NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

European Mathematical Society

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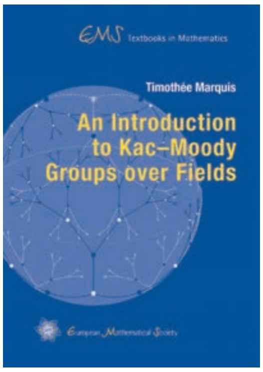
Feature
Kinetic Equations: A French History

Interview
Robert P. Langlands

Obituary
G. I. Barenblatt

Robert P. Langlands, The Abel Prize Laureate 2018. © Randall Hagadorn/Institute for Advanced Study

New books published by the European Mathematical Society

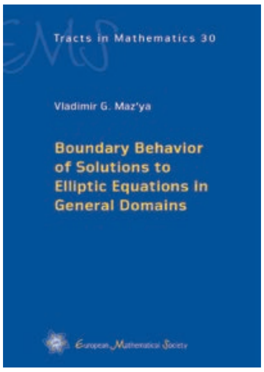


Timothée Marquis (Université Catholique de Louvain, Louvain-la-Neuve, Belgium)
An Introduction to Kac–Moody Groups over Fields (EMS Textbooks in Mathematics)

ISBN 978-3-03719-187-3. 2018. 343 pages. Hardcover. 16.5 x 23.5 cm. 48.00 Euro

The interest for Kac–Moody algebras and groups has grown exponentially in the past decades, both in the mathematical and physics communities, and with it also the need for an introductory textbook on the topic. The aims of this book are twofold:
- to offer an accessible, reader-friendly and self-contained introduction to Kac–Moody algebras and groups;
- to clean the foundations and to provide a unified treatment of the theory.

The book starts with an outline of the classical Lie theory, used to set the scene. Part II provides a self-contained introduction to Kac–Moody algebras. The heart of the book is Part III, which develops an intuitive approach to the construction and fundamental properties of Kac–Moody groups. It is complemented by two appendices, respectively offering introductions to affine group schemes and to the theory of buildings. Many exercises are included, accompanying the readers throughout their journey. The book assumes only a minimal background in linear algebra and basic topology, and is addressed to anyone interested in learning about Kac–Moody algebras and/or groups, from graduate (master) students to specialists.



Vladimir Maz'ya (Linköping University, Sweden and University of Liverpool, UK)
Boundary Behavior of Solutions to Elliptic Equations in General Domains (Tracts in Mathematics)

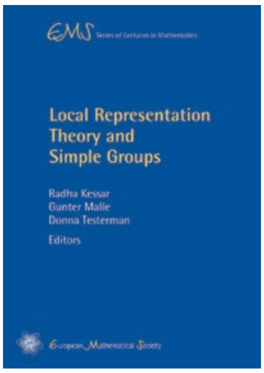
ISBN 978-3-03719-190-3. 2018. 441 pages. Hardcover. 17 x 24 cm. 64.00????? Euro

The present book is a detailed exposition of the author and his collaborators' work on boundedness, continuity, and differentiability properties of solutions to elliptic equations in general domains, that is, in domains that are not a priori restricted by assumptions such as "piecewise smoothness" or being a "Lipschitz graph". The description of the boundary behavior of such solutions is one of the most difficult problems in the theory of partial differential equations. After the famous Wiener test, the main contributions to this area were made by the author. In particular, necessary and sufficient conditions for the validity of imbedding theorems are given, which provide criteria for the unique solvability of boundary value problems of second and higher order elliptic equations. Another striking result is a test for the regularity of a boundary point for polyharmonic equations.
The book will be interesting and useful for a wide audience. It is intended for specialists and graduate students working in the theory of partial differential equations.

European Congress of Mathematics. Berlin, July 18–22, 2016
Volker Mehrmann and Martin Skutella (both Technical University Berlin, Germany), Editors

ISBN 978-3-03719-176-6. 2018. 901 pages. Hardcover. 16.5 x 23.5 cm. 118.00 Euro

The European Congress of Mathematics, held every four years, is a well-established major international mathematical event. Following those in Paris, Budapest, Barcelona, Stockholm, Amsterdam, and Kraków, the Seventh European Congress of Mathematics (7ECM) took place in Berlin, Germany, July 18–22, 2016, with about 1100 participants from all over the world.
Ten plenary, thirty-three invited and four special lectures formed the core of the program. As at all the previous EMS congresses, ten outstanding young mathematicians received the EMS prizes in recognition of their research achievements. In addition, two more prizes were awarded: The Felix Klein prize for a remarkable solution of an industrial problem, and – for the second time – the Otto Neugebauer Prize for a highly original and influential piece of work in the history of mathematics. The program was complemented by forty-three minisymposia with about 160 talks as well as contributed talks, spread over all areas of mathematics. These proceedings present extended versions of most of the plenary and invited lectures which were delivered during the congress, providing a permanent record of the best what mathematics offers today.



Local Representation Theory and Simple Groups (EMS Series of Lectures in Mathematics)
Radha Kessar (City University of London, UK), Gunter Malle (Universität Kaiserslautern, Germany) and Donna Testerman (EPF Lausanne, Switzerland), Editors

ISBN 978-3-03719-185-9. 2018. 369 pages. Softcover. 17 x 24 cm. 44.00 Euro

The book contains extended versions of seven short lecture courses given during a semester programme on "Local Representation Theory and Simple Groups" held at the Centre Interfacultaire Bernoulli of the EPF Lausanne. These focussed on modular representation theory of finite groups, modern Clifford theoretic methods, the representation theory of finite reductive groups, as well as on various applications of character theory and representation theory, for example to base sizes and to random walks. These lectures are intended to form a good starting point for graduate students and researchers who wish to familiarize themselves with the foundations of the topics covered here. Furthermore they give an introduction to current research directions, including the state of some open problems in the field.

