô, Gikhman and Skorokhod. This work formed a foundation for a variety of theories, including those of stochastic partial differential equations, stochastic quantisation of infinite particle systems and smooth measures on infinite-dimensional spaces.

Capri, 1993. From left to right: L.P. Daletskaya, Yu.L. Daletskii, R.A. Minlos.

Michigan, 1996. From left to right: Yu.L. Daletskii, A.V. Skorokhod.

Yuri's research was much concerned with differential-geometric aspects of global stochastic analysis on infinite-dimensional spaces. In a series of joint works with Ya. Belopolskaya, a concept of a stochastic differential equation on a Banach manifold endowed with a Hilbert–Schmidt structure was developed. The Itô stochastic differential was considered as a section of a special vector bundle (the *Itô bundle*), highlighting its geometric nature. This research was initiated in the paper "Diffusion processes in smooth Banach spaces and manifolds" (*Trans. Moscow Math. Soc.*,1978), and culminated in their book *Stochastic Equations and Differential Geometry*, which also covered the case of stochastic equations associated with quasi-linear parabolic equations on Banach spaces and manifolds.

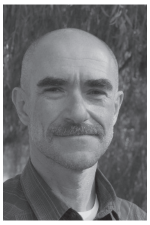
Other important contributions to the development of infinite-dimensional analysis were the theory of smooth measures on Banach manifolds and its connection with the notion of the Hitsuda–Skorokhod integral, the theory of Lévy-Laplacian, the theory of bi-orthogonal expansions and non-Gaussian distributions (with S. Albeverio, Yu. Kondratiev and L. Streit), to name but a few.

A lot of Yuri's work was done in collaboration with his students and colleagues. He successfully supervised 30 PhD students, some of whom became renowned scientists. A great lecturer and passionate teacher, he liked

Albert Einstein's quote "Student is not a container you have to fill but a torch you have to light up." He was always generous in sharing his ideas and supporting his students, collaborators and colleagues, and brave enough to stand up for them if needed. Always full of energy, with deep knowledge of history, literature and poetry, he liked being surrounded by people and felt comfortable in any company. His other favorite quote was that of the famous Soviet biologist Timofeev-Ressovsky: "Science is a jolly lady and does not tolerate spiderish seriousness."

Yuri passed away in December 1997, a couple of days before his 71st birthday. How is he remembered? The two decades after his departure have given us the answer: not only as a brilliant scientist and teacher, but also as a bright and cheerful man, loved by his family, friends and colleagues, who had lived his life to the full.

Yana Belopolskaya [yana@yb1569.spb.edu] is a Professor of Mathematics at the St.Petersburg State University of Architecture and Civil Engineering. She graduated from the Kiev Polytechnic Institute, then got her PhD in the Kiev Institute of Mathematics (1965) and her Doctor of Science degree in the Leningrad Department of the Steklov Mathematical Institute (1990). She works on stochastic differential equation theory, nonlinear PDE and systems theory and connections between these two theories.



Alexei Daletskii [alex.daletskii@york.ac.uk] obtained his PhD and Doctor of Science degrees from the Institute of Mathematics NAS, Kiev, Ukraine. He has held full-time positions in Bonn and Nottingham and works at present at the Mathematics Department of the University of York, UK. His main research interests are in stochastic analysis and its applications in mathematical physics. Alexei is Yuri Daletskii's son.

Exploring Leibniz's Nachlass at the Niedersächsische Landesbibliothek in Hanover

David Rabouin (CNRS – Université de Paris, France)

The archival situation of the works of Gottfried Wilhelm Leibniz (1646–1716) is specific in comparison with many other mathematicians. Apart from the dissertation *De arte combinatoria*, written in his youth (1666), he never published a single treatise of mathematics during his lifetime. Around sixty articles were rendered public, though, mainly on differential calculus (and a very few on binary arithmetic), but they were often very short, allusive and sometimes even hurried. This is the case, in particular, with the famous article in which he presented his differential algorithm in 1684, the *Nova methodus pro maximis et minimis* – which, for this reason, was simply not understood by most of his first readers. One can already find an overview of this published material in the third volume of Louis Dutens' *Gothofredi Guillelmi Leibnitii Opera omnia* (Leibniz 1768). In modern editions, such as that of Heß and Babin (Leibniz 2011), this represents around 500 pages – to be compared to the … 50,000 handwritten items (more than 7000 folios of mathematics) to be found in the Leibniz *Nachlass* (see Knobloch 2004). Brought back to the modern page format, a rough estimate is that Leibniz left around 99% of his written mathematical activity hidden during his lifetime!

Ironically enough, one can find a famous declaration in the sixth volume of Dutens' edition indicating the substantial limitations of this kind of enterprise: "who knows me only from what I published, does not know me" (*Qui me non nisi editis novit, non novit*, Letter to Vincent Placcius, 21 february /2 march 1696, Dutens VI, 1, 65). Indeed, it was already a well-known fact at the time. Just a few years before, Diderot wrote in the *Encyclopédie*: "Perhaps no man has ever read so much, studied so much, meditated so much, written so much as Leibnitz; yet there is no body of work by him; it is surprising that Germany, to which this man alone does as much honour as Plato, Aristotle and Archimedes together do to Greece, has not yet collected what has come out of his pen" (Diderot 1765, 379, as quoted in Poser 2012). "Jamais homme … n'a plus écrit": considering the nearly 100,000 folios and 20,000 letters which are kept in the department of manuscripts of the *Gottfried Wilhelm Leibniz – Bibliothek Niedersächsische Landesbibliothek* (GWLB) in Hanover, it does not seem to be too much of an exaggeration.

Soon, people felt compelled to look into the unpublished material, but mathematics was not well served. What one finds in the Raspe edition (Leibniz 1765) is only about logic, and even the Erdmann edition (Leibniz 1840) still only contains the logical calculi as a new topic that could be related to mathematics (as well as some methodological papers related to the project of a *scientia generalis*)[1]. Nothing, for example, on the famous project of *Analysis situs*, which was known through some allusions in published letters. Interestingly enough, mathematicians of the time thought that this material was lost and that their task was to reconstruct it – in the same way that early modern authors attempted to revive lost treatises from antiquity by some kind of *divinatio* (*Diophantus, Apollonius redivivus…*). As is well known, this is the way in which Grassmann or Moebius publicly presented their own projects of new geometrical calculi, which they considered as vindicating Leibniz's original ideas. It is only with Gerhardt's edition of the *Mathematische Schriften* (Leibniz 1849–1863), starting in the middle of the 19th century, that some of the hitherto unknown mathematical material was progressively discovered for the first time. This was especially the case with volume 5 (1858) containing texts related to "*Characteristica Geometrica. Analysis Geometrica propria. Calculus situs*", and with volume 7 (1863) dedicated, amongst other topics, to "*Initia mathematica*" or "*Mathesis universalis*". In the wake of these editions, authors such as Louis Couturat called for a reappraisal of Leibniz's ideas, in which the work in logic and mathematics should be put at the core of the entire philosophical system (Couturat 1901). Couturat himself completed Gerhardt's work in 1903 with his *Opuscules et Fragments inédits*, in which he gave access to many other unpublished mathematical works (Leibniz 1903).

From this overall picture, one might get the impression that for a century or so, we have been in possession of a good overview of Leibniz's unpublished work in mathematics. But this is far from true: in fact, to this day, more than half of Leibniz's mathematical work still remains completely unpublished. Moreover, when one looks at the published half, typically the editions of Gerhardt and Couturat, one soon realises that they were not accomplished according to modern scientific standards. This last point may seem incidental. We do not always need to have all the philological details of variants and

[1] It played a very important role in the history of logic as can be seen in the fact that Frege quoted this edition at the beginning of his *Grundlagen der Arithmetik* (1884) to vindicate an "analytic" reading of mathematical statements, which he attributed to Leibniz. On the edition of Leibniz mathematical texts in the 18th century, see Probst 2016.

marginalia to read a text. For the layperson who is foremostly interested in the content, such critical apparatus may even sometimes appear as an impediment to reading. The problem is unfortunately much deeper, because some texts edited by Gerhardt and Couturat have turned out to be mere artefacts. Let me give two talking examples of this difficulty taken from two of the main topics presented in volume 7 of Gerhardt's edition, namely *initia mathematica* and *mathesis universalis*:

- The very first text of volume 7 of the *Mathematische Schriften* is a much-celebrated piece of work on the "Metaphysical Foundations of Mathematics" (*Initia rerum mathematicarum metaphysica*). It has given rise to a wealth of commentaries not only from philosophers, but also from scientists, since Leibniz gives in it his definitions of time and space as pure systems of relations. The text is not easy to provide commentary on due to its non-linear structure. Many topics mentioned at the beginning are taken up again, in a slightly different manner, later in the text. When one looks at the manuscript, this comes as no surprise: as was discovered by Vincenzo De Risi when exploring the corpus related to *Analysis situs* (De Risi 2007), Gerhardt simply copied two different texts from two different periods and placed them side by side! The text, as published in the *Mathematische Schriften*, simply does not exist as such. It is Gerhardt's invention.
- Let us now take a look at another celebrated text: the *Elementa nova matheseos universalis*. It was published for the first time by Couturat in 1903 and also gave rise to a wealth of commentaries. It is one of the rare places where Leibniz expounded his programme of "new" universal mathematics (as opposed to Descartes', still centred on magnitude and algebra). Couturat's text is full of dots indicating lacunae in the manuscript (as specified at the beginning of the book: "*les points espacés indiquent les lacunes de notre copie*"). Yet, when one looks at the manuscript, one can easily see that there are no lacunae in it. Dots stand for passages Couturat did not take the trouble to copy. Maybe because they did not match the kind of interpretation he wanted to suggest! Indeed, since Couturat wanted to insist on the fact that Leibniz had a purely formal approach to what are now called "equivalence relations" and detailed their formal properties, he just cut the passage in which Leibniz derived this description from phenomenological considerations (according to Leibniz, geometrical relations are induced by the way in which we may or not be able to "discern" objects in perception). It was not until 1999, when the first scientific edition of the philosophical texts from the 1680s was issued, that scholars interested in this topic could read for the first time what Leibniz's programme really was.

There are many other examples showing that what we call "Leibniz's mathematical texts" is not a simple matter of one of the so-called Leibniz "editions". One reason is, of course, the enormous amount of papers preserved in the GWLB and the temptation that naturally arises for the editors confronted with such an overwhelming wealth of material to classify it in advance, neglect the "innocuous" variants or "unimportant" notes and select what they consider "interesting" passages, etc. Another reason is that the archive itself was catalogued (in the 19th century by Eduard Bodemann) in a very rough fashion: folios were only sorted according to general topics (sometimes as general as "mathematics") or vague similitude, and with no particular consideration of time period, precise relationship between the documents, etc. Hence the project, which did not arise until the beginning of the 20th century, to directly confront the difficulty and publish *all* of this material in chronological order and with all the variants. This is the origin of the so-called "*Akademie Edition*", which now constitutes the standard for any rigorous access to the Leibniz texts (Leibniz 1923–).

The first volume of the first series dedicated to the political and historical correspondence appeared in 1923 (*Allgemeiner politischer und historischer Briefwechsel*), soon to be followed, in 1926, by the first volume of the second series dedicated to the philosophical letters (*Philosophischer Briefwechsel*). When it came to the third series, however, the one dedicated to the mathematical correspondence, things became more complicated due to the expertise necessitated by such an enterprise (in Latin, in philology, in history of science and, of course, in mathematics). Dietrich Mahnke was the first to be in charge of this task and had completed almost half of it when he died unexpectedly in 1939. The work was then taken over by Joseph Ehrenfried Hofmann and was first published three years after his death in … 1976. At that time, the publication of the mathematical texts themselves (series VII) had not yet even begun! It is only with the efforts of Eberhard Knobloch that this gigantic enterprise could really start in 1976 (see Knobloch 2018 for an account of this enterprise). As a result, the first volume of the mathematical series, gathering texts of arithmetic (number theory), (elementary) geometry and algebra from the period of Leibniz's stay in Paris (1672–1676), did not appear until … 1990.

One of the reasons why Knobloch was (and still is!) enthusiastic about this thankless and arduous task was that he already knew that the papers left by Leibniz contained many mathematical treasures. He provided a first example in his PhD, in which he documented the many works Leibniz had done on combinatorics, in particular on what is now called symmetric functions, and his remarkable achievements in the constitution of determinant theory (these works were published as Knobloch 1973, Knobloch 1976 and Knobloch 1980). But there were many other treasures awaiting publication. One of them was a treatise on quadratures, which Leibniz wrote at the end of his Parisian sojourn under the title *Quadratura arithmetica circuli ellipseos et hyperbolae*. This text had been known for some time, because Leibniz often refers to it in his correspondence. Lucie Scholtz already provided a partial translation in her PhD (Scholtz 1934). Yet it was not until 1993 that a complete edition appeared (Leibniz 1993). Ten years before, the philosophical papers from the Parisian stay were published in volume

3 of series VI of the *Akademie Edition*. They gave access to many reflections Leibniz developed at the time of the creation of his differential algorithm on infinitely small and infinitely great quantities (see Arthur 2001 for an English translation). The result of these publications was a complete transformation of the prevailing view on the position Leibniz held on the foundation of infinitesimal techniques. While it was previously believed that his conception of the subject was somewhat fuzzy and that his recourse to the vocabulary of the "fictions utiles" was just a convenient way to escape the foundational difficulties raised by his opponents in the 1690s, it now appears that this "fictionalist" strategy was already achieved by 1676 (with no opponents to confront it!) and referred to some precise and rigorous proofs (for the detail of these proofs see Knobloch 2002; Rabouin 2015).

Eight volumes of the mathematical correspondence (series III) and seven volumes of the mathematical texts (series VII) have now appeared. Since the 2000s, the various volumes have been accessible as a digital edition as well as in print (https://leibnizedition.de/). Eberhard Knobloch, first assisted by Walter Contro (who passed away in 2017), was then joined by Nora Gädeke and later Siegmund Probst. A small team was progressively constituted for the edition of mathematical texts in the *Leibniz Archiv* in Hanover (see Knobloch 2018, 32, for a list of the contributors to the various volumes). Yet, this only gives access to the texts written in Paris and at the beginning of the stay in Hanover (1672–1677; recall that Leibniz died nearly 40 years later, in 1716). The classification of Leibniz's manuscripts made by Bodemann in the 19th century (for mathematics, essentially in the rubric noted LH XXXV), as well as the work done since, have led to an estimate of about … 22 mathematical volumes remaining to be published for the period 1677–1716 (the LH XXXV group represents around 7000 folios and each volume of the *Akademie Edition* generally gathers around 200 of these folios, which corresponds to 800 published pages *in quarto*). As we recall, only 25% of these texts have already been published. For half of them, there was simply no entry in the catalogue and for the other half, no precise indication of content. From 2012 to 2014, the data corresponding to the Leibnizian mathematical manuscripts was entered into the central publishing catalogue and preliminary dating work was undertaken by the Hanover publishers (http://mdb.lsp.uni-hannover.de/). The digitisation of the mathematical manuscripts was undertaken in parallel and went online in September 2016 – a resource of invaluable help for scholars working in the domain (http://digitale-sammlungen.gwlb.de/index.php?id=7).

Are there other mathematical treasures awaiting us in the ocean of unpublished manuscripts? Without a doubt! In 2010, a study group was launched in France, in close collaboration with German colleagues from the Leibniz Archiv, to analyse these unpublished manuscripts more systematically from a historical and epistemological point of view. It started with a modest goal: editing Leibniz texts on *mathesis universalis*, a topic I have already mentioned as having been mistreated by Couturat (and Gerhardt) and for which the corpus is fortunately not too vast. This resulted in the volume (Leibniz 2018), in which a new interpretation of the topic was proposed. In 2017, funding was obtained from the French *Agence Nationale de la Recherche* in order to develop these activities in close connection with the work done by Leibniz editors in Hanover[2]. The work was greatly helped by the advances made by the previous generations of scholars. Besides Knobloch, Siegmund Probst and their team in Hanover, one could mention work done by Javier Echeverría on *analysis situs* and geometry (which was (partially) published in Leibniz 1995), by Enrico Pasini on the foundations of differential calculus (which was (partially) published in Pasini 1993), by Emily Grosholz on the foundation of arithmetic (Grosholz and Yakira 1998) or by Mary Sol De Mora Charles on games (De Mora Charles 1992).

One could start with the last topic, since it shows nicely how many questions still remain open for the researchers exploring Leibniz's mathematical Nachlass. In 1992, De Mora Charles transcribed some unpublished mathematical texts on games, including a description of a game created by Leibniz: the "inverse solitary" (in which one has to fill the board instead of emptying it). This is an interesting piece, because Leibniz takes a similar example in one of his metaphysical texts from the same period, the famous *De rerum originatione radicali* (1697). In it, he compares God's act of creation with "certain games in which all the places on the board are to be filled according to definite rules, but unless we use a certain device, we find ourselves blocked out, in the end, from the difficult spaces and compelled to leave more places vacant than we needed or wished to" (Leibniz 1849–1863, t. VII, 303; English translation in Loemker 1989, 487). But there is more, since at the end of the text transcribed by De Mora Charles, Leibniz claims that one could treat this kind of game like "a" geometry. Notice that he does not speak of a geometrical treatment of the game, but of the fact that the game itself can be seen as "a kind of geometry". Accordingly, Leibniz claims, one could describe in it an equivalent of "straight lines, lines composed of lines, and simple and

2 ANR *Mathesis*, Édition et commentaires de manuscrits mathématiques inédits de Leibniz (2017–2021), N° ANR-17-CE27-0018-01 AAP GENERIQUE 2017. Members: Jean-Pascal Alcantara (ESPE/Université de Bourgogne), Andrea Costa (Centre Jean Pépin-UMR 8230), Valérie Debuiche (AMU, Centre Granger-UMR 7304, investigator for Aix-Marseille), Vincenzo De Risi (CNRS, SPHERE-UMR 7219), Baptiste Mélès (Archives Poincaré-UMR 7117), Anne Michel-Pajus (IREM, Paris Diderot), David Rabouin (CNRS, SPHERE-UMR 7219, principal investigator) & Claire Schwartz (University Paris Ouest, Nanterre). The project also includes four post-doctoral positions: Sandra Bella, Mattia Brancato, Davide Crippa and Miguel Palomo. It also includes three PhD: Morgan Houg, Vincent Leroux, Arilès Remaki.

3 "Pour rediger ce jeu en art, il faut le traiter comme une geometrie particuliere, par elemens. Il faut le moyen de former des lignes droites; des lignes cornposes de droites, et des figures simples et composees, qui sont toujours composees de lignes droites" (De Mora Charles 1992, 155).

composite figures”[3]. Such an abstract understanding of what “a geometry” could be, “straight line” being here a purely abstract object and figures being obtained in a combinatorial way, is striking for the modern reader and invites further inquiries into the *Nachlass*.

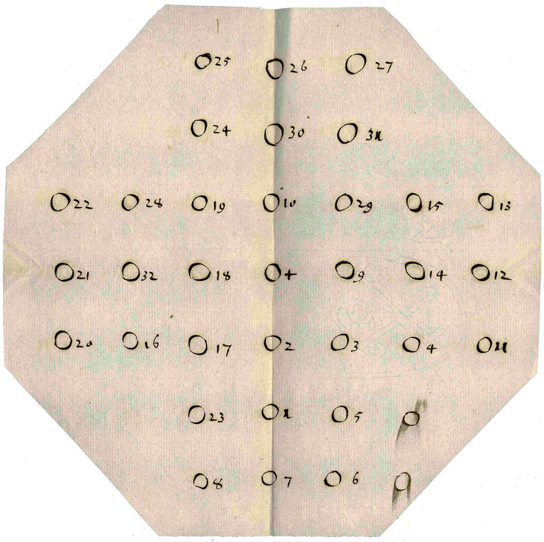
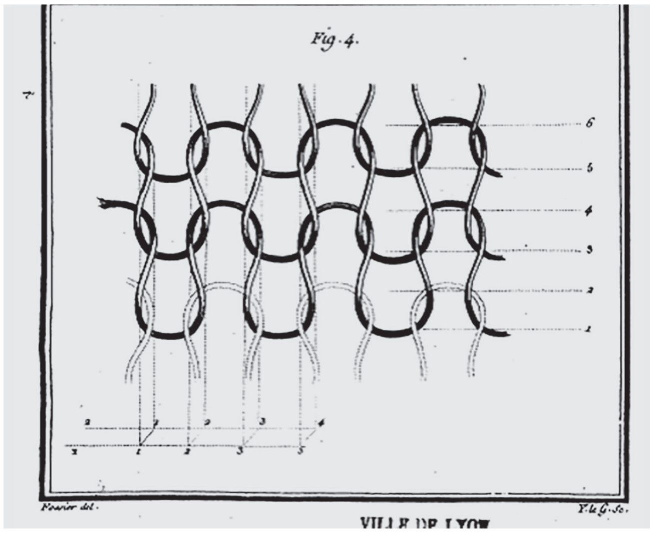
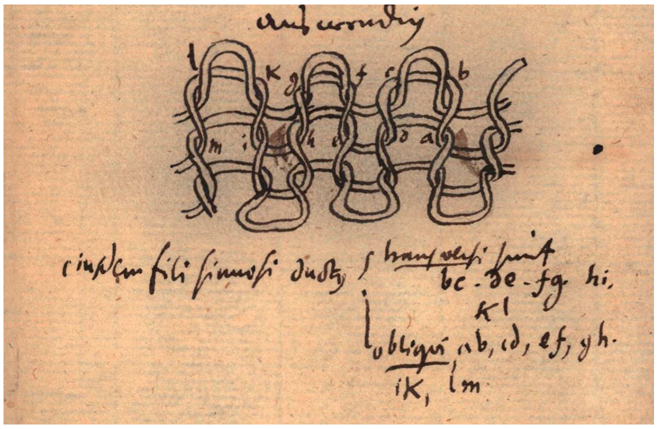


Diagram for the mathematical treatment of the Solitaire[4].

