NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Photo courtesy of Dyrol Lumbard, Oxford

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Wrinkles: From the Sea to Mathematics

Kardar–Parisi–Zhang Universality

Interview

Sir Andrew Wiles

History

Göttingen State and University Library

European Mathematical Society

September 2016
Issue 101
ISSN 1027-488X

Journals published by the European Mathematical Society

ISSN print 2313-1691
ISSN online 2214-2584
2016. Vol. 3
Approx. 600 pages. 21.0 x 28.0 cm
Price of subscription:
Online free of charge
198 € print (1 volume, hard cover)

Published by Foundation Compositio Mathematica and EMS

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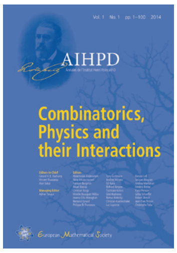

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ISSN online 2308-5835
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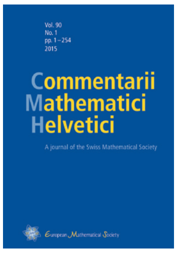

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ISSN online 1420-8946
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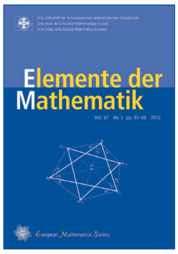

ISSN print 0013-6018
ISSN online 1420-8962
2017. Vol. 72. 4 issues
Approx. 180 pages. 17.0 x 24.0 cm
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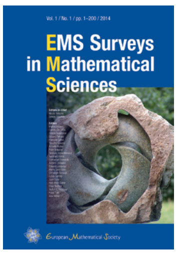

ISSN print 2308-2151
ISSN online 2308-216X
2017. Vol. 4. 2 double issues
Approx. 400 pages. 17.0 x 24.0 cm
Price of subscription:
198 € online only
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ISSN print 0013-8584
ISSN online 2309-4672
2017. Vol. 63. 2 double issues
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Price of subscription:
198 € online only
238 € print+online

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European Mathematical Society Publishing House
Seminar for Applied Mathematics, ETH-Zentrum SEW A27
Scheuchzerstrasse 70
CH-8092 Zürich, Switzerland
subscriptions@ems-ph.org
www.ems-ph.org

European Mathematical Society

Newsletter No. 101, September 2016

The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

ISSN 1027-488X

Published by the
EMS Publishing House
ETH-Zentrum SEW A27
CH-8092 Zürich, Switzerland.
homepage: www.ems-ph.org

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EMS Agenda

2016

4–6 November
EMS Executive Committee Meeting, Tbilisi, Georgia

2017

1–2 April
Presidents Meeting, Lisbon, Portugal

EMS Scientific Events

2016

18–22 September
7th International Conference on Advanced Computational Methods in Engineering, Ghent, Belgium
http://www.acomen.ugent.be/

26–30 September
10th Euro-Maghrebian Workshop on Evolutions Equations
Heinrich Fabri Institute, Blaubeuren, Germany
http://euromaghreb10.math.kit.edu

25 September–1 October
50th Jubilee Sophus Lie Seminar
Banach Conference Center, Bedlewo
Distinguished EMS Lecturer: Ernest Vinberg
http://50sls.impan.pl

3–4 November
Geometry and Lie theory. Applications to classical and quantum mechanics. Dedicated to 70th birthday of Eldar Straume.
NTNU Trondheim, Norway
https://www.math.ntnu.no/~mariusth/Eldar70/

14–16 November
ICERI2016 (9th Annual International Conference of Education, Research and Innovation), Seville, Spain
http://iated.org/iceri

8–12 December
International Conference Geometric Analysis and Control Theory, Sobolev Institute of Mathematics, Novosibirsk, Russian Federation
http://gct.math.nsc.ru/?event=geometric-analysis-and-control-theory

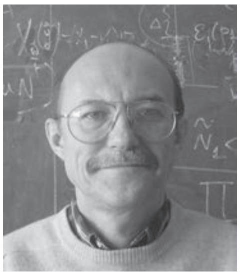
19–23 December
International Conference "Anosov systems and modern dynamics". Dedicated to the 80th anniversary of Dmitry Anosov.
Steklov Mathematical Institute, Moscow, Russian Federation
http://anosov80.mi.ras.ru/

2017

24–28 July
31st European Meeting of Statisticians
Helsinki
EMS-Bernoulli Society Joint Lecture: Alexander Holevo

Editorial

Valentin A. Zagrebnov (Institut de Mathématiques de Marseille, France), Editor-in-Chief of the EMS Newsletter



Dear EMS Newsletter Readers,

It is at once an honour and a great challenge for me to take over the editorship of the EMS Newsletter, starting with this September 2016 issue. The outgoing editor Lucia Di Vizio did a superb job of managing the publication of the newsletter over the previous four years. She has put a lot of energy into advancing the quality and visibility of the journal.

I hope to profit from her experience as a member of the editorial board. My personal thanks to Lucia are twofold. Firstly, I am grateful for her endless patience in the instructive and attentive way she guided me through the labyrinth of the production process of the previous newsletter issue (June 2016). Secondly, she armed me with a detailed and transparent memorandum covering the activity of the editor-in-chief of the Newsletter, which has served me well for the practical exercise of preparing this, my first issue of the Newsletter.

Lucia left me the EMS Newsletter in very good shape and I shall do my best to keep its high standards and reputation. In this role, I can only aspire to similar success. With the aid of the editorial board, I am optimistic that we can keep bringing to your attention insightful perspectives on research, news and other items of interest in mathematics. We hope that all readers will feel free to contact the editorial board whenever they have ideas for future articles, comments, criticisms or suggestions.

Report from the EMS Executive Committee Meeting in Stockholm, 18–20 March 2016

Richard Elwes, EMS Publicity Officer

This Spring, the Executive Committee gathered in Stockholm for a meeting at the historic Institut Mittag-Leffler of the Royal Swedish Academy of Sciences. To mutual regret, the institute's director (and former EMS President) Ari Laptev was unable to greet the assembled company in person. Instead, on Friday evening, the committee enjoyed a short film telling the fascinating story of the institute, which is the oldest mathematics research facility in the world.

Officers' Reports

The President Pavel Exner welcomed the Executive Committee to the meeting before relating his recent activities. He reminded the committee of the news that the Simons Foundation has decided to support mathematics in Africa through the EMS with 50,000 euros per year for five years. This development was enthusiastically welcomed.

The treasurer then presented his summary of the budget, income and expenditure of the society for 2015, reporting a surplus. The committee then discussed future financial strategy. As points of principle, it was agreed that the society should take a low risk approach to its investments and that financial gains should be reflected in increases in the budget for scientific activities. At the EMS council meeting in July, the treasurer will propose a detailed plan to enact the agreed strategy.

Membership

The committee expressed regret that the Belgian Statistical Society had cancelled its Class 1 Membership of the society. Meanwhile, the status of the Armenian Mathematical Union's application for Class 1 Membership is unclear. The invitation to present this application for approval at the forthcoming council meeting remains open.

At the start of 2016, the President wrote to a number of member societies who have not paid their dues for several years. The committee expressed sympathy for the financial difficulties in which learned societies may find themselves. However, member societies struggling to pay their dues are expected to contact the EMS to negotiate a way forward. The committee agreed that membership will be terminated for any society who neither pay their dues nor respond to letters from the President.

There was also concern about individual members who fail to pay their dues promptly, especially when these people are active in EMS bodies. The committee agreed to bring a proposal to the council to add to Rule 23 of the by-laws on sub-committee membership the sentence: "Each member should be an individual member of the EMS in good standing."

To make membership more appealing to younger mathematicians, the committee also agreed to propose to the council a modification of Rule 30 as follows: "Anyone who is a student at the time of becoming an individual EMS member, whether PhD or in a more junior category, shall enjoy a three-year introductory period with membership fees waived. All the standard benefits will be granted during this time, except printed copies of the Newsletter."

The committee was pleased to approve a list of 90 new individual members. It then discussed a proposal for a new lifetime membership, the major question being how to compute a fair fee. The treasurer Mats Gyllenberg proposed a formula (due to Euler!) that purportedly solves this problem. The Executive Committee agreed that Mats Gyllenberg will present his proposal at the forthcoming council meeting.

Scientific Meetings

The Executive Committee discussed the report from Volker Mehrmann on the preparations for the 7th European Congress of Mathematics in Berlin in July. The committee expressed its sincere appreciation to the local organisers and its confidence in the success of the congress.

Over the weekend, the committee heard presentations from both bids to host the 8th European Congress of Mathematics in 2020, in either Sevilla in Spain or Portorož in Slovenia. It was clear that both bids have merit and should proceed to the EMS Council in July, where a final decision will be taken by a simple majority vote. (The Executive Committee will neither issue a recommendation to the council nor provide evaluation criteria for the delegates.)

The President reported that memoranda of understanding have been signed for the seven Summer Schools that will be supported by the EMS in 2016.

Society Meetings

Preparations for the council meeting in Berlin in July are well underway. The committee agreed the draft agenda and heard that the nominations of delegates to represent associate, institutional and individual members at the council are now closed, with no need for elections. Associate members are represented by one delegate (the maximum permitted), institutional members are represented by three delegates (the maximum allowed is four) and individual members are represented by 25 delegates (the maximum allowed is 26). One task for the council will be the replenishment of the Executive Committee, with at least three new officers required. It was decided to organise the election in two rounds using an electronic voting system.

The next society meeting will be the meeting of the Presidents of Members Societies in Budapest in April (see separate report).

Standing Committees

The Executive Committee agreed to improve the database of past members/chairs of EMS committees (a task, it was noted, which may require some time and work to complete). It then considered reports from the Chairs of the Committees on Applied Mathematics, Developing Countries, Electronic Publishing, Ethics, European Solidarity and Raising Public Awareness of Mathematics.

Following the report from the ERCOM (European Research Centres On Mathematics) Committee, the Executive Committee was pleased to approve BCAM (the Basque Centre for Applied Mathematics in Bilbao) as a new ERCOM centre.

Projects

The European Digital Mathematics Library (EuDML) initiative is progressing at pace, with Volker Mehrmann to be the EMS representative on the EuDML Board.

Volker Mehrmann then reported on the activities and the new board make-up of EU-MATHS-IN (the European Service Network of Mathematics for Industry and Innovation), which is coordinating proposals for three infrastructure calls.

Publicity, Publishing and the Web

The committee discussed the report of the publicity officer and plans for a future letter-writing campaign to increase membership. As stocks of flyers and posters are now beginning to run low, it was also agreed that a new set of publicity materials will be designed and printed later in the year or early next year. The committee heard from the web team, including a discussion of ways to increase the EMS online presence. One possibility is the incorporation of a blog on the EMS website, perhaps with cooperation from member societies.

The committee continued its contemplation of the future direction of the EMS Publishing House and was pleased to hear that JEMS (the Journal of the EMS) is currently functioning well.

In a report on the EMS Newsletter, the committee heard that transition is underway between the outgoing Editor-in-Chief Lucia Di Vizio and her replacement Valentin Zagrebnov. The committee again thanked Lucia Di Vizio for her invaluable work.

A report from Olaf Teschke on Zentralblatt was received. The President recalled that Klaus Hulek will replace Gert-Martin Greuel as Editor-in-Chief of Zentralblatt from April 2016. The President thanked Gert-Martin Greuel for his hard work and offered the opinion that Zentralblatt has improved significantly under his tenure.

Relations with Political and Funding Bodies

The President reported on the latest developments regarding Horizon 2020, notably its running calls, its open consultation on mathematics and the installation of the High Level Group of Scientific Advisors of the EC Scientific Advice Mechanism, with Cédric Villani as a representative for mathematics.

The President reported on the latest developments regarding the new legal status of ISE (the Initiative for Science in Europe) and reminded the committee that the membership fee for the EMS is set to double to 3,000 euros. After two years, the Executive Committee will re-evaluate the benefits of ISE membership. The committee heard that, in July, Roberto Natalini (Chair of the Committee on Raising Public Awareness) will organise a session at ESOF (EuroScience Open Forum) in Manchester, UK, about Alan Turing, in which the EMS Publicity Officer Richard Elwes will also participate. Meanwhile, the President is also involved in an ESOF session about the ERC.

The committee regretted that attempts to secure funding for EMS Summer Schools from the COST programme were not successful. It was suggested that prospects for funding EMS activities through this programme may generally be poor.

The committee was delighted to hear that there will be a mathematics session at the World Economic Forum in Davos and considered what input the EMS might make.

Relations with Mathematical Organisations

The committee discussed news from the IMU, the ICIAM and other mathematical bodies. It then congratulated Andrew Wiles for his Abel Prize. Following a long tradition, EMS Vice-President Martin Raussen will interview the Abel Prize winner in May but confirmed that this will be his last year doing so.

The 2017 EMS-Bernoulli Society Joint Lecture will be delivered by Alexander Holevo at the 31st European Meeting of Statisticians in Helsinki, 24-28 July 2017.

Alice Fialowski presented a report on the activities of the TICMI (Tbilisi International Centre of Mathematics and Informatics). The Executive Committee will gather in November in Tblisi and it was agreed that EMS President Pavel Exner will meet the Director of TICMI.

Gert-Martin Greuel took the opportunity to advertise the Snapshot series from Oberwolfach on modern developments in mathematics, which is aimed at the general public (http://www.mfo.de/math-in-public/snapshots).

The committee was delighted to hear that agreements for collaboration and reciprocity membership have been signed between EMS and the Mathematical Society of Japan (MSJ). The committee then considered the possibility of holding a joint meeting.

Other Business

The Executive Committee was alarmed at developments in Turkey, which have seen a colleague imprisoned for signing the Academics for Peace petition. It was decided to write letters of concern to the Turkish prime minister and to the President of the European Parliament, and to follow developments closely. Copies of these letters were sent to our sister organisations at the European level and to the Presidents of our Member Societies.

Conclusion

The committee expressed its gratitude to the Institut Mittag-Leffler and, in absentia, to its director Ari Laptev, for the magnificent hospitality it enjoyed over the weekend.

Report on the Meeting of Presidents of Mathematical Societies in Budapest, 2–3 April 2016

Sjoerd Verduyn Lunel, Secretary of the EMS, and Richard Elwes, EMS Publicity Officer

Pavel Exner, President of the European Mathematical Society and Chair of the meeting, opened proceedings by greeting everyone present and thanking our hosts, the Alfred Rényi Institute of Mathematics of the Hungarian Academy of Sciences. After a *Tour De Table* in which the guests introduced themselves and their Society, Gyula Katona, President of the János Bolyai Mathematical Society delivered a presentation on its fascinating history as well as its current activities.

EMS Business

Pavel Exner gently reminded the assembled company of the importance of member societies and individuals paying their dues on time. (If they have difficulty doing so, they should get in touch with the EMS directly to negotiate an arrangement.)

The Chair reported on the EMS's wide range of scientific activities, and encouraged member societies to prepare proposals for scientific activities, such as Joint

Mathematical Weekends. (Proposals for 2017 can be submitted via the EMS website.)

After discussing the work of the EMS's eleven Standing Committees, he spoke about its online presence, a major way of keeping in contact with members and the broader mathematical community. He encouraged member societies to get involved, and to contact Richard Elwes, Publicity Officer of the EMS, with any new ideas in this area.

Presentations

The meeting then heard a number of presentations:

- Volker Bach described progress towards the 7th ECM (Berlin, 18–22 July 2016)
- Betül Tanbay, member of the EMS Ethics Committee, reported on its work on the application of the EMS Code of Practice and towards a new edition of the Code in due course. She welcomed comments and suggestions for additional areas to be addressed, for example, regarding unethical behaviour with respect to 'open access' publications.
- Waclaw Marzantowicz, President of the Polish Mathematical Society, reported on its recently established Mathematical Information Service.
- Mercedes Siles Molina, Vice-President of the Royal Spanish Mathematical Society (RSME), described connections between RSME and the mathematical societies of South America. She stressed that the RSME is keen to facilitate further mathematical interactions between Europe (via the EMS and its member societies) and South America.
- The rival bids for hosting the 8th ECM in 2020 were presented, with Klavdija Kutnar presenting the bid for Portorož, Slovenia, and Juan González-Meneses presenting that for Sevilla, Spain.

Discussion on Mathematical Education

The meeting then held an open discussion of Mathematical Education. A focus was the balance between challenging 'gifted' students on one hand, and, on the other, attaining a decent level for all students enrolled on mathematics courses. This latter group is increasing in size, because of the need for mathematical skills in subjects such as engineering and natural sciences. Approaches vary from country to country. For example, Hungary and Romania have special programs for gifted students. However, these countries also feel the effects of 'brain drain', with many high-achieving students opting to study abroad after high school.

With this shifting balance, and also as a consequence of the OECD Programme for International Student Assessment (PISA) survey, mathematics curricula are undergoing transformations in many countries. The meeting agreed that is vital to exchange information about this changing landscape.

In Germany, a catalogue has been compiled of the minimal requirements that students should meet when finishing secondary school, to study engineering or natural sciences. Of course, its contents are controversial. The meeting considered whether the EMS Education Committee might play a role in drafting standards of this kind, which could then be used as guidelines in national discussions. A number of presidents recommended becoming directly, personally involved in national curriculum committees.

Another area of significant difference between countries is that of teacher training. In some countries, this is completely separate from standard university education. In other countries, there are multiple tracks to become a teacher, via a university degree or via a teacher training school. In some countries there are special schemes to support a later career change into teaching. There was a general agreement that the Education Committee should become more directly involved with issues of mathematics education at secondary school, curriculum planning, and teacher training. It is also recommended that the Education Committee should work with the EMS sponsored EuroMath initiative, aimed at students between 9 and 18. (See www.euromath.org.)

Other Business

The Chair discussed the political situation in Turkey where a colleague was imprisoned for signing the Academics For Peace petition. He reported that he had written letters of concern, on behalf of the European Mathematical Society, to the Turkish prime minister and to the President of the European Parliament. The meeting was unanimously supportive of this initiative. It was agreed to follow developments closely and also to investigate what steps might be taken at a national level to express the strong concern of our community about the situation.

Closing

On behalf of all participants, the Chair thanked the local organizers for their excellent preparations, and for the generous hospitality offered by the Alfred Rényi Institute of Mathematics of the Hungarian Academy of Sciences.

The next meeting of Presidents of Mathematical Societies will be held in the Spring of 2017 in Lisbon, hosted by the Portuguese Mathematical Society.

2016 Kamil Duszenko Prize Goes to Kate Juschenko

Kate Juschenko of Northwestern University has been named the recipient of the 2016 Kamil Duszenko Award, awarded for outstanding research in Geometric Group Theory.

Kate Juschenko got her PhD in 2011 at Texas A& M. Her advisor was Gilles Pisier. She held positions at EPFL, Vanderbilt University and CNRS. She has been at Northwestern University since 2013.

Kate Juschenko started her mathematical education and work in Ukraine. She was then interested in operator algebras, and participated in exploring the notion and providing examples of *-wild C^*-algebras. She was also engaged in investigating multi-dimensional Schur multilpiers for C^*-algebras.

Her work connected to geometric group theory is concentrated on the notion of amenability. She was able (with various coauthors) to deeply explore it for groups acting on the Cantor set. To a homeomorphism T of the Cantor set one associates the topological full group $[[T]]$ consisting of homeomorphism piecewise given by powers of T. Matui showed that for suitably chosen T the commutator subgroup of $[[T]]$ is finitely generated and simple. Juschenko and Monod proved that these groups are also amenable, thus providing the first examples of finitely generated amenable simple groups. Later, with Nekrashevych and de la Salle she extended this result and devised a new method of proving amenability, settling some open cases and unifying many known ones. Moreover, Juschenko found a powerful method to demonstrate that many groups acting on trees are not elementary amenable.

The Kamil Duszenko Award has been set up in memory of an outstanding young mathematician who died in 2014 of acute lymphoblastic leukaemia at the age of 28. The webpage with more information is

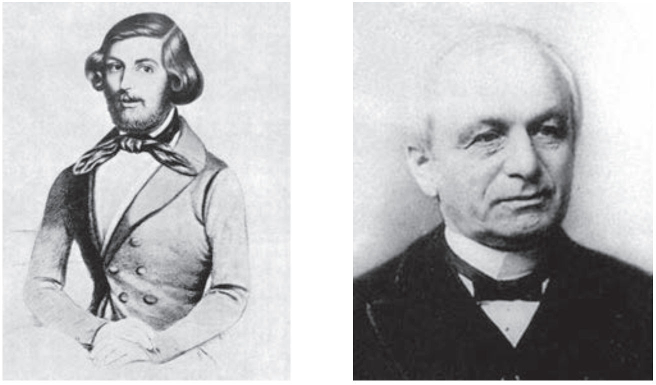
http://kamil.math.uni.wroc.pl/en/mathematics/about/

Elliptic Functions According to Eisenstein and Kronecker: An Update

Newly found notes of lectures by Kronecker

Pierre Charollois (Université Paris 6, France) and Robert Sczech (Rutgers University, Newark, USA)

This article introduces a set of recently discovered lecture notes from the last course of Leopold Kronecker, delivered a few weeks before his death in December 1891. The notes, written by F. von Dalwigk, elaborate on the late recognition by Kronecker of the importance of the "Eisenstein summation process", invented by the "companion of his youth" in order to deal with conditionally convergent series that are known today as Eisenstein series. We take this opportunity to give a brief update of the well known book by André Weil (1976) that brought these results of Eisenstein and Kronecker back to light. We believe that Eisenstein's approach to the theory of elliptic functions was in fact a very important part of Kronecker's planned proof of his visionary "Jugendtraum".



Gotthold Eisenstein (1823–1852) and Leopold Kronecker (1823–1891) (left to right)

1 Introduction

Born in 1823, Leopold Kronecker died in Berlin on 29 December 1891[1] at the age of 68, precisely 15 days after delivering the last lecture of his university course entitled "On elliptic functions depending on two pairs of real variables". This historical information was gathered from recently discovered lecture notes at the library of the University of Saarbrücken. Before discussing the content and the author of these handwritten notes, we wish to recall the circumstances of their discovery. The original discovery is due to Professor Franz Lemmermeyer, who started but did not finish the task of retyping the text written in old style German cursive handwriting. The manuscript then got lost during a library move. Being interested to learn more about Kronecker's work and gain insight into his so-called "liebster Jugendtraum", we decided to enlist the help of Simone Schulze at the library in finding it. After a search effort which lasted a few weeks, the complete set of notes was finally found and the library even produced a high quality digital copy, which is now available to the public [Cw]. We thank Ms Schulze for her help and we also thank Franz-Josef Rosselli, who undertook the effort to translate the cursive German handwriting (prevailing in the 19th century) into modern German typeface [Cw].

The manuscript was written by Friedrich von Dalwigk, a graduate student at the time, who was just finishing his dissertation on theta functions of many variables [vD]. Later, von Dalwigk became a professor of applied mathematics at the University of Marburg (1897–1923). Although the course only consisted of six lectures (due to the premature death of Kronecker), the whole manuscript is more than 120 pages, with many appendices, partly written by von Dalwigk, relying on published papers of Kronecker as well as unpublished papers from Kronecker's "Nachlass". Dalwigk explicitly mentions the word "Nachlass" in the manuscript. He is likely to have had access to that specific document from his colleague Hensel in Marburg, who was in possession of all the scientific papers of Kronecker at the time. As we learned from Hasse and Edwards [Ed], the personal papers of Kronecker were lost in the chaotic events surrounding World War II. Most probably, they were destroyed by a fire caused by exploding munitions in an old mine near Göttingen. That mine was used to store some of the collections of the Göttingen Library in 1945. This dramatic event is only part of the long-lasting spell put on the posterity of Eisenstein's ideas, as predicted by André Weil in his essay [We1].

At any rate, the care and the amount of detail included in the manuscript is extraordinary and leads us to think that it was written with the ultimate intention to publish it as a book.

We know from a letter of Kronecker to his friend Georg Cantor, who was the first president of the newly founded German Mathematical Association (DMV) and who invited Kronecker to deliver the opening address at the first annual meeting of the DMV in 1891, that Kronecker intended to talk at that meeting about the "forgotten" work of Eisenstein. In that letter, Kronecker apologises for not being able to attend the meeting due to the death of his wife Fanni. It is very likely that the lecture notes in question are an expanded version of his intended talk.

Kronecker was indeed a great mathematician who made fundamental contributions to algebra and number theory. We mention here only his "Jugendtraum", which, historically, gave rise to class field theory and to Hilbert's 12$^{\text{th}}$ prob-

1 According to the tombstone on his grave, Kronecker died on 23 December 1891.

lem (the analytic generation of all abelian extensions of a given number field), one of the great outstanding problems in classical algebraic number theory. The celebrated conjectures of Stark (published in a sequence of four papers during the 1970s) offer a partial solution to a problem with ultimate roots in the work of Kronecker and Eisenstein.

In passing, we wish to mention two standard references about the work of Kronecker in number theory. The first chapter of Siegel's Lectures on Advanced Number Theory [Si] is devoted to the so-called Kronecker limit formulas with applications (Kronecker's solution of Pell's equation). The second reference is the book by André Weil [We2] on elliptic functions according to Eisenstein and Kronecker. The lecture notes under review can be roughly classified as an extension and elaboration of the material discussed by Weil. Besides resurrecting the ideas of Eisenstein from final oblivion, the book of Weil is also a valuable source for many anecdotes about Eisenstein and Kronecker. Our favourite anecdote is the story that Kronecker, in the public defence of his PhD, claimed that mathematics is both science and art; his friend Eisenstein challenged him publicly by claiming that mathematics is art only.

2 Kronecker and the work of Eisenstein

Except for a letter to Dedekind dated 15 March 1880, there is no comprehensive statement of the Jugendtraum in the papers of Kronecker. In that letter, Kronecker reports on his recent progress towards a proof of his conjecture (the Jugendtraum) that all abelian extensions of an imaginary quadratic field F are generated by division values of suitable elliptic functions admitting complex multiplication by elements of F together with the corresponding singular moduli, that is, the values of the j-invariant of the corresponding elliptic curves. He expresses hope of completing the proof soon. In closing, he regrets having to postpone the problem of finding the analogue of singular moduli for arbitrary complex number fields (Hilbert's 12$^{\text{th}}$ problem) until the case of imaginary quadratic fields is completely resolved. Ten years later, in his lectures on elliptic functions, he does not mention his work on the Jugendtraum at all. Instead, he concentrates on reviewing and generalising the work of Eisenstein.

In what follows, we wish to give a hypothetical explanation of why the approach of Eisenstein may have been an important step in Kronecker's envisioned proof of his Jugendtraum. Namely, we are going to carry out Kronecker's programme in the simpler setting of abelian extensions of the field of rational numbers by modifying a basic example given by Eisenstein in his great paper [Eis2].

Let u be a complex number that is not an integer. Then, the coset $\mathbb{Z} + u \subset \mathbb{C} \setminus \mathbb{Z}$ does not contain the zero element so all the terms of the series

$$\phi(u) = \sum_{m \in \mathbb{Z}+u} \frac{1}{m}$$

are well defined but the series does not converge absolutely. It is therefore necessary to specify an order of summation. Following Eisenstein, we define

$$\phi(u) = \lim_{t \to +\infty} \sum_{\substack{m \in \mathbb{Z}+u, \\ |m|<t}} \frac{1}{m} = \frac{1}{u} + \sum_{n=1}^{\infty} \left(\frac{1}{u+n} + \frac{1}{u-n} \right)$$
$$= \frac{1}{u} + \sum_{n=1}^{\infty} \frac{2u}{u^2 - n^2}.$$

The last series on the right converges absolutely. The function ϕ is odd and 1-periodic, hence $\phi(\frac{1}{2}) = 0$. Next, we consider the special value $\phi(\frac{1}{4})$ and we obtain

$$\phi\left(\frac{1}{4}\right) = \sum_{m \in \mathbb{Z}+\frac{1}{4}} \frac{1}{m} = 4 \sum_{\ell \in 4\mathbb{Z}+1} \frac{1}{\ell}$$
$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots\right) = 4 \int_0^1 \frac{dx}{1+x^2} = \pi.$$

We are now ready to derive the fundamental property of the ϕ function, its addition formula. To this end, we let u, v, w be three complex numbers, none of which is an integer, such that $u + v + w = 0$. Then, the equation

$$p + q + r = 0 \tag{1}$$

with $p \in \mathbb{Z} + u, q \in \mathbb{Z} + v, r \in \mathbb{Z} + w$ has infinitely many solutions that can be obtained by letting p, q (or q, r or r, p) run independently. Since $pqr \neq 0$, Equation (1) becomes

$$\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp} = 0.$$

It is the starting point of Eisenstein's method, which he learnt from his high school teacher Schellbach, to average this rational identity over all solutions of Equation (1). Since the resulting double series are conditionally convergent, we again need to pay attention to the ordering of the series. To this end, we choose three non-zero fixed real parameters α, β, γ such that $\alpha + \beta + \gamma = 0$. Then,

$$\alpha p - \beta q = \gamma q - \alpha r = \beta r - \gamma p,$$

which allows us to write

$$\sum_{|\alpha p - \beta q| < t} \frac{1}{pq} + \sum_{|\gamma q - \alpha r| < t} \frac{1}{qr} + \sum_{|\beta r - \gamma p| < t} \frac{1}{rp} = 0.$$

It is easy to see that each of these series converges absolutely for every $t > 0$. In order to pass to the limit $t \to \infty$, we need the following lemma:

Lemma 2.1.

$$\lim_{t \to \infty} \left(\sum_{|\alpha p - \beta q| < t} \frac{1}{pq} \right) = \phi(u)\phi(v) + \pi^2 \operatorname{sign} \alpha \operatorname{sign} \beta.$$

For a generalisation and a proof of this lemma, we refer to [Scz3, Th.2]. As a corollary, we obtain

$$\phi(u)\phi(v) + \phi(v)\phi(w) + \phi(w)\phi(u) = \pi^2, \tag{2}$$

which is the addition formula for the function ϕ. To prove (2) without using Lemma 2.1, it is enough to show that the left side is independent of u, v because the specialisation at $u = v = \frac{1}{4}, w = -\frac{1}{2}$ provides the value π^2. That was the approach taken by Eisenstein in his original proof of (2).

To study arithmetic applications, we eliminate the period π by introducing the function

$$c(u) = \frac{\phi(u)}{\phi(1/4)} = \frac{\phi(u)}{\pi}$$

and its Cayley transform

$$\mathbf{e}(u) = \frac{c(u) + i}{c(u) - i}.$$

The addition formula (2) immediately implies

$$\mathbf{e}(u)\mathbf{e}(v) = \mathbf{e}(u + v).$$

From here, it follows that $\mathbf{e}(u) = e^{2\pi i u}$ and $c(u) = \cot(\pi u)$. Iterating the addition formula for the cotangent function, i.e.

$$c(u + v) = \frac{c(u)c(v) - 1}{c(u) + c(v)},$$

we obtain the formula

$$c(nu) = \frac{U_n(c(u))}{V_n(c(u))}, \ n = 1, 2, 3, 4, \ldots, \tag{3}$$

with polynomials $U_n, V_n \in \mathbb{Z}[t]$ given explicitly by

$$U_n(t) = \mathrm{Re}\,(t + i)^n, \ V_n(t) = \mathrm{Im}\,(t + i)^n.$$

A formula of type (3) was called a "transformation formula" in the 19th century. The elliptic analogues of (3) played a prominent role in the work of Kronecker. Taking $u \in \mathbb{Q}$ with denominator $n > 1$, we conclude that the numbers $c(u)$ are the roots of $V_n(t) = 0$, that is, $c(u)$ is a real algebraic number whenever u is a non-integral rational number. One can refine this result and give a more precise proof:

Proposition 2.2. The number $c(u)$ is an algebraic integer if and only if n is not the power of an odd prime. If $n = p^k$, with an odd prime p and $k > 0$, then $p\,c(u)$ is an algebraic integer. Moreover, $c(u)$ is a unit if and only if neither n nor $n/2$ is a power of an odd prime.

Theorem 2.3 (Kronecker-Weber)**.** The set of real numbers $c(\mathbb{Q} \setminus \mathbb{Z})$ generates the real subfield of the maximal abelian extension of $\mathbb{Q}$.

Kronecker was interested in generalising these results to the case of elliptic functions with period lattice being an ideal in an imaginary quadratic field. One of the difficulties he faced was that the addition formula for elliptic functions is in general algebraic and not rational, as in the case of the cotangent function. To the best of our knowledge, the proof of the Jugendtraum as envisioned by Kronecker has never been completed.[2] Except for the case of imaginary quadratic fields and the case of the rational number field (both closely related to the work of Kronecker), we do not even know whether Hilbert's 12th problem has a solution. This is partly related to a classical theorem of Weierstrass asserting that every meromorphic function with algebraic addition law is either elliptic, circular or rational.