Scan the QR code to go to the Newsletter web page:
http://euro-math-soc.eu/newsletter

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EMS Agenda

2018

30 October
AMC Committee Meeting
Paris, France

9–11 November
EMS Executive Committee Meeting, Barcelona, Spain

8 December
EMF Board of Trustees Meeting, Zurich, Switzerland

2019

23–24 March
EMS Meeting of the Presidents, Berlin, Germany

EMS Scientific Events

2018

8–12 October
International Workshop on Geometric Quantization
and Applications
CIRM Luminy, Marseille, France

11–14 December
Advances in Applied Algebraic Geometry
University of Bristol, UK

2019

7–10 January
Variational Problems in Geometry and Mathematical Physics,
UK-Japan Winter School
University of Leeds, UK

1–5 April
Imaging and Machine Learning
Institut Henri Poincaré, Paris, France

15–19 July
ICIAM 2019
Valencia, Spain

29 July–2 August
British Combinatorial Conference 2019
Glasgow, UK

9–13 July
SIAM Conference on Applied Algebraic Geometry
Bern, Switzerland

2020

5–11 July
European Congress of Mathematics
Portorož, Slovenia

Editorial – Report from the EMS Council Meeting in Prague, 23rd & 24th June 2018

Richard Elwes, EMS Publicity Officer

The EMS Council is our society's highest authority. It is made up of delegates representing individual, institutional, and society members, and meets every two years, most recently in 2016 in Berlin. This June, 78 delegates gathered with 11 guests for two days in the Czech National Library of Technology, in the magnificent surroundings of Prague, home city of the current EMS President Pavel Exner. The meeting's full agenda would include the appointment of his successor. At 2 pm on Friday afternoon, the President welcomed the assembled company and opened the meeting. The Council then paused for a few moments of silence, to honour the memory of those colleagues who have departed since the last meeting.

Officer reports and finance

The first order of business was a report from the President on his recent activities (several of the topics mentioned feature separately later in this report). This was followed by a report from EMS secretary Sjoerd Verduyn Lunel outlining the activities of the EMS Executive Committee. Over the last two years, the committee has officially met four times, in Tblisi, Bratislava, Portorož, and Rome. (A report from the Rome meeting can be found on page 5, and reports from other meetings have appeared in earlier editions of the Newsletter.) It also gathered for an informal retreat in Koli (Finland) in January 2018. EMS Treasurer Mats Gyllenberg next presented his report, noting that the society's finances are healthy. (We record two small contributing factors: the contribution of the EMS Publishing House which provides this Newsletter and the EMS Mathematical Surveys series at no cost to the society, and the University of Helsinki which generously houses the EMS Office.) The Treasurer was therefore able to propose an increase in the budget for scientific activities, with no accompanying rise in membership fees. A useful discussion then followed, about further activities the EMS might choose to support financially. The Council was pleased to approve the society budget for 2019-2020, and to agree the appointment of auditors.

Membership and publicity

The Council reviewed the society's membership statistics and the list of new EMS individual members, bringing the total to 2707. After a presentation by Tomaž Pisanski, President of the Slovenian Discrete and Applied Mathematics Society (SDAMS), the Council was pleased to approve SDAMS's application for full EMS membership. Cooperation agreements with the Chinese and Australian Mathematical Societies were also approved. Four new institutional members (already approved by the

Executive committee) were welcomed by the Council: the University of Lisbon, the Mathematical Institute of University of Oxford, the Mathematics Department of University of Pisa, and the Department of Mathematics "F. Casorati" of the University of Pavia; along with one new associate member: the society of European Women in Mathematics.

Although the EMS by its nature is an inclusive organisation, the council then took the sad decision to terminate the EMS memberships of the Malta Mathematical Society and the Association of Mathematicians of the Republic of Macedonia. Both of these societies have fallen severely into arrears on their dues, and have repeatedly failed to respond to letters from the EMS President.

After a lively debate, the council then approved a proposal to introduce life-membership. The new rules state that a member aged w where $w > 60$ may apply for life-time membership, for a once-and-for-all fee of $(77 - \frac{4w}{5}) \cdot \frac{y}{5}$ where y is the standard, undiscounted fee for annual EMS membership (currently y = €50).

The society's Publicity Officer Richard Elwes gave a presentation on EMS publicity, both in physical form (a new flyer has been produced, and promotional materials have regularly been present at mathematical meetings over the last two years) and online. EMS social media presence has significantly increased recently, and we currently have over 3000 followers on Twitter (@EMSNewsletter) and over 2000 on Facebook (@EMSNewsletter).

Elections to Executive Committee

It was now time for Council's most exciting moment. After presentations by the candidates, the council was delighted to elect Volker Mehrmann (Technische Universität Berlin) as EMS President and Betül Tanbay (Bogaziçi University) as Vice-President. Each will take up their new positions in January 2019. Mats Gyllenberg and Sjoerd Verduyn Lunel were each re-elected for second terms as (respectively) EMS Treasurer and Secretary.

The candidates just mentioned came on the recommendation of the executive committee, and these elections were uncontested and approved by large margins. Next there followed a contested election for the remaining position of member-at-large of the Executive Committee. After presentations by the candidates, a vote was held, and won by Jorge Buesco.

ECM2020 and EMS committees

Local organiser Klavdija Kutnar presented an update on preparations for the 8th European Congress of Mathematics, to be held 5–11 July 2020 in Portorož (Slovenia), birthplace of Giuseppe Tartini, the great 18th century composer, violinist, and amateur mathematician. There are now open calls for Minisymposia, Satellite Conferences, and Exhibitors. In the ensuing discussion council delegates enquired further about the plans, and made several suggestions to the local organisers.

The Council then heard reports from representatives of all the EMS's standing committees, starting with Stéphane Cordier, new Chair of the Applied Mathematics Committee, who discussed his committee's activities, including the current year of Mathematical Biology.

Leif Abrahamsson Chair of the Committee for Developing Countries, talked about its work, including the ongoing Simons Foundation for Africa scheme. We are currently in the second year of this programme, which is funded by the Simons Foundation at $50,000 per year for 5 years.

Jürg Kramer, the new Chair of the Education Committee outlined his committee's new plan of action, which includes applying for EU funding for mathematical educational projects.

Former EMS President Ari Laptev was next to address the council as Chair of the ERCOM Committee, comprising the directors of 26 European Research Centres On Mathematics Committee.