Interestingly enough, Vandermonde may have had some insight into this background when tackling the subject of the “géométrie de situation”. In the volume of *Histoire de l’Académie Royale* presenting his *Mémoire* on this topic (Histoire 1771), it is said, just after evoking the Leibnizian *Geometria situs*: “This idea of Leibniz has so far been very much neglected. I know in this kind only some Essays that Leibniz himself gave on the Game of Solitaire, and the Research of Mr. Euler on the walk of the knight in chess (…). M. Vandermonde endeavoured particularly to find, for this kind of analysis, a simple notation, and which could facilitate the calculations”. Vandermonde’s paper is about the way one can compute on knots, braids or the moving of a piece in chess – topics apparently unrelated to the Leibnizian project of *analysis situs.* Yet Leibniz mentions in a famous letter to Montmort that games could be classified into three types, the first one being about *situs* (Leibniz 1875–1889, vol. 3, 668). This is precisely where he mentions his work on the Solitaire[5]. Moreover, one finds in Leibniz’s *Nachlass* notes on *ars textoria* (see illustration below), which he discovered by reading Joachim Jungius and in which he points to his own project of a *characteristica situs*. These notes still await study and a more systematic exploration of the connection between games, knots and *analysis situs*.



Vandermonde 1771



From Leibniz’s notes on *ars textoria*.[6]

This gives an example of the complexity involved in working with Leibniz’s manuscripts and the dangers of selecting in advance where to look and what to look for. Although it was obvious for some of Leibniz’s heirs that the work on *analysis situs* had something to do with his work on games and on *geometria sartorum* – as was also pointed out by Vacca when undertaking his own research on the geometry of folding, on the basis of one of Leibniz’s unpublished manuscript (Friedman 2018, 320) – this knowledge was lost at some point in history. As a consequence, this kind of research is now classified outside of the mathematical series, amongst the “technical” texts (series VIII: *Naturwissenschaftliche, Medizinische und Technische Schriften*).

More generally, many aspects of the vast project of *analysis situs* still remain to be explored in greater detail. It was known, for example, since Echeverría and Parmentier’s edition (Leibniz 1995, 272) that Leibniz worked on geodesics (*lineae minimae*) on the sphere and the cylinder quite early on, but Vincenzo de Risi found

[4] Gottfried Wilhelm Leibniz Bibliothek, Niedersächsische Landesbibliothek, Leibniz-Handschriften zur Technica, LH 38, fol. 195 v. http://digitale-sammlungen.gwlb.de/resolve?id=00068537

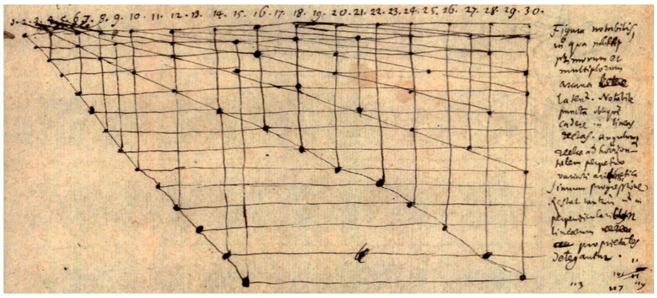
[5] Vincent Leroux is completing a PhD thesis on the topic of games in Leibniz’s ideas in the framework of our ANR project.

[6] Gottfried Wilhelm Leibniz Bibliothek, Niedersächsische Landesbibliothek, Leibniz-Handschriften zur Technica, LH 38, fol. 38r. http://digitale-sammlungen.gwlb.de/resolve?id=00068537

later texts in which Leibniz goes as far as posing the general problem of geodesics on curved surfaces (De Risi 2007, Appendix 6 and 7, 592–595). This question was taken into broader consideration. Indeed, if *linea minima* serves as a definition for "straight line", as has been proposed by some authors since antiquity, this would mean that one could have "straight lines" (in this case great circles) that are parallel (in the sense of making right angles with another line) and still intersecting – a fact which struck Leibniz very early as problematic for the definition of parallel lines. Following this thread, De Risi has uncovered very rich material related to several attempts at proving the parallel postulate. In these texts one can see Leibniz fighting against his own insight into the possibility of other "geometries". A very interesting result is that the philosopher realised that he could not provide such an exclusion by purely logical means (i.e. by resorting to the sole principle of contradiction) and was bound to invoke "superior principles", as he calls them, such as the "principle of sufficient reason" (De Risi 2016).

Another topic on which my colleagues Valérie Debuiche and Mattia Brancato are now working is Leibniz's studies on mathematical perspective. Like in the case of games, an interesting aspect of these unpublished studies is that they relate to a theme often mentioned in his metaphysics (each individual substance, or monads, is a "point of view" on the universe like various perspectives of the same town). Yet, the works Leibniz dedicated to *scientia perspectiva* are still not published, for reasons comparable to what happened with *ars textoria*: they belong, in some part, to technical papers which were not included in the mathematics series (see, for example, the notes on Bosse, Desargues, Aleaume and Dubreuil contained in the first volume of the technical series VIII; Debuiche 2013)[7]. One could also mention the many studies Leibniz dedicated to the role of tractional motion in the construction of curves and of which a first overview (for the beginning of the Hanoverian period) can now be accessed through the recent publication of the volume 7 of the series VII (http://www.gwlb.de/Leibniz/Leibnizarchiv/Veroeffentlichungen/VII7.pdf). As Davide Crippa has shown, this constituted a constant preoccupation for Leibniz, since he saw in tractional motion a way of enlarging the Cartesian standard of constructions by "continuous motion" and accordingly what could constitute a proper definition of what a "geometrical" curve was – as contrasted to Descartes' identification of "geometrical" to "algebraic" (see Crippa 2020).

But geometry is not the only topic in which many of Leibniz's gems are still waiting to be excavated. As pointed out in (Knobloch 2004), Leibniz dedicated some very interesting studies to the distribution of prime numbers during his Parisian stay, a topic which, at the time, was not of particular interest to mathematicians. In this context, he devised various diagrammatic representations with the hope of seeing a pattern emerging and by the same token developing a form of geometry of numbers – another brilliant and original idea for the time (Knobloch 2004)[8].



A study on the distribution of prime numbers.[9]

Let me finish by mentioning the example of Leibniz's work on the axiomatic foundations of algebra. As has been known since the beginning of the 19th century, Leibniz may be considered the first author to have developed formal axiomatics for logical calculi (as testified to in particular by the remarkable *Non inelegans specimen demonstrandi in abstractis* from 1687, which was already published by Erdmann and made such a great impression on Frege). For a long time, it was thought that he developed these axiomatic systems as a generalisation of his research in arithmetic and algebra (see, for example, Lenzen 1989). Thanks to a better dating of the manuscripts, we now know that it was the other way round: Leibniz first developed formal calculi for logic (and, more generally, for mereological relations) at the end of the 1680s and then turned to algebra. This evolution may be related to the fact that he stumbled upon what we now call the "idempotence" axiom as characterising logical operations – quite a remarkable result in and of itself. By the same token he realised that, contrary to his initial hopes, it was not possible to have one abstract calculus holding at the same time for "notions" and for "magnitudes" (Pajus and Rabouin 2017). This gives an interesting context for the study of the attempts he made to edify axiomatic foundations for algebra around 1700. In particular, it offers a new context for the reading of the famous derivation of "2 + 2 = 4" as a chain of definitions and substitutions (plus one axiom for equality), which Frege took as a basis for vindicating the idea that mathematical statements (at least on natural numbers) were purely logical statements (Leibniz 2018, 169–180).

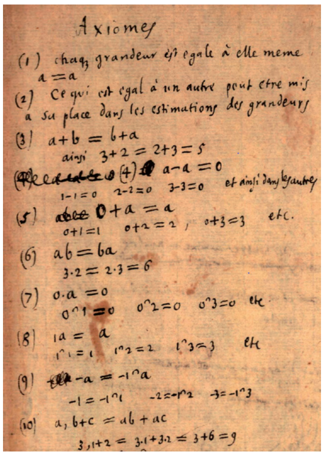
---

7 With our colleagues in Hanover we have prepared a preprint edition, which can be accessed at the following address: http://www.gwlb.de/Leibniz/Leibnizarchiv/Veroeffentlichungen/PreprintsReiheVII.htm

8 See also the PhD Morgan Houg just defended (December 2019) on Arithmetic during the Parisian Stay (soon to be accessible here: http://www.theses.fr/s180861). Arilès Remaki is also completing a PhD in which he also studies in great detail the role of diagrammatic representations for combinatorial thinking in Leibniz.

9 Gottfried Wilhelm Leibniz Bibliothek, Niedersächsische Landesbibliothek, Leibniz-Handschriften zur Mathematik, LH 35, 4, 17, fol. 5v. http://digitale-sammlungen.gwlb.de/resolve?id=00068014

Indeed, as was already noticed by Emily Grosholz, the derivation of "2 + 2 = 4" should be inserted in the context I just recounted and in which the central concept is not that of natural numbers, but that of magnitude (Grosholz and Yakira 1998). Accordingly, the "logicist" reading of these texts appears to be a truncated one. Many manuscripts from this period are still waiting to be transcribed and studied in more detail. Amongst them, I chose one which gives an idea of the kind of mathematics Leibniz achieved at the end of his life and which looks so familiar to the modern reader, although it was a kind of oddity in the context of the mathematics of the time. It gives a list of axioms in which any mathematician today would recognise the commutativity of addition and multiplication, the role of neutral elements (0 for addition, 1 for multiplication), the existence of inverse for both operations and the distributivity of the second over the first.



Elements du calcul.[10]

Axiomes

(1) chaque grandeur est égale à elle même
$a = a$

(2) ce qui est égal à un autre peut être mis a sa place dans les estimations des grandeurs

(3) $a + b = b + a$

(4) $a - a = 0$

(5) $0 + a = a$

(6) $ab = ba$

(7) $0 \cdot a = 0$

(8) $1a = a$

(9) $-a = -1^\frown a$

(10) $a, b + c = ab + ac$

Due to space constraints, I did not transcribe the numerical examples, nor did I reproduce item (11), which characterises the multiplicative inverse of any element by the axiom $\frac{a}{a} = 1$ (the next item in the list making it clear that Leibnitz interprets $\frac{a}{a}$ as $a \cdot \frac{1}{a}$ although he should have stated that a be different from 0). This is quite a remarkable way of characterising what will later be called an algebraic "structure". Although this is certainly not what Leibniz had in mind (one would not find any "structural" theorems in these texts), the kind of formal presentation which he reached at this occasion is nonetheless striking and had no equivalent at the time (and, more generally, before the 19th Century).

These are just a few examples of the treasures waiting to be studied in Leibniz's mathematical *Nachlass*. Let us hope more young researchers, historians as well as mathematicians, will be tempted to visit this temple in the future and contribute to the long-lasting effort of transcribing them. Leibniz, who lived just after the Thirty Years' War (he was born two years before the Peace of Westphalia) was a strong supporter of a political and scientific Europe, in which he saw the only way to overcome the dangers of religious extremism. One way to revive his legacy is certainly to maintain this spirit of European cooperation in science and the humanities.

[10] Gottfried Wilhelm Leibniz Bibliothek, Niedersächsische Landesbibliothek, Leibniz-Handschriften zur Mathematik, LH 35, 4, 12 fol. 2r. http://digitale-sammlungen.gwlb.de/resolve?id=00068009

References

Arthur 2001: Richard T. W. Arthur, *The labyrinth of the continuum: Writings on the continuum problem, 1672–1686*, New Haven: Yale University Press, 2001.

Couturat 1901: Louis Couturat, *La Logique de Leibniz*, Paris: Alcan, 1901.

Crippa 2020: David Crippa, "One String Attached: Geometrical Exactness in Leibniz's Parisian Manuscripts", *Leibniz and the Structure of Sciences. Modern Perspectives on the History of Logic, Mathematics, Dynamics*, ed. V. De Risi, Cham: Springer, 2020 (coll. Boston Studies in Philosophy and History of Science): 203–252.

Debuiche 2013: Valérie Debuiche, "Perspective in Leibniz's invention of Characteristica Geometrica: The problem of Desargues' influence", *Historia Mathematica*, vol. 40, Issue 4, 2013, 359–385.

De Risi 2007: Vincenzo De Risi, *Geometry and Monadology*, Dordrecht: Springer, 2007.

De Risi 2016: Vincenzo De Risi, *Leibniz on the Parallel Postulate and the Foundations of Geometry*, Cham: Springer, 2016.

Diderot 1765: Denis Diderot, article LÉIBNITZIANISME ou PHILOSOPHIE DE LÉIBNITZ, *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, Paris: Briasson, David l'Aîné, Le Breton, Durand, vol. IX (1765), p. 369–379.

Folkerts 2008: Menso Folkerts, "Die Leibniz-Edition zwischen Wissenschaft und Politik: zur Geschichte der mathematisch-naturwissenschaftlichen Reihen", *Kosmos und Zahl*, ed. Hartmut Hecht et al., Stuttgart: Franz Steiner, 2008: 23–45.

Friedman 2018: Michael Friedman, *A History of Folding in Mathematics: Mathematizing the Margins*, Basel: Birkhäuser, 2018 (Science Networks. Historical Studies, Band 59).

Grosholz and Yakira 1998: Emily Grosholz and Elhanan Yakira, *Leibniz's Science of the Rational*, Stuttgart: Franz Steiner, 1998 (Studia Leibnitiana, Sonderheft 26).

Knobloch 1973: Eberhard Knobloch, *Die mathematischen Studien von G.W. Leibniz zur Kombinatorik*, Wiesbaden: Franz Steiner, 1973 (Studia Leibnitiana Supplementa, Band XI)

Knobloch 1976: Eberhard Knobloch, *Die Mathematischen Studien von G.W. Leibniz zur Kombinatorik*. Textband, Stuttgart: Franz Steiner, 1976 (Studia Leibnitiana Supplementa, Band XVI).

Knobloch 1980: Eberhard Knobloch, *Der Beginn der Determinantentheorie. Leibniz nachgelassene Studien zum Determinantenkalkül*. Hildesheim: Gerstenberg, Arbor Scientiarum, Volume 2, 1980.

Knobloch 2004: Eberhard Knobloch, "Leibniz and the use of manuscripts: Text as a process", in *History of science, history of text*, ed. K. Chemla, Dordrecht: Springer, 2004: 51–79.

Knobloch 2018: Eberhard Knobloch, "Notes on Series VII and VIII of the Leibniz-Edition", *Mathematical Correspondences and Critical Editions. Trends in the History of Science*, ed. M. Borgato, E. Neuenschwander, I. Passeron, Cham: Birkhäuser, 2018, 27–47.

Leibniz 1765: *Oeuvres philosophiques, latines et françaises, de feu M. de Leibnitz*, publiées par M. Rud. Eric. Raspe, avec une préface de Mr. Kästner. Amsterdam et Leipzig: J. Schreuder, 1765.

Leibniz 1768: *Gothofredi Guillelmi Leibnitii Opera omnia*, ed. L. Dutens, Geneva: apud fratres de Tournes,1768.

Leibniz 1840: *God. Guil. Leibnitii Opera Philosophica quae exstant latina gallica germanica omnia*, ed. J. E. Erdmann,Berlin: Eicher, 1840

Leibniz 1849–1863: *Leibnizens Mathematische Schriften*, ed. C.I. Gerhardt, Berlin: A. Asher and Halle: H.W. Schmidt, 1849–1863.

Leibniz 1875–1889: *Gottfried Wilhelm Leibniz, Die philosophischen Schriften*, ed. C. Gerhardt, Halle; H. W. Schmidt, 1875–1889.

Leibniz 1903: *Opuscules et Fragments inédits de Leibniz: extraits des* *manuscrits de la Bibliothèque royale de Hanovre*, ed. Louis Couturat, Paris: Alcan, 1903.

Leibniz 1923: Gottfried Wilhelm Leibniz, *Sämtliche Schriften und Briefe*, herausgegeben von der Berlin-Brandenburgischen Akademie der Wissenschaften und der Akademie derWissenschaften zu Göttingen, Reihe 1–8, Darmstadt, Leipzig, Berlin, 1923–.

Leibniz 1993: Gottfried Wilhelm Leibniz, *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis,* critical edition and commentary by Eberhard Knobloch. Abhandlungen der Akademie der Wissenschaften in Göttingen, Mathematisch-physikalische Klasse 3. Folge, Nr. 43, Göttingen: Vandenhoeck & Ruprecht, 1993.

Leibniz 1995: Gottfried Wilhelm Leibniz, *La Caractéristique géométrique*, texte établi, introduit et annoté par Javier Echeverría, traduit, annoté et postfacé par M. Parmentier, Paris: Vrin, 1995.

Leibniz 2011: Gottfried Wilhelm Leibniz, *Die mathematischen Zeitschriftenartikel*, ed. von Heinz-Jürgen Heß u. Malte-Ludolf Babin, Hildesheim: Olms, 2011.

Leibniz 2018: Gottfried Wilhelm Leibniz, *Mathesis universalis. Écrits sur la mathématique universelle*, textes traduits, présentés et annotés sous la direction de David Rabouin, Paris: Vrin, 2018.

Lenzen 1989: Wolfgang Lenzen, "Arithmetical vs. 'Real' Addition – A Case Study of the Relation between Logic, Mathematics, and Metaphysics in Leibniz", *Leibnizian Inquiries*, ed. N. Rescher, Lanham: University Press of America, 1989: 149–157.

Loemker 1989: *Gottfried Wilhelm Leibniz, Philosophical Papers and Letters*, ed. Leroy E. Loemker, Dordrecht: D. Reidel, 1969.

Mora Charles 1992: Maria Sol de Mora Charles, "Quelques jeux de hazard selon Leibniz (manuscrits inédits)". *Historia Mathematica* 19 (1992): 125–157.

Pajus and Rabouin 2017: Anne Michel-Pajus and David Rabouin, "Logica Mathematica: Mathematics as Logic in Leibniz", *Leibniz and the Dialogue between Sciences, Philosophy and Engineering, 1646-2016. New Historical and Epistemological Insights*, ed. R. Pisano and al, London: The College's Publications, 2017: 309–330.

Pasini 1993: Enrico Pasini, *Il reale e l'immaginario. La fondazione del calcolo infinitesimale nel pensiero di Leibniz*, Torino: Edizioni Sonda, 1993.

Poser 2012: Hans Poser, "Die Frühphase der Leibniz-Edition", *Komma und Kathedrale. Tradition, Bedeutung und Herausforderung der Leibniz-Edition*, ed. Wenchao Li, Berlin: Akademie-Verlag, 2012, 23–35.

Probst 2016: Siegmund Probst, "Die postume Edition des mathematischen Schaffens von Leibniz und ihre Rezeption bis Ende des 18. Jahrhunderts", *Theatrum naturae et artium – Leibniz und die Schauplätze der Aufklärung*, ed. Daniel Fulda, Pirmin Stekeler-Weithofer, Stuttgart/Leipzig: S. Hirzel Verlag, 2019, 245–254.

Rabouin 2015: David Rabouin, "Leibniz's Rigorous Foundations of the Method of Indivisibles or How to Reason with Impossible Notions", *Seventeenth-Century Indivisibles Revisited*, ed. V. Jullien, Cham: Birkhäuser, 2015 (Science Networks. Historical Studies, Vol. 49): 347–364.

Scholtz 1934: Lucie Scholtz, *Die exakte Grundlegung der Infinitesimalrechnung bei Leibniz*, Marburg, 1934

David Rabouin is a Senior Research Fellow (CR1) at the French National Centre for Scientific Research (CNRS), in the research group SPHERE (UMR 7219, CNRS – Université de Paris). His interest is in the history of philosophy and mathematics in early Modern Times, with special focus on Descartes and Leibniz. He is the author of Mathesis universalis. L'idée de «mathématique universelle» d'Aristote à Descartes, *Paris, P.U.F., coll. «Épiméthée», 2009 and, in collaboration with the 'Mathesis' Group, of* Leibniz. Ecrits sur la mathématiques universelle, *Paris, Vrin, 2018. With K. Chemla and R. Chorlay, he co-edited the* The Oxford Handbook of Generality in Mathematics and the Sciences *(Oxford University Press, 2017).*

Publication of the Mathematical Works of René Thom in the Collection *Documents mathématiques* of the French Mathematical Society

Marc Chaperon (Université de Paris, France) and François Laudenbach (Université de Nantes, France)

Volume II has just appeared. We present the whole of the project, initiated by André Haefliger.

René Thom in brief

Thom was one of the first members of the mathematical school created around Henri Cartan at the end of the Second World War, and one of the most famous, but *singular* from the start: provincial, born outside of university circles, he had acquired from a very young age an intimate knowledge of differential calculus as conceived by its founders. Unusually for his time, he did not distrust geometry, where he had developed his intuition to the point of "seeing" in four dimensions. He finally followed Henri Cartan to Strasbourg as a young researcher in the CNRS,[1] remaining there after the departure of his master and benefitting in particular from the influence of Charles Ehresmann.