Instead of the series defining $\phi(u)$, Kronecker preferred to study the more general series

$$\phi(u, \xi) = \sum_{n=-\infty}^{\infty} \frac{\mathbf{e}(-n\xi)}{u + n} = 2\pi i \frac{\mathbf{e}(\xi u)}{\mathbf{e}(u) - 1}, \quad 0 < \xi < 1, \tag{4}$$

where u is again a complex number that is not an integer. The introduction of the second variable ξ is very natural and is suggested by Fourier analysis. Note that the limiting case

$$\lim_{\substack{\xi \to 0 \\ \xi > 0}} \phi(u, \xi) = \phi(u) - i\pi$$

relates $\phi(u, \xi)$ to the Eisenstein function ϕ. The right side of Equation (4) is essentially the generating function for the Bernoulli polynomials. Expanding the left side into a power series in u, one obtains the Fourier expansion of the Bernoulli polynomials. The above proof of the addition formula for the cotangent function applies to $\phi(u, \xi)$ as well and yields addition formulas for the Bernoulli polynomials. Due to the factor $\mathbf{e}(-n\xi)$ in the numerator, the convergence of this series is slightly better than that of the cotangent series but is still conditional. A substantial part of the lecture notes is devoted to the study of the elliptic analogue of (4), written in Kronecker's notation,

$$Ser\,(\xi, \eta, u, \tau) = \lim_{N \to +\infty} \lim_{M \to +\infty} \sum_{n=-N}^{N} \sum_{m=-M}^{M} \frac{\mathbf{e}(-m\xi + n\eta)}{u + n\tau + m}, \tag{5}$$

where ξ, η is a pair of real variables and τ is a point in the upper half space. The complex variable u must be restricted to the complement of the lattice $\mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C}$. Writing $u = \sigma\tau + \rho$ as a linear combination of τ and 1 with real coefficients σ, ρ, this series can be viewed as a function of two pairs of real variables $(\xi, \eta), (\sigma, \rho)$, the ones referred to in the title of the lecture notes.

Various alternative ways to sum the conditionally convergent series (5) are discussed in the manuscript. It is a remarkable fact that, in all cases, the value obtained for the sum is independent of the limiting process chosen. Kronecker's main result expresses these series in terms of Jacobi theta series. Let

$$\begin{aligned} \vartheta(z, \tau) = \vartheta_1(z, \tau) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \mathbf{e}\left(\frac{n^2\tau}{2} + n\left(z - \frac{1}{2}\right)\right) \\ &= 2\, q^{\frac{1}{8}} \sin(\pi z) \prod_{n \geq 1} (1 - q^n)(1 - q^n \mathbf{e}(z))(1 - q^n \mathbf{e}(-z)), \end{aligned}$$

with a complex variable z, a point τ in the upper half plane and $q = \mathbf{e}(\tau)$.

Theorem 2.4 (Kronecker)**.** Suppose $0 < \mathrm{Im}\ u < \mathrm{Im}\ \tau$ and $0 < \xi < 1$. Then,

$$Ser\,(\xi, \eta, u, \tau) = \mathbf{e}(\xi u) \frac{\vartheta'(0, \tau)\vartheta(u + \eta + \xi\tau, \tau)}{\vartheta(u, \tau)\vartheta(\eta + \xi\tau, \tau)}. \tag{6}$$

This result is reminiscent of the so-called limit formula of Kronecker, which is not discussed in the lecture notes but deserves to be stated here: let τ, τ' be two complex numbers with $\mathrm{Im}\ \tau > 0$ and $\mathrm{Im}\ \tau' < 0$ and let $0 \leq \xi, \eta < 1$ be two real numbers, not both equal to zero. Writing $u = \eta - \xi\tau, v = \eta - \xi\tau'$, the second limit formula is the identity

$$\begin{aligned} &\frac{(\tau - \tau')}{2\pi i} \sum_{m,n} \frac{\mathbf{e}(m\xi + n\eta)}{(m\tau + n)(m\tau' + n)} \\ &\qquad = -\log \frac{\vartheta(u, \tau)\vartheta(v, -\tau')}{\eta(\tau)\eta(-\tau')} - \pi i \frac{(u - v)^2}{\tau - \tau'}, \end{aligned} \tag{7}$$

where $\eta(\tau)$ refers to the Dedekind eta function and the term $(m, n) = (0, 0)$ needs to be excluded from the sum. Again, convergence is only conditional so a specific order of summation as in (5) needs to be observed.

Historically, the notion of complex multiplication of elliptic functions appeared for the first time in the work of Abel. Pages 64–67 of Kronecker's lecture notes are devoted to the task of expressing the elliptic functions used by Abel in terms

2 See the comment linked to interpretation (c) in [HH].

of the Kronecker series $Ser(\xi, \eta, u, \tau)$. This is perhaps one of the highlights of the manuscript and deserves special attention. It is very likely that Kronecker's conception of the Jugendtraum was the result of a close study of the work of Abel.

3 Recent developments

It would be wise to leave the complete discussion of the legacy of Eisenstein or Kronecker to serious historians. We offer, instead, a brief survey of several recent developments that feature Eisenstein's summation process and Kronecker's Theorem 2.4 for the series $Ser\,(\xi, \eta, u, \tau)$

Algebraicity and p-adic interpolation of Eisenstein–Kronecker numbers

It was already observed by Weil in his 1976 book that the combination of the methods of Eisenstein and Kronecker gives direct access to Damerell's classical result (1971) on the algebraicity of values of L-functions attached to a Hecke Grössencharakter of an imaginary quadratic field.

Weil also anticipated that their methods would extend to the investigation of the p-adic properties of these algebraic numbers. In the following 10 years, the works of Manin-Višik, Katz and Yager among others provided the expected p-adic interpolation of this family of special values. To give a taste of the results in question, we wish to introduce a recent work by Bannai-Kobayashi (2010), based on Kronecker's series, that enables a similar construction.

Our first task is to connect the series Ser to a series that includes an s-parameter pertaining to the style of Hecke. Let τ be a complex number in the upper half plane and let $A = \frac{\mathrm{Im}\tau}{\pi}$ be the area of the fundamental domain of the lattice $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ divided by π. Let ψ be the character $\psi(z) = e^{\frac{z-\bar{z}}{A}}$. For $a \geq 0$ an integer, we introduce the Kronecker-Eisenstein series as

$$K_a^*(z, w, s, \tau) = \sum_{\lambda \in \Lambda}^{*} \frac{(\bar{z} + \bar{\lambda})^a}{|z + \lambda|^{2s}}\, \psi(z\bar{w}), \quad \mathrm{Re}(s) > a/2 + 1,$$

where the $*$ means that the summation excludes $\lambda = -z$ if $z \in \Lambda$. It is a continuous function of the parameters $z \in \mathbb{C} \setminus \Lambda$, $w \in \mathbb{C}$ and it has meromorphic continuation to the whole s-plane, with possible poles only at $s = 0$ (if $a = 0$ and $z \in \Lambda$) and $s = 1$ (if $a = 0$ and $w \in \Lambda$). Moreover, it satisfies a functional equation relating the value at s to the value at $a + 1 - s$. Write $w = \eta + \xi\tau$ with real variables ξ, η and abbreviate the central value $K_1^*(z, w, 1, \tau)$ by $K(z, w)$ when there is no ambiguity on the lattice. This specific function is related to Kronecker's series by the identity

$$K(z, w) = K_1^*(z, w, 1, \tau) = Ser\,(\xi, \eta, z, \tau),$$

where $w = \eta + \xi\tau$, at least if $z, w \notin \Lambda$, using [We1, §5, p. 72]. In this new set of notations, Theorem 2.4 can be restated as

$$K(z, w) = e^{z\bar{w}/A}\, \Theta(z, w),$$

where

$$\Theta(z, w) = \frac{\vartheta'(0, \tau)\vartheta(z + w, \tau)}{\vartheta(z, \tau)\vartheta(w, \tau)} \tag{8}$$

denotes the meromorphic function appearing as the ratio in Equation (6). It will play a major role in the remainder of this text so we name it the "Kronecker theta function", in agreement with the terminology in [BK].

Given a pair of integers $a \geq 0, b > 0$ and $z_0, w_0 \in \mathbb{C}$, the Eisenstein–Kronecker numbers are defined as

$$e_{a,b}^*(z_0, w_0, \tau) = K_{a+b}^*(z_0, w_0, b, \tau).$$

When $b > a + 2$, these numbers include, in particular, the values of the absolutely convergent partial Hecke L-series

$$e_{a,b}^*(0, 0, \tau) = \sum_{\lambda \in \mathbb{Z}+\tau\mathbb{Z}} \frac{\bar{\lambda}^a}{\lambda^b},$$

which should be considered as elliptic analogues of the Bernoulli numbers. As such, the Eisenstein–Kronecker numbers can be packaged into a generating series that is the elliptic analogue of the cotangent function and its relative $\phi(u, \xi)$. To obtain the nice two-variables generating series, it is enough to translate and slightly alter the Kronecker theta function in order to define

$$\Theta_{z_0,w_0}(z, w) = e^{-\frac{(z_0+z)\overline{w_0}+w\overline{z_0}}{A}}\,\Theta(z_0 + z, w_0 + w).$$

Its Laurent expansion around $z = w = 0$ displays exactly the collection of Eisenstein–Kronecker numbers.

Proposition 3.1. Fix $z_0, w_0 \in \mathbb{C}$. We have the following Laurent expansion near $z = w = 0$:

$$\Theta_{z_0,w_0}(z, w) = \psi(w_0\overline{z_0})\frac{\delta(z_0)}{z} + \frac{\delta(w_0)}{w} + \sum_{a \geq 0, b > 0} (-1)^{a+b-1}\frac{e_{a,b}^*(z_0, w_0, \tau)}{a!A^a} z^{b-1}w^a, \tag{9}$$

where $\delta(u) = 1$ if $u \in \Lambda$ and 0 otherwise.

If, in addition, τ is a CM point and z_0, w_0 are torsion points over the lattice Λ then these coefficients are algebraic.

Theorem 3.2. Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice in $\mathbb{C}$. Assume that the complex torus $\mathbb{C}/\Lambda$ has complex multiplication by the ring of integers of an imaginary quadratic field k, and possesses a Weierstrass model $E : y^2 = 4x^3 - g_2x - g_3$ defined over a number field F. Fix $N > 1$ an integer. For z_0, w_0, two complex numbers such that $Nz_0, Nw_0 \in \Lambda$, the Laurent expansion in (9) has coefficients in the number field $F(E[4N^2])$. In particular, the rescaled Eisenstein–Kronecker numbers

$$e_{a,b}^*(z_0, w_0, \tau)/A^a$$

are algebraic.

We refer to [BK], Th. 1 and Cor. 2.11 for the proofs. The above construction enables Bannai and Kobayashi to recover Damerell's result along the way.

Let p be a prime number. Bernoulli numbers and Bernoulli polynomials satisfy a whole collection of congruences modulo powers of p, known as "Kummer congruences". These congruences are incorporated in the construction of the Kubota-Leopoldt p-adic zeta function $\zeta_p(s)$, $s \in \mathbb{Z}_p$, which interpolates the values of the Riemann zeta function at negative integers, as given by the Bernoulli numbers.

Similarly, Eisenstein–Kronecker numbers satisfy a collection of congruences that are the building blocks for the construction of p-adic L-functions of two-variables. Since Bannai-Kobayashi also have a generating series at their disposal, they can interpolate the Eisenstein–Kronecker numbers p-adically, not only when p splits in k, like in the work of Katz (ordinary case), but also when p is inert in k (supersingular case).

It seems appropriate to remark in passing that in this p-adic setting, the Kronecker limit formula (7) also has a counterpart. It has been originally obtained by Katz (1976) and proved to be crucial in the study of Euler systems attached to elliptic units.

To complete this p-adic picture and set the stage for a recurring theme for sections to come, we would like to mention a generalisation by Colmez-Schneps [CS] of the above construction of Manin-Višik and Katz. Colmez and Schneps consider the case where the Hecke character is attached to an extension F of degree n over the imaginary quadratic field k (the previous setting thus corresponds to $n = 1$). The field F might not necessarily be a CM field. Building on the techniques of [Co], they can construct a p-adic L-function for F by interpolating the algebraic numbers arising from the Laurent expansion of certain linear combinations of *products of n generating series* of Kronecker's type, each of them being evaluated at torsion points over the lattice Λ. The juxtaposition of n copies of Proposition 3.1-Theorem 3.2 allows them to bootstrap the case $n = 1$ to arbitrary $n \geq 1$ using their identity [CS, Eq. (31)].

Periods of Hecke eigenforms

Let (z_0, w_0) be a fixed pair of N-torsion points over the lattice Λ. As functions of the τ variable, the modular forms $e^*_{a,b}(z_0, w_0, \tau)$ are Eisenstein series for the principal congruence subgroup $\Gamma(N)$. In particular, the Laurent expansion (9) for the translated Kronecker theta function at $z = w = 0$ is a generating series for Eisenstein series of increasing weight and fixed level. Its decomposition under the action of the Hecke algebra thus possesses only Eisenstein components. From this perspective, interesting new phenomena start to appear when one considers *a product of two Kronecker theta functions*.

Such a product encodes all period polynomials of modular forms of all weights, at least in the level one case.[3] This is the content of the main result of Zagier's paper [Za1], which we now describe.

Let M_k be the $\mathbb{C}$-vector space of modular forms of weight $k \geq 4$ on $SL_2(\mathbb{Z})$ and let $S_k \subset M_k$ be the subspace of cusp forms, equipped with the Petersson scalar product (f, g) and its basis of normalised Hecke eigenforms $\mathcal{B}_k^{\mathrm{cusp}}$. The period polynomial attached to $f \in S_k$ is the polynomial of degree $\leq k-2$ defined by

$$r_f(X) = \int_0^{i\infty} f(\tau)(\tau - X)^{k-2}d\tau.$$

The Eichler-Shimura-Manin theory implies that the maps $f \mapsto r_f^{\mathrm{ev}}$ and $f \mapsto r_f^{\mathrm{od}}$ assigning to f the even and the odd part of r_f are both injective. Moreover, if f is a normalised Hecke eigenform then the two-variables polynomial

$$R_f(X, Y) = \frac{r_f^{\mathrm{ev}}(X)r_f^{\mathrm{od}}(Y) + r_f^{\mathrm{od}}(X)r_f^{\mathrm{ev}}(Y)}{(2i)^{k-3}(f, f)} \in \mathbb{C}[X, Y]$$

transforms under $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ as $R_{\sigma(f)} = \sigma(R_f)$, so R_f has coefficients in the number field generated by the Fourier coefficients of f. As a consequence, for each integer $k > 0$, the finite sum

$$C_k^{\mathrm{cusp}}(X, Y, \tau) = \frac{1}{(k-2)!}\sum_{f \in \mathcal{B}_k^{\mathrm{cusp}}} R_f(X, Y)f(\tau)$$

belongs to $\mathbb{Q}[X, Y][[q]]$. Zagier starts to complete this cuspidal term by a contribution that arises from the Eisenstein series in M_k, using the following convenient recipe. For any even $k > 0$, let B_k be the usual k-th Bernoulli number and let

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{m \geq 1}\Big(\sum_{d|m} d^{k-1}\Big)q^m$$

be the normalised weight k Eisenstein series for $SL_2(\mathbb{Z})$. Its pair of period functions in $X^{-1}\mathbb{Q}[X]$ is defined by the odd (and even) rational fractions

$$r_{E_k}^{\mathrm{od}}(X) = \sum_{h=0}^{k}\frac{B_h}{h!}\frac{B_{k-h}}{(k-h)!}X^{h-1}, \quad r_{E_k}^{\mathrm{ev}}(X) = X^{k-2} - 1$$

and they make up the contribution of Eisenstein series by the rule

$$C_k^{\mathrm{Eis}}(X, Y, \tau) = -\Big(r_{E_k}^{\mathrm{ev}}(X)r_{E_k}^{\mathrm{od}}(Y) + r_{E_k}^{\mathrm{od}}(X)r_{E_k}^{\mathrm{ev}}(Y)\Big)E_k(\tau).$$

The main identity of Zagier then establishes a remarkable closed formula for the generating series

$$C_\tau(X, Y, T) = \frac{(X+Y)(XY-1)}{X^2Y^2T^2} + \sum_{k=2}^{\infty}(C_k^{\mathrm{cusp}} + C_k^{\mathrm{Eis}})T^{k-2},$$

which factorises as a product of two Kronecker theta functions:

Theorem 3.3 (Zagier [Za1])**.** In $(XYT)^{-2}\mathbb{Q}[X, Y][[q, T]]$, we have

$$C_\tau(X, Y, T) = \Theta(XT', YT')\,\Theta(T', -XYT')/\omega^2, \tag{10}$$

where $\omega = 2\pi i$, $T' = T/\omega$.

Equation (10) shows that complete information on Hecke eigenforms of any desired weight for $SL_2(\mathbb{Z})$ and their period polynomial is encoded in the Laurent expansion at $T = 0$ of the right side. To further support that claim, Zagier explains in the sequel paper [Za2] how to deduce from Equation (10) an elementary proof of the Eichler-Selberg formula for traces of Hecke operators on $SL_2(\mathbb{Z})$.

The period polynomials satisfy a collection of linear relations under the action of $SL_2(\mathbb{Z})$. These cocycle relations are reflected using Equation (10) by relations satisfied by Kronecker's theta functions, e.g., [Za1, p. 461]. A typical example is

$$C_\tau(X, Y, T) + C_\tau\left(1 - \frac{1}{X}, Y, TX\right) + C_\tau\left(\frac{1}{1-X}, Y, T(1-X)\right) = 0, \tag{11}$$

which is the counterpart of the classical relation for the period polynomial

$$r_f \mid 1 + U + U^2 = 0,\ U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the remainder of this paper, we explain how the method of Eisenstein–Schellbach, when properly modulated, is the adequate tool to produce systematically general $(n-1)$-cocycle relations for $GL_n(\mathbb{Z})$ involving products of n Kronecker theta functions, including Equation (11) as a very special case.

3 A very recent preprint [CPZ] indicates that a similar result also holds for arbitrary squarefree level.

Trigonometric cocycles on GL_n

Let $0 < a, b < 1$ be rational numbers and $x, y \in \mathbb{C} \setminus \mathbb{Z}$ be complex parameters. We use the shorthand $0 < \{t\} < 1$ for the fractional part of a non-integral real number t. From a direct computation or a mild generalisation of Lemma 2.1, one deduces the following relation, which amounts to the addition formula for the function $\phi(u, \xi)$:

$$\frac{\mathbf{e}(xa)\mathbf{e}(yb)}{(\mathbf{e}(x)-1)(\mathbf{e}(y)-1)} - \frac{\mathbf{e}((x+y)a)\mathbf{e}(y\{b-a\})}{(\mathbf{e}(x+y)-1)(\mathbf{e}(y)-1)} - \frac{\mathbf{e}((x+y)b)\mathbf{e}(x\{a-b\})}{(\mathbf{e}(x+y)-1)(\mathbf{e}(x)-1)} = 0. \quad (12)$$

As pointed out by the second author in [Scz2], this identity can naturally be recast in terms of the cohomology of the group $SL_2(\mathbb{Z})$. The building blocks are products of two copies of the trigonometric function $\phi(u, \xi)$. Given two pairs $u = (u_1, u_2)$ and $\xi = (\xi_1, \xi_2)^t$, we set

$$\mathbf{\Phi}(u, \xi) = \phi(u_1, \xi_1)\phi(u_2, \xi_2).$$

To any matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ is associated $\sigma = \left(\begin{smallmatrix} 1 & a \\ 0 & c \end{smallmatrix}\right) \in M_2(\mathbb{Z})$. If $c \neq 0$, we define

$$\Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(u, \xi) = \operatorname{sign}(c) \sum_{\mu \in \mathbb{Z}^2/\sigma\mathbb{Z}^2} \mathbf{\Phi}(u\sigma, \sigma^{-1}(\mu + \xi)).$$

If $c = 0$ then $\Psi(A)(u, \xi) = 0$ by definition. The next proposition stands for the case $n = 2$ of a more general $(n-1)$-cocycle relation for the group $GL_n(\mathbb{Z})$ obtained in [Scz3, Cor. p. 598].

Proposition 3.4. Let $A, B \in SL_2(\mathbb{Z})$ be two matrices. For any u in a dense open domain in $\mathbb{C}^2$ and any non-zero $\xi \in \mathbb{Q}^2$, the following inhomogenous 1-cocycle relation holds:

$$\Psi(AB)(u, \xi) - \Psi(A)(u, \xi) - \Psi(B)(uA, A^{-1}\xi) = 0.$$

The addition law (12) corresponds to the choice of matrices $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in this proposition.

The coefficients of the Laurent expansion at $u = 0$ of the general $(n-1)$-cocycle Ψ carry a great deal of arithmetic information. According to the main result of [Scz3, Th. 1], the pairing of Ψ with an $(n-1)$-cycle, built out of the abelian group generated by the fundamental units of a totally real number field F of degree n, produces the values $\zeta_F(k)$ at non-positive integers k of the Dedekind zeta function of F and thereby establishes their rationality. This provides a new proof, deeply rooted in the Eisenstein–Schellbach method, of the Klingen–Siegel rationality result.

An integral avatar of the cocycle Ψ was later introduced by Charollois and Dasgupta to study the integrality properties of the values $\zeta_F(k)$, enabling them to deduce a new construction of the p-adic L-functions of Cassou–Noguès and Deligne–Ribet.

Combining their cohomological construction with the recent work of Spiess, they also show in [CD] that the order of vanishing of these p-adic L-functions at $s = 0$ is at least equal to the expected one, as conjectured by Gross. This result was already known from Wiles' proof of the Iwasawa Main Conjecture.

On elliptic functions depending on four pairs of real variables

Our goal in this last section is to propose a q-deformation of the trigonometric cocycle Ψ to construct an elliptic cocycle, where the role of the function $\phi(u, \xi)$ is now played by Kronecker's theta function. It will make it clear that the trigonometric relations we have encountered so far are a specialisation of the elliptic ones when $q \to 0$, i.e. $\tau \to i\infty$.

Kronecker has not been given the opportunity to implement the Eisenstein–Schellbach method in his own elliptic investigations. Let us now proceed by fixing τ in the upper half-plane and first perform in Equation (12) the change of variables

$$x \leftarrow n\tau + x,$$
$$y \leftarrow n'\tau + y.$$

We also aim to twist that equation by a pair of roots of unity $\mathbf{e}(nr)\mathbf{e}(n's)$ with $r, s \in \mathbb{Q}$ and then sum over $n, n' \in \mathbb{Z}$. We write $x_0 = a\tau + r, y_0 = b\tau + s, q = \mathbf{e}(\tau)$ so that the real and imaginary parts of x, x_0, y, y_0 make up four pairs of real variables. The first summand becomes

$$S := \mathbf{e}(xa)\mathbf{e}(yb) \sum_{n,n' \in \mathbb{Z}} \frac{\mathbf{e}(nr)q^{na}q^{n'b}\mathbf{e}(n's)}{(q^n\mathbf{e}(x)-1)(q^{n'}\mathbf{e}(y)-1)}.$$

The assumption $0 < a < 1$ ensures that $1 > |q^a| > |q|$, and similarly for q^b, so that this double series is absolutely convergent. The sum S naturally splits as a product, whose value is deduced from two consecutive uses of Kronecker's Theorem 2.4:

$$S = \mathbf{e}(xa)\mathbf{e}(yb)\,\Theta(x, x_0)\,\Theta(y, y_0).$$

A similar resummation process can be performed on the second term and the third term of Equation (12). After simplification by the common factor $\mathbf{e}(xa+yb)$, we obtain the identity

$$\Theta(x, x_0)\Theta(y, y_0) - \Theta(x+y, x_0)\Theta(y, y_0 - x_0) - \Theta(x+y, y_0)\Theta(x, x_0 - y_0) = 0, \quad (13)$$

which can be extended analytically to remove the restrictive assumptions made on the parameters. Equation (13) is just another form of the Riemann theta addition relation, also known as the Fay trisecant identity. It simultaneously implies Equation (12) when $\tau \to i\infty$ and Equation (11) under the choice of parameters $x = XT', x_0 = YT', y = -T', y_0 = XYT'$.

More generally, the resummation of the trigonometric $(n-1)$-cocycle Ψ gives rise to an elliptic $(n-1)$-cocycle that we name the Eisenstein–Kronecker cocycle κ on $GL_n(\mathbb{Z})$. One recovers Ψ from κ by letting $\tau \to i\infty$. The coefficients of the Laurent expansion of κ at zero now display modular forms, essentially sums of products of n Eisenstein series of various weights and levels, that are members of a compatible p-adic family.

When paired with a $(n-1)$-cycle built out of the fundamental units of a totally real number field F of degree n, the elliptic cocycle κ produces a generating series for the pullbacks of Hecke–Eisenstein series over the Hilbert modular group $SL_2(O_F)$, whose constant terms are the values $\zeta_F(k)$ at negative integers $k \leq 0$. These classical modular forms have already played a prominent role in the original proof by Siegel of the Klingen–Siegel theorem. More details on the construction of the Eisenstein–Kronecker cocycle κ and its properties will be given in [Ch].

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Wrinkles: From the Sea to Mathematics

François Laudenbach (Université de Nantes, France)

Wrinkles and sealing

The mathematical meaning of *wrinkle* and its real-life meaning are very similar, particularly when the word is used to describe features on the surface of the sea. Luckily, I was educated in sailing before learning advanced mathematics so I spent a lot of time looking at wrinkles on the sea. After a long, windless period, it is delightful to observe their birth and the forms they take. Long after learning to sail, I read that these forms bear great similarity to those predicted by R. Thom's singularity theory (see [15]). On one occasion, wrinkles appeared and grew so abruptly that they clearly announced a strong event; at the same time, the on-board radio issued gale warnings for the forthcoming night. Soon the fog cleared, allowing us to see an enormous cumulonimbus cloud, with its white anvil thousands of metres high, indicating the presence of fantastic thermodynamic machinery. Fortunately, this experience concluded safely.

Basics about wrinkles

Mathematical *wrinkles* are models for maps $\mathbb{R}^n \to \mathbb{R}^q$, $n \geq q$, restricted to some explicit compact subset. The very primitive wrinkle is the one-variable smooth function whose graph is shown in Figure 1. That function f, defined on $[a, b]$, has one maximum and one minimum, both non-degenerate, meaning that the second derivative does not vanish at these critical points; one says that f is a Morse function. It is unique up to reparametrisation of the domain and the range, which can alter, for example, the distance between critical points and critical values.

As $f(a) > f(b)$, the *maximum principle* for real continuous functions makes it impossible to deform f to a function without critical points if we insist on keeping f unchanged near the boundary points a and b. Indeed, there must exist a maximum value greater than $f(a)$ and a minimum value less than $f(b)$.

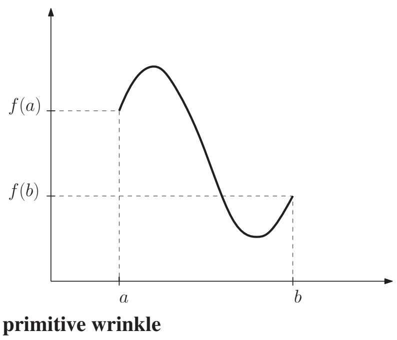

Figure 1. The primitive wrinkle

Nevertheless, the differential df can be deformed to a non-vanishing differential 1-form, denoted $\mathcal{R}df$, which coincides with df near a and b. Of course, the corresponding change in f would have to change the value at one endpoint, say b. This $\mathcal{R}df$ is called the *regularized differential* of f. It is unique up to deformation among the non-vanishing differential forms since the constraints near the endpoints define a convex set. The name and notation are due to Y. Eliashberg and N. Mishachev [2], the inventors of mathematical wrinkles.

Wrinkles and immersions

Historically, this first wrinkle (without its name) appeared soon after S. Smale's breakthrough of the *sphere eversion* [14], a counter-intuitive phenomenon where it is possible to turn a sphere inside out by allowing only self-intersections but no pinching, that is, by moving the sphere through *immersions*. The first person who figured out Smale's result was A. Shapiro (1961)[1] (see, in chronological order, [11], [6] and [9]). The eversion [11], of which Figure 2 shows one stage, exhibits a lot of wrinkles.

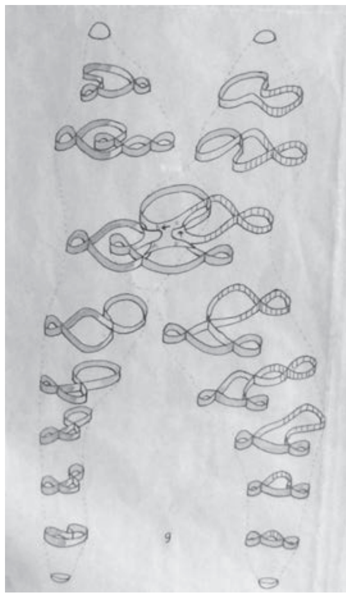

Figure 2. Original drawing for a sphere eversion (courtesy of Anthony Phillips)

1 Quoting the beginning of Francis & Morin's article: "We dedicate this article to the memory of Arnold Shapiro, who gave the first example of how to turn the sphere inside out, but never published it."

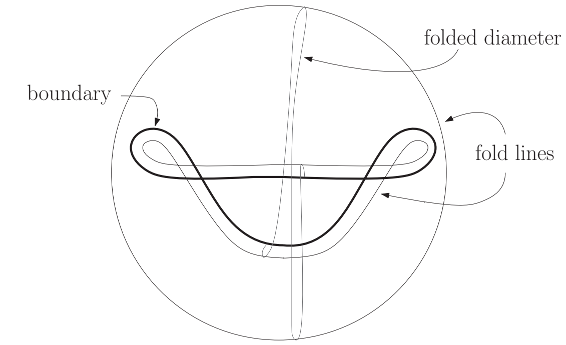

Figure 3. Folded immersion of the 2-disc and one folded diameter

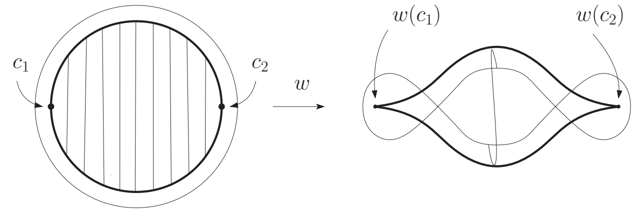

Figure 4. The *lips wrinkle* has cusps at c_1 and c_2

A natural question to ask is which immersions of the $(n-1)$-sphere into $\mathbb{R}^n$ extend to an immersion of the n-ball. Apart from some obvious homotopy theoretical obstructions, the problem remains open for $n > 2$. A complete (but difficult) answer in dimension 2 has been given by S. Blank (see [12]).

In 1966, V. Poenaru [13] showed that relaxing the immersion condition on the n-ball by allowing *folds* (such as, in local coordinates, $x \mapsto x^2$ in dimension one or $(x, y) \mapsto (x^2, y)$ in dimension two) made the problem of extension easily solvable. For instance, focusing on $n = 2$ to make drawings possible, every immersion $S^1 \to \mathbb{R}^2$ extends to a *folded immersion* of the disc to the plane as in Figure 3. In this example, without any fold line, there is no immersion of the disc extending the given immersion. Indeed, the immersion of the disc if it exists must enter the unbounded component of the complement of the given immersed boundary line. But the disc is compact and any point of its image lying at a maximal distance from the origin should be a critical point of the map, which is therefore not an immersion.

This result translates a sort of *flexibility* in a sense precisely defined by M. Gromov in his seminal book [8]. The fold lines that Poenaru introduces lie parallel to the boundary in a collar neighbourhood in the source disc. Thus, a pair of consecutive such fold lines may be thought of as a one-parameter family of primitive wrinkles. In the example shown in Figure 3, this family is made of folded diameters; one of the diameters is shown.

Wrinkles and the h-principle

The efficiency of what have been called primitive wrinkles above – mostly used in families – is already remarkable when applied to immersions (see W. Thurston's *corrugations* [16]). Of course, if one aims to apply wrinkles to very general classes of smooth mappings of manifolds, a more elaborate model of wrinkles has to be used. Fortunately, this model, which is *global* in essence, still involves the two local stable singularities of mappings from plane to plane only, namely the *fold* and the so-called *cusp*.

If the source and target are two-dimensional, a *wrinkle* W consists, as in Figure 4, of a neighbourhood of a disc D in the source, fibred in intervals over an interval I, with two fibres tangent to the boundary ∂D at points c_1 and c_2, and a smooth map $w : W \to \mathbb{R}^2$. This map is fibred over an interval; its singular locus consists of two cusps in c_1 and c_2 and of two fold lines, namely the two open arcs $\partial D \smallsetminus \{c_1, c_2\}$ (see Figure 4).

The image of the singular locus looks like *lips*, a famous figure in Cerf's analysis of *pseudo-isotopies* [1]. Using horizontal coordinate y in $[-1, 1]$ and x in the fibre over y,

$$w(y, x) = \big(y, f_y(x)\big),$$

where $\big(f_y(x)\big)_y$ is a 1-parameter family of primitive wrinkles from *birth* to *death* as y traverses the interval $[y(c_1), y(c_2)]$. For further extension to high dimension, one requires the symmetry $f_y = f_{-y}$. By regularising the differential df_y smoothly in y, we get a regularised differential:

$$\mathcal{R}dw : T_{(y,x)}W \to T_{w(y,x)}\mathbb{R}^2.$$

The word *regularisation* translates the fact that the rank of $\mathcal{R}dw$ is maximal on all fibres of the tangent space TW. This model extends to a wrinkle w with source and target of dimension q, simply by taking $y \in \mathbb{R}^{q-1}$ and rotating the 2-dimensional model about the axis $y = 0$. Then,

$$w(y, x) = (y, f_{|y|}(x))$$

and the regularised differential still exists. Finally, one can obtain a non-equidimensional model by enlarging the source with an $(n-q)$-ball (with coordinate z) and taking the map

$$w(y, x, z) = \big(y, f_{|y|}(x) + Q(z)\big),$$

where Q is a non-degenerate quadratic form.

With these models at hand, a *wrinkled map* $g : M^n \to N^q$, $n \geq q$, between (possibly closed) manifolds is a smooth map that coincides with wrinkle models in finitely many disjoint balls in M^n and has maximal rank elsewhere. Eliashberg & Mishachev [2] state the *flexibility* of wrinkled maps, in that an h-principle in the sense of Gromov holds true for them. To make this more precise, denote by TM and TN the respective tangent spaces and consider the set W(M,N), formed by the collection of wrinkled maps and completed by maps with so-called *embryos* or *unborn wrinkles*. This set embeds into the space of bundle epimorphisms $Epi(TM, TN)$ by the regularised differential operator $\mathcal{R}d$ and this embedding is a homotopy equivalence.