Chair of the Ethics Committee Jiří Rákosník then described the actions his committee is taking to defend our subject from unethical behaviour, and invited feedback in advance of a planned update to the EMS Code of Practice.

Alice Fialowski presented a report from Sandra Di Rocco, Chair of the European Solidarity Committee, which has an annual budget of €14000 to fund travel for researchers from European countries where such funding is difficult to obtain.

Martin Mathieu spoke on behalf of Ciro Ciliberto, Chair of the Meetings Committee which evaluates applications for EMS funding and makes recommendations to the Executive Committee. He noted that certain EMS activities, such as the joint mathematical weekends, have attracted few applications recently, and urged for more focussed advertising of these opportunities.

Since the last Council meeting, the Electronic Publishing and the Publication Committees have merged into a new Publications and Electronic Dissemination Committee. Thierry Bouche, Chair of this new committee, described its activities to date, which include working with the European Digital Mathematics Library and Zentralblatt Math (see separate later items). It has quickly become clear that issues around Open Access pose major questions in this area.

Silvia Benvenuti, Vice Chair of the Committee for Raising Public Awareness of Mathematics, gave a presentation on its work on behalf of Chair Roberto Natalini. She drew attention to their website Mathematics in Europe (www.mathematics-in-europe.eu/) which has published around 75 articles over the last two years, and discussed an ongoing series of interviews with female mathematicians (jointly undertaken with the Women in Mathematics Committee, see next item) for this Newsletter. She also outlined plans for an online Calendar-map of mathematical outreach events, and concluded by requesting the cooperation of national societies in advancing the committee's work.

Finally, Beatrice Pelloni discussed the committee for Women in Mathematics on behalf of Chair Alessandra Celletti, stressing the importance of encouraging diversity in the EMS's various activities. Besides the joint work with the RPA committee just mentioned, the committee's activities include coordinating with the European Women in Mathematics Society to schedule Summer Schools and other mathematical events.

To close this section, the President reiterated the critical role that all these standing committees play in the proper functioning of the EMS, and the council was pleased to join him in thanking all the current and outgoing Chairs and committee members for their hard work.

Mathematics in Europe

The Council heard updates from the President on proposals for Horizon Europe, the successor programme to Horizon 2020. We note that the proposed budget is smaller than expected, and puts Europe behind other advanced economies in terms of the proportion of GDP invested in science. A petition to increase it can be found online at double-ri.eu.

Jean-Pierre Bourguignon current President of the European Research Council (and former EMS President) was not able to attend, but sent a presentation which the EMS President discussed on the ERC's situation 10 years after it was founded, with a focus on how Mathematics fits within it.

Publications and projects

Valentin Zagrebnov, Editor-in-Chief of the EMS Newsletter, reported on the status of the newsletter and invited suggestions for new members of the Editorial Board. The Council took the opportunity to thank him, along with the editor of the EMS e-News Mireille Chaleyat-Maurel, for their excellent work. With the Managing Director of the EMS Publishing House unable to attend the meeting, the President summarised its current status and plans for its future.

On behalf of Klaus Hulek, Editor-in-Chief of Zentralblatt, Olaf Teschke gave a presentation on zbMATH. The President concluded the section with the view that Zentralblatt has improved significantly and is now a serious competitor amongst mathematical databases, and of course zbMATH is available free of charge for individual EMS members!

Jiří Rákosník gave a presentation on the European Digital Mathematical Library (www.eudml.org), a digital collection of Open Access mathematics libraries, of which the EMS is one sponsor. There are currently over 260,000 items uploaded and available online. He discussed recent innovations, such as the function to link posts on mathoverflow.net to EUDML articles. He also discussed the linked library eLibM (www.elibm.org) which supports a full journal production system, with a flagship journal Documenta Mathematica now sustainably produced as a Diamond Open Access publication.

With Springer having announced that it will no longer be involved in the online Encyclopedia of Mathematics (www.encyclopediaofmath.org), the EMS will assume the responsibility of hosting of this internet resource.

The EMS Vice-President (and new President-elect!) Volker Mehrmann delivered a presentation on EU-MATH-IN (European Service Network of Mathematics for Industry and Innovation), drawing attention to their recent vision paper on the increasingly commercially important theme of "Digital Twins". For readers who have not encountered this terminology, "a digital twin is a cross-domain digital model that accurately represents a product, production process or performance of a product or production system in operation. The digital twin evolves and continuously updates to reflect any change to the physical counterpart throughout the counterpart's lifecycle, creating a closed-loop of feedback in a virtual environment that offers companies the best possible design for their products and production processes."

The meeting was then brought to a close, with the Council warmly thanking our hosts: the Czech Mathematical Society, the Institute of Mathematics of the Czech Academy of Sciences, and the Faculty of Mathematics and Physics at Charles University. The next Council meeting will be 4 –5 July 2020 in Bled (Slovenia).

Report from the Executive Committee Meeting in Rome, 23–25 March 2018

Richard Elwes, EMS Publicity Officer

This is a slightly abbreviated report of March's Executive Committee meeting, as several matters were preparatory for, or superseded by, action at the subsequent EMS Council meeting. See page 3 for a longer report from that meeting.

There can be fewer more pleasant places to visit in Spring than Rome, where the Executive Committee (EC) gathered this March. Surprisingly, this was the EC's first visit to the Eternal City, and the committee members and guests were grateful for the generous hospitality they received at the Argiletum of Università Roma Tre.

On Friday evening, the committee was greeted by Ciro Ciliberto, President of the Italian Mathematical Union (UMI), our hosts for the weekend. He gave us a brief history of his society, which was founded in 1922, with Salvatore Pincherle as its first President. The UMI has always been headquartered in Bologna, where it hosted the 1928 International Congress of Mathematics. In 1954, the UMI founded the Centro Internazionale Matematico Estivo: an annual summer school of higher mathematics, through which thousands of participants from around the world have now passed. The UMI produces a number of publications on mathematics and associated cultural and educational topics. It currently has around 2000 members, with several active standing committees and working groups; the UMI awards mathematical prizes at all levels. Shortly after this meeting, the UMI would be hosting the 2018 European Girls' Mathematical Olympiad in Florence.