Thus blessed with a vision of the world complementary to that of the "Parisians", Thom resolved fundamental questions which others would doubtless not even have thought of. Some landmarks:

– From 1949 to 1956, Thom worked in algebraic topology and elaborated "a completely new way of studying differentiable manifolds" (Milnor), about which he thus obtained definitive results, giving birth to the theory of *cobordism*, for which he was awarded the Fields medal in 1958.
– From 1956 onwards, he concentrated on the singularities of differentiable maps, which appear naturally in his vision of differential topology. Following on from Whitney, he then studied stratifications and introduced the "natural stratification of mapping spaces" , defined later by John Mather.[2]
– From the middle of the 1960s, aware that singularities of maps and transversality aid in the understanding of all sorts of natural phenomena, he developed a *catastrophe theory*; this met with a period of extensive media coverage after the appearance of his book *Stabilité structurelle et morphogénèse* in 1972.
– At the end of the 1970s, the sometimes delirious infatuation with catastrophes came to a sudden halt, without doubt hardly more justified than the vagaries of fashion. Thom then moved away from mathematics in favour of philosophy and a fertile return to Aristotle.

These changes of orientation must not mask a profound unity of thought: already on his entry to the École Normale Supérieure, Thom was tempted by the philosophy of science, from which the director of the École had to dissuade him. His great mathematical results have a strong philosophical connotation, far from the "modern" pretension of separating mathematics from the question of meaning.

Birth of the project

André Haefliger's admiration for Thom went back to their common Strasbourg years (1954–1958).[3] Their relations only ceased after the disappearance of Thom in October 2002.

Already possessing documents from nearly half a century of exchange, Haefliger devoted much time from the end of 2010 onwards to the study of the Thom *archives*; sorted, inventoried and classified starting in mid-April 2011 in the basements of the IHES by its librarian Aurélie Brest, assisted, at the beginning, by Herminia Haefliger and himself. He discovered "veritable treasures," for example two quite amazing unpublished mathematical manuscripts, of which we will speak again. He involved Marc Chaperon in this discovery in September 2011, with the idea of setting up a "classical" paper publication, with his and Bernard Teissier's help, *annotated and commented on*, of the mathematical works of Thom.

Teissier and Étienne Ghys would have liked this edition to have been accompanied by the online publication of the complete works (mathematical or not) of Thom, published in the form of a CD-ROM at the beginning of 2003 by IHES.[4] But alas! this quite natural idea was halted by copyright problems.[5] As for the CD-ROM, perfectly usable despite the obsolescence of its search engine, it is no longer for sale.

[1] Centre National de la Recherche Scientifique (National Center of Scientific Research).

[2] This idea, to which Thom was much attached, provided for example the framework for Jean Cerf's work on pseudo-isotopy and for that of Victor Vassiliev on knot invariants.

[3] Numerous discussions between them about foliated manifolds had then led to the "concrete" part (analytic foliations) of the still famous thesis on which Haefliger worked under the direction of Charles Ehresmann, rarely present in Strasbourg during this period.

[4] Thanks to the efforts of its director, Jean-Pierre Bourguignon, and to the unfailing enthusiasm of Michèle Porte, director of this project begun in 1996 and in which Thom had actively collaborated.

[5] Those for the books reproduced were ceded by their publishers only for a limited number of copies.

A first editorial committee, constituted in October 2011, met at the beginning of the following month; as well as Haefliger, Teissier and Chaperon, it included Alain Chenciner. The project was submitted at the end of November by Teissier to Pierre Colmez and immediately accepted in the collection *Documents mathématiques* newly created by the French Mathematical Society (SMF). François Laudenbach, Jean Petitot, David Trotman and, for Volume I, Jean Lannes and Pierre Vogel quickly joined the editorial committee.

Overview

This publication does not pretend to be a substitute for the CD-ROM, but aims to complete it:

– It concentrates on the mathematical articles (or is at least classified as such in *Mathematical Reviews*), which are provided here with mathematical or historical commentaries, justified by later developments and the continued relevance of this often visionary work. Certain commentaries are to be found following the article which they relate to, others serve as introduction to several articles. After each article one finds shorter notes concerning precise details.

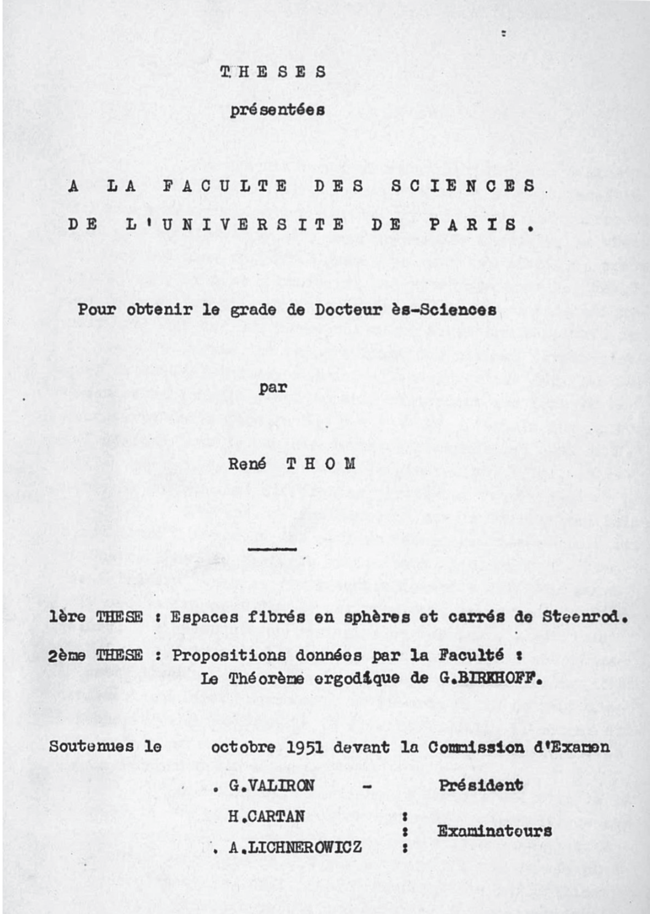
– We have chosen to reproduce the originals instead of transcribing them into modern mathematical typography, thus avoiding introducing errors and changing the page numbering.[6] An exception comprises unpublished texts (absent from the CD-ROM) whose typed version was not legible enough.
As well as these unpublished texts, we are publishing a certain number of documents for their historical interest, for example large extracts from the correspondance with Cartan which led to Thom's thesis[7] and fragments of letters written by him to his wife Suzanne during his stay in Princeton in 1951, which contain much information on the genesis of the work which followed.

– The bibliography, like the biographical notice, completes, corrects and enriches that of the CD-ROM, using earlier versions by Michèle Porte, Jean Petitot and Aurélie Brest, it covers all of Thom's writings, mathematical or not.

Volume I

This volume of 573 pages appeared at last in April 2017 as n° 15 of the collection *Documents mathématiques*. It covers Thom's publications from 1949 to 1959 except one, delayed until the beginning of the following volume. Given the dates, it is not surprising that it contains the major elements of Thom's work. We will evoke them a little more precisely.

The first article by Thom is a note in the Comptes Rendus de l'Académie des Sciences de Paris, 2 pages, entitled *Sur une partition en cellules associée à une fonction sur une variété*.


T H E S E S

présentées

A LA FACULTE DES SCIENCES

DE L'UNIVERSITE DE PARIS.

Pour obtenir le grade de Docteur ès-Sciences

par

René T H O M

1ère THESE : Espaces fibrés en sphères et carrés de Steenrod.

2ème THESE : Propositions données par la Faculté :
Le Théorème ergodique de G.BIRKHOFF.

Soutenues le octobre 1951 devant la Commission d'Examen

. G.VALIRON – Président

H.CARTAN : Examinateurs

. A.LICHNEROWICZ :

It concerns a Morse function, that is to say a function at least of class C^2 whose critical points have Hessians of maximal rank. This note has had a very rich descendance that we have made explicit by a commentary in volume I. Thom is curiously mute here about what will become, starting in 1954, one of his paradigms: the notion of *genericity*. There lacks indeed the supplementary hypothesis of genericity ensuring that the said partition into cells is what is called a *cellular decomposition*, with its specific properties concerning the attachment of the cells. It is Stephen Smale, ten years later, who will clarify this hypothesis on the gradient of the function in question, called the *Morse–Smale condition* today.

Next comes Thom's work on algebraic topology that we have preceded by a note of presentation written by J. Lannes and P. Vogel. This essentially covers the work by Thom for his thesis (1951) and his founding article on the *theory of cobordism* (1954). The thesis was published in the Annales de l'École Normale Supérieure (1952), 73 pages long.

The article from 1954, 69 pages long, has a title which looks to the future, *Quelques propriétés globales des variétés différentiables*. It is here that the notion of *transversality* appears for the first time, a property which is generically satisfied, "most of the time", by a sufficiently differentiable map. This article is important in two aspects. On the one hand it opens the way to a new branch of differential topology which will become *homotopic topology*. This starts from the following problem that Thom solves: under what condition on a homology class of a manifold it is realisable by a submanifold.

6 This choice, even if it has contributed to much delay in the publication of Volume I, was imposed all the more since Thom often published in journals or collections of articles which have become almost impossible to find.

7 We have unfortunately not been able to take into account in Volume I an inestimable document which is part of the archives donated since then by the Thom family to the library of the École Normale Supérieure: Chapter 6 on cobordism, written by Thom and excluded by Cartan, for whom it seemed to not yet make sense.

On the other hand, merely by the apparition of transversality – a few pages of this article – it opens the way to the study of singularities. With some exceptions, Thom will go down this second route. Let us see what happens.

Thom will realise that the way in which he proved his transversality theorem for a submanifold applies at once, without change, to certain situations that one now calls *transversality under constraint*. The first of these situations is the transversality to a submanifold in a *jet space*.

Jet bundles were discovered by Ehresmann. For a function f, let us say real for simplicity, k times differentiable and defined on a manifold M, its jet of order k at a point a of M is, in coordinates, the Taylor development of f at a. Of course, this polynomial depends on the coordinates, but the fact that two functions have the same Taylor development to order k does not depend on coordinates. One can then collect all the k-jets of functions at all points of M and we have thus created a new manifold $J^k(M, \mathbf{R})$ which fibres over M by indicating the point a where the k-jet is taken. A *section* of the bundle is rarely the collection of Taylor polynomials of a single function at all the points of M: there is said to be an *integrability* condition.

Despite this constraint, the theorem of transversality to a submanifold S of $J^k(M, \mathbf{R})$ holds, not in the space of all sections but among those which are integrable – and of course, one can replace $\mathbf{R}$ by any other manifold. However, the choice of S is decisive for the study of singularities of real functions. This is what Thom does in his article, *Les singularités des applications différentiables*, which appeared in the Annales de l'Institut Fourier (1956). Before this paper, a "cap" written by Haefliger and based on his own archives presents the work of Thom on singularities in the period 1956–1957. Moreover, with the agreement of the American Mathematical Society, we have reproduced the long report by W. S. Massey in Math. Reviews; this sheds a useful light on Thom's article.

Because we are in 1957, let us say a few words about two unpublished papers. The first is entitled: *Une démonstration d'un théorème de Lefschetz*, and the second *L'homologie des variétés de Stein*. Thom writes at the very beginning of the second:

> As a result we have the proof of a conjecture of J.-P. Serre: the homology groups $H_i(V, \mathbf{Z})$ of a Stein manifold of complex dimension n are zero for $i > n$.

Thom presented these in a seminar in Chicago in February 1957. Why did they remain unpublished? We will never have the answer. In any case, A. Andreotti and T. Frankel published an article, *The Lefschetz theorem on hyperplane sections*, Annals of Math. (1959), 14 pages. In their introduction one may read:

> Recently Thom has given a proof (unpublished) which, as far as we know, is the first to use Morse's theory of critical points. We present in §3, in a slightly more general setting, an alternate proof inspired by Thom's discovery.

Let us finish this look at Volume I with a last historical evocation. Thom gave a talk in a CNRS meeting in Lille in 1959. This talk was published in the Bulletin SMF with the title *Remarques sur les problèmes comportant des inéquations différentielles globales*. One cannot avoid relating this title to that of the book by M. Gromov, *Partial Differential Relations*, Springer-Verlag, 1986; moreover, Thom's article is cited there. In this article, Thom gives, along with general considerations, a precise statement on the homology of an open set in a jet space. Here we are ten years before Gromov's thesis (1969), who proves an analogous statement but, with an important difference, Gromov speaks of *homotopy*. As Gromov's statement will later be called an h-principle after the initial letter of *homotopy*, one could say that Thom's statement by is an h-principle (only homological) before its time.

The paradox of this affair is that Thom, at the time that S. Smale wrote his (brief) report for Math. Reviews, only believed in his theorem for the jet spaces of order one as shown by Smale's commentary. However, the commentary by D. Spring in our Volume I, p. 562, specifies that he himself knew how to give a proof using the holonomic approximation theorem of Eliashberg and Mishachev (L'Enseignement Math., 2001). Finally, there is a strong chance that Thom's idea of introducing small saw teeth in his simplexes can be carried out to the end.

Volume II

As in Volume I, Thom's texts are preceded here by a complete bibliography of his works.[8] It begins with the course on singularities given at Bonn in 1959, as recorded by Harold Levine. Annotated and commented by Haefliger, this lecture course is followed by a translation of the preface and the table of contents of the Russian edition by V. I. Arnold, then by a letter to Haefliger from February 1959; despite a casual error, this shows that Thom had quickly noticed the modules (*moduli*) which complicate the theory.

The volume then compiles the articles Thom published between 1962 and 1971,[9] in chronological order of their appearance. The help of the editorial committee was solicited more here than in Volume I.

- Many of the long articles contained in the latter, written under the gaze of Henri Cartan and his school, in fact required no further commentary than the mention of their whys and wherefores. After his Fields medal, Thom is more alone but does not lose his tranquil audacity, which results in often prophetic (and, very rarely, badly written) mathematical papers requiring completion[10] and, in this volume, commented on in more detail.
- This audacity led him to "step outside of the framework" with his catastrophe theory, of which the foundational articles figure in this Volume II. Sometimes containing very new mathematics, they are part of another story, notably in biology, of which it was important to give an idea: Sara Franceschelli and Jean Petitot took this on with much talent.

---

8 Mildly corrected with respect to Volume I, which sometimes changes the numbering.

9 Except for the Fermi Lectures *Modèles mathématiques de la morphogénèse* given at the Scuola Normale Superiore of Pisa in April 1971, kept for Volume III as the lectures in Bonn were kept for Volume II.

10 Mathematicians of the calibre of John Mather and Vladimir Arnold worked on this, but many exciting problems remain open.

Singularities

A large part of this volume is formed of essential papers on singularities of maps – notably on their topological stability and on stratifications – annotated and commented upon by Teissier and Trotman, experts in the field. This includes *La stabilité topologique des applications polynomiales*, which already sketches the celebrated isotopy theorems of Thom-Mather, then *Propriétés différentiables locales des ensembles analytiques (d'après H. Whitney)*, excellent exposition of fundamental results by Whitney on the stratification of analytic sets and a draft of the theory of stratified sets; this is further developed in *Local topological properties of differentiable mappings*, astonishingly clairvoyant, and followed by *On some ideals of differentiable functions*, focusing on the difference between the differentiable and the analytic.

Next there comes a "big part," *Ensembles et morphismes stratifiés* which, after inspiring Mather and a pleiad of other experts, has continued to be the source of active research for fifty years. *The bifurcation subset of a space of maps* introduces the absolutely fascinating idea of a natural stratification of function spaces, concerning which much remains to be done.

The volume concludes, in a way closer to the two unpublished papers of Volume I, with *Un résultat sur la monodromie*, commented on by Norbert A'Campo with Teissier, and with an unpublished manuscript on the monodromy[11] which, despite a *fatal error*, contains important and beautiful ideas. As an anecdote, Thom's article had been accepted in a prestigious journal, which had previously simultaneously refused the paper in which the young A'Campo had given a counterexample! Happily, Thom withdrew his text and A'Campo's work, which caused the greatest surprise, established his reputation.

Catastrophes

Preceded by a knowledgeable introduction of Petitot, this part contains five articles. *A dynamic theory for morphogenesis*, "lost" but recovered by Tadashi Tokieda, presents the theory for the first time, insisting on its mathematical aspects – this is in particular the first appearance of *universal unfoldings*; the commentary led us to interrogate Mather, whose response, of great interest, is partially reproduced.

Thom presents his ideas in more detail in *Une théorie dynamique de la morphogénèse*, a fundamental article followed by correspondence with the great biologist C. H. Waddington; the whole piece is commented on by Sara Franceschelli and Petitot, who begin with the relation to *The chemical basis of morphogenesis* by Alan Turing (1952).

Then comes *Topological models in biology*, then *A mathematical approach to morphogenesis: archetypal morphologies* and *Topologie et linguistique*, which illustrates the scale of the project.

Varia

The volume contains other "concrete" applications of the theory of singularities: *Sur la théorie des enveloppes*, written in 1960 to clean up a rather "dirty" domain, is not very readable – we have tried to remedy this in part without departing from Thom's ideas. *Sur les variétés d'ordre fini* sketches notably the proof that, save for rare exceptions, a compact submanifold M of dimension n in $\mathbf{R}^{n+\mu}$ cuts every affine μ-plane P in a finite number of points, bounded as P varies.

More ambitious, the article *Les symétries brisées en physique macroscopique et la mécanique quantique* is confronted by Valentin Poénaru with the later progress in physics. Like the talk *Travaux de Moser sur la stabilité des mouvements périodiques* on "KAM theory," it bares witness to Thom's variety of interests and to his exceptional insight.

The article *Sur l'homologie des variétés algébriques réelles*, commented on by Ilia Itenberg, is devoted to what is often called since the *Smith–Thom inequality*, very important in real algebraic geometry. This work dedicated to Marston Morse depends on Morse theory, of which a "foliated" version is proposed in *Généralisation de la théorie de Morse aux variétés feuilletées*.

Finally *Jets de Liapunov*, commented on by Krzysztof Kurdyka, examines the implications of the existence of a Liapunov function for a vector field in the neighbourhood of a point.

Volume III

Respecting the name of the collection, this *last* volume[12] essentially contains mathematical writings, a small minority of Thom's publications after 1971, but often remarkable. Let us point out some of them. The first text, the Fermi lectures *Modèles mathématiques de la morphogénèse*[13] is a clear exposition of the stakes of catastrophe theory and a lot of new mathematics, which have had an important progeny. A critical rereading was therefore in order.

Then comes *Sur le cut-locus d'une variété plongée*, a paper which is very rich mathematically (it takes the functional viewpoint of *The bifurcation subset of a space of maps*) and even beyond mathematics – one finds there quite an up-to-date model of visual perception.

Phase transitions as catastrophes which is mathematically substantial[14] proposes in particular a proof of the Gibbs phase rule based on the generic structure of *Maxwell sets*.[15] This notion was at the heart of the cut locus paper.

Sur les équations différentielles multiformes et leurs intégrales singulières provides among other things a marvellous introduction to singularities, contact geometry and Pfaffian systems.

Symmetries gained and lost analyses symmetry breakings, in connexion with the work of the physicist Louis Michel and as a continuation of the paper of Volume II.

Introduction à la dynamique qualitative, a mathematical-historical-philosophical text *à la* Thom, evidences his longstanding interest for this domain which was for a time not so popular in France, a situation which his seminar around

[11] In a modern typography of Duco van Straten.

[12] In preparation, we would like to see it published at the end of 2020.

[13] It provides the first three chapters of the book with the same title published in 1974 in the pocket collection 10–18.

[14] and bold, sometimes a little too much so!

[15] The set of values of parameters for which a function depending on those parameters attains its minimum in several points (if one counts them with multiplicities).

1970 had remedied, contributing to the orientation of future "leaders" in the subject, such as Michel Herman.

Gradients of analytic functions, absent from the CD-ROM and unearthed by our Iranian colleague Massoud Amini, notably states the "gradient conjecture" on the limits at a singular point of the tangents to integral curves of an analytic gradient.[16] Enthusiastic commentary by Kurdyka.

Tectonique des plaques et théorie des catastrophes is the trace of a daring incursion into Claude Allègre's territory.

The Les Houches lectures *Mathematical concepts in the theory of ordered media* is a follow-up to the work of Kléman and Toulouse on dislocations in crystalline media.

The note with Yannick Kergosien *Sur les points paraboliques des surfaces*, completed and corrected with Thomas Banchoff, deals with the apparent contours of a "generic" surface in euclidean space when the direction of projection varies.[17]

The two notes with Peixoto on *Le point de vue énumératif dans les problèmes aux limites pour les équations différentielles ordinaires*, illustrate in particular the very enlightening vision that Thom had of "well posed" problems: they are those in which the d-dimensional manifold of solutions of the differential system is transversal in the function space to the d-codimensional variety defined by the limit conditions of the problem.

Quid des stratifications canoniques? evidences again their extreme importance in the eyes of Thom.

Finally, the obituaries of Morse and Whitney written for the *Académie des sciences* are obviously of great interest, as well as *La théorie des jets et ses développements ultérieurs* published with Ehresmann's complete works.

