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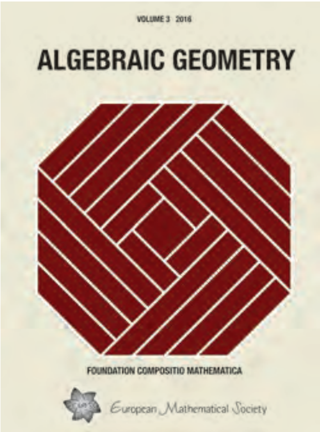

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EMS Executive Committee

EMS Agenda

2017

20 November
Applied Mathematics Committee Meeting, Amsterdam

24–26 November
EMS Executive Committee Meeting, Portorož, Slovenia

2 December
EMF Board of Trustees Meeting, Zürich, Switzerland

2018

13–15 April
Meeting of Presidents, Maynooth University, Dublin, Ireland

23–24 June
EMS Council, Prague, Czech Republic

29–30 July
IMU General Assembly, São Paulo, Brazil

EMS Scientific Events

2017

15–19 November
ASTUCON – The 2nd Academic University Student Conference (Science, Technology Engineering, Mathematics), Larnaca, Cyprus

2018

29 January–9 February
Winter School and Workshop "Riemann–Hilbert Correspondences", Università di Padova, Italy

4–6 April
Probability, Analysis and Dynamics '18, University of Bristol, UK

11–14 June
British Mathematical Colloquium 2018, University of St Andrews, UK

11–15 June
International Conference on Complex Analysis, Potential Theory and Applications, University College Dublin, Ireland

9–14 July
Young African Scientists in Europe (YASE), Toulouse, France

23–27 July
11th European Conference on Mathematical and Theoretical Biology (ECMTB 2018), Lisbon, Portugal

1–9 August
ICM 2018
Rio Centro Convention Center, Rio de Janeiro, Brazil

Editorial: One Year After…

Valentin Zagrebnov (Aix-Marseille Université, Marseille, France), Editor-in-Chief of the EMS Newsletter

This issue of the newsletter rounds off my first one-year period as the Editor-in-Chief of the EMS Newsletter. I would like to sum up this period of newsletter life by summarising the activity and evolution of the editorial board, as well as giving a description of some of our recent projects.

Editorial board and rotations

I would like to start with the almost permanent problem of rotation and of how to attract active enthusiasts to the board who are able to help the EMS Newsletter function effectively. The table below summarises the composition of the current editorial board.

As you can see, there are *four* rotations at the end of this year. Two of them need to be replaced and I already have suggestions concerning potential candidates from Eva Miranda and Olaf Teschke. For the other two (Volker Remmert and Jean-Paul Allouche), I am going to propose 4-year term extensions and then suggest their candidatures to the EMS Executive Committee for approval in November 2017. In addition, I would also like to invite two more people to join the board to ensure progress with some new projects that we have recently discussed and launched.

Projects

Since 2016, several ideas within the framework of "YMCo Future" have been proposed and then developed by Vladimir Salnikov. Note that the *EMS Newsletter* regularly publishes interviews with Abel Prize winners. A new idea is to publish, at the same regularity, interviews with EMS *Distinguished Speakers*. This could, for example, be run by YMCo (Young Mathematicians' Column), who would certainly provide the proper emphasis for a younger audience. YMCo is concerned with the position of young researchers in the scientific community so it welcomes material like this.

I would like to note that, besides features and discussion articles, interviews with Abel winners and distinguished speakers are the most demanded for republication and translation to other sources. The most important "consumer" of our publications is the *Mathematical Advances in Translation* of the Academy of Mathematics and Systems Science at the Chinese Academy of Sciences. Recently, *Mathematical Advances in Translation* requested authorisation from the *EMS Newsletter* to translate for republication in Chinese the interview with Abel Laureate Sir Andrew Wiles, which was published in *Newsletter* 101 (2016), and the interview (in *Newsletter* 102) with Ernest B. Vinberg, the EMS *Distinguished Speaker* for 2016.

A reciprocity agreement between the French Mathematical Society (SMF) *La Gazette de Mathématiciens* and the *EMS Newsletter* was approved by both sides in 2017. This agreement with the SMF supports the exchange of articles, as well as French/English translations by request. The first result of this cooperation is a translation for the current issue, by Javier Fresán, of the article "On the traces of operators (from Grothendieck to Lidskii)", written by D. Robert. Appreciating the

Name	Begin	End	Notes	Terms
Valentin A. Zagrebnov	2016-07-01	2020-06-30	Editor-in-Chief	4
Michael Th. Rassias	2017-01-01	2018-12-31	Problem Corner	2
Fernando P. da Costa	2017-01-01	2020-12-31	Societies	4
Dierk Schleicher	2017-01-01	2020-12-31	Features + Discussions	4+4
Volker Remmert	2013-09-30	2017-12-31	History	4
Eva Miranda	2010-03-30	2017-12-31	Research Centres	4+4
Jean-Paul Allouche	2014-03-30	2017-12-31	Book Reviews	4
Olaf Teschke	2010-06-30	2017-12-31	zbMATH	4+4
Vladimir Kostic	2014-09-30	2018-12-31	Social Media + YMCo	4
Jean-Luc Dorier	2015-01-01	2018-12-31	Education	4
Javier Fresán	2015-01-01	2018-12-31	YMCo + Contacts with SMF	4
Vladimir Popov	2015-01-01	2018-12-31	Features + Discussions	4
Vladimir Salnikov	2015-01-01	2018-12-31	YMCo	4
Ramla Abdelatif	2015-01-01	2018-12-31	Contacts with SMF + YMCo	4

enthusiasm of Javier Fresán and Fernando P. da Costa, who has translated an article for the next issue of the newsletter from *La Gazette de Mathématiciens*, I think we need to carry out this activity at a systematic level. In 2018, we need to reinforce the editorial board with francophones to take charge of the contact between the *EMS Newsletter* and the SMF and, in particular, with *La Gazette de Mathématiciens* (Editor-in-Chief Boris Adamczewski).

I would like to note that the reciprocity agreement between *La Gazette de Mathématiciens* and the EMS Newsletter is an example of bilateral cooperation involving French/English translations. On the other hand, republication of interesting articles is standard practice for the *EMS Newsletter*.

Another project is a new section called Archives. This was an initiative of Volker Remmert. It allows us to improve and to diversify presentations that were previously limited to the History Section. The current issue contains the second publication of the Archives Section.

In conclusion, I would like to thank the past and present members of the editorial board for helping run the *EMS Newsletter* and attracting the interest of the mathematical community of the world to our publication. Reactions and proposals of our readers are always very welcome and carefully considered.

Report from the Executive Committee Meeting in Bratislava, 17–19th March 2017

Richard Elwes, EMS Publicity Officer

This Spring, the Executive Committee (EC) of the EMS convened at the Slovak University of Technology in Bratislava, on the kind invitation of the Slovakian Mathematical Society. This meeting was the first sitting of the new committee, following elections at the Council in Berlin in Summer 2016. On Friday evening, the committee was welcomed by Martin Kalina, President of the Slovak Mathematical Society, who told us about his society. Established in 1969, the society is the mathematical branch of the Union of Slovak Mathematicians and Physicists, and welcomes both university mathematicians and maths teachers at primary and secondary level. Its activities include running Olympiads and summer/winter camps in the mountains for school pupils, competitions and events for undergraduate students, seminars for teachers, and domestic & international scientific conferences for researchers. It publishes the journal *Horizons of Mathematics, Physics and Informatics.*

Officers' reports and membership

The meeting got underway with a greeting from the Chair, EMS President Pavel Exner, who welcomed the members of the new committee, and reported on his activities since the last EC meeting at Tbilisi in November 2016. He began by looking back over 2016, observing that it was a significant and successful year in the life our society, noting especially the 7th European Congress of Mathematicians in Berlin. Laying the foundations for an equally successful 8th Congress in 2020 is an important current task. Beyond the society, the President remarked that 2016 was an unusually turbulent year politically (notably in USA, UK, and Turkey), and that this upheaval has seen an increase in the number of requests for the EMS to participate in political protests - requests which need to be considered carefully. (This topic was discussed further at the Presidents' meeting in Lisbon. See separate report below.)

The EMS Treasurer, Mats Gyllenberg, presented his report on the society's finances and discussed the budget approved by the Council for 2017-2019. Due to a healthy financial situation, it was agreed in March 2016 that the society will spend more on scientific activities; somewhat disappointingly, however, the number of suitable projects applying for funding has not increased correspondingly. There is thus a need to advertise for high-level applications for scientific activities, such as Joint Mathematical Weekends and Summer Schools. The Treasurer also proposed certain technical simplifications of the budget, which will be discussed further at the next Council. Finally, the Treasurer proposed (on the advice of the Society's bankers) to transfer more capital into investment funds. The committee agreed with this proposal, and adopted the rule that two-third of the society's assets will be kept in such funds. The committee thanked the Treasurer for his excellent work.

The committee was delighted to approve an application for institutional EMS membership from the University of Lisbon. Enquiries have also been received from two national societies; these may progress if they submit applications for corporate membership to be presented at the next Council. The committee was pleased to approve a list of 154 new individual members.

Scientific meetings

Volker Mehrmann, the chief local organiser of ECM7 in Berlin in July 2016, delivered an update on the congress's aftermath, confirming that it ended in deficit by 3000 Euros, and making several recommendations for the smoother running of future meetings. The committee reiterated their thanks to Volker and his team for a magnificent and well-run conference. The next meeting of the Executive Committee will be in November in Portorož (Slovenia) at the site of the 8th ECM, where arrangements for the next congress will be discussed in greater detail. The committee agreed that special attention should be given to gender balance in the congress's committees, and to matters of diversity in general. The committee will continue its ongoing deliberations about the mission and goals of the ECM in the changing mathematical landscape, with a view to using the ECM to build our community and make young mathematicians feel welcome and valued.

The President briefly discussed other forthcoming events including the five EMS Summer Schools to be held over the Summer, the joint EMS-Bernoulli Society Joint Lecture to be delivered by Alexander Holevo at the 31st European Meeting of Statisticians (Helsinki, July 24–28, 2017), and this year's EMS Distinguished speakers: Mireille Bousquet-Mélou at the Conference on Foundations of Computational Mathematics (Barcelona, 10–19 July 2017) and Kathryn Hess at the Meeting of the Catalan, Spanish and Swedish Math Societies (Umeå, 12–15 June, 2017). The committee also looked towards future major meetings: ICM 2018 in Rio de Janeiro, ICIAM 2019 in Valencia, and ICM 2022 to be held either in Paris or St Petersburg (and thus certainly in Europe). The committee approved funds for an EMS Joint Mathematical Weekend, 4–5 January 2018 in Joensuu (Finland) to mark the beginning of the 2018 Year of Mathematical Biology.

Society meetings & officers

The current EMS President's term, and those of Vice-President Volker Mehrmann and the other Officers, will all terminate at the end of 2018. Therefore, the next Council will need to elect a new President and Vice-President, and either to re-elect or replace the current Treasurer and Secretary. The search for a suitable presidential candidate is already underway.

The annual meeting of Presidents of EMS member societies will take place in Lisbon on 1–2 April 2017, hosted by the Portuguese Mathematical Society. (See separate report below.) In 2018, the equivalent meeting will take place in Dublin.

Standing committees & projects

Jiří Rákosník, Chair of the Electronic Publishing Committee, in attendance as a guest, presented a proposal to merge the Electronic Publishing and Publication Committees. The Executive Committee thanked Jiří for his work, and approved the merger. After some discussion, the new committee was named the *Publications and Electronic Dissemination Committee* (PED). The PED's new remit was also approved, and can be found on the EMS webpage (as can details of all EMS standing committees). The Executive Committee appointed Thierry Bouche as Chair and Olaf Teschke as Vice-Chair of PED, and as well as appointing a full quota of committee members, all for the term 2017–2020.

José Antonio Carrillo, Chair of the Applied Mathematics Committee (AMC), also present as a guest, delivered a short report about the AMC's activities, noting in particular the forthcoming Year of Mathematical Biology 2018. The committee thanked him for his efforts, and discussions followed regarding possible candidates for Chair, as his term will end this year (as will those of six AMC members).

Mats Gyllenberg, responsible EC member for the Committee for Developing Countries, delivered a short report on the first round of applications for funds from the EMS-Simons for Africa programme.

The new Chair Jürg Kramer and Vice-Chair Tine Kjedsen of the Education Committee have restarted that committee's work, and the EC looks forward to hearing more about its activities in due course. The President will consult member societies about useful lines of work for the Education committee.

Reports from the other standing committees (European Research Centres on Mathematics (ERCOM), Ethics, European Solidarity, Meetings, Raising Public Awareness of Mathematics, Women in Mathematics) were received and considered, and the Executive Committee reiterated its gratitude to all members of these committees, which are responsible for so much of the society's work.

Discussions followed on other projects the EMS is involved with, including the online Encyclopedia of Mathematics (www.encyclopediaofmath.org), EU-MATHS-IN (European Service Network of Mathematics for Industry and Innovation), the Global Digital Mathematics Library, Zentralblatt MATH (www.zbmath.org). The society's own newsletter, e-news, social media platforms, and other communications channels were also reviewed.

Funding, political, and scientific organisations

The committee discussed matters around Horizon2020 and the European Research Council (ERC). It was noted that the ERC budget is set to increase again next year, and mathematicians should be strongly encouraged to apply for funding. The President also called attention to the ERC's 10th anniversary this year, and to the various celebratory activities taking place around Europe. He then reported on recent developments regarding the new legal status of the Initiative for Science in Europe (ISE). The EMS's membership fee for the ISE is shortly set to double to 3000 Euros, at which point the Executive Committee will need to decide whether to continue with ISE membership.

Regarding the European Science Open Forum (ESOF), the next meeting will be in Toulouse in 2018, and it would be desirable for mathematics to be well represented. (At the 2016 meeting in Manchester, the EMS's

Raising Public Awareness committee successfully ran a special session on Alan Turing.)

Volker Mehrmann, also a member of the Board of ICIAM (International Council for Industrial and Applied Mathematics), reported on the latest developments there. The calls for ICIAM prizes are now open and the search for a new President to be elected at ICIAM 2019 in Valencia is underway.

The EMS's relationships with other societies, research centres, and prize committees was also discussed, with the EC deciding on several nominations for boards.

Conclusions

The committee will hold an informal retreat before the next official meeting, where members' visions for the society's future can be discussed without the pressure of a full agenda and a ticking clock. The committee's next official sitting will be in Portorož, 24–26 November 2017.

The meeting closed with sincere expressions of gratitude to our local hosts at the Slovak University of Technology in Bratislava and the Slovakian Mathematical Society, especially Martin Kalina and Mária Ždímalová, for a thoroughly well-organised and enjoyable meeting.

Report from the Meeting of Presidents of Mathematical Societies in Lisbon, 1–2 April 2017

Richard Elwes, EMS Publicity Officer

The annual meeting of Presidents of EMS Member Societies took place 1–2 April, in the elegant surroundings of Fundação Calouste Gulbenkian in sunny Lisbon. The Gulbenkian Foundation is of great cultural importance in Portugal (Jorge Buescu, President of the Portuguese Mathematical Society, told us that the Foundation is known informally as the "real" ministry of culture and education). It is also a special place in the history of the EMS, as the location for the first of the society's Joint Mathematical Weekends in 2003 (we are now on the 10th incarnation of this sequence).

The meeting was opened with words of welcome from the Chair, EMS President Pavel Exner, and addresses from two special guests. Guilherme d'Oliveira Martins, Administrator of the Gulbenkian Foundation, told us about its work and history. Manuel Heitor, Minister of Science and Technology of the Government of Portugal, then warmly welcomed all visitors to Lisbon, and spoke of his ambition for Portugal to become ever more a top-flight country of science, whilst calling for greater levels of scientific activism. He also drew our attention to Portugal's Position Paper on Horizon 2020, "On the evolving nature of EU research funding".

After a Tour de Table, in which everybody introduced themselves and their society, Jorge Buesco again welcomed the assembled company, and gave a short address about the Portuguese Mathematical Society (SPM). Founded in 1940, the SPM inhabited a "Schrödinger" existence for many years, due to Portugal's difficult political environment. It was fully reborn in 1977 and now represents a rapidly growing mathematical community (doubling in size each decade for the last 40 years), with a deep commitment to internationalism (including being a founding member of the EMS, as well as ICIAM, ICMI, ECMI, EU-MATHS-IN,…).

EMS business

The EMS President Pavel Exner delivered his report, noting the dramatic political changes the world has seen in the last year, in the USA, Turkey, UK, and elsewhere. He noted that there are increasing demands on the EMS to participate in political protest, and commented that this needs to be done selectively in order to be effective, remembering that one of the society's principal duties is to defend the mathematical community. For example, the EMS was a co-signatory in February 2017 of the open letter from European Science Organisations calling for the USA to stand by the principles of "transparency, open communication, and mobility of scholars and scientists, which are vital to scientific progress and to the benefit our societies". The meeting supported this stand, and agreed that future calls for political protest can only be considered on a case by case basis.

The President also took the opportunity to remember the physicist and former Portuguese Minister for Science and Technology, José Mariano Gago who died in 2015, one of the driving forces behind the creation of the European Research Council.

The year 2016 was also a significant one for the EMS, with the quadrennieal European Congress of Mathematics being held in Berlin. At the Council meeting beforehand, the Executive Committee enjoyed a major renewal after a properly contested election, surely a positive sign for our society. (All candidates for the Executive Committee that were not elected were invited to become members of one the EMS's standing committees.)

Pavel asked the assembled Presidents to spread the news that Council approved a change to the society's by-laws, which now allow students to enjoy a three-year introductory period of free membership. He drew attention to the fact that members of the EMS have special benefits at MathHire.org, a site to advertise and apply for mathematics jobs, supported by the EMS and German Mathematical Society.

In another recent development, the Simons Foundation is now supporting mathematics in Africa through the EMS, with a fund of 50,000 Euros per year for five years. A scheme to distribute this money has been developed by the Committee for Developing Countries (CDC).

The President continued his report on the Society's wide range of scientific activities, and encouraged member societies to submit proposals for activities such as Joint Mathematical Weekends and Summer Schools.

The EMS Treasurer (and mathematical biologist) Mats Gyllenberg then delivered a short presentation on the forthcoming Year of Mathematical Biology in 2018, a joint venture of the EMS and the European Society for Mathematical and Theoretical Biology.

The President completed his address with a round-up of recent developments around Horizon 2020 and ERC, notably running calls, the open EU consultation on Mathematics, and the High-Level Group of Scientific Advisors of the European Commission Scientific Advice Mechanism. He concluded with a brief discussion of bodies of which the EMS is itself a member: the Initiative for Science in Europe and International Mathematical Union.

Presentations

The meeting then enjoyed several presentations. First, Klavdija Kutmar delivered a report on preparations for the 8th ECM (Portorož, Slovenia, 5–11 July 2020). Ian Strachan, President of the Edinburgh Mathematical Society (EdMS), then told the meeting about that society. Founded in 1883, the EdMS was initially geared towards mathematics schoolteachers (although the knot theorist Peter Guthrie Tait was an early member). Reflecting on a tempestuous political period in Scotland (two referenda and a general election in the last 3 years), Ian ended his report with a challenge: How can we get members of our professional communities to engage with policy issues, funding mechanisms, and politicians?

Albert Erkip, Vice-President of the Turkish Mathematical Society, presented an update on the organization of the second Caucasian Mathematics Conference (CMC-II), to be held in Van (Turkey), 22–24 August 2017.

Waclaw Marzantowicz, President of the Polish Mathematical Society, announced a joint meeting of the Italian Mathematical Union, the Italian Society of Industrial and Applied Mathematics, and the Polish Mathematical Society, to be held in Wrocław (Poland), 17–20 September 2018.

Christian Kassel, representing the Société Mathématique de France, presented a report on that society's work. He announced an ambitious extension to CIRM (Centre International de Rencontre Mathématiques), already one of the largest mathematical conference centres in the world (co-funded by SMF, Centre National de la Recherche Scientifique (CNRS), and Aix-Marseille University). He reminded the meeting that CIRM's calls for proposals are already open. The SMF organises conferences for researchers and competitions for University students, and also campaigns for mathematics. For example, in the run-up to the 2017 French Presidential election, the SMF co-authored with other French scientific organisations a letter in Le Monde, calling on all candidates not to forget the importance of science to France.

Discussion

There followed a session of wide-ranging open discussion, which focussed first on Open Access publishing and related topics. With the publishing landscape having changed with innovations such as ArXiv.org, the principal role of journals is now to provide a trusted stamp of quality. Technology may offer new mechanisms, for instance via overlay journals such as episciences.org. However, there are dangers lurking in this changing landscape, notably in models where the author pays to publish, and in the fact that many learned societies currently derive much of their income from journals. The EMS's new Publications and Electronic Dissemination Committee (see separate report from the Executive Committee meeting in Bratislava) will be actively considering such matters, and member societies are invited to make their views known. Likewise, the EMS's Education committee will be contemplating the perennially controversial subject of mathematical education; ideas for suitable lines of investigation are again welcome from member societies.

Close

On behalf of all the participants, the Chair thanked the local organizers for their impeccable preparation and warm hospitality. Before departing, the participants enjoyed an exhibition at the Fundação Calouste Gulbenkian by the artist José de Almada Negreiros, whose giant mural Começar, 1968–69, contains many tantalising geometrical details.

The next meeting of Presidents will be held 13–15 April 2018, in Dublin.

Writing Positive Polynomials as Sums of (Few) Squares

Olivier Benoist (Université de Strasbourg, CNRS, France)

In 1927, Artin proved that a real polynomial that is positive semidefinite is a sum of squares of rational functions, thus solving Hilbert's 17th *problem. We review Artin's Theorem and its posterity, browsing through basic examples, classical results and recent developments. We focus on a question first considered by Pfister: can one write a positive semidefinite polynomial as a sum of* few *squares?*

1 Hilbert's 17th problem

A real polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is said to be *positive semidefinite* if $f(x_1, \ldots, x_n) \geq 0$ for all $x_1, \ldots, x_n \in \mathbb{R}$.

Artin's Theorem. Can one explain the positivity of such a polynomial by writing it as a sum of squares? The question (popularised by Hilbert as the 17th of his famous list of open problems that he proposed on the occasion of the 1900 International Congress of Mathematicians) was solved by Artin [1]:

Theorem 1.1 (Artin, 1927)**.** Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be positive semidefinite. Then, f is a sum of squares in $\mathbb{R}(X_1, \ldots, X_n)$.

Artin's proof of Theorem 1.1 was influential, fostering the development of real algebra. In collaboration with Schreier, and with Hilbert's 17th problem in mind, he had developed the theory of ordered fields [2]. A consequence of their work is that an element of a field K may be written as a sum of squares in K if and only if it is nonnegative with respect to all the orderings of K that are compatible with the field structure. It remains to show that if f is negative with respect to some ordering of $\mathbb{R}(X_1, \ldots, X_n)$, its evaluation at some point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is also negative. This "specialisation argument" is at the heart of the proof.

Sums of squares of polynomials. It would seem more natural to look for an expression of f as a sum of squares of polynomials, but this is too much to ask! It was Minkowski who convinced Hilbert, during his doctoral dissertation in 1885, that such a statement would be too strong. Three years later, in a seminal paper [14], Hilbert was able to show, by abstract means, the existence of counterexamples. Surprisingly, the first explicit counterexample appeared only much later, in 1967, and almost by chance. The polynomial

$$1 + X_1^2X_2^4 + X_1^4X_2^2 - 3X_1^2X_2^2, \qquad (1)$$

introduced by Motzkin [21] for other purposes, was recognised by Taussky–Todd to be the first concrete example of a positive semidefinite polynomial that is not a sum of squares of polynomials. It is positive semidefinite as a consequence of the arithmetic mean–geometric mean inequality and it satisfies the conclusion of Theorem 1.1 since it coincides with

$$\frac{(X_1^3X_2 + X_2^3X_1 - 2X_1X_2)^2(1 + X_1^2 + X_2^2) + (X_1^2 - X_2^2)^2}{(X_1^2 + X_2^2)^2}$$

but an elementary analysis of the low degree terms in a hypothetical expression of (1) as a sum of squares of polynomials quickly leads to a contradiction.

Sums of few squares. How many squares are needed in Theorem 1.1? A beautiful answer, surprisingly dependent only on the number of variables, was obtained by Pfister [22].

Theorem 1.2 (Pfister, 1967)**.** Let $f \in \mathbb{R}[X_1, \ldots, X_n]$ be positive semidefinite. Then, f is a sum of 2^n squares in $\mathbb{R}(X_1, \ldots, X_n)$.

Pfister's work is completely independent of Artin's. Indeed, what Pfister really proves is that any sum of squares in $\mathbb{R}(X_1, \ldots, X_n)$ is in fact a sum of 2^n squares. It is only in combination with Theorem 1.1 that Theorem 1.2 is obtained. His result stemmed from important progress in the algebraic theory of quadratic forms: the discovery of the so-called Pfister forms (which enjoy marvellous algebraic properties).

In three variables, Theorem 1.2 had previously been obtained by Ax. It is while reading Ax's manuscript that Pfister realised one could replace the cohomological tools of Ax by the use of Pfister forms, yielding a result in arbitrary dimension. It may not be a coincidence that Pfister forms later turned out to be the key to a cohomological classification of quadratic forms over fields, culminating in Voevodsky's proof of the Milnor conjecture [32].

We refer to [23] for a nice exposition of Theorems 1.1 and 1.2. Whether the 2^n bound in Theorem 1.2 may be improved or not (Question 2.6 below) is the main topic of this survey.

2 Polynomials of low degree or in few variables

Let us illustrate the theorems of Artin and Pfister, starting from basic cases. Let $\mathbb{R}[X_1, \ldots, X_n]_d$ be the space of polynomials of degree d. We consider a positive semidefinite polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]_d$. Of course, since an odd degree polynomial changes sign, the degree d of f must be even.

· **d = 2.** A degree 2 polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ may be homogenised to a quadratic form $F \in \mathbb{R}[X_0, X_1, \ldots, X_n]$ that is positive semidefinite if f is. Diagonalising the quadratic form F displays it as a sum of $n + 1$ squares of linear forms. Dehomogenising, we see that f is a sum of $n + 1$ squares of polynomials.

As soon as $n \geq 2$, this is a significant improvement on the 2^n bound of Theorem 1.2! Since it is not very impressive to write a quadratic form as a sum of squares, this particular case should not be viewed as representative of the general situation.

· **n = 1.** A polynomial $f \in \mathbb{R}[X]$ in one variable may be factored as a product of irreducible real polynomials:

$$f = \lambda \prod_i (X - a_i) \prod_j (X^2 + 2b_j X + c_j).$$

If f is positive semidefinite, the multiplicity of a_i as a root of f has to be even, and letting $X \to \infty$ implies that $\lambda \geq 0$. Completing the square $X^2 + 2b_j X + c_j = (X + b_j)^2 + (c_j - b_j^2)$ shows that f may be written as a product of sums of two squares of polynomials. The classical identity

$$(A^2 + B^2)(C^2 + D^2) = (AC + BD)^2 + (AD - BC)^2 \quad (2)$$

then implies that f is the sum of two squares of polynomials. We have recovered Theorems 1.1 and 1.2 in a stronger form: it was enough to consider sums of squares of polynomials!

The use of identity (2) is not innocent. In some sense, the contribution of Pfister in his proof of Theorem 1.2 was precisely to find a systematic way to produce identities analogous to (2) in more variables.

· **n = 2 and d = 4.** Degree 4 polynomials in two variables, or, as classical geometers say, *ternary quartics*, are particularly interesting. They have been studied in detail by Hilbert [14], who proved:

Theorem 2.1 (Hilbert, 1888)**.** Let $f \in \mathbb{R}[X_1, X_2]$ be positive semidefinite of degree 4. Then, f is a sum of 3 squares in $\mathbb{R}[X_1, X_2]$.

Not only is f a sum of squares of polynomials (rather than mere rational functions) but the 2^n bound of Theorem 1.2 may also be improved!

In contrast with the $d = 2$ and the $n = 1$ cases above, Theorem 2.1 is a non-trivial result. Hilbert's proof (one of the first applications of topology to algebra) is beautiful. His idea is to start with a ternary quartic that is obviously a sum of three squares of polynomials, such as $f_0 = 1 + X_1^4 + X_2^4$, to carefully choose a path from f_0 to f in the space of positive semidefinite ternary quartics and to deform the representation of f_0 as a sum of three squares to one for f.

We refer to [26] or [31] for modern accounts of Hilbert's proof and to [28] for recent developments.

· **n = 2 and d ≥ 6.** The behaviour of positive semidefinite polynomials in two variables stabilises from degree 6 onward. Hilbert proved in [15] that they are sums of squares in $\mathbb{R}(X_1, X_2)$ and Landau [18, p. 282], analysing Hilbert's proof, noticed that only 4 squares are needed.

Theorem 2.2 (Hilbert, 1893)**.** Let $f \in \mathbb{R}[X_1, X_2]$ be positive semidefinite. Then, f is a sum of 4 squares in $\mathbb{R}(X_1, X_2)$.

Theorem 2.2 is a particular case of Theorem 1.2 in two variables: Pfister's theorem is nothing but a higher-dimensional generalisation of Hilbert's classical result. Hilbert's argument, an elaboration of his proof of Theorem 2.1, is more difficult than Pfister's. It is also more precise. For instance, it allows one to control the denominators in an expression of f as a sum of 4 squares: if f has degree d, the denominators may be chosen to have degree $\leq \left\lfloor \frac{(d-2)^2}{8} \right\rfloor$.

We have already seen that, even in degree 6, one cannot expect to improve on Theorem 2.2 by requiring that f is a sum of squares of polynomials: Motzkin's polynomial (1) is a counterexample. It is natural to ask whether one could strengthen Theorem 2.2 by writing f as a sum of 3 squares. Again, the answer is negative when $d \geq 6$ and the first known counterexample, discovered by Cassels, Ellison and Pfister [7], was … Motzkin's polynomial!

Theorem 2.3 (Cassels-Ellison-Pfister, 1971)**.** Motzkin's polynomial (1) is not a sum of 3 squares in $\mathbb{R}(X_1, X_2)$.

Although it is elementary to verify that Motzkin's polynomial is not a sum of squares in $\mathbb{R}[X_1, X_2]$, showing that it is not a sum of 3 squares in $\mathbb{R}(X_1, X_2)$ requires a little bit of algebraic geometry. In [7], the authors use the precise form of (1) to produce an elliptic surface whose properties control the potential of writing Motzkin's polynomial as a sum of 3 squares and they study it in detail.

One is left to wonder how frequent the sums of 3 squares are. What does the subset of $\mathbb{R}[X_1, X_2]_d$ consisting of polynomials that can be written as sums of 3 squares in $\mathbb{R}(X_1, X_2)$ look like? The first result in this direction, due to Colliot-Thélène [9, 4.3], indicates that they are quite scarce.

Theorem 2.4 (Colliot-Thélène, 1993)**.** If $d \geq 6$, the degree d polynomials that are sums of 3 squares in $\mathbb{R}(X_1, X_2)$ form a meagre subset of measure 0 of $\mathbb{R}[X_1, X_2]_d$.

Hence, sums of 3 squares are negligible both from the topological (*meagre* means a countable union of nowhere dense subsets) and measure theory points of view (cf. Section 4 for an account of the proof).

Despite Theorem 2.4, sums of 3 squares turn out to be dense in the set of positive semidefinite polynomials [3].

Theorem 2.5 (2017)**.** Any degree d positive semidefinite polynomial $f \in \mathbb{R}[X_1, X_2]$ may be approximated by degree d polynomials that are sums of 3 squares in $\mathbb{R}(X_1, X_2)$.

The picture to have in mind is the following. The set of polynomials that may be written as sums of 3 squares of rational functions whose denominators have degree $\leq N$ is a closed subset of $\mathbb{R}[X_1, X_2]_d$. Taking the union on all integers N, we get a countable union of closed subsets and it is only this union that one may hope to be dense. In other words, when approximating a polynomial that is not itself a sum of 3 squares, the degrees of the denominators must grow to infinity. The author is unaware of a constructive approach to Theorem 2.5. In particular, can one write Motzkin's polynomial (1) explicitly as a limit of sums of 3 squares?

· **n ≥ 3 and d ≥ 4.**. In at least 3 variables (and degree ≥ 4), no further general result expressing a positive semidefinite polynomial as a sum of squares of polynomials holds true, as discovered by Hilbert [14]. It is hard to resist writing down a beautiful example, due to Lax and Lax [19], of a degree 4 positive semidefinite polynomial in 3 variables that is not a sum of squares of polynomials:

$$\sum_{i=1}^{5} \prod_{j \neq i} (X_i - X_j).$$

Its five variables are a smokescreen: it only depends on the four homogeneous variables $X_1 - X_2$, $X_2 - X_3$, $X_3 - X_4$ and $X_4 - X_5$, giving rise, after dehomogenisation, to a polynomial in three variables. A survey by Reznick [25] contains many more examples.

In contrast, whether the bound 2^n in Pfister's Theorem 1.2 is optimal or not remains completely mysterious.

Question 2.6 (Pfister)**.** Does there exist a positive semidefinite polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ that is not a sum of $2^n - 1$ squares in $\mathbb{R}(X_1, \dots, X_n)$?

This question was raised by Pfister immediately upon proving Theorem 1.2 and in general is still open today. It is arguably the most important problem of the subject. Defining the *Pythagoras number* $\mathrm{p}(K)$ of a field K to be the smallest $p \in \mathbb{N}$ such that every sum of squares in K is actually a sum of p squares (or $+\infty$ if no such integer exists), Question 2.6 may be reformulated as follows.

Question 2.7. Is $\mathrm{p}(\mathbb{R}(X_1, \dots, X_n))$ equal to 2^n?

To see the equivalence, one can reduce to studying polynomials by multiplying a rational function by the square of its denominator and use Artin's Theorem 1.1, according to which the polynomials that are sums of squares in $\mathbb{R}(X_1, \dots, X_n)$ are exactly those that are positive semidefinite.