Officers' reports and membership

After welcoming the EC and guests, EMS President Pavel Exner reported on his activities since the last formal EC Meeting, beginning with the successful informal retreat that the EC enjoyed in Koli, Finland, 5–7 January 2018. He highlighted important upcoming events including the forthcoming EMS Council meeting (see separate report on page 3) the ICM in August in Rio de Janeiro, and ICIAM 2019 in Valencia.

EMS Treasurer Mats Gyllenberg then presented his summary of the society's income and expenditure in 2017, reporting a healthy situation. The lower than desired expenditure on scientific projects continues to require attention; the EC discussed ways to make our calls for proposals more widely known within the mathematical community. The EC discussed instigating a new annual Summer School to be fully supported by the society, that can function as an EMS Flagship event.

The application for academic membership from the Department of Mathematics "F. Casorati" of the University of Pavia was approved, along with a list of 91 new individual members.

Scientific and society meetings

The EC held in-depth discussions on the membership of the scientific and prize committees for the next European Congress in 2020. The next step will be the formulation of the call for nominations for the prizes. The President gave a summary of this year's EMS supported scientific events, including 9 Summer Schools. It was agreed that Mats Gyllenberg will introduce Samuel Kou, the 2018 speaker for the EMS-Bernoulli Society Joint Lecture, at the European Conference on Mathematical and Theoretical Biology in Lisbon in July. The EC discussed preparations for the upcoming EMS Council in Prague, including agreeing the draft agenda. The nominations of Council delegates closed with no elections needed. The committee unanimously agreed to propose to the Council Betül Tanbay as next EMS Vice-President and Volker Mehrmann as next EMS President. (See separate council report.)

The President reminded the committee that at ICM2018 in Rio, the IMU's general assembly will choose the site for ICM2022, selecting between Paris and Saint Petersburg. The EMS is neutral between the two bids.

Publicity, committees, and projects

EMS Publicity Officer Richard Elwes presented his report along with a new flyer advertising the society and a new Powerpoint presentation to introduce the EMS at appropriate meetings. The Executive Committee welcomed these innovations, and unanimously approved his reappointment for a second term (2019–2022).

The EC discussed reports on the EMS standing committees (see Council meeting report for further discussion), before welcoming as guests Alessandra Celletti (Chair of the EMS Women in Mathematics committee) and Elena Resmerita representing the society of European Women in Mathematics (EWM). During a wide-ranging discussion, it was agreed that EWM would apply for EMS associate membership. This would improve communication between the two societies, and have the effect that issues related to gender will play a more prominent role in society events, such as the meetings of the EMS Council and Presidents of EMS Member Societies.

The EC considered the status of other EMS projects, including the European Digital Mathematics Library, the online Encyclopedia of Mathematics (noting that StatProb has successfully been incorporated), EU-MATHS-IN, the EMS Newsletter and Publishing House, and Zentralblatt. There was also discussion of the European Research Council and Horizon Europe. (See the Council report for further information on all of these.)

The meeting closed with grateful applause for our kind hosts at Università Roma Tre and the Italian Mathematical Union, especially UMI President Ciro Cilliberto.

Report on the Meeting of Presidents of Mathematical Societies in Maynooth, 14–15th April 2018

Sjoerd Verduyn Lunel, Secretary of the EMS, and Richard Elwes, EMS Publicity Officer

This is an abbreviated report of the Presidents' meeting, as several topics of discussion were also covered at the previous Rome Executive committee meeting (see page 5) or the subsequent Prague Council meeting (see page 3).

The meeting of the Presidents of EMS member societies is an annual society tradition. This year it gathered at Maynooth University in Ireland. Pavel Exner, EMS President and Chair of the meeting, welcomed the assembled company and thanked the organizers for the invitation to the Emerald Isle. After a tour de table in which the participants introduced themselves and their societies, Steve Buckley and Pauline Mellon, President and Vice-President (respectively) of our hosts the Irish Mathematical Society, delivered a presentation about their society. Founded at Trinity College Dublin in 1976, with a constitution based on that of the Edinburgh Mathematical Society, the Irish Mathematical Society holds regular mathematical events, including the *Groups in Galway* series which has run every spring since 1978, and an annual conference in September which covers both research and educational topics.

EMS business

The EMS President delivered a report on recent EMS activities. He reminded the meeting that ours is not principally a political body, and we cannot participate in all of the increasing number of public protests. At the same time, we must speak out when the life of our community is endangered. There is thus a balance to strike, and in line with this policy, he drew attention to an open letter of support for students arrested at Boğaziçi University in Turkey. (This can be found at www.bit.ly/2pUFPHu.)

The Chair asked the Presidents for help recruiting reviewers for the books section on the EMS webpage (www.euro-math-soc.eu/book-reviews), and recalled that members of the EMS have special benefits at www.MathHire.org, a website to find and advertise mathematical jobs, that is supported by the EMS. This Autumn the EMS will evaluate its cooperation agreement with MathHire, and the Chair invited any feedback on this service.

The Chair reported on the EMS's spectrum of scientific activities, and encouraged member societies to prepare proposals for scientific events, such as Joint Mathematical Weekends and Summer Schools. He then spoke about the EMS standing committees in turn (see Council report for more details), requesting the cooperation of member societies in a few cases. Firstly, with regard to the Applied Mathematics Committee, the President drew attention to those H2020 calls where Mathematics can play a role, and referred to the webpage of EU-MATHS-IN for further information. Secondly, a difficulty in raising money for the EMS's Committee for Developing Countries (CDC) is that increasing numbers of members pay their EMS dues through their local societies, rather than directly to the EMS, meaning they bypass the option to add a donation to the CDC. So the Chair requested that, where possible, this option also be advertised at the local level. Finally, the Chair passed a request from Roberto Natalini, Chair of the Raising Public Awareness committee (RPA), that the Presidents identify a contact for mathematical outreach activities within their own society, and send this information to the RPA.

Finally, the Chair informed the Presidents that the EMS has terminated its membership of Initiative for Science in Europe.