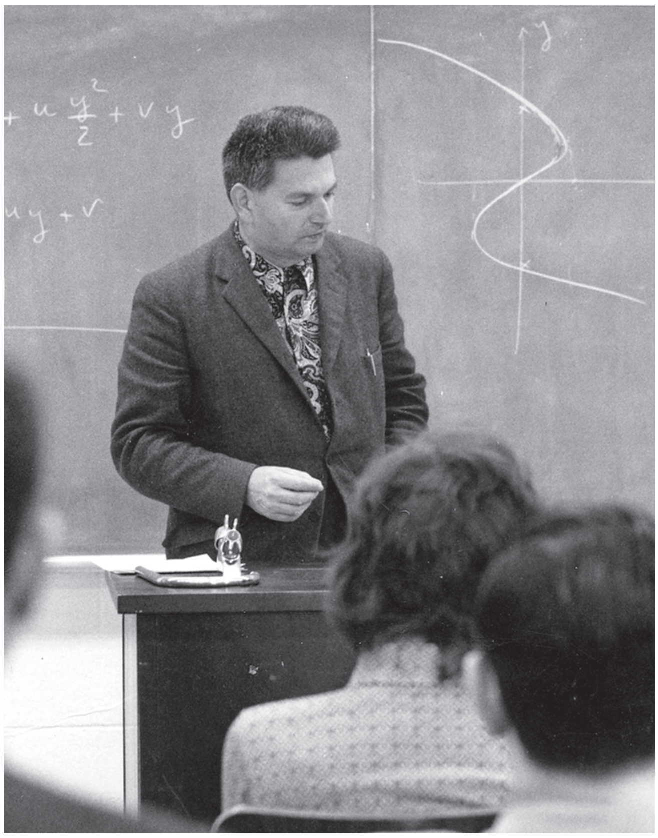
We also write a little about "the catastrophe of catastrophe theory" which took place in 1978 and without any doubt contributed to Thom distancing himself from mathematics, to its detriment.

Appendix: A modest overview of transversality according to Thom

Transversality plays an essential role in Thom's work, both in mathematics and in catastrophe theory, where a key idea is that one can only observe the phenomena that are stable under perturbation. The transversality lemma in jet spaces[18] amazed its first "guinea pig" Whitney, who at first could not believe this statement which throws almost all general position arguments into the same pot. That is why we thought it would be useful to present an overview of it.[19]

First elementary version

The following facts seem to correspond with our intuition. If, in $\mathbf{R}^3$, a point p lies on a surface S, jiggling[20] will push p outside of S. If a (compact) curve C is tangent to S at one or several of its points, jiggling will result in either C and S being disjoint or, at the possible common points, the tangent to C and the tangent plane to S will intersect transversally. Finally, if S' is a (compact) surface which at some points shares a tangent plane with S, jiggling of S' will put it in a transversal position with respect to S; at the possible common points the tangent planes will be secant; in fact – and this is less intuitive – if this last condition on the tangent planes is satisfied at each point of the intersection $S \cap S'$, this intersection will be a smooth curve (this last fact is guaranteed by the implicit function theorem).



Catastrophes, around 1975 (©Archives de l'IHÉS – Droits réservés)

Thom's first statement on transversality is a generalisation on all dimensions and all manifolds of those three examples. But one has still to specify what "jiggling" is. It is a diffeomorphism g of the ambient manifold M, arbitrarily close to the identity (in the C^∞ topology) which one applies to a (compact) submanifold N in order to make it *transversal* to another submanifold N' of M: at any point of $g(N) \cap N'$ the two tangent spaces $T_x g(N)$ and $T_x N'$ generate $T_x M$. In what follows all manifolds will be assumed to be C^∞ (we shall say "smooth").

Elementary transversality theorem (Thom)**.** *The property that* $g(N)$ *is transversal to* N' *is* generically *satisfied for* g *in the group* $\mathrm{Diff}^k(M)$ *of* C^k *diffeomorphisms of* M *if* $k > \max(0, \dim N - \operatorname{codim} N')$. *Here, a property is said to be* generic *if it is satisfied at all points of a countable intersection of dense open subsets of* $\mathrm{Diff}^k(M)$.

Remarks. 1) If N is compact, transversality is an open property in the C^1 topology and therefore in all finer topolo-

[16] A version of it has been proved by Kurdyka, Mostowski and Parusiński in a paper published in 2000 in the *Annals of Math.*

[17] The case of a single projection was very well treated in the paper *Sur les équations différentielles multiformes et leurs intégrales singulières*.

[18] See Thom's paper *Un lemme sur les applications différentiables* (1956), reproduced and commented in Volume I.

[19] Marc Chaperon had treated it very differently in issue 65 of the *Gazette de la SMF*, with many other applications.

[20] A generic small movement; we are borrowing this term from W. Thurston.

gies such as the C^∞ on the group $\mathrm{Diff}^\infty(M)$. Moreover, under this assumption, genericity is also true in the subgroup $\mathrm{Diff}^\infty_c(M)$ of diffeomeorphisms with compact support.

2) The same statement is still valid if N is singular (presence of double points, lack of tangent space) or even the image in M of a manifold N by a C^k map.

Proof of Thom's theorem based on theorems of A. Morse and A. Sard

Theorem (Morse–Sard)**.** *Let X be a manifold. If $f : X \to \mathbf{R}^q$ is C^k, $k > \max(0, \dim X - q)$, then, almost every y in $\mathbf{R}^q$, in the sense of the Lebesgue measure, is a* regular value *of f, which means that for every $x \in f^{-1}(y)$ the differential of f at x is of rank q.*

For instance, if $\dim X < q$, almost no y is in the image of f. By the implicit function theorem, the inverse image of a regular value is a C^k submanifold of codimension q.

The inequality on k in the statement above is necessary. For example, Whitney has built a function $f : \mathbf{R}^2 \to \mathbf{R}$ of class C^1 and an arc A in the plane where f is not constant but along which the differential of f is identically zero (the arc A is continuous but of course not C^1); the interval $f(A)$ contains no regular value of f although it is of positive length.

Here is how Thom uses Sard's theorem to prove his first transversality theorem. With its notations, let us consider the case $k = \infty$ and N compact and concentrate on the density part. Since $\mathrm{Diff}^\infty_c(M)$ is a group one easily shows that it suffices to prove that id_M can be approximated in the C^∞ topology by $g \in \mathrm{Diff}^\infty_c(M)$ such that $g(N)$ is transversal to N'.

Thom's idea is to build, for a certain integer r, a map $G : \mathbf{R}^r \times M \to M$, $(v, x) \mapsto G(v, x)$ of class C^∞, such that:

(1) $G(0, x) = x$ for all $x \in M$;

(2) For all small enough fixed v, $g_v := G(v, -)$ is a diffeomorphism of M with compact support;

(3) $\partial_v G(0, x)$ is of rank $n = \dim M$ for all $x \in N$.

We shall think of G as a family $\{g_v\}_{v\in\mathbf{R}^r}$ depending on r real parameters. Let us assume we have such a family and prove Thom's theorem. Since the rank condition is open, condition (3) implies that $\partial_v G(v, x)$ is of rank n for all (v, x) in a neighbourhood U of $\{0\} \times N$. Let us still denote by G its restriction to U. With the rank condition, the implicit function theorem tells us that $W := G^{-1}(N')$ is a submanifold of U of class C^∞.

Let $\Pi : W \to \mathbf{R}^r$ be the projection onto the parameter space. One now applies Sard's theorem to Π. Almost every (r-dimensional) parameter v is a regular value of Π. An easy lemma of linear algebra shows that for such a v, the diffeomorphism g_v pushes N to a position transversal to N'.

It remains only to build the family G. Let us begin with the case $M = \mathbf{R}^n$. We take as parameter space the vector space $\vec{\mathbf{R}}^n$ of translations of $\mathbf{R}^n$. By a well chosen partition of unity, one extends the translation of the compact N by the vector v into a diffeomorphism g_v with compact support which is id_M if $v = 0$. Conditions (1)–(3) are satisfied.

In the case where M is a manifold, using the compactness of N we cover it by a finite collection $\{B_i\}_{i=1}^s$ of balls, each of which is contained in a coordinate chart O_i of M.

One chooses the translations v_i in the coordinates of O_i small enough for $B_i + v_i$ to remain in a fixed compact subset of this chart. One easily extends this translation into a diffeomorphism g_{i,v_i} of M with compact support which is id_M if $v_i = 0$. Let $v := (v_1, \dots, v_s)$. Set $g_v = g_{1,v_1} \circ \cdots \circ g_{s,v_s}$. One easily verifies that conditions (1)–(3) are satisfied, which ends the proof.

Transversality and jet spaces

It is obviously in the jet spaces of order $r \geq 2$ that Thom will give the deepest applications of the transversality theorem to singularities of differentiable mappings. However, one can already realise the innovative aspect of this theorem with jets of order 1 (or 1-jets) of real functions of one real variable.

In order to start with a compact manifold, let us look at the 1-jets of functions on the interval $[0, 1]$, which is not exactly a manifold, but a manifold with boundary. The boundary creates no difficulty for understanding the space of real valued C^∞ functions.

If $f : [0, 1] \to \mathbf{R}$ is C^k, $k \geq 1$, we define its *jet of order one* $j^1 f(a) = (a, f(a), f'(a))$ at any point $a \in [0, 1]$.[21] The map $j^1 f$ is a section (that is, a right inverse) of the projection $(x, y, z) \mapsto x$ of $J^1([0, 1])$ to $[0, 1]$; thus, it identifies with its image which is a C^{k-1} submanifold of $J^1([0, 1])$.

Given $a \in [0, 1]$, any $P \in \mathbf{R}[T]$ of degree one is equal to $j^1_a f$, where $f(x) = P(x - a)$. However a global section $s(x) = (x, y(x), z(x))$ is of the form $j^1 f(x)$ if and only if

$$z(x) = \frac{dy}{dx}(x) \quad \text{(integrability condition).}$$

This condition is quite rarely satisfied. Thus, if Σ is a submanifold of $J^1([0, 1])$, the proof given for the transversality theorem (elementary version) will produce from $j^1 f$ at best a section transversal to Σ, with practically no chance to produce an integrable section.

We remember that everything lies in the choice of the family G of maximal rank with respect to the parameters. It is Thom's clairvoyance which made him see that the same proof worked in the subspace of integrable sections if we take as parameter space the (finite dimensional) space of degree one real polynomials, here in one variable. At the same time we shall replace the diffeomorphism group of $J^1([0, 1])$ by its subgroup, which one might call the *gauge group* consisting of those diffeomorphisms which preserve not only each fiber $J^1_x([0, 1])$ but also the distribution of *contact planes*, i.e., the kernels of the differential form $dy - z dx$.[22]

If P is a polynomial function of degree one, the translation (in the fibers)

$$(x, y, z) \mapsto (x, y + P(x), z + P'(x))$$
$$\text{in other words} \quad j^1 f(x) \mapsto j^1(f + P)(x) \quad (1)$$

is a gauge transformation, sending each integrable section $j^1 f$ to the integrable section $j^1(f + P)$. This family of translations verifies conditions (1)–(3), once truncated to obtain (2).

The same approach works in any dimension of base manifold and for any jet order.[23] Thom thus obtains the following theorem:

---

21 In other words $j^1 f(a) = (a, j^1_a f)$, where $j^1_a f \in \mathbf{R}[T]$ is the Taylor polynomial $f(a) + f'(a)T$.

22 A "contactologist" tradition exchanges y and z, giving $dz - y dx$.

23 Even replacing the target space $\mathbf{R}$ by another manifold.

Transversality theorem in a jet space (Thom). *Let N be an n-dimensional manifold, let $J^r(N)$ be its space of r-jets of real functions and let Σ be a submanifold of $J^r(N)$ of codimension q and of class C^ℓ, $\ell > \max\{0, n-q\}$. For any $f \in C^k(N, \mathbf{R})$, $k - r > \max\{0, n-q\}$, generically for g in the C^{k-r} gauge group, the submanifold $g(j^r f(N))$ is transversal to Σ.*

Here is a first application to Morse functions. At the beginning of the 1930s, Marston Morse showed the importance of the functions which now bear his name for the understanding of manifolds. Recall that a Morse function $f : N \to \mathbf{R}$ is a function, of class at least C^2, whose critical points (points where $df = 0$) are all non degenerate (in coordinates, the matrix of second partial derivatives is of maximum rank).

Corollary. *Morse functions of class C^k on N are dense in $C^k(N, \mathbf{R})$ for $k \geq 2$.*

Indeed, a simple computation shows that f is a Morse function if and only if $j^1 f$ is transversal to the zero section $\Sigma := \{j^1\varphi(x) : d_x\varphi = 0\}$ of the projection $j^1\varphi(x) \mapsto j^0\varphi(x) := (x, \varphi(x))$. According to the theorem (here with $q = n$ and $r = 1$), generically $j^1 f$ is transversal to Σ, hence the density.

Marc Chaperon [marc.chaperon@imj-prg.fr], currently a professor emeritus at the university of Paris, is a former student of the École normale supérieure. As a beginner, he received the advice of René Thom, Alain Chenciner and Michael R. Herman. His work in differential geometry deals with dynamical systems, singularities and symplectic geometry.

François Laudenbach [francois.laudenbach@univ-nantes.fr] is a topologist (Pseudo-isotopy, 2-spheres in 3-manifolds, generating functions in symplectic topology, Morse-Novikov theory). He was awarded Peccot lectures (Collège de France, 1973). He had professor positions successively in Université Paris-sud (Orsay), École Normale Supérieure de Lyon, École polytechnique, Université de Nantes where he retired as Professor emeritus.

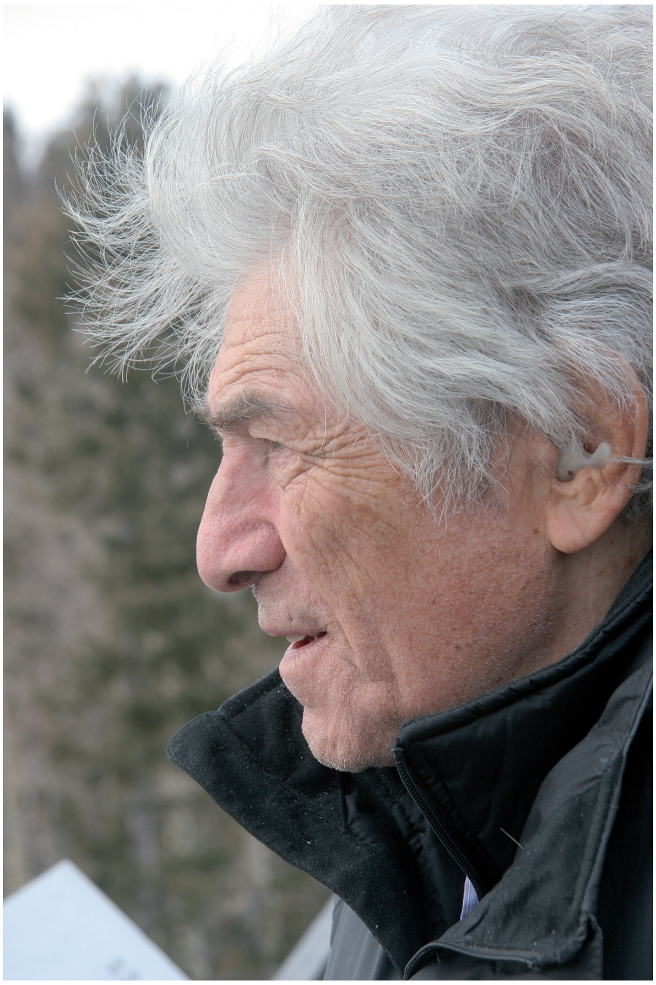
Reuben Hersh 1927–2020 Critic and Philosopher of Mathematics

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden)

Reuben Hersh died just after New Year this year, having just turned ninety-two a few weeks before. I first heard of him in the early 80s in connection with the book *The Mathematical Experience*, written along with Philip Davis. The book had just made a splash, as far as a mathematically oriented book can make a splash outside of the mathematical community. It had been reviewed by Martin Gardner in the *New York Review of Books*, which was considered an honour. And it also won a National Book Award. It is indeed a wonderful book, written by regular mathematicians and not philosophers or journalists. I recall a conversation at the IAS at the time when Armand Borel praised it, partly for that reason, and Borel would not praise without very good reason. Although I loved the book, I cannot, almost forty years later, really recall its contents. This is not atypical when you read something and thoroughly digest it. You may not be consciously aware of it, but it has worked its influence deeply. Flipping through the pages in connection with writing this text, I realised that many things I now know I had picked up there, and many things I thought I had made up myself, I could likewise find in its pages.

The book was a joint project with Philip Davis, a numerical analyst at Brown. As a reader you may be puzzled that it is written in the first person. Who is this person? Hersh or Davis? Sometimes Hersh and sometimes Davis, in fact. The explanation being that it is actually two books which were merged. Davis was working on a book on mathematics for the layperson, a genre in which he had some previous experience[1], while Hersh was writing on the philosophy of mathematics, so the overture of the book refers to the genesis of that project.[2]

Hersh told me that he did indeed give a course on the philosophy of mathematics in the time-honoured tradition of wanting to learn something new. As he did so, he realised that he did not have a philosophical point of view himself. Was he a Platonist, a Formalist or maybe even a Intuitionist? He wanted to find out, and what better way to do so than to write a book? The two became a team and went on to write one more joint book (*Descartes' Dream*). Hersh would then eventually strike out on his own and write books with titles such as *What is mathematics, Really?* and, jointly with his partner Vera John-Steiner, *Loving and Hating Mathematics*. Four books which form a loosely connected series. But this was of course not his last writing. He would go on writing a lot of articles as well as writing a biography on his mentor and advisor Peter Lax. Although it was the book *The Mathematical Experience* which brought their names to my and many other people's attention.

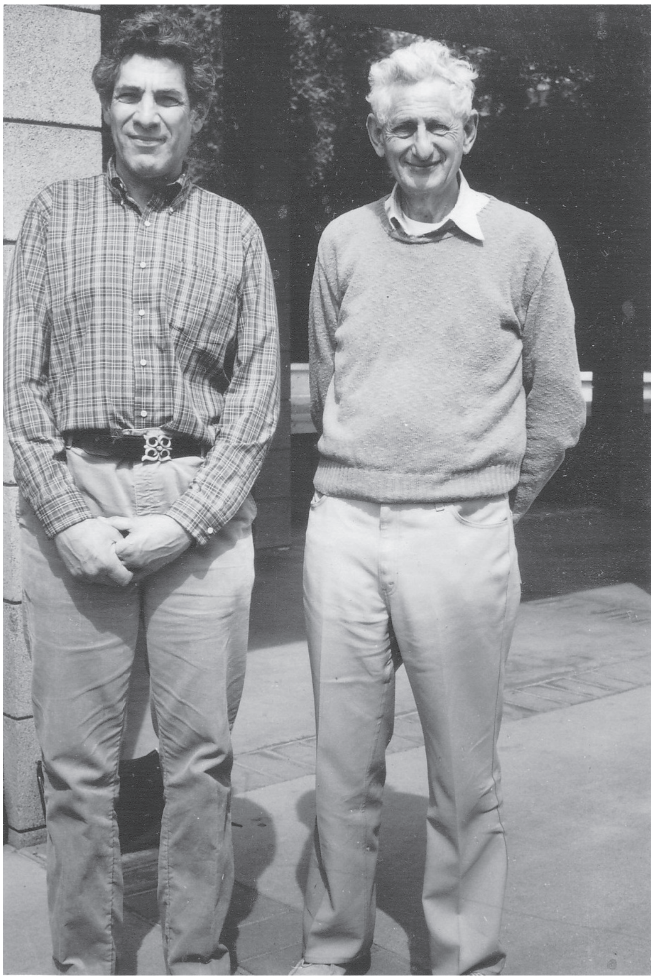


Reuben Hersh (February 2011).

The book has in many ways the character of a scrap book, as it consists of short sections (grouped into lengthier chapters) on a variety of topics, sometimes loosely tied together, sometimes tightly so. They are also

[1] In fact, in my early teens I read and enjoyed a Swedish translation of *The Lore of Large Numbers,* which I only recently realised was written by him.

[2] In *Experiencing Mathematics* Hersh writes: "I was fortunate in establishing a partnership with Philip J. Davis. He wanted to write math for the intelligent non-mathematician, while I wanted to write philosophy of math, so we agreed to work in parallel, for mutual support. Unfortunately, I was overwhelmed by deep personal problems. I wrote what I could, and then collapsed. Amazingly, Phil and Hadassah were able to sew together Phil's chapters and mine, into some kind of a coherent book. More amazingly, the book became a best seller, so far as math books go, and won a National Book Award."



Reuben Hersh and Philip Davis.

written in different styles, some very light-hearted in their forms of imaginary interviews, others more regular discourse. This makes the book easy to read, in fact a real page turner, as it does not put undue demands on your attention span, but instead invites you to happily race along, wondering what is going to happen next. It does contain a fair amount of mathematics, but although addressing a wider, mostly non-mathematical audience, it does not fall in the didactic trap which makes so much of mathematics get dumbed down, ostensibly for the benefit of the ordinary reader, and so painful for the professional mathematician to read. The secret is that most of the book, apart from playful interludes, is written in the manner of essays. And what is an essay? It is an attempt to find out what you think[3] and (optionally) letting others eavesdrop on you.