Applications and novelties

A spectacular application to pseudo-isotopy theory [3] is a strong generalisation of a theorem by K. Igusa (1984) [7], with no restriction on dimension. Namely, any family of smooth functions $\{f_u : M \times ([0, 1], 0, 1) \to ([0, 1], 0, 1)\}_{u \in S^k}$ contracts in the space of Morse functions with the same boundary condition, completed by the embryos (functions with one cubic singularity).

More recently, Eliashberg & Mishachev with S. Galatius [5] have shown that wrinkles apply in an area usually reserved to homotopy theorists. The question is to compute the *stable*[2] homology of the mapping class group of a Riemann surface.

Last but not least, thanks to the flexibility of a slightly different object called a *wrinkled embedding* [4], we witnessed an exceptional event: E. Murphy's breakthrough in high-dimensional *contact topology* [10]. Thanks to wrinkling techniques (large zig-zags in *front projections*[3]), she discovered the *loose* Legendrian embeddings into contact manifolds. Such an embedding makes the ambient contact structure flexible. The wrinkle story is clearly far from over.

Acknowledgement

I am very indebted to Frank Morgan who gave me many suggestions for writing in a simpler language for addressing a large audience. I also feel very grateful to Allyn Jackson for her help in some delicate situations. I express my thanks to Anthony Phillips who offered me a copy of one of his beautiful drawings and to Vincent Borrelli for his careful reading of this note.

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2 Quoted from [5, p. 412]: "stabilization with respect to the genus".

3 Near a Legendrian submanifold, the local model of a contact structure is given by the one-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ equipped with the $2n$-plane field whose equation reads $dz - \sum p_i \, dq^i = 0$ in canonical coordinates. The front projection is the projection onto the 0-jet space $(q, p, z) \mapsto (q, z)$.

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Kardar–Parisi–Zhang Universality

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1 Universality in random systems

Universality in complex random systems is a striking concept that has played a central role in the direction of research within probability, mathematical physics and statistical mechanics. In this article, we will describe how a variety of physical systems and mathematical models, including randomly growing interfaces, certain stochastic PDEs, traffic models, paths in random environments and random matrices, all demonstrate the same universal statistical behaviours in their long-time/large-scale limit. These systems are said to lie in the *Kardar–Parisi–Zhang (KPZ) universality class*. Proof of universality within these classes of systems (except for random matrices) has remained mostly elusive. Extensive computer simulations, non-rigorous physical arguments and heuristics, some laboratory experiments and limited mathematically rigorous results provide important evidence for this belief.

The last 15 years have seen a number of breakthroughs in the discovery and analysis of a handful of special *integrable probability systems*, which, due to enhanced algebraic structure, admit many exact computations and, ultimately, asymptotic analysis revealing the purportedly universal properties of the KPZ class. The structures present in these systems generally originate in representation theory (e.g., symmetric functions), quantum integrable systems (e.g. Bethe ansatz) and algebraic combinatorics (e.g., RSK correspondence) and the techniques in their asymptotic analysis generally involve Laplace's method, Fredholm determinants or Riemann-Hilbert problem asymptotics.

This article will focus on the phenomena associated with the KPZ universality class [3] and highlight how certain integrable examples expand the scope of and refine the notion of universality. We start by providing a brief introduction to the Gaussian universality class and the integrable probabilistic example of random coin flipping, as well as the random deposition model. A small perturbation to the random deposition model leads us to the ballistic deposition model and the KPZ universality class. The ballistic deposition model fails to be integrable; thus, to gain an understanding of its long-time behaviour and that of the entire KPZ class, we turn to the corner growth model. The rest of the article focuses on various sides of this rich model: its role as a random growth process, its relation to the KPZ stochastic PDE, its interpretation in terms of interacting particle systems and its relation to optimisation problems involving paths in random environments. Along the way, we include some other generalisations of this process whose integrability springs from the same sources. We close the article by reflecting upon some open problems.

A survey of the KPZ universality class and all of the associated phenomena and methods developed or utilised in its study is far too vast to be provided here. This article presents only one of many stories and perspectives regarding this rich area of study. To even provide a representative cross-section of references is beyond this scope. Additionally, even though we will discuss integrable examples, we will not describe the algebraic structures and methods of asymptotic analysis behind them (despite their obvious importance and interest). Some recent references that review some of these structures include [2, 4, 8] and references therein. On the more physics-oriented side, the collection of reviews and books [1, 3, 5, 6, 7, 8, 9, 10] provides some idea of the scope of the study of the KPZ universality class and the diverse areas upon which it touches.

We start now by providing an overview of the general notion of universality in the context of the simplest and historically first example – fair coin flipping and the Gaussian universality class.

2 Gaussian universality class

Flip a fair coin N times. Each string of outcomes (e.g., head, tail, tail, tail, head) has an equal probability 2^{-N}. Call H the (random) number of heads and let $\mathbb{P}$ denote the probability distribution for this sequence of coin flips. Counting shows that

$$\mathbb{P}(H = n) = 2^{-N}\binom{N}{n}.$$

Since each flip is independent, the expected number of heads is $N/2$. Bernoulli (1713) proved that H/N converges to $1/2$ as N goes to infinity. This was the first example of a *law of large numbers*. Of course, this does not mean that if you flip the coin 1000 times, you will see exactly 500 heads. Indeed, in N coin flips one expects the number of heads to vary randomly around the value $N/2$ in the scale $\sqrt{N}$. Moreover, for all $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(H < \tfrac{1}{2}N + \tfrac{1}{2}\sqrt{N}\,x\Big) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy.$$

De Moivre (1738), Gauss (1809), Adrain (1809), and Laplace (1812) all participated in the proof of this result. The limiting distribution is known as the Gaussian (or sometimes normal or bell curve) distribution.

A proof of this follows from asymptotics of $n!$, as derived by de Moivre (1721) and named after Stirling (1729). Write

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n dt = n^{n+1} \int_0^\infty e^{nf(z)} dz,$$

where $f(z) = \log z - z$ and the last equality is from the change of variables $t = nz$. The integral is dominated, as n grows, by the maximal value of $f(z)$ on the interval $[0, \infty)$. This occurs at $z = 1$; thus, expanding $f(z) \approx -1 - \frac{(z-1)^2}{2}$ and plugging this into the integral yields the final expansion

$$n! \approx n^{n+1} e^{-n} \sqrt{2\pi/n}.$$

This general route of writing exact formulas for probabilities in terms of integrals and then performing asymptotics is quite common to the analysis of integrable models in the KPZ universality class – though those formulas and analyses are considerably more involved.

The universality of the Gaussian distribution was not broadly demonstrated until work of Chebyshev, Markov and Lyapunov around 1900. The *central limit theorem* (CLT) showed that the exact nature of coin flipping is immaterial – any sum of independent, identically distributed (iid) random variables with finite mean and variance will demonstrate the same limiting behaviour.

Theorem 2.1. Let $X_1, X_2, \ldots$ be iid random variables of finite mean m and variance v. Then, for all $x \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\Big(\sum_{i=1}^{N} X_i < mN + v\sqrt{N}x \Big) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

Proofs of this result use different tools than the exact analysis of coin flipping and much of probability theory deals with the study of Gaussian processes that arise through various generalisations of the CLT. The Gaussian distribution is ubiquitous and, as it is the basis for much of classical statistics and thermodynamics, it has had immense societal impact.

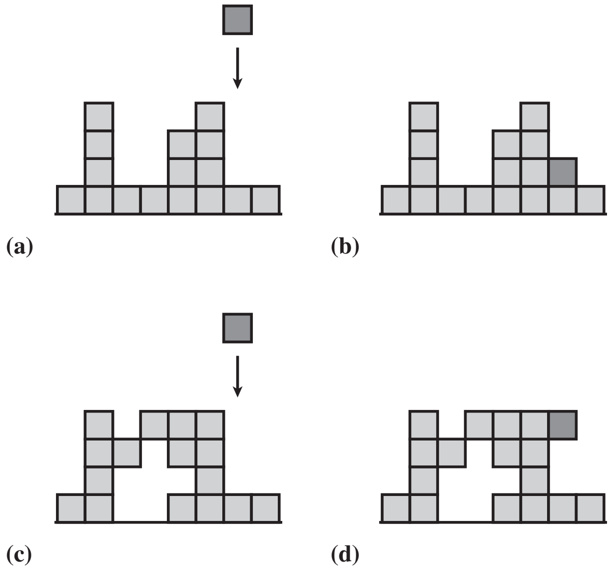


Figure 1. (A) and (B) illustrate the *random deposition model* and (C) and (D) illustrate the *ballistic deposition model*. In both cases, blocks fall from above each site with independent, exponentially distributed waiting times. In the first model, they land at the top of each column whereas in the second model they stick to the first edge to which they become incident.

3 Random versus ballistic deposition

The *random deposition model* is one of the simplest (and least realistic) models for a randomly growing one-dimensional interface. Unit blocks fall independently and in parallel from the sky above each site of $\mathbb{Z}$ according to exponentially distributed waiting times (see Figure 1). Recall that a random variable X has an exponential distribution of rate $\lambda > 0$ (or mean $1/\lambda$) if $\mathbb{P}(X > x) = e^{-\lambda x}$. Such random variables are characterised by the memoryless property – conditioned on the event that $X > x$, $X - x$ still has the exponential distribution of the same rate. Consequently, the random deposition model is *Markov* – its future evolution only depends on the present state (and not on its history).

The random deposition model is quite simple to analyse since each column grows independently. Let $h(t, x)$ record the height above site x at time t and assume $h(0, x) \equiv 0$. Define random waiting times $w_{x,i}$ to be the time for the i-th block in column x to fall. For any n, the event $h(t, x) < n$ is equivalent to $\sum_{i=1}^{n} w_{x,i} > t$. Since the $w_{x,i}$ are iid, the law of large numbers and central limit theory apply here. Assuming $\lambda = 1$,

$$\lim_{t\to\infty} \frac{h(t,x)}{t} = 1, \quad \text{and} \quad \lim_{t\to\infty} \frac{h(t,x) - t}{t^{1/2}} \Rightarrow N(x)$$

jointly over $x \in \mathbb{Z}$, where $\{N(x)\}_{x\in\mathbb{Z}}$ is a collection of iid standard Gaussian random variables. The top of Figure 2 shows a simulation of the random deposition model. The linear growth speed and lack of spatial correlation are quite evident. The fluctuations of this model are said to be in the Gaussian universality class since they grow like $t^{1/2}$, with Gaussian limit law and trivial transversal correlation length scale t^0. In general, fluctuation and transversal correlation exponents, as well as limiting distributions, constitute the description of a universality class and all models that match these limiting behaviours are said to lie in the same universality class.

While the Gaussian behaviour of this model is resilient against changes in the distribution of the $w_{x,i}$ (owing to the CLT), generic changes in the nature of the growth rules shatter the Gaussian behaviour. The *ballistic deposition (or sticky block) model* was introduced by Vold (1959) and, as one expects in real growing interfaces, displays spatial correlation. As before, blocks fall according to iid exponential waiting times; however, a block will now stick to the first edge against which it becomes incident. This mechanism is illustrated in Figure 1. This creates overhangs and we define the height function $h(t, x)$ as the maximal height above x that is occupied by a box. How does this microscopic change manifest itself over time?

It turns out that sticky blocks radically change the limiting behaviour of this growth process. The bottom of Figure 2 records one simulation of the process. Seppäläinen (1999) gave a proof that there is still an overall linear growth rate. Moreover, by considering a lower bound by a width two system, one can see that this velocity exceeds that of the random deposition model. The exact value of this rate, however, remains unknown.

The simulation in Figure 2 (as well as the longer time results displayed in Figure 3) also shows that the scale of fluctuations of $h(t, x)$ is smaller than in random deposition and that the height function remains correlated transversally over a long distance. There are exact conjectures for these fluctuations. They are supposed to grow like $t^{1/3}$ and demonstrate a non-trivial correlation structure in a transversal scale of $t^{2/3}$. Additionally, precise predictions exist for the limiting distributions. Up to certain (presently undetermined) constants c_1, c_2, the sequence of scaled heights $c_2 t^{-1/3}(h(t, 0) - c_1 t)$ should converge to the so-called *Gaussian Orthogonal Ensemble (GOE) Tracy-Widom* distributed random variable. The Tracy-Widom distributions can be thought of as modern-day bell curves and their names GOE or GUE (for *Gaussian*

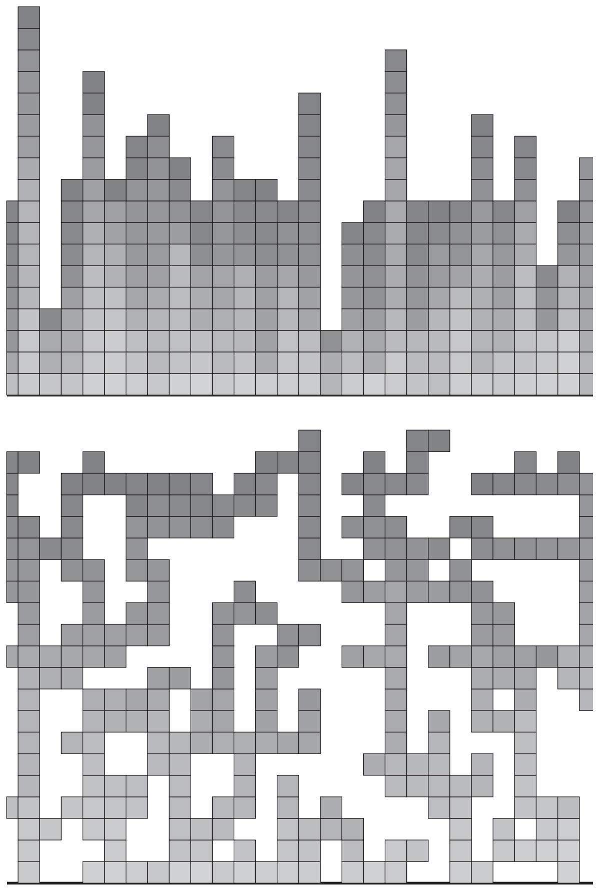


Figure 2. Simulation of random (top) versus ballistic (bottom) deposition models driven by the same process of falling blocks. The ballistic model grows much faster and has a smoother, more spatially correlated top interface.

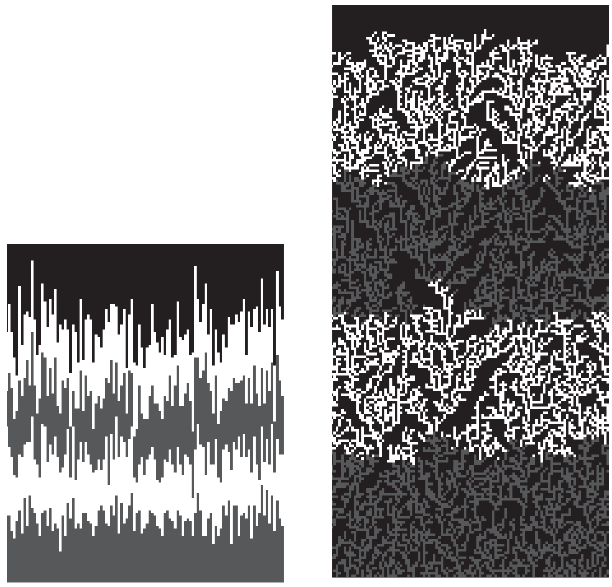


Figure 3. Simulation of random (left) versus ballistic (right) deposition models driven by the same process of falling blocks and run for a long time. The white and grey colours represent different epochs of time in the simulation. The sizes of the boxes in both figures are the same.

Unitary Ensemble) come from the random matrix ensembles in which these distributions were first observed by Tracy-Widom (1993, 1994).

Ballistic deposition does not seem to be an integrable probabilistic system so where do these precise conjectures come from? The exact predictions come from the analysis of a few similar growth processes that just happen to be integrable! Ballistic deposition shares certain features with these models that are believed to be key for membership in the KPZ class:

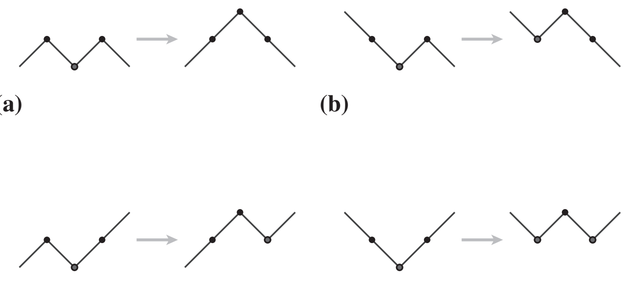
- Locality: height function change depends only on neighbouring heights.
- Smoothing: large valleys are quickly filled in.
- Non-linear slope dependence: vertical effective growth rate depends non-linearly on local slope.
- Space-time independent noise: growth is driven by noise, which quickly decorrelates in space and time and does not display heavy tails.

It should be made clear that a proof of the KPZ class behaviour for the ballistic deposition model is far beyond what can be done mathematically (though simulations strongly suggest that the above conjecture is true).

4 Corner growth model

We come to the first example of an integrable probabilistic system in the KPZ universality class – the *corner growth model*. The randomly growing interface is modelled by a height function $h(t, x)$ that is continuous, piecewise linear and composed of $\sqrt{2}$-length line increments of slope $+1$ or -1, changing value at integer x. The height function evolves according to the Markovian dynamics that each local minimum of h (looking like $\vee$) turns into a local maximum (looking like $\wedge$) according to an exponentially distributed waiting time. This happens independently for each minimum. This change in height function can also be thought of as adding boxes (rotated by 45°). See Figures 4 and 5 for further illustration of this model.

Wedge initial data means that $h(0, x) = |x|$ while *flat* initial data (as considered for ballistic deposition) means that $h(0, x)$ is given by a periodic saw-tooth function that goes between heights 0 and 1. We will focus on wedge initial data. Rost (1980) proved a law of large numbers for the growing interface when time, space and the height function are scaled by the same large parameter L.



(a) (b)

(c) (d)

Figure 4. Various possible ways that a local minimum can grow into a local maximum.

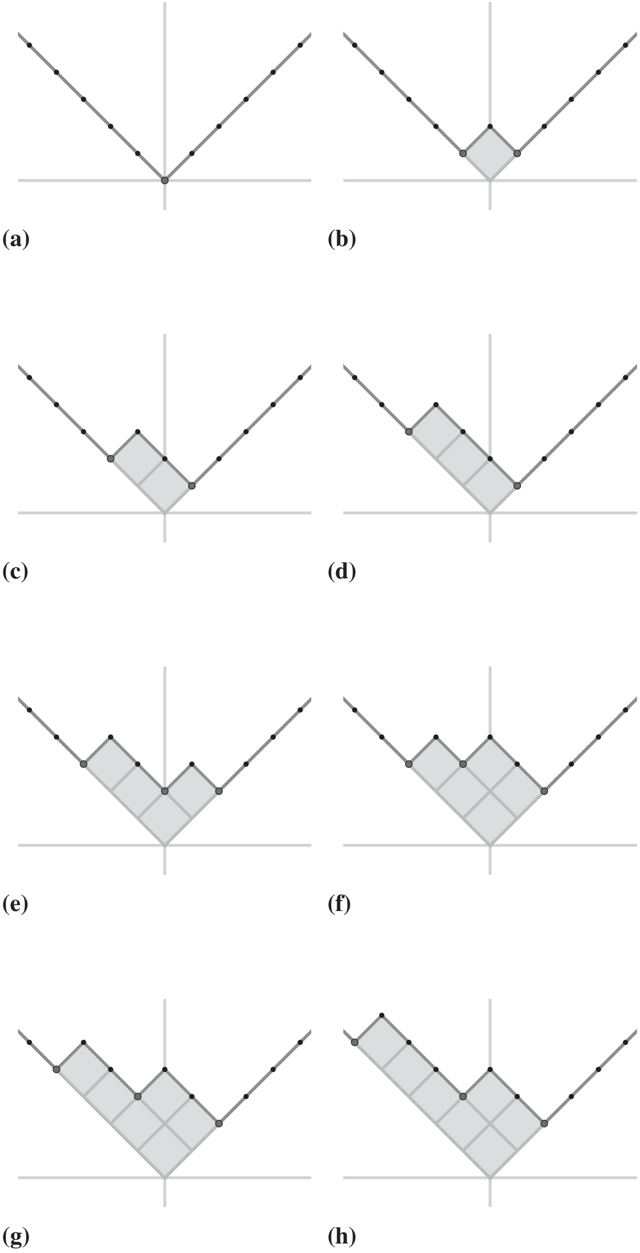


Figure 5. The corner growth model starts with an empty corner, as in (A). There is only one local minimum (the large dot) and after an exponentially distributed waiting time, this turns into a local maximum by filling in the site above it with a block, as in (B). In (B), there are now two possible locations for growth (the two dots). Each one has an exponentially distributed waiting time. (C) corresponds to the case when the left local minimum grows before the right one. By the memoryless property of exponential random variables, once in state (C), we can think of choosing new exponentially distributed waiting times for the possible growth destinations. Continuing in a similar manner, we arrive at the evolution in (D) through (H).

Theorem 4.1. For wedge initial data,

$$\lim_{L\to\infty} \frac{h(Lt, Lx)}{L} = \mathfrak{h}(t,x) := \begin{cases} t\,\frac{1-(x/t)^2}{2} & |x| < t, \\ |x| & |x| \geq t. \end{cases}$$

Figure 6 displays the result of a computer simulation wherein the limiting parabolic shape is evident. The function $\mathfrak{h}$ is the unique viscosity solution to the Hamilton-Jacobi equation

$$\frac{\partial}{\partial t}\mathfrak{h}(t,x) = \frac{1}{2}\Big(1 - \big(\frac{\partial}{\partial x}\mathfrak{h}(t,x)\big)^2\Big).$$

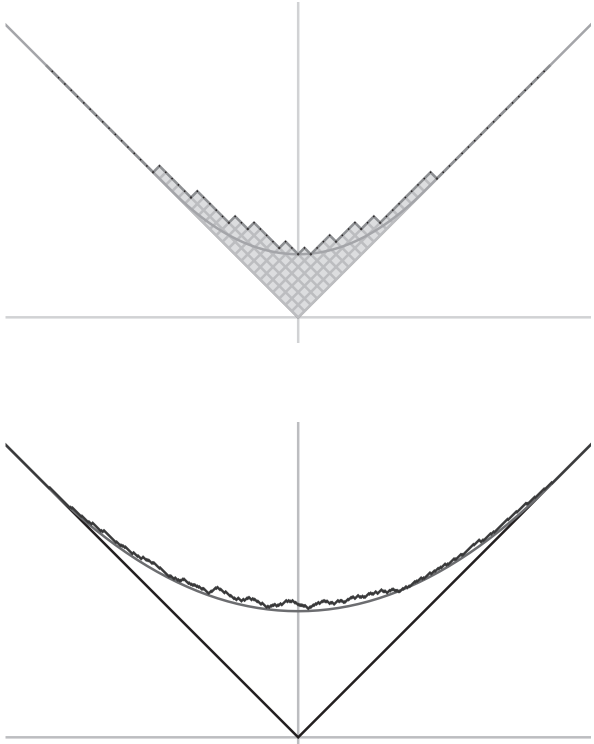


Figure 6. Simulation of the corner growth model. The top shows the model after a medium amount of time and the bottom shows it after a longer amount of time. The rough interface is the simulation while the smooth curve is the limiting parabolic shape. The simulation curve has vertical fluctuations of order $t^{1/3}$ and decorrelates spatially on distances of order $t^{2/3}$.

This equation actually governs the evolution of the law of large numbers from arbitrary initial data.

The fluctuations of this model around the law of large numbers are what is believed to be universal. Figure 6 shows that the interface fluctuates around its limiting shape on a fairly small scale, with transversal correlation on a larger scale. For $\epsilon > 0$, define the scaled and centred height function

$$h_\epsilon(t,x) := \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x) - \frac{\epsilon^{-1}t}{2},$$

where the *dynamic scaling exponent* $z = 3/2$ and the *fluctuation exponent* $b = 1/2$. These exponents are easily remembered since they correspond with scaling time : space : fluctuations like 3 : 2 : 1. These are the characteristic exponents for the KPZ universality class. Johansson (1999) proved that for fixed t, as $\epsilon \to 0$, the random variable $h_\epsilon(t,0)$ converges to a GUE Tracy-Widom distributed random variable (see Figure 7). Results for the related model of the longest increasing subsequence in a random permutation were provided around the same time by Baik-Deift-Johansson (1999). For that related model, two years later, Prähofer-Spohn (2001) computed the analogue to the joint distribution of $h_\epsilon(t,x)$ for fixed t and varying x.

The entire scaled growth process $h_\epsilon(\cdot,\cdot)$ should have a limit as $\epsilon \to 0$ that would necessarily be a fixed point under 3 : 2 : 1 scaling. The existence of this limit (often called the *KPZ fixed point*) remains conjectural. Still, much is known about the properties this limit should enjoy. It should be a stochastic process whose evolution depends on the limit of

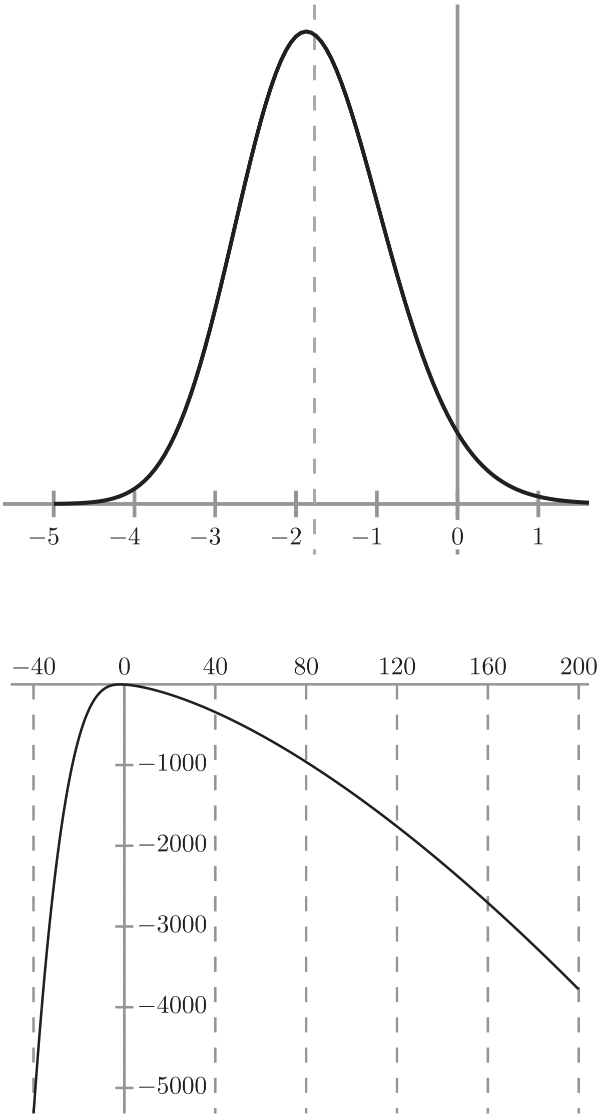


Figure 7. The density (top) and log of the density (bottom) of the GUE Tracy-Widom distribution. Though the density appears to look like a bell-curve (or Gaussian), this comparison is misleading. The mean and variance of the distribution are approximately -1.77 and 0.81. The tails of the density (as shown in terms of the log of the density in the bottom plot) decay like $e^{-c_-|x|^3}$ for $x \ll 0$ and like $e^{-c_+x^{3/2}}$ for $x \gg 0$, for certain positive constants c_- and c_+. The Gaussian density decays like e^{-cx^2} in both tails, with the constant c related to the variance.

the initial data under the same scaling. The one-point distribution for general initial data, the multi-point and multi-time distribution for wedge initial data and various aspects of its continuity are all understood. Besides the existence of this limit, what is missing is a useful characterisation of the KPZ fixed point. Since the KPZ fixed point is believed to be the universal scaling limit of all models in the KPZ universality class and since corner growth enjoys the same key properties as ballistic deposition, one is also led to the conjecture that ballistic deposition scales to the same fixed point and hence enjoys the same scalings and limiting distributions. The reason why the GOE Tracy-Widom distribution came up in our earlier discussion is that we were dealing with flat rather than wedge initial data.

One test of the universality belief is to introduce partial asymmetry into the corner growth model. Now we change local minima into local maxima at rate p, and turn local maxima into local minima at rate q (all waiting times are independent and exponentially distributed, and $p + q = 1$). See 8 for an illustration of this partially asymmetric corner growth model. Tracy-Widom (2007-2009) showed that so long as $p > q$, the same law of large numbers and fluctuation limit theorem holds for the partially asymmetric model, provided that t is replaced by $t/(p - q)$. Since $p - q$ represents the growth drift, one simply has to speed up to compensate for this drift being smaller.

Clearly, for $p \leq q$, something different must occur than for $p > q$. For $p = q$, the law of large numbers and fluctuations change nature. The scaling of time : space : fluctuations becomes 4 : 2 : 1 and the limiting process under these scalings becomes the stochastic heat equation with additive white noise. This is the *Edwards-Wilkinson* (EW) universality class, which is described by the stochastic heat equation with additive noise. For $p < q$, the process approaches a stationary distribution where the probability of having k boxes added to the empty wedge is proportional to $(p/q)^k$.

So, we have observed that for any positive asymmetry the growth model lies in the KPZ universality class while for zero asymmetry it lies in the EW universality class. It is natural to wonder whether by critically scaling parameters (i.e. $p - q \to 0$), one might encounter a crossover regime between these two universality classes. Indeed, this is the case and the crossover is achieved by the KPZ equation that we now discuss.

5 The KPZ equation

The *KPZ equation* is written as

$$\frac{\partial h}{\partial t}(t,x) = \nu \frac{\partial^2 h}{\partial x^2}(t,x) + \tfrac{1}{2}\lambda\Big(\frac{\partial h}{\partial x}(t,x)\Big)^2 + \sqrt{D}\xi(t,x),$$

where $\xi(t, x)$ is Gaussian space-time white noise, $\lambda, \nu \in \mathbb{R}$, $D > 0$ and $h(t, x)$ is a continuous function of time $t \in \mathbb{R}_+$ and space $x \in \mathbb{R}$, taking values in $\mathbb{R}$. Due to the white noise, one expects $x \mapsto h(t, x)$ to be only as regular as in Brownian motion. Hence, the non-linearity does not a priori make any sense (the derivative of Brownian motion has negative Hölder regularity). Bertini-Cancrini (1995) provided the physically relevant notion of solution (called the *Hopf–Cole solution*) and showed how it arises from regularising the noise, solving the (now well-posed) equation and then removing the noise and subtracting a divergence.

The equation contains the four key features mentioned earlier – the growth is local, depending on the Laplacian (smoothing), the square of the gradient (non-linear slope dependent growth) and white noise (space-time uncorrelated noise). Kardar, Parisi and Zhang introduced their eponymous equation and 3:2:1 scaling prediction in 1986 in an attempt to understand the scaling behaviours of random interface growth.

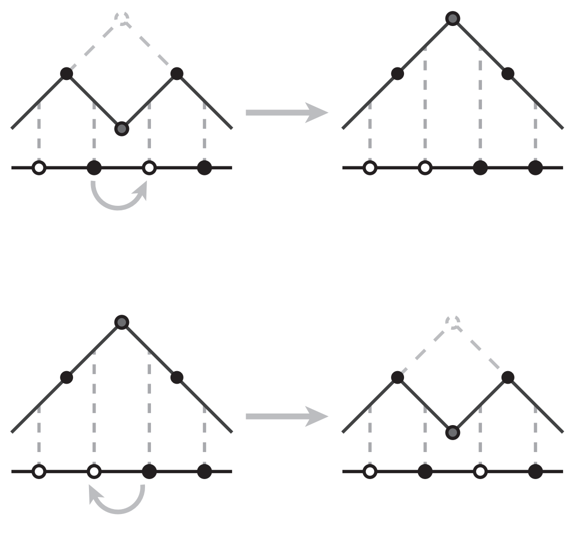
How might one see the 3:2:1 scaling from the KPZ equation? Define $h_\epsilon(t, x) = \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x)$; then, h_ϵ satisfies the KPZ equation with scaled coefficients $\epsilon^{2-z}\nu$, $\epsilon^{2-z-b}\frac{1}{2}\lambda$ and $\epsilon^{b-\frac{z}{2}+\frac{1}{2}}\sqrt{D}$. It turns out that two-sided Brownian motion is stationary for the KPZ equation; hence, any non-trivial scaling must respect the Brownian scaling of the initial data and thus have $b = 1/2$. Plugging this in, the only way to have no coefficient blow up to infinity and not to have every term shrink to zero (as $\epsilon \to 0$) is to choose $z = 3/2$. This suggests the plausibility of the 3:2:1 scaling. While this heuristic gives the right scaling, it does not provide for the scaling limit. The limit as $\epsilon \to 0$ of the equation (the inviscid Burgers equation where only the non-linearity survives) certainly does not govern the limit of the solutions. It remains something of a mystery as to exactly how to describe this limiting KPZ fixed point. The above heuristic says nothing of the limiting distribution of the solution to the KPZ equation and there does not currently exist a simple way to see what this should be.

It took just under 25 years until Amir-Corwin-Quastel (2010) rigorously proved that the KPZ equation is in the KPZ universality class. That work also computed an exact formula for the probability distribution of the solution to the KPZ equation – marking the first instance of a non-linear stochastic PDE for which this was accomplished. Tracy-Widom's work on the partially asymmetric corner growth model and the work of Bertini-Giacomin (1997) relating that model to the KPZ equation were the two main inputs in this development. See [3] for further details regarding this, as well as the simultaneous exact but non-rigorous steepest descent work of Sasamoto-Spohn (2010), and non-rigorous replica approach work of Calabrese-Le Doussal-Rosso (2010) and Dotsenko (2010).