We have already seen that Questions 2.6 and 2.7 have positive answers when $n = 1$ (as $1 + X_1^2$ is not a square) and when $n = 2$ (by Cassels-Ellison-Pfister's Theorem 2.3). When $n \geq 3$, the state of the art is the inequality

$$n + 2 \leq \mathrm{p}\left(\mathbb{R}(X_1, \dots, X_n)\right) \leq 2^n,$$

where the upper bound is Pfister's Theorem 1.2 and the lower bound is an easy consequence of the Cassels-Ellison-Pfister Theorem [23, p.97].

By analogy with Hilbert's Theorem 2.1, one may expect to obtain better bounds if one restricts to low degree polynomials. This point of view was taken in [4], yielding the following result.

Theorem 2.8 (2016)**.** Let $f \in \mathbb{R}[X_1, \dots, X_n]_d$ be positive semidefinite. If $n \geq 2$ and $d \leq 2n$, f is a sum of $2^n - 1$ squares in $\mathbb{R}(X_1, \dots, X_n)$, except maybe if $n \geq 7$ is odd and $d = 2n$.

This improvement on Theorem 1.2 seems incremental but is already new for degree 4 polynomials in three variables. In this case, it shows that a positive semidefinite polynomial is a sum of 7 squares. It is not known if this can be improved.

The hypothesis that $n \geq 2$ cannot be dispensed with, as attested by the polynomial $1 + X_1^2$. It is, however, likely that the result continues to hold when $n \geq 7$ is odd and $d = 2n$.

It may be expected that the degree range $d \leq 2n$ appearing in Theorem 2.8 is the correct one, in the sense that, from degree $d = 2n+2$ onward, there would exist positive semidefinite polynomials that are not sums of $2^n - 1$ squares, hence giving a positive answer to Question 2.6. We will give a geometric interpretation for this value of the degree at the end of Section 4.

3 A rich legacy

Hilbert's 17th problem has triggered developments in many other directions. A few will be listed here, without any attempt to be exhaustive.

Arithmetic base fields. What if the coefficients of f belong to a smaller field, say the field $\mathbb{Q}$ of rational numbers? Then, it was already known to Artin [1] that f is a sum of squares in $\mathbb{Q}(X_1, \dots, X_n)$. On the other hand, obtaining a bound à la Pfister on the number of squares involved is much harder. The best result to date is the following arithmetic geometry masterpiece.

Theorem 3.1 (Jannsen, 2016)**.** Let $f \in \mathbb{Q}[X_1, \dots, X_n]$ be positive semidefinite. If $n \geq 2$, f is a sum of 2^{n+1} squares in $\mathbb{Q}(X_1, \dots, X_n)$.

This theorem was found to follow from two outstanding conjectures by Colliot-Thélène and Jannsen [10]: the Milnor conjecture established by Voevodsky [32] and Kato's cohomological local-global principle eventually settled by Jannsen in [16].

The hypothesis $n \geq 2$ is necessary. When $n = 0$, the optimal statement is Euler's precursor of Lagrange's Theorem, according to which a non-negative rational number is a sum of 4 squares of rational numbers [12]. When $n = 1$, Pourchet [24] has proved that a positive semidefinite polynomial $f \in \mathbb{Q}[X]$ is a sum of 5 squares and his result is the best possible. When $n \geq 2$, it is not known whether Jannsen's bound is optimal. In the terminology introduced in Section 2, is the Pythagoras number $\mathrm{p}(\mathbb{Q}(X_1, \dots, X_n))$ equal to 2^{n+1} for $n \geq 2$?

Effectivity. Artin's proof of Theorem 1.1, relying on Zorn's lemma, is not constructive. The search for effective proofs was initiated by Kreisel, allowing one to derive bounds on the degrees of the rational functions involved. The history of this line of thought is explained in Delzell's survey [11]. Lombardi, Perrucci and Roy [20] have recently obtained the following theorem.

Theorem 3.2 (Lombardi, Perrucci, Roy, 2014)**.** A positive semidefinite polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ may be written as a sum of squares of rational functions of degree $\leq 2^{2^{2^{d^{4^n}}}}$.

The reader should not be intimidated by the formidable bound: it is a tremendous improvement on previous results!

Positivstellensätze. What can be said if the polynomial f is only known to be nonnegative on some domain $\Omega \subset \mathbb{R}^n$? The following theorem, due to Stengle [30] but close to independent earlier work of Krivine [17], is the prototype of such a result: a *Positivstellensatz*.

Theorem 3.3 (Krivine, Stengle, 1974)**.** Let $f \in \mathbb{R}[X_1, \dots, X_n]$ be positive on $\Omega := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_k(x) \geq 0\}$, where $g_1, \dots, g_k \in \mathbb{R}[X_1, \dots, X_n]$. Then, f belongs to the sub-semiring of $\mathbb{R}(X_1, \dots, X_n)$ generated by the g_i and the squares.

Given the many counterexamples to the variant of Hilbert's 17^{th} problem involving polynomials instead of rational functions, it came as a surprise when Schmüdgen [29] obtained a Positivstellensatz without denominators, at the expense of a compactness hypothesis.

Theorem 3.4 (Schmüdgen, 1991)**.** Under the assumptions of Theorem 3.3, if Ω is compact, f belongs to the sub-semiring of $\mathbb{R}[X_1, \dots, X_n]$ generated by the g_i and the squares.

The reader will find these statements and many more in a survey by Scheiderer [27].

The cone of sums of squares. What positive semidefinite polynomials $f \in \mathbb{R}[X_1, \dots, X_n]$ are sums of squares of polynomials? We have already seen in Section 2 that some positive semidefinite polynomials are not (unless $n = 1$ or $d = 2$ or $n = 2$ and $d = 4$) but which ones? More precisely, can one describe a set of linear inequalities defining the closed convex cone $\Sigma_{n,d} \subset \mathbb{R}[X_1, \dots, X_n]_d$ consisting of polynomials that are sums of squares of polynomials?

A full answer has been obtained by Blekherman [5] in the first two cases where $\Sigma_{n,d}$ does not coincide with the set of positive semidefinite polynomials: ternary sextics ($n = 2$ and $d = 6$) and quaternary quartics ($n = 3$ and $d = 4$). Surprisingly, the required linear inequalities are precisely those that were introduced by Hilbert [14] to show the existence of positive semidefinite polynomials that are not sums of squares of polynomials.

4 Sums of fewer squares

The theorems that express positive semidefinite polynomials as a sum of fewer squares than predicted by Pfister's Theorem 1.2, as well as those that show that it is impossible, use tools from algebraic geometry. More precisely, they rely on the study of *algebraic cycles*, that is, of the algebraic subvarieties of a fixed algebraic variety.

The first indication of such a link was Cassels, Ellison and Pfister's proof of Theorem 2.3. Its main step is the computation of the Mordell-Weil group of an elliptic curve over a function field [7, §7], a problem essentially equivalent to the determination of all algebraic curves lying on an elliptic surface.

The relation between sums of squares and algebraic cycles is much more transparent in Colliot-Thélène's proof of Theorem 2.4 [9]. We devote the greater part of this section to explaining its principle. The way algebraic cycles enter Theorems 2.5 and 2.8 is similar and we will comment on these proofs too. Our main goal is to understand how algebraic geometry governs the dependence of the properties of a positive semidefinite polynomial on its degree.

Sums of 3 *squares in* $\mathbb{R}[X_1, X_2]$. Fix a degree d positive semidefinite polynomial $f \in \mathbb{R}[X_1, X_2]_d$. We want to understand under which conditions f is a sum of 3 squares in $\mathbb{R}(X_1, X_2)$ and to deduce, following Colliot-Thélène [9], that this is a rare phenomenon when $d \geq 6$.

We reformulate this property. Let Y be a square root of $-f$ and consider the quadratic field extension $K := \mathbb{R}(X_1, X_2)[Y]$ of $\mathbb{R}(X_1, X_2)$.

Lemma 4.1. That f is a sum of 3 squares in $\mathbb{R}(X_1, X_2)$ is equivalent to -1 being a sum of 2 squares in K.

Proof. This is elementary and we only verify the direct implication, which is the one we actually use. If $f = a^2 + b^2 + c^2$ in $\mathbb{R}(X_1, X_2)$, dividing out by $-f = Y^2$ yields an identity of the form $-1 = r^2 + s^2 + t^2$ in K. Applying (2) cleverly, one gets

$$-1 = \left(\frac{rs + t}{1 + r^2}\right)^2 + \left(\frac{s - rt}{1 + r^2}\right)^2. \qquad \square$$

We are reduced to understanding when -1 is a sum of 2 squares in K. To do so, we introduce the geometric incarnation of K, that is, the algebraic surface S defined as a set by

$$\{(x_1, x_2, y) \in \mathbb{C}^3 \mid y^2 + f(x_1, x_2) = 0\}. \tag{3}$$

To be more precise, what we will really denote by S is the compactification of (3) obtained by adding "points at infinity". We will also assume that f has been chosen so that S has no singularities. Elements of K may be viewed as functions on S (which may not be defined everywhere: there may be poles). For this reason, K is called the *function field* of S.

The proof of Theorem 2.4 depends on the knowledge of algebraic curves on S, that is, of the subsets $C \subset S$ of complex dimension 1 that are defined by polynomial equations. There are obvious algebraic curves on S, defined by a single polynomial equation $g \in \mathbb{C}[X_1, X_2, Y]$:

$$C = \{(x_1, x_2, y) \in S \mid g(x_1, x_2, y) = 0\}. \tag{4}$$

There may, however, be more! This happens, for instance, if the restriction of f to some complex line in $\mathbb{C}^2$ is the square of a polynomial. Indeed, suppose that the line is defined, say, by the equation $X_2 = 0$ and that $f(X_1, 0) = h(X_1)^2$ for some $h \in \mathbb{C}[X_1]$. Then, the algebraic curve $C = \{(x_1, x_2, y) \in S \mid x_2 = 0\}$ splits into two algebraic curves in S:

$$\begin{cases} C_+ = \{(x_1, x_2, y) \in S \mid x_2 = 0 \text{ and } y = +ih(x_1)\}, \\ C_- = \{(x_1, x_2, y) \in S \mid x_2 = 0 \text{ and } y = -ih(x_1)\}, \end{cases}$$

which are not individually of the form (4). Moreover, one can check that, if $d \leq 4$, there always exist such lines in $\mathbb{C}^2$, so that not all curves in S are of the form (4). When $d \geq 6$, the situation is completely different.

Theorem 4.2 (Noether–Lefschetz)**.** If $d \geq 6$ and f is very general, all algebraic curves $C \subset S$ are of the form (4).

Here, *very general* is the algebraic geometers' way to indicate a generic behaviour: it means that the statement holds for all f except maybe for those that belong to a countable union of algebraic subvarieties of the parameter space $\mathbb{C}[X_1, X_2]_d$. It implies that the set of those f for which the statement fails is meagre and of measure 0.

We may now conclude the proof of Theorem 2.4. Suppose that $-1 = u^2 + v^2$ is a sum of two squares in K. Then, it can be claimed that the algebraic curve

$$\Gamma = \{(x_1, x_2, y) \in S \mid u = 0 \text{ and } v = i\} \subset S$$

is not of the form (4). By the Noether–Lefschetz Theorem 4.2, this can only happen for very particular choices of f, and Theorem 2.4 is proven.

Let us give a hint for the purely topological argument proving the above claim. Remember that our polynomial f has real coefficients. Consequently, the complex conjugation

$$\sigma : (x_1, x_2, y) \mapsto (\overline{x_1}, \overline{x_2}, \overline{y})$$

induces an involution of S. Triangulating the curve Γ yields a class $[\Gamma] \in H_2(S, \mathbb{Z})$ in the homology of S that is called the fundamental class of Γ (indeed, as a complex curve, Γ is topologically a surface). One can verify that this homology class cannot be realised by a 2-cycle that is invariant under the action of the complex conjugation, whereas the fundamental classes of all the algebraic curves of the form (4) can!

Why degree 6*?* The above proof indicates a reason why positive semidefinite polynomials in two variables exhibit different behaviours when $d \leq 4$ (Theorem 2.1) and $d \geq 6$ (Theorem 2.4). This is due to the Noether–Lefschetz Theorem! When $d \leq 4$, the associated surface S has a rich geometry and contains plenty of algebraic curves but when $d \geq 6$, a typical S contains only obvious algebraic curves.

Still, our understanding is not yet complete: why is the Noether–Lefschetz Theorem only valid in degree ≥ 6? Since it will also be important in our discussion of Theorem 2.5, we explain this now. The main tool is *Hodge theory*.

We need to understand when the surface S contains unexpected algebraic curves that are not of the form (4). To do so, we fix a homology class $\gamma \in H_2(S, \mathbb{Z})$ and we consider the question: when is γ the fundamental class of an algebraic curve $\Gamma \subset S$ or, rather, a linear combination with integral coefficients of such classes?

A necessary condition is that if ω is a holomorphic 2-form on S (for every $s \in S$, ω_s is an alternating $\mathbb{C}$-bilinear form on the tangent space of S at s varying holomorphically with s), the integral $\int_\gamma \omega$ needs to vanish. This is a simple dimension argument: ω vanishes in restriction to algebraic curves on S because there are no non-zero alternating $\mathbb{C}$-bilinear forms on a one-dimensional $\mathbb{C}$-vector space. This condition also turns out to be sufficient. Denoting by $\Omega^2(S)$ the space of holomorphic 2-forms on S, this is the famous Lefschetz (1, 1) Theorem.

Theorem 4.3 (Lefschetz (1, 1))**.** A class $\gamma \in H_2(S, \mathbb{Z})$ is a linear combination of classes of algebraic curves on S if and only if $\int_\gamma \omega = 0$ for every $\omega \in \Omega^2(S)$.

If $d \leq 4$, one can compute that $\Omega^2(S) = 0$. Consequently, Theorem 4.3 predicts the existence of many algebraic curves on S, in particular of curves not of the shape (4).

On the other hand, if $d \geq 6$, one can check that $\Omega^2(S) \neq 0$. The Lefschetz (1, 1) Theorem then gives non-trivial obstructions to the existence of algebraic curves on S and one can verify that, for most values of f, these obstructions prevent the existence of any curve not of the form (4). This proves Theorem 4.2.

This completely explains why the properties of positive semidefinite polynomials $f \in \mathbb{R}[X_1, X_2]_d$ change when $d \geq 6$. It is the influence of the geometry of the associated surface S that carries non-zero holomorphic 2-forms if and only if $d \geq 6$.

Density. Now that we have understood why there are few semidefinite polynomials that are sums of 3 squares (in degree $\geq$ 6), let us explain why these are dense in the set of positive semidefinite polynomials (Theorem 2.5). Recall that we have associated to a degree d polynomial $f \in \mathbb{C}[X_1, X_2]_d$ an algebraic surface S defined by (3):

$$y^2 + f(x_1, x_2) = 0,$$

and explained that if f is a sum of 3 squares in $\mathbb{R}(X_1, X_2)$, the surface S carries more algebraic curves than expected. The archetype of the density result we need has essentially been obtained by Ciliberto, Harris, Miranda and Green [8].

Theorem 4.4 (Ciliberto, Harris, Miranda, Green)**.** The set of $f \in \mathbb{C}[X_1, X_2]_d$ such that the associated surface S contains algebraic curves not of the form (4) is dense in $\mathbb{C}[X_1, X_2]_d$.

Of course, this cannot imply Theorem 2.5 because it says nothing about density in $\mathbb{R}[X_1, X_2]_d$. Proving Theorem 2.5 requires an adaption over $\mathbb{R}$, carried out in [3], of the techniques of [8].

Let us explain what enters the proof of Theorem 4.4 and of its real variant yielding Theorem 2.5. One has to analyse how the obstructions to the existence of algebraic curves on S that are provided by Theorem 4.3 vary with $f \in \mathbb{C}[X_1, X_2]_d$. This amounts to understanding the variation with f of the integrals $\int_\gamma \omega = 0$, called the *periods* of the surface S. Since the work of Griffiths, this very classical topic has been known as the study of *infinitesimal variations of Hodge structures*. Both [8] and [3] rely on these modern tools.

More variables. To study a positive semidefinite polynomial $f \in \mathbb{R}[X_1, \dots, X_n]$ in $n \geq 3$ variables, it is still useful to introduce the algebraic variety X defined by the equation

$$y^2 + f(x_1, \dots, x_n) = 0 \tag{5}$$

and its function field $K = \mathbb{R}(X_1, \dots, X_n)[Y]$, where $Y = \sqrt{-f}$. An analogue of Lemma 4.1 holds: f is a sum of $2^n - 1$ squares in $\mathbb{R}(X_1, \dots, X_n)$ if and only if -1 is a sum of 2^{n-1} squares in K. When $n \geq 3$, this is not as elementary as Lemma 4.1 and relies on Pfister's work on quadratic forms.

Relating the latter property to algebraic cycles on X depends on the far-reaching enhancement of Pfister's work provided by Voevodsky's proof of the Milnor conjecture [32]. This allows a cohomological reformulation: letting the group $G := \mathbb{Z}/2\mathbb{Z}$ act on X by the complex conjugation

$$\sigma : (x_1, \dots, x_n, y) \mapsto (\overline{x_1}, \dots, \overline{x_n}, \overline{y}),$$

there exists a class $\alpha \in H^n_G(X, \mathbb{Z}/2\mathbb{Z})$ in G-equivariant cohomology such that -1 is a sum of 2^{n-1} squares in K if and only if α vanishes in the complement of an algebraic subvariety of X. Intuitively, this will happen if and only if X contains many algebraic subvarieties.

Proving Theorem 2.8 requires one to show that if $d \leq 2n$, the variety X contains many algebraic subvarieties in this sense. The Hodge theory arguments are of no use now and [4] relies on other methods, such as Bloch-Ogus theory. Let us explain the origin of the hypothesis $d \leq 2n$ on the degree. As before, it reflects a geometric property of the algebraic variety X. Namely, it ensures that X is *rationally connected*: that there exist enough algebraic maps $\mathbb{P}^1 \to X$ to connect any two points $p, q \in X$. It has been known since Bloch and Srinivas [6] that this geometric information gives strong control on the cohomology of X, which is exactly what is needed.

In contrast, when trying to answer Question 2.6, one has to show that the algebraic variety X may contain only few algebraic subvarieties if the degree d is high enough (maybe if $d \geq 2n + 2$?). When $n = 3$, which is the smallest value for which Question 2.6 is open, the required statement is a variant of a classical question asked by Griffiths and Harris in [13]. To give a flavour of what is needed, we state a slightly different question, closer to the one raised in [13]. Recall that the *degree* of an algebraic curve $C \subset X$ is the cardinality, taking multiplicities into account, of the set

$$\{(x_1, \ldots, x_n, y) \in C \mid x_1 = 0\}.$$

Question 4.5. Let $f \in \mathbb{C}[X_1, X_2, X_3]_d$ and X be defined by (5). If f is very general and $d \geq 10$, are all algebraic curves in X of even degree?

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Interview with Abel Laureate Yves Meyer

Bjørn Ian Dundas (University of Bergen. Norway) and Christian Skau (Norwegian University of Science and Technology, Trondheim, Norway)

Professor Yves Meyer, congratulations on being awarded the Abel Prize 2017 for your pivotal role in the development of the mathematical theory of wavelets. You will receive the Abel Prize from His Majesty the King of Norway in a ceremony tomorrow. The history of wavelets is fascinating and some aspects of it are old, but before we delve deeper into the mathematical side of things, perhaps you could share a bit of your personal journey.

Becoming a mathematician

You spent most of your childhood in Tunis. You attended the Lycée Carnot, which we understand was a very intellectually stimulating environment. But you were interested in many subjects. Why did you turn toward mathematics?
Yes, that was not an obvious choice because I was more interested in humanities. I was in love with Socrates and Plato, and I am still reading Plato right now, day after day, night after night. I am no longer reading Plato in Greek but I used to do that. I would say my main interest is literature. The point is that I am a bad writer. That is my bad side. So, I took mathematics because I was gifted – I was unusually gifted in mathematics. I cannot explain that. I understood mathematics from the inside in a very natural way. When I was in high school, I understood mathematics by myself and not by listening to my teachers.

So, you did not have any role models you found inspiring?
I had very good teachers and the teachers assessed me as being gifted in mathematics. The teachers were a stimulation but I had my own perception of mathematics. I had naive misconceptions. For example, I was thinking that all functions were continuous. And for me, you know, it was obvious and my mathematics was the mathematics of the time of Euler. So, not only were all functions continuous but they were all analytic.

Euler was also Abel's teacher! Abel learned mathematics from reading Euler.
That's beautiful – so we are back to Abel! It took me a while to understand that mathematics was not the toy I was playing with in my childhood. There were distinct subjects, so I had to mature to that fact when I was 19. That was rather difficult because, for me, mathematics was obvious. I always found a solution of a problem but with my own way of thinking, which was not canonical.

Yves Meyer received the Abel Prize from King Harald of Norway. (Photo: Thomas Brun)

So, in 1956, as a 17 year old, when you returned to France in order to prepare for the entering exam to the elite school l'École Normale Supérieure, you had mathematics as a career choice, would you say?
No, I was still hesitating. I took mathematics as a major because I was more gifted in mathematics than in humanities. Also, of course, I had to earn my living so I took mathematics for getting a job.

In 1957, after only one year of training at what in France is called "Classes de Préparation aux Grandes Écoles", you entered l'École Normale Supérieure in Paris, coming first in the entrance examination in mathematics. Could you give us a glimpse of your years there?
When I was at l'École Normale Supérieure, we were mixed with people in humanities. We were about 40 scientists – maths and physics – and 40 kids in humanities. And most of the time, I was discussing with my schoolmates in humanities, spending hours and hours. There was a Japanese student that was admitted as a foreign student, Abe Yoshio was his name – he is dead now. To my great surprise he wrote a memoir about the times at l'École Normale Supérieure. I read very recently the page of his memoir where I was described. According to Abe Yoshio, I am described as the only scientist to whom he could talk. So, for him, I was different, and I felt about myself that I was different. I was not obsessed by science. Now I have completely changed; now I am completely obsessed by science. But that took a part of my life to come around to. But in the beginning – because you always have a certain inertia in your life – since I

had focused on humanities until my Baccalaureate (that is, the end of high school), the influence of humanities remained for about 10 years before I was convinced that mathematics was something absolutely marvellous. In the beginning, it was, in a sense, a little bit too easy for me to pass the exams, which was not doing mathematics at a research level. Then it could not be serious or such a big deal, I thought.

After three years as a teacher at Prytanée National Militaire (an experience we hope we can come back to when we talk about teaching in general), you moved to Strasbourg. Can you tell us something about those years and how you ended up doing your thesis in harmonic analysis?
The atmosphere at the Department of Mathematics at Strasbourg was absolutely marvellous. Because it was a very small department, there were 14 full professors. I was a teaching assistant and there were altogether 14 teaching assistants in the department. All the teaching assistants worked in just one office – a large office – and everyone was smoking. It was impossible to work, so we were just discussing. We were in complete freedom, so we could choose the subject of our PhD just by our own inclination, without a supervisor, so I decided upon my choice of thesis after reading the book by Antoni Zygmund: *Trigonometric Series*. I found the book fascinating and I asked myself what were the important problems in this subject? So I decided what were the important problems and I tried to solve the problems. I wrote 12 chapters of my thesis, my wife typed these 12 chapters and then I asked: "Who could be a supervisor of the thesis?" Pierre Cartier, who was a professor at the Université de Strasbourg, advised me to contact Jean-Pierre Kahane. So I took the train, brought to Jean-Pierre Kahane the 12 chapters and asked him to give me a PhD subject. And he said: "It is ridiculous – you have already written a PhD." And so I got a PhD that way. But if you do it that way it means that you are either stupid or arrogant. The penalty came immediately: at exactly the time I was submitting my thesis, Elias Stein proved a much better theorem. Elias Stein was still at the University of Chicago working with Alberto Calderón and they had made much more progress on the same problem I was doing.

Stein had much stronger tools, didn't he?
Yes, he had much stronger tools.

Number theory and quasicrystals

Is that why you decided to move to Diophantine approximations?
Yes. I was hired at the Université d'Orsay and then I was influenced by Jean-Pierre Kahane. He had a very good influence on me. The idea was that, in general, after you get a PhD, you should change subject because you should not remain under the influence of your supervisor. In my case, I had no supervisor but I decided to change subject anyway. At that time, the book by Jean-Pierre Kahane and Raphaël Salem, *Ensembles parfaits et séries trigonométriques*, appeared. I read the book and I fell in love with it. I decided to solve one of the main problems that Salem could not solve because he died prematurely. That took me about three or four years; it was a problem in number theory.

The keywords here are Pisot and Salem numbers. Actually, the concept of Pisot number was first introduced by the Norwegian mathematician Axel Thue, in connection with Diophantine approximations. A Pisot number is a real algebraic integer θ greater than 1 such that the conjugate numbers to θ have absolute values less than 1. A Salem number has the same definition except that the absolute values should be less than or equal to 1 with at least one having absolute value 1. There is a very fascinating story about these numbers. You won the Salem prize the third time it was awarded in 1970 and that was because you proved a theorem that Salem had posed, which you already alluded to. Could you describe it?
It's a fascinating story. I discovered quasicrystals by accident because they were a tool for solving this problem. The concept of quasicrystals did not exist at that time but it turned out they were exactly the correct tool for solving the problem raised by Salem. In solving this problem, I proved the following theorem, which is my favourite result. I can explain that almost with my bare hands. So, you have something that is now called a Meyer set. I called these sets "quasicrystals" – a precursor of this concept was my definition of a "model set" from as far back as 1972 – but Robert Moody later called them "Meyer sets". So a Meyer set is a set of points in $\mathbb{R}^n$ – so if n is 2 we are in the plane – that has two conflicting properties: the set is spread uniformly, which means that there is a radius R so large that each ball of radius R, whatever the location, contains at least one point, so the points are spread uniformly; but there are no concentrations, which means there is a small radius r such that each ball of radius r, whatever its location, contains at most one point.

This is what is called a Delone set, right?
Delone set, exactly! So, a Meyer set can be defined just by the following property – the definition is due to Jeffrey Lagarias, which improves a little bit on my definition – it is a Delone set Λ such that the set Λ-Λ of all differences is still a Delone set. That is a Meyer set. So that is something I introduced with a seemingly more restrictive definition but Lagarias proved that my definition is identical to this one. And then you ask yourself, is it possible that a Meyer set Λ will be self-similar in the sense that for θ a real number, θ larger than 1, $\theta\Lambda$ would be contained in Λ? For instance, if Λ is an ordinary lattice and θ is a real number then $\theta\Lambda$ is contained in Λ if and only if θ is an integer. Amazingly, for a general Meyer set Λ, this is true if and only if θ is either a Pisot or a Salem number.

That is remarkable!
That is the most beautiful theorem I have proved. I love this theorem! It combines, you know, geometry and number theory. There is no analysis in this theorem, which is

truly remarkable! And the converse is true: that is, if you are given a Pisot or a Salem number θ, there is always a Meyer set Λ such that $\theta\Lambda$ is contained in Λ.

What is the connection with quasicrystals, more specifically?
People discovered that in a very, I would say, accidental way. Once they understood the construction rule of a quasicrystal, which is the construction that I have given in my book *Algebraic Numbers and Harmonic Analysis* from 1972, they observed that there is what they call "inflation" of a quasicrystal, that a quasicrystal is self-similar. But they did not know that at the time because my book was pulped by the publisher Elsevier.

You mean it was not accessible or was out of print?
No, no, not out of print. It was destroyed! Elsevier wrote me a letter asking me for permission to destroy the copies that were left because there were too many copies and it was impossible to sell that garbage anymore, so they thought, I imagine. Of course I accepted because I was doing something else. I was no longer interested in what I had written; it was already remote past, you know.

What a fascinating story! Your book contains material that can aptly be described as a precursor – which went unnoticed for a long time – of quasicrystals. In fact, it essentially contains the abstract theory of the cut-and-project method, in the full generality of locally compact abelian groups. To cut a long story short, Roger Penrose subsequently introduced his tilings in 1974, and later came Dan Shechtman, in 1982, who discovered that you find quasiperiodic crystals in nature (for which he received the Nobel Prize for Chemistry in 2011). Who made the connection with what you had done?
I think Enrico Bombieri made the connection and then Robert Moody, who was an important person in this. Bombieri suspected that there was a connection and it was understood completely by Moody. Moody is a very fair person, a remarkably open-minded person. He read my book in full detail. And he observed that everything was predicted in some sense. Like Nostradamus in some obscure language! One more thing should be mentioned concerning Meyer sets and tilings. If Λ is a Meyer set in $\mathbb{R}^n$ and $V(\Lambda)$ is the associated Voroni cells – these cells are simply connected polytopes – then $V(\Lambda)$ is a tiling of $\mathbb{R}^n$. What is remarkable is that there are only a finite number of Voronoi cells up to translation and so one gets a translation tiling of $\mathbb{R}^n$ by a finite number of prototiles.

Calderón's conjecture

We move to the next big chapter in your mathematical discoveries and that is the solution of Alberto Calderón's conjecture. There is a long story behind that but the crowning achievement was the paper you wrote jointly with Ronald Coifman and Alan McIntosh that was published in 1982. Could you tell the story of this cooperation?
Oh yes, the story is so beautiful. It is, in many ways, an accidental story. It is a story I like very much because it relates to my younger years in Tunis. Arabic people have a tendency to be fatalists. They will say everything is written in the Book. You cannot avoid your fate, your destiny. It was a time when my colleagues in Orsay for some political reasons refused to give graduate courses. They were objecting to a decision by the Minister of Education or something. I hate to follow the crowd and so I decided to give a graduate course anyway, just to prove that I do not follow the crowd. So I gave the graduate course and there was a student following the course who was completely distinct from the other students and who seemed to be much older. So I spoke to this person. He was from Australia and his name was Alan McIntosh. I invited him to have lunch at the end of the course every week. After three weeks, he explained what he was trying to find – his programme. His programme was exactly what I was trying to do with Coifman, but he was a student of another mathematician Tosio Kato. Kato (he is dead now) was working in operator theory but from a very abstract viewpoint. Kato had a general conjecture from which Calderón's conjecture would follow as a simple corollary. Calderón did not know Kato and Kato did not know Calderón. When they were in the US, Calderón was at the University of Chicago and Kato was at Berkeley. McIntosh explained that the problem I was trying to solve could be rephrased in the terminology of Kato. As soon as I got this information, I discussed with Coifman the possibility of solving the problem through this new formulation. Coifman was excited and wrote a kind of draft version of the solution. Then I returned to France and I managed to find the missing points. So, without my discussion with McIntosh, who knows if the problem would have been solved by me? McIntosh did not play any further role but he knew that the problem had a double meaning, that it could be rephrased inside another completely distinct theory, and with this new perspective on the problem, the problem could be solved. So that is the reason why the paper is signed with the three names. Elias Stein was the Editor-in-Chief of the *Annals of Mathematics* at the time and he asked me to write the paper in French because it was clear to Stein that I had solved the problem and that just hints were given by McIntosh and Coifman. But I am very proud to have included McIntosh and Coifman as co-authors. Sadly, McIntosh died from cancer recently.

So this actually sprang out of a graduate course?
Yes, exactly, and just because I dislike following the crowd.

This must be a prime example of solving a problem through rephrasing it in a new mathematical language?
Exactly, exactly. When this happens, it always gives me an intense feeling of happiness. This also shows that personality plays an important role in your mathematical life. The fact that I dislike following the crowd. Otherwise, I would never have met McIntosh. By the way, McIntosh worked with my students for about 30 years before he

died, so it was a great collaboration with the group. I was then doing something else.

Before we drop the Calderón programme and his conjectures, could you tell us briefly what it was all about?
In order to explain what Calderón's conjectures are, let me begin with the end of the story. The goal is the computation of the analytic capacity of a compact set $\mathbb{K}$ in the complex plane $\mathbb{C}$. The analytic capacity of $\mathbb{K}$ is 0 if and only if every function f on $\mathbb{C}\backslash\mathbb{K}$ that is holomorphic and bounded on $\mathbb{C}\backslash\mathbb{K}$ is constant. By Riemann's theorem, the analytic capacity of a single point is 0. The analytic capacity of an arc of a smooth curve is not 0. A problem raised by Painlevé is to find a geometric characterisation of compact sets with vanishing analytic capacity. This has been achieved by Xavier Tolsa and the best reference is the Proceedings of ICM 2006. Tolsa's work relies on what was achieved on Calderón's programme. Calderón asked the following. Let Γ be a closed rectifiable Jordan curve in the complex plane. Let $\mathbb{U}$ be the bounded domain limited by Γ and $\mathbb{V}$ be the exterior domain delimited by Γ. The Hardy space $\mathbb{H}^2(\mathbb{U})$ is the closure of the polynomials in $\mathbb{L}^2(\Gamma, ds)$, while the Hardy space $\mathbb{H}^2(\mathbb{V})$ is the closure in $\mathbb{L}^2(\Gamma, ds)$ of polynomials in the variable $1/z$ vanishing at infinity. Here, ds is the arc-length on Γ. Calderón wanted to know whether $\mathbb{L}^2(\Gamma, ds)$ is the direct sum of $\mathbb{H}^2(\mathbb{U})$ and $\mathbb{H}^2(\mathbb{V})$. I proved this fact when Γ is a Lipschitz curve. Then, Guy David proved it when Γ is a Lavrentiev curve. Finally, David solved the Painlevé problem in a joint effort with Tolsa (David did the first half and Tolsa the second half).

We will now segue from Calderón to wavelet theory, the connecting thread being the so-called Calderón's reproducing identity, which you were intimately familiar with. But before we do that, tell us a little about your encounter with Calderón on a personal level.
I loved discussing with Calderón, also because we could speak in Spanish. I am fluent in Spanish and Calderón was from Argentina. After discussing mathematics, we talked about literature and other expressions of Spanish culture that we appreciated. I liked Calderón very much. He was like a spiritual father for me. He was joking about my political ideas because he was right-wing and I was left-wing, and we talked about Argentina and its political conflicts, which were serious. But even if we disagreed about politics, it was a delight to discuss with him. I have kept in contact with his second wife Alexandra Bellow and from time to time she says that Calderón viewed me as his spiritual son. Yes, I cherish Calderón in a very strong sense.