So when the authors expound on some mathematics, they do so in order to look at it critically and philosophically with fresh eyes, thus putting themselves at the same level as the reader, viewing the material both with the innocence of the ignorant and the retroactive wisdom of the expert. This does not mean that the book is perfect, I personally find that the short section on non-Euclidean geometry leaves much room for improvement, even without having to be extended in bulk. But the real interest of the book, as well as its greatest impact, has to do with Hersh's take on traditional mathematical philosophy. In the past there really was little distinction between a scientist and a philosopher, you tended to be a little bit of both in as far as you were at least a little bit of either. In the 19th century it was not uncommon for a physicist to become a philosopher in old age, examples are Mach and to some extent Boltzmann. Einstein's approach to relativity was as much that of a philosopher as a mathematician. And at the turn of the century, mathematicians such as Hilbert, Weyl and Brouwer were also philosophers. On the other hand, philosophers such as Russell and Wittgenstein were not mathematicians, although Russell made valiant efforts. Nowadays it is rare to find a mathematician with not only a philosophical interest but also an active engagement with it. With this book, Hersh proved himself to be one of the rare birds.[4]

In the book, Hersh looks at Platonism, formalism and intuitionism, and rejects them all as being far too restrictive. Platonism he finds downright embarrassing with its references to an abstract world beyond space and time, and thus not much above ancient superstition; but he agrees that most working mathematicians tend to be naive Platonists, although they may not admit it openly. Formalism, introduced by Hilbert[5], Hersh dismisses as a caricature of mathematical practice, turning it into a meaningless game. Intuitionism, on the other hand, he is very sympathetic to, yet finds it in many ways too extreme in its radical rejection of the exclusion of the middle third. While most mathematicians are Platonist at heart, although when pressed may make references to the formalist foundation of mathematics, few mathematicians are in fact practicing intuitionists, a movement Hersh sees in terms of being an heretic sect, although he does appreciate many of its points and thus they act like a salt in the mathematical community. He concludes that mathematics is too rich and mysterious to be straitjacketed into one of those three slots. Each category admittedly provides a view of mathematics, but a very limited one. He compares it to the various views of a cube projected onto the plane. The projections do look like different pictures; only when you can fathom the full 3-dimensional cube do they make sense taken together. The same is true with mathematics. Incidentally a very Platonic metaphor. Along with the rejection of the three 'isms' of mathematical philosophy, he takes exception

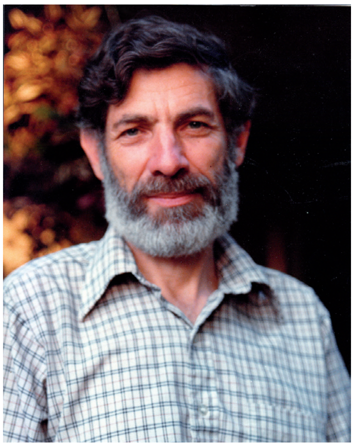
[3] I picked this up from the American diplomat George Kennan, who reveals in his autobiography that the real reason he wrote all those dispatches was not for the ostensible recipient – the State Department, which he did not expect to read them, but for himself, in order to find out what he really thought.

[4] Another rare bird was Gian-Carlo Rota, who was in fact a card-carrying phenomenologist and an enthusiastic follower of Edmund Husserl. Rota incidentally provides the book with a foreword.

[5] But Hilbert was never a formalist at heart, formalism he espoused only for very specific technical purposes; namely the dream of mathematically proving the consistency of mathematics, and thus keeping the paradise safe.

to the undue emphasis on foundations in contemporary philosophy of mathematics. These questions do not really interest him, in fact he finds them sterile dead-ends. Mathematics provides such a wealth of interesting food for philosophical thought, so there is no need to fixate yourself on scholastic speculations. When it comes to his own philosophy of mathematics, he takes as his point of departure two apparently irreconcilable facts. The first fact is that mathematics is done and created by humans, the second fact is that, unlike art, which is also done and created by humans, it is nevertheless a science. As every working mathematician experiences, the concepts he or she encounters and maybe even creates are not arbitrary, but are forced on you. The mathematical reality kicks back at you and becomes very palpable in spite of the abstractness of its mental concepts[6]. There is no subjectivity when it comes to mathematics, unlike in the arts, and there is also a remarkable consensus when it comes to mathematics, a consensus you would never find in any other human endeavour. The formalists would attribute this to the ironclad rules of logic, but it is a well-kept secret among mathematicians that the consensus and unity of mathematics goes deeper than mere logical compulsion. Deductive reasoning does play an important role in mathematics, similar to that of election in democracies, but it is far from the full story; I am fond of comparing the logical foundation of mathematics with presenting a picture pixel by pixel.

How can we reconcile these very different aspects of mathematics? We simply have to, because mathematics is there. Out there? We may provocatively ask him. No, Hersh replies, not out there but within us. Not within individuals as much as within mankind itself, there is no reason to look for it anywhere else. To me this seems very close in spirit to Carl Jung's notion of the collective unconsciousness. But never mind, the Jungian vision may strike the reader as mere mysticism, however, I think it nevertheless has great potential. Yet to explain mathematics by it presents an even greater challenge than that of explaining art, which was the original intention of Jung. Hersh does not refer to Jung (although he uses the term 'collective consciousness' a few times), instead he refers to Karl Popper and his World3 of mental constructs. Maybe that constitutes a 'collective consciousness' born out of a mysterious 'collective unconsciousness'? Hersh, by the way, is no stranger to ascribing a measure of mystery to many mathematical facts. The reference to Popper is significant, as Popper did not count mathematics as a science, as its results were based on deduction and thus not straddled with the problems of induction. In this respect, I think Popper was mistaken, and one of his students – Imre Lakatos – actually made a serious start on viewing mathematics from a Popperian point of view (actually a subject for one of the sections in the book). But the legacy of Hersh does not consist of viewing mathematics as a science (let alone the horrible thought of merely being a language for the same), but putting it firmly in the humanistic camp. As a philosopher, he was not accepted by the community of professional philosophers, while he on the other hand did not think that they really understood mathematics.



Reuben Hersh (1983).

So how could one pursue an interesting and fruitful philosophical study of mathematics? Hersh does not formulate any definite answers, but the book and his other writings make suggestions. Maybe it would be more instructive to view Hersh not so much as a philosopher of mathemetics, but as a cultural critic of it. Personally, I think that there is a need for mathematical critics who do the critical work that is done in literature and the arts, especially in view of the bureaucratic way mathematical achievement tends to be judged academically in matters of promotion. But how to start? I am tempted to refer to the British philosopher and historian R.G. Collingwood and his view on the relationship between philosophy and history. Collingwood took exception to the analytic philosophy practiced in his days, and instead stressed the moral responsibility of philosophy and thus how its study related to man. Thus history was his main philosophical concern. Not history of the trivial 'cutting and pasting' variety, but history which puts human thought at its centre, and thus makes intelligible its acts as being the outcome of human intentions. It is exactly the presence of human intentions which makes history as we know it differ profoundly from natural history, in which mankind has no role. This view by Collingwood has been criticised, among others by Popper, as being too subjective, but there is a very strong objective component to human thought (as exemplified by the notion of intention), which is the one that interests Collingwood and to which Hersh refers, at least implicitly, as manifested by mathematics. Pursuing a history of mathematics putting the mental history at its core would be, I believe, one fruitful avenue to exploring Hersh's humanistic view of mathematics. As Collingwood puts it, you cannot move the past into the present, not even fragments of it (pace Marcel Proust), but you can reconstruct it in the context of the present, just as we reconstruct mathematical ideas when we ponder a proof. Thus a history of mathematics should not be so much a documentation of who did what and when, but an exploration of the evolution of ideas and how they interact and their dependence on actual time.

In 2007 I picked up Hersh's *What is mathematics, really?*, the title being an obvious take on the book by Courant and Robbins *What is mathematics?* I contacted

[6] I would personally argue that, just as reality becomes palpable due to the consistency of all our senses in perceiving it, mathematics thrives on the consistency of so many different approaches, which makes up for a remarkable degree of interrelatedness.

the author and we soon established a delightful e-mail exchange which would continue up to his death. I met him only once in person, when I was dispatched to his home in New Mexico in early 2011.[7] I stayed over at his and his partner Vera's adobe house for a few days, during which time we had a continuous conversation. I wrote some of it down in the form of an interview with his active assistance, the link to which will be listed below. We touched upon a great variety of topics, as his interests ranged widely, and he was never short on thoughts or words. He made me feel very welcome by claiming that it was so nice to be able to talk and discuss matters that mattered to him deeply and which would only bore those around him. He was already an old man when we met, in fact well into his eighties, and his appearance with a full head of white hair made me think of an old American Indian chief (later on, I learned from pictures, he grew a long beard and my associations turned to Walt Whitman). He was fit enough to beat me at ping-pong and, apart from some hearing loss, he showed few if any of the signs of aging, in particular none of the befuddlement which often degrade the final years of the elderly. His mind was lucid then and would continue to be so almost until the end, I have been told. At the time of my visit, he was promoting his book with Vera on loving and hating mathematics, which focused on the psychology of mathematicians or more precisely how encountering mathematics affects you. When I made some ironic remarks about his concerns for marketing it, he told me that you look out for your books as if they were your children and I should just wait and see until I was publishing my first book. Unfortunately, the book did not receive the attention that the authors had hoped for, and some of the reviews were even hostile. I wrote one review and sent it to Hersh, who responded very warmly; as I suspect it was not entirely due to politeness and thus constituting some sort of confirmation that it at least gave a fair description of the book, I include a link below.

We both left for the airport in a shared taxi before dawn. I had to fly back to the East Coast, he had to meet with his advisor Peter Lax in connection with his work on the above-mentioned biography. The early hour of the day in no way impaired his volubility, on the contrary, being on the eve of the conclusion of our get-together it was instead enhanced. We never met again, but our e-mail exchanges continued, refreshed by our personal meeting.

We did not always agree on our views of mathematics, anything else would of course have been surprising, not to say disturbing. The greatest source of dissension was our view on Platonism in mathematics. He was puzzled by my stand, naturally, as his conception of Platonism was a bit too crude and literal in my opinion; on the other hand I never managed to present a version of Platonism that he could embrace. This could be construed as an unbridgeable division as supposedly between deists and atheists; but in fact it had very little impact on our mutual sympathy and respect and was to be seen merely as a matter of rhetoric rather than of any real substance. What mattered was what we shared.

No one lives forever, not even the long-living, and it is with sadness I note that I have lost a most stimulating discussion partner. His legacy, as noted above, is that he was a very articulate proponent for the human side of mathematics and that the philosophy of mathematics was too rich and important to be left to the professionals.

Remarks and links

A memorial site can be found at https//sites.google.com/view/in-memory-of-reuben-hersh.

The interview is accessible at the memorial site or directly from https://sites.google.com/view/in-memory-of-reuben-hersh/interview-with-ulf-persson-2011
It contains some biographical material.

The review of *Loving and Hating Mathematics* can be found on http://www.math.chalmers.se/~ulfp/remat.html

In 2008 I invited a few mathematicians to expound on mathematical Platonism in the Newsletter. Some wrote in favour, others critically, and among them were Hersh: https://www.ems-ph.org/journals/newsletter/pdf/2008-06-68.pdf (page 17)

Ulf Persson is a member of the Editorial Board of the Newsletter. A detailed biography can be found in previous Newsletter issues, e.g. NL 107, March 2018.

[7] Sent by the Swedish institution NCM (National Center for Mathematics, an institution devoted to the teaching of mathematics) by its director Bengt Johansson.

The Spanish Society of Statistics and Operations Research

Jesús López Fidalgo (Universidad de Navarre, Pamplona, Spain), President of the Spanish Society of Statistics and Operations Research and Director of DATAI: the Institute of Data Science and Artificial Intelligence

www.seio.es / @SEIO_ES

The Spanish Statistics and Operations Research Society (SEIO) was first created on the 12th of February 1962 under the name of the Spanish Operations Research Society. Its first meeting took place in the rooms of the Institute of Statistical Research of the Spanish Upper Council of Scientific Research (CSIC) and was attended by 52 people interested in methods of operations research and the propagation of the theory and practice in the field throughout Spain.

In a general meeting held on the 30th of June 1976, the scope of the SEIO was extended to include statistics and computing. In the general meeting held on the 20th of December 1984 it was decided to modify its statutes. The name was changed to the current one of the Spanish Society of Statistics and Operations Research. In March 1999, some of the statutes were further amended and the very latest change came in 2012, which are the ones that govern the Society today. At present, the SEIO has nearly 700 individual members plus a number of institutional members. One special member is the Spanish Statistical Institute (INE), part of the Spanish administration for the management of official statistics. In 1989, both institutions signed an agreement to the effect that there is always a representative of INE in the executive committee of the SEIO and a representative of SEIO in the National Council of Statistics. Since then there has been a symposium of public statistics within the conference of the Society, held every one and a half years. In that symposium a prestigious prize for public statistics is awarded. There are a number of agreements with other national and international societies.

The following is a chronological list of those who have held the position of president of the SEIO since its inauguration: Fermín de la Sierra, Sixto Ríos García, Pilar Ibarrola Muñoz, Marco A. López Cerdá, Daniel Peña Sánchez de Rivera, Elías Moreno Bas, Jesús T. Pastor Ciurana, Rafael Infante Macías, Pedro Gil Álvarez, Domingo Morales González, Ignacio García Jurado, José Miguel Angulo Ibáñez, Leandro Pardo Llorente, Emilio Carrizosa Priego and Jesús López Fidalgo.

Today, the SEIO is in excellent health, and enjoys playing a special role in the promotion of these two popular disciplines, which are very much related, both in the national and international arena. Having statistics and operations research in a single society is not common in many countries. We believe this is a powerful added value with the upcoming new era of data science and artificial intelligence. This places our Society in an important position for modern science.

Our conference every one and a half years is one of our main activities. There were around 400 attendees at the last one. Most of them come from all over Spain and there is a significant number of people who come from abroad, normally invited by one of our working groups. Since 2018 we also have a workshop for our young members in the period between the main conferences. There have been two editions so far and the quality of the contributions and participants has been remarkable.

Another important potential are our journals. *TEST* is the statistical journal and *TOP* is the operations research

General meeting held on the 30th of June 1976.

Prof. Daniel Peña was a plenary speaker at the last conference in Alcoi, 2019.

one. Both are indexed in the Web of Science and Scopus, among other databases. BEIO (Bulletin of Statistics and Operations Research) publishes scientific outreach articles on statistics and operations research. Such articles aim to treat relevant topics in a way which is accessible to the majority of professionals in statistics and operations research without sacrificing scientific correction in the treatment of the subject in question.

At the moment we own nineteen working groups covering most of the areas of our disciplines. The purpose of the working groups is to promote the communication and research between members of the SEIO working on common themes in statistics and/or operations research. They hold some meetings or workshops within the periods between the SEIO conferences. They also organise high quality sessions during the SEIO conference.

Helping young people and promoting women's participation are also big priorities in our Society. We have been making special efforts in primary, secondary and high school teaching of our disciplines. Transference with industry, private companies and other institutions is also an important goal.

SEIO is part of the Spanish National Committee for Mathematics (CEMat). At this moment one of our representatives holds the presidency of the Committee.

Jesús López Fidalgo [fidalgo@unav.es] has an MSc in mathematics and a PhD in statistics from the University of Salamanca. The PhD awarded a second prize of the Spanish Royal Academy of Doctors. He is the current president of the Spanish Society of Statistics and Operations Research (SEIO) and director of the Institute of Data Science and Artificial Intelligence. He is editor-in chief of the journal TEST *and associate editor and reviewer of a number of journals.*

EMS Prize winners for 2020 announced

Every four years, during the ECM, the **EMS** awards **10 EMS Prizes**, the **Felix Klein Prize** and the **Otto Neugebauer Prize**.

All prize winners are usually announced at the Opening Ceremony at the beginning of the congress. Unfortunately, this year the tradition called for an alternative.

All prize lectures will take place at the 8ECM in 2021, from 20 – 26 June in Portorož, Slovenia.

See https://8ecm.si/prizes to find out more about the prize winners. The organizers of the 8ECM extend their sincere congratulations to all the winners and look forward to welcoming them at the 8ECM in 2021!

EMS Prize winner
Karim Adiprasito

EMS Prize winner
Kaisa Matomäki

EMS Prize winner
Ana Caraiani

EMS Prize winner
Phan Thành Nam

EMS Prize winner
Alexander Efimov

EMS Prize winner
Joaquim Serra

Felix Klein Prize winner
Arnulf Jentzen

EMS Prize winner
Simion Filip

EMS Prize winner
Jack Thorne

Otto Neugebauer Prize winner
Karine Chemla

EMS Prize winner
Alexander Logunov

EMS Prize winner
Maryna Viazovska

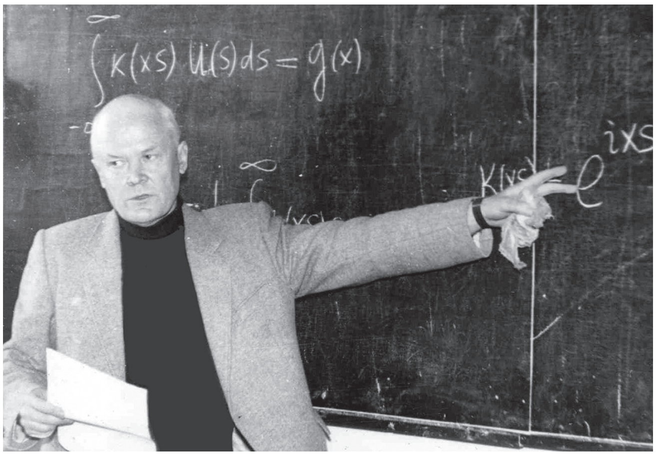
8th European Congress of Mathematics - 8ECM

International Workshop on "Equations of Convolution Type in Science and Technology" (Miskhor, Crimea, September, 2019)

Vladimir Lukianenko (V.I. Vernadsky Crimean Federal University, Simferopol, Russian Federation)

The international workshop on convolution-type equations ECTST-2019 was held in September 2019 (www.ectst.ru), dedicated to the 90th anniversary of the birth of Yuri Iosifovich Chersky. The conference was organised by the Crimean Federal University (Simferopol) and was held in the Park Hotel "Gloria", which is located in the territory of the former estate of the Naryshkin princes in the famous park "CHAIR" (Mishor, Yalta city district). This region not only boasts a unique climate, there are also many famous attractions along the southern coast of the Crimea: Lastochkino gnezdo, Vorontsov and Yusupov palaces, and the passenger cable car to the Ai-Petri mountain.

Park hotel "Gloria" (Mishor, Yalta city district).



Yuri Iosifovich Chersky.

The workshop was attended by more than 100 renowned and young scientists from Russia, Armenia, Germany, Mexico, Norway and France. Reports were presented (25 plenary, 36 poster and sectional) on a wide range of studies of pure and applied mathematics relating to the results and ideas of Yu.I. Chersky.

Yuri Iosifovich Chersky was born on 8 December 1929 in Kazan, USSR, to a family of medical doctors. In 1952 he graduated with honours from the Faculty of Mathematics of Kazan University. In his student years he was engaged in scientific work under the scientific supervision of Fyodor Dmitrievich Gakhov. After finishing post-graduate school in 1956, he defended his PhD thesis on the topic "Integral Equations of Convolution Type". After 1955 he successively became a lecturer, a senior lecturer, an associate professor of the Department of Mathematical Analysis, and from 1961 to 1964 an associate professor of the Department of Computational Mathematics of the Rostov State University in the south of the Soviet Union. After defending his doctoral dissertation on "Integral equations of convolution type and some of their applications" in 1964, Yu.I. Chersky was invited to Odessa University, where he worked until 1972 as the head of the Department of Mathematical Physics. From 1972 to 1977 he was the head of the Department of Differential and Integral Equations of the Simferopol State University in Crimea, and from 1977 to 1983 the head of the Department of Integral Equations of the Institute of Applied Problems of Mathematics and Mechanics of the USSR Academy of Sciences in L'vov. In 1983 Yu.I. Chersky returned to Odessa as the head of the Department of Advanced Mathematics of the Odessa Institute for Marine Engineers. From 1995 he was a professor of the Odessa State Academy of Construction and Architecture, and the head of the Interuniversity Scientific Seminar.

The main areas of scientific interest to Yu.I. Chersky lie in the field of boundary value problems; the theory of analytic functions; and convolution-type equations and their applications, which cover a wide range of problems. His theoretical results are usually combined with appropriate applications.

Yu.I. Chersky is considered to be the first mathematician to reduce the boundary value problem of mathematical physics to the Riemann boundary value problem on the real axis. He created an effective method of incomplete factorisation, developed an abstract theory of the Riemann problem and constructed a solution for convolution-type equations in the space of generalised functions.

Yu.I. Chersky researched new classes of convolution-type equations. He introduced the smooth transition equation, its analogs and generalisations, and studied them in detail. He pioneered the formulation and solution of a number of extremal problems from the theory of analytic functions. The main results of Yu.I. Chersky are collected in two monographs.

Scientific research directions connected with the ideas of Yuri Iosifovich are intensively developed by many mathematical research centres and they are widely presented in the reports of the ECTST-2019 workshop. The conference topics covered a wide range of problems on convolution-type equations and boundary value problems in the theory of analytic functions; on differential equations, mathematical physics, geometry; on nonlinear convolution-type equations; but also on nonlinear dynamical systems, incorrect and inverse problems, optimisation methods, extremum problems and approximate solution methods.

Prof. A.F. Voronin delivered a plenary report on "Integral equations of convolution type on a finite interval and boundary value problems for analytic functions theory", Associate Prof. V.A. Lukyanenko – the report on "Smooth transition equations", Prof. S.N. Askhabov – the report on "Stability of solutions of nonlinear equations of convolution type in cones", Prof. Kh.A. Khachatryan (Armenia) – the report on "On some classes of nonlinear multidimensional integral equations of convolution type in the mathematical theory of the geographical spread of epidemics".

The section is working.

Interesting reports were also presented on differential equations and dynamical systems. Prof. V.A. Zagrebnov (France) spoke about solving the non-autonomous Cauchy problem in normed spaces; Prof. G.S. Osipenko about calculating the averaging spectrum over pseudo-trajectories of dynamical systems; Prof. M.M. Shumafov and Prof. V.B. Tlyachev reported on the stability of random processes defined by the second-order differential equations; Prof. V.B. Vasiliev talked about elliptic boundary value problems; Prof. A.V. Kochergin about the escape velocity of cylindrical mapping orbits; and Prof. O.V. Anashkin presented a report on the algorithm for investigating the stability of a periodic pulse system in a critical regime.