The proof that the KPZ equation is in the KPZ universality class was part of an ongoing flurry of activity surrounding the KPZ universality class from a number of directions such as integrable probability [4], experimental physics [10] and stochastic PDEs. For instance, Bertini-Cancrini's Hopf–Cole solution relies upon a trick (the Hopf–Cole transform) that linearises the KPZ equation. Hairer (2011), who had been developing methods to make sense of classically ill-posed stochastic PDEs, focused on the KPZ equation and developed a direct notion of solution that agreed with the Hopf–Cole one but did not require use of the Hopf–Cole transform trick. Still, this does not say anything about the distribution of solutions or their long-time scaling behaviours. Hairer's KPZ work set the stage for his development of regularity structures in 2013 – an approach to construction solutions of certain types of ill-posed stochastic PDEs – work for which he was awarded a Fields Medal.

6 Interacting particle systems

There is a direct mapping (see Figure 8) between the partially asymmetric corner growth model and the *partially asymmetric simple exclusion process* (generally abbreviated ASEP). One can associate to every -1 slope line increment a particle on the site of $\mathbb{Z}$ above which the increment sits, and to every $+1$ slope line increment an empty site. The height function then maps onto a configuration of particles and holes on $\mathbb{Z}$, with at most one particle per site. When a minimum of the height function becomes a maximum, it corresponds to a particle jumping right by one into an empty site and, likewise, when a maximum becomes a minimum, a particle jumps left by one into an empty site. Wedge initial data for corner growth corresponds to having all sites to the left of the origin initially occupied and all to the right empty – this is often called *step initial data* due to the step function in terms of particle density. ASEP was introduced in biology literature in 1968 by MacDonald-Gibbs-Pipkin as a model for RNA's movement during transcription. Soon after, it was independently introduced within the probability literature in 1970 by Spitzer.



(a)

(b)

Figure 8. Mapping the partially asymmetric corner growth model to the partially asymmetric simple exclusion process. In (A), the local minimum grows into a local maximum. In terms of the particle process beneath it, the minimum corresponds to a particle followed by a hole and the growth corresponds to said particle jumping into the hole to its right. In (B), the opposite is shown as the local maximum shrinks into a local minimum. Correspondingly, there is a hole followed by a particle and the shrinking results in the particle moving into the hole to its left.

The earlier quoted results regarding corner growth immediately imply that the number of particles to cross the origin after a long time t demonstrates KPZ class fluctuation behaviour. KPZ universality would have that generic changes to this model should not change the KPZ class fluctuations. Unfortunately, such generic changes destroy the model's integrable structure. There are a few integrable generalisations discovered over the past five years that demonstrate some of the resilience of the KPZ universality class against perturbations.

TASEP (the totally asymmetric version of ASEP) is a very basic model for traffic on a one-lane road in which cars (particles) move forward after exponential rate one waiting times, provided the site is unoccupied. A more realistic model would account for the fact that cars slow down as they approach the one in front. The model of q-TASEP does just that (Figure 9). Particles jump right according to independent exponential waiting times of rate $1 - q^{\text{gap}}$, where gap is the number of empty spaces to the next particle to the right. Here $q \in [0, 1)$ is a different parameter than in the ASEP, though when q goes to zero, these dynamics become those of TASEP.

Another feature one might include in a more realistic traf-

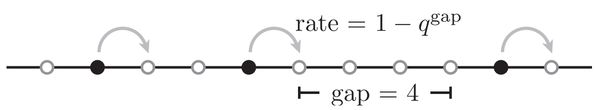


Figure 9. The q-TASEP, whereby each particle jumps one to the right after an exponentially distributed waiting time with rate given by $1-q^{\text{gap}}$.

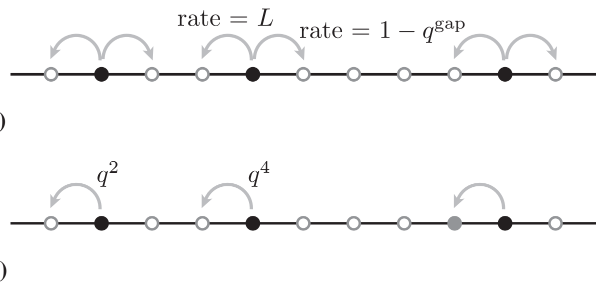


Figure 10. The q-pushASEP. As shown in (A), particles jump right according to the q-TASEP rates and left according to independent exponentially distributed waiting times of rate L. When a left jump occurs, it may trigger a cascade of left jumps. As shown in (B), the right-most particle has just jumped left by one. The next particle (to its left) instantaneously jumps left by one with probability given by q^{gap}, where gap is the number of empty sites between the two particles before the left jumps occurred (in this case gap = 4). If that next left jump is realised, the cascade continues to the next-left particle according to the same rule, otherwise it stops and no other particles jump left in that instant of time.

fic model is the cascade effect of braking. The q-pushASEP includes this (Figure 10). Particles still jump right according to q-TASEP rules; however, particles may now also jump left after exponential rate L waiting times. When such a jump occurs, it prompts the next particle to the left to likewise jump left, with a probability given by q^{gap}, where gap is the number of empty spaces between the original particle and its left neighbour. If that jump occurs, it may likewise prompt the next left particle to jump, and so on. Of course, braking is not the same as jumping backwards; however, if one goes into a moving frame, this left jump is like a deceleration. It turns out that both of these models are solvable via the methods of Macdonald processes as well as stochastic quantum integrable systems and it has thus been proved that, just as for ASEP, they demonstrate KPZ class fluctuation behaviour (see the review [4]).

7 Paths in a random environment

There is yet another class of probabilistic systems related to the corner growth model. Consider the totally asymmetric version of this model, starting from wedge initial data. An alternative way to track the evolving height function is to record the time when a given box is grown. Using the labelling shown in Figure 11, let us call $L(x, y)$ this time, for x, y positive integers. A box (x, y) may grow once its parent blocks $(x-1, y)$ and $(x, y-1)$ have both grown – though even then it must wait for an independent exponential waiting time that we denote by $w_{x,y}$. Thus, $L(x, y)$ satisfies the recursion

$$L(x, y) = \max\left(L(x-1, y), L(x, y-1)\right) + w_{x,y},$$

subject to boundary conditions $L(x, 0) \equiv 0$ and $L(0, y) \equiv 0$. Iterating yields

$$L(x, y) = \max_{\pi} \sum_{(i,j)\in\pi} w_{i,j},$$

where the maximum is over all up-right and up-left lattice paths between box $(1, 1)$ and (x, y). This model is called *last passage percolation* with exponential weights. Following from the earlier corner growth model results, one readily sees that for any positive real (x, y), for large t, $L(\lfloor xt\rfloor, \lfloor yt\rfloor)$

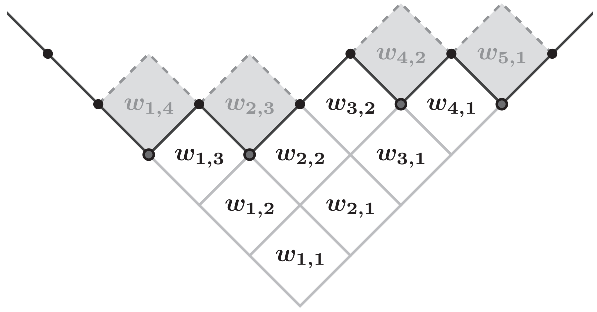


Figure 11. The relation between the corner growth model and last passage percolation with exponential weights. $w_{i,j}$ are the waiting times between when a box can grow and when it does grow. $L(x, y)$ is the time when box (x, y) grows.

demonstrates KPZ class fluctuations. A very compelling and entirely open problem is to show that this type of behaviour persists when the distribution of $w_{i,j}$ is no longer exponential. The only other solvable case is that of geometric weights. A certain limit of the geometric weights leads to maximising the number of Poisson points along directed paths. Fixing the total number of points, this becomes equivalent to finding the longest increasing subsequence of a random permutation. The KPZ class behaviour for this version of last passage percolation was shown by Baik-Deift-Johansson (1999).

There is another related integrable model that can be thought of as describing the optimal way to cross a large grid with stop lights at intersections. Consider the first quadrant of $\mathbb{Z}^2$ and to every vertex (x, y) assign waiting times to the edges leaving the vertex rightwards and upwards. With a probability of $1/2$, the rightward edge has waiting time zero, while the upward edge has waiting time given by exponential rate 1 random variables; otherwise, reverse the situation. The edge waiting time represents the time needed to cross an intersection in the given direction (the walking time between lights has been subtracted). The minimal passage time from $(1, 1)$ to (x, y) is given by

$$P(x, y) = \min_{\pi} \sum_{e\in\pi} w_e,$$

where π goes right or up in each step and ends on the vertical line above (x, y) and w_e is the waiting time for edge $e \in \pi$. From the origin, there will always be a path of zero waiting time, whose spatial distribution is that of the graph of a simple symmetric random walk. Just following this path, one can get very close to the diagonal $x = y$ without waiting. On the other hand, for $x \neq y$, getting to $(\lfloor xt\rfloor, \lfloor yt\rfloor)$ for large t requires some amount of waiting. Barraquand-Corwin (2015) demonstrated that as long as $x \neq y$, $P(\lfloor xt\rfloor, \lfloor yt\rfloor)$ demonstrates KPZ class fluctuations. This should be true when π is restricted to hit exactly (x, y), though that result has not yet been proved. Achieving this optimal passage time requires some level of omnipotence as you must be able to look forward before choosing your route. As such, it could be considered as a benchmark against which to test various routing algorithms.

In addition to maximising or minimising path problems, the KPZ universality class describes fluctuations of 'positive temperature' versions of these models in which energetic or probabilistic favouritism is assigned to paths based on the sum of space-time random weights along its graph. One such sys-

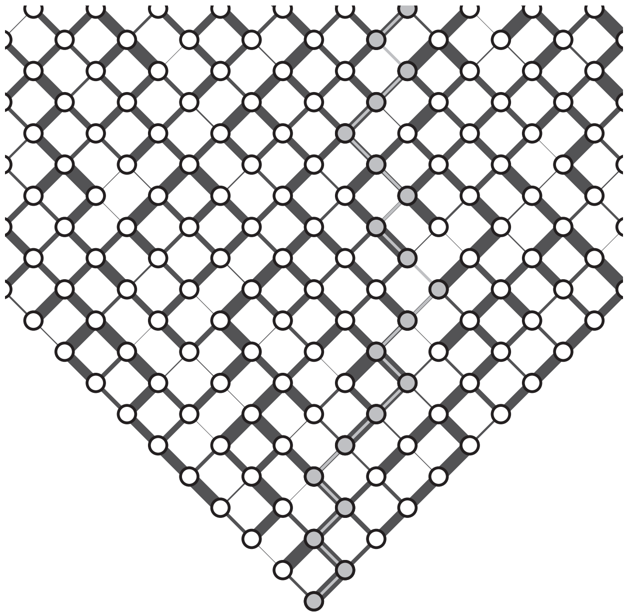


Figure 12. The random walk in a space-time random environment. For each pair of up-left and up-right pointing edges leaving a vertex (y, s), the width of the edges is given by $u_{y,s}$ and $1 - u_{y,s}$, where $u_{y,s}$ are independent uniform random variables on the interval $[0, 1]$. A walker (the grey highlighted path) then performs a random walk in this environment, jumping up-left or up-right from a vertex with probability equal to the width of the edges.

tem is called *directed polymers in random environment* and is the detropicalisation of LPP where one replaces the operations of $(\max, +)$ by $(+, \times)$ in the definition of $L(x, y)$. Then, the resulting (random) quantity is called the *partition function* for the model and its logarithm (the *free energy*) is conjectured for very general distributions on $w_{i,j}$ to show KPZ class fluctuations. There is one known integrable example of weights for which this has been proved – the inverse-gamma distribution, introduced by Seppäläinen (2009) and proved in the work of Corwin-O'Connell-Seppäläinen-Zygouras (2011) and Borodin-Corwin-Remenik (2012).

The stop light system discussed above also has a positive temperature lifting of which we will describe a special case (see Figure 12 for an illustration). For each space-time vertex (y, s), choose a random variable $u_{y,s}$ distributed uniformly on the interval $[0, 1]$. Consider a random walk $X(t)$ that starts at $(0, 0)$. If the random walk is in position y at time s then it jumps to position $y - 1$ at time $s + 1$ with probability $u_{y,s}$ and to position $y + 1$ with probability $1 - u_{y,s}$. With respect to the same environment of u's, consider N such random walks. The fact that the environment is fixed causes them to follow certain high probability channels. This type of system is called a *random walk in a space-time random environment* and the behaviour of a single random walker is quite well understood. Let us, instead, consider the maximum of N walkers in the same environment $M(t, N) = \max_{i=1}^{N} X^{(i)}(t)$. For a given environment, it is expected that $M(t, N)$ will localise near a given random environment dependent value. However, as the random environment varies, this localisation value does as well in such a way that for $r \in (0, 1)$ and large t, $M(t, e^{rt})$ displays KPZ class fluctuations.

8 Big problems

It took almost 200 years from the discovery of the Gaussian distribution to the first proof of its universality (the central limit theorem). So far, KPZ universality has withstood proof for almost three decades and shows no signs of yielding.

Besides universality, there remain a number of other big problems for which little to no progress has been made. All of the systems and results discussed herein have been 1 + 1 dimensional, meaning that there is one time dimension and one space dimension. In the context of random growth, it makes perfect sense (and is quite important) to study surface growth, i.e. 1 + 2 dimensional. In the isotropic case (where the underlying growth mechanism is roughly symmetric with respect to the two spatial dimensions), there are effectively no mathematical results though numerical simulations suggest that the $1/3$ exponent in the $t^{1/3}$ scaling for corner growth should be replaced by an exponent of roughly 0.24. In the anisotropic case there have been a few integrable examples discovered that suggest very different (logarithmic scale) fluctuations such as observed by Borodin-Ferrari (2008).

Finally, despite the tremendous success in employing methods of integrable probability to expand and refine the KPZ universality class, there still seems to be quite a lot of room to grow and new integrable structures to employ. Within the physics literature, there are a number of exciting new directions in which the KPZ class has been pushed, including out-of-equilibrium transform and energy transport with multiple conservation laws, front propagation equations, quantum localisation with directed paths and biostatistics. What is equally important is to understand what type of perturbations break out of the KPZ class.

Given all of the rich mathematical predictions, one might hope that experiments would have revealed KPZ class behaviour in nature. This is quite a challenge since determining scaling exponents and limiting fluctuations require immense numbers of repetitions of experiments. However, there have been a few startling experimental confirmations of these behaviours in the context of liquid crystal growth, bacterial colony growth, coffee stains and fire propagation (see [10] and references therein). Truly, the study of the KPZ universality class demonstrates the unity of mathematics and physics at its best.

Acknowledgements

The author appreciates comments on a draft of this article by A. Borodin, P. Ferrari and H. Spohn. This text is loosely based on his "Mathematic Park" lecture entitled *Universal phenomena in random systems* and delivered at the Institut Henri Poincaré in May 2015. The author is partially supported by the NSF grant DMS-1208998, by a Clay Research Fellowship, by the Poincaré Chair and by a Packard Fellowship for Science and Engineering.

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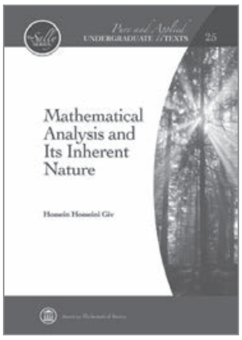

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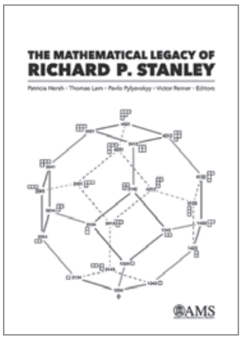

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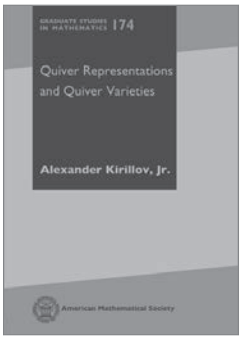

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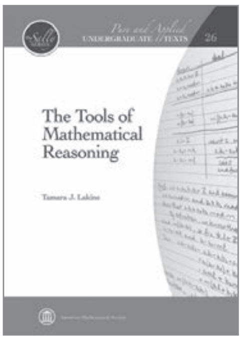

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Interview with Abel Laureate Sir Andrew Wiles

Martin Raussen (Aalborg University, Denmark) and Christian Skau (Norwegian University of Science and Technology, Trondheim, Norway)

The interview took place in Oslo on 23 May 2016.

Professor Wiles, please accept our congratulations for having been selected as the Abel Prize Laureate for 2016. To be honest, the two of us had expected this interview to take place already several years ago!

You are famed not only among mathematicians, but also among the public at large for, and now we cite the Abel Committee: "the stunning proof of Fermat's Last Theorem, by way of the Modularity Conjecture for elliptic curves, opening a new era in number theory". This proof goes back to 1994, which means that you had to wait for more than 20 years before it earned you the Abel Prize. Nevertheless, you are the youngest Abel Prize Laureate so far.

After you finished your proof of Fermat's Last Theorem you had to undergo a deluge of interviews, which makes our task difficult. How on earth are we to come up with questions that you have not answered several times before? Well, we will try to do our best.

Sir Andrew Wiles received the Abel Prize from Crown Prince Haakon of Norway. (Photo Audun Braastad)

Fermat's Last Theorem: A historical account

We have to start at the very beginning, with a citation in Latin: "…nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere", which means: "it is impossible to separate any power higher than the second into two like powers". That is in modern mathematical jargon: The equation $x^n + y^n = z^n$ has no solution in natural numbers for n greater that two. And then it continues: "cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet", which means: "I have discovered a truly marvellous proof of this, which this margin is too narrow to contain". This remark was written in the year 1637 by the French lawyer and amateur mathematician Pierre de Fermat [1601–1665] in the margin of his copy of Diophantus' Arithmetica. He certainly did not expect that it would keep mathematicians, professionals and amateurs alike, busy for centuries trying to unearth the proof.

Could you please give us a short account of some of the attempts towards proving Fermat's Last Theorem up until the time you embarked on your successful journey? Furthermore, why was such a simple-minded question so attractive and why were attempts to prove it so productive in the development of number theory?

The first serious attempt to solve it was presumably by Fermat himself. But unfortunately we know nothing about it except for what he explained about his proofs in the specific cases of $n = 3$ and $n = 4$.[1] That is, he showed that you can't have the sum of two cubes be another cube, or the sum of two fourth powers being a fourth power. He did this by a beautiful method, which we call infinite descent. It was a new method of proof, or at least a new way of presenting proofs, in arithmetic. He explained this method to his colleagues in letters and he also wrote about it in his famous margin, which was big enough for some of it at least. After the marginal notes were published by Fermat's son after his father's death, it lay dormant for a while. Then it was picked up by Euler [1707–1783] and others who tried to find this truly marvellous proof. And they failed. It became quite dramatic in the mid-19th century – various people thought they could solve it. There was a discussion concerning this in the French Academy – Lamé [1795–1870] claiming he was just about to prove it – and Cauchy [1789–1857] said he thought he could too, and so on.

In fact it transpired that the German mathematician Kummer [1810–1893] had already written a paper where he explained that the fundamental problem was what is known now as the fundamental theorem of arithmetic. In our normal number system any number can be factorized in essentially one way into prime factors. Take a number like 12; it is 2 times 2 times 3. There is no other way of breaking it up. But in trying to solve the Fermat problem you actually want to use systems of numbers where this uniqueness does not

[1] Strictly speaking Euler was the first to spell out a complete proof in the case $p = 3$.

hold. Every attempt that was made to solve the Fermat problem had stalled because of this failure of unique factorization. Kummer analysed this in incredible detail. He came up with the most beautiful results, and the end product was that he could solve it for many, many cases. For example for $n \leq 100$ he solved it for all primes except for 37, 59 and 67. But he did not finally solve it. His method was based on the idea that Fermat had introduced – the method of infinite descent – but in these new number systems.

The new number systems he was using spawned algebraic number theory as we see it today. One tries to solve equations in these new systems of numbers instead of solving them with ordinary integers and rational numbers. Attempts in the style of Fermat carried on for a while but somewhat petered out in the twentieth century. No one came up with a fundamentally new idea. In the second half of the twentieth century number theory moved on and considered other questions. Fermat's problem was all but forgotten by the professionals.

Then in 1985, Gerhard Frey, a German mathematician, came up with a stunning new idea where he took a hypothetical solution to the Fermat problem and rewrote it so that it made what is called an elliptic curve. And he showed, or suggested, that this elliptic curve had very peculiar properties. He conjectured that you can't really have such an elliptic curve. Building on this a year later an American mathematician, Kenneth Ribet, demonstrated, using this Frey curve, that any solution of Fermat would contradict another well-known conjecture called the Modularity Conjecture. This conjecture had been proposed in a weak form by Taniyama [1927–1958] and refined by Shimura, but the first real evidence for it came from André Weil [1906–1998] who made it possible to check this precise form of the Modularity Conjecture in some detail. And a lot of evidence was amassed showing that this should certainly be true. So at that point mathematicians could see: Yes, Fermat is going to be true. Moreover, there has to be a proof of it.

What happened was that the Modularity Conjecture was a problem that mathematics could not just put to one side and go on for another five hundred years. It was a roadblock right in the middle of modern mathematics. It was a very, very central problem. As for Fermat you could just leave it aside and forget it almost forever. This Modularity Conjecture you could not forget. So at the point when I heard that Ribet had done this I knew that this problem could be solved and I was going to try.

Concerning speculations about Fermat's claimed proof: Do you think he had the same idea as Lamé, assuming, wrongly it turned out, that the cyclotomic integers have unique factorization?

No, I don't think so, though the idea might be in there somewhere. It is very hard to understand. André Weil wrote about this. All the other problems Fermat considered had to do with curves that were of genus zero or genus one. And suddenly he is writing down a curve that has higher genus. How is he going to think about it?

When I was trying this myself as a teenager, I put myself in Fermat's frame of mind because there was hardly anything else I could do. I was capable of understanding his mathematics from the 17th century, but probably not much beyond that. It seemed to me that everything he did came down to something about quadratic forms, and I thought that might be a way of trying to think about it. Of course, I never succeeded, but there is nothing else that suggests Fermat fell into this trap with unique factorization. In fact, from the point of view of quadratic forms he understood that sometimes there was unique factorization and sometimes there was not. So he understood that difference in his own context. I think it is unlikely that that was the mistake.

In the same book by André Weil that you referred to, titled "Number Theory: an approach through history from Hammurapi to Legendre", it is mentioned that Fermat looked at the equation a cube minus a square equal to 2 [$x^3 - y^2 = 2$], and he showed that it has essentially only one solution, namely $x = 3$ and $y = \pm 5$. André Weil speculates that Fermat at the time looked at the ring $\mathbb{Z}[\sqrt{-2}]$, which does have unique factorization.

Yes, he used unique factorization but the way he did it was in terms of quadratic forms. And I think he also looked at quadratic forms corresponding to $\mathbb{Z}[\sqrt{-6}]$ where there is not unique factorization. So I think he understood. It was my impression when I thought about it that he understood the difference.

A mathematical education

You were apparently interested in mathematical puzzles already as a quite young boy. Have you any thoughts about where this interest came from? Were you influenced by anyone in particular?

I just enjoyed mathematics when I was very young. At the age of ten I was looking through library shelves devoted to mathematics. I would pull out books and at one point I pulled out a book of E.T. Bell [1883–1960] titled "The Last Problem", which on its cover describes the Fermat equation, the Wolfskehl Prize and the romantic history of the problem. I was completely captivated by it.

Were there other things that fascinated you in this book by Eric Temple Bell?

It is entirely about that one equation, really. And it is actually quite wordy. So there is less mathematics in some sense than you might think. I think it was more the equation. Then, when I found this equation I looked for other elementary books on number theory and learned about congruences and solved congruences and so on, and looked at other things that Fermat did.

You did this work besides your ordinary school work?

Yes, I don't think my school work was too taxing from that point of view.

Was it already clear for you at that time that you had an extraordinary mathematical talent?

I certainly had a mathematical aptitude and obviously loved to do mathematics, but I don't think I felt that I was unique. In fact, I don't believe I was in the school I attended. There were others who had just as strong a claim to be future mathematicians, and some of them have become mathematicians, too.

Did you plan to study mathematics and to embark on a mathematical career already at that age?
No, I don't think I really understood you could spend your life doing mathematics. I think that only came later. But I certainly wanted to study it as long as I could. I'm sure that as far as my horizon extended it involved mathematics.

You started to study mathematics as a student at Oxford in 1971. Can you tell us a little bit about how that worked out? Were there any particular teachers, any particular areas that were particularly important for you?
Before I went to college, actually in high-school, one of my teachers had a PhD in number theory. He gave me a copy of Hardy and Wright's *An Introduction to the Theory of Numbers*, and I also found a copy of Davenport's *The Higher Arithmetic*. And these two books I found very, very inspiring in terms of number theory.

So you were on track before you started studying?
Yes, I was on track before. In fact, to some extent I felt college was a distraction because I had to do all these other things, applied maths, logic and so on, and I just wanted to do number theory. You were not allowed to do number theory in your first year. And you could not really get down to it before your third year.

But you were not interested in geometry, not as much as in algebra and number theory, anyway?
No, I was primarily interested in algebra and number theory. I was happy to learn these other things, but I really was most excited about number theory. My teachers arranged for me to take extra classes in number theory, but there was not that much on offer.

At one point I decided that I should put all the years of Latin I had done at school to good use and try to read some of Fermat in the original, but I found that was actually too hard. Even if you translated the Latin, the way they wrote in those days wasn't in the algebraic symbols I was used to; so it was quite difficult.

It must have been a relief when you were done and came to Cambridge to start studying number theory for real, with John Coates as your supervisor?
That's right. I had a year, a preliminary year, in which I just studied a range of subjects, and then I could do a special paper. John Coates was not yet at Cambridge, but I think he helped me – maybe over the summer. Anyway, that summer I met him and started working with him right away, and that was just wonderful. The transition from undergraduate work, where you were just reading and studying, to research, that was the real break for me. It was just wonderful.

Elliptic curves

We assume it was John Coates who initiated you to work on elliptic curves, and to Iwasawa theory?
Absolutely. He had some wonderful ideas and was generous to share them with me.

Did you tell John Coates that you were interested in the Fermat problem?
Perhaps I did. I don't remember. It is really true that there hadn't been any new ideas since the 19th century. People were trying to refine the old methods, and, yes, there were refinements. But it didn't look like these refinements and the solution were going to converge. It was just too hard that way.

At the time you started to work with John Coates, you had no idea that these elliptic curves were going to be crucial for the solution of Fermat's Last Theorem?
No, it's a wonderful coincidence. The strange thing is that, in a way, the two things that are most prominent in Fermat that we remember today are his work on elliptic curves and his famous last theorem. For example, this equation you mentioned, $y^2+2=x^3$, is an elliptic curve. And the two strands came together in the proof.

Could you explain what an elliptic curve is and why elliptic curves are of interest in number theory?
For a number theorist the life of elliptic curves started with Fermat as equations of the form y^2 equals a cubic polynomial in x with rational coefficients. Then the problem is to find the rational solutions to such an equation. What Fermat noticed was the following: Sometimes you can start with one or even two rational solutions, and use them to generate infinitely many others. And yet sometimes there are no solutions. This latter situation occurs, for example, in the case $n=3$ of Fermat's Last Theorem, the equation being in fact an elliptic curve in disguise. Sometimes you can show there are no rational solutions. You could have infinitely many and you could have none. This was already apparent to Fermat.

In the early 19th century one studied these equations in complex numbers. Abel [1802–1829] himself came in at this point and studied elliptic functions and related these to elliptic curves, implying that elliptic curves have a group structure. They were very well understood in terms of doubly periodic functions in the early 19th century. But that is what underlies the complex solutions, solutions to the equation in complex numbers.

The solutions to the equation in rational numbers were studied by Poincaré [1854–1912]. What's now known as the Mordell–Weil theorem was proved by Mordell [1888–1972] and then Weil in the 1920s, answering a question of Poincaré. In our setting it says that the K-rational points on an elliptic curve over a number field K, in particular for K equal to the rationals, form a finitely generated abelian group. That is, from Fermat's language you can start with a finite number of solutions and using those generate all the solutions by what he called the chord-and-tangent process.

Birch and Swinnerton–Dyer, Tate–Shafarevich, Selmer…

By now you know the structure, it is a very beautiful algebraic structure, the structure of a group, but that does not actually help you find the solutions. So no one really had any general methods for finding the solutions, until the conjectures of the 1960s, which emerged from the Birch and Swinnerton–Dyer Conjecture. There are two aspects to it; one is somewhat analytic and one is in terms of what is called the Tate–Shafarevich group. Basically the Tate–Shafarevich group gives you the obstruction to an algorithm for finding the solutions. And the Birch and Swinnerton–Dyer Conjecture tells you that there is actually an analytic method for analysing this so-called Tate–Shafarevich group. If you combine all this together, ultimately it should give you an algorithm for finding the solutions.

You worked already on the Birch and Swinnerton–Dyer Conjecture when you were a graduate student together with John Coates?
Yes, that is exactly what he proposed working on. We got the first result in certain special families of elliptic curves on this analytic link between the solutions and what is called the L-function of the elliptic curve.

These were curves admitting complex multiplication?
Exactly, these were the elliptic curves with complex multiplication.

Was this the first general result concerning the Birch and Swinnerton–Dyer Conjecture?
It was the first one that treated a family of cases rather than individual cases. There was a lot of numerical data for individual cases, but this was the first infinite family of cases.

This was over the rational numbers?
Yes.

We should mention that the Birch and Swinnerton–Dyer Conjecture is one of the Clay Millennium Prize Problems which would earn a person who solves it one million dollars.
That's right. I think it's appealing, partly because it has its roots in Fermat's work, just like the Fermat problem. It is another 'elementary to state' problem, concerned with equations – in this case of very low degree – which we can't master and which Fermat initiated. I think it is a very appealing problem.

Do you think it is within reach? In other words, do we have the necessary tools for somebody daring enough to attack it and succeed? Or do we have to wait for another three hundred years to see it solved?
I don't suppose it will take three hundred years, but I don't think it is the easiest of the Millennium Problems. I think we are still lacking something. Whether the tools are all here now, I am not sure. They may be. There are always these speculations with these really difficult problems; it may be that the tools simply aren't there.

I don't believe that anyone in the 19th century could have solved Fermat's Last Theorem, certainly not in the way it was eventually solved. There was just too big a gap in mathematical history. You had to wait another hundred years for the right pieces to be in place. You can never be quite sure about these problems whether they are accessible to your time. That is really what makes them so challenging; if you had the intuition for what can be done now and what can't be done now you would be a long way towards a solution!

You mentioned the Tate–Shafarevich group and in that connection the Selmer group appears. Selmer [1920–2006] was a Norwegian mathematician, and it was Cassels [1922–2015] who is responsible for naming this group the Selmer group. Could you say a few words about the Selmer group and how it is related to the Tate–Shafarevich group, even if it's a little technical?
It is technical, but I can probably explain the basic idea of what the Selmer group is. What you are trying to do is to find the rational solutions on an elliptic curve. The method is to take the rational points on the elliptic curve – suppose you have got some – and you generate field extensions from these. So when I say generate extensions, I mean that you can take roots of those points on the elliptic curve. Just like taking the n'th root of 5 or the cube root of 2. You can do the same thing on an elliptic curve, you can take the n'th root of a point. These are all points which *added to themselves n times* gives you the point you started with. They generate certain extensions of the number field you started with, so in our case of the rational number field $\mathbb{Q}$.

You can put a lot of restrictions on those extensions. And the Selmer group is basically the smallest set of extensions you can get putting on all the obvious restrictions.

Let me summarize this. You've got the group of points. They generate some extensions; that's too big, you don't want all extensions. You cut that down as much as you can using local criteria, using p-adic numbers; that's called the Selmer group. And the difference essentially between the group generated by the points and the Selmer group is the Tate–Shafarevich group. So the Tate–Shafarevich group gives you the error term if you like, in trying to get at the points via the Selmer group.

Selmer's paper, which Cassels refers to, studied the Diophantine equation, $3x^3+4y^3+5z^3=0$ and similar ones. Selmer showed that it has just a trivial solution in the integers, while modulo n it has non-trivial solutions for all n. In particular, this curve has no rational points. Why did Cassels invoke Selmer's name in naming the group?
Yes, there are quite subtle relationships between these. What happens is you are actually looking at one elliptic curve, which in this case would be $x^3+y^3+60z^3=0$. That is an elliptic curve, in disguise, if you like, and the Tate–Shafarevich group involves looking at other ones

like it, for example $3x^3+4y^3+5z^3=0$, which is a genus one curve, but which has no rational points. Its Jacobian is the original elliptic curve, $x^3+y^3+60z^3=0$. One way of describing the Tate–Shafarevich group is in terms of these curves that have genus one but don't have rational points. And by assembling these together you can make the Tate–Shafarevich group, and that is reflected in the Selmer group. It is too intricate to explain in words but it is another point of view. I gave it in a more arithmetic terminology in terms of extensions. The more geometric terminology was in terms of these twisted forms.

The Modularity Conjecture

What you proved in the end was a special case of what is now called the Modularity Conjecture. In order to explain this one has to start with modular forms, and how modular forms can be put in relation with elliptic curves. Could you give us some explanations?