Wavelets

We now come to a very exciting part of your research centred on wavelets – ondelettes in French. There is a very fascinating story of how you got into this and how your familiarity with some of Calderón's work turned out to be propitious. Could you tell us about this?
My discovery of wavelets is also completely accidental. It came about through the Head of the Department of Mathematical Physics at l'École Polytechnique, Jean Lascoux. I was teaching at l'École Polytechnique and I soon became a friend of Lascoux. Jean Lascoux was spending almost all his life at the photocopying machine. Mathematicians and mathematical physicists were sharing the same Xerox machine. He was making several copies of everything, absolutely everything, and distributing copies around. If you needed to make a copy, you had to wait until he had finished. Instead of being irritated, I liked discussing with Jean Lascoux and we soon became friends, and every time Jean had a mathematical problem, he was asking me for an idea or suggestion. And one day – this was in 1984 – he said: "Yves, you should have a look at this paper. I am sure you will be interested." It was a preprint by Jean Morlet and Alex Grossmann about wavelets. What they proved in that paper was a simple version of a theorem by Calderón that I immediately recognised, namely Calderón's reproducing identity. They had the fantastic idea that this could be a revolution in signal processing. So that was a fantastic step. I was immediately excited by the paper and by the way it was written. They were working at the Centre de Physique Théorique in Marseille. So I took the first train to Marseille and I joined the group. I observed that they were using a very clumsy algorithm. They had a continuous version so they wanted a digital version and were just taking Riemann sums and so on and so forth. And then I began discussing with Ingrid Daubechies, who already belonged to the group. The three of them – Morlet, Grossmann and Daubechies – were in a sense ahead of me in their work on wavelets. So I was the "Quatrième Mousquetaire". They were Les Trois Mousquetaires – you know d'Artagnan was joining the group – so I was d'Artagnan. I discussed with Ingrid and then I had the idea to try to find an orthonormal basis of wavelets, which would make everything trivial on the algorithmic level. It took me three months of intense work but that is nothing compared to the seven years I spent proving Calderón's conjecture. In just three months, I found the basis.

The wavelet you found was in the space of rapidly decreasing functions, that is, it was in the Schwartz class, right?
That was in the Schwartz class. Then, a year later, I realised that Jan-Olov Strömberg had found another basis some years before. He was, at that time, working in Tromsø. Tromsø is a beautiful city in Norway north of the Polar Circle.

The wavelet Strömberg found was a spline function and so it was not in the Schwartz class.
No, it could not be in the Schwartz class. Neither Ingrid Daubechies nor Grossmann nor Morlet were aware of Strömberg's paper because it looked very technical.

We have to interrupt you right there because Strömberg gave a talk about these spline functions of his at a memorial for Zygmund. And you sat and listened to this.
Yes, exactly. I have to confess to that! That was in March 1981 and I was working madly on Calderón's conjecture.

I was so obsessed with Calderón's conjecture, which I solved in May, that I could not even remember his talk. But it is true – I should be ashamed. My construction is completely distinct from Strömberg's and my solution paved the way for all the other solutions. The solution by Strömberg was more tricky. By the way, Strömberg also had the idea of multiresolution analysis. When I discovered Strömberg's paper, I sent a telegram to Tromsø – emails were hardly used at the time – telling Jan-Olov that he is the father and I am no longer the father of wavelets.

Let us stop for a moment and catch up on what we have been talking loosely about. Could you tell us briefly what is an analysing wavelet and what is a so-called mother wavelet, and how does Calderón enter the picture?

Roughly speaking, the wavelets mimic an orthonormal basis for $L^2(\mathbb{R}^n)$ and the reproducing identity is like an expansion of an arbitrary vector in this Hilbert space. In Calderón's formula, one begins with two functions $g(x)$ and $h(x)$ defined on $\mathbb{R}^n$ and satisfying the following identity

$$\int_0^\infty \hat{g}\,(tu)\,\hat{h}\,(tu)\,\frac{dt}{t} = 1 \qquad (*)$$

for all u in $\mathbb{R}^n$ distinct from zero, where $\hat{g}$ and $\hat{h}$ are the Fourier transforms of g and h, respectively. One denotes by G_t and H_t the convolution operators associated to g_t and h_t, respectively, where $g_t(x) = t^{-n} g\,(t^{-1}x)$, and h_t is defined similarly. Finally, one obtains the identity

$$\int_0^\infty G_t H_t \frac{dt}{t} = 1, \qquad (**)$$

which is Calderón's reproducing identity. In Morlet's approach, $h(x) = \overline{g(x)}$ and (*) is precisely the compatibility condition he imposed on a wavelet. As in the one-dimensional case, the functions $t^{-n/2} g\,(t^{-1}(x – x_0))$ are called wavelets, the function g being the analysing wavelet. Let's for simplicity assume we are in the one-dimensional case. A mother wavelet is a function $\psi(x)$ such that its set of siblings $\{\psi_{k,j}\}$, where k and j are integers, and

$$\psi_{k,j}(x) = 2^{j/2}\,\psi\,(2^j x - k)$$

is an orthonormal basis for $L^2(\mathbb{R})$. So the siblings are obtained from $\psi(x)$ by dilations and translations.

But then you took the story further to multiresolution analysis. Perhaps you could say something about that?

Yes. Multiresolution analysis is more natural than wavelets. It is my fault that I have always attributed the discovery of multiresolution analysis to my joint work with Stéphane Mallat, while it is due to my joint work with Coifman. So, multiresolution analysis is something completely trivial from the viewpoint of image processing: it is just to zoom in and zoom out – to see an image at distinct scales. Wavelets are the difference between two successive views of the image. So, once we have got multiresolution analysis, all those other constructions were very natural. In analysing an image it is very natural to get another viewpoint, or a better perspective – you zoom in to see some details. It is like the difference between a sequence and a series: multiresolution analysis is a sequence of numbers or a sequence of views of an image; wavelets are the corresponding series, which corresponds to the difference between two terms of the sequence. So it is very natural.

Gauss gave four different proofs of the fundamental theorem of algebra that every polynomial over the complex numbers has a complex root. And he had more than six proofs of the quadratic reciprocity theorem. For the basic theorem within wavelet theory, there exist several proofs. Is it important to have different proofs?

Yes, it is very important because it gives distinct perspectives. It is also important from the viewpoint of the psychology of scientists. For example, there are some people who prefer wavelets visually, having the shape of an oscillating character and so on. Some other people prefer the viewpoint of multiresolution analysis. To the wavelet room, so to say, you can enter through distinct doors and it is good for the public. It was very good to have distinct approaches to wavelets.

Is it true that quadrature mirror filters are closer to applications?

Yes, and that is the great insight of Stéphane Mallat. Mallat wrote a PhD thesis in signal processing from the viewpoint of the electrical engineering community. So he belongs somehow to another community. He knew what quadrature mirror filters were. And he was 24 years old when he made this fundamental discovery that wavelets and quadrature mirror filters were telling the same story. That had a fantastic impact because all electrical engineering people were despising wavelets, saying that it is just a foolish theory by crazy mathematicians. Immediately, they changed their opinions, saying that we were all doing wavelets. But my student Albert Cohen discovered that there are some quadrature mirror filters that were used that cannot produce wavelets because when they are iterated you have some kind of instability. People could not explain that within the community of electrical engineering. When you iterated those filters, they did not converge to a wavelet. The good quadrature mirror filters were those that, once iterated, converged to a wavelet. So it illuminated the whole theory. So the discovery of Mallat played a fundamental role.

You have to tell us how wavelets were used for a really spectacular detection. We are thinking of gravitational waves that were discovered a couple of years ago.

Yes, that is also a funny story. It illustrates something I like about science: nothing is exactly the way you dream about it. So, the discovery of gravitational waves does not use my wavelets at all. They use another brand of wavelets that were dreamed about long before I worked on the subject. The first person who dreamed about such wavelets was Dennis Gabor. Dennis Gabor was a Hungarian physicist who won the Nobel Prize for Physics in 1971 for his invention of holography. He was an emigrant from Hungary to Great Britain because of Nazism and he wrote a fantastic paper in 1951 about digital speech processing. So this was in 1951, a few years after the transistor was invented. He was already anticipating the digital revolution and the idea that modern telecommunication would transform speech processing into a sequence of 0s and 1s. So, for that purpose, he guessed that there should be a basis in which each signal could be written as a series, a simple series, and it would suffice to transmit the coefficients of the series. That would be enough and that would be the fastest and the most efficient way to transmit speech and sound. But the basis he proposed was completely incorrect and another Nobel Prize winner Kenneth Wilson proposed a slightly different solution than the solution of Gabor. Wilson, incidentally, won the Nobel Prize for Physics in 1982 for his theory about renormalisation. Then, Ingrid Daubechies became aware of the paper by Kenneth Wilson. She was, at the time, working with two of my students, Stéphane Jaffard and Jean-Lin Journé, and they solved the problem. That means transforming the intuition of Kenneth Wilson into a mathematical theorem. So they proposed an algorithm and, by that, both Gabor and Wilson were justified in a sense. It is this algorithm that was used by Sergey Klimenko in his detection of gravitational waves. So, it is a parallel theory of wavelets but they are not the same wavelets as the ones I introduced. It is not zooming into finer and finer scales; it is a problem of catching the right frequency at the right time. It is like hearing a sonata and then writing the score, which is a completely distinct problem. They are both called wavelets but they are solving distinct physical problems.

Other research interests

You made certain forays into the Navier–Stokes equation. Could you tell us about this?
Oh, yes. That was also marvellous because it was a scientific disaster! Yes, but with a good ending. There was a paper written by Guy Battle and Paul Federbush claiming that using wavelets, time-scale wavelets, zooming into finer and finer scales, you could solve Navier–Stokes. Then, Jacques-Louis Lions, the father of Pierre-Louis Lions, asked me: "Yves, what do you think about this paper; you should read this paper and tell me the true story." So, with my Italian student Marco Cannone, we decided to accept this challenge and to read the paper. And, as usual, when a mathematician reads a paper he just forgets the paper he is reading and tries another tool for solving the problem. We first observed that using the Littlewood-Paley decomposition, which was known already in the 1930s, the proof of the paper could be much simplified. So wavelets did not have to play any role in the paper by Battle and Federbush. And then both of us became interested in Navier–Stokes regardless of wavelets; we just forgot about wavelets. We wanted to see what could be proved, what better theorem could be proved in the programme of Federbush. We obtained some interesting results and we were conjecturing that the best result should be so and so – it is technical. We were unable to prove the best result. The best result was proved by Herbert Koch and Daniel Tataru. So we gave up when we were reaching the final point. The good point is that I had three students working on Navier–Stokes (because as soon as I became interested I was able to convince other people to work in that direction). These three students are excellent (Fabrice Planchon, Lorenzo Brandolese and the already mentioned Marco Cannone) and after completing their PhDs they worked on some other aspects of non-linear PDEs. So, during my Navier–Stokes period, I did not prove anything really interesting. In June, we had a day at l'École Normale Superieure de Cachan for explaining my mathematics to the students. I refused to have someone explaining what I did on Navier–Stokes because I am slightly ashamed. But the beginning was good, you know: I wanted to answer the problem raised by Jacques-Louis Lions. And at the end, there were three excellent PhDs and the three people are now full professors, and that is fine.

Together with Coifman you did some important work related to pseudo-differential operators, which inspired J.-M. Bony's theory of so-called paradifferential operators and paraproducts. Could you tell us a little about this?
It is true that Bony's paraproducts are an example of the general theory developed by Coifman and I. Nevertheless, in Bony's hands, these operators yielded fantastic estimates on the regularity of solutions of non-linear PDEs (something Coifman and I never thought about).

An intellectual nomad

You have made contributions in several other fields of mathematics that we have not touched upon. This provokes a meta-question. You have been through various phases. You started in harmonic analysis, you went through number theory for a while and you worked on the Calderón problem and wavelets… Is there a common thread through what you are doing?
No, I have asked myself your question. Of course, the theorem I was describing on Pisot and Salem numbers and Meyer sets has absolutely nothing to do with Navier-Stokes. No, I like discovering another country. For example, this morning I woke up rather early and decided to explore Oslo by myself. That is just fantastic. I feel I am reborn when I explore a new city without a guide.

In mathematics, there is also this human aspect. You talk to people and you get input from them, and perhaps that changes your direction?
Yes, and most of the time my change is just accidental and under the influence of another person. But the idea to be born again, to start to learn… When I began working on Navier–Stokes, I felt like I was a child because I did all the mistakes you do in the beginning. That is something absolutely fantastic.

Even though you have switched fields several times, your main research thrust has been in what is broadly

called harmonic analysis. You are the second one to win the Abel Prize in harmonic analysis; the first was Lennart Carleson.
Yes, of course I admire him very much. Lennart Carleson for me is like God, you know. I cannot be compared to Lennart Carleson; he is so much above.

We are not going to compare anybody. But you did use some results by Lennart Carleson at the time?
Yes, of course. I used what are called Carleson measures in a very deep way in the solution of Calderón's conjecture. I worked in a very intense way on his paper on the convergence of trigonometric series. I admire his style very much, not only the deepness of the results but also his style and irony. I am very different from Carleson but he is a model. I wouldn't say that I was close to Carleson the way I was close to Calderón. But it might be because with Calderón I was sharing the Spanish language and the Spanish culture and that helped a lot.

Carleson said in the interview we had with him that he was a problem solver. He was not interested in building theories. Do you count yourself as a problem solver or are you in-between?
I am in-between. For example, in the work on quasicrystals or on wavelets I was more building a theory. For quasicrystals, my book gave something very systematic and when it was rediscovered there were, I would say, hundreds of papers written on Meyer sets. I gave a basis of a theory but once I had done that I got immediately bored and changed subject. I leave it to students. Now, that explains why I have 50 PhD students. One year at the Université Paris-Dauphine, I had 19 students simultaneously.

How did you manage?
Some of them were finishing but I spent three hours every week discussing with the students, and at that time I lost five kilos! Yes, that was the worst. But I love transmitting the fire to the students and then doing something else. So, it is a way of cheating because it means that it will be their responsibility to make a building from my ideas, while I can escape. Like people who invite their friends and then disappear.

Teaching and outreach

Actually, you have a very varied teaching experience, from the Prytanée National Militaire all the way to the Grandes Écoles. How has your philosophy about teaching evolved over time?
My teaching evolved very much; my teaching reflects my personality – I am eager to transmit my visions. When you write, you are very cautious. When you are teaching, you can make some slight exaggerations or, you might say, you can be less cautious. And that is very good because, being less cautious, you can take bets on the future of the subject.

You can give your own gut feelings?

From left to right: Yves Meyer, Christian Skau and Bjørn Ian Dundas. (Photo: Eirik Furu Baardsen)

Yes, and I think the oral aspect of teaching will disappear completely with the new way of courses prepared as electronic versions. It is too controlled.

So, you are sceptical of recording lectures?
Yes, exactly. I am sceptical because teaching is always an improvisation. It is like a performer: he never plays it exactly the same way twice. When teaching, you can convey the fact that making mistakes could be a benefit for the listener because mistakes can be creative in some way. But that is good for some students and bad for others. Everyone has a way of teaching that is beneficial to a part of the group and negative to the other part. My way of teaching is a way of trying to inspire. I like that people can react and be challenged. The idea is that the group should be challenged: begin to think either one or the other way, even if this is to criticise the view I am trying to convey to them. It is a kind of Socratic experience.

On a higher level, you have been quite clear on your views on the French model for higher education. In view of recent developments in France, do you have a new take on that?
Yes, this is a very important problem because there are several theories about how to improve the teaching of mathematics in France. I was very moved by the presentation of Hanan Mohamed Abdelrahman [the winner of the Holmboe prize 2017] this morning. She made a very important point: give the same challenges to all students. In France, we have the tendency to say that we should not be as demanding with this group as with that group. But this is a way of underestimating the group to which you are less demanding. "This poor person coming from the North of Africa is so unhappy that we should not demand too much." But that is terrible for them!

Are you thinking of distinctions between the universities and the Grand Écoles?
That is another point, yes. I was speaking of high school level. It is a very difficult problem – it is a problem that cannot be solved in a theoretical way. For example, in the beginning, there existed an École Normale Supérieure de jeunes filles (for young girls) and an École Normale

Supérieure de garçons (for boys). So, every year, in mathematics, there were 14 girls admitted to the exam for girls. And then a lot of very bright women in mathematics – faculty members – were women coming from l'École Normale Supérieure. They decided that this was unfair, that it underestimated women and that we should unify. Now it is unified and every year the total number is 44 in mathematics: there are 40 boys and 4 women (at most). And sometimes there are no women at all. It is a complete disaster!

It is difficult to find one solution for a big problem. The fact is that all the young students are distinct; they have distinct needs, they have distinct demands and they have distinct abilities. Should we say that their level is equal by definition and that we should impose the same burden on all or should we have an honours class? This is very difficult.

We wanted to talk about outreach. What do you think about the importance of popularising mathematics, like your own work? Is that futile?
On that I do not know. I can just give you an interesting example. In Tunis, cultural life was rather narrow because it was so far away from Paris at that time. Planes hardly existed and we took the boat to go to France. So, when a person was coming far away from France, it was a local attraction. As a high school student, I went to a talk given by Jean-Pierre Kahane. I remember the subject he was talking about was very interesting; it was a problem of trigonometric series he was trying to solve. He gave a talk – and he is a very good speaker – in such a way that I understood what he was talking about. I was a student in high school. I was truly fascinated. I was fascinated by his personality. Later, I went to Orsay and was there for about 15 years, and he had had a great influence on my work. What he did when he came to Tunis was a kind of popularisation of mathematics: going to Tunis, giving a talk for a general audience about his research.

It was quite exceptional and I would like to say that this influenced my work. I cannot prove that it truly influenced my work. It might have been just something accidental but I love the story.

On the topic of popularising mathematics, Ludvig Sylow was a Norwegian mathematician and, in his eulogy at Sophus Lie's funeral, he said the following: "It is the mathematician's misfortune more than the other scientists, that his work cannot be presented or interpreted for the educated general public, in fact, hardly for a collection of scientists from other fields. One has to be a mathematician to appreciate the beauty of a proof of a major theorem or to admire the edifice erected by mathematicians over thousands of years." That was Sylow's attitude.
I slightly disagree. Because the point is that there is nothing special about mathematics. Take difficult literature or poetry, for example. I would say that I do not understand the living French poets. I try to read their poems and I do not understand them. The problem with mathematics is that people do not even understand the language. In the case of poetry, to be completely honest, I understand the words but I do not understand the language. It means that, for every aspect of art, the difficulties are the same. Like modern music – have you heard a work by Xenakis?

Yes, I have.
But you did not understand it!

No…
No, but people never say that, you know. They think they understand music but they do not understand music either! And nobody talks about that.

Private passions

Perhaps concluding the interview, are there aspects that are not regularly touched upon? Some passions – private passions?
I have private passions. Yes, I have several passions. I am a passionate person.

People, I would say. I like people. I like discussing with people – meeting, admiring people. I would say the pleasure to do mathematics is related to the pleasure of joint work. Let me single out Raphy Coifman. I have been working with him for 40 years. He is like a brother and he is viewing me as a brother. I like his personality. I like his life.

I like people, and everything that is related to literature. My first addiction was literature – I took humanities as a child. I am still enjoying Plato with delight.

In Greek?
I am no longer reading Plato in Greek. I used to. And I was still doing that at l'École Normale Supérieure. In that way, I was admired by students in humanities and despised by scientists because a true scientist does not read Plato. I also love reading the Bible.

Both the Old and the New Testament?
Only the Old Testament. It is more spicy, you know: David and Bathsheba, and the relationship of David and Jonathan. It is completely fascinating because there is a smell of homosexuality. And the mourning of David when Jonathan dies – it is beautiful.

That is great poetry also.
Yes, it is completely marvellous that David said that their friendship was more important than the love of a woman. It is completely fascinating.

Also the story with Abraham sacrificing Isaac. You know that ...
Kierkegaard…

Kierkegaard, exactly! Søren Kierkegaard was extremely fascinated by the story about Abraham and Isaac. His book "Fear and Trembling" ("Frygt og Bæven") is centred on this story.
Beyond mathematics, my very deep world is literature.

Also Russian literature, we understand?

Yes! Vasily Grossman, for example, and Aleksandr Solzhenitsyn and Anton Chekhov. I know by memory Tolstoy's *Anna Karenina*.

We heard the story that you even found wavelets in Russian literature.

Yes, in Solzhenitsyn's *The First Circle*. There you find not wavelets exactly but time frequency analysis. Solzhenitsyn was a physicist and then moved to literature – because of the war, because he was sent to a concentration camp. And he could not resist writing in *The First Circle* a chapter on time frequency analysis. I will not describe it – it is too long – but there is a page that I read each time I give a course on signal processing because it is so beautiful. He is describing exactly the problem that I will be describing on Wednesday: to catch something inside a signal, to catch a pattern. The problem in the detection of gravitational waves was to catch a specific pattern that would be the signature of the gravitational wave. The signal is completely noisy and the noise is a thousand times larger than the signal. So, we have to capture these very small, short-lived patterns. In *The First Circle*, it is an audio signal, a recording of the voice of someone, and the group has to detect the person through finding the characteristic patterns of the person, patterns that would be for the person the equivalent of fingerprints – the patterns of a voice. Solzhenitsyn calls that "voice-prints". He is describing the problem truly as a physicist, using the correct words and so on. It is completely fascinating. So, my interest in Russian and Soviet literature is related to my research work, as everything is… Of course, for students – but I am not teaching anymore – the problem when you speak about Solzhenitsyn today is that they don't know Solzhenitsyn, and the two of them who do know Solzhenitsyn have never read *The First Circle*. And then, when I am reading a page of *The First Circle*, they just fall asleep.

Do you have other interests beside mathematics and literature?

I like music – I am very fond of music. And I love painting.

Some special painters?

Oh, yes. But that changes from age to age. I would put at the very top two Spanish painters: Goya and Velázquez. I have special ties with Spain. But that is very personal. I wouldn't say that they are the greatest painters in the world but I love Goya.

On behalf of the Norwegian Mathematical Society and the European Mathematical Society, thank you very much for this interview. It has been most interesting.

Bjørn Ian Dundas is a professor of mathematics at the University of Bergen. His research interests are within algebraic K-theory and algebraic topology.

Christian Skau is a professor emeritus of mathematics at the Norwegian University of Science and Technology (NTNU) at Trondheim. His research interests are within C^*-algebras and their interplay with symbolic dynamical systems. He is also keenly interested in Abel's mathematical works, having published several papers on this subject.

Bringing a New Light on Old Problems – Interview with Laure Saint-Raymond

Roberto Natalini (Consiglio Nazionale delle Ricerche, Rome, Italy, Chair of the Raising Public Awareness Committee of the EMS)

Laure Saint-Raymond is a French mathematician working in partial differential equations, fluid mechanics and statistical mechanics. She is a professor at École Normale Supérieure de Lyon. In 2008, she was awarded the EMS Prize and, in 2013, when she was 38 years old, she became the youngest member of the French Academy of Sciences.

Roberto: Let me start with a very trivial question: when did you become interested in mathematics?
Laure: Actually, it was quite late. In high school, I was a good student but somehow I was more interested in music. But, being good in maths, as was the norm in France, I entered the so-called "Classes préparatoires" (which is preparation for entrance selection for the "Grandes Écoles") and then the École Normale Supérieure (ENS) in Paris. Here, I found very enthusiastic teachers and so my interest for mathematics started.

How were your parents involved in your interest in mathematics? Did you have an important teacher before university?
I had a maths teacher during the "Classes préparatoires" with a strong passion for mathematics and, in particular, for logic. However, even though my father is a mathematician, I was not really pushed by my parents to go in this direction. I was quite free to make my choice.

In the ENS, I found many inspirational professors, like the physicist Yves Pomeau, who used to introduce baby models to catch important physical phenomena such as the growth of trees. On the mathematical side, I should mention Yann Brenier, with his very original way of seeing all things, and Henry Berestycki. And, finally, it was with François Golse that I really discovered the connection between mathematics and physics or, to say it better, how to couple the rigour of maths with the inspiration arising from physics.

What are your main fields of interest in mathematics and how and why did you start to work on them?
I started my research in plasma theory, looking at the qualitative behaviour of beams of charged particles in strong magnetic fields. The approach was driven by kinetic theory methods, with a deep interplay of mathematics and physics. In collaboration with my PhD advisor François Golse, we solved one part of Hilbert's sixth problem. This problem consists of developing mathematically "the limiting processes [merely indicated in Boltzmann's work] which lead from the atomistic view to the laws of motion of continua".

Laure Saint-Raymond at ENS Paris, 2014. (Pboto: CNRS)

What we established is the rigorous transition from the Boltzmann kinetic description, where the gas is considered as a collection of interacting particles described statistically, to a fluid description given by the Navier–Stokes equations, where the flow is described only through macroscopic quantities such as the average speed or the pressure of the fluid.

What have been your main original ideas in proving the limit from Boltzmann to the Navier–Stokes equations?
Actually, I have contributed in both collecting and organising in an original way many existing techniques, and in developing some new mathematical tools, such as the so-called L^1 velocity averaging lemma related to dispersion and mixing. The Boltzmann equation describes the state of a gas using a distribution function that depends on space, velocity and time. It expresses a balance between two mechanisms: the transport and the collisions. This equation has no regularising effect and so, if we have a singularity in the solution, we keep it forever. And

this is a problem when you study the fast relaxation limit (i.e. the asymptotic behaviour when the relaxation to local equilibrium due to collisions is much faster than the transport that correlates close positions) because you need some compactness.

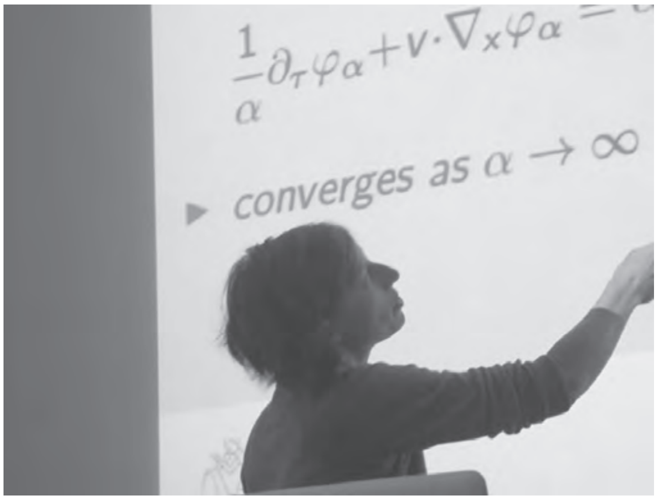
It was noticed by Golse, Lions, Perthame and Sentis that observables, which are obtained by taking averages with respect to the velocity variable, are more regular than the solution itself. We were able to combine this result with hypoelliptic properties of the transport to prove that if you gain some nice behaviour in the velocity then you can gain something also in the space variable. This was one of the main tools to prove our convergence result.

What about some other problems you have considered?
The other part of my work is concerned with large-scale geophysical flows where the Coriolis force is dominant, taking into account the dominating influence of the Earth's rotation. Classical methods for linear singular perturbation problems fail when the oscillations cannot be described explicitly because one does not even know whether the waves will be captured or dispersed. For instance, close to the equator, the spatial variations of the Coriolis acceleration cannot be neglected. The spectral structure of the propagator is completely modified and one can prove that fast oscillations are trapped in a thin band of latitudes.

Another challenging problem is to understand the interaction with the boundaries, which is responsible for most energy exchanges (forcing and dissipation), even though it is concentrated in very thin layers close to the bottom and the surface.

Now, I try to understand the propagation of internal and inertial waves in the ocean, in regions with a variable topography. I collaborate with physicists to understand how to separate the different time and space scales, neglecting the very complex dynamics at small scales but keeping the qualitative behaviour of the solutions.

Are you still working on Hilbert's sixth problem?
Yes, of course! More recently, mainly in collaboration with Isabelle Gallagher and Thierry Bodineau, I have worked on the full problem, namely, to make a rigorous derivation of fluid models from particle models, which I think is a much more difficult problem. A very challenging question is to explain the appearance of irreversibility at the macroscopic level. At this stage, there is no general theory but some special results have been obtained. For instance, we were able, under some specific scaling assumptions, to obtain the Stokes equations directly as the limit of particle models. It is not the optimal result but it is the first rigorous derivation of fluid equations from Newton's mechanics. Our starting point is solely the deterministic collisions of hard spheres, coupled with a suitable entropy bound. However, it is quite clear that we cannot hope to obtain the full result, i.e. the convergence to the Navies–Stokes equations, using the same ideas. So, we are looking around for some new ideas.



Laure at the 23rd International Conference on Discrete Simulation of Fluid Dynamics, July 29, 2014, École Normale Supérieure, Paris.

You have been awarded with many prizes. Which one is the most important for you?
First, I have to say that when you receive a prize, you then receive a lot of them, which does not mean that you have more merit. Of course prizes come as recognition from the mathematical community and I am very proud of the EMS Prize I received in 2008. But I think that prizes should be overall understood as an encouragement to go further and maybe to take more risks and more responsibilities.

Speaking of responsibility, I remember your intervention in 2015 about publications, during the event for the 25th anniversary of the EMS at the Institut Poincaré in Paris.
Yes, I am really concerned by this point. I believe that, as a mathematical community, we really publish too much and that senior people with accomplished careers should be more careful and selective when submitting papers. Most of the time, nobody reads these papers and it is even difficult to find somebody to do a good peer review. Myself, I have adopted as a rule to referee at least twice the number of papers that I publish each year. It is crucial to properly review the papers and also to read and discuss articles from other researchers. This is the only way to be a scientific community.

I believe that science is a common project and not an isolated enterprise. On the other side, unfortunately, we are faced by all these national and international rankings, which are very often quite meaningless and based on quantitative metrics. Nobody is interested in what people are really doing and I think it is bad for mathematics.

How much in your work is intuition and how much is just hard work?
The starting point of each of my papers is about trying to bring a new light on a problem. Unfortunately, many of my papers are a mess of technical details but still we try to explain one or two new ideas. In this sense, my works

are not only technical but there is always some intuition to be made rigorous. You have an idea and then you try to work out the details and you struggle with some problems. And to solve these problems, you have to understand something that you missed before. You don't fully understand it until you have a complete proof. This is, in my opinion, the essence of mathematical work.

How do you organise your work? Do you follow a routine or does it vary a lot according to external conditions?

I work most of the time with the same collaborators, since it takes a lot of time to share the same language, the same feelings on the topics and so on. I'm not the kind of person who goes to a conference, meets some people and immediately starts a new collaboration.

Two years ago, I spent a sabbatical in the US, where I had a lot of time and no duties. It was really quiet and I had a great time working with no constraints but somehow it was not long enough to develop new collaborations.

How has it been important for you to be in Paris for many years?

For a very long time, we didn't leave Paris so as to stay close to our parents, who helped a lot with the children, and, I have to say, I didn't quite realise the great opportunity I had. Actually, in Paris, it is possible to discuss and collaborate with a lot of people with different backgrounds and ideas.

Out of Paris, you are maybe not exposed to such a large mathematical community but somehow it gives you more opportunities to meet people doing something really different and to go into new research directions. I have now moved to Lyon where I am very happy.

In France, women in mathematics are not so common, even if some things have changed in recent years. Could you explain the difficulties that women can sometimes experience in having a satisfactory career in mathematics?

Actually, I have to say that, in my experience, I didn't feel any discrimination against women. My impression is that somehow the problem is more in our society. One reason why women are not following scientific careers is maybe the French system of education based on selection and competition, which can discourage women following this path.

On a family holiday in the Alps.

Also, there is the dominant model of family, where men are choosing their jobs and women are following their husbands. In academic careers, it is very often hard to stay together.

And how did you manage to face these problems? You have a large family with six children. How is it possible to work so hard with a lot of children and commitments?

My husband is just great and makes everything at home [she smiles]. Also, for many years, our parents helped us taking care of the kids very often. Besides, the French school system (starting at the age of 3) is helpful in this regard.

But, nevertheless, for a long time, I needed to be at home at 5 pm almost every day … and I wrote fewer papers than most of my colleagues!

What do you do outside maths? Do you have hobbies? What do you like to do?

I do plenty of things like hiking and skiing and this is also one of the reasons why I like very much being in Lyon. Also, I enjoy music, playing the cello. Sometimes I even play chamber music with colleagues.

A last question: what is your bedtime reading?

It is hard to say; sometimes I just sleep [laughing]. But, for instance, I like very much Eric-Emmanuel Schmitt for his positive attitude about life. More generally, I look for books where I find a supplement of energy to live, something that helps to find the positive side of our lives.

Since 2014, Roberto Natalini has been Director of the Istituto per le Applicazioni del Calcolo "Mauro Picone" of the National Research Council of Italy. His research interests include fluid dynamics, road traffic, semiconductors, chemical damage of monuments and biomathematics. He is Chair of the Raising Awareness Committee of the European Mathematical Society.

On the Traces of Operators (from Grothendieck to Lidskii)

Didier Robert (Université de Nantes, France)

1 Presentation

In this paper, the reader's attention is drawn to some notions that are classical in linear algebra but become more subtle to deal with in the context of infinite-dimensional vector spaces (endowed with a norm or a vector space topology). The Fredholm theory of integral equations, which will be mentioned at the end of the article, shares many common points with the systems of linear equations taught in the first year of undergraduate studies, except that the endomorphisms operate on Banach spaces of the form $C(K)$ (the space of continuous functions on a compact K) or $L^p(\Omega)$ for a given measure on Ω. In Fredholm theory, the notions of trace and determinant, as well as their relations to eigenvalues, of course play the same role as in finite dimension. Fredholm's seminal article "Sur une classe d'équations fonctionnelles" [Acta Mathematica, 27, pp. 365–390] goes back to 1903, a time when functional analysis was poorly developed. It was the starting point of many works that partially motivated the advances in the field throughout the 20th century (Hilbert, Banach, Fréchet, Dieudonné, Schwartz, Grothendieck, Sobolev, Gelfand, Krein and many others).