Presentations

The meeting then heard several presentations. First, Barbara Kaltenbacher delivered a report on the work of the Austrian Mathematical Society. Next, Stanislav Smirnov spoke about the Saint Petersburg Mathematical Society. Jorge Buesco talked about the Portuguese Mathematical Society, before Caroline Series presented a report on the EMS's newest member society, European Women in Mathematics. Thierry Horsin spoke about the work of the Société de Mathématiques Appliquées et Industrielles (SMAI), followed by Stéphane Seuret on the Société Mathématique de France.

Catalin Gherghe presented a short film about the International Mathematical Olympiad (IMO). The IMO started in 1959 in Romania and this July returned there for its 59th edition, with more than 2000 youngsters participating.

Klavdija Kutnar provided an update on preparations for the 8th European Congress of Mathematics to be held in Portorož (Slovenia), 5–11 July 2020. She especially drew attention to the open calls for Satellite Conferences and Minisymposia.

Finally, Carlos Vázquez presented a report on preparations for the International Congress on Industrial and Applied Mathematics (ICIAM) in 2019 in Valencia (Spain).

Discussion

At every Presidents' meeting, time is set aside for open discussion. While participants are free to raise any topic (and do), it is helpful to have a theme for the conversation, which this time was the relationship between national societies and the EMS. There was extensive discussion about what should be done at a national level and what at the European level, and more generally how can we learn from and strengthen each other. It is not possible to summarise the entire discussion, but we mention a few items that were raised.

Firstly, the quality of mathematical high school education is a pan-European concern. Here, the EMS (and its Education committee in particular) can serve as a platform for the exchange of best practices.

In general, we should always seek maximise the use of EMS communication channels such as the E-news, to keep information flowing between national societies and the EMS. On a related point, the President conveyed a message from the Editor-in-Chief of the EMS Newsletter, Valentin Zagrebnov, that he would be pleased to receive proposals for articles.

The Presidents continued discussing assorted challenges, from the relationship between Mathematics and Computer Science to life-long learning opportunities for mathematics teachers.

After this lively and wide-ranging conversation, the Chair drew the meeting to a close, with enthusiastic thanks to the Irish Mathematical Society and, particularly Steve Buckley, for their excellent organization, and to Maynooth University for the friendly welcome to their beautiful campus.

12th International Vilnius Conference on Probability Theory and Mathematical Statistics and 2018 IMS Annual Meeting on Probability and Statistics

Remigijus Leipus (Vilnius University, Lithuania)

The 12th Vilnius Conference on Probability Theory and Mathematical Statistics and the 2018 IMS Annual Meeting took place in Vilnius, one of the most beautiful cities in the Old Continent and capital of Lithuania, on 2–6 July 2018. The time of the conference is linked to an important date in the history of Lithuania – on 16 February 1918, the Act of Independence declared an independent State of Lithuania.

The Vilnius Conference on Probability and Mathematical Statistics has a long and successful history. The first Vilnius Conference on Probability Theory and Mathematical Statistics was organised in June 1973, becoming the first big international meeting on probability theory and mathematical statistics in the former Soviet Union. Now the conference has established itself as a must-attend, international, quadrennial event for many researchers in the field.

Many prominent probabilists and statisticians attended the conference. There were more than 180 invited talks. The Wald Lectures were delivered by Luc Devroye, the 2018 Le Cam Lecturer was Ruth Williams, the Neyman Lecturer was Peter Bühlmann and the Schramm Lecturer was Yuval Peres. Six Medallion Lectures were given at this meeting, by Jean Bertoin, Svante Janson,

Alfredas Račkauskas, the Minister of Science and Education of Lithuania Jurgita Petrauskienė, Remigijus Leipus and the Rector of Vilnius University Artūras Žukauskas at the opening ceremony.

General view of the conference.

Thomas Mikosch, Sonia Petrone, Richard Samworth and Allan Sly. The Vilnius Lecture was delivered by Lithuanian probabilist Liudas Giraitis.

The EMS co-sponsored this event, supporting young scientists from developing countries. The EMS also established the EMS Gordin Prize to honour the memory of Mikhail Gordin. It was awarded to Mateusz Kwaśnicki (Wrocław University of Science and Technology) for his outstanding contributions to the spectral analysis of Lévy processes.

The organisers of the conference were the Lithuanian Mathematical Society, Vilnius University and the IMS. The programme co-chairs were Peter Bühlmann (IMS) and Vygantas Paulauskas (Vilnius). The organising committee co-chairs were Erwin Bolthausen (IMS) and Remigijus Leipus (Vilnius). The homepage of the conference is at http://ims-vilnius2018.com.

Mikhail Lifshits, Mateusz Kwaśnicki and Pavel Exner after the 2018 EMS Gordin Prize Award.

Remigijus Leipus is professor of Statistics and Probability at the Institute of Applied Mathematics, Vilnius University, Lithuania. His research interests are in probability theory, statistics, time series, insurance and financial mathematics, financial econometrics. At present he is the President of the Lithuanian Mathematical Society.

Kinetic Equations: A French History

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1 Introduction

In this article, we aim to describe the evolution of the mathematical study of kinetic equations between 1970 and 2000, covering French activity in particular. Of course, the subject being broad, we are going to focus on certain aspects. Particular attention will be paid to the role of the "averaging lemmas" as a crucial tool in classical kinetic theory. In this context, we are also going to present the Wigner transform in quantum kinetic theory, as well as the link to classical physics, in the "semi-classical" limit $\hbar \to 0$.

Asymptotic analysis will be a central topic in the following. For this reason, we are going to trace the development from ("mesoscopic") kinetic equations, like the Boltzmann equation, to ("macroscopic") fluid equations, like the Euler or Navier-Stokes equations, when certain parameters (Strouhal and Knudsen numbers) tend to zero. In order to place things in their context, it seemed important to us to have a look back and recall the pioneers. From the decade of 2000 on, however, the subject in a way "exploded" so that many additional papers would be necessary to expose it adequately.