Yes; an elliptic curve (over the rationals) we have described as an equation $y^2=x^3+ax+b$, where the a and b are assumed to be rational numbers. (There is also a condition that the discriminant should not vanish). As I said, at the beginning of the 19th century you could describe the complex solutions to this equation. You could describe these very nicely in terms of the Weierstrass $\wp$-function, in terms of a special elliptic function. But what we want is actually a completely different uniformization of these elliptic curves which captures the fact that the a and b are rational numbers. It is a parametrization just for the rational elliptic curves. And because it captures the fact that it is defined over the rationals it gives you a much better hold on solutions over the rationals than the elliptic functions do. The latter really only sees the complex structure.

And the place it comes from are modular forms or modular curves. To describe modular functions first: we are used to functions which satisfy the relation of being invariant under translation. Every time we write down a Fourier series we have a function which is invariant under translation. Modular functions are ones which are invariant under the action of a much bigger group, usually a subgroup of $SL_2(\mathbb{Z})$. So, you would ask for a function $f(z)$ in one complex variable, usually on the upper half plan, which satisfies $f(z)$ is the same as $f((az+b)/(cz+d))$; or more generally, is that times a power of $cz+d$.

These are called modular functions and were extensively studied in the 19th century. Surprisingly they hold the key to the arithmetic of elliptic curves. Perhaps the simplest way to describe it is that because we have an action of $SL_2(\mathbb{Z})$ on the upper half plane H – by the action z goes to $(az+b)/(cz+d)$ – we can look at the quotient H modulo this action. You can then give the quotient the structure of a curve. In fact, it naturally gets the structure of a curve over the rational numbers.

If you take a subgroup of $SL_2(\mathbb{Z})$, or more precisely what is called a congruence subgroup, defined by the c value being divisible by N, then you call the curve a modular curve of level N. The Modularity Conjecture asserts that every elliptic curve over the rationals is actually a quotient of one of these modular curves for some integer N. It gives you a uniformization of elliptic curves by these other entities, these modular curves. On the face of it, it might seem we are losing because this is a high genus curve, it is more complicated. But it actually has a lot more structure because it is a moduli space.

And that is a very powerful tool?

That is a very powerful tool, yes. You have function theory, you have deformation theory, geometric methods etc. You have a lot of tools to study it.

Taniyama, the young Japanese mathematician who first conjectured or suggested these connections, his conjecture was more vague, right?

His conjecture was more vague. He didn't pin it down to a function invariant under the modular group. I've forgotten exactly what he conjectured; it was invariant under some kind of group, but I forget exactly which group he was predicting. But it was not as precise as the congruence subgroups of the modular group. I think it was originally written in Japanese so it was not circulated as widely as it might have been. I believe it was part of notes compiled after a conference in Japan.

It was an incredibly audacious conjecture at that time, wasn't it?

Apparently, yes.

But then it gradually caught the attention of other mathematicians. You told us already about Gerhard Frey, who came up with a conjecture relating Fermat's Last Theorem with the Modularity Conjecture.

That's right. Gerhard Frey showed that if you take a solution to the Fermat problem, say $a^p+b^p=c^p$, and you create the elliptic curve $y^2=x(x-a^p)(x+b^p)$, then the discriminant of that curve would end up being a perfect p'th power. And if you think about what that means assuming the Modularity Conjecture – you have to assume something a bit stronger as well (the so called epsilon conjecture of Serre) – then it forces this elliptic curve to have the level N that I spoke about to be equal to one, and hence the associated congruence subgroup is equal to $SL_2(\mathbb{Z})$. But H modulo $SL_2(\mathbb{Z})$ is a curve of genus zero. It has no elliptic curve quotient so it wasn't there after all, and hence there can't be a solution to the Fermat problem.

The quest for a proof

That was the point of departure for your own work, with crucial further ingredients due to Serre and Ribet making this connection clear. May we briefly summarize the story that then follows? It has been told by you many times, and it is the focus of a BBC-documentary.

You had moved to the United States, first to Harvard, then to Princeton University, becoming a professor there. When you heard of Ribet's result you devoted all your research time to prove the Modularity Con-

jecture for semistable elliptic curves over the rationals. This work went on for seven years of really hard work in isolation. At the same time you were working as a professor in Princeton and you were raising small kids.

A proof seems to be accomplished in 1993, and the development culminates in a series of three talks at the Isaac Newton Institute in Cambridge back in England, announcing your proof of Fermat's Last Theorem. You are celebrated by your peer mathematicians. Even the world press takes an interest in your results, which happens very rarely for mathematical results.

But then when your result is scrutinized by six referees for a highly prestigious journal, it turns out that there is a subtle gap in one of your arguments, and you are sent back to the drawing board. After a while you send for your former student, Richard Taylor, to come to Princeton to help you in your efforts. It takes a further ten months of hard and frustrating work; we think we do not exaggerate by calling it a heroic effort under enormous pressure. Then in a sudden flash of insight you realize that you can combine some of your previous attempts with new results to circumvent the problem that had caused the gap. This turns out to be what you need in order to get the part of the Modularity Conjecture that implied Fermat's Last Theorem.

What a relief that must have been! Would you like to give a few comments to this dramatic story?

With regard to my own work when I became a professional mathematician working with Coates I realized I really had to stop working on Fermat because it was time-consuming and I could see that in the last hundred years almost nothing had been done. And I saw others, even very distinguished mathematicians, had come to grief on it. When Frey came out with this result, I was a bit sceptical that the Serre part of the conjecture was going to be true, but when Ribet proved it then, okay, this was it!

And it was a long hard struggle. In some sense it is irresponsible to work on one problem to the exclusion of everything else, but this is the way I tend to do things. Whereas Fermat is very narrow, I mean it is just this one equation, whose solution may or may not help with anything else, yet the setting of the modular conjecture was one of the big problems in number theory. It was a great thing to work on anyway, so it was just a tremendous opportunity.

When you are working on something like this it takes many years to really build up the intuition to see what kinds of things you need and what kinds of things a solution will depend on. It's something like discarding everything you can't use and won't work till your mind is so focused that even making a mistake, you've seen enough that you'll find another way to the end.

Funnily enough, concerning the mistake in the argument that I originally gave, people have worked on that aspect of the argument and quite recently they have actually shown that you can produce arguments very like that. In fact, in every neighbouring case arguments similar to the original method seem to work but there is this unique case that it doesn't seem to work for, and there is not yet any real explanation for it. So the same kind of argument I was trying to use, using Euler systems and so on, has been made to work in every surrounding case but not the one I needed for Fermat. It's really extraordinary.

You once likened this quest for the proof of the Modularity Theorem in terms of a journey through a dark unexplored mansion. Could you elaborate?

I started off really in the dark. I had no prior insights how the Modularity Conjecture might work or how you might approach it. One of the troubles with this problem – it's a little like the Riemann Hypothesis but perhaps even more so with this one – is you didn't even know what branch of mathematics the answer would be coming from.

To start with, there are three ways of formulating the problem, one is geometric, one is arithmetic and one is analytic. And there were analysts – I would not understand their techniques at all well – who were trying to make headway on this problem.

I think I was a little lucky because my natural instinct was with the arithmetic approach and I went straight for the arithmetic route, but I could have been wrong. The only previously known cases where the Modularity Conjecture were known to hold were the cases of complex multiplication, and that proof is analytic, completely analytic.

Partly out of necessity, I suppose, and partly because that's what I knew, I went straight for an arithmetic approach. I found it very useful to think about it in a way that I had been studying in Iwasawa theory. With John Coates I had applied Iwasawa theory to elliptic curves. When I went to Harvard I learned about Barry Mazur's work, where he had been studying the geometry of modular curves using a lot of the modern machinery. There were certain ideas and techniques I could draw on from that. I realized after a while I could actually use that to make a beginning – to find some kind of entry into the problem.

Before you started on the Modularity Conjecture, you published a joint paper with Barry Mazur, proving the main theorem of Iwasawa Theory over the rationals. Can you please tell us what Iwasawa Theory is all about?

Iwasawa theory grew out of the work of Kummer on cyclotomic fields and his approach to Fermat's Last Theorem. He studied the arithmetic, and in particular the ideal class groups, of prime cyclotomic fields. Iwasawa's idea was to consider the tower of cyclotomic fields obtained by taking all p-power roots of unity at once. The main theorem of Iwasawa theory proves a relation between the action of a generator of the Galois group on the p-primary class groups and the p-adic L-functions. It is analogous to the construction used in the study of curves over finite fields where the characteristic polynomial of Frobenius is related to the zeta function.

And these tools turned out to be useful when you started to work on the Modularity Conjecture?

They did, they gave me a starting point. It wasn't obvious at the time, but when I thought about it for a while I realized that there might be a way to start from there.

Parallels to Abel's work

We want to read you a quotation: "The ramparts are raised all around but, enclosed in its last redoubt, the problem defends itself desperately. Who will be the fortunate genius who will lead the assault upon it or force it to capitulate?"

It must been E.T. Bell, I suppose? Is it?

No, it's not. It is actually a quote from the book "Histoire des Mathématiques" by Jean-Étienne Montucla [1725–1799], written in the late 18th century. It is really the first book ever written on the history of mathematics. The quotation refers to the solvability or unsolvability of the quintic equation by radicals.

As you know Abel [1802–1829] proved the unsolvability of the general quintic equation when he was 21 years old. He worked in complete isolation, mathematically speaking, here in Oslo. Abel was obsessed, or at least extremely attracted, to this problem. He also got a false start. He thought he could prove that one could actually solve the quintic by radicals. Then he discovered his mistake and he finally found the unsolvability proof.

Well, this problem was at that time almost 300 years old and very famous. If we move fast forward 200 years the same quotation could be used about the Fermat problem, which was around 350 years old when you solved it. It is a very parallel story in many ways. Do you have comments?

Yes. In some sense I do feel that Abel, and then Galois [1811–1832], were marking a transition in algebra from these equations which were solvable in some very simple way to equations which can't be solved by radicals. But this is an algebraic break that came with the quintic. In some ways the whole trend in number theory now is the transition from basically abelian and possibly solvable extensions to insolvable extensions. How do we do the arithmetic of insolvable extensions?

I believe the Modularity Conjecture was solved because we had moved on from this original abelian situation to a non-abelian situation, and we were developing tools, modularity and so on, which are fundamentally non-abelian tools. (I should say though that the proof got away mostly with using the solvable situation, not because it was more natural but because we have not solved the relevant problems in the general non-solvable case).

It is the same transition in number theory that he was making in algebra, which provides the tools for solving this equation. So I think it is very parallel.

There is an ironic twist with Abel and the Fermat Problem. When he was 21 years old, Abel came to Copenhagen to visit Professor Degen [1766–1825], who was the leading mathematician in Scandinavia at that time. Abel wrote a letter to his mentor in Oslo, Holmboe [1795–1850], stating three results about the Fermat equation without giving any proofs – one of them is not easy to prove, actually. This, of course, is just a curiosity today.

But in the same letter he gives vent to his frustration, intimating that he can't understand why he gets an equation of degree n^2, and not n, when dividing the lemniscate arc in n equal pieces. It was only after returning to Oslo that he discovered the double periodicity of the lemniscate integral, and also of the general elliptic integral of the first kind.

If one thinks about it, what he did on the Fermat equation turned out to be just a curiosity. But what he achieved on elliptic functions, and implicitly on elliptic curves, turned out later to be a relevant tool for solving it. Of course, Abel had no idea that this would have anything to do with arithmetic. So this story tells us that mathematics sometimes develops in mysterious ways.

It certainly does, yes.

Work styles

May we ask for some comments about work styles of mathematicians in general and also about your own? Freeman Dyson, a famous physicist and mathematician at IAS in Princeton, said in his Einstein lecture in 2008: "Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects and they solve problems one at a time".

Freeman Dyson didn't say that birds were better than frogs, or the other way around. He considered himself a frog rather than a bird.

When we are looking at your work, it seems rather difficult for us to decide where to place you in his classification scheme: among the birds, those who create theories, or among the frogs, those who solve problems. What is our own perception?

Well, I don't feel like either. I'm certainly not a bird – unifying different fields. I think of frogs as jumping a lot. I think I'm very, very focused. I don't know what the animal analogy is, but I think I'm not a frog in the sense that I enjoy the nearby landscape. I'm very, very concentrated on the problem I happen to work on and I am very selective. And I find it very hard to even take my mind off it enough to look at any of the flowers around, so I don't think that either of the descriptions quite fits.

Based on your own experience could you describe the interplay between hard, concentrated and persevering work on the one side, and on the other side these sudden flashes of insights that seemingly come out of nowhere, often appearing in a more relaxed setting. Your mind must have worked unconsciously on the problem at hand, right?

I think what you do is that you get to a situation where you know a theory so well, and maybe even more than one theory, so that you have seen every angle and tried a lots of different routes.

It is this tremendous amount of work in the preparatory stage where you have to understand all the details, and maybe some examples, that is your essential launch pad. When you have developed all this, then you let the mind relax and then at some point – maybe when you go away and do something else for a little bit – you come back and suddenly it is all clear. Why did you not think of that? This is something the mind does for you. It is the flash of insight.

I remember – this is a trivial example in a non-mathematical setting – once someone showed me some script, it was some gothic script, and I couldn't make head or tail of it. I was trying to understand a few letters, and I gave up. Then I came back half an hour later and I could read the whole thing. The mind somehow does this for you and we don't quite know how, but we do know what we have to do to set up the conditions where it will happen.

This is reminiscent of a story about Abel. While in Berlin he shared an apartment with some Norwegian friends, who were not mathematicians. One of his friends said that Abel typically woke up during the night, lighted a candle and wrote down ideas that he woke up with. Apparently his mind was working while asleep.

Yes, I do that except I don't feel the need to write them down when I wake up with it because I know I will not forget it. But I am terrified if I have an idea when I am about to go to sleep that I would not wake up with that idea, so then I have to write it down.

Are you thinking in terms of formulas or in terms of geometric pictures, or what?

It is not really geometric. I think it is patterns, and I think it is just parallels between situations I have seen elsewhere and the one I am facing now. In a perfect world, what is it all pointing to, what are the ingredients that ought to go into this proof, what am I not using that I have still in my pocket? Sometimes it is just desperation. I assemble every piece of evidence I have and that's all I've got. I have got to work with that and there is nothing else.

I often feel that doing mathematics is like being a squirrel and there are some nuts at the top of a very tall tree. But there are several trees and you don't know which one. What you do is that you run up one and you think, no, it does not look good on this one, and you go down and up another one, and you spend your whole life just going up and down these trees but you've only got up to thirty feet. Now if someone told you the rest of the trees – it's not in them, you have only one tree left, then you would just keep going until you found it. In some sense it is ruling out the wrong things that is really crucial. And if you just believe in your intuition, and your intuition is correct, and you stick with your one tree then you will find it.

Problems in mathematics

Felix Klein [1849–1925] once said: "Mathematics develops as old results are being understood and illuminated by new methods and insights. Proportionally with a better and deeper understanding new problems naturally arise." And David Hilbert [1862–1943] stressed that "problems are the lifeblood of mathematics". Do you agree?

I certainly agree with Hilbert, yes. Good problems are the lifeblood of mathematics. I think you can see this clearly in number theory in the second half of the last century. For me personally obviously the Modularity Conjecture, but also the whole Langlands Program and the Birch and Swinnerton–Dyer Conjecture: These problems give you a very clear focus on what we should try to achieve. We also have the Weil Conjectures on curves and varieties over finite fields and the Mordell Conjecture and so on.

These problems somehow concentrate the mind and also simplify our goals in mathematics. Otherwise we can get very, very spread out and not sure what's of value and what's not of value.

Do we have as good problems today as when Hilbert formulated his twenty-three problems in 1900?

I think so, yes.

Which one do you think is the most important problem today? And how does the Langlands program fit in?

Well, I think the Langlands program is the broadest spectrum of problems related to my field. I think that the Riemann Hypothesis is the single greatest problem from the areas I understand. It is sometimes hard to say exactly why that is, but I do believe that solving it would actually help solve some of these other problems. And then of course I have a very personal attachment to the Birch and Swinnerton–Dyer Conjecture.

Intuition can lead us astray sometimes. For example, Hilbert thought that the Riemann Hypothesis would be solved in his lifetime. There was another problem on his list, the 7th, that he never thought would be solved in his lifetime, but which was solved by Gelfond [1906–1968] in 1934. So our intuition can be wrong.

That is right. I'm not surprised that Hilbert felt that way. The Riemann Hypothesis has such a clear statement and we have the analogue in the function field setting. We understand why it is true there, and we feel we ought to be able to translate it. Of course, many people have tried and failed. But I would still myself expect it to be solved before the Birch and Swinnerton–Dyer Conjecture.

Investing in mathematics

Let's hope we'll find out in our lifetimes!

Classical mathematics has, roughly speaking, two sources: one of them coming from the physical sciences and the other one from, let's for simplicity call it num-

ber theoretical speculations, with number theory not associated to applications.

That has changed. For example, your own field of elliptic curves has been applied to cryptography and security. People are making money with elliptic curves nowadays! On the other hand, many sciences apart from physics really take advantage and profit from mathematical thinking and mathematical results. Progress in industry nowadays often depends on mathematical modelling and optimization methods. Science and industry propose challenges to the mathematical world.

In a sense, mathematics has become more applied than it ever was. One may ask whether this is a problem for pure mathematics. It appears that pure mathematics sometimes is put to the side lines, at least from the point of view of the funding agencies. Do you perceive this as a serious problem?

Well, I think in comparison with the past one feels that mathematicians two, three hundred years ago were able to handle a much broader spectrum of mathematics, and a lot more of it touched applied mathematics than would a typical pure mathematician do nowadays. On the other hand that might be because we only remember the very best and most versatile mathematicians from the past.

I think it is always going to be a problem if funding agencies are short-sighted. If they want to see a result in three years then it is not going to work. It is hard to imagine a pure development and then the application all happening within three to five years. It is probably not going to happen.

On the other hand, I don't believe you can have a happily functioning applied maths world without the pure maths to back it up, providing the future and keeping them on the straight and narrow. So it would be very foolish not to invest in pure mathematics.

It is a bit like only investing in energy resources that you can see now. You have to invest in the future; you have to invest in fusion power or solar power or these other things. You don't just use up what is there and then start worrying about it when it is gone. It is the same with mathematics, you can't just use up the pure mathematics we have now and then start worrying about it when you need a pure result to generate your applications.

Mathematical awards

You have already won a lot of prizes as a result of your achievements, culminating in proving Fermat's Last Theorem. You have won the Rolf Schock Prize, given by the Swedish Academy, the Ostrowski Prize, which was given to you in Denmark, the Fermat Prize in France, the Wolf Prize in Israel, the Shaw Prize in Hong Kong – the prize that has been named the Nobel Prize of the East; and the list goes on, ending with the Abel Prize tomorrow. May we ask you whether you enjoy these awards and the accompanying celebrations?

I certainly love them, I have to say. I think they are a celebration of mathematics. I think with something like Fermat it is something people are happy to see in their lifetime. I would obviously be very happy to see the Riemann Hypothesis solved. It is just exciting to see how it finally gets resolved and just to understand the end of the story. Because a lot of these stories we won't live to see the end of. Each time we do see the end of such a story it is something we naturally will celebrate. For me I learned about the Fermat problem from this book of E.T. Bell and about the Wolfskehl Prize attached to it. The Wolfskehl Prize was still there – only just I may say – I only had a few years left before the deadline for it expired.

This gives us the lead to talk a little about that prize. The Wolfskehl Prize was founded in 1906 by Paul Wolfskehl [1856–1906], who was a German physician with an interest in mathematics. He bequeathed one hundred thousand Reichmarks (equivalent to more than one million dollars in today's money) to the first person to prove Fermat's Last Theorem. The prize was, according to the testament, valid until September 13, 2007, and you received it in 1997. By then, due in part to hyperinflation Germany suffered after World War I, the prize money had dwindled a lot.

For me the amount of money was unimportant. It was the sentimental feeling attached to the Wolfskehl Prize that was important for me.

Graduate students

You have had altogether twenty-one PhD-students and you have attracted very gifted students. Some of them are really outstanding. One of them, Manjul Bhargava, won the Fields medal in 2014. It must have been a pleasure to be advisor to such students?

Yes, I don't want to take too much credit for it. In the case of Manjul I suggested a problem to him but after that I had nothing much more to do. He was coming up with these absolutely marvellous discoveries. In some sense you get more credit if you have very gifted students, but the truth is that very gifted students don't really require that much help.

What is the typical way for you of interacting with graduate students?

Well, I think the hardest thing to learn as a graduate student is that afterwards you need to carry on with the rest of your professional life; it's hard to pick problems. And if you just assign a problem and they do it, in some sense that hasn't given them terribly much. Okay, they solved that problem, but the hard thing is then to have to go off and find other problems! So I prefer it if we come to a decision on the problem together.

I give them some initial idea and which area of mathematics to look at, having not quite focused on the problem. Then as they start working and become experts they can see a better way of pinning down what the right question is. And then they are part of the process of choosing the problem. I think that is a much better investment for their future. It doesn't always work out that way, and sometimes the initial problem you give them turns out to be the right thing. But usually it is not that way, and usually it's a process to find the right problem.

Hobbies and interests

We always end the Abel interviews by asking the laureate what he enjoys doing when he doesn't work with mathematics. What are your hobbies and interests outside of mathematics?

Well, it varies at different times. When I was doing Fermat, and being a father with young children, that combination was all-consuming.

I like to read and I like various kinds of literature, novels, some biographies, it is fairly balanced. I don't have any other focused obsessions. When I was in school I played on chess teams and bridge teams, but when I started to do serious mathematics I completely lost interest in those.

What about music; are you fond of music?

I go and listen to concerts, but I am not myself actively playing anything. I enjoy listening to music, classical, preferably.

Are you interested in other sciences apart from mathematics?

I would say somewhat. These are things I do to relax, so I don't like them to be too close to mathematics. If it is something like animal behaviour or astrophysics or something from a qualitative point of view, I certainly enjoy learning about those. Likewise about what machines are capable of, and many other kinds of popular science, but I'm not going to spend my time learning the details of string theory. I'm too focused to be willing to do that. Not that I would not be interested, but this is my choice.

We would like to thank you very much for this wonderful interview. That is first of all on behalf of the two of us, but also on the behalf of the Norwegian, the Danish and the European Mathematical Society.

Thank you so much!

Thank you very much!

From left to right: Sir Andrew Wiles, Martin Raussen and Christian Skau. (Photo: Eirik F. Baardsen, DNVA.)

Martin Raussen is a professor with special responsibilities (mathematics) at Aalborg University, Denmark. Christian Skau is a professor of mathematics at the Norwegian University of Science and Technology at Trondheim. Together, they have held interviews with all the Abel Laureates since 2003.

Göttingen's SUB as Repository for the Papers of Distinguished Mathematicians

David E. Rowe (University of Mainz, Germany)

For over a century, the Göttingen State and University Library (SUB) has served as an important repository for writings, correspondence and other documents of value for the history of mathematics. These collections date back to the early years of the university when the polymath mathematician Abraham Gotthelf Kästner and the astronomer Tobias Mayer were among its more prominent faculty members. Today, the Department of Special Collections (Spezialsammlungen) in the old library of the SUB houses over 50 collections of papers as bequests (Nachlässe) of prominent mathematicians, most of whom were associated with the university. Since 1997, information relating to these holdings can be found in the databank HANS (http://hans.sub.uni-goettingen.de/), an acronym for Handschriften, Autographen, Nachlässe and Sonderbestände.

Many of these collections are relatively recent acquisitions awaiting future investigation. In 1992, the SUB entered into an agreement with the Deutsche Mathematiker-Vereinigung (DMV, German Mathematical Society) in order to create the Central Archive for German Mathematics Bequests (Patterson/Rohlfing/Schappacher 2003). For several years, the Göttingen Academy of Sciences was able to provide financial support that helped enable these newer acquisitions to be catalogued and made available to researchers. Some of the newly acquired Nachlässe, for example those of Erich Hecke and Emil Artin who were colleagues in Hamburg, document the lives and works of mathematicians who spent most of their careers outside Göttingen. As such, the Central Archive at the SUB plays a role similar to that of the library at the University of Texas in Austin, which houses an even more extensive collection in its Archives of American Mathematics (AAM) – http://www.cah.utexas.edu/collections/math.php. With one exception, there would seem to be virtually no overlap between the mathematicians represented by documents in these two archives. If that impression is correct then that distinction goes to Max Dehn, one of Hilbert's star pupils whose posthumous papers were donated to the AAM in 1979 by Wilhelm Magnus, acting on behalf of Dehn's widow, Toni. Several letters from Dehn can be found among Hilbert's papers but also in other Nachlässe in the SUB.

It was surely not an accident that the DMV chose the SUB in Göttingen as the ideal site for a central archive. No doubt there were many practical matters to consider but one factor would have been obvious, namely the rich archival resources already located there. These included the Nachlässe of Gauss, Riemann, Dedekind, Klein and Hilbert, five central figures associated with the Göttingen mathematical tradition. Taking a glance backward to the late 18th century, we should briefly observe that Göttingen's library had long played a significant role in shaping its early mathematical culture.

When the young Carl Friedrich Gauss had to account for why he chose to study in Göttingen rather than at the local university in Helmstedt, he gave a simple reason: books (Küssner 1979, 48). Already as a pupil at the Collegium Carolinum in his native Braunschweig, Gauss was reading works as difficult as Newton's *Principia*, which he was able to purchase in 1794. Thanks to a stipend from Duke Karl Wilhelm Ferdinand, he was later granted the opportunity to spend three years pursuing his intellectual interests, which were by no means confined to mathematics (Goldstein /Schappacher/Schwermer 2007). Still, he knew that to become an educated and versatile mathematician he would have to have ready access to current scientific literature. Helmstedt, though a much older university than Göttingen's Georgia Augusta, simply lacked the resources of the newer institution, which, almost from its founding in 1737, had a far better library, not to mention faculty. This was no mystery: the university's patron was the Duke of Hanover, better known in the English-speaking world as George II, King of Great Britain and Ireland, and Prince-Elector of the Holy Roman Empire. So Gauss had good reason to believe he would be pleased with the scholarly apparatus housed in Göttingen's university library. Travelling the 90 km by foot, he arrived in October 1795 for the Winter semester. A short time later, he wrote to his former teacher, Eberhard A. W. Zimmermann, that he was impressed with the library holdings and had already begun studying several volumes from the Proceedings of the Petersburg Academy. He thereafter made regular use of it (Küssner 1979).

The following Spring, Gauss recorded the first entry in his mathematical diary (30 March 1796): "Principia quibus innititur sectio circuli, ac divisibilitas eiusdem geometrica in septemdecim partes etc." (The principles upon which the division of the circle depends, and geometrical divisibility of the same into seventeen parts, etc.) (Klein 1903, Gray 1984). He had cracked the problem of determining which regular polygons can be constructed by straightedge and compass alone, including the first non-classical case: the 17-gon. Commentators could henceforth note that the theory set forth by Euclid in Book IV of the *Elements* had now been completed. This breakthrough was quickly followed by an incredible cascade of new discoveries, including a proof of the $n=3$ case of Fermat's conjecture, according to which every number can be written as the sum of at most n n-gonal numbers. Gauss thus wrote on 10 July 1796: "ΕΥΡΗΚΑ! num = Δ + Δ + Δ." By the end of the year, though still not yet 20 years of age, Gauss had already filled his diary with 49 entries for results he had obtained during the preceding nine months!

Today, any viewer can glance through the original pages of Gauss' diary just by clicking on the link: http://webdoc.sub.gwdg.de/ebook/e/2005/gausscd/html/kapitel_tagebuch.htm. Oddly enough, however, the introductory text begins by asserting that this booklet was first discovered in the Gauss Nachlass in 1928! Perhaps this error will have been corrected by the time this text appears in print but, in any case, the story behind Gauss' diary should certainly be better known. In fact, it was first found three decades earlier, as Felix Klein reported in his lectures on 19th century mathematics (Klein 1926, 30). Klein was, in fact, the first to publish Gauss' diary along with some preliminary commentary. This appeared in a special Festschrift in 1901, which was then reprinted two years later in *Mathematische Annalen* (Klein 1903). The discovery itself was made by Paul Stäckel in 1899 when he visited a grandson of Gauss in Hameln.

Although this tiny booklet has since taken its unique place in the annals of mathematics, its survival was largely a matter of lucky circumstance. Shortly after Gauss' death in 1855, his Nachlass was divided so that the "scientific portion" could be sold to the Hanoverian Government and acquired by the Göttingen Scientific Society. Responsibility for ordering the documents and publishing the Gauss *Werke* was first placed in the hands of his former student, Ernst Schering. Work proceeded slowly, however, which was a source of frustration for Klein, who had to await Schering's death in 1897 before he could begin to manage this project. He quickly came to realise that Gauss' papers had been sorted out and divided decades earlier, a circumstance that led to the loss of his diary, which had been classified as a non-scientific work! Thereafter, Klein threw himself into this project with passion; between 1898 and 1921, he published no fewer than 14 separate reports on the course of the work on the Gauss edition. This undertaking went on long afterwards and was only completed in 1933. For those who can afford the luxury, reprints of the 12 volumes of the Gauss *Werke* can now be purchased from the Cambridge Library Collection. As a far more affordable alternative, the SUB provides cost-free access to all of these volumes via http://gdz.sub.uni-goettingen.de/dms/load/toc/?PID=PPN235957348.

Felix Klein's career in Göttingen marks a watershed period, during which he saw himself as the grand impresario who would usher in the new while paying homage to the old (Parshall & Rowe 1994). His grand aim and vision was to build a bridge joining the tradition of Gauss and Riemann with the modern era of 20th century research mathematics. As the principal architect of several innovative reforms, he took an active part in cultivating the legacies of his distinguished predecessors. He did so by drawing on the resources of the Göttingen Scientific Society (forerunner of the Göttingen Academy) but also by engaging a small army of allies and assistants. These local efforts in Göttingen can be seen as part of a broader modernisation process in Prussia, part of which affected the state libraries. In fact, Klein's appointment in Göttingen in 1886 coincided with that of Karl Dziatzko as director of the university library. As Göttingen's first professor of library science, Dziatzko helped implement new standards that would be adopted elsewhere. Thus, like Klein, he took part in sweeping reforms that would eventually transform the Prussian universities under the aegis of the ministerial official Friedrich Althoff (Rowe 1989).

Klein's work in connection with the Gauss Nachlass was primarily organisational in nature, whereas his interest in Riemann's ideas and work ran far deeper. Moreover, a quite different set of circumstances surrounded Riemann's posthumous papers. Riemann only taught for a few years in Göttingen before his declining health forced him to take long periods of leave in Italy. He died there in 1866 at the age of 40. Riemann's Nachlass was thus far smaller than that of Gauss, though it presented special challenges for the three experts who first worked through it: Richard Dedekind, Alfred Clebsch and Heinrich Weber. Dedekind's efforts led to the posthumous publication of three unknown papers, including the famous lecture of 1854 on the foundations of geometry (Riemann 1868). Following the sudden and unexpected death of Clebsch in 1872, Weber agreed to complete the task of preparing Riemann's *Werke*, which appeared a few years later (Riemann 1876). By the 1890s, however, Klein was intent on recovering additional documents, particularly lecture notes taken by some of Riemann's former auditors. This initiative eventually brought to light some 20 sets of lecture notes (Neuenschwander 1988). These were then carefully studied by Max Noether and Wilhelm Wirtinger when they prepared the Nachträge for the expanded edition of Riemann's Collected Works, which was then published (Riemann 1902).

Scholarly interest in Riemann's original works remained strong throughout the 20th century. Erich Bessel-Hagen, a protégé of Klein and close friend of Carl Ludwig Siegel, was actively involved in preserving documents relating to Riemann's life and work. In 1932, Siegel published a number of new results on the Riemann zeta-function based on formulas he had found scattered across papers in the Nachlass (Siegel 1932). Though he discovered no hints in the Nachlass of an Ansatz that might lead to a proof of the Riemann conjecture, Siegel did find and then derive what is today called the Riemann-Siegel formula. He also emphasised how the notes he had found bore witness to Riemann's analytical powers, thereby taking issue with Klein, who often promoted a picture of Riemann as an intuitive genius. Siegel's paper was originally published in the short-lived journal *Quellen und Studien zur Geschichte der Math-*

ematik, Astronomie und Physik (Studien) 2(1932): 45-80. It was later reprinted both in his *Collected Works* (Siegel 1979) as well as in the latest and most comprehensive edition of Riemanns *Gesammelte Mathematische Werke* (Riemann 1990).

After the turn of the century, Klein had moved away from research mathematics to pursue other goals. As his younger colleagues in Göttingen – Hilbert, Minkowski and Schwarzschild – took on the more specialised courses, Klein found time to promote historical studies. Thus, beyond his interest in documenting the scientific achievements of Gauss and Riemann, he also began offering courses on several broader developments that shaped modern mathematics. This culminated with his wartime lectures on 19th century mathematics, which began with a survey of Gauss' work. Typescript copies of these lectures circulated fairly widely after the war, though the first volume (Klein 1926), published posthumously by Richard Courant and Otto Neugebauer, omits some of Klein's more polemical remarks.

As an offshoot of these interests, Klein also established the so-called Mathematiker-Archiv, which collected documents relating to modern mathematical developments. Three of its holdings are particularly noteworthy. Descendents of the geometer Carl Georg Christian von Staudt, who studied under Gauss and later taught for many years in Erlangen, donated a number of manuscripts from his papers. Klein, who held Staudt in high esteem, later added to this collection with a small number of Staudt's handwritings that he presumably acquired in the early 1870s when he taught in Erlangen.