More recently, microlocal analysis and the theory of pseudo-differential operators have allowed major progress in understanding the non-self-adjoint operators that describe the instability of certain systems in fluid or quantum mechanics. Indeed, the location in the complex plane of the eigenvalues with non-zero imaginary part enables one to get qualitative information about the resonance of the system (see Zworski's paper [18] for an illustration of these phenomena). The recent work [16] by Sjöstrand on Weyl formulas for randomly perturbed non-self-adjoint operators clearly shows the interest of determinants in infinite dimension. By exploiting their subtle properties as entire functions on the complex plane, one can obtain information about the eigenvalues. Lidskii's trace formula plays a pivotal role in these studies.

The aim here is to tell the history of this formula and review its role in the developments of functional analysis, as well as its revival in recent years. At the end, a sketch is presented of a proof that is close to the original one.

2 Introduction

On a complex vector space $\mathcal{E}$, of finite dimension n, the trace and the determinant of an endomorphism A have the following two basic properties: they are invariant under conjugation by automorphisms and can naturally be expressed in terms of the eigenvalues of A (using a basis for which A is triangular).

Let us briefly recall some well known results from linear algebra. Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathcal{E}$ and let $\{e_1^*, \cdots, e_n^*\}$ be the dual basis of the dual vector space $\mathcal{E}^*$. We denote by $\mathcal{L}(\mathcal{E})$ the $\mathbb{C}$-vector space of endomorphisms of $\mathcal{E}$.

The trace of $A \in \mathcal{L}(\mathcal{E})$ is defined by the equality

$$\mathrm{Tr}(A) = \sum_{1\le j\le n} e_j^*(Ae_j). \tag{1}$$

It is a linear form on $\mathcal{L}(\mathcal{E})$.

Let us now turn to the determinant. We write $u \wedge v$ for the exterior product of two linear forms u and v on $\mathcal{E}$. We denote the alternating n-linear form on $\mathcal{E}^n$ by $L_n = e_1^* \wedge e_2^* \cdots \wedge e_n^*$, the symmetric group on $\{1, \dots, n\}$ by $\mathfrak{S}_n$ and the sign of σ by ε_σ. Then, we have:

$$L_n(x_1, \cdots, x_n) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon_\sigma e_1^*(x_{\sigma(1)}) \cdots e_n^*(x_{\sigma(n)})$$

for all $(x_1, \cdots, x_n) \in \mathcal{E}^n$. The determinant of an endomorphism A of E is defined by the following equality:

$$\begin{aligned}\det A &= L_n(Ae_1, \cdots, Ae_n)\\ &= \sum_{\sigma\in\mathfrak{S}_n} \varepsilon_\sigma e_1^*(Ae_{\sigma(1)}) \cdots e_n^*(Ae_{\sigma(n)}).\end{aligned} \tag{2}$$

It is the unique complex number such that, for all alternating n-linear forms f on $\mathcal{E}^n$ and all $(x_1, \cdots, x_n) \in \mathcal{E}^n$, one has

$$f(Ax_1, \cdots, Ax_n) = (\det A) f(x_1, \cdots, x_n). \tag{3}$$

It follows that $\det(AB) = (\det A)(\det B)$ for all $A, B \in \mathcal{L}(\mathcal{E})$.

In particular, $\det A$ does not depend on the chosen basis. By picking a basis for which A is triangular, we deduce that $\det A = \lambda_1\lambda_2 \cdots \lambda_n$, where λ_j are the eigenvalues of A. Therefore, the characteristic polynomial of A can be written as

$$D_A(z) = \det(A - z\mathbb{1}) = \prod_{1\le j\le n} (\lambda_j - z).$$

In the above formula, the eigenvalues are repeated according to their multiplicity.

Thanks to (2), we obtain a formula for the coefficients of the characteristic polynomial involving traces. As we shall see, it extends to infinite dimension. For this, we introduce the tensor powers $\otimes^k\mathcal{E}$ ($k \ge 1$) and the antisymmetrisation operator Π_a, defined for $x_1, \cdots, x_k \in \mathcal{E}$ by

$$\Pi_a(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = \frac{1}{k!} \sum_{\sigma\in\mathfrak{S}_k} \varepsilon_\sigma x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(k)}.$$

Given $A_j \in \mathcal{L}(\mathcal{E})$, $1 \le j \le k$, we define an endomorphism of $\otimes^k\mathcal{E}$ by

$$A_1 \wedge A_2 \cdots \wedge A_k = \Pi_a(A_1 \otimes A_2 \cdots \otimes A_k)\Pi_a.$$

Setting $\wedge^k A = A \wedge A \overset{\text{k times}}{\cdots} \wedge A$, one has:

$$D_A(-z) = z^n + z^{n-1}\mathrm{Tr}(A) + \cdots + z^k \mathrm{Tr}(\wedge^k A) + \cdots + z^0 \det A. \tag{4}$$

Notice that $\det A = \mathrm{Tr}(\wedge^n A)$. The traces in $\mathcal{L}(\otimes^k \mathcal{E})$ are computed in the basis

$$\{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k},\ (j_1, j_2, \cdots, j_k) \in \{1, \cdots, n\}^k\}.$$

In particular, using the property analogous to the one we saw for the determinant, one has $\mathrm{Tr}(B^{-1}AB) = \mathrm{Tr}(A)$ for all invertible $B \in \mathcal{L}(\mathcal{E})$, and $\mathrm{Tr}A = \lambda_1 + \cdots + \lambda_n$.

Finally, the trace is the unique linear form on $\mathcal{L}(\mathcal{E})$ that is invariant under conjugation up to multiplication by a constant: if f is a linear form on $\mathcal{L}(\mathcal{E})$ such that $f(B^{-1}AB) = f(A)$ for all $A, B \in \mathcal{L}(\mathcal{E})$, with B invertible, then there exists $\mu \in \mathbb{C}$ such that $f(A) = \mu\mathrm{Tr}A$ for all $A \in \mathcal{L}(\mathcal{E})$. The proof of this property is left to the reader as an exercice (hint: first consider rank one endomorphisms).

In finite dimension, there are, of course, several ways to obtain these results; the above presentation has the advantage of remaining valid in infinite dimension. In that case, the notions of trace and determinant are harder to deal with, even for Hilbert spaces, since there is *a priori* no equivalent technique to the triangulation for arbitrary operators.

To analyse the spectrum of non-diagonalisable compact operators (which appear, for instance, in the study of dissipative systems and in the Fredholm theory of integral equations), it is very useful to be able to define a trace and a determinant with reasonable properties, analogous to the ones we just recalled in finite dimension.

Let us consider a compact operator A on $\mathcal{H}$. We know that, besides 0, the spectrum of A consists of a sequence of eigenvalues $\{\lambda_j(A)\}_{j\geq 1}$ with finite multiplicity. The multiplicity $\mu(\lambda)$ of the eigenvalue $\lambda \neq 0$ is equal to

$$\mu(\lambda) = \dim[\mathcal{E}_\lambda(A)], \quad \text{where } \mathcal{E}_\lambda(A) := \bigcup_{k\geq 1} \ker(A - \lambda \mathbb{1})^k$$

is the generalised eigenspace. The convention used here will be to write the sequence $\{\lambda_j(A)\}_{j\geq 1}$ with each eigenvalue repeated according to its multiplicity.

In 1959, the Russian mathematician V. B. Lidskii proved [9] that if $\mathcal{H}$ is a separable Hilbert space and A is a trace class operator (as defined below) on $\mathcal{H}$ then, for each orthonormal basis $\{e_n\}$ of $\mathcal{H}$, one has

$$\sum_{n\geq 1} \langle e_n, Ae_n\rangle = \sum_{j\geq 1} \lambda_j(A). \tag{5}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathcal{H}$, which is assumed to be anti-linear with respect to the first argument. We write $\|\cdot\|$ for the norm on $\mathcal{H}$ defined by the scalar product.

Equality (5) might look like a trifling extension of the finite-dimensional case. However, it was not until 1959 that a proof was published (although Grothendieck implicitly had all that was needed, he did not state the result explicitly). As we shall see, Lidskii's proof relies on classical, yet tricky, arguments involving subtle properties of entire functions.

To begin with, we recall a definition of the Hilbert–Schmidt and the trace classes (the terminology "nuclear operator" instead of "trace class operator" is also used).

Definition 2.1. An operator A on $\mathcal{H}$ is in the Hilbert–Schmidt class if there is an orthonormal basis $\{e_n\}_{n\geq 0}$ of $\mathcal{H}$ such that

$$\sum_{n\geq 1} \|Ae_n\|^2 < +\infty. \tag{6}$$

One shows that the left side of (6) is independent of the chosen orthonormal basis and that this condition implies that A is compact. We will say that A is a trace class operator if there exists a decomposition $A = A_1A_2$, where A_1 and A_2 are in the Hilbert–Schmidt class. A positive operator A is in the trace class if and only if $\sum_{j\geq 1} \lambda_j(A) < +\infty$.

The sets of Hilbert–Schmidt and trace class operators are denoted by $\mathfrak{S}_2(\mathcal{H})$ and $\mathfrak{S}_1(\mathcal{H})$ respectively. We write $\mathfrak{S}_\infty(\mathcal{H})$ for the set of compact operators and $\mathcal{L}(\mathcal{H})$ for the set of bounded operators. As the reader will have already guessed, there exist classes $\mathfrak{S}_p(\mathcal{H})$ for all real numbers $p > 0$: an operator A belongs to $\mathfrak{S}_p(\mathcal{H})$ if and only if $(A^*A)^{p/2}$ is a trace class operator (here, A^* denotes the Hermitian adjoint of A). For $p \geq 1$, these are Banach spaces with respect to some natural norms $\|\cdot\|_p$. We shall often use the convention $\|\cdot\|_\infty = \|\cdot\|$ (uniform norm for bounded operators). The spaces $\mathfrak{S}_p(\mathcal{H})$ are normed two-sided ideals in the C^*-algebra $\mathcal{L}(\mathcal{H})$ (see below).

The spaces $\mathfrak{S}_p(\mathcal{H})$ were introduced by von Neumann and Schatten and carry the name of Schatten class operators (see [4] for their properties). They share common features with the Lebesgue spaces L^p (for a space with a measure). The functional trace "Tr" plays the role of the integral, as one has, for instance, the relation $\mathrm{Tr}(A^*A) = \|A\|_2^2$.

It is then clear that, if A is a trace class operator, one can define its trace by the natural formula

$$\mathrm{Tr}A = \sum_{n\geq 1} \langle e_n, Ae_n\rangle. \tag{7}$$

The fact that the series converges absolutely is a consequence of the Cauchy-Schwarz inequality. One can easily check that Tr is a continuous linear form on $\mathfrak{S}_1(\mathcal{H})$ that is independent of the chosen orthonormal basis. It satisfies $\mathrm{Tr}A^* = \overline{\mathrm{Tr}A}$ and $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for all $A \in \mathfrak{S}_1(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$.

It has been known since H. Weyl (see [4]) that if A is a trace class operator then the series $\sum_j \lambda_j(A)$ is absolutely convergent (we shall give more details below). We can now rigorously state the result proved by Lidskii.

Theorem 2.2 (Lidskii [9])**.** For all trace class operators A, equality (5) holds.

This can be considered as the fundamental theorem in spectral analysis of non-self-adjoint operators. Indeed, the eigenvalues of an operator are, in general, difficult to access (even in the self-adjoint case). One way to control them is to write a trace formula

$$\mathrm{Tr}(f(A)) = \sum_{j\geq 1} f(\lambda_j(A)) \tag{8}$$

in such a way that the left side can be analysed and estimated for a suitable family of functions f depending on a real or a complex parameter. One then concludes by a tauberian-type argument (see for instance [1]).

Furthermore, if A is assumed to be a normal operator, meaning that $AA^* = A^*A$, then Lidskii's theorem is of course trivial, since A is then diagonalisable. However, in the general case, we are faced with a subtle and deep theorem. Due to the instability of the spectrum of compact non-self-adjoint operators, it is not easy to go from finite to infinite dimension.

Detailed proofs can be found in a number of books [3, 4, 10, 15]. Notice that all of them except [3] explicitly

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attribute the result to Lidskii. His original paper (1959), written in Russian, was translated in 1965, whereas [3] appeared in 1963. Lidskii's original proof was revisited and simplified in [4]. The basic ideas will be explained at the end of this article. The proofs mentioned above use the properties of determinants as entire functions of one complex variable. Other proofs, more algebraic in nature, are based on the search for triangular forms in infinite dimension [12].

In his monumental thesis, published in [5], Grothendieck considerably developed the question of defining a trace (and a determinant) for general classes of operators on Banach or Fréchet spaces. He wrote his *thèse d'État* under the supervision of J. Dieudonné and L. Schwartz at Nancy University and defended it in 1953. The reader is invited to look at the four pages that L. Schwartz dedicates to A. Grothendieck in his memoir book "A mathematician grappling with his century" [pp. 282–286, Birkhäuser 2001]. Recall that Grothendieck received the Fields Medal in 1966 for his work in algebraic geometry.

One of the motivations of his doctoral research was to establish a general setting for Fredholm theory, as an extension of the Schwartz kernel theorem [6]. Nevertheless, one question seemed not to have been solved: when applied to the particular case of Hilbert spaces, Grothendieck's results definitely give a proof of (5) but only for a more restrictive class of operators than the natural trace class $\mathfrak{S}_1(\mathcal{H})$, namely $\mathfrak{S}_{2/3}(\mathcal{H})$.

In [10, Section (27.4.11)], A. Pietsch obtained a sufficient condition for equality (5) to hold in an arbitrary Banach space. It contains the trace formulas of Grothendieck and Lidskii as particular cases. Grothendieck's approach to establish (5) for Banach spaces is also discussed in [11].

3 Grothendieck's approach to the trace equality (5)

Grothendieck's thesis [5] is devoted to locally convex topological vector spaces, as well as the various classes of operators between them. Here, we shall limit ourselves to Banach spaces (see also [6]).

In this section, $\mathcal{E}$ denotes a complex Banach space, $\mathcal{L}(\mathcal{E})$ denotes the algebra of bounded operators on $\mathcal{E}$ and $\mathcal{E}'$ denotes the topological dual of $\mathcal{E}$. We write $\langle \cdot, \cdot \rangle$ for the duality (note that the bracket is bilinear) and $\mathcal{L}_F(\mathcal{E})$ for the ideal of finite rank operators, which we identify with the tensor product $\mathcal{E}' \otimes \mathcal{E}$ via the linear map $J(x' \otimes x)y = \langle y, x' \rangle y$.

We define the ideal $\mathcal{N}(\mathcal{E})$ of nuclear operators by introducing on $\mathcal{E}' \otimes \mathcal{E}$ the so-called projective norm

$$\|u\|_\pi = \inf \left\{ \sum_{j \geq 1} \|x'_j\| \|x_j\|,\ u = \sum_{j \geq 1} x'_j \otimes x_j \right\}$$

and denote by $\mathcal{E}' \hat{\otimes}_\pi \mathcal{E}$ the completion with respect to this norm.

The canonical injection J extends to a continuous linear map J_π from $\mathcal{E}' \hat{\otimes}_\pi \mathcal{E}$ to $\mathcal{L}(\mathcal{E})$. In general, J_π is not injective (see [5]) but it is for Hilbert spaces. Most of the Banach spaces that one actually uses have this property (for example, the spaces L^p for any measure μ and any $1 \leq p \leq +\infty$), which is related to the approximation property (see [5]). The first counterexample is due to P. Enflo [1973].

The set $\mathfrak{S}_1(\mathcal{E})$ of nuclear operators on $\mathcal{E}$ is the image of $\mathcal{E}' \hat{\otimes}_\pi \mathcal{E}$ in $\mathcal{L}(\mathcal{E})$ by J_π. It is a two-sided ideal and a Banach space with respect to the quotient norm on $\mathcal{E}' \hat{\otimes}_\pi \mathcal{E} / \ker J_\pi$.

In what follows, we always assume that J_π is injective.

For any A of finite rank, the trace is naturally defined as $\mathrm{Tr} A = \sum x'_j(x_j)$ if $A = J(u)$ and $u = \sum_j \langle x_j \otimes x'_j \rangle$. One can easily show that it extends by continuity to $\mathfrak{S}_1(\mathcal{E})$, yielding a linear form such that $|\mathrm{Tr}(A)| \leq \|A\|_1$ for all $A \in \mathfrak{S}_1(\mathcal{E})$. Moreover, the trace is invariant on $\mathfrak{S}_1(\mathcal{E})$, in that $\mathrm{Tr}(TA) = \mathrm{Tr}(AT)$ for all $T \in \mathfrak{S}_1(\mathcal{E})$ and all $A \in \mathcal{L}(\mathcal{E})$.

We introduce, on the Banach space $\mathcal{E}$, a family $\mathfrak{S}_p(\mathcal{E})$ of ideals for all real numbers $p > 0$. For this, let A be a continuous linear operator on $\mathcal{E}$.

Definition 3.1. We say that the operator A is p-summable if there exist sequences x_j in $\mathcal{E}$ and x'_j in $\mathcal{E}'$ with $\|x_j\| = \|x'_j\| = 1$, and a sequence of positive real numbers $\sigma = \{\sigma_j\}$ such that $\sum_{j \geq 1} \sigma_j^p < +\infty$ and

$$A(u) = \sum_{j \geq 1} \sigma_j \langle u, x'_j \rangle x_j,\ \forall u \in \mathcal{E}. \tag{9}$$

We denote by $\mathfrak{S}_p(\mathcal{E})$ the ideal of p-summable operators. Clearly, one has $\mathfrak{S}_p(\mathcal{E}) \subseteq \mathfrak{S}_q(\mathcal{E})$ if $p \leq q$. The 1-summable operators are exactly the trace class (or nuclear) operators. The trace is then given by

$$\mathrm{Tr}(A) = \sum_{j \geq 1} \sigma_j \langle x_j, x' \rangle. \tag{10}$$

For this definition to make sense, the right side of (10) must be independent of the representation (9) of A. This is the case whenever $\mathcal{E}$ has the approximation property, since J_π is then injective.

Clearly, every p-summable operator for $p > 0$ is compact. Let λ_j be the non-zero eigenvalues of A represented as many times as their algebraic multiplicities. In [5] (Ch. I, pp. 171–177 and Ch. II, p. 20), Grothendieck obtained the following result.

Theorem 3.2 (Grothendieck)**.** Suppose that A is 2/3-summable. Then, the following equality holds:

$$\mathrm{Tr}(A) = \sum_{j \geq 1} \lambda_j(A). \tag{11}$$

Theorem 3.2 is optimal for general Banach spaces: for example, in the Banach space ℓ_1 of summable sequences, there exists an operator N that is p-summable for all $p > 2/3$ and such that $N^2 = 0$ and $\mathrm{Tr}(N) = 1$ ([10], paragraph 10.4.5).

It might come as a surprise that Grothendieck did not tackle the particular case of Hilbert spaces in a more explicit way. In Chapter II of his thesis (p. 13), he claims: "If $p \leq 1$ then the Fredholm determinant of u has genus 0." In our notation, $u = A$ is assumed to be p-summable. The next section recalls the properties of Fredholm determinants, as well as the definition of the genus of an entire function (see below, following formula (17)).

At the end of [9], Lidskii adds that he learned about Grothendieck's work [5] (including the above quotation) when his article was in press. Although the combination of the genus zero property, the results on Fredholm theory [6] and the factorisation theorem of entire functions by Weierstrass-Hadamard [7] give a proof of Theorem 2.2, Grothendieck neither stated the result nor pursued the argument to the end. This observation was noted in several publications, in particular in [11].

Theorems 2.2 and 3.2 were unified by Pietsch [10] as follows. We denote by ℓ_p the space of sequences of complex numbers whose p-th powers are summable and we denote by p' the real conjugate of p. In [10], the author studies many families of ideals of operators. In particular, he introduces the following ideals in the Banach space $\mathcal{B}$.

Definition 3.3. Let r, p, q be three real numbers such that $r > 0$ and $1 + \frac{1}{r} \geq \frac{1}{p} + \frac{1}{q}$. The operator $A \in \mathcal{L}(\mathcal{B})$ is called (r, p, q) nuclear if there exists a factorisation $A = S D_{iag}(\sigma) R$ such that $R \in \mathcal{L}(\mathcal{B}, \ell_{q'})$, $S \in \mathcal{L}(\ell_p, \mathcal{B})$ and $\sigma \in \ell_r$, where $D_{iag}(\sigma)$ denotes the diagonal operator associated with σ.

We write $\mathfrak{N}_{(r,p,q)}(\mathcal{B})$ for the set of (r, p, q) nuclear operators of $\mathcal{B}$. Note that A belongs to $\mathfrak{N}_{(r,p,q)}(\mathcal{B})$ if and only if A admits a representation

$$A = \sum_{j \geq 1} \sigma_j x'_j \otimes x_j,$$

where $\sigma \in \ell_r$, $(x_j) \in \ell_{p'}(\mathcal{B})$ and $(x'_j) \in \ell_{p'}(\mathcal{B}')$. Here, $\ell_p(\mathcal{B})$ denotes the space of sequences of $\mathcal{B}$ that are weakly in ℓ_p. In particular, $\mathfrak{N}_{(r,1,1)}$ agrees with the set of r-summable operators on $\mathcal{B}$ and, given a Hilbert space $\mathcal{B} = \mathcal{H}$, one has $\mathfrak{N}_{(1,1,2)}(\mathcal{H}) = \mathfrak{S}_\infty(\mathcal{H})$.

Theorem 3.4 (Pietsch [10]). Let $A \in \mathfrak{N}_{(1,1,2)}(\mathcal{B})$ be an operator. Then, $\sum_j |\lambda_j(A)| < +\infty$ and

$$\mathrm{Tr} A = \sum_{j \geq 1} \lambda_j(A).$$

It is easy to see that $\mathfrak{N}_{(2/3,1,1)}(\mathcal{B}) \subseteq \mathfrak{N}_{(1,1,2)}(\mathcal{B})$. This theorem contains the trace equalities of Grothendieck and Lidskii.

Remark 3.5. Note that, in his proof, Pietsch does not use the approximation property for $\mathcal{B}$.

In 1988, Pisier [11] introduced a class of "weak Hilbert" Banach spaces that are characterised by a condition on the weak type and cotype. He shows that, in these spaces, one has $\mathrm{Tr} A = \sum_{j \geq 1} \lambda_j(A)$ for any nuclear operator A satisfying $\sum_{j \geq 1} |\lambda_j(A)| < +\infty$.

4 Traces and invariant functions

One can easily show that any invariant continuous linear form f on $\mathfrak{S}_1(\mathcal{B})$ is a multiple of the trace Tr (see the introduction). It is natural to look for other invariant functions on $\mathfrak{S}_1(\mathcal{B})$, particularly for polynomial functions on other ideals of $\mathcal{L}(\mathcal{B})$.

Let us mention that Dixmier studied another property of the trace: normality. Let $\mathcal{L}_+(\mathcal{H})$ be the cone of positive operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$. We call a trace any function $f\colon \mathcal{L}_+(\mathcal{H}) \to [0, +\infty]$ that is positive, additive and homogeneous. We say that f is a normal trace if it is, moreover, completely additive: if $A = \sum_{n \geq 1} A_n$, with $A_n \in \mathcal{L}_+(\mathcal{H})$, then $f(A) = \sum_{n \geq 1} f(A_n)$. It is easy to see that any normal trace is proportional to the usual trace Tr.

Dixmier proved that $\mathcal{L}_+(\mathcal{H})$ possesses a non-normal trace Tr_{D}, nowadays called a Dixmier trace. This trace is identically zero for finite rank operators. In his book "Non-commutative geometry" [Academic Press, Inc., 1994], A. Connes reproduces the article by Dixmier and gives an application to perturbative field theory.

We are now going to consider the invariant polynomial functions that appear naturally in the Fredholm theory of determinants.

Definition 4.1. A two-sided ideal $\mathfrak{S}$ of $\mathcal{L}(\mathcal{E})$ is said to be normed if it is equipped with a norm $\|\cdot\|_{\mathfrak{S}}$ such that

$$\|RAS\|_{\mathfrak{S}} \leq \|R\| \, \|A\|_{\mathfrak{S}} \|S\|.$$

A continuous function f with complex values on the ideal $\mathfrak{S}$ is invariant if $f(T^{-1}AT) = f(A)$ for all $A \in \mathfrak{S}$ and all $T \in \mathcal{L}(\mathcal{E})$. This property amounts to $f(AT) = f(TA)$ for all $T \in \mathcal{L}(\mathcal{E})$.

The spaces $\mathfrak{S}_1(\mathcal{E})$, $\mathfrak{S}_\infty(\mathcal{E})$ and $\mathfrak{S}_p(\mathcal{H})$, for $1 \leq p < +\infty$, are all normed ideals.

In this section, we shall determine all polynomial functions that are invariant under a normed ideal $\mathfrak{S}_1(\mathcal{E})$, as well as under the Schatten classes $\mathfrak{S}_p(\mathcal{H})$ in the Hilbert case. This computation was carried out independently in [13] and [2] for different purposes: in [13], it was to justify a numerical method to find the eigenvalues of systems of elliptic partial differential equations, initiated by Fichera in the work "Linear elliptic systems and eigenvalue problems" [Lecture Notes in Math. No. 8, Springer-Verlag-1965); and in [2], it was to study classifying spaces of vector bundles.

We start by computing the invariant polynomial functions on the ideal $\mathcal{L}_F(\mathcal{E})$. Recall that a homogeneous polynomial function of degree n on a Banach space $\mathcal{B}$ is a map Φ from $\mathcal{B}$ to $\mathbb{C}$ defined by a continuous n-linear symmetric form $\tilde{\Phi}$ such that $\Phi(A) = \tilde{\Phi}\,(A, \overset{\text{n times}}{\ldots}, A)$ ($\tilde{\Phi}$ is unique). Following [6], we obtain the fundamental invariant forms by a tensor computation.

We denote by $\otimes^n \mathcal{E}$ the n-th tensor power of $\mathcal{E}$ and we use the natural identification between $\otimes^n(\mathcal{E}')$ and $(\otimes^n \mathcal{E})'$. Let Λ_n (resp. Λ^n) be the antisymmetrisation operator on $\otimes^n \mathcal{E}$ (resp. $\otimes^n(\mathcal{E}')$). Then, Λ_n is a projector on $\otimes^n \mathcal{E}$ and $\Lambda'_n = \Lambda^n$. Given $A_j \in \mathcal{L}(\mathcal{E})$, for $j = 1, \cdots, n$, one defines

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n = \Lambda_n (A_1 \otimes A_2 \cdots \otimes A_n) \Lambda_n.$$

The assignment $(A_1, \cdots, A_n) \mapsto A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is n-linear and symmetric. Moreover, $\wedge^n \mathcal{L}_F(\mathcal{E}) \subseteq \mathcal{L}_F(\otimes^n \mathcal{E})$.

One can check that, for any two integers $s, n \geq 1$, the function

$$A \mapsto \mathrm{Tr}(\wedge^s(A^n)) := \mathcal{J}_s^n(A)$$

is an invariant polynomial function of degree sn on $\mathcal{L}_F(\mathcal{E})$. In fact, $\mathcal{J}_s^n$ is an elementary invariant of type (s, n). Setting $\mathcal{J}_0^n = 1$, the following recurrence relation holds:

$$\mathcal{J}_s^n(A) = \frac{1}{s}\sum_{q=1}^{q=s} \mathcal{J}_1^{nq}(A)\mathcal{J}_{s-q}^n(A). \tag{12}$$

Using an inequality for determinants due to Hadamard [3, (p. 1018)], as well as the Stirling formula, one obtains:

$$|\mathcal{J}_s^n(A)| \leq \gamma_s \|A\|_1^{ns}. \tag{13}$$

In the above, $\gamma_s \leq C(e^2/s)^{\frac{s+1}{2}}$ for a universal constant C in the general case and $\gamma_s = 1/s!$ for Hilbert spaces. It follows that $\mathcal{J}_s^n(A)$ extends by continuity to an invariant function on $\mathfrak{S}_1(\mathcal{E})$, which satisfies, in particular, (12) and (13).

Theorem 4.2 ([2,13])**.** Assume that $\mathcal{E}$ has infinite dimension. The vector space $\mathcal{P}_n$ of homogeneous polynomial functions of degree $n \geq 1$ that are invariant under $\mathfrak{S}_1(\mathcal{E})$ has finite dimension $p(n)$, equal to the number of partitions of n as a sum of positive integers. Moreover, each of the following two families is a basis for $\mathcal{P}_n$:

$$\left\{(\mathcal{J}_1^1)^{r_1}(\mathcal{J}_1^2)^{r_2}\cdots(\mathcal{J}_1^n)^{r_n},\right\}_{r_1+2r_2+\cdots+nr_n=n}, \tag{14}$$

$$\left\{(\mathcal{J}_1^1)^{r_1}(\mathcal{J}_2^1)^{r_2}\cdots(\mathcal{J}_n^1)^{r_n},\right\}_{r_1+2r_2+\cdots+nr_n=n}.$$

In the case of Hilbert spaces ($\mathcal{E} = \mathcal{H}$), there is a similar statement for the Schatten classes $\mathfrak{S}_p(\mathcal{H})$, $1 \leq p < +\infty$ ([13]). In particular, any invariant polynomial function of degree $< p$ is identically zero.

The elementary invariants can be expressed in an arbitrary orthonormal basis $\{e_k\}$ of $\mathcal{H}$. For this, it is convenient to introduce the Hilbert tensor product defined as follows: if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, the sesquilinear form given by $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle\langle x_2, y_2\rangle$ defines a scalar product on $\mathcal{H}_1 \otimes \mathcal{H}_2$. The Hilbert tensor product is the Hilbert space $\mathcal{H}_1\hat{\otimes}_2\mathcal{H}_2$ obtained as the completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let A be such that A^n is in the trace class. Then, $\wedge^s A^n$ is in the trace class in the Hilbert space $\hat{\otimes}_2^s\mathcal{H}$ and one has

$$\mathcal{J}_s^n(A) = \mathrm{Tr}(\wedge^s A^n).$$

From this, we derive

$$\mathcal{J}_s^n(A) = \frac{1}{s!}\sum_{k_1,\cdots,k_s} \det_{1\leq i,j\leq s}\langle e_{k_j}, A^n e_{k_i}\rangle.$$

Theorem 2.2 yields an expression of $\mathcal{J}_s^n(A)$ in terms of the eigenvalues of A, namely, $\mathcal{J}_s^n(A) = \mathcal{T}_s^n(A)$ if

$$\mathcal{T}_s^n(A) = \sum_{j_1<j_2<\cdots<j_s} \lambda_{j_1}(A)^n\lambda_{j_2}(A)^n\cdots\lambda_{j_s}(A)^n.$$

From these properties, there are two possible ways to introduce a determinant. Since A is a trace class operator, following Grothendieck's presentation [6], one can consider the Fredholm determinant

$$\det(\mathbb{1} - zA) := \sum_{k\geq 0}(-1)^k z^k \mathcal{J}_k^1(A). \tag{15}$$

It is an entire function of $z \in \mathbb{C}$ of order 1 (in a Hilbert space). This follows from inequality (13).

For the other definition, one begins directly with the eigenvalues and the infinite product

$$D_A(z) := \prod_{j\geq 1}(1 - z\lambda_j). \tag{16}$$

We know (19) that $\sum_{j\geq 1}|\lambda_j| \leq \|A\|_1$, hence the infinite product defines an entire function of order 1. Theorem 2.2 amounts to showing that $\det(\mathbb{1} - zA) = D_A(z)$ for all $z \in \mathbb{C}$.

To go further, it is useful to recall the Weierstrass factorisation theorem for entire functions (see W. Rudin, Real and Complex Analysis). The Weierstrass factors are the entire functions $E_0(z) = (1-z)$ and $E_p(z) = (1-z)\exp(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p})$ for $p \geq 1$. Let f be an entire function. We denote by m the multiplicity of 0 if $f(0) = 0$ and by $\{z_n\}_{n\geq 1}$ the sequence of non-zero complex numbers such that $f(z_n) = 0$, ordered by increasing modulus and repeated according to their multiplicity. The Weierstrass factorisation theorem says that f admits a (non-unique) factorisation of the form

$$f(z) = z^m e^{g(z)}\prod_{n=1}^{\infty} E_{p_n}(\tfrac{z}{z_n}), \tag{17}$$

with g an entire function and $\{p_n\}$ a sequence of integers.

We say that f has genus $\leq \mu$ if there exists a Weierstrass decomposition such that $p_n \leq \mu$ for all n and g is a polynomial of degree $\leq \mu$. The genus is the smallest positive integer that has this property. Therefore, being an entire function of genus 0 means that f trivially factors over its zeros:

$$f(z) = az^m\prod_{n=1}^{\infty}\left(1 - \tfrac{z}{z_n}\right),$$

where a is a constant. For more information on entire functions and their zeros, the reader is referred to, for example, B. Levin "Distribution of zeros of entire functions" [AMS Transl-1964].

In [6, Théorème 3], Grothendieck shows that the zeros of the entire function $F_A(z) = \det(\mathbb{1} - zA)$ are exactly the eigenvalues $\lambda_j(A)$ (counted with multiplicity). By the Weierstrass factorisation theorem, the equality $D_A = F_A$ holds, provided we can show that F_A has genus 0. This is how Grothendieck proceeds for general Banach spaces. Indeed, he proves [5, pp. 13–19] that, for all 2/3-summable operators A, the function F_A has genus 0. Theorem 3.2 follows from this.

Let $p \geq 2$ be an integer. Given an operator $A \in \mathfrak{S}_p(\mathcal{H})$, we introduce the regularised determinant ([3])

$$\det{}_p(\mathbb{1} - zA) = \prod_j (1 - z\lambda_j)R_p(z\lambda_j), \tag{18}$$

$$R_p(z) = \exp\left[z + \frac{z^2}{2} + \cdots + \frac{z^{p-1}}{p-1}\right].$$

The function $\det_p(\mathbb{1} - zA)$ is an entire function of the variable $z \in \mathbb{C}$ whose zeros are the inverses of the non-zero eigenvalues of A. Therefore, if A lies in $\mathfrak{S}_p(\mathcal{H})$ with $p \geq 1$, there exists $r > 0$ such that the equality

$$\det{}_p(\mathbb{1} - zA) = \exp\left(\sum_{k\in\mathbb{N}} \frac{\mathcal{T}_1^{p+k}(A)}{p+k} z^{p+k}\right) \tag{19}$$

holds for $|z| < r$. This formula goes back to Poincaré.