To situate kinetic equations in a broader perspective, it seems advisable to remark that it took humanity thousands of years to invent, with Leibniz in 1684 and Newton in 1713, differential equations, well suited for the description of movement, while the invention of partial derivatives, describing continuous media, then took only 30 years, for example with d'Alembert in 1747 and Euler in 1775. In this context, for rarefied media, the notion of a "particle density" with a speed v at a point x is due to Maxwell in 1866, who also proposed the speed distribution that today bears his name. The "kinetic" equations, dedicated to the evolution of that density, appear subsequently with Boltzmann in 1872 and Lorentz in 1905. The importance of Boltzmann's ideas may not have been recognised immediately but they have played an essential role in mathematics since the beginning of the 20th century, through Hilbert's 6^{th} problem, exposed by him at the International Congress of Mathematicians (ICM) in 1900 in Paris, as well as through several contributions made by Einstein.

However, it was not before 1970 that mathematicians, and in particular French mathematicians, developed a genuine interest in the subject. As an example, at the ICM in Nice in 1970, there was only one single talk (by Guiraud) related to this problem. One may also quote, regarding that period, some articles by specialists in mechanics, like Choquet-Bruhat and Bancel in 1973 and, of course, Cabannes in 1962, who was subsequently going to play a major role.

The subject gained growing importance in our community and in particular in France, as proved by the Fields Medals for P.-L. Lions (1994) and for Villani (2010), as well as numerous other international prizes.

This is certainly, among other reasons, due to the fact that kinetic equations appear in a broad variety of sciences: astrophysics, spaceflight (especially regarding the re-entry of vehicles into the atmosphere), interaction between fluids and particles, nuclear physics, semi-conductor technology and biology (for modelling cell evolution in immunology and chemotaxis). In addition, research in this field requires both pure and applied mathematics, with results of a geometric nature or related to harmonic analysis, probability and numerical methods. This explains why the subject is part of the expansion and globalisation of mathematical research. Many Summer (or Winter) schools have been dedicated to it, leading to diverse international collaborations, for example the GdR SPARCH (led by Raviart) and, in particular, the European network HYKE (HYperbolic and Kinetic Equations, led by N. M.), which unified the communities of "kinetic equations" and "hyperbolic conservation laws". It is certainly impossible to describe, in detail, all the mentioned topics at once, therefore we, as two active witnesses of the evolution in France and Europe, chose to focus on the years 1970–2000, with the history of "averaging lemmas" being a central thread.

2 The time of the physicists and prehistory

The kinetic equations concern quantities $f(x, v, t)$, with $f \geq 0$ representing a density (in the sense of a probability) of particles situated at a point x and time t, additionally depending on a kinetic variable v. In the initial examples, v represents the speed of the particles, so f is the density of particles at point x with speed v at time t. We call $\mathbb{R}^d_x \times \mathbb{R}^d_v$ "the phase space" in d space dimensions ($d = 3$ or $d = 2, 1$ for systems with symmetry or confinement). The speed v can be replaced by a "momentum" ξ, etc., or a length and a direction when the objects are polymers or biological cells.

The kinetic equations contain by nature a "free transport" term (= "advection term"): $\partial_t f + v \cdot \nabla_x f$, which represents the evolution of the particle density in the absence of exterior forces. In this simple case, for an initial datum $f_0 = f(t = 0)$, the solution of the equation

$$\partial_t f + v \cdot \nabla_x f = 0$$

will be

$$f(x, v, t) = f_0(x - vt, v).$$

In the presence of an external force F (gravity, electric force, etc.), we have to add to this free transport a term that corresponds to Newton's second law $\frac{d}{dt} v = F$ (mass $m = 1$). If this force comes from a potential, i.e., $F = -\nabla_x V$, we have a so-called Liouville equation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = 0\,,\ f(x, v, 0) = f_0(x, v)\,, \qquad (1)$$

which, with the Poisson bracket

$$\{H, f\} = \nabla_v H \cdot \nabla_x f - \nabla_x H \cdot \nabla_v f \qquad (2)$$

of the Hamiltonian $H(x,v)$, i.e., the energy $\frac{|v|^2}{2} + V(x)$, and the function f, can also be expressed in its symplectic form: $\partial_t f + \{H, f\} = 0$. If the force F depends on the solution f, Equation (1) becomes nonlinear, which is the case of the Vlasov equation (see Section 6).

With the kinetic equations describing intermediate regimes between the dynamics of particles and macroscopic observables, certain parameters (measuring, for instance, the rarefaction of the environment or the time scale) appear naturally. In particular, the Strouhal number, St, gives the timescales, while the Knudsen number, Kn, describes the density of the medium (also called the "mean free path"). Finally, a mathematical analysis leads to the introduction of a small reference parameter, denoted by ϵ, to be compared to the other parameters.

We consider three equations in order to illustrate this analysis. We consider the Boltzmann equation

$$\mathrm{St}\partial_t f + v \cdot \nabla_x f = \frac{1}{\epsilon} C(f,f), \tag{3}$$

the Lorentz equation

$$\epsilon\partial_t f + v \cdot \nabla_x f = -\frac{1}{\epsilon}\int_{\mathbb{R}^d} k(x,v,w)(f(t,x,v) - f(t,x,w))\, d\mu(w) \tag{4}$$

and, with S^2 being the unit sphere of $\mathbb{R}^3$, an equation for a simplified model of the transport:

$$\epsilon\partial_t f + \omega \cdot \nabla_x f = -\frac{1}{\epsilon}\Big(f - \int_{S^2} f(x,\omega',t)d\omega'\Big). \tag{5}$$

In the Boltzmann equation (3), $C(f,f)$ represents the changes in particle speed due to elastic binary collisions, i.e., conserving the mass, linear momentum and the energy of the two colliding particles. In the scope of this article, we omit details of its structure; it is sufficient to keep in mind that one has:

$$\int_{\mathbb{R}^d_v} C(f,f)dv = 0\,, \quad \int_{\mathbb{R}^d_v} C(f,f)v dv = 0, \quad \int_{\mathbb{R}^d_v} C(f,f)\frac{|v|^2}{2}dv = 0, \tag{6}$$

as well as the decrease of the "entropy production", which vanishes for a Maxwellian distribution. It follows that

$$\int_{\mathbb{R}^d_v} C(f,f)\log f dv \le 0 \tag{7}$$

and, moreover, if

$$\int_{\mathbb{R}^d_v} C(f,f)\log f dv = 0$$

then

$$f(x,v,t) = M_{\rho,u,\theta}(x,v,t) = \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} e^{-\frac{|v-u|^2}{2\theta}}, \tag{8}$$

where $\rho(x,t)$, $u(x,t)$ and $\theta(x,t)$ are the macroscopic densities characterising the Maxwellian distribution.