Unfortunately, only a few documents relating to the career of Hermann Minkowski exist today. Several of his letters to Hilbert did survive and can be found in the Hilbert Nachlass; these were published in (Minkowski 1973). In 1915, Arnold Sommerfeld published Minkowski's lecture "Das Relativitätsprinzip" (Minkowski 1915), which was delivered in November 1907 to the Göttingen Mathematical Society. This was Minkowski's original presentation of his new approach to relativity, which became famous from his lecture "Raum und Zeit" of the following year. After his death in early 1909, the text of the first lecture was buried among his papers until Sommerfeld decided to make it better known, publishing it in both the *Annalen der Physik* and in the *Jahresbericht der DMV*. He also persuaded Minkowski's widow to donate her husband's manuscripts on electrodynamics and relativity to the Göttingen archives. Aside from these, the Jewish National and University Library at Hebrew University has custody of several notebooks that Minkowski used for lecture courses presented in Göttingen. Scans of these are available online as part of the ECHO project of the Max Planck Institute for History of Science in Berlin (http://echo.mpiwg-berlin.mpg.de/content/modernphysics/jnul).

The third particularly important collection of documents in the Mathematiker-Archiv consists of an extensive set of letters posted to Adolf Hurwitz before his death in 1919. Hurwitz was not only Klein's star pupil but he also exerted a strong influence on Hilbert during the latter's student days in Königsberg. The published correspondence between Hilbert and Klein (Hilbert/Klein 1985), along with the letters from Hurwitz to Klein and to Hilbert preserved in their respective Nachlässe, brings together a rich set of material that throws much light on the mutual relations between these three figures.

Alongside such famous names, a mention should be made of one of several nearly forgotten figures from this era, Conrad Müller, who worked closely for several years with Klein. Müller's dissertation dealt with the mathematics curriculum in Göttingen during the pre-Gaussian years. He afterwards worked as a librarian in Göttingen until 1910, when he was appointed to a professorship in Hanover. As a student in Göttingen, Müller learned to read Sanskrit. He later exploited that skill in order to study early Indian mathematics, about which he published several articles. Famous for his colossal memory, he also played an invaluable role in supporting Klein's work on one of the era's massive undertakings, the German Encyclopedia of the Mathematical Sciences, which was only completed in 1935.

Unlike Gauss and Riemann, Klein was in the somewhat more enviable position of having the time and resources to prepare his own collected works, which were published in three volumes by Springer, one per year (Klein 1921–23). By then, he was old and ailing, so he was unable to revise his historical wartime lectures. Courant delegated that task to Neugebauer and Stephan Cohn-Vossen and then published these edited versions in his yellow series (Klein 1926, 1927). Courant also gained funding from the Göttingen Academy so that Bessel-Hagen could order Klein's scientific and personal papers. These today form the Klein Nachlass at the SUB, a true goldmine for the history of mathematics from 1870 to 1925 (the year of his death). The larger part of Klein's personal library was soon thereafter sold to Hebrew University, an arrangement that was probably facilitated by Edmund Landau.

After the First World War, Courant assumed Klein's former chair as well as his unofficial position as the head of operations in mathematics. As a Göttingen product, he was naturally deferential toward his former mentors and no one did more to uphold Hilbert's name and reputation throughout the 1920s. Like many others, he failed to recognise the enormity of what Hitler's ascent to power in 1933 meant for the future of Germany. It took Courant some time to realise that the Nazi Government had no interest in supporting mathematicians and scientists – no matter how patriotic they might be – if their political orientation and racial background made them misfits for the new order. After the removal of "Jewish influences" and the dissolution of the Göttingen Institute under his regime, Courant reluctantly took a position at New York University. There, he began building what was at first largely an exile community but which eventually became one of the leading mathematical centres in the United States (Reid 1976).

After the war, Courant regularly visited Göttingen, where his former protégé, Franz Rellich, headed the institute. Courant also kept a watchful eye on Hilbert's papers, which were only transferred to the university library in 1967. Whereas Klein's Nachlass covers more territory, Hilbert's papers provide invaluable insights into his research and teaching activities. Knowing this, Courant kept some of the letters from it in Rellich's home, where several still remained even after the rather late transfer of the Nachlass to the SUB. Thus, for many years, the original letters and postcards Einstein sent to Hilbert in November 1915 were missing from the Hilbert Nachlass at the SUB. These concern one of the most important chapters in the early history of general relativity and though photocopies of the extant writings were available, no one knew the whereabouts of the originals. In fact, they only turned up about a decade ago when these and other documents were discovered in the attic of Rellich's home. These and other documents originally in the Hilbert Nachlass have since been acquired by the SUB and are available for study in the Department of Special Collections.

The circumstances that surrounded the temporary disappearance of the Einstein writings from 1915 were surely related to Courant's ongoing interest in preserving important sources related to Hilbert's life. During the late 1960s, he was cooperating with Constance Reid, who was then working on her biography of Hilbert (Reid 1970). After completing that project, she was assisted by Courant's friend and former collaborator, K.O. Friedrichs, who asked her to write a book about Courant's life (Reid 1976). For both books, Reid interviewed many mathematicians associated with Göttingen. Transcripts and tapes from these interviews, along with other documentary material from Reid's posthumous papers, were recently acquired by the SUB and have now been catalogued and so await future study.

The 1970s and 1980s saw a fresh outburst of interest in the history of modern mathematics, in particular with regard to foundational research. Two pioneering figures in this regard were Richard Dedekind and Georg Cantor. A substantial collection from Dedekind's scientific papers and correspondence was obtained by the SUB from the family in 1931. In addition to these, in 1966, Ludwig Bieberbach donated a set of five supplementary manuscripts in Dedekind's hand. It was not until 1982, however, that the Nachlass was put in good order by Winfried Scharlau. In the meantime, the SUB had acquired a portion of what remained of Cantor's Nachlass, a resource soon thereafter exploited by several historians of mathematics. Interest in the work and influence of Cantor and Dedekind will surely remain strong for a long time to come. Unfortunately, however, most of Cantor's Nachlass was lost under circumstances that are no longer clear. After his death in 1918, most of Cantor's papers remained in his home in Halle and though they apparently survived the Second World War, they disappeared soon thereafter.

At first it seemed that the same fate had befallen the famous letters Cantor wrote to Dedekind during the 1870s and 1880s, parts of which had been published in 1937 in a booklet prepared by Emmy Noether and Jean Cavaillès (Cantor/Dedekind 1937). Noether had co-edited Dedekind's *Werke* (Dedekind 1930–32) in the early 1930s but soon thereafter lost her position in Göttingen. Like Courant, she fell victim to the very first wave of dismissals after the Nazis gained power. Soon thereafter, she gained an appointment at Bryn Mawr College outside Philadelphia but, in 1935, died unexpectedly from complications following an operation. In 1968, Clark Kimberling was researching her life, in the course of which he received a surprising letter from a retired attorney in Philadelphia, who happened to have been involved in the handling of Noether's estate after her death. His firm had been unable to locate the rightful owners of a large collection of older letters found in her possession at that time and so these documents were eventually shelved away and simply forgotten. When Kimberling opened the package containing these letters, he soon realised that he had struck gold.

This fortunate turn of events led not only to the recovery of Cantor's letters to Dedekind but also to the discovery of a large number of letters that Dedekind had received from Heinrich Weber and Georg Frobenius. This collection was first transcribed by Walter Kaufmann-Bühler and then later by Ralf Haubrich. The latter transcription circulated for some time in various hands but to date these letters have not been published. Nevertheless, Haubrich's transcription was carefully studied by Thomas Hawkins, who used these letters as a crucially important source for his groundbreaking study, *The Mathematics of Frobenius in Context: A Journey through 18th to 20th Century Mathematics* (Hawkins 2013). The correspondence between Dedekind and Weber, on the other hand, was published in a volume that appeared just last year (Scheel 2015). Documentation relating to the recovery of these documents can be found in the Kimberling papers, one of the smaller Nachlässe available at the SUB.

Several of the most recent acquisitions in the expanded collection housed in the Göttingen library relate to the era of National Socialism and the post-war period (Siegmund-Schultze 2009). Here we should recall that Hilbert retired from his professorship in 1930, just about the time that the new institute building was opened; for information on Courant's role in creating the new institute, see (Siegmund-Schultze 2001). He was succeeded by Hermann Weyl, who quickly sought refuge at the Institute for Advanced Study in Princeton after the Nazi takeover. A year later, Helmut Hasse was appointed in his place and he remained head of the Göttingen Institute throughout the Nazi period (Schappacher 1987). Although he was a close friend of Emmy Noether and other Jewish mathematicians, Hasse clearly sympathised with many of the goals of the NS-government. His political views were decidedly nationalistic, enough so that he could curry favour with officials of the new regime. After his death in 1979, his papers fell into the possession of the Göttingen algebraist Martin Kneser, who had married Hasse's daughter Jutta. Kneser then donated the main part of Hasse's extensive Nachlass to the SUB in 1980. Since then, it has been supplemented by further materials but the Hasse papers have yet to receive close scrutiny by historians of the NS period.

Martin Kneser took a serious interest in the history of mathematics and he encouraged younger people who were interested in the field. Kneser was no doubt partly influenced by his own personal experiences. Not only was he the son-in-law of Helmut Hasse but he was also the son of a distinguished mathematician, Helmuth Kneser, and the grandson of another, Adolf Kneser. All three made important contributions to quite different fields of mathematics. The eldest Kneser, to whom his grandson bore a striking physical resemblance, was best known for his work on the calculus of variations. He also wrote a very interesting historical account of the principle of least action, *Das Prinzip der kleinsten Wirkung von Leibniz bis zur Gegenwart* (1928), an important source for (Schramm 1985).

Helmuth Kneser, who taught in Tübingen after 1937, was, like his father, a very versatile mathematician. He did important work in topology, which included his discovery that Max Dehn's original argument for Dehn's lemma was inadequate. His son later speculated that this roadblock may well have played a role in his father's decision to take up other areas of mathematics after 1930. Martin Kneser took his doctorate in Berlin in 1950 under Erhard Schmidt, one of Hilbert's most distinguished pupils. The youngest Kneser's most important research was in algebra, particularly the theory of quadratic forms. In 1963, he was appointed to a professorship in Göttingen, where he remained till his retirement in 1993. Martin Kneser died in 2004 but he saw to it that all three Nachlässe, his own and those of his father and grandfather, found their place in the Central Archive for German Mathematics Bequests at the SUB.

For the immediate post-war period, the Nachlass of Franz Rellich surely deserves close scrutiny. These papers contain 38 letters from Courant to Rellich, written between 1945 and 1955 (the year of Rellich's death). He was then only 49 years of age. An Austrian by birth, Rellich was one of Courant's close protégés during the early 1930s. After 1933, he lost his position in Göttingen but eventually found a job in Dresden. In 1946, he returned to Göttingen, where he was appointed head of the institute. Courant held out high hopes that he and Rellich would be able to form a new Göttingen-NYU alliance in the future, particularly after Siegel rejoined the Göttingen faculty in 1951. Siegel officially retired in 1961 but continued to teach afterward. A portion of his Nachlass is also available in the collection at the SUB. After 1955, Courant's ties with Göttingen gradually weakened, even though he remained in close contact with Brigitte Rellich, Franz's widow. He had the greatest admiration for Siegel but rightly regarded him as strictly a pure mathematician in the mould of Hilbert. In his obituary for Rellich (Courant 1957), Courant repeated his familiar warning that mathematicians in general needed to reorient their research toward applications in the sciences.

These brief remarks obviously only scratch the surface of a very large subject. Hopefully they at least give a small idea of the rich resource materials held in the SUB's Department of Special Collections. Since I have not even mentioned several important Nachlässe in Göttingen, interested readers should consult the partial list compiled in the appendix below. A full list can be found by going to the webpage of the Central Archive for German Mathematics Bequests, which is linked with the Göttingen archive. Appointments are needed to order materials from the Department of Special Collections so it is always advisable to make these arrangements beforehand. Speaking from personal experience from over many years, this institution offers a most pleasant working atmosphere, supported by very helpful and friendly staff. Serious archival research, on the other hand, requires much time and patience, which are scarce commodities in today's world. Thankfully, plans are afoot to digitise some of the archival holdings in the Nachlässe at the SUB. One can only hope this will be possible so that future researchers can find it easier to explore this vast collection of documentary material, which will surely bring new insights for the history of mathematics.

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David E. Rowe has recently retired as professor of history of mathematics at Mainz University. He is the author/editor of over 100 books and articles and since 2002 has edited the column "Years Ago" for The Mathematical Intelligencer. *He is presently preparing a sourcebook documenting the life and writings of Otto Blumenthal, who was the managing editor of* Mathematische Annalen *for over 30 years.*

Appendix
A Partial List of Nachlässe in the SUIB

Artin, Emil [1898–1962]
Bernstein, Felix [1878–1956]
Bieberbach, Ludwig [1886–1982]
Brauer, Richard [1901–1977]
Cantor, Georg [1845–1918]
Dedekind, Richard [1831–1916]
Gaier, Dieter [1928–2002]
Gauß, Carl Friedrich [1777–1855]
Gerardy, Theo [1908–1986]
Grötzsch, Herbert [1902–1993]
Hasse, Helmut [1898–1979]
Hecke, Erich [1887–1947]
Heegner, Kurt [1893–1965]
Herglotz, Gustav [1881–1953]
Hilbert, David [1862–1943]
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Club de Mathématiques Discrètes Lyon Discrete Math Circle

Bodo Lass (Université de Lyon, France)

History

In 2002, Laurent Demonet, Jean-Christophe Novelli and Dimitri Zvonkine suggested that it would be nice to have the International (Russian) Tournament of Towns organised not only in Paris (and all the Russian and the many Canadian, German and Australian towns) but also in Lyon. One of the first participants in Lyon was Benjamin Scellier, a high school student living near Grenoble. He thought that we should not only have two very beautiful mathematics competitions every year but also regular mathematical club meetings – and the "Club de mathématiques discrètes"

http://math.univ-lyon1.fr/~lass/club.html

was born, quickly arriving at its actual format: two Sundays every month from 11:30 to 18:30 and three entire weeks during the holidays. The club has become attractive for more and more high school students. It started with a single student (Benjamin Scellier) and is now around 60 (15 of whom are girls) in two groups: beginners level and advanced level. Almost half of them are now from the Lyon area but most students still come from far away: Grenoble, Montpellier, Clermont-Ferrand, Dijon, Genève and, last but not least, Paris.

Mathematical Instruction

We start from high school mathematics and make excursions into all kinds of subjects. One dimensional complex (Möbius transformations) and two dimensional real projective geometries are very popular in the club. One of our favourite results is the fact that if we look at the $a \times b$ intersection points of curves of degree a and b and if we know that $a' \times b$ of them lie on a curve of degree a' then the other $a'' \times b$ of them lie on a curve of degree a'', $a' + a'' = a$. In analysis, we mainly study inequalities (Jensen, Hölder, Chebyshev, Schur, Newton and Maclaurin) and the method of Lagrange multipliers, etc. In number theory, it is the Chinese remainder theorem, Hensel's lemma, Lucas' theorem, quadratic and cubic residues, primitive roots, Pell's equation, cyclotomic polynomials (their irreducibility and arithmetic properties), etc. We study functional equations and polynomials (uniqueness of decomposition into irreducible forms in one or several variables) and, last but not least, even some combinatorics: classical graph theory (theorems of König, Hall, Menger, Mader, Gallai-Milgram, etc.) and some enumerative combinatorics. A particular highlight was a Winter School in 2009, financed by the ENS Lyon, with excursions into linear programming, game theory, hyperbolic geometry and dynamical systems.

Mathematical Competitions

The aim of the club is not to prepare a competition but to do beautiful, elegant, amusing, efficient, profound, important and passionate mathematics. There are participants who do not like competitions and it is important that there is no selection in the club: every high school student who likes doing mathematics (and who wants to ask me a mathematical question per week by email) is always welcome and is free to choose their group (beginners or advanced).

Nevertheless, most participants of the club take part in mathematical competitions like the International Mathematical Olympiad (IMO) but this is based on their individual activities and totally independent of the club. Benjamin Scellier got a lot of publicity in the Grenoble area; another very talented high school student from Grenoble, Jean-François Martin, continued this and, in 2010, half of the French IMO team were students from Grenoble's International High School. The IMO is not too far away from the Tournament of Towns but the problems of the Tournament of Towns are even more beautiful, suggested by Russian mathematicians like Kontsevich, Razborov, etc. Therefore, the IMO entered naturally into the club's life. France always has to send a team of six high school students to this competition and the Lyon maths club has become more and more involved with it. In 2007, only one of the six French IMO team members (Benjamin Scellier) came from the club but he won a gold medal and 37% of the points of the French team. In 2008, the Lyon club provided three of the six team members and received 52% of the French points. In 2009, it was four of the six team members and 70% of the points and in 2010 we had five of the six team members and 95% of the French points. Last year, all six French IMO participants came from the Lyon maths club and the French team got the best result of Western Europe (and the 14th worldwide). This Summer, however, 'only' five students from our club were selected for the French team but each of them won 17–28 points, whereas the sixth student got 11 points. Moreover, in 2015 and 2016 all the French girls selected for the EGMO (European Girls' Mathematical Olympiad) were participants of our "club de mathématiques discrètes". In particular, Lucie Wang won a gold medal.

Teachers

The most regular teachers in the club are Gabriel Dospinescu (CNRS and ENS Lyon), Theresia Eisenkölbl (Lyon University) and Bodo Lass (CNRS and Lyon University). Moreover, many others have helped: Marc Pauly, Pierre Dehornoy, Alexander Thomas, Petru

Mironescu, Xiaolin Zeng, Nathanaël Courant, Hsueh-Yung Lin, Jean-François Martin, Jill-Jênn Vie, Daniel El-Baz, Marko Radovanović, Antoine Michon, Matthew Fitch, Victor Vermès, Yassine Hamdi, Julien Alamelle, Christophe Barré and Christine Fenoglio.

Acknowledgements
The LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the programme "Investissements d'Avenir" (ANR-11-IDEX-0007) and operated by the French National Research Agency (ANR), helps disadvantaged students to participate in the club during the holidays.

Bodo Lass studied mathematics, computer science and operations research in Bonn and Aachen. He has been enamoured with combinatorics since the German preparation of the IMOs in China and Sweden (where he won silver medals) and he loves set functions (learned from András Frank). He prepared his thesis with Eberhard Triesch and Dominique Foata and received his PhD in Strasbourg. He is now a Chargé de Recherche at CNRS.

Moscow University Maths Department for Schoolchildren

Alexander V. Begunts and Anton E. Pankratiev (Lomonosov Moscow State University, Russia)

Minor Department of Mechanics and Mathematics of Lomonosov Moscow State University (Malyi Mechmath)

The Minor Department of Mechanics and Mathematics ("Malyi Mechmath" – "Junior Maths Department") comprises study groups for motivated schoolchildren keen on mathematics. It is hosted by the Department of Mechanics and Mathematics of Lomonosov Moscow State University and was founded in the late 1970s through the initiative of some young mathematicians (and later professors) S.B. Gashkov, I.N. Sergeev and Ya.V. Tatarinov, who managed to unite independent evening classes for residents and distance learning courses. Now, the resident classes pertain to the Evening Division and the distance learning courses have been reorganised into the Correspondence Division of Minor Mechmath.

Originally, the Correspondence Division of Minor Mechmath worked via the postal system. Students received printed material containing theoretical background on a subject, examples of problem solutions and a set of problems to be solved on their own. Nowadays, this form of studying employs modern educational technology and attracts schoolchildren from all over the world, including countries such as Germany and the United States.

Resident classes are held on a weekly basis (on Saturdays) from late September to the middle of May. They are oriented at schoolchildren aged from 8 to 18 and involve over 3,000 attendees of various mathematical skill levels. Attendance is free of charge and there are no entrance exams, nor are there any other constraints for participation in the classes. Such massive involvement and availability is due to the enthusiastic and thorough work of undergraduates and postgraduates of MSU under the supervision of highly experienced tutors. The basic principles of Minor Mechmath are an individual approach for every attendee, an activation of their creativity, a stimulation of their interest in mathematics and an encouragement of independent thinking.

Attendees of the classes are offered interesting, non-standard problems in various fields of elementary mathematics. They learn to conduct rigorous logical reasoning and come to understand and feel the beauty and harmony of mathematics.

The lesson topics are quite diverse and they have little in common with the standard mathematical programmes of secondary schools. Subjects of the lessons include parity, divisibility, mathematical induction, backward reasoning, the principle of the extreme element, colourings, cuts and tessellations, graphs, combinatorics, compass and ruler geometry, decomposition-equal figures, motions of the plane, metrics, elementary topology, probability, the pigeonhole principle, inequalities, centre of mass, invariants, games, logic problems, pouring and weighting problems and amounts of information. Topics of the lessons are intentionally chosen so as not to rely upon previously learnt material; therefore, missing one lesson or several lessons does not hinder attendees from understanding ideas discussed at future lessons.

Most of the participants of Minor Mechmath are 11 to 14 years old, which corresponds to 5th-8th grade. Work with this age category of children is organised as follows. Attendees of each grade are grouped into a so-called parallel. This means that all children are offered the same set of problems and the classes are conducted in several rooms (each room admits up to 30 attendees taught by 3–5 tutors). Schoolchildren individually discuss their ideas and approaches to problems with the tutors.

Thereby, schoolchildren learn to substantiate and defend their points of view while their arguments are being tested for consistency by the tutors. Also, they learn to look fairly at their mistakes and make due corrections. In turn, the form of a private dialogue allows the tutors to suggest hints and cast a light on possible routes to a solution.

An important form of popularisation activity at Minor Mechmath is a series of lectures in mathematics, mechanics and other related disciplines given by outstanding scientists who work at various departments of Moscow University. They present, in a comprehensible way, the most brilliant modern results and illustrate areas of actual scientific research.

Studying at Minor Mechmath effectively reveals the mathematical abilities of children and allows them to continue their education at specialised schools offering enhanced programmes in mathematics.

Mechmath classes at Moscow School 54

In 1988, the Department of Mechanics and Mathematics of Lomonosov Moscow State University launched a new form of educational activity. In collaboration with Secondary School 54 (Moscow), experienced scientists of the department started teaching schoolchildren an enhanced course of mathematics, physics and programming. The main idea was to raise the general level of Moscow schoolchildren who intended to continue their studies at Moscow University, to enlighten them on contemporary scientific achievements and involve them in actual research projects. The new educational approach was based on the experience of Minor Mechmath, Moscow Mathematical Olympiads and other intellectual competitions. Tutorials involved scientists of the department elaborating on fundamental school disciplines. The collaboration was backed by an agreement between the department and the school. The school formed new classes and the department selected candidates from all over Moscow.

The declared aim of the classes is a comprehensive development of each pupil's personality, a widening of the scope of their interests and an improving of their mathematical culture. The four-year course (8th-11th grade) empowers pupils and enables them to compete and win various top-level intellectual competitions, as well as successfully pass enrolment exams at all high-rank universities.

The style of presentation and training of maths in the classes is highly original; it has nothing in common with conventional and routine teaching, which reduces to solving particular types of standard problem. The main tool of teaching is "the method of guided insights": the teachers offer a series of auxiliary problems, which guide the students and prepare them to unveil fundamental mathematical results themselves. Thereby, substantial mathematical facts appear to be discovered by children themselves rather than being obtruded by the teacher. This approach takes considerable time compared to the standard methods of teaching but it is worth the effort, as students gain indispensable experience of scientific research.

A distinctive feature of the academic programme practised in the classes is the idea of the educational spiral. According to this idea, fundamental concepts and facts appear throughout the entire course several times, each time at a higher level of comprehension. Owing to this approach, the students are permanently surrounded by all of the essential mathematical topics. This is in perfect agreement with recent achievements in the area of teenager psychology: a teenager as an individual is likely to put their surrounding reality to the test and to rediscover facts they are already aware of. And as soon as the facts have been re-established, they fit well into the pupil's current worldview.

The fact that the main disciplines in the classes are taught by actively working scientists encourages pupils to work hard and invest in mathematics. Moreover, children are offered additional, complicated research problems to work on, under the supervision of professional mathematicians. Every year, research projects are presented at the annual school conference, which gathers schoolchildren of all grades. Pupils present their projects to their schoolmates and teachers, thereby improving their presentation skills. In turn, the audience gets a chance to learn about many additional areas of elementary mathematics that usually fall outside the scope of the academic programme.

However, along with the attention given to teaching mathematics, physics and programming, the main educational efforts are focused on harmonious development of the children's personalities. Traditionally, schoolchildren go hiking and visit picturesque regions of the country. They also enjoy sightseeing in historical places of Russia and other European countries. Children go to theatres, exhibitions, museums and keep up with the cultural life of Moscow.

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Anton Evgenievich Pankratiev [anton.pankratiev@gmail.com] is an associate professor at Lomonosov Moscow State University, where he received his PhD in mathematics in 2001. The area of his interests includes algebra, computer science and information security. He is also involved in working with high school students and helping to organise various Olympiads. He is the head of "Malyi Mechmath".

Mathematics in Kolmogorov's School

Nikolay Salnikov and Kirill Semenov (Moscow State Lomonosov University, Russia)

More and more activities aimed at high school students are being organised by universities in Europe and all over the world. The reason of this trend is twofold: to raise interest in scientific education and to increase the level of students eventually passing entrance exams or other forms of selection to enter higher educational institutions. Russian experience of more than 50 years can be useful to create an operational model of a specialised school connected with a higher educational institution.

The first school specialising in mathematics and physics was created in 1963 in collaboration with Moscow State University through the initiative of the world famous mathematician and member of the Soviet Academy of Sciences, Andrei Kolmogorov. Almost at the same time, similar schools were created in Novosibirsk and then in Leningrad, Kiev and Alma-Ata, and later – right before the dissolution of the Soviet Union – in Minsk and Sverdlovsk. Together with Kolmogorov, the initial idea belonged to academician Isaak Kikoin, Rector of Moscow University Ivan Petrovskiy and President of SAS Mstislav Keldysh.

The creation of specialised high schools at prominent universities has allowed the training of students of higher grades who have shown interest in mathematics and physics; professors from these universities have participated in the development of the programmes for the students and the realisation of the programmes. On top of that, students of such schools have had access to scientific research at a very early stage of their "careers". University professors have acted as their scientific advisors – this approach has been applied since then and it has produced promising results.

Traditionally, these specialised schools welcome students who pass entrance examinations after the 9th grade (at an age of 15–16). These students are selected from all over the country, with the possibility of being lodged at the school's dormitory. At this age, they already have some ideas about their professional orientation and specific talents start showing up – one just needs to help the students acquire the necessary level of specialised knowledge. They are also autonomous enough to be able to live away from their families. So, another important aim has been in allowing them to socialise with other talented teenagers – in such schools they are surrounded by their peers. They not only study together but also interact in their spare time. In a specialised school, all students are offered advanced learning programmes, complicated optional and special courses, and projects and research activities but also various leisure activities coherent with their high intellectual potential.

In 1988, the school at MSU was transformed into the Advanced Education and Science Centre (AESC) – Kolmogorov's boarding school – which obtained the status of a department of MSU. This was a unique pedagogical experiment aimed at carrying out concrete work with talented children – a practical realisation of the ideas of Kolmogorov about selection, instruction and upbringing of gifted students.

Nowadays, AESC MSU works with students from the 10th and 11th grades (the last two grades in the Russian system), specialising in physics and mathematics, information technology, chemistry and biology. Regardless of the chosen specialisation, a very high level of mathematical education is maintained.

The school consists of six departments, five of them related to the specialisations of mathematics, physics, chemistry, informatics and biology; there is also a department of humanities. The classes follow the university scheme: lectures and exercise classes, optional and special courses, and exam sessions at the end of each semester – precisely like first and second year university students.

Concerning mathematics, three disciplines are taught at the school: mathematical analysis (calculus), geometry and algebra. The programmes of these disciplines are not precisely fixed; they reflect the personal tastes and experiences of each lecturer. The course of mathematical analysis is traditional and resembles a first year at university. The general goals of the geometry course are rather close to a high school programme, studying the properties of geometric shapes on a plane and in space but using more advanced techniques and results. The course of algebra is supposed to establish the foundations necessary for related disciplines. The advanced level means solving more complicated problems, considering some chapters not presented in classical programmes and addressing various topics usually discussed at university. All this arouses an interest for mathematics in general and its applications to other fields. It should also provide a basis for applying mathematical and computer modelling in science.

Andrei Kolmogorov with students at his school.

An important supplement to general mathematics courses is the programme of mathematical practicum. Within this framework, students solve computational and constructive geometric problems and conduct data analysis in order to classify mathematical objects and their properties. Sometimes this leads to the possibility of realising a project and even getting engaged in research activities; in these cases, the university professors act as scientific advisors. Results of such projects are then presented at various contests and even scientific conferences; sometimes they give rise to serious publications.

Let us stress again that special courses are delivered by university professors and actual researchers. This provides students with up-to-date information on recent scientific advances and allows them to make well-founded choices in deciding upon their own directions of research and education.

Amongst recent trends, let us mention that professors from AESC MSU also participate in educational projects for other students. Various evening, weekend and Summer courses are held and the distance learning platform is up and running. The school organises internet Olympiads, the "Mathematical multiathlon" tournament[1] and the "Kolmogorov's readings" conference.[2] This allows future students of AESC MSU and MSU to be addressed directly (including students of lower ages) whilst not forcing them to change their usual school and family atmosphere. Within the framework of these activities, the profile subjects are taught clearly with an important emphasis on mathematics.

To conclude, let us stress again that during the years of its existence, the pedagogical team of Kolomogorov's school have obtained enormous experience of working with talented and highly motivated students. This experience may be interesting for those looking to start similar initiatives.

[1] http://internat.msu.ru/doc/mathematical-multiathlon-2015-en.pdf.
[2] http://internat.msu.ru/16th-kolmogorov-readings/.

Nikolay Salnikov [nikolay.m.salnikov@gmail.com]graduated from Kolmogorov's school in 1977 and studied at the Department of Mechanics and Mathematics of Moscow State University. For more than 15 years, he taught applied mathematics to students of that department as well as those specialising in information technology and economics. He continued as principal at a high school in Moscow and managed its transformation to a specialised lyceum. Currently he is Vice-Principal for Development at AESC MSU.

Kirill Semenov [ksemen@mech.math.msu.su] graduated from AESC MSU (Kolmogorov's school) in 1990 and continued at the Department of Mechanics and Mathematics of Moscow State University. He has been working at the department as an assistant and then associate professor for nearly 20 years. Recently, he has taken on the responsibility of Principal of AESC MSU.

Ludus' 10th Anniversary

Jorge Nuno Silva (Ludus & University of Lisbon, Portugal)

It is not unusual for the Museu Nacional de História Natural e da Ciência (MUHNAC) to host open extension events on a Saturday. On 25 June 2016, however, the reason for the festivities was new: Associação Ludus (AL)[1] was celebrating its 10th anniversary!

Along one of the avenues leading to Jardim Botânico, several tables were prepared for games. Besides the Rubik's cube and origami, one could find traditional games, such as Mahjong, but the boards were mainly intended for abstract (i.e. mathematical) games. Visitors could play Hex, Amazons, Dots & Boxes, Traffic Lights, etc. These, among others, can be found at the National Championship of Mathematical Games,[2] a competition that has been attracting over 100,000 young people each year since 2004. In these games, luck plays no part. To beat the opponent, a player must be better at thinking with rigour and creativity.

Final of the 11th National Championship of Mathematical Games, Vila Real, 6 March 2015.

We believe that mathematics should be promoted not only by calling attention to its applications but also by emphasising what is at its core: the art and pleasure of thinking! Abstract board games are, in this way, very mathematical. As Hardy puts it in his *Apology*: "Chess problems are the hymn-tunes of mathematics."

In the crowd, one could spot some young people in orange overalls. These are members of the Mathematical Circus (MC),[3] showing their tricks as real buskers. The Circus performs regularly in schools, museums, science centres and also in the street. It is a very successful project of AL. As W.F. White has written, "Amusement is one of the fields of applied mathematics"[4] and we follow suit.

MC's repertoire includes several mathematical (i.e. self-working) card tricks. After a few years of practice and study, we launched a book on mathemagic with cards.[5] Jorge Buescu introduced the work to a very interested audience.

An early team of the Mathematical Circus.

Suddenly, *Project Bug*, a jazz band from Mafra, starts performing, its members flowing among the visitors, some of whom start dancing. People are having a very good time… Recreational mathematics and good music go very well together.

AL takes recreational mathematics very seriously. We honour Martin Gardner every year around 21 October, being part of the global Celebration of Mind event.[6] On odd numbered years, we organise the Recreational Mathematics Colloquia – Gathering for Gardner, Europe.[7] Of course, on even numbered years, the leading event happens in Atlanta, USA.

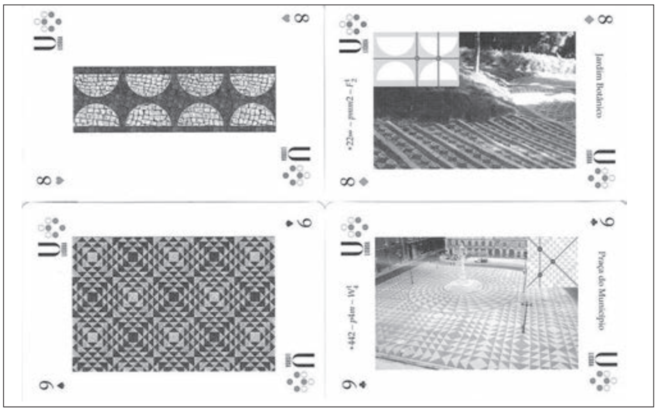
Poster of the Recreational Mathematics Colloquium V.