Recent research on optimisation developed under the guidance of Prof. A.V. Gasnikov and associate Prof. F.S. Stonyakin (mirror descent method for conditional problems of convex optimization, Distributed and Parallel

Participants of the ECTST-2019.

Optimization) was also presented at the conference, as well as a report by D.V. Lemtyuzhnikova on suboptimal solutions of sparse discrete optimisation problems.

The workshop facilitated the exchange of views and scientific contacts in interdisciplinary research between teams of scientists from Simferopol, Novosibirsk, Moscow, Grozny, Rostov-on-Don, Nizhny Novgorod, Belgorod and mathematicians from Armenia, France, etc.

According to the participants, the workshop ECTST-2019 was held at a high scientific level; its results have been published in the *Collection of Abstracts* and in the issues 2–4 (2020) of the journal *Dynamical Systems*" (published by V.I. Vernadsky Crimean Federal University).

In September 2020, it is planned to continue discussions and exchange of scientific information at the workshop "Equations of convolution type, optimisation, forecasting" as part of the International Conference "Dynamic systems in Science and Technology" DSST-2020 (www.dsst.su)

The organising committee has the pleasure of inviting readers of the EMS Newsletter to contribute and to participate in the International Conference DSST-2020, organised by the V.I. Vernadsky Crimean Federal University (Simferopol) with the participation of St. Petersburg State University, the Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences (St. Petersburg), the Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences (Moscow)and the Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences (Novosibirsk).

The topics of interest include, but are not limited to:

1. Mathematical theory of dynamical systems. Regular and chaotic dynamics.
2. Dynamical systems in mechanics.
3. Dynamical systems in control and information technologies.
4. Equation of Convolution type, optimisation, prediction.

Vladimir Lukianenko [art-inf@yandex.ru, LukianenkoVA1@gmail.com] is an associate professor at the Department of Differential Equations and Geometry at V.I. Vernadsky Crimean Federal University in Simferopol, Republic of Crimea. His research interests include convolution type equations, boundary value problems, extreme and incorrect problems, nonlinear equations, mathematical modeling. He is chairman of the organising committee of the International workshop ECTST-2019.

ICMI Column

Jean-Luc Dorier (Université de Genève, Switzerland)

ICME-14 postponed to July 2021

Due to the global pandemic caused by the new coronavirus disease (COVID-19), ICMI and ICME-14 have decided, after careful discussion and consultation, to postpone ICME-14 by one year.

The details relating to the conference organisation, including information for those already registered, will be announced as soon as possible. A "frequently asked questions" factsheet can be found on the ICME-14 and ICMI websites (https://www.icme14.org/).

While the one-year postponement of ICME-14 poses many challenges, ICMI and ICME-14 believe this is the solution to ensure the fully fledged and successful realisation of this important conference.

ICMI and ICME-14 sincerely apologise to all concerned for the consequences of this unfortunate and dramatic development.

ICMI General Assembly is maintained

Although ICME-14 has been postponed to July 2021, a main item of the General Assembly, which was supposed to occur just before the opening ceremony of ICME-14, is the election of the new ICMI Executive Committee (EC). This must take place this year to enable the new EC to take up office on 1 January 2021.

Given the current global situation, the only option ICMI has is to hold the election electronically. The current ICMI EC has thus decided to hold the election in July 2020 as planned. Since all previous elections have taken place during a General Assembly, some adjustments will need to be made this time; simultaneously ensuring the adherence to the Terms of Reference within which ICMI operates. Following the procedures carefully will ensure that we will hold a fair, transparent and secure process.

ICMI Awardees Multimedia Online Resources (AMOR) making the main trends in math education more visible

http://icmiamor.org/

This project initiated by Jean-Luc Dorier within the Executive Committee of ICMI aims at building online resources reflecting highly significant and influential research in mathematics education at an international level, that could serve as a reference not only for researchers in the field, but also for educators, teachers, curriculum developers and policy makers and other agents in the field. In particular, the project could serve as a basis for a PhD training program and induction into mathematics education research.

ICMI was formed more than a century ago (see https://www.mathunion.org/icmi/organization/historical-sketch-icmi) and international events like the International Congress on Mathematics Education (ICME https://www.mathunion.org/icmi/conferences/icme-international-congress-mathematical-education) have been held since 1969. Mathematics education research does not have an equally long history and is a rather young field of research, with its roots often embedded in local contexts. However, one can say that the field is now at a turn of its history when there is a need for a set of references with theories, methodologies, results and fields of investigation that the community can claim as the most important trends.

Of course, this begs the question as to how to select which are the most important trends to be represented, at least initially. Since 2003 ICMI has honored outstanding individuals every second year with two awards (https://www.mathunion.org/icmi/awards/icmi-awards):

- The Felix Klein Award (https://www.mathunion.org/icmi/awards/felix-klein-award-life-time-achievement-mathematics-education-research), named after the first president of ICMI (1908–1920), honours a lifetime achievement.

- The Hans Freudenthal Award (https://www.mathunion.org/icmi/awards/hans-freudenthal-award-outstanding-contributions-individuals-theoretically-well), named after the eighth president of ICMI (1967–1970), recognizes a major cumulative program of research.

In order to build our resources ICMI has decided to focus on each Klein and Freudenthal ICMI Awardee, through what we have called the AMOR (Awardees Multimedia Online Resources) project (A more recent award Emma Castelnuovo is not strictly research oriented but more on practice so we have not included it in our quest).

Here is a table of all the awardees to date (following this link https://www.mathunion.org/icmi/awards/recipients-icmi-awards you can access the citation of all awardees).

	Felix Klein Award	Hans Freudenthal Award
2003	Guy Brousseau	Celia Hoyles
2005	Ubiratan D'Ambrosio	Paul Cobb
2007	Jeremy Kilpatrick	Anna Sfard
2009	Gilah Leder	Yves Chevallard
2011	Alan Schoenfeld	Luis Radford
2013	Michèle Artigue	Frederick Leung
2015	Alan Bishop	Jill Adler
2017	Deborah Ball	Terezinha Nunes
2019	Tommy Dreyfus	Gert Schubring

ICMI AMOR project is a long-term project that is built for the prosperity and history of Mathematics Education research. In 2017, and for practical reasons, Jean-Luc Dorier started with the three French awardees. He contacted: Annie Bessot and Claire Margolinas for Guy Brousseau, Marianna Bosch for Yves Chevallard and Michèle Artigue herself. The five of them have since worked regularly in order to give shape to the project. They made different choices and decided on the frame of the project. Recently, Abraham Arcavi joined the project and has begun work on the Anna Sfard Unit with Anna herself.

Each unit is devoted to one awardee and consists of a series of 8 to 12 modules between 10 and 30 minutes up to a total of 120–180 mins of videos. One module is basically a slide presentation with a speaker visible in the right below corner (sometimes full screen). The speaker can be, but is not necessarily, the awardee him/herself. There can also be variation from one module to the other or even within the same module. A range of additional multimedia, including films, animations can be used. No doubt new media formats will emerge over time.

The idea is that each module gives some keys to help reading some research papers which are given as much as possible as free access resources attached to each module. There is also a global selected bibliography of the awardee's work and of connected researchers.

An introductory module (Module 0) on each awardee presents some biographical and scientific elements of the background of the awardee. All the videos are in English, like most of the text. But some texts in other languages will sometimes be available.

At the moment, Michèle Artigue Unit is complete. Guy Brousseau Unit has more than half of its planned content online. Yves Chevallard and Anna Sfard units are on their way and some contacts have been made with other awardees. We will try build the many units and modules as fast as possible but as can be imagined, the development of a unit takes time and not all awardees will have their resources online immediately.

ERME Column

Caterina Primi (University of Florence, Italy), Aisling Leavy (University of Limerick, Ireland) and Jason Cooper (Weizmann Institute of Science, Israel)

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME) holds a biennial conference (CERME), in which research is presented and discussed in Thematic Working Groups (TWG). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME Thematic Working Group 5 – Probability and Statistics Education

Group leaders: Caterina Primi, Aisling Leavy, Pedro Arteaga, Daniel Frischemeier, Orlando Rafael Gonzalez, Sibel Kazak

1. COVID-19: The relevance of Statistical and Probabilistic Reasoning

As we write this article, most of the world is in isolation or lockdown due to the COVID-19 pandemic. There is a proliferation of data representations and models of the pandemic which are keeping us attuned to emerging trends and developments. The term ‘flattening the curve’ has become a part of everyday parlance, even amongst those who would not normally engage in statistical reasoning. These recent developments make the following words of H.G Wells (cited in Huff, 1954) all the more relevant, when he states that ‘Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write’.

2. Probability and Statistics Education Thematic Working Group (TWG5)

Much like the field of statistics and probability education, the *Probability and Statistics Education* Thematic Working Group (TWG5) is a vibrant and energetic community within the CERME biennial conference. The working group’s responsiveness to an evolving and growing field of scholarship is evident in the transformation in CERME9 from its previous nomenclature of the *Stochastic Thinking* working group to encompass probability education in the newly named *Probability and Statistics Education* Thematic Working Group. The growth in international research in statistics and probability education (evident in the proliferation of publications, conferences and discipline-specific journals) was reflected in the increase in submissions to the working group and subsequent necessity to expand into two subgroups at the most recent CERME11 conference. Consequently, there were 27 papers and 9 posters presented by over 42 participants reaching far beyond the borders of Europe.

The research studies presented at the conference focused predominantly on three themes: teacher education, reasoning about data, and statistical and probabilistic thinking and reasoning. Although they may appear as discrete themes, there are overlaps and similarities across subthemes, for example, research focusing on teacher education also explores teachers’ reasoning about data and their statistical and probabilistic thinking and reasoning. Furthermore, reasoning about data and statistical and probabilistic thinking and reasoning are not seen in a disjoint way – which means, in both of these subthemes we can of course also find submissions with connections between the “data world” and the “probability world”.

At the outset of the CERME conferences, we endeavour to focus participants on linking their own research with the field of scholarship in general, and on considering ways in which their work can advance scholarship in the fields of statistics and probability education. To this end, we pose a series of questions generated from our review of the work being presented alongside current developments in the fields of statistics and probability research. These questions are discussed in an initial plenary session by all participants, are revisited in the subgroups throughout the duration of the conference and then re-examined in our culminating session at the final TWG5 plenary. Examples of focus questions explored over the past two CERME conferences are:

- How should tasks and learning environments be designed to enhance reasoning about data and statistical and probabilistic thinking and reasoning for learners (primary students, high school students, teachers, etc.)?
- Which research methods are and should be used to explore students´ or learners´ reasoning with data and statistical and probabilistic thinking and reasoning?
- How can we build bridges between data analysis, probability and inference? What role does context play?
- In what ways can the use of digital tools enhance statistical reasoning (reasoning about data and statistical and probability reasoning)?
- What about the handling of big data in the upcoming data science era? How can we (statistics education) profit from the availability of big and open data? What implications does it have for statistics education?

Discussions of various papers addressing *teacher education* revealed a number of questions and themes regarding how to prepare teachers to address and foster students’ reasoning and thinking about statistics and probability. As a field, we are still grappling with ways to enhance the pedagogical content knowledge[1] facets of statistical knowledge for teaching, and not just focus on the subject matter knowledge facets. We recognise the need to make teachers familiar with performing statistical tasks and investigations from a procedural, interpretative and contextual perspective, while making connections among fundamental ideas of statistics (e.g., data and randomness in sampling), chance, relevant and appealing real-life contexts, software and technology. There is growing appreciation among the participants of TWG5 for attending to the development of statistical ideas (both formal and informal) in early years education, and to preparing teachers to address and foster this emerging understanding. In supporting teachers, our research also explores and recognises the features of successful Professional Learning Communities (i.e., communities of practice within schools and beyond), which support teachers in developing understanding and skills in a range of areas such as task design, assessment methods and other contributors to high-quality statistics and probability instruction.

A number of key ideas and research foci have been identified from discussions around the theme of *reasoning about data*. The emergence of the era of data science is an exciting development. The exploration and analysis of big data and open data will be a fundamental aspect in reasoning about data, and these areas are ripe for research and inquiry. Furthermore, there are other data collection methods which are becoming more relevant (e.g., with sensors, web scraping, etc.) and which provide new types of data. We are more cognisant of the necessity to cooperate with other disciplines such as computer science, social science and citizen science, and such cooperation is becoming increasingly evident. Concomitantly, the use of digital tools when exploring data is fundamental, and we need to consider and make the distinction between educational software tools (such as TinkerPlots or Fathom) and professional software tools (e.g. R, Python), and recognise the potential of open source and online tools such as Gapminder and CODAP (Common Online Data Analysis Platform), which can help develop reasoning about data.

3. Conclusion

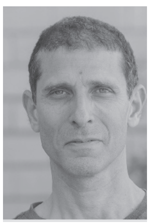
The research findings presented in TWG5 cover a wide range of learners, from children to university students – particularly pre-service mathematics teachers – and their thinking and reasoning about data and chance. These can inform teacher education programmes at the universities in terms of designing courses for pre-service teachers to enhance their content knowledge as well as pedagogical content knowledge regarding statistics and probability. From our discussions of various papers focusing on statistical and probabilistic thinking and reasoning, we acknowledge the critical role played by the use of context, real problems and visualisation tools in supporting statistical and probabilistic thinking and reasoning in data and chance explorations. We have noticed a shift in focus on misconceptions of learners towards recognising nascent understandings through building on learners' intuitions and emerging conceptions. Further research is necessary here as this reconceptualisation may prove to be more useful for supporting the early development of thinking and reasoning in data and chance. We also acknowledge the need to move the field forward through consideration of design-based research as a methodology to further inform the community on how to build learning environments that support learners' statistical and probabilistic thinking and reasoning through iterative cycles of design and research.

We conclude with a reminder to remain mindful of the need to diversify our research design and approaches to inquiry – incorporating attention to fine-grained studies, eye-tracking research, larger cross-comparative studies and experimental design, to name but a few – all in an effort to advance our understanding of the complex and evolving areas of statistics and probability education.

[1] See in Shulman, L.S. (1986) Those who understand: Knowledge growth in teaching. *Educational Researcher* 15(2), 4–14.

Caterina Primi is associate professor in psychometrics at the School of Psychology, University of Florence, and she is responsible for the psychometrics lab at the NEUROFARBA Department. She is currently conducting research into a number of topics related to test development and adaptation. She is the director of the specialisation course "Item Response Theory Models". She is involved in collaborative works with national and international research groups. She is associate editor for Frontiers Quantitative Psychology and Measurement.

Aisling Leavy is head of the STEM Education Department at Mary Immaculate College, University of Limerick. She is a mathematics and statistics educator and her research interests include children's mathematical thinking, the development of statistical reasoning and the mathematics preparation of pre-service teachers. She is currently joint general editor of Irish Educational Studies *and associate editor of* Statistics Educational Research Journal.



Jason Cooper is associate staff scientist at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including roles of advanced mathematical knowledge in teaching mathematics and contributions of research mathematicians to the professional development of mathematics teachers.

The Transition of zbMATH Towards an Open Information Platform for Mathematics

Klaus Hulek (Gottfried Wilhelm Leibniz Universität Hannover, Hannover, Germany) and Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

Introduction

Mathematics has been both a driving force and a beneficiary of digitisation from the very beginning. Computers entered mathematical institutes early on, enabling previously unattainable calculations, and mathematical research literature was among the first to be made available electronically – both through new electronic journals and extensive retro-digitisation efforts. Review databases have long provided an overview of the state of research, and extensive collections of mathematical objects such as simulation data, integer sequences or special functions have been available electronically for more than twenty years. Platforms are widely used for collaborative work and for rapid dissemination and discussion of results.

At the same time, mathematics depends more than almost any other science on the reliability and completeness of its body of knowledge. The intrinsic interdependence of mathematical concepts and theorems means that errors in the knowledge corpus might propagate dramatically. It is a characteristic feature of mathematics that results which date far back in time are often highly relevant. The half-life of mathematical knowledge is considerable and typically significantly longer than that of most other disciplines [KTW12, AT17, AHT17]. The growing quantity, and above all the diversity of results, pose a great challenge for safeguarding the standards established in the past. Clearly, a close connection of the various digital resources is of great value for mathematical research – it would both allow much more efficient work (e.g., by avoiding duplicated efforts or reducing routine tasks) and increase the chances for completely new insights. Due to traditional business models, many valuable resources have grown monolithically as closed systems and, as a result, many desirable links have not yet been created and are difficult to establish.

zbMATH[1] (formerly Zentralblatt für Mathematik) has long been a comprehensive service with information on mathematical publications, authors and references, as well as more recently, on mathematical software. However, the reusability of this data for networked services or research purposes has been severely restricted by the traditional licensing model. Following a recommendation of the 2017 evaluation of FIZ Karlsruhe, the Joint Science Conference of the German Federal Government and the German States in 2019 decided to support the transformation of zbMATH into an open access platform. As an immediate effect, the zbMATH database will become open access from 2021, but the benefits for the mathematical community will certainly go beyond this, facilitating openly available data and the resulting networked information. Here we want to outline the next steps in this transition, as well as possible long-term prospects.

Mathematical information systems: Status quo and needs

Mathematical information has traditionally been stored primarily in publications, but mathematical data is also available today in a variety of other forms, such as mathematical software and research data, community platforms, non-textual materials or educational content. Such information is characterised by a high degree of formalisation and longevity, whereby the linking of the information requires a high quality of indexing, not least due to the mathematical formulae. Indeed, publications are still the most important medium for exchanging and securing scientific results. With zbMATH and MathSciNet[2] there are two independent, high-quality reference systems, thus providing excellent access to the published literature. A growing proportion of mathematical publications are now available via open services like such as arXiv[3] or EuDML[4].

Classical publications are being supplemented by an increasing amount of research data, which, however, are still available in mostly isolated form and have hardly been indexed. Nevertheless, they are produced and used by many mathematicians. Examples are the number sequences in the On-Line Encyclopedia of Integer Sequences[5], functions in the NIST Digital Library of Mathematical Functions[6], Calabi–Yau data[7] or the L-Functions and Modular Forms Database[8]. Mathematical modelling and simulation require and generate terabytes of numerical data.

[1] https://zbmath.org

[2] https://mathscinet.ams.org/mathscinet/index.html
[3] https://arxiv.org
[4] https://eudml.org
[5] OEIS, https://www.oeis.org
[6] DLMF, https://dlmf. nist.gov
[7] http://hep.itp.tuwien.ac.at/~kreuzer/CY/
[8] LMFDB, https://www.lmfdb.org

Community platforms play an increasingly important role in the generation and exchange of information. An excellent example is the Q&A forum MathOverflow[9]. This is a source of information, especially for young researchers, that should not be underestimated. Another example are the PolyMath projects[10], in which mathematical questions are explored collaboratively. Wikipedia[11] and Imaginary[12] reach a broad audience beyond the field of mathematics and make an essential contribution to the dissemination of mathematical knowledge, the popularisation of mathematics and scientific education.

The last two examples in particular are also an excellent illustration of the strength of open platforms: information of a different nature and difficulty can be integrated collaboratively, linked, and made available to the general public. In general, the mathematical community has been committed to open access to knowledge from the very beginning, for example through extensive use of the arXiv, early establishment of open journals or initiatives such as TheCostofKnowledge[13]. As the survey [NRW] shows, open access to mathematical knowledge continues to have a very high status in the community – only (and by a clear margin) quality assurance is considered more important. This is also important with respect to the change of the publication landscape, in particular to both secure openness and to avoid quality problems caused by commercial incentives. Above all, the outstanding importance of assured results for mathematical research can only be realised with the engagement of the community, which expects to have the results of their efforts openly available in return.

In order to meet these needs, the broadest possible free access to the various quality-assured, linked and tapped resources will be an indispensable basis for mathematical research in the future. A broad vision of such an infrastructure has been formulated by the IMU as the Global Digital Mathematics Library [GDML14]. Such a service must support mathematicians in their search for and evaluation of information, and enable them to recognise and analyse connections. The special nature of mathematical information must also be taken into account: mathematics has developed its own language in the form of mathematical formulae, which in a very compact form ensure a high degree of precision. Often the information contained in formulae cannot be found in any other form. This applies to all mathematically relevant data, and especially to research data, which is essentially formula-based and therefore requires a formula search for indexing, and further tools tailored to the management of mathematical knowledge in the longer term.

[9] https://mathoverflow.net/
[10] https://polymathprojects.org, see also [P14]
[11] https://www.wikipedia.org
[12] https://imaginary.org; see also [GMM14]
[13] http://thecostofknowledge.com

What obstacles have blocked the realisation of such an interlinked information system?

An important task for the future is to connect the resources described above with each other, develop their content and build a coherent, comprehensive, open and sustainable platform. This platform must contain information of different levels in order to guarantee a comprehensive knowledge transfer. A standardised evaluation of research data increases its reproducibility, which further serves to assure the quality of publications. Additionally, a systematic citation of research data must be made possible, which in turn ensures the recognition of the research work contained within it. It is also essential to make highly formalised mathematical content such as formulae accessible. Even though these developments are still in their infancy, they indicate the great potential of modern information technology for mathematics.

Free access to interconnected and annotated resources will be an indispensable basis of mathematical research in the future. This will help mathematicians to search for and evaluate this information; connections can be clarified and analysed. The high quality of all available information remains an essential prerequisite. However, a large part of the available information described above is currently not freely available or is not or only poorly developed. As a result, information is sometimes not findable or is lost due to the lack of interconnection.