The relation (7) yields, in particular, the decrease of entropy

$$\frac{d}{dt}\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} f(x,v,t)\log f(x,v,t)dxdv \le 0,$$

which is the famous H-theorem, a subject of controversies in Boltzmann's time as it seemed to be in apparent contradiction with Poincaré's recurrence principle. One can say that, at present, this paradox has been solved. The Lorentz equation (4) describes a situation in which the dominant process is the interaction of particles with an environment while the interaction between the particles is neglected. This explains why the equation is linear. It was introduced by Lorentz in 1905 for the evolution of electrons between atoms. Subsequently, it has played an essential role in the study of the interaction between neutrons and atomic nuclei. Here, $k(x,v,w)$ represents a positive and symmetric nucleus while $d\mu(w)$ is a probability on $\mathbb{R}^d_v$. The parameter ϵ is introduced to validate macroscopic approximations.

The "simplified transport model" (5) is an adaptation of the Lorentz equation (4) and corresponds to a measure $d\mu$ only supported on the unit sphere, so the absolute value of the speed is not affected by the interaction.

Concerning the Boltzmann equation, one obtains from the entropy balance that

$$\frac{d}{dt}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(x,v,t)\log f_\epsilon(x,v,t)dxdv + \frac{1}{\epsilon}\frac{d}{dt}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(x,v,t)C(f_\epsilon,f_\epsilon)\log f_\epsilon(x,v,t)dxdv = 0.$$

From Equation (8), one also infers that, for $\epsilon \to 0$, any adherence value of the family f_ϵ is a local Maxwellian, i.e., a Gaussian M given by (8). By inserting this expression into the Boltzmann equation and by using Equation (6) of momentum conservation, one additionally deduces that the macroscopic parameters are the solution of the Euler equations of compressible fluids:

$$\begin{aligned}
&\mathrm{St}\partial_t\rho + \nabla_x\cdot(\rho u) = 0,\\
&\mathrm{St}\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u + \rho\theta) = 0,\\
&\mathrm{St}\partial_t\Big(\rho\frac{|u|^2}{2} + \frac{d}{2}\rho\theta\Big) + \nabla_x\cdot\Big(u\Big(\rho\frac{|u|^2}{2} + \frac{d+2}{2}\rho\theta\Big)\Big) = 0.
\end{aligned} \tag{9}$$

This derivation (in a more modern form, of course) motivated Hilbert when he announced his 6th problem at the ICM in Paris in 1900:

> Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua.

On the other hand, we observe that Equations (9) expresses neither the viscosity nor the thermal diffusivity of the fluid (fundamental quantities in fluid mechanics). To solve the problem, Hilbert suggests in 1916 to express these quantities in the second term in ϵ of a formal development of the function f_ϵ. Independently, Chapman in 1916 and Enskog in 1917 established a more direct connection between the Boltzmann equation and the macroscopic equations. Rather than considering the first two terms of the development in ϵ, they introduce a local Maxwellian (8), which depends on the parameter

$$M_\epsilon(x,v,t) = \frac{\rho_\epsilon}{(2\pi\theta_\epsilon)^{\frac{d}{2}}} e^{-\frac{|v-u_\epsilon|^2}{2\theta_\epsilon}}, \tag{10}$$

and they prove that, for M_ϵ to be a solution of the Boltzmann equation up to an ϵ^2-order term, it is necessary and sufficient that $\rho_\epsilon, u_\epsilon, \theta_\epsilon$ are solutions of the "macroscopic" Navier-

Stokes equations

$$\mathrm{St}\partial_t\rho_\epsilon + \nabla_x(\rho_\epsilon u_\epsilon) = 0\,,$$

$$\begin{aligned}&\mathrm{St}\partial_t(\rho_\epsilon u_\epsilon) + \nabla_x(\rho_\epsilon u_\epsilon \otimes u_\epsilon + \rho_\epsilon\theta_\epsilon)\\ &\qquad = \epsilon\nabla_x \cdot \Big\{\nu(\theta_\epsilon)(\sigma(u_\epsilon) - \frac{2}{d}\nabla_x\cdot\sigma(u_\epsilon)\mathrm{I})\Big\},\end{aligned} \tag{11}$$

$$\begin{aligned}&\mathrm{St}\partial_t\Big(\rho_\epsilon\frac{|u_\epsilon|^2}{2} + \frac{d}{2}\rho_\epsilon\theta_\epsilon\Big) + \nabla_x\Big(u_\epsilon\Big(\rho\frac{|u_\epsilon|^2}{2} + \frac{d+2}{2}\rho_\epsilon\theta_\epsilon\Big)\Big)\\ &\qquad = \frac{\epsilon\nu(\theta_\epsilon)}{2}\sigma(u_\epsilon) : \sigma(u_\epsilon) + \epsilon\nabla_x\cdot[\kappa(\theta_\epsilon)\nabla_x\theta_\epsilon],\end{aligned}$$

with

$$\sigma(u_\epsilon) = \frac{\nabla u_\epsilon + \nabla^t u_\epsilon}{2}, \quad \sigma(u_\epsilon) : \sigma(u_\epsilon) = \mathrm{Trace}\,\sigma(u_\epsilon)\otimes\sigma(u_\epsilon).$$

Thus, the viscosity $\epsilon\nu(\theta_\epsilon)$ and the thermal diffusivity $\epsilon\kappa(\theta_\epsilon)$, two macroscopic quantities that in Boltzmann's or Maxwell's time were already experimentally measurable, are, in the above derivations, quantities that are proportional to the Knudsen number. They depend on the macroscopic temperature according to the power laws, which can be intuited (and even computed) from the interaction between the molecules. The accordance between the experimentally obtained measurements and the formulas deduced from the above calculation allowed the confirmation of the hypotheses on the dynamics of those molecules.