From Theorem 2.2, we derive the relations

$$\mathcal{J}_s^p(A) = \sum_{1\leq k\leq s} \frac{(-1)^{k+s}}{k!}\left(\sum_{r_1+r_2+\cdots+r_k=s} \frac{\mathcal{J}_1^{pr_1}(A)\cdots\mathcal{J}_1^{pr_k}(A)}{r_1\cdots r_k}\right).$$

Alexander Grothendieck and Laurent Schwartz

Examples

Let us now consider the case $\mathcal{H} = L^2(\Omega, \mu)$, where μ is a Borel measure on a locally compact space Ω. An operator A on $\mathcal{H}$ is in the Hilbert–Schmidt class if and only if there exists an integral kernel $K \in L^2(\Omega\times\Omega, \mu^{\otimes 2})$ such that, for $u \in L^2(\Omega, \mu)$, one has $Au(x) = \int_\Omega K_A(x, y)u(y)dy$. Then,

$$\|A\|_2^2 = \mathrm{Tr}(A^*A) = \int_{\Omega\times\Omega} \left|K_A(x, y)\right|^2 d\mu(x)d\mu(y).$$

Let A be a trace class operator of the form $A = A_1A_2$, where A_1 and A_2 are in the Hilbert–Schmidt class. It follows that A has an integral kernel

$$K_A(x, y) = \int_\Omega K_{A_1}(x, z)K_{A_2}(z, y)d\mu(z).$$

To simplify, we suppose that K_A is continuous on $\Omega\times\Omega$. Using Fubini's theorem, we obtain

$$\mathrm{Tr}(A) = \int_\Omega K_A(x, x)d\mu(x),$$

as well as the following formula for each $s \geq 1$:

$$\mathcal{J}_s^1(A) = \frac{1}{s!}\int_{\Omega^s}\left(\det_{\substack{1\le i\le s\\ 1\le j\le s}} K_A(x_i, x_j)\right)d\mu^{\otimes s}(x_1, \cdots, x_s).$$

These are classical expressions in the Fredholm theory of integral equations, which one can find, for instance, in Goursat's "Cours d'analyse mathématique" [Vol III, Gauthiers-Villars-1943].

Pseudo-differential operators form an important class of examples of integral operators. On a Riemannian compact manifold M of dimension d, a pseudo-differential operator A [14] admits an integral kernel K_A that can be written locally as

$$K_A(x, y) = (2\pi)^{-d}\int_{\mathbb{R}^d} a(x, \xi)\mathrm{e}^{i(x-y)\cdot\xi}d\xi,$$

where a is a smooth function with complex values (it is called the symbol of A). We assume that a is a classical elliptic symbol of order m. The principal symbol a_m is then a homogeneous function on the cotangent space $T^*(M)$ that has degree m in ξ. If $m \le 0$ then A is bounded on $\mathcal{H} := L^2(M)$. If $m < -d/p$ then A is in the Schatten class $\mathfrak{S}_p(\mathcal{H})$. For example, in the case of the Laplace-Beltrami operator $\triangle_M$ on M, the operator $(-\triangle_M + 1)^{-s}$ belongs to $\mathfrak{S}_p$ if $s > \frac{d}{2p}$. If $m < -d$ then the trace of A is given (locally) by the formula

$$\mathrm{Tr}A = (2\pi)^{-d}\int_{T^*(M)} a(x, \xi)dxd\xi.$$

Observe that, in general, the symbol a is not globally defined on a manifold: only the principal symbol (the homogeneous part of highest degree) is well defined.

Symbolic calculus on pseudo-differential operators allows one to obtain information about the eigenvalues of elliptic differential operators A (not necessarily self-adjoint) on a compact variety. The work of Seeley [14] is at the origin of many developments of this topic. The paper [1] contains an example of how to use these techniques to obtain asymptotic formulas for the spectrum of non-self-adjoint elliptic operators.

In the self-adjoint case, one can go much further in the study of trace formulas. For instance, the spectrum of the Laplacian on M is related to geometry (Selberg, Gutzwiller); a vast literature is devoted to this subject. There are also extensions of the notion of trace to operators that are not in the trace class, for example, the relative traces (and determinants) introduced by Krein [4] and analytic continuation of a generalised zeta function [14]. As an illustration, when $A = -\triangle_M + 1$, the function $\zeta_A(s) := \mathrm{Tr}A^{-s}$ is holomorphic on the complex half-plane $\{s,\ \Re s > d/2\}$ and extends to a meromorphic function on $\mathbb{C}$ whose poles belong to the sequence $s_j = (d - j)/2$. Moreover, ζ_A is regular at the integers [14].

5 Sketch of the proof of Lidskii's theorem

We first present a basic tool in the study of the eigenvalues of non-self-adjoint operators: Weyl inequalities ([4, pp. 35–41]).

Let A be a compact operator on $\mathcal{H}$. We consider the sequence $\{\lambda_j(A)\}$ of non-zero eigenvalues (if any) ordered by increasing modulus and repeated according to their multiplicities. Let $s_j(A)$ be the sequence of eigenvalues of $|A| := \sqrt{A^*A}$ greater than zero, the so-called singular (or characteristic) values of A. For any integer $N \geq 1$ and any real numbers $p, r > 0$, the following inequalities hold:

$$\left|\lambda_1(A)\lambda_2(A)\cdots\lambda_N(A)\right| \le s_1(A)s_2(A)\cdots s_N(A),$$
$$\sum_{1\le j\le N}\left|\lambda_j(A)\right|^p \le \sum_{1\le j\le N}(s_j(A))^p,$$
$$\prod_{1\le j\le N}(1 + r|\lambda_j(A)|) \le \prod_{1\le j\le N}(1 + rs_j(A)).$$

Moreover, $\|A\|_1 = \sum_{j\ge 1} s_j(A)$.

Recall that $D_A(z) := \prod_{j\ge 1}(1 - z\lambda_j)$. By the third Weyl inequality, one has

$$\left|D_A(z)\right| \le \mathrm{e}^{|z|\|A\|_1} \tag{20}$$

for all $z \in \mathbb{C}$. We first assume that A is in the trace class and does not have any non-zero eigenvalue. This is equivalent to $\lim_{n\to+\infty}\|A^n\|^{1/n} = 0$ (i.e. A is quasi-nilpotent).

An elementary example of a quasi-nilpotent operator is the integration operator $Ku(x) = \int_0^x u(y)dy$, which is defined on $\mathcal{H} = L^2([0, 1])$ for the Lebesgue measure. One can immediately check that K does not have eigenvalues (it is injective). It is not in the trace class but it is in the $(1 + \varepsilon)$-Schatten class for all $\epsilon > 0$. In particular, K^2 is a trace-class quasi-nilpotent operator.

Let Π_N be an increasing sequence of orthogonal projectors of rank N in $\mathcal{H}$ that converges strongly to the identity. Then, $A_N := \Pi_N A \Pi_N$ converges to A in $\mathfrak{S}_1(\mathcal{H})$ and hence in $\mathcal{L}(\mathcal{H})$. We set $\lambda_j^{(N)} = \lambda_j(A_N)$ and $D_N = D_{A_N}$. It follows that

$$\lim_{N\to+\infty} \left|\lambda_1^{(N)}\right| = 0.$$

On the other hand, computing the logarithmic derivative of $D'_N(z)/D_N(z)$, one can show that

$$D_N(z) = \exp\left(-\sum_{k\in\mathbb{N}} \frac{\mathcal{T}_1^{1+k}(A_N)}{1+k} z^{1+k}\right).$$

Recall that, since A_N has finite rank N, the following equality holds for any integer s:

$$\mathcal{T}_1^s(A_N) = \mathrm{Tr}(A_N^s) = \sum_{j=1}^N (\lambda_j^{(N)})^s.$$

But the second Weyl inequality yields

$$\left|\mathcal{T}_1^{1+k}(A_N)\right| \le \|A_N\|_1 \left|\lambda_1^{(N)}\right|^k,$$

from which it follows, writing $a = \mathrm{Tr}(A)$ and using the continuity of the trace, that

$$\lim_{N\to+\infty} D_N(z) = \mathrm{e}^{-az}. \qquad (21)$$

One can now prove that $\mathrm{Tr}A = 0$ by arguing that if $a \neq 0$ then D_N has polynomial growth, uniform with respect to N, which contradicts (21). Indeed, from the Weyl inequalities and $s_j(A_N) \le s_j(A)$, one can obtain

$$|D_N(z)| \le \prod_{j\ge 1} (1 + s_j(A)|z|)$$
$$\le \prod_{1\le j\le M} (1 + s_j(A)|z|) \exp\left(|z| \sum_{j\ge M+1} s_j(A)\right).$$

Choosing M such that $\sum_{j\ge M+1} s_j(A) \le |a|/2$ and $z = \mathrm{e}^{-ia\arg a} r$, with $r > 0$, we derive the inequality

$$\mathrm{e}^{r|a|/2} \le \prod_{1\le j\le M} (1 + s_j(A)r)$$

and hence the contradiction.

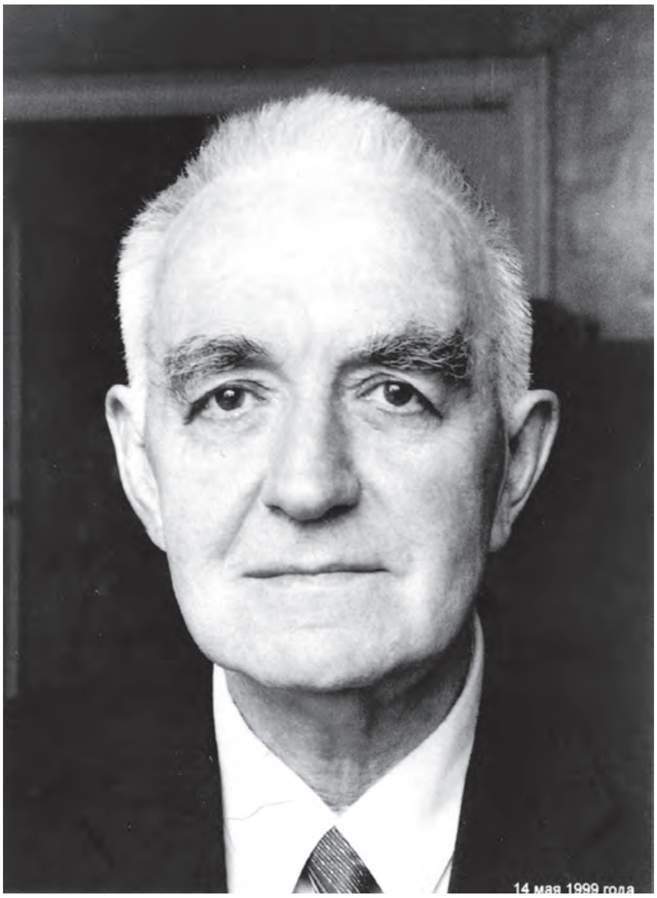
Lidskii's theorem is thus proved for quasi-nilpotent operators in the trace class. For the general case, one can decompose the Hilbert space into two orthogonal subspaces

$$\mathcal{H} = \mathcal{H}_D \bigoplus \mathcal{H}_N, \quad \mathcal{H}_D = \bigoplus_{j\ge 1} \mathcal{E}_{\lambda_j}(A)$$

(sum of the generalised eigenspaces for all non-zero eigenvalues). Let P denote the orthogonal projector onto $\mathcal{H}_D$ and $P^\perp = \mathbb{1} - P$. Then,

$$A = PAP + PAP^\perp + P^\perp AP + P^\perp AP^\perp.$$

Note that $\mathrm{Tr}(P^\perp AP) = \mathrm{Tr}(PAP^\perp) = 0$ (invariance of the trace) and $P^\perp AP^\perp$ is quasi-nilpotent. Indeed, P commutes with A and hence A^* commutes with $P^\perp$. Since A^* is quasi-nilpotent, the same holds for $P^\perp AP^\perp = (P^\perp A^* P^\perp)^*$. Finally, PAP is a sum of Jordan blocks containing exactly the eigenvalues of A repeated according to their multiplicities. By the linearity of the trace, this concludes the proof of Theorem 2.2.



Victor Borisovich Lidskii, May 14, 1999

6 Who was Lidskii?

We close with a few biographical notes, referring the interested reader to the introduction of *Operator Theory and its Applications* [AMS Transl-2010] for more details. Edited by two of his former students, M. Levitin and D. Vassiliev, this book is devoted to Lidskii's mathematical work.

Victor Borisovich Lidskii was born in 1924 in Odessa and died in Moscow in 2008. He defended his PhD thesis "Conditions for the completeness of the system of root subspaces of non-self-adjoint operators with discrete spectra" in Moscow University in 1954, under the supervision of I. M. Gelfand. He was a professor at Moscow's Fiz Tech University from 1961 to 2008, as well as at the Institute for Problems of Mechanics of the USSR Academy of Sciences. Fiz Tech University was created in 1946 in the suburbs of Moscow to encourage research in physics (connected to the nuclear and space programmes of the USSR); it had a favoured status.

Together with Gohberg and Krein, Lidskii was one of the pioneers of spectral analysis of non-self-adjoint operators. He became a recognised expert in the field, both for theoretical aspects and applications, especially in mechanics in areas such as elasticity and hydroelasticity equations and thin-shell theory. He accomplished important work in these subjects. His best known result is certainly Theorem 2.2. A renewed interest in his works arose from the recent developments on the spectrum of non-self-adjoint operators and the pseudo-spectrum (see [17] for an overview).

Another celebrated result by Lidskii deals with the extension of Weyl inequalities to eigenvalues of Hermitian matrices. Let A and B be two n by n Hermitian matrices. We denote by $\{\lambda_j(A), 1 \leq j \leq n\}$ the sequence of eigenvalues of A *in increasing order.*

Theorem 6.1 (Lidskii's inequalities, 1950)**.** For each subset $J \subseteq \{1, 2, \cdots, n\}$ of cardinal k, one has

$$\sum_{j \in J} \lambda_j(A+B) \leq \sum_{j \in J} \lambda_j(A) + \sum_{j=1}^{k} \lambda_j(B). \qquad (22)$$

This important result was obtained after work by Berezin and Gelfand on Lie groups [8]. Inequality (22) is related to geometric properties concerning Schubert varieties and representation theory. A pedagogical introduction may be found in the paper by R. Bhatia, "Linear algebra to quantum cohomology: the story of Horn's inequality" [Amer. Math. Monthly-2001]. Inequality (22) also has applications to numerical analysis. But this would take us too far …

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The Archives of American Mathematics

Carol Mead (University of Texas at Austin, USA) and Albert C. Lewis (The Initiative for Mathematics Learning by Inquiry, Austin, USA)

The Archives of American Mathematics (AAM) was established by an agreement between the Mathematical Association of America (MAA) and the University of Texas at Austin in 1978. It was formed not only as the official repository for the association but also to serve as a nucleus for additional gifts of mathematical archival material. It has since become a de facto national centre for original sources in the history of mathematics, accepting personal and institutional collections. It also encourages other archives to offer homes for collections relating to the mathematical sciences.

Origin of the AAM

An American researcher using the archives at Göttingen described earlier in the *EMS Newsletter* (Rowe 2016) will be impressed with the richness of its representation of the principal mathematicians who helped make Göttingen the centre of the mathematical universe in the 19th century, including Gauss, Riemann, Klein, Dedekind and Hilbert. That researcher might wish that there were an equally rich collection for American mathematicians who brought the United States to modern prominence. For this early period in the US, the publications of historians, especially Parshall and Rowe (1994), demonstrate that substantial original sources do exist for exploring the lives and works of the most prominent mathematicians, such as Benjamin Peirce, J.W. Gibbs, E.H. Moore, J.J. Sylvester, O. Veblen, R.L. Moore and G.D. Birkhoff. However, of these mathematicians, it can be argued that only Veblen and R.L. Moore are represented by collections comparable to those in Göttingen with respect to the depth and breadth of coverage of their life and work. The Veblen collection is at the Library of Congress in Washington, D.C., and R.L. Moore's is at the University of Texas at Austin. It is the latter collection that formed the starting point for the AAM.

For most of the first half of the 20th century and somewhat beyond, the University of Texas mathematics department was dominated by R.L. Moore (1882–1974). Renowned as a mathematician, he became even better known for his method of teaching, often called the "Moore method". His students became the leading set theory topologists of the century and perhaps the most distinguished group of research mathematicians to be taught by one professor (Parker 2005). After his death in 1974, his extensive "papers" (to use the common but ambiguous American archival term), or Nachlass, were preserved at the university and an effort was made to see that the Nachlass of his students was also preserved, if not at Texas then at least at suitable institutions elsewhere. Several of his students had been presidents of one or both of the major mathematical organisations: the Mathematical Association of America (MAA) and the American Mathematical Society (AMS). One of these students, R.L. Wilder, who had been president of both groups and who was a mathematician as well as an historian, pointed out that neither of these organisations had an archival repository for preserving their history. He received a positive response from Texas when he suggested that perhaps they would be interested in being that repository.

Others in the mathematical community picked up the cause and, though only the MAA agreed to make Texas their official archival repository, the AMS arranged a separate agreement with Brown University, a neighbour of the society in Providence, Rhode Island. Inspired by this activity, the American Statistical Association formed an archival agreement with Iowa State University not long afterward.

Today, with nearly 130 collections of papers and records of mathematicians and mathematical organisations, the AAM offers an abundance of potential avenues for research. Culture, art and organisational history, among others, have all been explored in the collections. In pursuit of these topics, the archives primarily attract historians of mathematics focused on the history of American mathematics and mathematicians, and their influence and roles in society. As a part of this, the collections reflect the vital influence that European mathematicians had on developments in the USA.

European connections

Among Hilbert's correspondence at Göttingen are letters from 1900 to 1906 from George Bruce Halsted, R.L. Moore's mathematics professor at Texas. Most of Halsted's original letters no longer exist but he did write to Moore that he had sent a copy of his book, *Rational Geometry: A Textbook for the Science of Space; Based on Hilbert's Foundations* (Halsted 1904), to Hilbert. Since the book was dedicated to Hilbert, who had politely given his approval for Halsted's project, Hilbert gave it to his assistant Max Dehn to review. Dehn published a severe criticism of it, particularly focusing on the textbook's results that depended on continuity, in spite of Halsted's explicit statement that he was not assuming the Archimedean axiom.

While Europe has largely been able to retain a rich archival history in modern times, wars have taken their toll. Hermann G. Grassmann (1809–1877) is an example: his Nachlass was listed in his *Gesammelte mathematische und physikalische Werke* (1894–1911) but then apparently lost during World War I (Petsche et al. 2009). Much of Cantor's Nachlass met a similar fate around the time of World War II, as is mentioned by Rowe. What exists from this period is often fragmentary and scattered. This, perhaps not unexpectedly, is the case for Max Dehn, given his torturous journey after leaving Germany in 1939 before eventually settling in the USA. Prior to leaving Germany, Dehn taught at Frankfurt University, where one of his students was Wilhelm Magnus, who received his degree in 1931. As indicated in Rowe (2016), Magnus encouraged Dehn's widow, Toni, to deposit whatever papers she had of her husband in the archives. The contents range from 1900, when he was at Göttingen, to 1950, when he was living in the USA in North Carolina. Many of the documents are in German, though some are in English, and include lectures he delivered, as well as correspondence, manuscripts, original drawings by Dehn, published works by him and others and a few photographs. It includes correspondence with Helmuth Kneser, items relating to Ernst Hellinger and C.L. Siegel, and single letters from Emmy Noether, Ernst Zermelo and Leo Frobenius, to mention just a few names.

Other collections of Dehn's papers in the USA, of varying sizes and dates, can be found, for example, in the Western Regional Archives in North Carolina, the Idaho State University in Pocatello and the Library of Congress, where Oswald Veblen's papers contain two folders of Dehn's documents.

What is interesting about the Texas collection is the broad coverage of Dehn's life and his interest in the intersection of art and mathematics. Marjorie Senechal (a professor emerita of Smith College and Editor-in-Chief of the Mathematical Intelligencer) and Philip Ording (a professor at Sarah Lawrence College) are collaborating with a group of about 12 mathematicians and historians of mathematics to write a composite biography of Dehn.

In an email to one of the authors, Senechal described the AAM collection, compared to others in the USA, as providing "a deeper, broader, richer picture, of Dehn the mathematician, Dehn the teacher, Dehn the father of three children, and his and [Toni's] journey across America, from San Francisco to Black Mountain College" (Senechal 2017). With this broader understanding of Dehn, and with information she gathered from other archives, Senechal proposes to write about the Dehns' journey.

In a talk at an Oberwolfach mini-workshop in 2016, Senechal described the Dehns' adventures in America starting from their arrival to its shores. They landed in San Francisco, California, on 1 January 1941, after escaping Nazi Germany via Oslo, Siberia and Japan. From there, the Dehns' already-uprooted lives continued in a similar fashion, moving from one state to another in an effort to find a more permanent home. It was not until Dehn found a teaching position at Black Mountain College in North Carolina that they settled down. There, as Senechal notes, Dehn indulged his many interests outside of mathematics by teaching ancient Greek, philosophy, elementary mathematics and a course he called "geometry for artists". Dehn retired in 1952 and died a few weeks later (Senechal 2016).

Philip Ording approaches Dehn from the perspective of art. In 2013, on a visit to the AAM, he found the "Geometry for Artists" folder, connected to the course mentioned above, to be especially inspiring. He included a drawing from the "Geometry" folder as the cover for an essay called "A Definite Intuition", which appeared in the *Bulletins of the Serving Library* (Ording 2013). The folder contains mathematical artwork created by Dehn and his students in 1948. As part of his description of the figure, Ording writes: "The points of the beautifully penned ellipse … emerge not by coordinates determined by an equation, but rather by the intersections of the construction lines produced by the interaction of a pair of concentric circles".

Ernst Hellinger, another figure whose Nachlass hardly exists as a whole, is, in addition to the Dehn collection, also represented in the papers of William T. Reid. A colleague of Hellinger at Northwestern University in the 1940s and 1950s, Reid was evidently bequeathed papers that Hellinger brought with him from Germany to Northwestern in 1939, including lecture notes from his teaching at Frankfurt from as early as 1919.

A further Göttingen connection is through Isaac Jacob Schoenberg (1903–1990), who moved to the USA in 1930 with many of his personal papers, including correspondence and student notebooks from his studies at Jassy [Iaşi] in Moldavia and in Berlin and Göttingen, mainly from the 1920s. Especially interesting are the lecture notes of Edmund Landau, Harald Bohr and Issai Schur, all of whom are represented in his correspondence.

Naturally, there is correspondence with European mathematicians in the Nachlässe of other American mathematicians. At the AAM, this would particularly include Moore (corresponding with Maurice Fréchet, Bronisław Knaster, Kazimierz Kuratowski and Wacław Sierpiński) and the number theorist H. S. Vandiver (corresponding with Helmut Hasse, Erich Bessel-Hagen, Henri Fehr, Dragoslav S. Mitrinović and B.L. Van der Waerden). Also, in the pre-World War II period, R.L. Moore's students, J.R. Kline and R.G. Lubben, visited members of the Polish school of topologists in Warsaw; Lubben's group photograph of himself with Samuel Dickstein, K. Kuratowski, Kline, K. Zarankiewicz and S. Masurkiewicz must be a rare artefact from that time.

More recently, the international connections of AAM were enhanced by the addition of the Nachlass of the prolific English historian Ivor Grattan-Guinness (1941–2014), whose world-ranging activities give unique insight into the state of the history of mathematics in the late 20th century.

American stories

Researchers, of course, typically employ multiple collections to achieve their aims. One such researcher is Michael Barany, currently a postdoctoral fellow at Dartmouth College, whose project concerns the "history of intercontinental research and disciplinary institutions in twentieth-century mathematics". During a week-long visit to the archives, Barany surveyed three collections: the Paul R. Halmos Papers, the R.L. Wilder Papers and the MAA Records.

The Nachlass of Paul Halmos (1916–2006) alone provided several perspectives relevant to his project. The

AAM holds his complete archive, including his papers and his photographic collection of some 14,000 images. His papers cover his long and distinguished career, revealing a level of professional involvement that is almost breathtaking in its scope: teaching (including visiting appointments in the USA and abroad), committee work, editorships, articles and books, extensive correspondence, conferences and national and international travel.

Barany was especially interested in a 1951 visiting appointment in which Halmos, along with his wife, Virginia ("Ginger"), spent a year at the University of Montevideo in Uruguay. Halmos, as was his custom, kept a diary of the excursion, which gave Barany background information that will assist him in arguing that Halmos played "an important role in the consolidation of professional mathematics in South America". Barany goes on to note that other papers in the collection will help him to explain Halmos' important role in international mathematics as it developed in the 20th century (Barany 2017).

While Wilder's career was also outstanding, he remained for most of it at one institution: the University of Michigan in Ann Arbor. Nonetheless, he "was a well-connected participant in the major [American] mathematical organizations" (Barany 2017). Wilder, a pioneer in the field of topology, was heavily involved in numerous organisations and, as noted above, originated the idea of the AAM (Raymond 2003). Through this collection, Barany was able to "identify new consequences and connections from those [organisational] events" of interest.

For Barany, the MAA Records provided supplementary information. Both Wilder and Halmos were active participants in the MAA and contributed to its mid-century evolution and character. Surveying officer records and correspondence, he found that "these formal documents show the regular hum of activity at a prominent organization of professional mathematicians…".

Another historian of mathematics recently made a discovery that demonstrates how the AAM's archival collections may enrich our understanding of the history of mathematics and American cultural history while also demonstrating the ties among American mathematicians. As she was working with the R.L. Wilder Papers, Karen Parshall (a professor of history and mathematics at the University of Virginia) stumbled on correspondence between Wilder and an aspiring African-American mathematician, William Schieffelin Claytor. The result of her discovery is her article, "Mathematics and the Politics of Race: The Case of William Claytor" (Parshall 2015).

From the R. L. Wilder and the R. L. Moore Nachlässe, she writes a story of a "'family' of topologists" and of the cultural realities of the day. Moore corresponded extensively with notable mathematicians, many of whom had been his students. Among the latter were Wilder and J.R. Kline. Kline had been Moore's first doctoral student, obtaining his degree at the University of Pennsylvania in 1916. Serving on the faculty of his alma mater from 1920 to 1955, Kline supervised 19 doctoral students, two of whom were black.

When Claytor wrote his first letter to Wilder in 1931, he was a young graduate student at the University of Pennsylvania under Kline's tutelage. Claytor came to Kline after having earned his Master's degree under Woodard at the historically black institution of Howard University in 1930. Recognising strong potential in Claytor, Woodard encouraged him to apply to Pennsylvania's graduate programme in order to study with Kline.

By mentoring and supervising Woodard and Claytor, Kline enlarged the Moore family, though in a way in which Moore may not have approved. As Parshall notes, Moore was a "product of his time and place", which is to say that he was a dedicated segregationist from the American South. The University of Texas did not admit black people through most of Moore's long career there from 1920 to 1969.

William Claytor with his wife Mae Pullins Claytor, Harpers Ferry, West Virginia, 1967. (Raymond Louis Wilder Papers, Dolph Briscoe Center for American History)

Claytor could undoubtedly have pursued a career in research, which was his primary interest. Parshall notes that the 1920s to the 1960s were a "golden age" in point-set topology and Claytor "contributed to enhancing its shine". Unfortunately, despite the efforts of Kline, Wilder and others, he was not inclined to risk experiencing more discrimination than he had already in other situations by following up on these efforts. He took a teaching position at Howard University in Washington, D.C., and remained there for the rest of his career. He died in 1967.

Parshall shared her information about Claytor with Dr Sibrina Collins, a chemist and historian of science, technology, engineering and mathematics (STEM) at The Marburger STEM Center at Lawrence Technological University in Southfield, Michigan. Collins was writing a profile of Claytor for the online journal, Undark, which focuses on the "intersection of science and society". The resulting essay, "Unsung: William Claytor" (Collins 2016), relays Claytor's academic story beyond the mathematical community.

The huge topic of mathematics education is represented by one of the largest AAM collections, the School Mathematics Study Group (SMSG), a national education reform project launched in 1958 with a major grant from the National Science Foundation. Headquartered at Stanford University, it oversaw an extensive programme of textbook writing, teacher training and educational filmmaking, as well as promulgating, throughout the 1970s, the so-called "new math" movement. This archive has been a valuable source for researchers interested in mathematics education and its history, Phillips

(2015) being a recent example. One of the precursors of SMSG was the University of Illinois Committee on School Mathematics led by Max Beberman. AAM has 47 film reels of his talks on teaching high school mathematics, produced in the 1950s and viewable on the AAM website.

Photographs, films, oral histories

Images, of course, supplement published discourse in most subject areas and mathematics is no exception. Two of the AAM's oft-used collections are the Paul R. Halmos Photograph Collection and the Marion Walter Photograph Collection, both of which depict distinguished 20th century mathematicians (American and international). While Halmos' collection spans more than 9 decades from 1907 to 1998, his snapshots of mathematicians begin in 1940, when he served as John von Neumann's assistant at the Institute of Advanced Study in Princeton, New Jersey. Marion Walter's photographs span the years from 1952 to the 1980s.

As noted above, the Halmos Collection contains approximately 14,000 photographs, mostly taken by Halmos himself (though he also accumulated images from other people). In 1987, Halmos published a small proportion of his pictures in his book, *I Have a Photographic Memory* (Halmos 1987). He notes in the preface that "I have been a snapshot addict for more than 45 years, and I have averaged one snapshot a day". A number of his subjects, such as John Nash, were rising stars at the time Halmos captured them, while others, like Arthur Erdélyi, were in the latter years of long, successful careers.

Halmos photographing himself, Varenna, Italy, 1960. (Paul Halmos Photograph Collection, Dolph Briscoe Center for American History)

Knowing that interest in the Halmos snapshots will be high, the AAM is digitising over 8,300 of them and will put them on the web. In fact, in collaboration with the MAA, 342 photographs have already been digitised and added to their online periodical, *Convergence*, with the title "Who's That Mathematician? Images from the Paul R. Halmos Photograph Collection" (Beery 2012).

While significantly smaller than the Halmos collection, the Marion Walter collection – with some 60 or so images instead of thousands – contains subjects that are no less eminent. Among them are A.A. Albert, Paul Erdős, Olga Taussky-Todd and H.S.M. Coxeter.

Now retired from the University of Oregon, Eugene, Walter came to the USA in 1948 from England, where she, her sister and her parents had lived after escaping Nazi Germany in 1939. Walter then pursued her mathematical degrees, first a Master's of science in 1954 from New York University and eventually her PhD in 1967 from Harvard University. George Pólya influenced her interest in education and he became one of her teachers at Stanford University, where she held a fellowship in 1960.

Walter took her photographs during various events that she attended between the 1950s and the 1980s. For example, in 1952, she attended the Institute for Numerical Analysis at the University of California, Los Angeles, where she snapped Ernst G. Straus, and went for tea at the home of Alexander Ostrowski, whom she also photographed. At a picnic in 1953 with other mathematicians from New York University, she captured Fritz John and Wilhelm Magnus and, during her stay at Stanford, she took a snapshot of Pólya.

Left to right: Tobias Danzig, J.G. van der Corput, Marion Walter and Magnus Hestenes at a tea party at the home of Alexander Ostrowski, Los Angeles, CA, 3 August 1952. (Marion Walter Photograph Collection, Dolph Briscoe Center for American History)

Both Halmos and Walter captured mathematicians (both well known and less well known) during unscripted moments: at conferences, chatting at parties, teaching a class or sitting around a pool. In the same informal, personal vein, the AAM has several hundred voice and video recorded interviews with mathematicians, which are the product of a number of projects. They bring the history of mathematics to life. Indeed, this is true for all the AAM's collections.

The AMS-MAA Joint Archives Committee acts as an unofficial liaison between the AAM and the two mathematical organisations. It has encouraged the preservation of Nachlässe and has promoted the awareness of archives, for example, through the committee project (Batterson et al. 2003). The AAM has been supported by the University of Texas at Austin through its Briscoe Center for American History, where it is housed and made available, and also through generous seed-money grants from the Educational Advancement Foundation.

To learn more about the AAM's Nachlass, please visit: http://www.cah.utexas.edu/collections/math_finding aids.php.

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Carol Mead is the Archivist for the Archives of American Mathematics at the Briscoe Center for American History at the University of Texas at Austin. She began her career in 1999 in the Denver Public Library's archives. In 2002, she received her library degree from the University of Texas at Austin. She remained in Austin and founded the archives for a local company, National Instruments. She has been in her current position since 2007.

Albert C. Lewis is an historian of mathematics who has contributed to the scholarly editions of the works of Bertrand Russell and Charles S. Peirce. Soon after obtaining a doctorate at the University of Texas at Austin, he helped to found the Archives of American Mathematics. He is currently a member of the board of directors and a volunteer at a public foundation called 'The Initiative for Mathematics Learning by Inquiry'.

Maryam Mirzakhani (1977–2017): Her Work and Legacy

Valentin Zagrebnov (Aix-Marseille Université, Marseille, France), Editor-in-Chief of the EMS Newsletter

International Congress of Mathematicians, Seoul 2014. (Credit ?)

In 2014, Maryam Mirzakhani, a Harvard-educated mathematician and professor at Stanford University in California, was one of four Fields Medal winners announced by the International Congress of Mathematicians (ICM) at its conference in Seoul.

The award recognised Mirzakhani's sophisticated and highly original contributions to the fields of geometry and dynamical systems, particularly in understanding the symmetry of curved surfaces such as spheres.

Maryam Mirzakhani has made striking and highly original contributions to geometry and dynamical systems. Her work on Riemann surfaces and their moduli spaces bridges several mathematical disciplines – hyperbolic geometry, complex analysis, topology and dynamics – and influences them all in return. She gained widespread recognition for her early results in hyperbolic geometry and her most recent work constitutes a major advance in dynamical systems.