Functions and contents of an open platform must go far beyond the current possibilities of closed systems such as zbMATH, MathSciNet, Google Scholar[14], ResearchGate[15] or WolframAlpha[16]. Currently, a large part of the existing information described above is neither available via open access nor as open data, nor is it sufficiently indexed or findable. As a result, much information is lost due to the lack of interconnection. Each individual information service provides individual building blocks of information which, if correctly combined and analysed, will lead to new findings. Figure 1 (see next page) shows a selection of possible different sources of information, each of which must be linked within the individual segments and also with each other.

These links are only the beginning: open interfaces give the community the opportunity to use existing data for their own research and to subsequently make research results retrievable. These can be publications or research data as well as tools for using or analysing the data. In this way, the community itself makes innovative contributions to the development of a comprehensive mathematical research system. All experience so far shows that this would uncover considerable and often unpredictable potential for innovation.

There are several kinds of obstacles which currently delay the realisation of such an ecosystem. Firstly, business models which exist for historically good reasons (not least to create resources needed for ensuring scientific

[14] https://scholar.google.com/
[15] https://www.researchgate.net/
[16] https://www.wolframalpha.com/

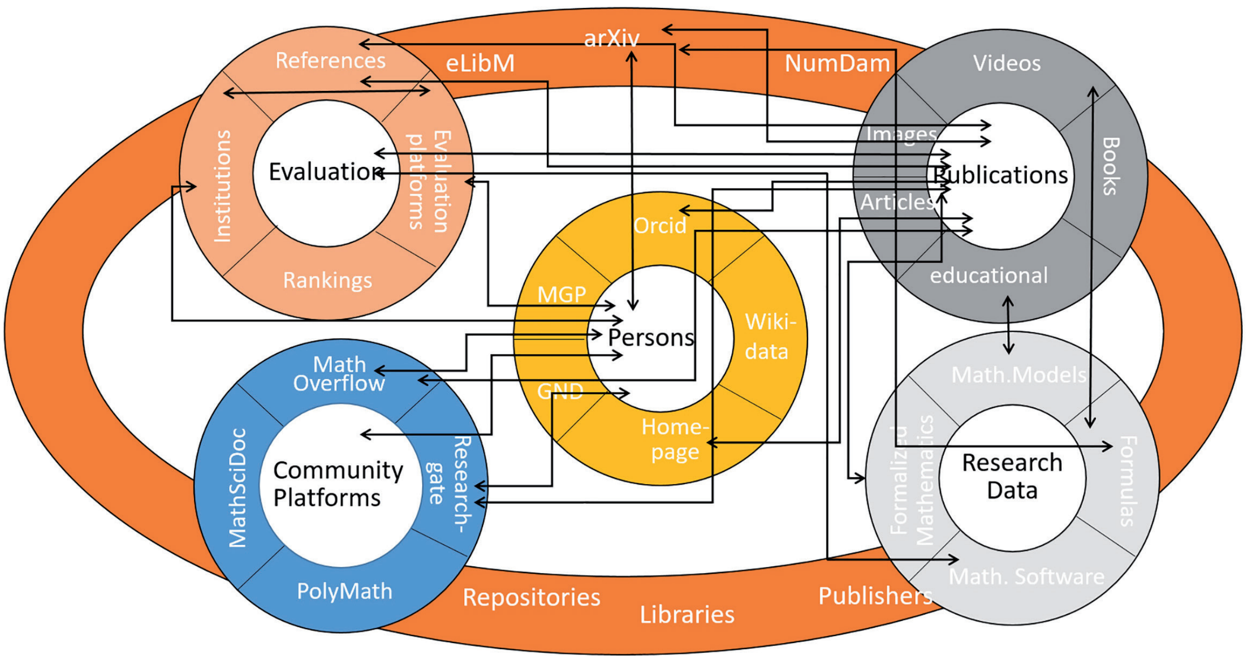


Figure 1. Components of an interlinked information system for mathematics.

quality) are in conflict with open solutions. Secondly, several techniques desirable for automated interlinking, indexing and knowledge management of digital mathematics are not yet mature or even existing for full-scale implementation (e.g., formula OCR, automated verification tools). Thirdly, many facets of such a networked, collaborative system would require extensive community engagement (like it has grown over decades for reviewing activities). Without doubt, this is the most critical, scarce and valuable resource, which requires manifest benefits to become activated.

What would the next steps be for zbMATH Open?

Due to the support of the German government, zbMATH will be able to offer its services open access from 2021 and thus overcome the previous limitations due to its traditional business model. Most of the data will become open via a CC-BY-SA license (with the exception of third-party content such as author summaries, which may be provided under a different license).

These open data will be freely available for independent research and development. As it is, there are already many projects on, for example, the history of mathematics, which make use of zbMATH data. Such projects and the resulting services will be much facilitated by open data. Another example is the use of zbMATH data in the ISC project Gender Gap In Science[17], led by the IMU. Part of the project results is the visualisation platform Gender Publication Gap[18]. In order to become valuable resources, these projects face the challenge that they require long-term maintenance. A key task will be to identify important project results involving zbMATH data, to integrate them into its services if possible, and to make them sustainably available.

Publications are still the crucial core of mathematics research. In the last decades, a growing portion has become openly available (see [T18] for a qualitative and quantitative overview). Many developments accelerate this: the growing share of arXiv submissions [MT16B], implementation of moving wall policies, open retrodigitisation projects, transformative publisher agreements and initiatives like Plan S[19]. However, the technical level (as well as the used license) of the results is very different, ranging from plain scans to available LaTeX sources which already facilitate full-text formula search [MT16A]. Integration of these diverse resources with the help of open zbMATH data will greatly improve the current situation, and create a framework which will finally make full-text related services for indexing and retrieval available.

As outlined in [HMST19], mathematical research data will become an issue of strongly growing relevance. However, these data are very diverse by nature, and it is a great challenge to make them available sustainably according to the FAIR principles[20]. Again, their integration into a linked system involving open publication data will be a big step forward. This will enhance their retrieval and visibility and will allow mechanisms to be set up to ensure their quality. Some years ago, such an approach was started for mathematical software which evolved into the free swMATH service[21]. A similar service for the entirety of mathematical research data is a task that is far too complicated to be achieved within the zbMATH Open framework alone. Instead, we will pursue this for several tech-

[17] https://gender-gap-in-science.org/
[18] https://gender-publication-gap.f4.htw-berlin.de/
[19] https://www.scienceeurope.org/our-priorities/open-access
[20] Findability, Accessibility, Interoperability, and Reusability; see https://www.go-fair.org/fair-principles/
[21] https://swmath.org, see also [BGS13, CDNS17]

nology-ready resources, and make the results available to further networked activities like those envisioned in the currently formed MaRDI consortium[22] in the framework of the German National Research Data Infrastructure[23].

As in the case of mathematical software, efforts to create, develop and maintain research data collections are often not adequately appreciated. Such an interlinked platform would also greatly help its creators to obtain recognition for contributions not directly related to publications. When their widespread usage becomes visible in a linked system, it will be much easier for the mathematicians, as well as their institutions, to obtain the appropriate recognition and reputation for their efforts, and hopefully to obtain sufficient funding for services which are currently often pursued on a voluntary and therefore not always sustainable basis.

Likewise, the role of non-textual material such as videos has changed significantly – something we are experiencing right now when we try to maintain mathematics dissemination and education under lockdown circumstances. Even after the threat of SARS-CoV-2 is reduced, experiences from the forced distancing will probably have a lasting impact on communicating mathematics research. It seems unlikely (and maybe even undesirable, since, e.g, the much larger threat of global warming still remains) to quickly restore the same overfilled conference calendars as before. Instead, we will surely make further use of remotely communicating mathematics. Resources created in this course, however, will require the same efforts of integration, interlinking, sustainable preservation, and indexing as research data.

Similar work must be done for the vast amount of mathematical knowledge assembled by the already mentioned digital mathematical communities. While first steps have been taken, e.g., interlinking Wikidata information into zbMATH author profiles with the benefit of timeline creation, or the interlinking of publication with MathOverflow discussions [MST19], much more will be possible by empowering the creativity of mathematicians with appropriate tools. It is upon us all to define what are the most needed tools which make use of the open data and can be implemented as the next steps – your suggestions and feedback to editor@zbmath.org will be highly appreciated!

References

[AT17] A. Bannister, O. Teschke: An update on time lag in mathematical references, preprint relevance, and subject specifics. *Eur. Math. Soc. Newsl.* 106, 37–39 (2017; Zbl 1384.01099)

[ATH17] A. Bannister, K. Hulek, O. Teschke: Das Zitationsverhalten in mathematischen Arbeiten. Einige Anmerkungen. *Mitt. Dtsch. Math.-Ver.* 25, No. 4, 208–214 (2017; Zbl 1390.01104)

[BGS13] S. Bönisch, G.-M. Greuel, W. Sperber, The software information service swMATH – release of the first online prototype. *Eur. Math. Soc. Newsl.* 87, 48–49 (2013; Zbl 1296.68186).

[BTW12] T. Bouche, O. Teschke, K. Wojciechowski: Time lag in mathematical references. *Eur. Math. Soc. Newsl.* 86, 54–55 (2012; Zbl 1263.01012)

[22] Mathematical Research Data Initiative, https://wias-berlin.de/mardi/

[23] NFDI, https://www.dfg.de/foerderung/programme/nfdi/

[CDNS17] H. Chrapary, W. Dalitz, W. Neun, W. Sperber: Design, concepts, and state of the art of the swMATH service. *Math. Comput. Sci.* 11, No. 3–4, 469–481 (2017; Zbl 1382.68320)

[GDML14] *NAS Committee on Planning a Global Library of the Mathematical Sciences: Developing a 21st Century Global Library for Mathematics Research* (2014), doi: 10.17226/18619

[GMM14] G.-M. Greuel, A. Matt, A. Mey: IMAGINARY – mathematics communication for the 21st century. *Eur. Math. Soc. Newsl.* 92, 3–6 (2014; Zbl 1302.00008)

[HMST19] K. Hulek, F. Müller, M. Schubotz, O. Teschke: Mathematical research data – an analysis through zbMATH references. *Eur. Math. Soc. Newsl.* 113, 54–57 (2019; Zbl 1422.01015)

[MST19] F. Müller, M. Schubotz, O. Teschke: References to research literature in QA forums – a case study of zbMATH links from MathOverflow. *Eur. Math. Soc. Newsl.* 114, 50–52 (2019; Zbl 1428.68357)

[MT16A] F. Müller, O. Teschke: Full text formula search in zbMATH. *Eur. Math. Soc. Newsl.* 102, 51 (2016; Zbl 1366.68355)

[MT16B] F. Müller, O. Teschke: Will all mathematics be on the arXiv (soon)? *Eur. Math. Soc. Newsl.* 99, 55–57 (2016; Zbl 1345.68267)

[NRW17] C. Neylon, D. M. Roberts, M. C. Wilson: Results of a worldwide survey of mathematicians on journal reform. *Eur. Math. Soc. Newsl.* 103, 46–49 (2017; Zbl 1366.01071)

[P14] D.H.J. Polymath et al.: The "Bounded gaps between primes" Polymath project – a retrospective analysis. *Eur. Math. Soc. Newsl.* 94, 13–23 (2014; Zbl 1315.01078)

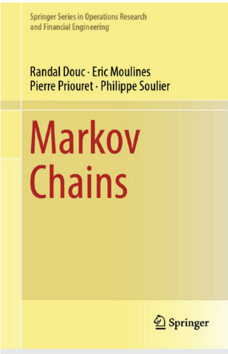
[T18] O. Teschke: Green, gold, platinum, nickel: on the status of open access in mathematics. *Eur. Math. Soc. Newsl.* 110, 60–63 (2018; Zbl 1403.01045)

Photo: Sebastian Gerhard

Klaus Hulek studied mathematics at the University of Munich and the University of Oxford. He completed his PhD in Erlangen in 1979 and spent the academic year 1982/83 at Brown University, Providence, RI, USA. He received his habilitation in Erlangen in 1984. Klaus Hulek was a professor of mathematics in Bayreuth from 1985 to 1990, when he moved to Hanover. He held the position of Vice-President of Research of Leibniz Universität Hannover from 2005 to 2014, and from 2007 to 2014 he represented the German Rectors' Conference (HRK) at the Research Policy Working Group of the European University Association (EUA). Klaus Hulek was a member of the Institute for Advanced Study (IAS) in Princeton in 2015. He is currently Vice-President of the German Mathematical Society (DMV). His field of research is algebraic geometry.

Olaf Teschke studied mathematics at Humboldt University Berlin, and completed his PhD in algebraic geometry there. He moved to FIZ Karlsruhe in 2008, and has been working there since 2009 as the head of the Mathematics Department and Managing Editor of zbMATH (including a short intermediate term as Editor-in-Chief). Since 2017, he has been serving as the Vice Chair of the EMS Committee on Publications and Electronic Dissemination. His main occupation is information infrastructure for mathematics.

Book Reviews



Randal Douc, Eric Moulines, Pierre Priouret and Philippe Soulier
Markov Chains

Springer, 2018
xviii, 757 p.
ISBN 978-3-319-97703-4

Reviewer: Andrew Wade

The Newsletter thanks zbMATH and Andrew Wade for permission to republish this review, which originally appeared as Zbl 06928217.

From the publisher's synopsis: "This book covers the classical theory of Markov chains on general state-spaces as well as many recent developments. The theoretical results are illustrated by simple examples, many of which are taken from Markov chain Monte Carlo methods. The book is self-contained while all the results are carefully and concisely proven. Bibliographical notes are added at the end of each chapter to provide an overview of the literature."

The book concerns discrete-time Markov chains taking values in a general measurable space, almost exclusively in the time-homogeneous case. The authors acknowledge the influence of the books of E. Nummelin [*General irreducible Markov chains and non-negative operators*. Cambridge: Cambridge University Press (1984; Zbl 0551.60066)] and, especially, S.P. Meyn and R.L. Tweedie [*Markov chains and stochastic stability*. Berlin: Springer-Verlag (1993; Zbl 0925.60001)] on their work. The broad technical approach to general state-space Markov chains follows these previous books, in that the splitting technique is used to reduce the general case to the case of chains with an atom, and then regeneration structure is exploited. An outline of the contents of the book is provided by the chapter titles:

Part I. Foundations. 1. Markov chains: basic definitions. 2. Examples of Markov chains. 3. Stopping times and the strong Markov property. 4. Martingales, harmonic functions and Poisson-Dirichlet problems. 5. Ergodic theory for Markov chains.

Part II. Irreducible chains: basics. 6. Atomic chains. 7. Markov chains on a discrete state space. 8. Convergence of atomic Markov chains. 9. Small sets, irreducibility, and aperiodicity. 10. Transience, recurrence, and Harris recurrence. 11. Splitting construction and invariant measures. 12. Feller and T-kernels.

Part III. Irreducible chains: advanced topics. 13. Rates of convergence for atomic Markov chains. 14. Geometric recurrence and regularity. 15. Geometric rates of convergence. 16. (f,r)-recurrence and regularity. 17. Subgeometric rates of convergence. 18. Uniform and V-geometric ergodicity by operator methods. 19. Coupling to irreducible kernels.

Part IV. Selected topics. 20. Convergence in the Wasserstein distance. 21. Central limit theorems. 22. Spectral theory. 23. Concentration inequalities.

Appendices cover weak convergence, total variation distances, martingales, and mixing coefficients. Each chapter concludes with exercises, solutions for many of which appear at the end of the book, and bibliographical notes. A more detailed description of the arrangement of the material is as follows.

Chapter 1 includes general definitions of Markov transition kernels, invariant measures, and reversibility. Chapter 2 introduces random iterated functions, and some Markov chain Monte Carlo algorithms, including variations on Metropolis-Hastings and Gibbs samplers. Chapter 3 treats the existence of a Markov chain as a canonical process, introduces stopping times, discusses the strong Markov property, and gives the representation of invariant measures via excursion occupations. Chapter 4 introduces (super)harmonic functions for Markov kernels, shows how they give rise to (super)martingales, and discusses the potential kernel and maximum principle. The Dirichlet and Poisson boundary-value problems and their relevance for hitting probability functions are discussed. Exercises include the Riesz decomposition of a finite superharmonic function, and treatment of the gambler's ruin problem via Dirichlet and Poisson problems. Chapter 5 treats ergodicity, starting with invariant events and a proof of Birkhoff's ergodic theorem, and giving the characterization of triviality of the invariant σ-algebra via bounded harmonic functions.

Chapter 6 defines atoms for a Markov chain, recurrence and transience (for sets) and the role played by atoms and accessibility. Periodicity and positive/null recurrence are defined for atoms. Excursions away from an atom are shown to be i.i.d. Ratio limit theorems, and laws of large numbers and central limit theorems for additive functionals are given in the atomic case. Chapter 7 defines irreducibility, recurrence, and transience for Markov chains on discrete state spaces. Notably, irreducibility is defined here as existence of an accessible state, rather than all states being accessible. Foster-Lyapunov drift conditions for recurrence, transience, and positive-recurrence are presented, and, in the positive-recurrent case, coupling is used to prove convergence to stationarity in total variation norm. Chapter 8 extends the convergence in the positive-recurrent case to atomic chains, using Blackwell's and Kendall's renewal theorems. Chapter 9 defines small sets, and takes as the definition of irreducibility the existence of an accessible small set, analogously to the discrete state-space approach in Chapter 7. The equivalence to the more usual definition, in terms of an irreducibility measure, is established under the condition that the σ-algebra of the state space is countably generated. Periodicity and petite sets are also introduced. Chapter 10 develops the relationship between recurrence, transience and petite sets, defines Harris recurrence, and gives Foster-Lyapunov drift criteria for transience and Harris recurrence. Chapter 11 introduces the main technical apparatus of splitting, which reduces the study of irreducible chains to that of chains with atoms, and uses this to establish total-variation convergence for Harris recurrent chains. Geometric convergence is discussed. An appendix presents an alternative proof of convergence for Harris chains using the tail σ-algebra and ideas of Blackwell and Orey. Chapter 12 considers the case where the state space is a metric space endowed with its Borel σ-algebra, introduces the Feller property, and interprets some of the preceding considerations in a topological context, such as topological recurrence.

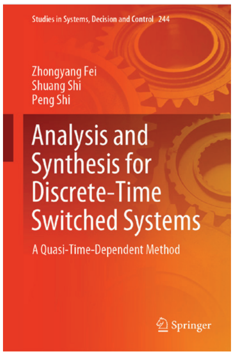
Chapter 13 generalizes the results of Chapter 8 on rates of convergence for atomic chains to the case of subgeometric convergence. Coupling is the key tool to give total-variation bounds, and f-norm bounds are also given. Chapter 14 quantifies recurrence through expectations of weighted excursion sums, providing drift conditions for f-geometric recurrence. Chapter 15 specializes to the case where the rate of convergence is geometric, discussing geometric ergodicity, and uniformity in the initial state via concepts of uniform ergodicity. Chapter 16 introduces the notion of (f,r)-recurrence, which generalizes f-geometric recurrence, and Chapter 17 studies (f,r)-ergodicity and subgeometric rates of convergence to stationarity. Chapter 18 gives an alternative approach to geometric uniform ergodicity (cf. Chapter 15) based on operator methods and a contraction theorem. The Dobrushin coefficient and Doeblin condition are introduced. Chapter 19 deals with coupling techniques for Markov chains on general state spaces. Couplings of probability measures and Markov kernels are introduced, related to norms, and maximal couplings defined.

Chapter 20 treats the metric space setting and convergence in the Wasserstein distance, with a particular view to non-irreducible chains. Sufficient conditions are given for geometric and subgeometric convergence rates. Chapter 21 deduces central limit theorems for additive functionals of Markov chains under various conditions, applying a martingale central limit theorem to chains in stationarity and then extending to other initial distributions. Alternative approaches, using the Poisson or resolvent equations, provide central limit theorems under conditions that are close to optimal. Chapter 22 deals with spectral theory and the consequences of a spectral gap, focusing on reversible chains. Cheeger's inequality is introduced, and spectral descriptions of the limiting variances in Markov chain central limit theorems are given. An appendix covers prerequisite operator theory. Chapter 23 covers concentration inequalities for functions of bounded differences. For independent arguments, the inequalities of Hoeffding and McDiarmid are established, and analogues are obtained for uniformly ergodic Markov chains. Sub-geometric concentration results for geometrically ergodic chains are also given.

The choice of topics in the book is oriented towards the wealth of applications in modern computational statistics and machine learning, where a central interest is the convergence properties of sampling and optimization algorithms that can be formulated as Markov chains on general state spaces. Many of the results in the text are illustrated by a variety of Markov chain Monte Carlo algorithms, in addition to more classical processes such as random walks, time series models, urn models, or branching processes. The nature of the applications that the authors have in mind means that the natural focus is on a deep analysis of ergodic chains; there is comparatively little on transient chains.

This book is a rich and timely addition to the literature on Markov chains, presenting a very thorough treatment of some of the key aspects of the modern theory, oriented towards applications from computational statistics. It presents the state of the art on a number of interesting topics, and should serve as an excellent reference for researchers working with Markov chains.

Andrew Wade is a probabilist working at Durham University (UK). His research interests include random walks, interacting particle systems, and random spatial structures.



Zhongyang Fei, Shuang Shi and Peng Shi
Analysis and Synthesis for Discrete-Time Switched Systems. A Quasi-Time-Dependent Method

Springer, 2020
xxiii, 210 p.
ISBN 978-3-030-25812-2

Reviewer: Mikhail I. Krastanov

The Newsletter thanks zbMATH and Mikhail I. Krastanov for permission to republish this review, which originally appeared as Zbl 1426.93001.