1 http://ludicum.org.
2 http://ludicum.org/cnjm.
3 https://www.facebook.com/CircoMatematico/.
4 In *A scrap book of elementary mathematics*.
5 Silva, A., Freitas, P., Silva, J. N., Hirth, T., Matemagia. Ludus 2016, ISBN 978-989973468-5.
6 http://www.celebrationofmind.org/wordpress_site/.
7 http://ludicum.org/ev/rm.

AL publishes an online journal on this subject, the *Recreational Mathematics Magazine*.[8] We invite you to browse through its articles and we are sure you will not be disappointed.

The mathematics of combinatorial games is particularly interesting and an active area of research. Every other year, AL organises an international meeting, the Combinatorial Game Theory Colloquia,[9] where most of the leading experts gather together.

Games in general are mysterious and universal; their history and culture is also a subject that deserves increasing academic attention. AL promotes the online publication of the prestigious journal *Board Game Studies*.[10]



A deck of cards with 26 puzzles based on all the symmetries of the traditional Portuguese pavement, and 26 matching answers. A special card (a mirror) helps in figuring out the nature of each frieze/pattern/rosace. (Published by Ludus in 2015.)

[8] http://www.degruyter.com/view/j/rmm.
[9] http://cgtc.eu/.
[10] http://www.degruyter.com/view/j/bgs.

From the early hours, a pig was being prepared and roasted, together with many other delicacies (Portuguese wine included).

The meal and the cake with anniversary candles had to wait. It was time for the conference and Henrique Leitão's '*The mathematics of the calendar*'. We all enjoyed learning about the complexities of calendrical calculations. Sophisticated mathematics insinuated itself into this practical/religious topic. The festivities were at a high point!

After the talk came the meal. It was worth waiting for the delicious roasted boar. People shared ideas about the activities of the day while enjoying a nice dinner.

Last but not least, we had a live broadcast of the Croatia-Portugal football match (UEFA Euro championships, last 16 knockout stage). Portugal won and moved on to become European champions.

A perfect day!

Jorge Nuno Silva [jnsilva@cal.berkeley.edu] is a professor at the Department of History and Philosophy of Science, University of Lisbon. He received his PhD in mathematics in 1994 (UC Berkeley). His interests include mathematics, history of mathematics, board games and their history, and magic. He is President of Associação Ludus. *He is married to Galina and has two children, Manuel and Laura. A cat, Pantera, and a dog, Proust, complete the family.*

e-Math Workshops: a Forum for Exchanging Experiences of Mathematics e-Learning at University Level

Antonio F. Costa (UNED, Madrid, Spain), Fernando P. da Costa (UAb, Lisbon, Portugal), Maria Antonia Huertas (UOC, Barcelona, Spain)

We report here about a series of international workshops on e-learning of mathematics at university level, which have been jointly organised by the three publicly funded open universities in the Iberian Peninsula and which have taken place annually since 2009. The history, achievements and prospects for the future of this initiative will be addressed.

Introduction

Distance learning already has a long and remarkably successful history in enabling access to higher education for populations that, for whatever reasons, are not able to attend classes at a traditional institution [1].

Distance learning programmes have always been dependent on the most up-to-date communication technology of the day, from the widespread generalisation of regular mail distribution using the rail network in 19th century Europe, which was the underpinning of the "correspondence courses" phase of distance learning, to the use of radio and television capabilities in the 20th century, to the use of electronic mail at the turn of the millennium, until the present day internet based e-learning platforms for teaching online.

These technological changes in distance learning have naturally implied a corresponding change in the way materials in the various disciplines are produced and presented to students. In the case of mathematics, the ongoing pace of change in information, communication and computing technologies, with a strong impact in the newly available informatics tools, also requires a constant actualisation in technological and pedagogical issues from mathematicians teaching at universities offering online courses, either completely (e-learning) or partially (b-learning). These actualisation requirements have resulted in a number of initiatives aimed at fostering the interaction and exchange of experiences between them.

e-math workshops and online learning of mathematics in Spain and Portugal

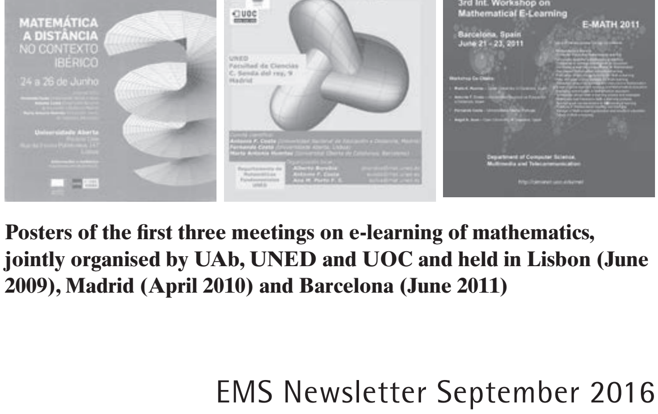
The Iberian Peninsula has three state funded universities entirely dedicated to offering online courses: in Spain the *Universidad Nacional de Educación a Distancia* (with its central site in Madrid, founded in 1972) and the *Universitat Oberta de Catalonia* (in Barcelona, founded in 1995) and in Portugal the *Universidade Aberta* (in Lisbon, founded in 1988). In spite of the close geographical proximity (about 1,000 kilometres separate Lisbon and Barcelona, with Madrid almost exactly in the middle), the fact is that close collaboration between mathematicians of our three universities only started after we met for the first time while attending the 5th JEM Workshop – "ICT on the Teaching of Mathematics and on the Mathematics Curriculum" in Paris, organised in November 2008 by the late Mika Seppälä [2].

The recognition of common challenges facing the teaching of university mathematics over the internet and the need for learning from each other's experiences, both from the successes and the failures, led us to start a regular series of meetings, jointly organised by our three universities, the first of which took place in Lisbon, at UAb, in the Summer of 2009. The success of this first meeting resulted in the organisation of the next two meetings, in Madrid (2010) and Barcelona (2011). At this stage, the name of these annual meetings finally stabilised to "e-math workshops". The cyclic nature of the organisation of the meetings has been maintained and last May the meeting achieved the middle of its third cycle with the organisation of the 8th meeting in Madrid.

Mathematical e-learning has important specific characteristics implying particular use of technology and educational strategies. Therefore, the subjects in the meetings include news on the technology of mathematical communication, innovations in mathematical e-learning strategies, assessment technology, educational data analysis, videos, MOOCS, good educational practices, student results presentations, programmes and tools for teaching mathematics and many, many more.

e-math workshops as an international maths e-learning forum

Although the origin of the e-math workshops was the desire to exchange experiences about e-learning of mathe-



Posters of the first three meetings on e-learning of mathematics, jointly organised by UAb, UNED and UOC and held in Lisbon (June 2009), Madrid (April 2010) and Barcelona (June 2011)

matics in our three Iberian distance learning universities, from the very beginning we were aware that our collaboration must not lack attention to what was going on in other educational institutions in our countries, and to the developments of mathematics e-learning in Europe and the world beyond the Pyrenees.

A reflection of this awareness is the inclusion of invited guest speakers to the workshops from outside Spain and Portugal: Hans Cuypers (Eindhoven University of Technology, the Netherlands), Sven Trenholm (Loughborough University, UK), Miryam Berezina (Ort Braude College, Israel) Kellie Grasman (Missouri University of Science and Technology, USA) and Karan Sagar (Massachusetts Institute of Technology, USA) in Barcelona, 2011; David Brannan (Open University, UK) in Lisbon, 2012; Magally Martínez Reyes and René Cruz Flores (Universidad Autónoma del Estado de México) and Olga Caprotti (Project Manager of the Thematic Network: Joining Educational Mathematics) in Madrid, 2013; and Mika Seppälä (University of Helsinki, Finland) in Barcelona 2014, by videoconference.

Another indication of the openness of the e-math workshops to the wider mathematical community is that they continue to attract participants from other institutions of both our countries that use and develop online teaching tools as an aid to their more traditional face-to-face teaching. Some of those who have presented oral communications are José Antonio Seijas (Universidade da Coruña, Spain), José Santos and Alexandre Trocado (Instituto GeoGebra de Portugal), Juan Medina (University of Cartagena, Spain), Juan Moreno García (UDIMA – Universidad a Distancia de Madrid, Spain), Laura Bujalance (Universidad Camilo José Cela, Spain), Marta Ginovart and Mónica Blanco (Polytechnic University of Catalonia, Spain), Rui Paiva (Leiria Polytechnic, Portugal) and Ruy Costa (Universidade Nova de Lisboa, Portugal). In several meetings, we have also had participation from scientists in areas such as physics, chemistry and economy (e.g. Mª Isabel Gómez del Rio, Chemistry of UNED, Madrid, 2013). And, of course, there have been many who, although not presenting, have attended the talks and contributed to the discussions. There have also been participants with interests in other educational levels, including secondary schools, adult access to university, lifelong learning and permanent teacher training, revealing the importance of e-learning methodology in teaching mathematics.

Participants at the 8th e-math workshop, Madrid (May 2016)

On another level, the e-math workshops have also attracted the interest of a technological company dedicated to mathematical e-learning activity – Wiris (Maths and More, Barcelona), with representatives present at most of our meetings and talks by Carles Aguiló and Ramón Eixarch.

Last but not least, an extended dossier on "Mathematical e-Learning", with papers directly related to the 3rd e-math workshop, has been published in the "Universities and Knowledge Society Journal" [3].

e-math impacts and future prospects

In addition to the important aspects of exchanging ideas, practices and experiences on how to teach mathematics online, there have been a number of other tangible consequences of the e-math meetings. A major one is the supervision of students for Master's degrees and PhDs. Several MSc students have worked for their dissertations under the supervision of professors of one of the other universities, using for that accompanying work the technological tools pertaining to distance learning. Another type of collaborative initiative between teaching staff of UAb, UOC and UNED has also taken place, with postgraduate (MSc and PhD) courses and lifelong learning programmes of one of the universities being taught through an online platform by a professor of another, with no need for them to leave their office. Distance learning can also serve students through "delocalisation" of instructors, who, like the students themselves, can be located anywhere in the world and still be effective in their teaching roles!

Due to the support of a number of financing sources, among them an Erasmus+ Agreement, some of the students involved in the joint supervision effort have afterwards been able to present their work at later e-maths meetings, which, in this way, have also provided future mathematicians a way to gain experience in communicating their new results.

All e-math conferences conclude in a round table about possible collaborations and, on all occasions, new ideas arise, some turning into a reality by the time of the next meeting.

In the last two e-math workshops (Porto 2015 and Madrid 2016), we have been happy to have the organisational support of the Real Sociedad Matemática Española, the Societat Catalana de Matemàtiques and the Sociedade Portuguesa de Matemática, who all provided remarkable help in spreading information about the workshops to their associates and to the Iberian public at large. To bring e-math to the attention of a wider European constituency is one of the goals of this paper, and the natural role of the European Mathematical Society in supporting this objective is, we believe, invaluable.

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forschung der Carl von Ossietzky Universitat Oldenburg, Band 11, BIS-Verlag der Carl von Ossietzky Universitat Oldenburg, Oldenburg, 2005.
[2] JEM-Joining Educational Mathematics: eContentPlus Thematic Network: http://webalt.mobi/jem/index.html.
[3] Juan, Ángel A.; Huertas, María Antonia; Cuypers, Hans; Loch, Birgit eds. (January 2012). "Mathematical e-learning". *Universities and Knowledge Society Journal (RUSC)*. Vol. 9, No. 1, ISSN 1698-580X. http://dx.doi.org/10.7238/rusc.v9i1.1431.

Annex: internet addresses of past e-math workshops

4th e-math, Lisbon 2012:
https://sites.google.com/site/4emath2012/

5th e-math, Madrid 2013:
https://sites.google.com/site/5emath2013madrid/

6th e-math, Barcelona 2014:
http://www.uoc.edu/portal/en/symposia/emath_2014/

7th e-math, Porto 2015:
https://sites.google.com/site/7themath2015/

8th e-math, Madrid 2016:
https://sites.google.com/site/8emath2016madrid/

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Fernando P. da Costa [fcosta@uab.pt] teaches at the Department of Sciences and Technology of Universidade Aberta, Lisbon, and is a researcher at the Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, University of Lisbon, Portugal. His research interests are analysis and differential equations, particularly dynamical aspects. He received his PhD at Heriot-Watt University, Edinburgh, in 1993. He was Vice-President of the Portuguese Mathematical Society from 2012 to 2014 and President from September 2014 until August 2016.

Maria Antonia Huertas [mhuertass@uoc.edu] teaches at the Department of Computer Science of Universitat Oberta de Catalunya, Barcelona. She is currently leading the Technology-Enhanced Knowledge and Interaction Group (TEKING) at the e-learn centre (UOC) in Barcelona, Spain. Her research interests include logic, knowledge representation, technology enhanced learning and mathematical e-learning. She is organiser of the annual workshops on Technology-Enhanced Assessment, Analytics and Feedback (TEAAF) and takes part in several interest groups on mathematical education.

New zbMATH Atom Feed

Fabian Müller (FIZ Karlsruhe, Berlin, Germany)

zbMATH has recently launched a new way of pursuing its aim of keeping researchers up-to-date with current developments in their areas of interest. Whilst the traditional usage of the printed Zentralblatt edition – browsing the huge yellow volumes in search of papers and reviews in your area of interest – has been declining for a long time, resulting in the discontinuation of the print version in 2010, there is now the equivalent option of getting an update of newly included items and reviews in electronic form every two weeks. The access is managed through an Atom Feed, a web feed format that is an enhanced version of the classic RSS format and is supported by all modern news readers. By entering a URL like 'https://zbmath.org/atom/cc/14' in your browser or any other news reader program, you can easily find the articles that were added in the most recent volume, in this case for MSC field 14 (Algebraic Geometry). Each entry displays (depending on your news reader) the title, the authors and a review or summary. Included is a link to the zbMATH article page containing further infor-

Atom feed replaces tedious browsing through huge volume rows.

mation like MSC classification, references and full text links.

At present, the news feed can be accessed via two facets: author and MSC. To keep up-to-date with the articles written by your favourite author you can use the author ID, which is displayed at the top of every author profile. The Atom Feed URL for an author then has the form 'https://zbmath.org/atom/ai/gray.jeremy-j', where gray.jeremy-j is the author ID (in this case belonging to the recent winner of the EMS Otto Neugebauer Prize).

The MSC facet is accessible in various levels of granularity, which can be arbitrarily mixed. Thus, for example, if your field of specialty is quantum theory, you might be interested in the whole of MSC 81 (Quantum theory) as well as the more finely grained 17B (Lie algebras and Lie superalgebras) and 57R56 (Topological quantum field theories). The corresponding feed URL can then be very easily constructed as simply

https://zbmath.org/atom/cc/81,17B,57R56.

We think that this service will become a valuable tool in your research. If you have any feedback or suggestions for improvement, please do not hesitate to contact us.

Fabian Müller [fabian.mueller@fiz-karlsruhe.de] studied mathematics and computer science at Humboldt-Universität, Berlin. After finishing his doctoral studies in algebraic geometry in 2013, he started working at zbMATH, where he is responsible for coordinating IT development efforts.

MSC2020 – Announcement of the Plan to Revise the Mathematics Subject Classification

Edward G. Dunne (American Mathematical Society, Ann Arbor, USA) and Klaus Hulek (Gottfried Wilhelm Leibniz Universität Hannover, Germany)

Mathematical Reviews (MR) and *zbMATH* cooperate in maintaining the Mathematics Subject Classification (MSC), which is used by these reviewing services, as well as publishers and others, to categorise items in the mathematical sciences literature. The current version, MSC2010, consists of 63 areas classified with two digits, refined into over 5000 three- and five-digit classifications. Details of MSC2010 can be found at www.msc2010.org, www.ams.org/msc/msc2010.html or zbmath.org/classification/.

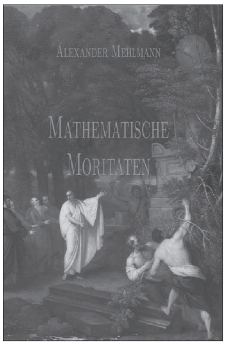
MSC2010 was a revision of the 2000 subject classification scheme developed through the collaborative efforts of the editors of zbMATH and MR, with considerable input from the community. zbMATH and MR have initiated the process of revising MSC2010, with an expectation that the revision will be used beginning in 2020. From the perspective of MR and zbMATH, the five-digit classification scheme is an extremely important device that allows editors and reviewers to process the literature. Users of the publications of zbMATH and MR employ the MSC to search the literature by subject area. In the decade since the last revision, keyword searching has become increasingly prevalent, with remarkable improvements in searchable databases. Yet, the classification scheme remains important. Many publishers use the subject classes at either the time of submission of an article, as an aid to editors, or at the time of publication, as an aid to readers. The arXiv uses author-supplied MSC codes to classify submissions and as an option in creating alerts for the daily listings. Browsing the MR or zbMATH database using a two- or three-digit classification search is an effective method of keeping up-to-date with research in specific areas.

Based in part on some thoughtful suggestions from members of the community, the editors of MR and zbMATH have given preliminary consideration to the scope of the revision of the MSC. We do not foresee any changes at the two-digit level; however, it is anticipated that there will be refinement of the three- and five-digit levels.

At this point, zbMATH and MR welcome additional community input into the process. Comments can be submitted through the Web at www.msc2020.org. You may also send email to feedback@msc2020.org. All information about the MSC revision is jointly shared by MR and zbMATH. This input will be of great value as the process moves forward.

Edward Dunne, Executive Editor,
Mathematical Reviews
Klaus Hulek, Editor-in-Chief,
zbMATH

Book Reviews



Alexander Mehlmann

Mathematische Moritaten

CreateSpace Independent
Publishing Platform, 2016
188 p.
ISBN 978-1519708434

Reviewer: Jean-Paul Allouche

Alexander Mehlmann is a now retired mathematician working inter alia in Game Theory, who is also interested in the theme Mathematics and Literature. In the book *Mathematische Moritaten*" (Mathematical Ballades) A. Mehlmann proposes poems that are either about mathematics or have a mathematical construction. The book has five chapters, entitled "Die Magier der Zahlen", "Apokryphe Parabeln", "Schattenwörter", "Pamphlete und Poeme", and "Translationen", i.e., "The Magicians of Numbers", "Apocryphal Parables", "Shadow Words", "Pamphlets and Poems", and a last chapter consisting of translations [in English and in Romanian] of some of these poems.

Except for the last chapter the poems are in German, but even if one knows only a few words in German, one can hear the music and rhythm of these nice texts. Here is an example:

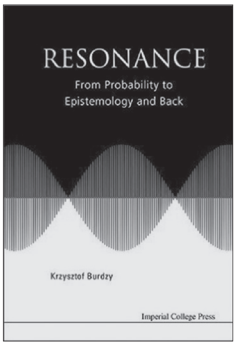
Gelehrten-Kalender

Gelehrten Kalender
Gehenkter Ländler.

Generell Kathedern
Entgehen! Klar edler
Kegelnde Altherren
Ernten Kahle Gelder.

Helden; Kartenleger
Erhellter Gedanken.

The reader might have noted that each verse is an anagram of the title, but also that only two vowels were used. A classical question for art with constraints (in particular literature with constraints, as the works of the famous Oulipo – these contraints are often of mathematical nature) asks whether the result is a piece of art due to constraints or despite constraints. The answer might be more complicated, since the artist is sometimes tempted (willingly or unconsciously) to bypass the rules. As Paul Klee said: "Das Genie ist der Fehler im System" ("Genius is the error in the system")…



Krzysztof Burdzy

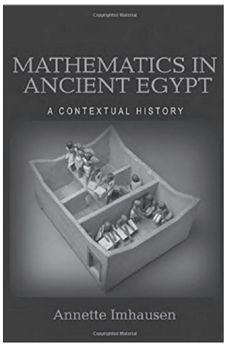
Resonance – From Probability to Epistemoloy and Back

Imperial College Press, 2016
428 p.
ISBN 978-1783269204

Reviewer: Jean-Paul Allouche

It seems that only a few mathematicians really get involved with epistemological questions about mathematics or their own fields of research. On the other hand, in the particular case of probability theory, questions often arise about the "real" meaning of certain concepts or about seemingly paradoxical statements. The author of the book under review tries to analyse the foundations of probability theory, starting from the most well known "philosophical" theories of probability, namely *frequency theory* and *subjective theory*. The former (very roughly speaking) identifies probability and long-run frequency, while the latter (again very roughly speaking) claims that probability is personal opinion. In both theories, the question is what the probability of a single event means. Furthermore, the author confronts these two approaches with the two main branches of statistics, namely frequency statistics and Bayesian statistics.

It is "probably" not possible to say more about this book in a few lines without betraying its contents. The real way of understanding all the issues discussed is to read and re-read the author's explanations. This would certainly be useful even for mathematicians for whom probability theory is just a branch of measure theory. In particular, this might lead them to make a distinction between measure and probability; for example, it would be interesting to see their reaction to Axiom L6 in Burdzy's book: *An event has probability 0 if and only if it cannot occur. An event has probability 1 if and only if it must occur.*



Annette Imhausen

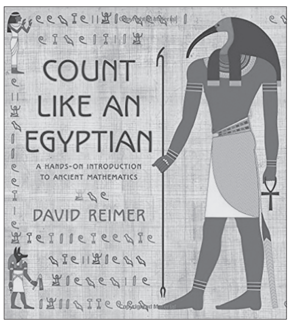
Mathematics in Ancient Egypt
A Contextual History

Princeton University Press, 2016
245 p.
ISBN 978-0-691-11713-3

Reviewer: Victor Pambuccian

The Newsletter thanks zbMATH and Victor Pambuccian for the permission to republish this review, originally appeared as Zbl 1336.01010.

This is the third book to appear within a short period of time on the subject of Egyptian mathematics. Surprisingly enough, there is hardly any overlap with the other two (M. Michel, *Les mathématiques de l'Egypte ancienne. Numération, métrologie, arithmétique, géométrie et autres problèmes*. Bruxelles: Éditions Safran (2014; Zbl 1314.01001)) and (D. Reimer, *Count like an Egyptian. A hands-on introduction to ancient mathematics.* Princeton, NJ: Princeton University Press (2014; Zbl 1296.01002)). The reason for this is the difference in emphasis. The current book's emphasis is rather on Egyptian society throughout its ancient history and the position of mathematics and of scribes inside that society. While most histories of Egyptian mathematics focus on the few extant mathematical texts, chiefly the Rhind papyrus, the Moscow papyrus, the mathematical leather roll, and perhaps the Lahun mathematical fragments, the description of their content is dealt with in this book in the space of less than forty pages. A mathematical intricacy, such as the *mathematical* reason for choosing the Egyptian fraction decomposition of the famous $2/n$ table for values up to 101, is considered to be an ill-posed problem, for it is likely that "the table is a result of experience and (presumably) trial and error rather than a systematic execution of a set of rules" (p. 96). The presentation is strictly chronological. It starts with prehistory and the early dynastic period, the invention of writing and of number notation, the uses of numbers and their context in predynastic and early dynastic times. It moves on to the Old Kingdom, the documents from which a use of mathematical notions could be inferred, documents that show the presence of metrological systems, plans and guidelines for constructions that must have involved some mathematics. The most richly documented time, in particular as far as mathematics is concerned, the Middle Kingdom, is treated next, not only by presenting the extant mathematical texts, but also the practical contexts in which mathematics was used (distribution of ratios, architecture, land measurement). For the New Kingdom the reader is presented, besides two mathematical texts, Ostraca Senmut 153 and Turin 57170, with several papyri that are not usually encountered in a "mathematics in ancient Egypt" context. These are administrative or literary texts that throw some indirect light on mathematics and the (privileged) life of scribes: papyrus Harris I, papyrus Wilbour, papyrus Anastasi I and V, papyrus Lansing, papyrus Leyden 348, the *Teaching of Amenemope*, the *Duties of the Vizier*. The book ends with the Greco-Roman periods, the demotic texts, the possible interactions with Mesopotamian and Greek mathematics. Throughout the book, there is an implicit and explicit animus against mathematicians who transgress into the history of Egyptian mathematics. Regarding Michel's book [loc. cit.], the author writes that "despite the author's knowledge on recent publications on Egyptian mathematics" it "reverts to writing an assessment of Egyptian mathematics from a modern mathematician's point of view" (p. 4). The author believes that older historiography was unaware of "the insight that mathematics is not culturally independent, that it is not constantly evolving in a linear way toward the next level", an insight "gained much later" (p. 6), for which she credits *S. Unguru*'s (*Arch. Hist. Exact Sci*. 15, 67–114 (1975; Zbl 0325.01002)). This attribution and the late date is remarkable, given that the author is certainly well aware that Oswald Spengler had stated exactly that thesis, with a wealth of arguments, more than 50 years earlier. After quoting a passage from a book by the Egyptologist Kurt Sethe of 1916, in which he professes his disbelief that someone could have come up with such a system of fractions, the author states that "although it has become accepted that mathematics down to its roots (i.e. numbers) has this social dimension, Egyptian fraction reckoning has not been reassessed" (p. 53). Reimer's book [loc. cit.] is precisely the reassessment the author was looking for, which turns Egyptian fractions into tools that are, in many respects, *superior* to our own. This reassessment comes from a mathematician, but it was published too late to be considered here.



David Reimer

Count like an Egyptian
A hands-on introduction to ancient mathematics

Princeton University Press, 2014
237 p.
ISBN 978-0-691-16012-2

Reviewer: Victor Pambuccian

The Newsletter thanks zbMATH and Victor Pambuccian for the permission to republish this review, originally appeared as Zbl 1296.01002.

This is a wonderful book, very well written, filled with illustrations on every page, witty, addressing anyone interested in grade school arithmetic, including "anyone at the junior-high level and up." Its subject matter is Egyptian mathematics, for the most part Egyptian arithmetic, followed by a chapter on base-based systems of numeration and the relative advantages and disadvantages of the Egyptian system when compared to the system currently taught in schools. Far from seeing Egyptian computation as "awkward and primitive", as it is often framed by a cursory look at the extant documents, the author finds it very rational, very well adapted to its purpose, and in many respects superior to the drudgery called arithmetic in the curricula of contemporary schools. "To someone who's mastered it, Egyptian mathematics is beautiful. It scorns memorization and rote algorithms while it favors insight and creativity. Each problem is a puzzle that can be solved in many ways." (p. ix) The largest part of the book is understandably taken by explaining Egyptian fractions. And the explanation is enlightening, leaving nothing exotic or backward in the system of Egyptian fractions. There is no mystery left as to why they wouldn't add more than one copy of a unit fraction, why there is an exception for $1/3$, for which there is a symbol for $2/3$, and why the system "carefully balance[s] the two primary requirements of number systems: the ability to approximate values and the need for exact values." (pp. 206–207) Most of the book is thus dealing, indirectly, with solving the great puzzle of Egyptian fractional arithmetic, namely the question how the scribe Ahmose of the Rhind Mathematical Papyrus came to his table of decompositions of $2/n$ for odd $3 \leq n \leq 101$. Given that the book has no bibliography, and that the author succeeds in giving the impression that he came to learn Egyptian arithmetic by the Moore method, as it were, it is not clear whether he has looked into the reconstructions of the reasons behind that $2/n$-table to be found in the literature, from 1926 to 2011, in O. Neugebauer, *Die Grundlagen der ägyptischen Bruchrechnung* [Berlin: J. Springer (1926; JFM 52.0004.02)]; *Quell. Stud. Gesch. Math. B* 1, 301–380 (1930; JFM 56.0802.03); B.L. van der Waerden, *ibid*. 4, 359–382 (1938; Zbl 0019.24203); *Centaurus* 23, 259–274 (1980; Zbl 0431.01001); R.J. Gillings, *Mathematics in the time of the Pharaohs* [Cambridge, MA: The MIT Press (1972; Zbl 0491.01004)]; *Hist. Math.* 5, 221–227 (1978; Zbl 0389.01002); M. Bruckheimer and Y. Salomon, *ibid*. 4, 445–452 (1977; Zbl 0389.01001); E.M. Bruins, *Centaurus* 19, 241–251 (1975; Zbl 0338.01001); *Janus* 68, 281–297 (1981; Zbl 0478.01002); A.A. Abdulaziz, *Hist. Math*. 35, No. 1, 1–18 (2008; Zbl 1221.01008); T. Asai, *Bull. Nara Univ. Ed. Natur. Sci.* 60, No. 2, 1–13 (2011).

Solved and Unsolved Problems

Themistocles M. Rassias (National Technical University, Athens, Greece)

I am not really doing research, just trying to cultivate myself.

Alexander Grothendieck (1928–2014)

With this opportunity, I would like to express my deepest thanks to Professor Martin Raussen, who appointed me as a member of the Editorial Board of the Newsletter of the EMS in charge of the problem corner in 2005. I would also like to express my gratitude to Professor Krzysztof Ciesielski for proposing in 2004 that I write an article in the Newsletter of the EMS, which subsequently initiated my communication with Martin Raussen, with whom I have had a wonderful and productive collaboration. I note that editors generally serve for four years and I feel deeply honoured that my membership as the problem column editor has lasted for more than 10 years. Thus, I wish to express my sincere thanks to Professors Vicente Munoz and Lucia Di Vizio, who served as Editors-in-Chief after Martin Raussen; I continued to have a wonderful collaboration with them.

The preparation of this column has been very stimulating and a source of great pleasure. From the very beginning, the "Problem Corner" has appeared in two issues per year (the March and September issues) with six proposed problems and two open problems. In every subsequent issue in which the problem corner has appeared, the solutions of the previous proposed problems have appeared together with the names of additional problem-solvers. In total, 170 problems have appeared in the problem column while I have served as its editor. Mathematicians from all over the world have participated in this effort. Going through the issues of the Newsletter of the EMS, one can see problems proposed or solved by mathematicians from Australia, Canada, China, Denmark, England, Germany, Greece, Hong-Kong, Iran, Ireland, India, Italy, Poland, Portugal, Romania, Russia, Sweden, Ukraine, USA and others.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

163. Find all positive integers m and n such that the integer

$$a_{m,n} = \underbrace{2\ldots2}_{m\ \text{time}}\underbrace{5\ldots5}_{n\ \text{time}}$$

is a perfect square.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

164. Prove that every power of 2015 can be written in the form $\frac{x^2+y^2}{x-y}$, with x and y positive integers.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

165. Find the smallest positive integer k such that, for any $n \geq k$, every degree n polynomial $f(x)$ over $\mathbb{Z}$ with leading coefficient 1 must be irreducible over $\mathbb{Z}$ if $|f(x)| = 1$ has not less than $\left[\frac{n}{2}\right] + 1$ distinct integral roots.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

166. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing (f not necessarily continuous). If $f(0) > 0$ and $f(100) < 100$, show that there exists $x \in \mathbb{R}$ such that $f(x) = x$.

(Wing-Sum Cheung, The University of Hong Kong, Pokfulam, Hong Kong)

167. Show that, for any $a, b > 0$, we have

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

168. Let $f : I \to \mathbb{C}$ be an n-time differentiable function on the interior $\mathring{I}$ of the interval I, and $f^{(n)}$, with $n \ge 1$, be locally absolutely continuous on $\mathring{I}$. Show that, for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in \mathbb{R}\setminus\{0,1\}$, we have the representation

$$\begin{aligned} f(x) &= (1-\lambda) f(a) + \lambda f(b) \\ &+ \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(a)(x-a)^k + (-1)^k \lambda f^{(k)}(b)(b-x)^k\right] \\ &+ S_{n,\lambda}(x,a,b), \end{aligned} \quad (1)$$

where the remainder $S_{n,\lambda}(x,a,b)$ is given by

$$\begin{aligned} &S_{n,\lambda}(x,a,b) \\ &:= \frac{1}{n!}\left[(1-\lambda)(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds \right. \\ &\left. +(-1)^{n+1}\lambda(b-x)^{n+1}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds\right]. \end{aligned} \quad (2)$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

II Two new open problems

169*. Find all functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$[f(x) - f(y)]\,k(x+y) = [g(x) - g(y)]\,h(x+y) \quad (3)$$

for all $x, y \in \mathbb{R}$.

Remark. The above open problem appeared in the book of Sahoo and Riedel (see Section 2.7, page 80 in [2]). In a recent paper, Balogh, Ibrogimov and Mityagin [1] have given a partial solution to this open problem.

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[1] Z. M. Balogh, O. O. Ibrogimov and B. S. Mityagin, Functional equations and the Cauchy mean value theorem, *Aequat. Math.*, (2016), DOI 10.1007/s00010-015-0395-6.
[2] P. K. Sahoo and T. Riedel, Mean Value Theorems and Functional Equations, World Scientific, Singapore, 1998.

(Prasanna K. Sahoo, University of Louisville, Louisville, USA)

170*. Consider the operators

$$P_n^\alpha(f,x) = \sum_{k=0}^{\infty} v_{n,k}^\alpha(x) f(k/n),$$

where

$$v_{n,k}^\alpha(x) = \binom{n+k-1}{k}\frac{1^{[n,-\alpha]}x^{[k,-\alpha]}}{(1+x)^{[n+k,-\alpha]}}, \quad \text{for } x \in [0,\infty),$$

and $x^{[k,-\alpha]} = x(x+\alpha)\cdots(x+(k-1)\alpha)$. In the case $\alpha = 1/n$, we can write this in an alternative form as

$$\begin{aligned} v_{n,k}^{1/n}(x) &= \binom{n+k-1}{k}\frac{(nx)_k.(2n)!}{2(n!)(nx+n)_{n+k}} \\ &= \frac{(n)_k}{k!}.\frac{(nx)_k.(2n)!}{2(n!)(nx+n)_n(nx+2n)_k}. \end{aligned}$$

If we denote the m-th order moment by

$$T_{n,m}(x) = \sum_{k=0}^{\infty} v_{n,k}^{1/n}(x)\left(\frac{k}{n}\right)^m \quad (4)$$

then, by simple computation, we have $T_{n,0}(x) = 1, T_{n,1}(x) = \frac{nx}{n-1}$. Examine whether a recurrence relation can be obtained for $T_{n,m}(x)$ between the moments.