"This is a great honour. I will be happy if it encourages young female scientists and mathematicians," Maryam said. "I am sure there will be many more women winning this kind of award in coming years."

Born on 3 May 1977 and raised in Tehran, Maryam Mirzakhani initially dreamed of becoming a writer but, by the time she started high school, her affinity for solving mathematical problems and working on proofs had shifted her sights. "It is fun – it's like solving a puzzle or connecting the dots in a detective case," she said when she won the Fields Medal. "I felt that this was something I could do, and I wanted to pursue this path." Maryam said she enjoyed pure mathematics because of the elegance and longevity of the questions she studied.

"It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks and, with some luck, you might find a way out," she added.

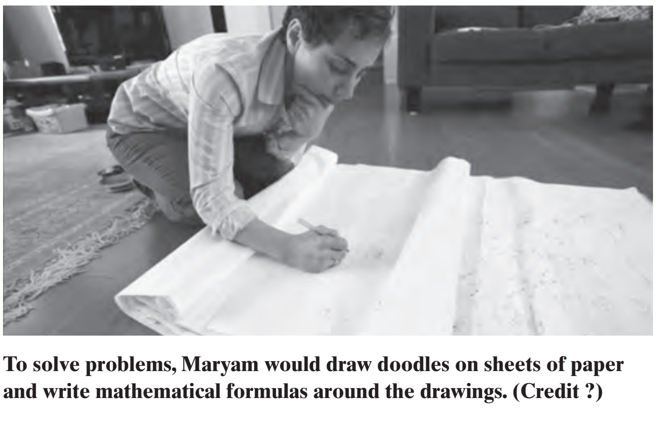
Maryam Mirzakhani became known on the international mathematics scene as a teenager, winning gold medals at both the 1994 and 1995 International Mathematical Olympiads – finishing with a perfect score in the latter competition.

She also won the 2009 Blumenthal Award for the Advancement of Research in Pure Mathematics and the 2013 Satter Prize of the American Mathematical Society.

In 2008, she became a professor of mathematics at Stanford, where she lived with her Czech husband and her daughter born in 2011.

In recent years, Maryam Mirzakhani has explored other aspects of the geometry of moduli space. Non-closed geodesics in moduli space are very erratic and even pathological and it is hard to obtain any understanding of their structure and how they change when perturbed slightly. However, Maryam and co-authors have proved that complex geodesics and their closures in moduli space are in fact surprisingly regular, rather than irregular or fractal. It turns out that while complex geodesics are transcendental objects defined in terms of analysis and differential geometry, their closures are algebraic objects defined in terms of polynomials and therefore have certain rigidity properties.

Because of its complexities and inhomogeneity, moduli space has often seemed impossible to work on directly, but not to Maryam. She had a strong geometric intuition that allowed her to grapple directly with the geometry of moduli space. Fluent in a remarkably diverse range of mathematical techniques and disparate mathematical cultures, she embodied a rare combination of superb technical ability, bold ambition, far-reaching vision and deep curiosity.



To solve problems, Maryam would draw doodles on sheets of paper and write mathematical formulas around the drawings. (Credit ?)

Maryam Mirzakhani, the first woman to win the coveted Fields Medal, died on 15 July 2017 in a Californian hospital after a battle with cancer. She was only 40.

Maryam's friend Firouz Naderi announced her death on Saturday 15 July on Instagram.

"A light was turned off today. It breaks my heart ... gone far too soon," wrote Naderi, a former director of Solar Systems Exploration at NASA.

"A genius? Yes. But also a daughter, a mother and a wife," he added in a subsequent post.

La Maison des Mathématiques et de l'Informatique. The House of Mathematics and Computer Science in Lyon

Régis R.G. Goiffon (Université de Lyon 1, Villeurbanne, France)

The role played by the mathematics laboratories in the dissemination of scientific culture to the public is difficult to overestimate.

It is crucial to enable a larger number of people to comprehend different aspects of mathematics, to find the "Ariadne's Thread" to a better understanding of the science that is becoming more and more subtle and powerful, and is often the "cornerstone" of knowledge in the modern world, which in turn is getting continuously more complex.

La Maison des Mathématiques et de l'Informatique (MMI – the House of Mathematics and Computer Science) was created in 2012 through the initiative of Etienne Ghys, Bertrand Remy and Vincent Borrelli[1] within the framework of the "LabEx MILyon"[2] which groups together the scientific communities of mathematics and fundamental informatics of Lyon. The aim of this laboratory of excellence is "creating a synergy between mathematics and fundamental computer science in order to found a house of mathematics and computer science to attract the best researchers in these fields". The idea to put an emphasis on dissemination (which is unique and original in such a context) marks the will of the participants of the project to react to the strong demand for better understanding of the rapidly developing world. The mathematicians wanted to put together and amplify the numerous dissemination initiatives that have been developing locally over the last 20 or 30 years, especially in the direction of a younger audience, such as lectures, meetings with researchers, presentations during large events and an international exhibition "Pourquoi les mathématiques?" at the Museum.[3]

The main goal is to arouse the interest of the largest possible number of young people towards mathematics and informatics, to show them the picture of a science in the process of active development and to reorient the representations, often negative, that one comes across regarding mathematics.

Since its creation, the MMI has favoured an open approach, which helps to mix and intertwine science, arts, music, history, etc.

This is both a space of mediation, with a large opportunity for dissemination, and a meeting space for all those who are curious and seek for an approach to mathematics and informatics that is simultaneously alive, entertaining and multidisciplinary. The House of Mathematics and Computer Science develops new projects and accompanies pre-existing projects, giving the audience a better overview and visibility of the available opportunities.

Let us, from the very beginning, stress two important points:

1 In 2015, Étienne Ghys received the first edition of the "Clay Award for Dissemination of Mathematical Knowledge" for "his important personal contributions to research in mathematics and his remarkable work for promoting mathematics" (see http://perso.ens-lyon.fr/ghys/accueil/).
Bertrand Remy was the first head of the LabEx MILyon and he is currently a professor at the École Polytechnique (see http://bremy.perso.math.cnrs.fr/maths.html).
Vincent Borrelli was the first director of the MMI (see http://math.univ-lyon1.fr/homes-www/borrelli/).

2 The LabEx is one of the instruments of the "investment to the future" programme launched in France in 2011 with the goal of supporting the research activities of all the teams within a given scientific subject. Located in the heart of the second scientific hub of France, MILyon aims to establish the Lyon – Saint-Etienne hub as one of the leading French research and higher education centres for mathematics and computer science and to become an international reference point. It pulls together more than 350 researchers and three research centres that are internationally recognised for their tradition of excellence:
- the Institut Camille Jordan (ICJ) – which, in turn, includes participants from the Claude Bernard Lyon 1 University, École Centrale de Lyon, INSA de Lyon and the Jean Monnet University of Saint Etienne;
- the Computer Science and Parallelism Laboratory (LIP) and the
- Laboratory of Pure and Applied Mathematics of the ENS (UMPA).

3 See "Why Maths?", Régis Goiffon, EMS Newsletter, September 2015, pp. 59–60.

House of Mathematics and Computer Science, 1 Place de l'École, Lyon.

- The MMI is fully piloted by researchers and professors, out of their passion and on a voluntary basis,[4] via a directing committee that meets every month.
- The aim of the MMI is to become a permanent establishment, going beyond the lifetime of the LabEx. The analysis of the structures to be organised and of future financial tools is a constant preoccupation of the directing committee and of the LabEx. As a future educational and cultural actor, the MMI wants to claim its position with the local decisionmaking authorities and ensure long-lasting collaborations.

The main mission of the MMI is to:

- Provide a place specifically conceived for the actions of dissemination and exhibitions.
- Unite and amplify the actions of dissemination of mathematics and informatics in Lyon and its greater region.
- Support the actions and associations of popularisation of mathematics and informatics.
- Develop actions aimed at supporting and promoting mathematics and informatics in the public and political arenas.

Some actions realised since 2012

A place for dissemination and exhibitions

The House of Mathematics and Computer Science has been installed in a space rented at the ENS,[5] situated in a rapidly developing district in the heart of one of the scientific poles of Lyon, close to the recent museum of the "Confluences". This space of 450 m^2 includes an exhibition hall of 200 m^2, a conference room for 40 people with multimedia equipment, the "ateliers" area (a hands-on workshop), administration space with a meeting room, offices and storage space also available for our partners (like the IFE[6] and the associations like "Plaisir Maths", "ÉbulliScience", "Math à modeler", "Les Bricodeurs" and "Mix-TeeN"). It welcomes school students for the whole year, and a general audience on Wednesday and Saturday afternoons and during the school holidays, for seminars and conferences.

A view of the exhibition "Surfaces".

Mix-TeeN association activities.

To stress the presence of mathematics and informatics in all branches of the society, the MMI develops annual on-the-spot exhibitions (which are always original) and, for several years, has invited artists working on the topic "art and mathematics/informatics":

- In 2013–2014, "Entropie / Néguentropie". Guest: artist Sophie Pouille.[7]
- In 2014–2015, "Surfaces". Guest: artist-mathematician Pierre Gallais.[8]
- In 2015–2016, "Musimatique", an interactive visual exhibition with sound, highlighting the convergences between music, mathematics and informatics. Guest: Denys Vinzant (composer and visual artist) and the GRAME national centre for musical creation.
- In 2016–2017, "Magimatique", a show-exhibition mixing magic and science and showing aspects of magic based on mathematics and computer science. Guest: Yves Doumergue, French champion of illusion.

The MMI also exhibits permanent exhibitions on its premises, such as a Turing machine in Lego® bricks, ellipsoids and a flat torus.

The MMI has also presented some of its exhibits "out of the walls", in city-halls, university sites and state libraries, as well as associated parts of the show (for example, the scientific fairytale "Lune", directed by Marie Lhuissier).

As mentioned, it welcomes classes during the week and a general audience (with more than 2000 visitors in 2016–2017) on Wednesdays, Saturdays and during school

[4] The dissemination of knowledge is formally a part of the mission of professors and researchers at the universities, as well as at CNRS and INRIA.
[5] Ecole Normale Supérieure de Lyon – one of the French "Grandes Écoles", the leading research and teaching institutions in France (see http://www.ens-lyon.fr).
[6] L'Institut Français de l'Éducation (French Institute for Education) is an entity within the ENS de Lyon that succeeded the National Institute for Pedagogic Research (INRP): http://ife.ens-lyon.fr/ife.
[7] Sophie Pouille continued her collaboration with the MMI for the exhibition "Formes élémentaires, mouvements et géométries de la pensée" (Médiathèque Jean Rousselot de Guyancourt, 10 October–1 December 2013) and the exhibition "Espaces intuitifs" (Abbaye Espace d'Art contemporain d'Annecy le Vieux, September to December 2016): http://www.sophiepouille.com.
[8] Pierre Gallais has also developed several original "ateliers" with the MMI that have been integrated into the exhibition "MathαLyon" (see http://institutdemathologie.fr).

Club de Mathématiques Discrètes – 90 participants of the last course, in the Square Evariste Galois.

breaks. For schoolchildren, from kindergarten to final grade level, about 20 "ateliers" are offered.

The privileged axes are research actions and an approach using games and hands-on material.

In parallel to these actions, the MMI offers, for a general audience, within its walls, a "ludothèque" (entertaining activities, run by Plaisir Maths) and some activity-based clubs in subjects like maths and magic, electronics, robotics and mathematical origami.

"Math α Lyon: Meet the Mathematicians!" is a particular action of the MMI. Initiated by the UMPA and the ICJ in 2008 (and supported by the Institute for Research on Teaching Mathematics of Lyon and the Department of Mathematics of the Claude Bernard Lyon 1 University) to respond to the demand formulated right after the presentation of the UNESCO exhibition "Pourquoi les mathématiques?" in the museum (which welcomed more than 7000 visitors in two months), this is one of the key actions of the MMI. Researchers intervene in classes at high school to present about 20 hands-on stands. Some of them have been developed by PhD students for Math α Lyon (for example, the workshop "Peaucellier-Lipkin mechanism" or "The hanging of paintings", a workshop that aims to solve a concrete problem using algebraic topology, which is a current field of research). On the other hand, three or four students in mathematics accompany the researchers during the presentations for two years. It is, for them, a first experiment in the dissemination of mathematics. Every year, about 5000 schoolchildren (and not necessarily from scientific classes) profit from this very popular action (the waiting time is currently two years).

Unite and amplify the actions of dissemination of mathematics and computer science}

Since its creation, the MMI has participated in major events like the Fête de la Science (Science Festival), which is a national event (in partnership with the ICJ, the UMPA and the LIP), and the week of mathematics, as well as supporting "girls in maths".

Concerning the week of mathematics, on top of lectures and interactive exhibitions in schools, in 2016 the MMI organised a "Forum des Mathématiques Vivantes", which takes place in the heart of the city (a mathematics rally for all ages in the old town of Lyon, lectures, interactive workshops in the Academy of Sciences, Letters and Arts of Lyon and mathematical competitions at the ENS). Its successful reception by the public (about 2000 participants) resulted in it being repeated, with similar success, in 2017.

Support the popularisation of mathematics and informatics}

The MMI has organised and piloted summer schools for young mathematicians: International Summer School of Mathematics for Young Students (ISSMYS), 20–30 August 2012 (110 participants), and Modern Mathematics International Summer School for Students (MoMISS), 20–29 August 2014 (eight series of lectures with 81 participants from 38 countries).

In 2016, the MMI organised "MathInfoLy", which gathered together 96 students who were 15 to 18 years old with good potential from 13 countries.[9] The goal was to encourage these students by suggesting mini-courses and scientific activities guided by real researchers. MathInfoLy concluded with presentations of the posters of each group (of six to eight students).

"Les Soirées Mathématiques de Lyon" (Mathematical Evenings): these are co-organised by the MMI and the mathematicians of the ENS de Lyon (UMPA), the INSA de Lyon, the Lycée du Parc and the Claude Bernard Lyon 1 University (ICJ). In these sessions, renowned mathematicians deliver popular lectures (that take place in one of the organising institutions) aimed at an audience of students in mathematics.

"Hippocampe Camp on Robotics" internship on robotics: high school students learn to "construct the intellect", i.e. how to create artificial intelligence. There are activities initiating research (lasting from a couple of days to a week) to understand what a robot is and to build it. This initiation of mathematics research covers common subjects but with several ways of reflecting on

[9] Applications were submitted by 260 candidates from 21 countries (Algeria, Morocco, Congo, Romania, Togo, Switzerland, Italy, Tunis, Senegal, Czech Republic, USA, Thailand, Dubai, Laos, Germany, Ethiopia, Canada, Saudi Arabia and France). The selection was carried out by specialists in mediation.

solving the problem. The students work in small groups and give an oral presentation on their results, which they also formalise on a poster. The subjects are suggested by the supervisors.

"Séminaire de la détente mathématique" (mathematics entertainment): this seminar is organised by PhD students of the ENS de Lyon. It takes place in the MMI, mathematicians meeting in an informal and relaxed setting to discuss fun mathematics. Every week, there is a new lecture, a new speaker, a new subject and new mathematical perspectives. The speakers can be students, PhD candidates and researchers at one of the mathematical laboratories in Lyon (e.g. UMPA or ICJ).

The MMI obviously supports the "Lyon Discrete Math Circle", run by Bodo Lass for 15 years, which prepares high school students who are passionate about mathematics for maths contests, particularly the International Mathematical Olympiad.[10]

Develop the actions aimed at supporting and promoting the position of mathematics and informatics

The MMI gives its support to:

- The Mathematics Rally of the academy of Lyon[11] (every year since 2005, it has gathered together almost 900 classes from about 200 educational institutions, with 25,000 students taking part).
- The "Statistic's Café" (an original initiative that takes place in a big café in Lyon every month).
- Students of the PiDay association, who, in 2017, presented their mathematical and musical show in Marseille, Lyon and Paris.

For the general public, in partnership with l'Université Ouverte (the Open University) of Lyon 1 University, the MMI organises conferences on subjects as varied as algorithms, artificial intelligence, history of mathematics, mathematics and literature, etc.

The MMI presents a stand at the Salon de la Culture et des Jeux Mathématiques (Mathematical Culture and Games Event in Paris), which is organised by the CIJM.[12] The MMI supports the "filles et maths" (girls and maths) days, which promote the parity and equality of the sexes in mathematics and informatics. In 2014 and 2016, in partnership with IREM, the MMI supported the organisation of the regional congress Math.en.Jeans, which groups together teams from the south-east quarter of France for three days. Math.en.Jeans is an association that permits students from colleges and lyceums to do research in mathematics, supervised over a period of six months.

In collaboration with the University Theatre Astrée of Lyon 1 University, the MMI presents a series of performances combining dance, theatre and music that are also an opportunity for the general public to explore science.

So, at its halfway point, the summary of the MMI in terms of initiated actions, as well as impact on the scholarly world and the general audience, is very encouraging. Here are some key figures:

- In 2014, 20,000 hours were spent with schoolchildren and the general public.
- 150 classes benefit from "Math α Lyon" workshops every year (about 5000 students).
- 1500 students have so far come to the House of Mathematics and Computer Science to attend workshops.

This success shows that the response of the specialists in mathematics and computer science to the demand from the audience is being well received and corresponds to a defined need.

Maison des mathématiques et de l'informatique (MMI)
1 Place de l'École, 69007 Lyon, France

Webpages: http://www.mmi-lyon.fr/ and http://milyon.universite-lyon.fr/en/mediation/maison-mathematiques-informatique/.

Open to a general audience:
Wednesday from 11:30 to 18:00
and Saturday from 13:30 to 18:00.
Open to all and free of charge.

Régis Goiffon is a researcher associated to the Institut Camille Jordan and is one of the three Vice-Directors of the House of Mathematics and Computer Science (together with Natacha Portier (LIP) and Alexei Tsygvintsev (UMPA)). For several years, he has been involved in the dissemination of mathematics and, together with Vincent Calvez (ICJ), Thomas Lepoutre (ICJ) and Adrien Kassel (ENS), he manages "Math α Lyon".

All the pictures were taken by Régis Goiffon. For more pictures, the reader is encouraged to visit the webpage of the MMI.

[10] See "Club de Mathématiques Discrètes" – Lyon Discrete Math Circle, Bodo Lass, EMS Newsletter, September 2016, pp. 45–46 (see http://mmi-lyon.fr/?site_club=club-de-mathematiques-discretes and http://math.univ-lyon1.fr/~lass/club.html).

[11] Academy is the name of the high school administration in France.

[12] Comité International des Jeux Mathématiques, International Committee for Mathematical Games (see http://www.cijm.org/salon).

La Real Sociedad Matemática Española

Vicente Muñoz (Universidad Complutense de Madrid, Spain) and Antonio Rojas León (Universidad de Sevilla, Spain)

The Royal Spanish Mathematical Society (Real Sociedad Matemática Española, RSME) has recently celebrated its first centennial. It was officially founded, at the time as "Sociedad Matemática Española", in Madrid in 1911. Its first president was José Echegaray, a civil engineer who is best known for having won the Nobel Prize in Literature in 1904 but who was also a professor of mathematical physics at the University Central at Madrid (now the Universidad Complutense de Madrid). Other preeminent names of presidents of the society include Zoel García de Galdeano, Leonardo Torres Quevedo and Julio Rey Pastor.

Poster of the exposition of the bibliographical legacy of Zoel García de Galdeano, held at the library of the University of Zaragoza during the days of the last biennial congress of the RSME.

As of 2016, the society has around 1600 individual members and 100 institutional members (such as schools, departments and research institutes). It has reciprocity agreements with many other Spanish, European and international mathematical societies (in particular, with the European Mathematical Society), which allow its members to join these other societies for a reduced membership fee. The society itself is a member of several international mathematical organisations, such as the EMS, the IMU (via the Spanish Committee of Mathematics, which acts as the Spanish Adhering Organisation between the IMU and the Spanish mathematical community), CIMPA and ICIAM.

The main tasks of the society are the promotion and dissemination of mathematics and its applications, and fostering research and teaching in all mathematical areas and educational levels. Through its conferences, meetings and publications, the society tries to serve as a central reference and meeting point for all the Spanish mathematical community.

The governing board of the society consists of the president (currently Francisco Marcellán), who is elected for a period of three years and for a maximum of two terms, two vice-presidents, a secretary, a treasurer, a general editor and 10 elected members (also for a period of three years, renewable for a second term). The board decides on all important matters related to the society. The general assembly takes place at least once a year, at which the society members approve the annual budget and discuss other matters. There are also several specialised thematic committees, such as the Scientific Committee, the Dissemination Committee, the Education Committee, International Relations, and Women and Mathematics. We would like to point out the parity in composition of all of them, according to gender policy.

The RSME is signing collaboration agreements with Spanish Universities to foster collaboration in the promotion and dissemination of mathematics. As of August 2017, we have signed agreements with 17 institutions in order to support activities of our members, as well as organising official RSME scientific meetings.

The most important scientific event organised by the society is the biennial congress, in which some 400-500 mathematicians meet every two years in a different Spanish city to discuss the latest development in their research areas. These conferences consist of plenary talks, parallel thematic sessions, poster sessions and discussion panels and are usually complemented by several cultural events related to mathematics in the hosting city. The latest biennial congress took place in Zaragoza in February 2017 and the next one will be in Santander in 2019.

Another important scientific event organised by the society is the congress for young researchers, which also takes place every two years and mimics the structure of the biennial congress. The members of the organising and scientific committees and the participants of these congresses are promising young mathematicians, who will lead mathematical research in Spain in the coming years. In September 2017, this meeting will take place in Valencia.

Besides these, the RSME has organised over the last few years many joint congresses with mathematical societies of other countries: with the AMS (in 2003, with more than 1000 participants), with the Belgian and Luxembourg Mathematical Societies (twice since 2012), with the French Mathematical Society (2007), with the Swedish and Catalan Mathematical Societies (2017), with the Italian and other Spanish Mathematical Societies (2014), with the Mexican Mathematical Society (four times since 2009), with the Brazilian Mathematical Societies (2015) and with the Argentinian Mathematical Union (2017), in addition to the well-established biennial Iberian Meeting, organised jointly with the Portuguese Mathematical Society. This is an excellent way to foster interactions between the mathematical communities of Spain and these other countries. Also organised by the society is the summer school of mathematical research *Lluís Santaló*, which has taken place in the Universidad Internacional Menéndez Pelayo since 2002.

Mathematical education is another important topic that the society is concerned with and actively involved in. Together with the Spanish Federation of Mathematics Teachers, the Education Committee of the RSME organises the school *Miguel de Guzmán*, a biennial meeting that deals with all sorts of topics related to mathematical education.

Logo of the Spanish Mathematical Olympiad.

The society (and, in particular, its Olympiads Committee) has also been responsible for the organisation of the local and national phases of the Spanish Mathematical Olympiad since 1964. In the national phase, the high school students that will be part of the Spanish team in the International and Iberoamerican Mathematical Olympiads are selected from some 70–80 participants who have won the local phases in their provinces. In 2004, the Iberoamerican Mathematical Olympiad was organised in Castellón and in 2008, the International Mathematical Olympiad took place in Madrid, a report of which can be found in EMS Newsletter No. 69, September 2008.

The *Bulletin of the RSME* is a weekly newsletter that is distributed to all members of the society (originally in print and nowadays only electronically) with the most important news of the society and the Spanish and International mathematical communities from the previous week. It contains news, conference announcements and links to websites of mathematical interest. The other main publication is *La Gaceta de la RSME*, a printed journal published by the society (currently three times a year) and freely distributed to all members. Within this journal, one can find mathematical research articles, biographical and historical articles, interviews with members of the international mathematical community, book reviews, problems and much more.

Front cover of the last issue of *La Gaceta de la RSME*. It contains pictures from the RSME-Imaginary Exhibition.

The society also edits the *Revista Matemática Iberoamericana*, a peer-reviewed scientific journal currently published by the EMS Publishing House. It also publishes mathematical books in several collections. It has agreements with the AMS and with Springer, among other publishers, in order to publish joint volumes in some of their collections. Among the non-periodic publications, one should note the facsimile editions of the treatises *Introductio in Analysin infinitorum* by Leonhard Euler and *De Analysi per Aequationes Numero Terminorum Infinitas* by Isaac Newton, as well as Archimedes' Collected Works, all of them under the editorial supervision of A. J. Durán. The book series "*Publications of the RSME*" includes proceedings of conferences supported by the society.

The *José Luis Rubio de Francia Prize* is one of the most important prizes in mathematics and the highest distinction awarded by the RSME. It is awarded to young researchers in mathematics who are Spanish or have done their work in Spain. Its first edition was in 2004 and it is awarded annually. The list of prize winners is: Joaquim Puig, Javier Parcet, Santiago Morales, Pablo Mira, Francisco Gancedo, Álvaro Pelayo, Carlos Beltrán, Alberto Enciso, María Pe, Ángel Castro, Nuno Freitas, Roger Casals and Xavier Ros-Otón.

The *Prize Vicent Caselles* is an annual distinction for young Spanish researchers whose doctoral work is of high standard and internationally recognised. The first edition was in 2015 and there are six awards annually. Every year since 2015, the RSME Medals have acknowledged three relevant mathematicians from Spain for their commitment to the community and their contributions to scientific advancement, education or the dissemination of mathematics.

Award ceremony of the Vicent Caselles Prizes and the RSME Medals, 2017. Photograph courtesy of Fundación BBVA.

RSME is very concerned with the dissemination of mathematics. The *Divulgamat* website "http://www.divulgamat.net/" is a virtual centre of popularisation of mathematics. The *Arbolmat* website (Arbol de las Matemáticas, translated as Tree of Mathematics) "http://www.arbolmat.com/" contains biographies of many Spanish mathematicians. The RSME has also organised the Spanish version of Imaginary, an itinerant exhibition focused on interactive geometry open to the general public (see "https://imaginary.org/").

The society is also involved with Real Academia Española in the renewal and insertion of new mathematical words in its dictionary, as well as with Fundación Museo Thyssen-Bornemisza in a programme on *Art and Mathematics*, including mathematical paths in the permanent collection of this prestigious museum in Madrid.

Vicente Muñoz [vicente.munoz@mat.ucm.es] is a professor in geometry and topology at the Universidad Complutense de Madrid. He is a member of the Executive Committee of the EMS and of the Governing Board of the RSME. His research areas are differential geometry, algebraic geometry and algebraic topology.

Antonio Rojas León [arojas@us.es] is a lecturer in algebra at the Universidad de Sevilla. He is the President of the Committee of International Affairs of RSME and a member of the Governing Board of the RSME. His research areas are algebraic geometry and number theory.

ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

Call for Intention to Bid to Organise and Host ICME-15 in 2024

The ICMI is hereby inviting its state representatives, national/regional organisations and academic institutions to consider organising and hosting the International Congress on Mathematical Education in July/August 2024.

At present, the ICMI is inviting a *declaration of intent,* which should be received by 1 December 2017. *The full bid document* should be submitted by 1 November 2018.

When considering and preparing the submission of a declaration of intent to organise and host this conference, the ICMI advises potential candidates to consider the following (which will be required as part of a full bid document due by 1 November 2018):

- Provide a statement explaining why the ICME should take place in the proposed state. Please point out particular highlights but also address honestly any potential weaknesses or difficulties.
- State that the bid is presented in agreement with the ICMI Country Representative.
- Provide a list of national and regional organisations (professional associations, universities, governmental/non-governmental organisations and others) and prominent mathematicians and/or mathematics educators who support the idea of organising and hosting the conference and who will contribute to the organisational efforts.
- Nominate the convenor of the conference and the head of the Local Organising Committee, prepare a brief CV for each of these two persons and provide a personal letter of intent signed by them. Bear in mind that whereas all the members of the Local Organising Committee are appointed by the organisers, the members of the International Programme Committee (the IPC - in charge of the scientific components of the conference) are appointed by the ICMI.
- Provide a statement confirming that participants from all over the world (regardless of their nationality) will be allowed freedom of entrance to the hosting state (except for the possible need for a visa).
- Provide a concise description of the venue (and its facilities) available to host the academic activities of the conference (with an expected attendance of 2500-3000 participants).
- Provide a description of the amount/type of accommodation that can be offered, including an adequate amount of inexpensive lodging. Provide some information about distances to the venue and availability of convenient transportation.
- Provide an estimate of the budget and list possible sources of funding (including intentions to approach commercial, governmental or philanthropic entities). Bear in mind that the registration fees to be collected from the participants should be within the range of the fees charged for previous ICMEs. Please take into account: personnel costs, publication costs (proceedings, website, programme and photocopies), rental of the venue, equipment, social events (reception, farewell, happy hour, excursion and coffee breaks), invited participants (travel and accommodation of plenary speakers), costs of the organisation of two IPC meetings (travel and accommodation for 15-20 members for two one-week periods), insurance and miscellaneous expenses.
- Provide an estimated timeline for the publication of the proceedings.

The ICMI recognises that not all states have similar conditions to mount a potentially successful bid. Nevertheless, the ICMI acknowledges that every bid will have its own advantages and highlights as well as its own weaknesses and difficulties. Therefore all states are encouraged to consider bidding according to the guidelines. The Executive Committee of the ICMI will judiciously weigh the weak and strong points of all bids, taking into special consideration proposals from regions in which ICMEs have not been held in the past and for which the conference will considerably boost mathematics education.

The ICMI warmly recommends potential bidders to approach previous conference convenors in order to gain first-hand information about the character and scope of the task.

All members of the Executive Committee of the ICMI, and certainly the President and the Secretary General, will be open for consultation toward the preparation of the proposal.

Please provide your letter of intent (acknowledging each of the above points) by 1 December 2017. Address the letter and/or any related questions to:

Abraham Arcavi, Secretary General of the ICMI
Abraham.arcavi@weizmann.ac.il
ICMI_Secretary-General@mathunion.org

First annual meeting of the new ICMI Executive Committee

The first annual meeting of the Executive Committee of the ICMI was held at the University of Geneva on 8–10 June 2017, hosted by member at large Jean-Luc Dorier.

The Executive Committee discussed all the issues concerning ICMI-related activities, made some decisions about the future of ongoing activities and planned to launch some new ones, which will be announced in future issues of the ICMI Newsletter.

From left to right: Lena Koch, Zahra Gooya, Binyan Xu, Yuriko Yamamoto Baldin, Merrilyn Goos, Jean-Luc Dorier, Shigefumi Mori, Abraham Arcavi, Jill Adler, Helge Holden, Luis Radford and Anita Rampal. Ferdinando Arzarello was unable to attend in person but participated in parts of the meeting via Skype.

ERME Column

Jason Cooper (Weizmann Institute of Science, Rehovot, Israel), Irene Biza (University of East Anglia, Norwich, UK) and Alejandro S. González-Martín (Université de Montréal, Canada)

Report on CERME10

The 10th Congress of European Research in Mathematics Education (CERME10) took place in Dublin (Ireland), 1–5 February 2017 (see *EMS Newsletter* 100, p. 57). This conference has been growing steadily and, as usual, it was the largest to date, with 772 registered participants involved in 24 parallel thematic working groups (TWGs), where a total of 478 research papers and 98 posters were accepted for presentation and discussion. A major aim of ERME is to promote communication, cooperation and collaboration in research in mathematics education throughout Europe. Thus, its conference, CERME, is not merely a platform for presentation of research but also an opportunity for researchers to discuss and advance each other's work in common research domains. Ideally, researchers should leave CERME not only with information on the research that is taking place in their field but also with an improved version of their contribution to the conference proceedings, new ideas for extending their research and leads for future collaborations.

CERME Thematic Working Groups

We wish to make use of this column to present the kind of research that is taking place in the ERME community, focusing on ways in which this research may be interesting and/or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs. In this issue, we begin with TWG14, the topic of which is University Mathematics Education (UME), arguably the most relevant TWG for readers of this column.

Introducing CERME's Thematic Working Group 14 – University Mathematics Education

Group leaders: Alejandro S. González-Martín and Irene Biza

This TWG is concerned with the teaching and learning of mathematics at university. It was the largest TWG in CERME10, contributing 41 full-length papers and 17 short papers to the conference proceedings. Its steady growth since its inception in CERME7 (2011) reflects the increasing popularity of UME research in Europe and internationally, grounded in the realisation that teaching and learning mathematics at all levels can benefit from systematic research. Research in this field touches on, among other topics, general teaching challenges and learning difficulties at university level, such as the transition from school to university mathematics and the transition from university to the workplace. It also touches on the particulars of specific mathematical topics, such as calculus and analysis, linear algebra, proof and proving, mathematical logic and group theory. Contexts include the education of students specialising in mathematics or in other fields, such as economy, engineering, physics or biology. Of particular interest is the mathematical education of mathematics teachers, who are learning "mathematics for teaching", which is often considered a field of applied mathematics. Activities of the TWG have led to the creation of the *International Network for Didactic Research in University Mathematics* (INDRUM), with a biannual ERME topic conference (see INDRUM 2016, https://indrum2016.sciencesconf.org/, and INDRUM 2018, https://indrum2018.sciences conf.org/), and to the publication of a special issue in the highly regarded journal *Research in Mathematics Education*, summarising and extending some of the work presented during CERME7 and CERME8, with a focus on the use of institutional, sociocultural and discursive approaches to research in university mathematics education [1].

Though most of the participants in this TWG are from the field of mathematics education, there is a growing participation of professional mathematicians, who come to share insights, such as novel teaching practices, and to learn from the experience of others. The work presented and discussed at the conference requires some familiarity with the theories and methodologies of mathematics education (a field of the social sciences). Often, mathematicians team up with researchers in mathematics education. Collaboration between researchers from these two distinct yet related communities has been stretching the boundaries of the field. Research is being carried out on what these communities can learn from each other – both mathematically and pedagogically. Another line of research investigates mutual influences between research and teaching practices in mathematics departments. The growing involvement of this community in the activities of the TWG is a welcome trend and is strongly encouraged.