The book presents the so called quasi-time-dependent technique for analysis and synthesis of discrete-time switched systems. Its first part includes Chapters 2–8 and concerns switched systems with mode-dependent average dwell-time, while switched systems with mode-dependent persistent dwell-time are studied in the second part (cf. Chapters 9–12). A brief description of each chapter is written below.

The stability analysis, presented in Chapter 2, gives a family of time-scheduled multiple Lyapunov-like functions. Stability criteria are obtained for the corresponding class of discrete-time switched systems. Based on these criteria, conditions in the form of linear matrix inequalities are deduced for linear switched systems. Finally, numerical examples illustrate the proposed approach.

The disturbance attenuation performance in the sense of weighted l_2-gain is studied in Chapter 3. Based on the stability criteria from Chapter 2, sufficient conditions are obtained for globally uniformly asymptotic stability (GUAS) with a weighted l_2-gain performance. A numerical example is also presented.

$\mathcal{H}_\infty$ control issue is considered in Chapter 4. Both state and output feedback controllers are designed to ensure that the corresponding closed-loop system is GUAS and has a prescribed l_2-gain performance. Numerical examples are also included.

$\mathcal{H}_\infty$ filtering issue is considered in Chapter 5. The designed filters ensure that the resulting filtering error system is GUAS and has a prescribed l_2-gain performance.

Switched systems under asynchronous switching are studied in Chapter 6. First, a criterion for stability and l_2-gain performance is proposed. Then, asynchronous $\mathcal{H}_\infty$ control and filtering issues are discussed.

Ellipsoidal outer approximations of the reachable set of discrete-time switched systems are obtained in Chapter 7 by using the Lyapunov functions' approach.

The finite-time performance is studied in Chapter 8. First, a sufficient condition is proved for finite-time boundedness of a switched system with prescribed $\mathcal{H}_\infty$ performance. Based on this result, control and filtering schemes are proposed. Numerical examples are also given.

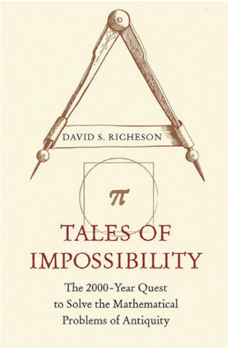
Stability and non-weighted l_2-gain analysis are provided in Chapter 9 for discrete-time switched system with mode-dependent persistent dwell-time switchings. These results are applied for a class of linear discrete-time switched uncertain systems. Illustrative examples are also presented.

$\mathcal{H}_\infty$ control issue is studied in Chapter 10 for discrete-time switched uncertain systems. Non-fragile state feedback control as well as non-fragile output feedback control are proposed.

$\mathcal{H}_\infty$ filtering issue is studied in Chapter 11 for a class of linear discrete-time switched systems. The designed time-scheduled filters ensure that the resulting filtering error system is GUAS and has a prescribed l_2-gain performance.

Finite-time control and filtering problems are studied in Chapter 12. The obtained results are based on a sufficient condition for finite-time boundedness of a class of switched discrete-time systems.

Mikhail Ivanov Krastanov is a Full Professor of Operations Research at the Faculty of Mathematics and Informatics at the Sofia University "St. Kliment Ohridski". He works in the area of nonlinear system theory.



David S. Richeson
Tales of Impossibility.
The 2000-Year Quest to Solve the Mathematical Problems of Antiquity

Princeton University Press , 2020
xii, 436 p.
ISBN 978-0-691-19296-3

Reviewer: Victor V. Pambuccian

The Newsletter thanks zbMATH and Victor V. Pambuccian for permission to republish this review, which originally appeared as Zbl 1429.01001.

By spreading a rather wide net around the history of the Greek geometric construction problems, the author has written a particularly readable history, aimed at a general audience, of a significant part of ancient Greek geometry, as well as of algebra all the way to the first half of the 19th century. The story of the ancient geometrical constructions problems can be told in a few words, by stating what they asked for, the successful attempts in antiquity to solve them with means surpassing ruler and compass, followed by Gauss' discovery of more constructible regular n-gons, P. Wantzel's 1837 paper proving the impossibility of cube duplication and angle trisection, as well as the proof that there are no other constructible regular n-gons, besides those mentioned by Gauss, to be followed by Lindemann's 1882 proof that π is a transcendental number. Add to this Hippocrates's squarable moons problem, with three squarable moons found by Hippocrates himself, two additional squarable moons discovered by D. Wijnquist in 1766 (to be rediscovered by Euler and Clausen), and the proof that there are no other moons with rational ratio of the angles formed between the center of each circle and the two points of intersection of the two circles forming the moon, spread out over the work of Landau (1902), Tschakaloff (1929), Chebotarëv (1935), and Dorodnov (1947). A theorem of A. Baker (1966) implies, as pointed out by Girstmair (2003), that there can be no other squarable moons, even if the ratio of the angles mentioned above is not assumed to be rational.

The author's achievement consists in skillfully weaving many other strands into this story. The reader finds out about the discovery of incommensurable magnitudes, as well as Theodorus' proof of irrationalities of the square roots of odd non-squares up to 17, a fairly comprehensive history of π, down to the latest fast-converging series (and thus convenient for computing many exact digits of π), as well as interesting asides surrounding π, such as its connection with Buffon's needle or with the probability that two numbers picked at random are relativley prime, a visit to Archimedes' workshop, the *neusis* construction, the quadratrix, the conchoid, the limaçon of Pascal, the spiral of Archimedes, compass-only constructions and the Mohr-Mascheroni theorem, constructions with a rusty compass, the Poncelet-Steiner theorem, origami, a history of algebra from Diophantus to Viète, Descartes's *Géométrie*, the history of complex numbers and of transcendental numbers.

The reviewer found it odd that the author states that there are (only) four ancient Greek construction problems. Since he mentions on pages 99–103 Hippocrates' squarable moon problem and provides the complete story of its solution (partly in footnotes), he must consider it part of the squaring of the circle problem, which it is not, given that the solution in 1882 of the latter did not solve the former, which took much longer (1966) to be settled. Had he considered that problem as a self-standing one (even though its origin lies in the squaring of the circle), he would have stated that there are *five* major ancient construction problems.

When mentioning constructions with a "rusty compass", the author could have stated that all ruler-and-compass constructions can be performed with a rusty compass alone, as proved by J. Zhang et al. [*Geom. Dedicata* 38, No. 2, 137–150 (1991; Zbl 0722.51017)]. When mentioning Cauer's work on straightedge-only constructions, he could have mentioned the correction brought by C. Gram [*Math. Scand*. 4, 157–160 (1956; Zbl 0070.16102)] ([A. Akopyan and R. Fedorov, *Proc. Am. Math. Soc.* 147, No. 1, 91–102 (2019; Zbl 1407.51023)] appeared too late to be included). Given his extraordinarily good bibliography, J. Schönbeck [*Centaurus* 46, No. 3, 208–229 (2004; Zbl 1073.01008)] and C. Lathrop and L. Stemkoski [*The MAA Tercentenary Euler Celebration* 5, 217–225 (2007; Zbl 1172.01305)] should be added to it and referred to when mentioning Clausen's rediscovery of additional squarable moons. Regarding the whole squarable moons story, it is worth pointing out that C. J. Scriba [*Mitt. Math. Ges. Hamb.* 11, No. 5, 517–539 (1988; Zbl 0639.01001)] presents the complete story. Finally, when mentioning on page 96 a (modern) method for turning a rectangle into a square of the same area, one could also mention the ancient way of accomplishing that feat. A. Seidenberg [*Arch. Hist. Exact Sci.* 18, 301–342 (1978; Zbl 0392.01002)] has pointed out that the ancient method for doing so is, rather mysteriously, the same in Euclid's *Elements* II.5 (together with the Pythagorean theorem) and in the Śulbasūtras. These should be taken as suggestions for a second edition, not as criticism of this book.

Victor Pambuccian is a Professor of Mathematics at Arizona State University, specialising in the axiomatic foundation of geometry. He recently edited and translated into English a bilingual anthology of avant-garde and avant-garde inspired Romanian poetry, Something is still present and isn't, of what's gone, *Aracne Editrice, Rome, 2018.*

Personal Column

Please send information on mathematical awards and deaths to newsletter@ems-ph.org.

Awards

Hillel Furstenberg from the Hebrew University of Jerusalem received the **Abel Prize 2020** awarded by the Norwegian Academy of Science and Letters.

Anna Erschler (DMA, CNRS-ENS Paris) was awarded the **CNRS 2020 Silver Medal** for her research.

Irène Waldspurger (CEREMADE, CNRS/Université Paris Dauphine) and **Susanna Zimmerman** (LAREMA, CNRS/Université d'Angers) were awarded the **CNRS 2020 Bronze Medal** for their research.

The **Shaw Prize in Mathematical Sciences 2020** was awarded in equal shares to **Alexander Beilinson**, David and Mary Winton Green University Professor, The University of Chicago, and **David Kazhdan**, Professor of Mathematics, The Hebrew University of Jerusalem, for their huge influence on and profound contributions to representation theory, as well as many other areas of mathematics.

The **Ferran Sunyer i Balaguer Prize 2020** was awarded to **Urtzi Buijs** (Universidad de Málaga), **Yves Félix** (Institut de Mathématiques et Physique, Université catholique de Louvain), **Aniceto Murillo** (Universidad de Málaga) and **Daniel Tarré** (Université de Lille) for their monograph *Lie models in topology*, and to **Giovanni Catino** (Politecnico di Milano) and **Paolo Mastrolia** (Università degli Studi di Milano) for their monograph *A perspective on canonical Riemannian metrics*.

In 2020 the Polish Mathematical Society (PTM) awards its two highest prizes.
Yuriy Tomilov (Institute of Mathematics, Polish Academy of Sciences) is a recipient of **The Stefan Banach Main Prize 2019** for deep and broadly cited results in the field of operator theory and semigroups of operators on Banach spaces and their applications in the theory of differential equations and ergodic theory.
Adam Bobrowski (Lublin University of Technology) is a recipient of **The Hugo Steinhaus Main Prize 2019** for overall achievements in the field of mathematics applications (functional analysis and stochastic processes).

The **2019 Jacques-Louis Lions Prize** of the French academy was awarded to **Maria Estéban** (CEREMADE, CNRS/Université Paris Dauphine).

The **PTM Prize for Young Mathematicians 2019** was awarded to **Agnieszka Hejna** (University of Wroclaw) for a cycle of five works from harmonic analysis published in renowned mathematical journals and to **Dominik Burek** (Jagiellonian University in Cracow) for a series of four papers on the subject of the Calabi–Yau variety.

The **2019 "Young Mathematician"** prize of the St. Petersburg Mathematical Society was awarded to **Yulia Petrova** (St. Petersburg State University) for a cycle of works "Exact asymptotics for L2-small finite-dimensional perturbations" and **Maria Platonova** (St. Petersburg Department of the Steklov Mathematical Institute, RAS) for a cycle of works "Evolutional equations and related stochastic processes".

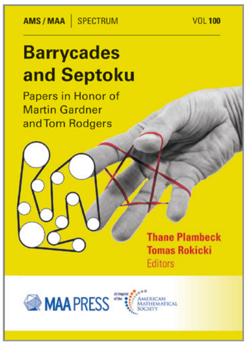
The Austrian Academy of Sciences awarded the **Erwin Schrödinger prize 2019** to **Karlheinz Gröchenig** from the University of Vienna for his outstanding achievements in the field of harmonic analysis. This prize is among the highest scientific prizes in Austria and awardees range over all scientific disciplines.

Deaths

We regret to announce the deaths of:

Ernest Borisovich Vinberg (12 May 2020, Moscow, Russia)
Stevo Komljenović (20 April 2020, Belgrade, Serbia)
Łybacka Krystyna (20 April 2020, Poznań, Poland)
John Horton Conway (11 April 2020, Princeton, USA)
Vladimir Koroliuk (4 April 2020, Kiev, Ukraine)
Jean Ginibre (26 March, 2020, Bligny, France)
Boris Tsirelson (21 January 2020, Basel, Switzerland)
Blagovest Sendov (19 January 2020, Sofia, Bulgaria)
Reuben Hersh (3 January 2020, Santa Fe, USA)
Edward Smaga (31 January 2020, Cracow, Poland)
Oleg Ivanov (28 December 2019, St. Petersburg, Russia)
Jan Stanisław Lipiński (20 December 2019, Gdańsk, Poland)
Sergey Slavyanov (4 November 2019, St. Petersburg, Russia)
Józef Krasinkiewicz (2 November 2019, Warsaw, Poland)
Abram Zinger (11 October 2019, St. Petersburg, Russia)
Bolesław Gleichgewicht (26 September 2019, Wrocław, Poland)
Alexander Plotkin (24 August 2019, St. Petersburg, Russia)
Janina Śladkowska-Zahorska (2 August 2019, Gliwice, Poland)
Irena Łojczyk-Królikiewicz (22 July 2019, Wieliczka, Poland)
Zdzisław Denkowski (28 June 2019, Cracow, Poland)
Antoni Wiweger (19 June 2019, Warsaw, Poland)

AMERICAN MATHEMATICAL SOCIETY



BARRYCADES AND SEPTOKU

Papers in Honor of Martin Gardner and Tom Rodgers

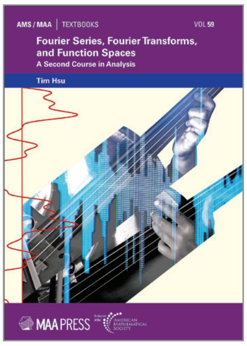
Edited by Thane Plambeck, Counterwave, Inc. & Tomas Rokicki

Consists of papers originally presented at the Gathering 4 Gardner meetings. Recreational mathematics is prominent with games and puzzles, including new Nim-like games, chess puzzles, coin weighings, coin flippings, and contributions that combine art and puzzles or magic and puzzles. Anyone who finds pleasure in clever and intriguing intellectual puzzles will find much to enjoy in *Barrycades and Septoku*.

Spectrum, Vol. 100

MAA Press

May 2020 234pp 9781470448707 Paperback €75.00



FOURIER SERIES, FOURIER TRANSFORMS, AND FUNCTION SPACES

A Second Course in Analysis

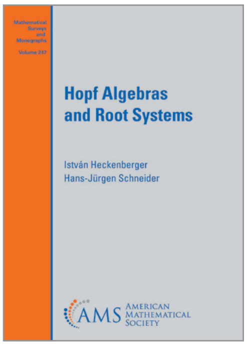
Tim Hsu, San Jose State University

A textbook for a second course or capstone course in analysis for advanced undergraduate or beginning graduate students. By assuming the existence and properties of the Lebesgue integral, this book makes it possible for students who have previously taken only one course in real analysis to learn Fourier analysis in terms of Hilbert spaces.

AMS/MAA Textbooks, Vol. 59

MAA Press

May 2020 354pp 9781470451455 Hardback €91.00



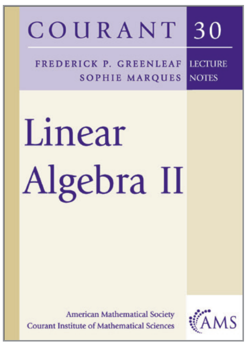
HOPF ALGEBRAS AND ROOT SYSTEMS

István Heckenberger, Philipps Universität & Hans-Jürgen Schneider, Ludwig-Maximilians-Universität, Mathematisches Institut

Provides an introduction to Hopf algebras in braided monoidal categories with applications to Hopf algebras in the usual sense. The main goal is to present from scratch and with complete proofs the theory of Nichols algebras (or quantum symmetric algebras) and the surprising relationship between Nichols algebras and generalized root systems.

Mathematical Surveys and Monographs, Vol. 247

Jun 2020 582pp 9781470452322 Hardback €161.00



LINEAR ALGEBRA II

Frederick P. Greenleaf, Courant Institute, New York University & Sophie Marques, Stellenbosch University

This book is the second of two volumes on linear algebra for graduate students in mathematics, the sciences, and economics, who have: a prior undergraduate course in the subject; a basic understanding of matrix algebra; and some proficiency with mathematical proofs. The book provides a varied selection of exercises; examples are well-crafted and provide a clear understanding of the methods involved.

Courant Lecture Notes, Vol. 30

Jun 2020 312pp 9781470454258 Paperback €68.00

AMS is distributed by EUROSPAN
Free delivery at **eurospanbookstore.com/ams**

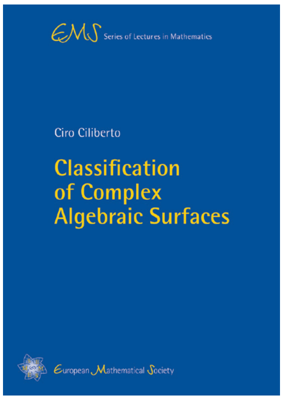
CUSTOMER SERVICES:
Tel: +44 (0)1767 604972
Fax: +44 (0)1767 601640
Email: eurospan@turpin-distribution.com

FURTHER INFORMATION:
Tel: +44 (0)20 7240 0856
Fax: +44 (0)20 7379 0609
Email: info@eurospan.co.uk

New books published by the

European Mathematical Society

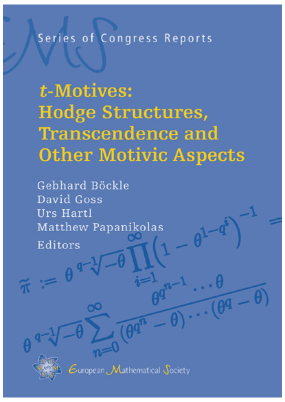
Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.



Ciro Ciliberto (Università di Roma Tor Vergata, Italy)
Classification of Complex Algebraic Surfaces (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-210-8. 2020. 143 pages. Softcover. 17 x 24 cm. 36.00 Euro

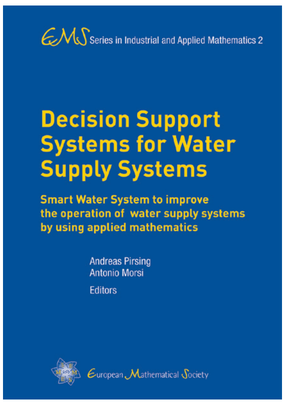
The classification of complex algebraic surfaces is a very classical subject which goes back to the old Italian school of algebraic geometry with Enriques and Castelnuovo. However, the exposition in the present book is modern and follows Mori's approach to the classification of algebraic varieties. The text includes the P_{12} theorem, the Sarkisov programme in the surface case and the Noether–Castelnuovo theorem in its classical version.
This book serves as a relatively quick and handy introduction to the theory of algebraic surfaces and is intended for readers with a good knowledge of basic algebraic geometry. Although an acquaintance with the basic parts of books like *Principles of Algebraic Geometry* by Griffiths and Harris or *Algebraic Geometry* by Hartshorne should be sufficient, the author strove to make the text as self-contained as possible and, for this reason, a first chapter is devoted to a quick exposition of some preliminaries.



***t*-Motives: Hodge Structures, Transcendence and Other Motivic Aspects**
(EMS Series of Congress Reports)
Edited by Gebhard Böckle (Universität Heidelberg, Germany), David Goss, Urs Hartl (Universität Münster, Germany) and Matthew Papanikolas (Texas A&M University, College Station, USA)

ISBN 978-3-03719-198-9. 2020. 473 pages. Hardcover. 17 x 24 cm. 89.00 Euro

This volume contains research and survey articles on Drinfeld modules, Anderson t-modules and t-motives. Much material that had not been easily accessible in the literature is presented here, for example the cohomology theories and Pink's theory of Hodge structures attached to Drinfeld modules and t-motives. Also included are survey articles on the function field analogue of Fontaine's theory of p-adic crystalline Galois representations and on transcendence methods over function fields, encompassing the theories of Frobenius difference equations, automata theory, and Mahler's method. In addition, this volume contains a small number of research articles on function field Iwasawa theory, 1-t-motifs, and multizeta values.
This book is a useful source for learning important techniques and an effective reference for all researchers working in or interested in the area of function field arithmetic, from graduate students to established experts.



Decision Support Systems for Water Supply Systems. Smart Water System to Improve the Operation of Water Supply Systems by Using Applied Mathematics
(EMS Series in Industrial and Applied Mathematics, Vol. 2)
Edited by Andreas Pirsing (Siemens AG, Berlin, Germany) and Antonio Morsi (Universität Erlangen-Nürnberg, Germany)

978-3-03719-207-8. 2020. 243 pages. Hardcover. 17 x 24 cm. 69.00 Euro

Operating water supply systems is complex. Engineers must ensure that consumers are reliably supplied with a sufficient quantity and quality of water, as well as a sufficient water pressure at all times – all while maintaining reasonable prices. This book summarizes the results of the German BMBF (Federal Ministry of Education and Research) funded joint research project, EWave (Project ID: 02WER1323F), that was initiated to develop an innovative Decision Support System (DSS) for water supply companies. For decision making and operational support, the EWave system uses newly developed integrated optimization modules. As a result, the user receives operating schedules on a 15 minute scale. To achieve this, mixed-integer linear and nonlinear mathematical optimization methods are combined. First, a mixed-integer optimization model is solved in order to derive all discrete decisions (primarily pump schedules). The aim is to approximate the physics by piecewise linear relaxations sufficiently to optimize decisions. EWave then uses nonlinear optimization and simulation methods to verify the physics. The process is iterated as necessary. This approach enables globally optimal solutions within an a priori given quality tolerance.
Optimization results obtained in real time yield a potential of energy savings of up to 4–6% daily for the waterworks in the pilot area.
This book was written for automation experts in water supply companies as well as mathematicians who work for infrastructure companies.