(Vijay Gupta, Netaji Subhas Institute of Technology, New Delhi, India)

III Solutions

152. Let G be an arbitrary group written multiplicatively. Let $\sigma : G \to G$ be an anti-homomorphism (i.e., $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$) satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Let $\mathbb{C}$ be the field of complex numbers.

(i) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(xy) + f(\sigma(y)x) = 2f(x) \quad (5)$$
for all $x, y \in G$.

(ii) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(xy) - f(x\sigma(y)) = 2f(y) \quad (6)$$
for all $x, y \in G$.

(iii) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(x\sigma(y)) = f(x)f(y) \quad (7)$$
for all $x, y \in G$.

(*Prasanna K. Sahoo, University of Louisville, Louisville, USA*)

Solution of problem 152 (i), (ii) and (iii). Let $Hom(G,\mathbb{C})$ be the set of all homomorphisms from group G to the additive group $(\mathbb{C},+)$ of $\mathbb{C}$ and $Hom(G,\mathbb{C}^\star)$ be the set of all homomorphisms from group G to the multiplicative group of non-zero complex numbers $\mathbb{C}^\star$. A function $f : G \to \mathbb{C}$ is said to be σ-even if and only if $f(\sigma(x)) = f(x)$ for all $x \in G$. Similarly, a function $f : G \to \mathbb{C}$ is said to be σ-odd if and only if $f(\sigma(x)) = -f(x)$ for all $x \in G$. A function $f : G \to \mathbb{C}$ is called a central function if and only if $f(xy) = f(yx)$ for all $x, y \in G$.

First, we determine all central functions $f : G \to \mathbb{C}$ satisfying functional equations (5) and (6) respectively. Then, we find all functions $f : G \to \mathbb{C}$ that satisfy functional equation (7).

Solution of problem (i). The central solution of functional equation (5) is of the form

$$f(x) = \phi(x) + \alpha, \quad \forall\, x \in G, \tag{6}$$

where ϕ is a σ-odd function in $Hom(G, \mathbb{C})$ and $\alpha \in \mathbb{C}$ is an arbitrary constant. The converse is also true.

It is easy to verify that f given by (6) satisfies (5). It is left to show that (6) is the only solution of (5). Let a, b and c be three arbitrary elements in G. Letting $x = ab$ and $y = c$ in (5), we have

$$f(abc) + f(\sigma(c)ab) = 2f(ab). \tag{7}$$

Next, letting $x = \sigma(c)a$ and $y = b$ in (5), we obtain

$$f(\sigma(c)ab) + f(\sigma(cb)a) = 2f(\sigma(c)a). \tag{8}$$

Use of (7) in (8) yields

$$2f(ab) - f(abc) + f(\sigma(cb)a) = 2f(\sigma(c)a). \tag{9}$$

Using (5), we see that $f(\sigma(cb)a) = 2f(a) - f(acb)$ and $f(\sigma(c)a) = 2f(a) - f(ac)$. In view of these, equation (9) gives rise to

$$f(abc) + f(acb) = 2f(ab) + 2f(ac) - 2f(a).$$

Letting $a = e$ (the identity in group G), we have

$$f(bc) + f(cb) = 2f(b) + 2f(c) - 2f(e).$$

Defining $\phi : G \to \mathbb{C}$ by $\phi(x) := f(x) - \alpha$, where $\alpha := f(e)$, the last equation reduces to

$$\phi(bc) + \phi(cb) = 2\phi(b) + 2\phi(c).$$

Since f is central, ϕ is also central and hence we have $\phi \in Hom(G, \mathbb{C})$. From the definition of ϕ, we obtain $f = \phi + \alpha$. Using this form of f in equation (5), we have $\phi(\sigma(y)) + \phi(y) = 0$ for all $y \in G$. Hence, ϕ is σ-odd.

Solution of problem (ii). If $f : G \to \mathbb{C}$ is any central function that satisfies functional equation (6) for all $x, y \in G$ then f is a σ-odd function in $Hom(G, \mathbb{C})$. The converse is also true.

It is easy to check that any σ-odd homomorphism f from G to $\mathbb{C}$ satisfies functional equation (6). Next, we show that it is the only solution of (6). Let a, b and c be any three arbitrary elements in G. With $x = ab$ and $y = c$ in (6), we get

$$f(abc) - f(ab\sigma(c)) = 2f(c). \tag{10}$$

Next, substitute $x = a$ and $y = b\sigma(c)$ in (6) to obtain

$$f(ab\sigma(c)) - f(ac\sigma(b)) = 2f(b\sigma(c)). \tag{11}$$

Adding (10) and (11), we see that

$$f(abc) - f(ac\sigma(b)) = 2f(c) + 2f(b\sigma(c)). \tag{12}$$

Using (6), we get $f(ac\sigma(b)) = f(acb) - 2f(b)$ and $f(b\sigma(c)) = f(bc) - 2f(c)$. Hence, (12) can be rewritten as

$$2f(bc) + f(acb) - f(abc) = 2f(c) + 2f(b).$$

Letting $a = e$ in the last equation, we obtain $f(bc) + f(cb) = 2f(b) + 2f(c)$ and, since f is central, we have $f \in Hom(G, \mathbb{C})$. Since $f \in Hom(G, \mathbb{C})$, from equation (6) we have $f(x) + f(y) - f(x) - f(\sigma(y)) = 2f(y)$, which proves that f is a σ-odd function in $Hom(G, \mathbb{C})$.

Remark 1. (a) Note that in (i) and (ii) the group G can be replaced by a unital semigroup S.

(b) We have provided the solution of (i) and (ii) assuming f to be a central function. Without this assumption on f, we do not know the solutions of (i) and (ii).

Solution of problem (iii). Every function $f : G \to \mathbb{C}$ that satisfies functional equation (7) is either a zero function or a σ-even function in $Hom(G, \mathbb{C}^\star)$. The converse of this is also true.

It is easy to verify that a zero function or every non-zero σ-even function in $Hom(G, \mathbb{C}^\star)$ satisfies functional equation (7). Next, we show that these are the only solutions of (7).

If f is a constant function then, from (7), we get $f = 0$ or $f = 1$. If $f = 1$ then $f \in Hom(G, \mathbb{C}^\star)$. Further, this f is σ-even. Next, assume that f is a non-constant function. For arbitrary elements $a, b, c \in G$, letting $x = a$ and $y = b\sigma(c)$ in (7), we have $f(ac\sigma(b)) = f(a)f(b\sigma(c))$. Using (5), the last equality can be rewritten as $f(ac)f(b) = f(a)f(b)f(c)$. Thus, $f(b)[f(ac) - f(a)f(c)] = 0$. Since f is non-constant, this implies that $f(ac) = f(a)f(c)$. Hence, $f \in Hom(G, \mathbb{C}^\star)$. Since $f \in Hom(G, \mathbb{C}^\star)$ and non-constant, we have from (7) that f is σ-even.

Remark 2. Note that in (iii) the group G can also be replaced by a unital semigroup. □

Notes.

1. John N. Daras, (pupil, Lyceum of Filothei, Athens, Greece) also solved problems 131 and 142.
2. G. C. Greubel (Newport News, Virginia, USA) also solved problems 149, 153* and 154*.

155. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, with $a, b \in \mathring{I}$ (interior of I), $a < b$ and $\nu \in [0, 1]$. Show that

$$\begin{aligned}(0 \le)\nu(1-\nu)(b-a)\left[f'_+((1-\nu)a+\nu b) - f'_-((1-\nu)a+\nu b)\right] \quad (8)\\ \le (1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b)\\ \le \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right],\end{aligned}$$

where $f'_\pm$ are the lateral derivatives of the convex function f.
In particular, for any $a, b > 0$ and $\nu \in [0, 1]$, show that the following reverses of Young's inequality are valid:

$$(0 \le)(1-\nu)a + \nu b - a^{1-\nu}b^\nu \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{9}$$

and

$$(1 \le)\ \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{10}$$

where K is Kantorovich's constant *defined by*

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{11}$$

(*Sever S. Dragomir, Victoria University, Melbourne City, Australia*)

Solution by the proposer. The case $\nu = 0$ or $\nu = 1$ reduces to equality in (8).

Since f is convex on I, it follows that the function is differentiable on $\mathring{I}$ except at a countable number of points, the lateral derivatives $f'_\pm$ exist at each point of $\mathring{I}$, they are increasing on $\mathring{I}$ and $f'_- \le f'_+$ on $\mathring{I}$.

For any $x, y \in \mathring{I}$, we have, for the Lebesgue integral,

$$\begin{aligned} f(x) &= f(y) + \int_y^x f'(s)\,ds \\ &= f(y) + (x-y)\int_0^1 f'((1-t)y + tx)dt. \end{aligned} \tag{12}$$

Assume that $\nu \in (0,1)$. By (12), we have

$$\begin{aligned} &f((1-\nu)a+\nu b) \\ &\quad = f(a) + \nu(b-a)\int_0^1 f'\Big((1-t)a + t((1-\nu)a+\nu b)\Big)dt \end{aligned} \tag{13}$$

and

$$\begin{aligned} &f((1-\nu)a+\nu b) \\ &= f(b) - (1-\nu)(b-a)\int_0^1 f'((1-t)b + t((1-\nu)a+\nu b))\,dt. \end{aligned} \tag{14}$$

If we multiply (13) by $1-\nu$, (14) by ν and add the obtained equalities then we get

$$\begin{aligned} f((1-\nu)a+\nu b) &= (1-\nu)f(a) + \nu f(b) \\ &\quad + (1-\nu)\nu(b-a)\int_0^1 f'\Big((1-t)a + t((1-\nu)a+\nu b)\Big)dt \\ &\quad - (1-\nu)\nu(b-a)\int_0^1 f'\Big((1-t)b + t((1-\nu)a+\nu b)\Big)dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b) \\ &= (1-\nu)\nu(b-a) \\ &\qquad \times \int_0^1 \Big[f'\Big((1-t)b + t((1-\nu)a+\nu b)\Big) \\ &\qquad\qquad - f'\Big((1-t)a + t((1-\nu)a+\nu b)\Big)\Big]dt. \end{aligned} \tag{15}$$

This is an equality of interest in itself.

Since $a < b$ and $\nu \in (0,1)$, we have $(1-\nu)a + \nu b \in (a,b)$ and

$$(1-t)a + t((1-\nu)a+\nu b) \in [a, (1-\nu)a+\nu b]$$

while

$$(1-t)b + t((1-\nu)a+\nu b) \in [(1-\nu)a+\nu b, b]$$

for any $t \in [0,1]$.

By the monotonicity of the derivative, we have

$$f'_+((1-\nu)a+\nu b) \le f'\Big((1-t)b + t((1-\nu)a+\nu b)\Big) \le f'_-(b) \tag{16}$$

and

$$f'_+(a) \le f'\Big((1-t)a + t((1-\nu)a+\nu b)\Big) \le f'_-((1-\nu)a+\nu b) \tag{17}$$

for almost every $t \in [0,1]$.

By integrating the inequalities (16) and (17), we get

$$f'_+((1-\nu)a+\nu b) \le \int_0^1 f'((1-t)b + t((1-\nu)a+\nu b)\Big)dt \le f'_-(b)$$

and

$$f'_+(a) \le \int_0^1 f'\Big((1-t)a + t((1-\nu)a+\nu b)\Big)dt \le f'_-((1-\nu)a+\nu b),$$

which implies that

$$\begin{aligned} &f'_+((1-\nu)a+\nu b) - f'_-((1-\nu)a+\nu b) \\ &\qquad \le \int_0^1 f'\Big((1-t)b + t((1-\nu)a+\nu b)\Big)dt \\ &\qquad\qquad - \int_0^1 f'\Big((1-t)a + t((1-\nu)a+\nu b)\Big)dt \\ &\qquad \le f'_-(b) - f'_+(a). \end{aligned}$$

Making use of equality (15), we obtain the desired result (8).

If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$ then, for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$, we have

$$\begin{aligned} &(0 \le)(1-\nu)f(a) + \nu f(b) - f((1-\nu)a+\nu b) \\ &\quad \le \nu(1-\nu)(b-a)[f'(b) - f'(a)]. \end{aligned} \tag{18}$$

If we write inequality (18) for the convex function $f : \mathbb{R} \to (0,\infty)$, $f(x) = \exp(x)$, then we have

$$\begin{aligned} &(0 \le)(1-\nu)\exp(x) + \nu\exp(y) - \exp((1-\nu)x + \nu y) \\ &\quad \le \nu(1-\nu)(x-y)[\exp(x) - \exp(y)] \end{aligned} \tag{19}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $a, b > 0$. If we take $x = \ln a$, $y = \ln b$ in (19) then we get the desired inequality (9).

Now, if we write inequality (18) for the convex function $f : (0,\infty) \to \mathbb{R}$, $f(x) = -\ln x$, then we get

$$(0 \le)\ln((1-\nu)a+\nu b) - (1-\nu)\ln a - \nu\ln b \le \nu(1-\nu)\frac{(b-a)^2}{ab},$$

namely

$$\ln\left[\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\right] \le \nu(1-\nu)\frac{(b-a)^2}{ab}.$$

This is equivalent to the desired result (10). □

Also solved by Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Socratis Varelogiannis (National Technical University of Athens, Greece)

156. Evaluate

$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})}{\sqrt{e}}\right]^n.$$

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. Recall that

$$\lim_{x\to 0}\frac{\ln(1+x) - x}{x^2} = -\frac{1}{2}.$$

Hence, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every real number x with $|x| < \delta$, we have

$$-\frac{1}{2} - \varepsilon < \frac{\ln(1+x) - x}{x^2} < -\frac{1}{2} + \varepsilon.$$

Choose an integer n_0 such that, for $n \ge n_0$,

$$\frac{1}{n} < \delta.$$

Therefore, we have

$$\frac{k}{n^2} \le \frac{n}{n^2} = \frac{1}{n} < \delta,$$

implying, for $n \ge n_0$, that

$$-\frac{1}{2} - \varepsilon < \frac{\ln(1+\frac{k}{n^2}) - \frac{k}{n^2}}{\frac{k^2}{n^4}} < -\frac{1}{2} + \varepsilon, k = 0, 1, \ldots, n.$$

Then,
$$-\frac{1}{2}-\varepsilon < \frac{\sum_{k=1}^{n}[\ln(1+\frac{k}{n^2})-\frac{k}{n^2}]}{\sum_{k=1}^{n}\frac{k^2}{n^4}} < -\frac{1}{2}+\varepsilon.$$
Hence,
$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}[\ln(1+\frac{k}{n^2})-\frac{k}{n^2}]}{\sum_{k=1}^{n}\frac{k^2}{n^4}} = -\frac{1}{2}. \tag{20}$$
On the other hand, using the well-known formula
$$\sum_{k=1}^{n}\frac{k^2}{n^4}=\frac{n(n+1)(2n+1)}{6n^4},$$
we have
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3}=\frac{1}{3}$$
and from (20) we obtain
$$\lim_{n\to\infty} n\cdot\sum_{k=1}^{n}\left[\ln(1+\frac{k}{n^2})-\frac{k}{n^2}\right]=-\frac{1}{6},$$
that is,
$$\lim_{n\to\infty} n\cdot\left[\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})-\frac{n+1}{2n}\right]=-\frac{1}{6}.$$
Hence,
$$\lim_{n\to\infty}\left[\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})^n-\frac{n+1}{2}\right]=-\frac{1}{6}.$$
It follows that
$$\lim_{n\to\infty}\left[2\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})^n-n\right]=1-\frac{1}{3}=\frac{2}{3}$$
and we obtain
$$\lim_{n\to\infty}\left[\ln\prod_{k=1}^{n}(1+\frac{k}{n^2})^n-\ln(\sqrt{e})\right]=\frac{1}{3}.$$
The last relation is equivalent to
$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})}{\sqrt{e}}\right]^n=\sqrt[3]{e}.$$
□

Also solved by Ulrich Abel (University of Applied Sciences, Friedberg, Germany), Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Mihaly Bencze (Brasov, Romania), Albero Bersani (Sapienza Universita di Roma, Italy), John N. Daras, (pupil, Lyceum of Filothei, Athens, Greece), Jorge Mozo Fernandez, (Universidad de Valladolid, Spain), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Edward Omey (KU Leuven, Brussels, Belgium), Angel Plaza (University of Las Palmas de Gran Canaria, Spain) Socratis Varelogiannis (National Technical University of Athens, Greece).

157. Let X be a compact space and $f : X \to X$ be continuous and expansive, that is,
$$d(f(x),f(y)) \ge d(x,y) \quad \forall\, x,y\in X\,.$$
What can be said about the function f?

(*W. S. Cheung, University of Hong Kong, Pokfulam, Hong Kong*)

Solution by the proposer.

(i) Observe that f is clearly $1-1$.

(ii) $f^{-1}: f(X)\to X$ is continuous.
In fact, $\forall\,\varepsilon>0$, let $\delta:=\varepsilon$. Then,
$$d(f(x),f(y))<\delta \Rightarrow d(x,y)\le d(f(x),f(y))<\delta=\varepsilon\,.$$

(iii) f is onto.
In fact, since X is compact and f is continuous, $f(X)$ is compact. If there exists $x\in X\setminus f(X)$, we must have
$$d(x,f(X)) = \text{some positive number } d>0\,.$$
For any $m,n\in\mathbb{N}$,
$$d\le d(x,f^m(x))\le d(f(x),f^{m+1}(x))\le\cdots\le d(f^n(x),f^{n+m}(x))$$
and therefore $\{f^n(x)\}$ is a sequence in X without accumulation point, which violates the assumption that X is compact.

Combining (i), (ii) and (iii), f is a homeomorphism. □

Also solved by Vincenzo Basco (Universita degli Studi di Roma "Tor Vergata", Italy), Mihaly Bencze (Brasov, Romania), Jorge Mozo Fernandez (Universidad de Valladolid, Spain), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Socratis Varelogiannis (National Technical University of Athens, Greece).

158. Find all differentiable functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy the equation
$$xf'(x)+kf(-x)=x^2 \quad \forall\,x\in\mathbb{R},$$
where $k>0$ is an integer.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*)

Solution by the proposer. We prove that such functions are of the following form:
$$f(x)=\begin{cases}\frac{x^2}{k+2} & \text{if } k \text{ is even},\\ \frac{Cx^k}{2}+\frac{x^2}{k+2}, \quad C\in\mathbb{R} & \text{if } k \text{ is odd}.\end{cases}$$
We replace x by $-x$ and we have
$$\begin{cases}xf'(x)+kf(-x)=x^2,\\ -xf'(-x)+kf(x)=x^2\end{cases}$$
and this implies, by subtraction, that
$$x(f'(x)+f'(-x))+k(f(-x)-f(x))=0.$$
Let $g:\mathbb{R}\to\mathbb{R},\ g(x)=f(x)-f(-x)$. The previous equation implies that
$$xg'(x)-kg(x)=0,\ \forall\,x\in\mathbb{R}.$$
This implies that $g(x)=Cx^k$, for all $x\in\mathbb{R}$. It follows that
$$f(x)-f(-x)=Cx^k,\ \forall\,x\in\mathbb{R}.$$
Replacing $f(-x)=f(x)-Cx^k$ in the initial differential equation, we get that
$$xf'(x)+kf(x)=Ckx^k+x^2.$$
We multiply this equation by x^{k-1} and we get that
$$\left(x^kf(x)\right)'=Ckx^{2k-1}+x^{k+1},\ \forall\,x\in\mathbb{R},$$
which implies that
$$x^kf(x)=\frac{Cx^{2k}}{2}+\frac{x^{k+2}}{k+2}+C_1.$$

We let $x = 0$ in the previous equality and we get that $C_1 = 0$. This implies that

$$f(x) = \frac{Cx^k}{2} + \frac{x^2}{k+2}, \quad C \in \mathbb{R}. \tag{21}$$

Now we check that if k is an odd integer, functions of the form (21) verify the differential equation and if k is an even integer then functions in (21) verify the differential equation if $C = 0$. The problem is solved. □

Also solved by Ulrich Abel (University of Applied Sciences, Friedberg, Germany), Mihaly Bencze (Brasov, Romania), Jorge Mozo Fernandez (Universidad de Valladolid, Spain), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), Sotirios E. Louridas (Athens, Greece), Socratis Varelogiannis (National Technical University of Athens, Greece).

159. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$ (interior of I). If there exist the constants d, D such that

$$d \le f''(t) \le D \text{ for any } t \in \mathring{I}, \tag{22}$$

show that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\,d\,(b-a)^2 &\le (1-\nu)\,f(a) + \nu f(b) - f\big((1-\nu)\,a + \nu b\big)\\ &\le \frac{1}{2}\nu(1-\nu)\,D\,(b-a)^2\end{aligned} \tag{23}$$

for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$.
In particular, for any $a, b > 0$ and $\nu \in [0,1]$, show that the following refinements and reverses of Young's inequality are valid:

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\}\\ &\le (1-\nu)\,a + \nu b - a^{1-\nu}b^{\nu}\\ &\le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\le \frac{(1-\nu)\,a + \nu b}{a^{1-\nu}b^{\nu}}\\ \le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right].\end{aligned} \tag{25}$$

(*Sever S. Dragomir, Victoria University, Melbourne City, Australia*)

Solution by the proposer. We consider the auxiliary function $f_D : I \subset \mathbb{R} \to \mathbb{R}$ defined by

$$f_D(x) = \frac{1}{2}Dx^2 - f(x).$$

The function f_D is differentiable on $\mathring{I}$ and $f_D''(x) = D - f''(x) \ge 0$, showing that f_D is a convex function on $\mathring{I}$.

By the convexity of f_D, we have, for any $a, b \in \mathring{I}$ and $\nu \in [0,1]$, that

$$\begin{aligned}0 &\le (1-\nu)f_D(a) + \nu f_D(b) - f_D((1-\nu)a + \nu b)\\ &= (1-\nu)\Big(\frac{1}{2}Da^2 - f(a)\Big) + \nu\Big(\frac{1}{2}Db^2 - f(b)\Big)\\ &\qquad - \Big(\frac{1}{2}D((1-\nu)a + \nu b)^2 - f_D((1-\nu)a + \nu b)\Big)\\ &= \frac{1}{2}D[(1-\nu)a^2 + \nu b^2 - ((1-\nu)a + \nu b)^2]\\ &\qquad - (1-\nu)f(a) - \nu f(b) + f_D((1-\nu)a + \nu b)\\ &= \frac{1}{2}\nu(1-\nu)D(b-a)^2 - (1-\nu)f(a) - \nu f(b) + f_D((1-\nu)a + \nu b),\end{aligned}$$

which implies the second inequality in (23).

The first inequality follows in a similar way by considering the auxiliary function $f_d : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_d(x) = f(x) - \frac{1}{2}dx^2$, which is twice differentiable and convex on $\mathring{I}$.

If we write inequality (23) for the convex function $f : \mathbb{R} \to (0,\infty)$,

$$f(x) = \exp(x),$$

then we have

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)(x-y)^2 \min\{\exp x, \exp y\}\\ &\le (1-\nu)\exp(x) + \nu\exp(y) - \exp((1-\nu)x + \nu y)\\ &\le \frac{1}{2}\nu(1-\nu)(x-y)^2 \max\{\exp x, \exp y\}\end{aligned} \tag{26}$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $a, b > 0$. If we take $x = \ln a$, $y = \ln b$ in (26) then we get the desired inequality (24).

Now, if we write inequality (23) for the convex function $f : (0,\infty) \to \mathbb{R}$, $f(x) = -\ln x$, then we get, for any $a, b > 0$ and $\nu \in [0,1]$, that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}} &\le \ln((1-\nu)a + \nu b) - (1-\nu)\ln a - \nu\ln b\\ &\le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}.\end{aligned} \tag{27}$$

Now, since

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2$$

and

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2,$$

we have that (27) is equivalent to the desired result (25). □

Also solved by Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea).

160. Let p be the partition function (counting the ways to write n as a sum of positive integers), extended so that $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Prove that, for $n \geqslant 0$,

$$1 \leqslant \frac{2p(n+2) - p(n+3)}{p(n)} \leqslant \frac{3}{2}.$$

(*Mircea Merca, University of Craiova, Romania*)

Solution by the proposer. To prove this double inequality, we consider the generating function of $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \qquad |q|<1,$$

and Euler's identity

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |q|,|z|<1,$$

where

$$(a;q)_n = \begin{cases} 1 & \text{for } n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{for } n>0 \end{cases}$$

and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

The left side of the double inequality is equivalent to

$$p(n)-2p(n-1)+p(n-3)\leqslant 0, \qquad n\neq 0.$$

To prove this inequality, we need to show that the coefficient of q^n in the series

$$\sum_{n=0}^{\infty}(p(n)-2p(n-1)+p(n-3))q^n = \frac{1-2q+q^3}{(q;q)_\infty}, \quad |q|<1,$$

is non-positive for $n>0$. We have

$$\begin{aligned}\frac{1-2q+q^3}{(q;q)_\infty} &= \frac{(1-q)(1-q-q^2)}{(1-q)(q^2;q)_\infty} \\ &= \frac{1}{(q^3;q)_\infty} - \frac{q}{(q^2;q)_\infty} \\ &= \sum_{n\geq 0}\frac{q^{3n}}{(q;q)_n} - \sum_{n\geq 0}\frac{q^{2n+1}}{(q;q)_n} \\ &= 1-q+\sum_{n\geq 2}\frac{q^{2n+1}}{(q;q)_n}\left(q^{n-1}-1\right) \\ &= 1-q-\sum_{n\geq 2}\frac{q^{2n+1}}{(q;q)_{n-2}(1-q^n)}\end{aligned}$$

and we see, for $n>0$, that the coefficient of q^n is non-positive. We have invoked the fact that

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n, \quad |q|<1.$$

The right side of the double inequality is equivalent to

$$p(n)-2p(n-1)+\frac{3}{2}p(n-3)\geqslant 0, \quad n\neq 1.$$

Moreover, considering the trivial inequality

$$2p(n-1)-p(n)\geqslant 0, \quad n>0,$$

we can write

$$\begin{aligned}&p(n)-2p(n-1)+\frac{3}{2}p(n-3) \\ &\qquad \geqslant p(n)-2p(n-1)+2p(n-3)-p(n-4), \quad n\neq 3.\end{aligned}$$

We show that, except for the coefficient of q, all the coefficients in the series

$$\begin{aligned}&\sum_{n=0}^{\infty}(p(n)-2p(n-1)+2p(n-3)-p(n-4))q^n \\ &\qquad = \frac{1-2q^2+2q^3-q^4}{(q;q)_\infty}, \quad |q|<1,\end{aligned}$$

are non-negative. We have

$$\begin{aligned}\frac{1-2q^2+2q^3-q^4}{(q;q)_\infty} &= \frac{(1-q)^2(1-q^2)}{(q;q)_\infty} \\ &= \frac{1-q}{(q^3;q)_\infty} \\ &= (1-q)\sum_{n=0}^{\infty}\frac{q^{3n}}{(q;q)_n} \\ &= 1-q+\sum_{n=1}^{\infty}\frac{q^{3n}}{(q^2;q)_{n-1}}.\end{aligned}$$

Clearly, the coefficient of q^0 is 1, the coefficient of q^1 is -1 and, for $k>1$, all the coefficients of q^k are non-negative. In other words, the inequality

$$p(n)-2p(n-1)+2p(n-3)-p(n-4)\geqslant 0$$

is valid for $n\neq 1$. This concludes the proof. □

Also solved by Mihaly Bencze (Brasov, Romania)

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *mathematical analysis*.

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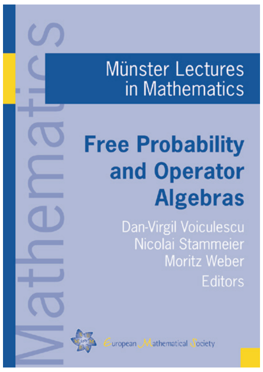
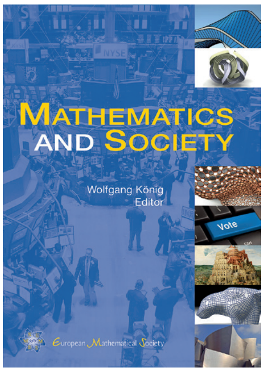


Free Probability and Operator Algebras (Münster Lectures in Mathematics)
Dan-Virgil Voiculescu (University of California, Berkeley, USA), Nicolai Stammeier (University of Oslo, Norway) and Moritz Weber (Universität Saarbrücken, Germany), Editors

ISBN 978-3-03719-165-1. 2016. 148 pages. Softcover. 17 x 24 cm. 32.00 Euro

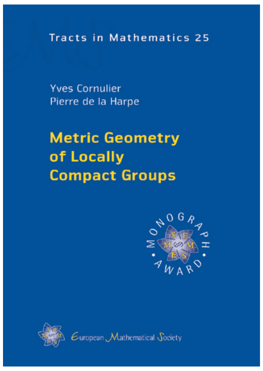
Free probability is a probability theory dealing with variables having the highest degree of noncommutativity, an aspect found in many areas (quantum mechanics, free group algebras, random matrices etc). Thirty years after its foundation, it is a well-established and very active field of mathematics. Originating from Voiculescu's attempt to solve the free group factor problem in operator algebras, free probability has important connections with random matrix theory, combinatorics, harmonic analysis, representation theory of large groups, and wireless communication.
These lecture notes arose from a masterclass in Münster, Germany and present the state of free probability from an operator algebraic perspective. This volume includes introductory lectures on random matrices and combinatorics of free probability (Speicher), free monotone transport (Shlyakhtenko), free group factors (Dykema), free convolution (Bercovici), easy quantum groups (Weber), and a historical review with an outlook (Voiculescu). In order to make it more accessible, the exposition features a chapter on basics in free probability, and exercises for each part.
This book is aimed at master students to early career researchers familiar with basic notions and concepts from operator algebras. The Handbook is addressed to researchers and to graduate students.



Mathematics and Society
Wolfgang König (WIAS Berlin and Technical University Berlin, Germany), Editor

ISBN 978-3-03719-164-4. 2016. 314 pages. Hardcover. 17 x 24 cm. 42.00 Euro

The ubiquity and importance of mathematics in our complex society is generally not in doubt. However, even a scientifically interested layman would be hard pressed to point out aspects of our society where contemporary mathematical research is essential. Most popular examples are finance, engineering, wheather and industry, but the way mathematics comes into play is widely unknown in the public. And who thinks of application fields like biology, encryption, architecture, or voting systems?
This volume comprises a number of success stories of mathematics in our society – important areas being shaped by cutting edge mathematical research. The authors are eminent mathematicians with a high sense for public presentation, addressing scientifically interested laymen as well as professionals in mathematics and its application disciplines.

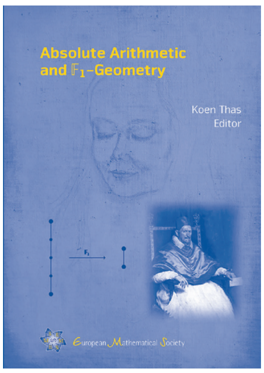


Yves Cornulier (Université Paris-Sud, Orsay, France) and Pierre de la Harpe (Université de Genève, Switzerland)
Metric Geometry of Locally Compact Groups (EMS Tracts in Mathematics, Vol. 25)

Monograph Award

ISBN 978-3-03719-166-8. 2016. 241 pages. Hardcover. 17 x 24 cm. 62.00 Euro

The main aim of this book is the study of locally compact groups from a geometric perspective, with an emphasis on appropriate metrics that can be defined on them. The approach has been successful for finitely generated groups, and can favourably be extended to locally compact groups. Parts of the book address the coarse geometry of metric spaces, where 'coarse' refers to that part of geometry concerning properties that can be formulated in terms of large distances only. This point of view is instrumental in studying locally compact groups.
Basic results in the subject are exposed with complete proofs, others are stated with appropriate references. Most importantly, the development of the theory is illustrated by numerous examples, including matrix groups with entries in the the field of real or complex numbers, or other locally compact fields such as *p*-adic fields, isometry groups of various metric spaces, and, last but not least, discrete group themselves.
The book is aimed at graduate students and advanced undergraduate students, as well as mathematicians who wish some introduction to coarse geometry and locally compact groups.



Absolute Arithmetic and $\mathbb{F}_1$-Geometry
Koen Thas (University of Gent, Belgium), Editor

ISBN 978-3-03719-157-6. 2016. 404 pages. Hardcover. 17 x 24 cm. 68.00 Euro

It has been known for some time that geometries over finite fields, their automorphism groups and certain counting formulae involving these geometries have interesting guises when one lets the size of the field go to 1. On the other hand, the nonexistent field with one element, $\mathbb{F}_1$, presents itself as a ghost candidate for an absolute basis in Algebraic Geometry to perform the Deninger–Manin program, which aims at solving the classical Riemann Hypothesis.
This book, which is the first of its kind in the $\mathbb{F}_1$-world, covers several areas in $\mathbb{F}_1$-theory, and is divided into four main parts – Combinatorial Theory, Homological Algebra, Algebraic Geometry and Absolute Arithmetic. Topics treated include the combinatorial theory and geometry behind $\mathbb{F}_1$, categorical foundations, the blend of different scheme theories over $\mathbb{F}_1$ which are presently available, motives and zeta functions, the Habiro topology, Witt vectors and total positivity, moduli operads, and at the end, even some arithmetic.
Each chapter is carefully written by experts, and besides elaborating on known results, brand new results, open problems and conjectures are also met along the way. The diversity of the contents, together with the mystery surrounding the field with one element, should attract any mathematician, regardless of speciality.

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