References

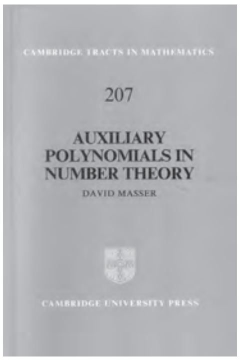
[1] Nardi, E., Biza, I., González-Martín, A. S., Gueudet, G., and Winsløw, C., Institutional, sociocultural and discursive approaches to research in university mathematics education, *Research in Mathematics Education*, 16 (2014), 91–94.

Jason Cooper is a research fellow at the University of Haifa's Faculty of Education. He is also a researcher at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including roles of advanced mathematical knowledge in teaching, and contributions of research mathematicians to the professional development of teachers. He has been a member of the ERME Board since 2015.

Irene Biza is a lecturer in mathematics education at the School of Education and Lifelong Learning at the University of East Anglia. Her research focuses on mathematical learning at university and upper secondary levels, on pedagogical use of information technology and on mathematics teachers' beliefs and knowledge.

Alejandro S. González-Martín is a professor in the Didactics Department of the Faculty of Education at the Université de Montréal. His research focuses on the teaching and learning of mathematics at post-secondary levels, with a special interest in textbook analysis, calculus for engineers and faculty with different academic backgrounds.

Book Reviews



David Masser

Auxiliary Polynomials in Number Theory

Cambridge University Press,
Cambridge, 2016
xviii + 348 pp.
ISBN: 978-1-107-06157-6

Reviewer: Peter Jossen

Opening this book is as easy as opening any other. Closing it is much less so. Start reading and you will quickly find yourself sitting next to the author while he is recollecting story after story about this and that kind of auxiliary polynomial and their use in transcendence theory and other applications. Right from page one, the author makes it clear that he does not intend to deliver a formal lecture. He is not trying to describe any abstract method of auxiliary polynomials. Instead, the conversation dives straight into examples.

Irrationality and transcendence

One name and one function that keeps showing up throughout the first half of the book is Kurt Mahler and the power series $f(z) = z + z^2 + z^4 + z^8 + z^{16} + \cdots$. Mahler's series satisfies the functional equation $z + f(z^2) = f(z)$, it is transcendental over $\mathbb{C}(z)$ and, for $|z| < 1$, it converges rapidly. Together, these three properties are used to prove that, for any rational α with $0 < |\alpha| < 1$, the value $f(\alpha)$ is irrational. The auxiliary polynomial makes its appearance in the form of $P \in \mathbb{Z}[X, Y]$ of low degree, with the property that $P(z, f(z))$ vanishes with a comparatively high order at $z = 0$. Successively, the toolkit is expanded in order to define and say something useful about the irrationality measure of $f(\alpha)$, and ultimately show that $f(\alpha)$ is transcendental for any algebraic α with $0 < |\alpha| < 1$. Why do we care? Arguably, Mahler's series is a bit artificial, yet its treatment is exemplary for the general transcendence theorems about special values of transcendental functions. Another example in this category is the Hermite-Lindemann theorem stating that e^{α} is transcendental for nonzero algebraic α, which is also proven, and its huge generalisation in the form of the Siegel-Shidlovskii theorem, which is only mentioned much later.

Diophantine approximation and counting problems

Transcendence theory being one parent of auxiliary polynomials, the other one is Diophantine approximation (these progenitors are at least second degree relatives, say what you will about their mutual child). Still in the first half of the book, we learn about Runge's method for proving that certain Diophantine problems only have finitely many solutions and that in certain families of polynomials only finitely many are reducible. During the same estimating frenzy, we also encounter Stepanov's method for counting rational points on elliptic curves over finite fields (it is not assumed that the reader knows what an elliptic curve is) and the closely related estimations of exponential sums.

The height machine

Although we encounter several proto-versions up to that point, heights make their first official appearance about halfway through the book. Chapter 14 is an excellent introduction to heights, which are not only a convenient tool but also indispensable for the formulation of many results to come. And heights bring with them a case of intriguing problems to the ongoing auxiliary polynomials party. After tasting a few of these – among them Dobrowolski's theorem about lower bounds for heights, Bilu's equidistribution theorem and a counting theorem due to Bombieri and Pila – we come to the central results in the second half of the book: the Gelfond–Schneider–Lang theorem and its elliptic variant. Once the main tools are in place and the theorems are proven, a lot of amusement with elliptic integrals ensues. We prove that: "animals like

$$\int_4^5 \frac{(X-8)\mathrm{d}X}{\sqrt{X^3 - 7X + 6}}$$

are transcendental! Wonderful things!" Apparently, the editor wasn't too amused, as the author eventually got fired.

Exercises

A prominent place is taken by *exactly* 700 exercises (strictly counting 20.87 and 20.88 as two separate exercises and including A.34), ranging from procrastinatory to recreational, to educational, to serious. And then, if you're up for a challenge, there are those exercises ending with "I don't know" or "Nobody knows".

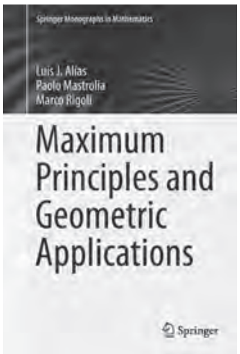
Throughout the book, the exposition is kept elementary. With only very few leaps of faith, an advanced undergraduate student can read the book from start to finish. Here and there, there are hints to the more experienced reader explaining results from a more detached point of view.

All chapters end with a short discussion and overview of developments to date. Without claiming to be comprehensive, these overviews constitute a very helpful guide to the literature. This makes the book even more interesting to mathematicians in general and, in particular, to algebraic geometers with an interest in transcendence and an irrational fear of auxiliary polynomials.

This is not a serious book

David Copperfield, George Clooney, Woody Allen and Mike Tyson are quite a silly cast for a maths book, not to mention Monty Python, the Fat Lady and a cameo by Honecker's first wife. And it's not only the cast – in the book, we also find sledgehammers, top hats, a battleship and weapons of mass destruction. What is this mockery of the language God has written the Universe in? It is blasphemy. I suggest you buy the book and, in the name of St. Bourbaki, write a furiously angry letter to the author about it.

Peter Jossen [peter.jossen@math.ethz.ch] is currently an assistant professor at ETH Zürich. He is mainly interested in motives and their application to arithmetic geometry and number theory.



Luis J. Alías, Paolo Mastrolia and Marco Rigoli

Maximum Principles and Geometric Applications

Springer 2016
xxvi + 570 pp.
ISBN: 978-3-319-24335-1

Reviewer: Alina Stancu

The Newsletter thanks zbMATH and Alina Stancu for the permission to republish this review, originally appeared as Zbl 1346.58001.

This is a very well-written book on an active area of research appealing to geometers and analysts alike, whether they are specialists in the field, or they simply desire to learn the techniques. Moreover, the applications included in this volume encompass a variety of directions with an accent on the geometry of hypersurfaces, while the high number of references dating from 2000 or later are a testimonial of the state of the art developments presented in this volume.

Historically, the starting point of this research can be traced back to the Omori-Yau maximum principle. At the basis of the Omori maximum principle lies a very natural idea. Suppose that a C^2 function u on $\mathbb{R}^n$, endowed with the Euclidean structure $\langle , \rangle$, has a finite supremum u^*. Then there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that, for each k, we have

$$u(x_k) > u^* - \frac{1}{k}, \quad |\nabla u|(x_k) < \frac{1}{k}, \quad \mathrm{Hess}(u)(x_k) < \frac{1}{k}\langle , \rangle. \qquad (1)$$

For a complete Riemannian manifold $(M, \langle , \rangle)$, *H. Omori* [J. Math. Soc. Japan 19, 205–214 (1967; Zbl 0154.21501)] showed that (1) holds under a lower bound condition on the sectional curvature. *S.-T. Yau* [Commun. Pure Appl. Math. 28, 201–228 (1975; Zbl 0291.31002)] proved some outstanding geometric problems using a relaxed maximum principle, referred here as the Omori–Yau maximum principle:

$$u(x_k) > u^* - \frac{1}{k}, \quad |\nabla u|(x_k) < \frac{1}{k}, \quad \Delta(u)(x_k) < \frac{1}{k}, \quad \forall k \in \mathbb{N}. \qquad (2)$$

Additionally, he provided a validity result of (2) on complete Riemannian manifolds based on a weaker assumption, namely a lower bound on the Ricci curvature.

From a different point of view, this statement can be regarded as a result on the Laplace-Beltrami operator thus it should perhaps come as no surprise that it generated high quality results in fields as geometry of submanifolds, harmonic maps, conformal geometry and elliptic equations. For many geometric applications, it suffices in fact to have an even weaker principle, called the weak maximum principle (WMP). We say that WMP holds on the manifold M for the operator Δ if for each $u \in C^2$, such that $u^* = \sup_M u < +\infty$, there exists a sequence $\{x_k\}_k \subset M$ for which

$$u(x_k) > u^* - \frac{1}{k}, \quad \text{and} \quad \Delta(u)(x_k) < \frac{1}{k}, \quad \forall k \in \mathbb{N}. \qquad (3)$$

A remarkable fact about the latter principle, proved by *M. Rigoli* et al. [Rev. Mat. Iberoam. 21, No. 2, 459–481 (2005; Zbl 1110.58022)], is that the condition (3) is equivalent to the stochastic completeness of the manifold M. The manifold M need not be geodesically complete anymore. The weak maximum principle has also been at the basis of groundbreaking results among which the positive answer to two of Calabi's conjectures on minimal submanifolds of the Euclidean space by *L. J. Alías* et al. [Math. Ann. 345, No. 2, 367–376 (2009; Zbl 1200.53050)].

The interest in geometric analysis for similar maximum principles related to other operators generated new research in obtaining maximum principles for classes of differential operators and, with it, many other applications. In fact, the interplay between the maximum principles and the geometric applications is as much the subject of this book as the maximum principles themselves. In this book, the authors decided to focus their attention on the applications to the geometry of submanifolds, in particular hyersurfaces as mentioned earlier, although some applications to elliptic PDEs and the geometry of Ricci solitons are also presented.

Regarding the structure of the book, Chapter 1 is devoted to Riemannian geometry via the method of the moving frame, and some classical comparison results on the Laplacian, respectively the Hessian operator, with the attained goal of making the book self-contained. In Chapter 2, the Omori–Yau maximum principle, as well as the weak maximum principles, are presented in detail following a different formulation than the original one. Using a function theoretic approach will prove advantageous in later chapters when the classical results are extended to other operators. In particular, the weak maximum principle is presented with the aid of an auxiliary function, which makes more natural the transition to the proofs of the weak maximum principle, and the Omori-Yau maximum principle, for a class of linear operators in Chapter 3. In the third chapter, the authors present a type of sufficient conditions for the two principles to hold for a class of linear operators, while in the next chapter, another type of sufficient conditions for the weak maximum principle to hold are provided for operators in divergence form. This latter type condition is expressed in terms of the volume growth of geodesic balls with a fixed center on M. Auxiliary elements for treating the case of nonlinear operators, considerably more delicate, are treated in the last section of Chapter 3 where versions of Omori-Yau and weak maximum principles for nonlinear operators are presented. It is worth mentioning the applications of Chapter 4 to second order a priori estimates for solutions of certain differential inequalities as well as those pertaining to the uniqueness problem for positive solutions of certain Lichnerowicz-type equations.

Chapter 5 is devoted to applications of the maximum principles of Chapters 3 and 4, particularly improving on a non-immersability result of a manifold M into cones of the Euclidean space that belongs to Omori who was led to his maximum principle precisely by the study of this problem. A fair amount of the chapter treats cylindrically bounded submanifolds related to a conjecture by Calabi which is known to be false in its original formulation, but still pursued in modified form, followed by some consequences on the Gauss map of submanifolds of Euclidean space. Chapters 6 and 7 focus on applications to the geometry of hypersurfaces. The accent shifts from surfaces with constant mean curvature to hypersurfaces with constant scalar curvature. The proofs in the latter case rely on the Omori-Yau maximum principle for the Cheng and Yau operator. This operator is part of a series of operators which can be defined for hypersurfaces in general Riemannian manifolds that will also be used in Chapter 7 in which the aim is to study Alexandrov type embedding results for hypersurfaces with constant higher order mean curvatures. The object of Chapter 8 is the study of Ricci solitons that became famous with work of Hamilton and Perelman, and for which the main investigation aims at classifying them. The chapter benefits from the expertise of the authors, see [*P. Mastrolia* et al., Commun. Contemp. Math. 15, No. 3, Article ID 1250058, 25 p. (2013; Zbl 1335.53062)], even if the presentation is by no means exhaustive. Among other things, the validity of the maximum principle on solitons is presented. In Chapter 9, the authors select applications of the maximum principles to spacelike hypersurfaces in Lorentzian spacetimes. For example, the weak maximum principle applied to a certain operator in divergence form leads to some properties for space-like graphs in a generalized Robertson-Walker spacetime.

The book benefits of a detailed presentation of even the most technical parts, easing the reader's way through forefront developments of this area of geometric analysis.

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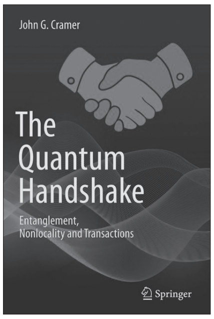

John G. Cramer

The Quantum Handshake. Entanglement, Nonlocality and Transactions

Springer, 2016
xxv, 218 p.
ISBN 978-3-11-027964-1

Reviewer: Carlos Pedro Gonçalves

The Newsletter thanks zbMATH and Carlos Pedro Gonçalves for the permission to republish this review, originally appeared as Zbl 1358.81001.

The book *The quantum handshake. Entanglement. nonlocality and transactions*, by John G. Cramer, is a fundamental work on the transactional interpretation of quantum mechanics, which was proposed by Cramer himself and that provides a different way to look at quantum mechanics. The work is, thus, of major interest for physicists, researchers working on quantum technologies, students learning quantum mechanics as well as researchers working on the foundations of quantum mechanics. The book is organized in 10 chapters and four appendices, providing an extensive review of the transactional interpretation, addressing in particular:

- Its conceptual foundations;
- Its application to major quantum experiments and paradoxes of quantum mechanics;
- Its theoretical value as a way to address fundamental physics and as a support for new theory development.

Chapter 1 provides a perspective on how Cramer came formulate the Transactional Interpretation (TI), influenced by his early reading, as a Physics student, of Wheeler and Feynman's work on a time-symmetric formalism for electrodynamics, using retarded waves that propagate from an emitter in the present and are answered by advanced waves from absorbers in the future, with a "handshake" between the retarded and the advanced waves taking place for the transfer of energy and momentum, so that the absorber's electric charge responds to the retarded field and the emitter's charge responds to the advanced field.

This approach to electrodynamics fitted Cramer's view on the importance of boundary conditions and their role in physical processes linked to the formulation and solving of differential equations applied to physical problems. In the case of (classical) electrodynamics, Maxwell's equations allow for the existence of two independent solutions: the retarded wave solution which arrives after it departs and the advanced wave solution which arrives before it departs.

In Chapter 1, Cramer recounts how, while teaching modern physics to a class of undergraduates at the University of Washington, he came to the main intuition that gave rise to his formulation of the TI. When describing "Einstein's Bubble paradox" to the students, Cramer had what he calls an insight linking Wheeler-Feynman's handshake to quantum mechanics, which led him to introduce an interpretation of quantum mechanics based on the expansion of the concept of a Wheeler-Feynman handshake to the quantum setting, by considering the wave functions ψ as retarded waves and their complex conjugates ψ^* as advanced waves *participants in handshakes across space-time* (p. 5), where the formation of a transaction takes place along a worldline and therefore one no longer considers a wave function collapse as taking place at a given point in time, that is, one no longer needs to assume a *mid-flight* change of the propagating wave, collapsing to an alternative at the instant of measurement (p. 81). Regarding this point, Cramer argues for a better consistency with the treatment of time in special relativity avoiding problems with the concept of *instantaneous*.

By introducing the concept of a Wheeler–Feynman handshake to the quantum setting, Cramer is also able to derive Born's rule from within the formalism itself rather than assuming it as a postulate in the theory, a point that is addressed along the book.

The author introduces an underlying physical process as a possible explanation for Born's rule, with the rule coming out as a result from a physical process taking place at the quantum level, which is considered by Cramer using the concept of a *transaction* process as a fundamental quantum mechanical interaction where a retarded physical wave with spatial extension is emitted, represented by a time-dependent Schrödinger wave function ψ, taking the role of an *offer wave* for an exchange of energy and momentum, and the response to this *offer wave* comes from the advanced *confirmation wave* propagating in the reverse time direction and arriving back to the source at precisely the instant of emission, delivering a retarded/advanced echo with an amplitude of $\psi\psi^*$, a hierarchical stochastic selection then takes place at the source with regard to possible transactions, where the hierarchy proceeds from the echoes coming from the smallest space-time intervals of separations to the echoes from the larger separations.

In each case, the choice is between forming a transaction or no transaction at all. When a transaction is selected an alternative is actualized guaranteeing that the physical conservation conditions are met. This process, as described by Cramer, takes place across space-time, so that Born's rule is met, avoiding the problem of determining when the collapse takes place. In this sense, and throughout the book, Cramer places the TI as an alternative to the Copenhagen Interpretation. A point that Cramer stresses at the end of Chapter 1, where the author defends that although the Copenhagen Interpretation is consistent, it is also an unsatisfying way of approaching the quantum world, and the TI provides a better alternative (p. 8).

Chapter 2 reviews the history of quantum mechanics. Even if the reader is already familiar with that history, this is a Chapter not to be skipped over. Not only does Cramer address the key points that led to modern quantum theory, but also the main approach and issues that are worked in the rest of the book can be traced to this Chapter, namely, Cramer reviews how the Copenhagen Interpretation came to form and main interpretational problems that it raises, problems that are addressed by the author along the book. Of particular interest, in Chapter 2, is the problem of nonlocality. The TI fully assumes the nonlocality, however, it is not a nonlocal hidden variables proposal.

Chapter 3 delves further into this issue, addressing quantum entanglement and nonlocality. At the beginning of Chapter 3, Cramer explicitly states that quantum mechanics is nonlocal, in the sense that: ``the component parts of a quantum system may continue to influence each other, even when they are well separated in space and out of speed-of-light contact'' (p. 39). In Chapter 3, Cramer links nonlocality with conservation laws and complementarity related to quantum uncertainty, as expressed by Heisenberg's uncertainty principle, arguing that entanglement works as a solution to the problem of guaranteeing that actualization preserves conservation laws.

The argument developed by Cramer in Chapter 3 is linked directly with both the subtitle of this book *Entanglement, Nonlocality and Transactions* and a major point of the TI that Cramer develops in subsequent chapters. Since the transaction process is related to the preservation of conserved quantities, the interpretation must account for entanglement and nonlocality within the quantum handshake dynamics.

Chapter 4 deals with the Wheeler–Feynman absorber theory. Together with the previous 3 chapters, it provides the core of what can be looked at as a first part of the book, giving the reader the fundaments for understanding the foundations of the TI. Chapters 5 and 6 constitute a second part of the book and these are core chapters: Chapter 5 reviews the TI and Chapter 6 deals with the applications of the interpretation to major quantum paradoxes and experiments. While the reader may get a basic understanding of the interpretation with Chapter 5, Chapter 6 provides the richest material in terms of getting deeper into the interpretation and its implications.

Chapters 7 to 9 constitute a third part of the book dealing with different subject matters. Chapter 7 addresses nonlocal signaling, Chapter 8 addresses quantum technologies' related applications and Chapter 9 addresses the nature of time.

While the TI assumes nonlocality, Cramer shows in Chapter 7, main dynamics that block nonlocal signaling, namely: switchable interference patterns that might offer the possibility of signaling are blocked by accompanying anti-interference patterns, thus, preventing nonlocal signaling. The author also links signal blocking to orthogonal basis transformation.

While Cramer leaves open the possibility that some future generalization of quantum theory's formalism might lead to some solution for nonlocal signaling, he also points out that there is, presently, no guarantee that this may take place.

In regards to quantum technologies, the two dominant interpretations are the Copenhagen Interpretation and the Many Worlds Interpretation (MWI), Cramer reviews the later in Chapter 8 about Deutsch's work on quantum computing theory. Cramer proposes an alternative perspective on quantum computation, which actually implies a different definition of what constitutes a quantum computation.

If we follow the MWI or even, more generally, an Everett relative state-based interpretation, then, the quantum computation corresponds to a unitary state transition which encodes the different computational gates, and such that each computation changes the state of a quantum register system, implying, in particular a change in the weight with which the state vector projects over different orthogonal patterns of systemic activity (for instance, different spin states), corresponding to different possible configurations of the quantum register system.

In the MWI these are considered to be different worlds, so that each alternative worldline is projected and actualized in an alternative world configuration. Geometrically the "worlds" actually correspond to orthogonal dimensions of a Hilbert space, with each dimension corresponding to a different configuration. The term used however, in the MWI, is parallel worlds.

In Cramer's interpretation, the computation is not defined in terms of the unitary propagation, corresponding to the implementation of the "quantum program", but rather to a more complex process, namely as Cramer (p. 158) states: "The programming of the quantum computer that sets up the problem has created a set of conditions such that the quantum mechanical wave function can only form a transaction and collapse by solving the problem (…)".

Thus, under the TI, the computation is not identified with the unitary transformation from an input state to an output state but is, instead, associated with the transaction itself, that is, the computing system is programmed in such a way that it must select a solution to a computing problem, so that the advanced and retarded waves propagate through the computer seeking a final multi-vertex quantum handshake that solves the problem.

The quantum computer functions, in this case, as a form of artificial cognitive system, a form of Artificial Intelligence (AI), that uses the advanced and retarded waves to evaluate the computational problem and select a solution. This interpretation of quantum computation may be useful for the research on quantum AI and quantum machine learning, since the advanced waves already incorporate a form of quantum backpropagation. On the other hand, in the quantum computational setting, the hierarchical transaction selection process, described by Cramer, is akin to a computational solution search process.

While Chapter 8 is of primary interest to those working on quantum technologies. Chapter 9 deals, in turn, with a fundamental point of the TI, which is the concept of time, since the interpretation describes quantum mechanics with waves propagating in both time directions, the origins of a preferred time direction pointing towards the future, the so-called *arrow of time*, becomes a main point that the interpretation needs to address.

Cramer begins by presenting four (macroscopic) arrows of time: the subjective arrow of time, the electromagnetic arrow of time, the thermodynamic arrow of time, the cosmological arrow of time and the charge-parity (CP)-violation arrow of time.

The author reviews two alternative hierarchies for these arrows of time, which differ in terms of the relation between the thermodynamic, the electromagnetic and the subjective arrows of time. Cramer argues that the electromagnetic arrow of time takes precedende over the thermodynamic and the later over the subjective arrow of time, as opposed to the alternative in which the thermodynamic arrow of time takes precedence over the electromagnetic and over the subjective arrows of time.

After reviewing the four arrows of time, and presenting an argument in favor of the above hierarchical precedence, Cramer then introduces one more arrow, which the author calls the quantum mechanical arrow of time, stating that, while advanced waves may be involved in quantum processes to enforce conservation laws no net advanced effects are allowed (p. 164). In this sense, Cramer argues that there is a similarity between the quantum mechanical and the electromagnetic arrows of time, since there is a dominance of retarded over advanced waves, otherwise the principle of causality would not be met.

A second point that Cramer stresses regards the way in which the future is addressed within the interpretation, namely the statement that the TI may require an Einsteinian block universe to pre-exist in order [the author, *The quantum handshake – entanglement*. Singapore: Springer (2015)] for the future to be able to affect the past in a transactional handshake. While a block-universe determinism is consistent with Cramer's interpretation of quantum mechanics, the author rejects it as a necessary condition for the TI to hold, in the sense that a part of the future is emerging into a fixed local existence with each transaction, the future is not, however, determining the past and the two are not rigidly locked.

The third point about time, addressed by Cramer, regards the "plane of the present". Namely, while in a standard mechanical perspective the present moment can be addressed as a moving plane, the transaction formation process introduces a different geometrical perspective where, rather than a plane, a fractal-like surface stitching back and forth between past and present and between present and future is more appropriate, according to Cramer.

The author provides a view then of how a unique present comes out from the transaction process comparing it to a progressive formation of frost crystals on a cold windowpane, such that, as the forst pattern expands, there is no clear freeze-line, but, instead, a moving boundary, with fingers of frost reaching out beyond the general trend, until the whole window pane is frozen into a fixed pattern. This provides a picture of the transaction process, involving what the author calls a lacework of connections with the future and the past, which insure that the conservation laws are respected and the balances of energy and momentum are preserved.

This introduces a different way to address time within physics. A matter that may be explored further by physicists and those working with the connections between the foundations of quantum mechanics and fundamental physical notions such as time, space and dynamics.

Chapter 10 concludes the book, with a summary of main points that were made along the entire work. In its whole, the book is a key work on quantum theory, providing for an alternative interpretation of quantum mechanics and dealing in-depth with the main arguments, criticisms and applications to quantum experiments.

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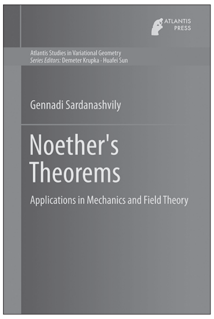

Gennadi Sardanashvily

Noether's Theorems. Applications in Mechanics and Field Theory

Atlantis Press, 2016
xvii, 297 p.
ISBN 978-94-6239-170-3

Reviewer: Miguel Paternain

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This book contains a systematic and detailed approach to Lagrangian systems through the variational bicomplex providing a very general geometric and algebraic view of the variational calculus. Noether's theorems can be seen as the guiding theme for this algebraic approach. This standpoint is specially apt for the geometric formulation of Lagrangian field theory in which fields are represented by sections of fibre bundles. The calculus of variations of Lagrangians on fibre bundles can be algebraically formulated in terms of the variational bicomplex of differential forms on a jet manifold of sections. In this setting the Lagrangian is a horizontal density.

Chapter 1 contains a description of the space of jets of sections of fibre bundles, the fundamental cohomology properties of the variational bicomplex and the Euler-Lagrange operator. It is also shown that the cohomology of the variational bicomplex yields the variational formula. In chapter 2 it is shown, by using the variational formula, that for any Lagrangian symmetry there is a conserved current whose total differential vanishes on-shell, generalizing Noether's first theorem from classical mechanics. Chapter 3 deals with first order Lagrangians and polysymplectic Hamiltonians on Legendre bundles because of their relevance in the physical models. The relationship between both theories is examined under the assumption of regularity conditions. Chapter 4 is devoted to Lagrangian and Hamiltonian non relativistic mechanics. In this chapter the case of the classical particle motion is seen as the special case in which the base manifold is one dimensional. Particular attention is given to the Kepler problem in chapter 5, providing a detailed analysis of the symmetries characterizing the system. In chapter 6 Grassmann-graded algebraic calculus, graded manifolds and graded bundles are addressed. The theory of Grassmann-graded Lagrangians on graded bundles is developed in terms of a graded variational bicomplex. This complex provides the appropriate variational formula and gives a very general version of Noether's first theorem. Chapter 7 contains a detailed analysis of Noether identities in the graded setting. In this context, Noether's inverse and direct second theorems associate to the Noether identities the gauge symmetries of the Grassmann-graded Lagrangian theory. Chapters 8 and 9 are devoted to applications to Yang–Mills gauge theory on principal bundles and supersymmetric gauge theory on principal graded bundles. Chapter 10 treats gravitation in the setting of gauge theory on natural bundles and it is shown that in this case the conserved current is the energy-momentum current. Chapters 11 and 12 deal with topological field theories, namely, Chern-Simons and BF theory.

The book is very well written in a clear and direct style containing very helpful cross references. In addition the book under review includes 4 appendices which make the presentation self contained. Appendix A covers differential calculus over commutative rings, Appendix B includes differential calculus on fibre bundles and appendix C contains a very readable explanation of sheaf cohomology. This material is especially helpful for the most advances chapters. The last appendix covers the Noether identities of differential operators in the homological context.

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Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

The world is continuous, but the mind is discrete.

David Mumford

The problem column in this issue is devoted to discrete mathematics. This beautiful and highly applicable area of mathematics deals with the study of discrete structures and phenomena. The structures studied in discrete mathematics consist of sequences of individual steps. This is in contrast to other areas of mathematics such as differential calculus, where the concept of a continuous process plays an integral role.

Discrete mathematics covers several subjects, among them the theory of sets and relations, mathematical logic, combinatorics and graph theory, as well as some aspects of number theory. Combinatorics and graph theory have a prominent place in the world of discrete mathematics.

It is worth mentioning that in our modern society, discrete models – and thus the techniques to study them – have a wide range of applicability. Apart from the fact that the notion of enumeration appears so naturally in our everyday life, another essential reason for the increased applicability of such models is their intimate connection to computers, which have become so deeply integrated into our culture.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

179. Let $p = p_1 p_2 \cdots p_n$ and $q = q_1 q_2 \cdots q_n$ be two permutations. We say that they are *colliding* if there exists at least one index i so that $|p_i - q_i| = 1$. For instance, 3241 and 1432 are colliding (choose $i = 3$ or $i = 4$), while 3421 and 1423 are not colliding. Let S be a set of *pairwise colliding* permutations of length n. Is it true that $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

180. Let us say that a word w over the alphabet $\{1, 2, \cdots, n\}$ is *n-universal* if w contains all $n!$ permutations of the symbols $1, 2, \ldots, n$ as a subword, not necessarily in consecutive positions. For instance, the word 121 is 2-universal as it contains both 12 and 21, while the word 1232123 is 3-universal. Let $n \geq 3$. Does an n-universal word of length $n^2 - 2n + 4$ exist?

(Miklós Bóna, Department of Mathematics, University of Florida, Gainesville, FL 32608, USA)

181. Given natural numbers m and n, let $[m]^n$ be the collection of all n-letter words, where each letter is taken from the alphabet $[m] = \{1, 2, \ldots, m\}$. Given a word $w \in [m]^n$, a set $S \subseteq [n]$ and $i \in [m]$, let $w(S, i)$ be the word obtained from w by replacing the j^{th} letter with i for all $j \in S$. The Hales–Jewett theorem then says that for any natural numbers m and r, there exists a natural number n such that every r-colouring of $[m]^n$ contains a monochromatic combinatorial line, that is, a monochromatic set of the form $\{w(S, 1), w(S, 2), \ldots, w(S, m)\}$ for some $S \subseteq [n]$. Show that for $m = 2$, it is always possible to take S to be an interval in this theorem, while for $m = 3$, this is not the case.

(David Conlon, Mathematical Institute, University of Oxford, Oxford, UK)

182. (A) Let $A_1, A_2, \ldots$ be finite sets, no two of which are disjoint. Must there exist a finite set F such that no two of $A_1 \cap F$, $A_2 \cap F, \ldots$ are disjoint?
(B) What happens if all of the A_i are the same size?

(Imre Leader, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK)

183. The following is from the 2012 Green Chicken maths contest between Middlebury and Williams Colleges. A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \ldots, 2012\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus, 30 and 1593 are connected by an edge as 3 divides each but 30 and 49 are not. The colouring number of a graph is the smallest number of colours needed so that each vertex is coloured *and* if two vertices are connected by an edge then those two vertices are not coloured the same. The Green Chicken says the colouring number of this graph is at most 9. Prove he is wrong and find the correct colouring number.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

184. There are n people at a party. They notice that for every two of them, the number of people at the party that they both know is *odd*. Prove that n is an odd number.[1]

(Benny Sudakov, Department of Mathematics, ETH Zürich, Zürich, Switzerland)

II Open Problems: Two combinatorial problems by Endre Szemerédi

(Renyi Alfred Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary. This work was supported by the ERC-AdG. 321104 and OTKA NK 104183 grants.)

185* (Erdős' unit distance problem). In 1946, Erdős [7] published a short paper in the *American Mathematical Monthly*, in which he suggested a very natural modification of the Hopf-Pannwitz question. Let P be a set of n points in the plane. What happens if we want to determine or estimate $u(n)$, the largest number of unordered pairs $\{p, q\} \subset P$ such that p and q are at a fixed distance, which is not necessarily the largest distance between two elements of P? Without loss of generality, we can assume that this distance is the *unit distance*. This explains why Erdős' question is usually referred to as the *unit distance problem*. That is,

$$u(n) = \max_{P \subset \mathbb{R}^2, |P|=n} \left| \{\{p, q\} \subset P \, : \, |p - q| = 1\} \right|.$$

(a) Using classical results of Fermat and Lagrange, Erdős showed that one can choose an integer $x \leq n/10$ that can be written as the sum of two squares in at least $n^{c/\log\log n}$ different ways, for a suitable constant $c > 0$. Thus, among the points of the $\sqrt{n} \times \sqrt{n}$ integer lattice, there are at least $(1/2)n^{1+c/\log\log n}$ pairs whose distance is $\sqrt{x}$. Scaling this point set by a factor of $1/\sqrt{x}$, we obtain a set of n points with at least $(1/2)n^{1+c/\log\log n}$, i.e. with a superlinear number of unit distance pairs.
Erdős proved that

$$n^{1+c_1/\log\log n} \leq u(n) \leq c_2 n^{3/2},$$

for some $c_1, c_2 > 0$, and he conjectured that the order of magnitude of $u(n)$ is roughly $n^{1+c/\log\log n}$. In spite of many efforts to improve on the upper bound, 70 years after the publication of the paper in *Monthly*, the best known upper bound is still only slightly better than the above estimate. Erdős' upper bound was first improved to $o(n^{3/2})$ by Józsa and Szemerédi [13], and ten years later to $O(n^{13/9})$ by Beck and Spencer [2]. In a joint paper with Spencer and Trotter [17], I proved $u(n) = O(n^{4/3})$, which is currently the best known result.

(b) We say that n points in the plane are in *convex position* if they form the vertex set of a convex polygon.
Erdős and Moser [9] conjectured that the number of unit distances, $u_{\text{conv}}(n)$, among n points in convex position in the plane satisfies $u_{\text{conv}}(n) = \frac{5}{3}n + O(1)$. They were wrong: Edelsbrunner and P. Hajnal [6] exhibited an example with $2n - 7$ unit distance pairs, for every $n \geq 7$. It is widely believed that $u_{\text{conv}}(n) = O(n)$ and perhaps even $u_{\text{conv}}(n) = 2n + O(1)$. The best known upper bound is due to Füredi [11], who proved by a forbidden submatrix argument that $u_{\text{conv}}(n) = O(n \log n)$. A very short and elegant inductional argument for the same bound can be found in [4].

Erdős suggested a beautiful approach to prove that $u_{\text{conv}}(n)$ grows at most linearly with n. He conjectured that every convex n-gon in the plane has a vertex from which there are no $k + 1$ other vertices at the same distance. Originally, he believed that this is also true with $k = 2$ but Danzer constructed a series of counterexamples. Later, Fishburn and Reeds [10] even found convex polygons whose unit distance graphs are 3-regular, that is, for each vertex there are precisely three others at unit distance. If Erdős' latter conjecture is true for some integer k then this immediately implies by induction that $u_{\text{conv}}(n) < kn$.

186* Some problems on Sidon sets. $A \subset [1, n]$ is called a *Sidon sequence* if all sums $a + a'$, $a, a' \in A$ are different.

(a) Prove or disprove that

$$|A| < n^{1/2} + O(1).$$

The best result is due to B. Lindström [15], who proved that $|A| < n^{1/2} + n^{1/4}$.

(b) Prove or disprove that

$$|A| < n^{1/2} + o(n^{1/4}).$$

* * *

$A = \{a_1 < a_2 < a_3 < \ldots\}$ is an *infinite Sidon sequence* if all sums $a + a'$, $a, a' \in A$ are distinct. $A(n)$ denotes the number of elements of A in $[1, n]$.

(c) For every $\epsilon > 0$, construct (the construction can be a random construction) an infinite Sidon sequence with $A(n) > n^{1/2-\varepsilon}$.
The best bound is due to I. Ruzsa [16] and J. Cilleruelo [5]. They constructed an infinite Sidon sequence with $A(n) > n^{\sqrt{2}-1+o(1)}$. I. Ruzsa's construction was a random one and J. Cilleruello's construction was a deterministic one.

* * *

$A_h = \{a_1 < a_2 < a_3 < \ldots\}$ is a sequence such that all sums $a_1 + a_2 + \ldots + a_h$, $a_1, a_2, \ldots, a_h \in A$ are distinct. $A_h(n)$ is the number of elements of A_h in $[1, n]$.

(d) Prove or disprove that for $h = 3$,

$$A_3(n) = o(n^{1/3}).$$

Here, there are no results. $A_3(n) = O(n^{1/3})$ is trivial.

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III Solutions

171. Prove that every integer can be written in infinitely many ways in the form

$$\pm 1^2 \pm 3^2 \pm 5^2 \pm \cdots \pm (2k+1)^2$$

for some choices of signs + and −.

(Dorin Andrica, Babeşş Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. The proof uses induction by step 16. In this respect, we note that we have, for any positive integer m, the identity

$$16 = (2m-1)^2 - (2m+1)^2 - (2m+3)^2 + (2m+5)^2 \qquad (1)$$

and the representations

$$0 = -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2,$$
$$1 = 1^2,$$
$$2 = 1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2,$$
$$3 = 1^2 - 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 - 15^2 - 17^2 - 19^2 + 21^2,$$
$$4 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 - 17^2 + 19^2,$$
$$5 = 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 - 17^2 - 19^2 - 21^2 - 23^2 - 25^2 + 27^2 + 29^2,$$
$$6 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 + 11^2,$$
$$7 = 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 - 19^2 + 21^2 - 23^2 - 25^2 - 27^2 + 29^2,$$
$$8 = -1^2 + 3^2,$$
$$9 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 - 17^2 + 19^2 - 21^2 - 23^2 - 25^2 - 27^2 + 29^2 + 31^2 + 33^2,$$
$$10 = 1^2 + 3^2,$$
$$11 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 + 17^2 - 19^2 - 21^2 + 23^2 + 25^2,$$
$$12 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 + 15^2 + 17^2,$$
$$13 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 - 15^2 + 17^2,$$
$$14 = -1^2 - 3^2 - 5^2 + 7^2,$$
$$15 = -1^2 - 3^2 + 5^2.$$

For example, to write 16 in this form, we use the representation of 0 and we consider $m = 9$ in identity (1) to get $16 = 17^2 - 19^2 - 21^2 + 23^2$. We obtain

$$\begin{aligned} 16 &= 0 + 16 \\ &= -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 + 17^2 \\ &\quad - 19^2 - 21^2 + 23^2. \end{aligned}$$

To show that there are infinitely many such representations, we observe that, from (1), we have $16 = (2m+7)^2 - (2m+9)^2 - (2m+11)^2 + (2m+13)^2$. Hence, for any positive integer m, the following identity holds:

$$\begin{aligned} 0 &= (2m-1)^2 - (2m+1)^2 - (2m+3)^2 + (2m+5)^2 - (2m+7)^2 \\ &\quad + (2m+9)^2 + (2m+11)^2 - (2m+13)^2. \end{aligned}$$

In this way, we can add 0 to a representation for a suitable value of m to get a new representation and then continue. □

Also solved by José Harnández Santiago (Morelia, Michoacán, Mexico) and Alexander Vauth (Lübbecke, Germany)

172. Show that, for every integer $n \geq 1$ and every real number $a \geq 1$, one has

$$\frac{1}{2n} \leq \frac{1}{n^{a+1}} \sum_{k=1}^{n} k^a - \frac{1}{a+1} < \frac{1}{2n}\left(1 + \frac{1}{2n}\right)^a.$$

(László Tóth, University of Pécs, Hungary)

Solution by the proposer. We prove by induction on n. For $n = 1$, we have

$$\frac{1}{2} \leq 1 - \frac{1}{a+1} < \frac{1}{2}\left(\frac{3}{2}\right)^a.$$

Here, the first inequality is equivalent to $a \geq 1$, which holds true by the condition. For the second one, if $1 \leq a < 3$ then $1 - \frac{1}{a+1} < \frac{3}{4} \leq \frac{1}{2}\left(\frac{3}{2}\right)^a$; if $a \geq 3$ then $1 - \frac{1}{a+1} < 1 < \frac{1}{2}\left(\frac{3}{2}\right)^a$.

Let $S_a(n) = 1^a + 2^a + \cdots + n^a$ and assume that the inequalities hold true for n, that is,

$$\frac{1}{2}n^a + \frac{1}{a+1}n^{a+1} \leq S_a(n) < \frac{1}{2}\left(n + \frac{1}{2}\right)^a + \frac{1}{a+1}n^{a+1}, \qquad (2)$$

and prove (2) for $n+1$.

Adding $(n+1)^a$ to (2), we get

$$\begin{aligned} \frac{1}{2}n^a + \frac{1}{a+1}n^{a+1} + (n+1)^a &\leq S_a(n+1) \qquad (3) \\ &< \frac{1}{2}\left(n + \frac{1}{2}\right)^a + \frac{1}{a+1}n^{a+1} + (n+1)^a. \end{aligned}$$

Applying the inequalities

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(t)\,dt \leq \frac{f(x)+f(y)}{2}$$

for the convex function $f(t) = t^a$ with $a \geq 1$ and for $x = n$, $y = n+1$, we deduce

$$\left(n + \frac{1}{2}\right)^a \leq \frac{1}{a+1}\left((n+1)^{a+1} - n^{a+1}\right) \leq \frac{n^a + (n+1)^a}{2}. \qquad (4)$$

Now, by the second inequality of (4), we obtain

$$\frac{1}{a+1}(n+1)^{a+1} - \frac{1}{2}(n+1)^a \leq \frac{1}{2}n^a + \frac{1}{a+1}n^{a+1},$$

that is,

$$\frac{1}{a+1}(n+1)^{a+1} + \frac{1}{2}(n+1)^a \leq \frac{1}{2}n^a + \frac{1}{a+1}n^{a+1} + (n+1)^a. \qquad (5)$$

By (5) and (3), we deduce

$$\frac{1}{2}(n+1)^a + \frac{1}{a+1}(n+1)^{a+1} \le S_a(n+1). \qquad (6)$$

On the other hand, the first inequality of (4) gives

$$\frac{1}{a+1}n^{a+1} \le \frac{1}{a+1}(n+1)^{a+1} - \left(n+\frac{1}{2}\right)^a \qquad (7)$$

and, by using the well known inequality

$$\left(\frac{x+y}{2}\right)^a < \frac{x^a+y^a}{2}, \quad (a>1, x\neq y)$$

for $x = n+\frac{1}{2}$, $y = n+\frac{3}{2}$, we have

$$(n+1)^a < \frac{1}{2}\left(n+\frac{1}{2}\right)^a + \frac{1}{2}\left(n+\frac{3}{2}\right)^a. \qquad (8)$$

Now, by summing the inequalities (7) and (8),

$$\frac{1}{2}\left(n+\frac{1}{2}\right)^a + \frac{1}{a+1}n^{a+1} + (n+1)^a < \frac{1}{2}\left(n+\frac{3}{2}\right)^a + \frac{1}{a+1}(n+1)^{a+1} \quad (9)$$

and finally, by (9) and (3),

$$S_a(n+1) < \frac{1}{2}\left(n+\frac{3}{2}\right)^a + \frac{1}{a+1}(n+1)^{a+1}. \qquad (10)$$

Taking into account inequalities (6) and (10), we conclude that (2) holds true for $n+1$ and the proof is complete. □

Remarks. 1. The equality holds if and only if $a = 1$ and $n \ge 1$ is arbitrary.

2. We deduce by these inequalities the following well known results:

$$\lim_{n\to\infty} \frac{1^a+2^a+\cdots+n^a}{n^{a+1}} = \frac{1}{a+1},$$

$$\lim_{n\to\infty} n\left(\frac{1^a+2^a+\cdots+n^a}{n^{a+1}} - \frac{1}{a+1}\right) = \frac{1}{2},$$

valid for every fixed real $a \ge 1$.

Also solved by Mihály Bencze (Brasov, Romania) and Panagiotis T. Krasopoulos (Athens, Greece)

173. Let $c_n(k)$ denote the Ramanujan sum, defined as the sum of kth powers of the primitive nth roots of unity. Show that, for any integers n, k, a with $n \ge 1$,

$$\sum_{d|n} c_d(k) a^{n/d} \equiv 0 \pmod n.$$

(László Tóth, University of Pécs, Hungary)

Solution by the proposer. The proof is based on the congruence

$$M_n(a) := \sum_{d|n} \mu(d) a^{n/d} \equiv 0 \pmod n, \qquad (11)$$

represented several times in the literature (see, for example, [1] and [2]), and on Hölder's relation,

$$c_n(k) = \sum_{\delta|(n,k)} \delta\mu(n/\delta).$$

We obtain

$$R_n(k,a) := \sum_{d|n} c_d(k) a^{n/d} = \sum_{d|n}\left(\sum_{\delta|(d,k)} \delta\mu(d/\delta)\right) a^{n/d},$$

where, by denoting $k = \delta a$, $d = \delta b$, $n = dj$ and regrouping the terms,

$$R_n(k,a) = \sum_{\substack{\delta bj=n\\ \delta a=k}} \delta\mu(b)a^j = \sum_{\substack{\delta m=n\\ \delta a=k}} \delta \sum_{bj=m} \mu(b)a^j = \sum_{\delta|(n,k)} \delta M_{n/\delta}(a).$$

We have from (11) that, for any δ, $M_{n/\delta}(a)$ is a multiple of n/δ, hence $\delta M_{n/\delta}(a)$ is a multiple of n. This shows that $R_n(k,a)$ is a multiple of n. □

Remarks 1. If $k = 0$ then $c_n(0) = \varphi(n)$ is Euler's totient function and we have, as a consequence,

$$\sum_{d|n} \varphi(d) a^{n/d} \equiv 0 \pmod n,$$

which is also known in the literature.

2. For $k = 1$, one has $c_n(1) = \mu(n)$, the Möbius function, and the given congruence reduces to (11).

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Also solved by Mihály Bencze (Brasov, Romania) and Sotirios E. Louridas (Athens, Greece)

174. Prove, disprove or conjecture:

(1) There are infinitely many primes with at least one 7 in their decimal expansion.

(2) There are infinitely many primes where 7 occurs at least 2017 times in their decimal expansion.

(3) There are infinitely many primes where at most one-quarter of the digits in their decimal expansion are 7s.

(4) There are infinitely many primes where at most half the digits in their decimal expansion are 7s.

(5) There are infinitely many primes where 7 does not occur in their decimal expansion.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

Solution by the proposer. (1) This follows from Dirichlet's Theorem of Primes (if a and b are relatively prime then there are infinitely many primes congruent to a modulo b), as 7 and 10 are relatively prime. The same is true for part (2).

We tackle part (4) first as it is easier than (3). Assume it is not true. Let us count how many numbers in $[10^k, 10^{k+1})$ have at least half their digits as 7s and, if k is large, there will be no primes in this interval with at most half their digits as 7s (by assumption). How many numbers are there? For each $\ell \in [k/2, k]$, we have $\binom{k}{\ell}$ ways to choose which ℓ of k digits are 7s and thus the number of such numbers is

$$\sum_{\ell=k/2}^{k} \binom{k}{\ell} 1^\ell 9^{k-\ell} \le \frac{k}{2}\binom{k}{k/2} 9^{k/2}$$

because the largest binomial coefficient is in the middle. While we could use Stirling, note $\binom{k}{k/2} < (1+1)^k = 2^k$ by the binomial theorem. Thus, the number of numbers in $[10^k, 10^{k+1})$ with at least half their digits as 7s is at most $k2^k 9^{k/2} = k \cdot 6^k$; as there are more than 10^k such numbers, we see the percentage of numbers that have at least half their digits as 7s is at most $k6^k/10^k = k(6/10)^k$, which tends to zero VERY rapidly. By Chebyshev's Theorem we have that there are at least $.9\pi(10^{k+1}) - 1.1\pi(10^k)$ primes in this interval, or at least $10^{k-2}/k$ primes. Thus, even if every number with at least half its digits as 7s were prime, there wouldn't be enough such numbers to account for all the primes in $[10^k, 10^{k+1})$. Thus, there are infinitely many primes with at most half their digits as 7s. It is not unreasonable to expect that a typical large prime has about 10% of its digits as 7s. *Do you expect there to be infinitely many primes where there are at most c% of the digits as 7s, where c is any number strictly less than* $1/7$*?*

One can argue similarly for (3) but it is a little more involved. I found it easiest to break the counting into primes with between $k/4$ and $k/3$ of their digits as 7s and then $k/3$ and $k/2$ of their digits as 7s (by (4), we don't need to worry about more than $k/2$ of their digits as 7s or we could just look directly at $k/3$ to k of their digits as 7s). The proof follows from estimating the sums – Stirling was useful for $k/4$ to $k/3$. Let's analyse from $k/3$ to $k/2$. Arguing as above, we have

$$\sum_{\ell=k/3}^{k/2} \binom{k}{\ell} 1^\ell 9^{k-\ell} \ \le\ \frac{k}{6}\binom{k}{k/2} 9^{2k/3} \ \le\ \frac{k}{6} 2^k (9^{2/3})^k \ \le\ k(2\cdot 9^{2/3})^k.$$

As $2\cdot 9^{2/3} \approx 8.65 < 10$, the number of such numbers tends to zero so rapidly that, arguing as in part (4), there just aren't enough of these numbers to matter. We are thus left with $\ell \in [k/4, k/3]$:

$$\sum_{\ell=k/4}^{k/3} \binom{k}{\ell} 1^\ell 9^{k-\ell} \ \le\ \frac{k}{12}\binom{k}{k/3} 9^{3k/4}.$$

If we used $\binom{k}{k/3} \le 2^k$, we would find the above is at most $k(2\cdot 9^{3/4})^k$ but $2\cdot 9^{3/4} \approx 10.39 > 10$; this is why we must be more careful and why we have to split up into different ranges. By Stirling,

$$\begin{aligned}\binom{k}{k/3} &\sim \frac{k^k e^{-k}\sqrt{2\pi k}}{(k/3)^{k/3} e^{-k/3}\sqrt{2\pi k/3}\cdot (2k/3)^{2k/3} e^{-2k/3}\sqrt{2\pi 2k/3}} \\ &\ll \frac{1}{(1/3)^{k/3}\cdot (2/3)^{2k/3}\sqrt{k}} \\ &\le \left(\frac{3}{2^{2/3}}\right)^k.\end{aligned}$$

Substituting this in above gives

$$\sum_{\ell=k/4}^{k/3} \binom{k}{\ell} 1^\ell 9^{k-\ell} \ \ll\ k\cdot\left(\frac{3}{2^{2/3}}\right)^k 9^{3k/4} \ \le\ k\cdot\left(\frac{3\cdot 9^{3/4}}{2^{2/3}}\right)^k.$$

As $3\cdot 9^{3/4}/2^{2/3} \approx 9.82 < 10$, arguing as before, we see that there are negligibly many numbers of this form. I really like this problem, as it highlights how careful we must be. We just need to get a number less than 10, so we keep splitting things up into different regions and using different estimates in each. We always replace the binomial coefficients with their largest value in the interval (which is at the right end point for $\ell \le k/2$) and the $9^{k-\ell}$ term with its largest value (which is at the left end point for $\ell \le k/2$). It would be interesting to do a more careful analysis and not bound things so crudely but this is what we number theorists do whenever possible: arguing as crudely as possible to get the required result.

Finally, part (5) is interesting. It's natural to conjecture that there are infinitely many. The following is a very common heuristic. Assume the two events are independent, namely having no 7s and being a prime. Let us label all such numbers $a_1, a_2, a_3, \ldots$. The probability a number x is prime is essentially $1/\log x$, thus the expected number of numbers at most x that are prime and 7-free is $\sum_{a_i\le x} 1/\log a_i$. We break this into sums of $a_i \in [10^k, 10^{k+1})$. There are 9^k numbers in this interval that are 7-free. We obtain an upper bound for the sum by replacing each a_i with 10^k and a lower bound by replacing with 10^{k+1}. This yields

$$\sum_{k=1}^{K} \frac{9^k}{(k+1)\log 10} \ \le \sum_{a_i\le 10^{K+1}} \frac{1}{\log a_i} \ \le\ \sum_{k=1}^{K} \frac{9^K}{k\log 10}.$$

Both the upper and lower bounds clearly tend to infinity with K, though much more slowly than $\pi(10^{K+1}) \approx 10^{K+1}/(K+1)\log 10$. As an aside, there are some sequences that are so sparse that we do not expect infinitely many primes. The standard example is the Fermat numbers: $F_n = 2^{2^n}+1$. It is conjectured that only the first four are prime; see, for example, tinyurl.com/yarhbtu3. Using $a_n = 2^{2^n}+1 \approx 2^{2^n}$, we find that the expected number of prime Fermat numbers is about

$$\sum_{n=0}^{\infty} \frac{1}{\log 2^{2^n}} \ \approx\ \sum_{n=0}^{\infty} \frac{1}{2^n\log 2} \ \approx\ \frac{2}{\log 2} \ \approx\ 3.$$

Returning to the problem at hand, it was recently successfully resolved by James Maynard; see his arXiv post "Primes with restricted digits", available at https://arxiv.org/pdf/1604.01041v1, where he shows there are infinitely many primes base 10 omitting any given digit. □

Also solved by Mihály Bencze (Brasov, Romania), Cristinel Mortici (Targoviste, Romania) and Socratis Varelogiannis (National Technical University of Athens, Greece)

175. Show that there is an infinite sequence of primes $p_1 < p_2 < p_3 < \cdots$ such that p_2 is formed by appending a number in front of p_1, p_3 is formed by appending a number in front of p_2 and so on. For example, we could have $p_1 = 3$, $p_2 = 13$, $p_3 = 313$, $p_4 = 3313$, $p_5 = 13313, \ldots$. Of course, you might have to add more than one digit at a time. Find a bound on how many digits you need to add to ensure it can be done.

(Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA, USA)

Solution by the proposer. One way to solve this problem is to use Dirichlet's Theorem for Primes in Arithmetic Progression, which states that if a and m are relatively prime then there are infinitely many primes congruent to a modulo m. Start with any prime number, and call that p_1, and define the function $g(n)$ to be the number of digits of n. By Dirichlet's theorem, since p_1 and $10^{g(p_1)}$ are relatively prime, there are infinitely many primes congruent to p_1 modulo $10^{g(p_1)}$; note that all of these primes will have their final $g(p_1)$ digits as p_1, and thus are constructed by appending digits in front of p_1. For definiteness, take the smallest such prime and call that p_2. We continue by induction. If we have formed p_m then p_{m+1} is obtained by applying Dirichlet's result to the pair $p_m, 10^{g(p_m)}$.

Unfortunately, as usually stated, Dirichlet's theorem is not constructive; it just states that there are infinitely many primes but says nothing about how far we must go before we find the first such prime. Fortunately, with a bit more work, one can find upper bounds on how far we must search. Interestingly, however, what we need is the *second* smallest prime in arithmetic progressions, and thus many of the results in the literature are not directly applicable. If we wish to use them, however, we can easily modify our work. Start off with a prime p_1, append a 1 to the front of it and then construct p_2 by choosing the first prime congruent to $10^{g(p_1)} + p_1$ modulo $10^{g(p_1)+1}$, and so on. It is conjectured that the first prime congruent to a modulo m can be found by going up to $C_\epsilon m^{1+\epsilon}$ (where, for each $\epsilon > 0$, there is some C_ϵ) but this is far from known. Linnik proved in 1944 that there are c and L such that the first prime is found before cm^L, though he didn't provide a value for L. The best current value is $L = 5$, which is due to Xylouris.

As an aside, this problem bears some similarity to searches for Cunningham chains, which are sequences of primes with specific relations between terms. A Cunningham chain of the first kind is a set of primes where $p_n = 2p_n + 1$ (the second kind is $p_n = 2p_n - 1$). It is believed that there are Cunningham chains of arbitrarily long length and this follows from standard conjectures (the world record of either is 19, which is due to Wroblewski from 2014). □

Also solved by Mihály Bencze (Brasov, Romania) and Sotirios E. Louridas (Athens, Greece)

176. Consider all pairs of integers x, y with the property that $xy - 1$ is divisible by the prime number 2017. If three such integral pairs lie on a straight line on the xy–plane, show that both the vertical distance and the horizontal distance of at least two of such three integral pairs are divisible by 2017.

(W. S. Cheung, Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer. By assumption, there are real numbers a, b, $c \in \mathbb{R}$ with $(a, b, c) = 1$ and integers k_i, $i = 1, 2, 3$, such that

$$ax_i + by_i = c \tag{1}$$

and

$$x_i y_i = 1 + k_i \cdot 2017 \tag{2}$$

for all $i = 1, 2, 3$. Without loss of generality, we may assume that $2017 \nmid b$. By (1), we have

$$a(x_1 - x_i) = b(y_i - y_1)\,, \qquad i = 1, 2, 3\,.$$

By (2),

$$x_i y_i = x_1 y_1 + (k_i - k_1) \cdot 2017\,, \qquad i = 1, 2, 3\,.$$

Hence,

$$\begin{aligned}(x_1 - x_i)by_i &= bx_1 y_i - bx_i y_i\\ &= bx_1(y_i - y_1) - b(k_i - k_1) \cdot 2017\\ &= ax_1(x_1 - x_i) - b(k_i - k_1) \cdot 2017\end{aligned}$$

and so we have

$$(x_1 - x_i)(ax_1 - by_i) = b(k_i - k_1) \cdot 2017\,.$$

Hence,

$$2017 \mid (x_1 - x_i)(ax_1 - by_i)\,. \tag{3}$$

Observe that this forces that at least one of

$$2017 \mid (x_1 - x_2)\,, \qquad 2017 \mid (x_1 - x_3) \qquad \text{and} \qquad 2017 \mid (y_2 - y_3)$$

should hold. In fact, if $2017 \nmid (x_1 - x_2)$ and $2017 \nmid (x_1 - x_3)$, by (3), we must have

$$2017 \mid (ax_1 - by_2)\,, \qquad 2017 \mid (ax_1 - by_3)\,,$$

and so

$$2017 \mid b(y_2 - y_3)\,.$$

Since $2017 \nmid b$, we have $2017 \mid (y_2 - y_3)$.

Take, for example, $2017 \mid (y_2 - y_3)$. Then, by

$$2017 \mid x_2(y_2 - y_3)\,, \qquad 2017 \mid (1 - x_2 y_2)\,, \qquad 2017 \mid (x_3 y_3 - 1)\,, \tag{4}$$

we have

$$2017 \mid (x_3 - x_2)y_3\,.$$

By (4), $2017 \nmid y_3$, so we have $2017 \mid (x_2 - x_3)$. Hence, both the vertical and horizontal distances of (x_2, y_2) and (x_3, y_3) are divisible by 2017. The remaining cases can be proven analogously. □

Also solved by Cristinel Mortici (Targoviste, Romania) and Panagiotis T. Krasopoulos (Athens, Greece)

Remark 1. The following much shorter solution to Problem 164 (Newsletter, March 2017, Issue 103) was provided by Panagiotis T. Krasopoulos (Greece), Hans J. Munkholm (Denmark) and Ellen S. Munkholm (Denmark).

Any power of 2015, say $P = 2015^n$, has the form $P = 5k$ with k a positive integer. Therefore

$$P = 5k = \frac{k^2}{k} \cdot \frac{2^2 + 1^2}{2 - 1} = \frac{(2k)^2 + k^2}{2k - k}\,.$$

Remark 2. Problems 163, 164, 166 and 167 (Newsletter, March 2017, Issue 103) were also solved by Dimitrios Koukakis (Greece).

We would like you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions either by ordinary mail to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, or by email to michail.rassias@math.uzh.ch.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Fundamentals of Mathematical Analysis*.

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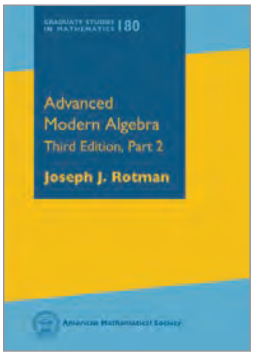

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Joseph J. Rotman, University of Illinois at Urbana-Champaign

This is the second part of the new edition of *Advanced Modern Algebra*. Compared to the previous edition, the material has been significantly reorganized and many sections have been rewritten. The book presents many topics mentioned in the first part in greater depth and in more detail. The five chapters of the book are devoted to group theory, representation theory, homological algebra, categories, and commutative algebra, respectively. The book can be used as a text for a second abstract algebra graduate course, as a source of additional material to a first abstract algebra graduate course, or for self-study.

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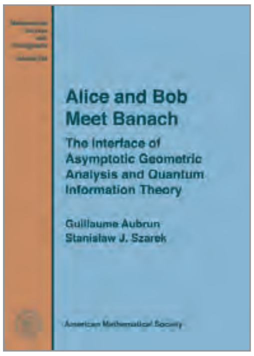

ALICE AND BOB MEET BANACH

The Interface of Asymptotic Geometric Analysis and Quantum Information Theory

Guillaume Aubrun, Université Claude Bernard Lyon 1 & Stanislaw J. Szarek, Case Western Reserve University

Alice and Bob Meet Banach is aimed at multiple audiences connected through their interest in the interface of QIT and AGA: at quantum information researchers who want to learn AGA or apply its tools; at mathematicians interested in learning QIT, especially the part that is relevant to functional analysis/convex geometry/random matrix theory and related areas; and at beginning researchers in either field. Moreover, this user-friendly book contains numerous tables and explicit estimates, with reasonable constants when possible, which make it a useful reference even for established mathematicians generally familiar with the subject.

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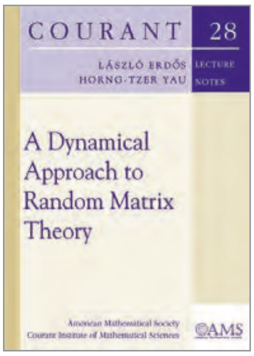

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László Erdös, Institute of Science and Technology Austria & Horng-Tzer Yau, Harvard University

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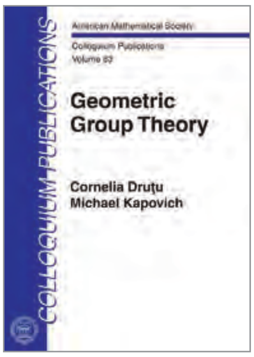

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Cornelia Drutu, Mathematical Institute, Oxford & Michael Kapovich, University of California

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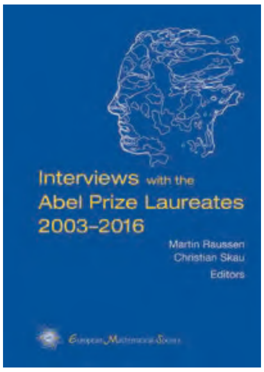
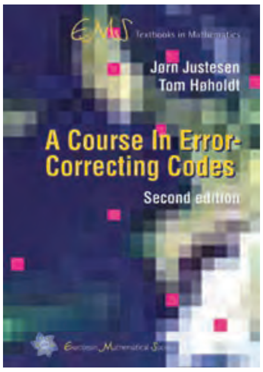


Interviews with the Abel Prize Laureates 2003–2016
Martin Raussen (Aalborg University, Denmark) and Christian Skau (Norwegian University of Science and Technology (NTNU), Trondheim, Norway), Editors

ISBN 978-3-03719-177-4. 2017. 301 pages, 66 photographs. Softcover. 17 x 24 cm. 24.00 Euro

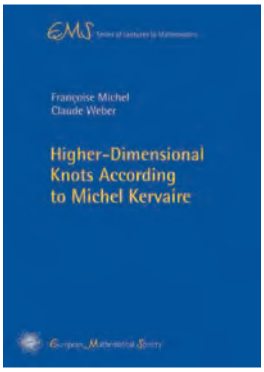
The Abel Prize was established in 2002 by the Norwegian Ministry of Education and Research. It has been awarded annually to mathematicians in recognition of pioneering scientific achievements.
Since the first occasion in 2003, Martin Raussen and Christian Skau have had the opportunity to conduct extensive interviews with the laureates. The interviews were broadcast by Norwegian television; moreover, they have appeared in the membership journals of several mathematical societies.
The interviews from the period 2003 – 2016 have now been collected in this edition. They highlight the mathematical achievements of the laureates in a historical perspective and they try to unravel the way in which the world's most famous mathematicians conceive and judge their results, how they collaborate with peers and students, and how they perceive the importance of mathematics for society.



Jørn Justesen and Tom Høholdt (both Technical University of Denmark, Lyngby, Denmark)
A Course In Error-Correcting Codes. Second edition (EMS Textbooks in Mathematics)

ISBN 978-3-03719-179-8. 2017. 226 pages. Hardcover. 16.5 x 23.5 cm. 39.50 Euro

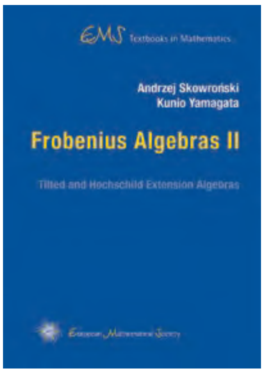
This book, updated and enlarged for the second edition, is written as a text for a course aimed at 3rd or 4th year students. Only some familiarity with elementary linear algebra and probability is directly assumed, but some maturity is required. The students may specialize in discrete mathematics, computer science, or communication engineering. The book is also a suitable introduction to coding theory for researchers from related fields or for professionals who want to supplement their theoretical basis. The book gives the coding basics for working on projects in any of the above areas, but material specific to one of these fields has not been included. The chapters cover the codes and decoding methods that are currently of most interest in research, development, and application. They give a relatively brief presentation of the essential results, emphasizing the interrelations between different methods and proofs of all important results. A sequence of problems at the end of each chapter serves to review the results and give the student an appreciation of the concepts. In addition, some problems and suggestions for projects indicate direction for further work. The presentation encourages the use of programming tools for studying codes, implementing decoding methods, and simulating performance. Specific examples of programming exercises are provided on the book's home page.



Françoise Michel (Université Paul Sabatier, Toulouse, France) and Claude Weber (Université de Genève, Switzerland)
Higher-Dimensional Knots According to Michel Kervaire (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-180-4. 2017. 144 pages. Softcover. 17 x 24 cm. 32.00 Euro

Michel Kervaire wrote six papers which can be considered fundamental to the development of higher-dimensional knot theory. They are not only of historical interest but naturally introduce to some of the essential techniques in this fascinating theory.
This book is written to provide graduate students with the basic concepts necessary to read texts in higher-dimensional knot theory and its relations with singularities. The first chapters are devoted to a presentation of Pontrjagin's construction, surgery and the work of Kervaire and Milnor on homotopy spheres. We pursue with Kervaire's fundamental work on the group of a knot, knot modules and knot cobordism. We add developments due to Levine. Tools (like open books, handlebodies, plumbings, …) often used but hard to find in original articles are presented in appendices. We conclude with a description of the Kervaire invariant and the consequences of the Hill–Hopkins–Ravenel results in knot theory.



Andrzej Skowroński (Nicolaus Copernicus University, Toruń, Poland) and Kunio Yamagata (Tokyo University of Agriculture and Technology, Japan)
Frobenius Algebras II. Tilted and Hochschild Extension Algebras (EMS Textbooks in Mathematics)

ISBN 978-3-03719-174-3. 2017. 629 pages. Hardcover. 17 x 24 cm. 58.00 Euro

This is the second of three volumes which will provide a comprehensive introduction to the modern representation theory of Frobenius algebras. The first part of the book is devoted to fundamental results of the representation theory of finite dimensional hereditary algebras and their tilted algebras, which allow to describe the representation theory of prominent classes of Frobenius algebras.
The second part is devoted to basic classical and recent results concerning the Hochschild extensions of finite dimensional algebras by duality bimodules and their module categories. Moreover, the shapes of connected components of the stable Auslander-Reiten quivers of Frobenius algebras are described.
The only prerequisite in this volume is a basic knowledge of linear algebra and some results of the first volume. It includes complete proofs of all results presented and provides a rich supply of examples and exercises.
The text is primarily addressed to graduate students starting research in the representation theory of algebras as well mathematicians working in other fields.The book is accessible to advanced students and researchers of complex analysis and differential geometry.
The first volume (ISBN 978-3-03719-102-6) has appeared under the title *Frobenius Algebras I. Basic Representation Theory.*