Some of the above calculations are very formal: as all these equations are nonlinear, studying them rigorously was only possible with the modern tools of functional analysis. In particular, the Euler equations support singular solutions (with discontinuities) and under natural conditions of compression, regular solutions become singular at finite time. They are good examples of notions of derivatives being used in the sense of distributions and, hence, the solutions are weak. We observe that the relation of conservation of the macroscopic entropy can be derived from these equations:

$$\partial_t\Big(\rho\log\frac{\rho^{\frac{2}{3}}}{\theta}\Big) + \nabla_x\cdot\Big(\rho u\log\frac{\rho^{\frac{2}{3}}}{\theta}\Big) = 0\,. \tag{12}$$

However, this type of computation is no longer valid in the presence of discontinuities. One can also demonstrate that, if the weak solutions of the compressible Euler equations are weak limits, in a convenient sense, of solutions to Navier-Stokes equations or moments of solutions to the Boltzmann equation, they only satisfy Relation (12) in the sense of an inequality (it turns out that the left side is negative or zero). In one space dimension, this constraint ensures the uniqueness and stability of the corresponding solutions. In more than one dimension, however, the recent works of Chiodaroli, De Lellis and Kreml [8], following up on the results obtained by De Lellis and Szkelyhidi, have proven a total instability (even an infinite number of entropic solutions) for the Euler equations.

Extracting what can be rigorously proven (considering the available tools) is the aim of the research activity of the years 1970–2000, which mainly involves incompressible solutions of Navier-Stokes and fluctuations of renormalisable solutions (in the sense of DiPerna–Lions) of the Boltzmann equation. This is a central object of this article (see Section 4).

3 The CEA and the transport equation

As mentioned above, after Hilbert and during the interwar period, we hardly find any research on this subject led by French mathematicians. They seemed to have forgotten, for example, Poincaré as well as Hilbert's 6th problem. On the other hand, there is at least a physicist, Jacques Yvon, who had already formulated problems in a new language, the interest of which would be understood only many years later by the mathematicians. In 1936 (see [23]), he introduced, in order to examine a gas of N molecules, in addition to the densities of $n \leq N$ particles $f_n^N(x_1, x_2, \ldots, x_n, v_1, v_2, \ldots, v_n)$, the joint probability to have the first particle at position x_1 with speed v_1 , the second at position x_2 with speed v_2, etc., and he made explicit the relations between those densities. Every density f_n^N is, for $n < N$, a solution of an equation evolving in its second term the density f_{n+1}^N. And, of course, f_N^N coincides with the solution of the Liouville equation determined by the initial system. He demonstrated that the solutions of the Boltzmann equation produce, by factorisation, approximate solutions of this system of equations. He also invented a hierarchy of equations that would be rediscovered 11 years later by Kirkwood, Born and Green, as well as by Bogoliubov, and would therefore be named BBGKY hierarchy. Besides its very concrete interpretation in physics, this hierarchy was going to play an important role in mathematical proofs, starting with the works of Grad in 1949 and Lanford in 1974 on the rigorous derivation of the Boltzmann equation.

The "Commissariat à l'Energie Atomique" (CEA) was founded in 1945 and Yvon entered it in 1946, first as a collaborator, secondly as an "external member" in 1949 and finally becoming high-commissioner from 1970 to 1975.

On the transport equation of neutrons, he wrote: "I soon understood (1946) that Boltzmann's integro-differential equation, slightly modified, would serve as an arsenal for the new mathematical physicists." As mentioned above, it is, in fact, an application of the Lorentz equation to interactions (absorption and re-emission) of neutrons with surrounding atoms. This equation, which of course does not have an explicit solution, forms part of the challenges that the CEA would face with the help of computers and more mathematics. It is the programme of Amouyal and Horowitz, defined under the influence of Yvon and within this framework, where Dautray was to become, in 1955, part of the CEA's mathematical physics group. That's where J.-L. Lions met him as an external collaborator introduced by Lattes.

Thanks to the participation and financial support of the CEA and industry partners like EDF and Dassault, and with the collaboration of some former students of J.-L. Lions, this group played a major role in the expansion of applied mathematics in the 1950s, where kinetic equations were important, initially through the organisation of Summer schools like the CEA-EDF-INRIA at Bréau Castle (following the example of the physics schools of Houches and, later on, Cargèse), which are very popular in our community.

His position as a university professor allowed J.-L. Lions to invite some of the leading mathematicians in the domain of kinetic equations to extended stays in France: Nishida to Orsay in 1974, Ukai to Orsay and Paris 13 in 1977, Nicolaenko to Orsay in 1977 and Papanicolaou to the INRIA and the Summer school at Bréau Castle in 1978.

Finally, there is an important publication that should be explicitly highlighted. Dautray took the initiative to coordinate, together with J.-L. Lions, the elaboration of a treatise in applied mathematics "Analyse Mathématique et Calcul Numérique pour les Sciences et la Technologie". The first of the 9 volumes was published in 1984 by Masson, Paris, and there were subsequently various versions and translations, in particular one published by Springer in English.

The initiative was inspired by the work of the Bourbaki group on one hand and Courant-Hilbert, a seminal book of mathematical physics, on the other. J.-L. Lions and Dautray followed the structure of this reference book regarding its subjects, its volumes and chapters but they delegated, even more than Courant–Hilbert (where the contributions of Friedrichs, John, Lax, Nirenberg and others are mentioned), the writing of chapters and whole volumes of the work to younger researchers, quoting them entirely. Following Bourbaki's tradition in optimising editing, Dautray gathered the writers and made the different contributions circulate for reciprocal reviews.

At the time, he arranged convenient conditions for such an initiative. He had ensured the collaboration of various CEA researchers, among them Sentis and Kavenoky. He had Cessenat, a CEA engineer, working full-time on the project and could count on permanent consultants such as P.-L. Lions and Perthame. He had also ensured the collaboration of scientists doing their military service (which was still mandatory at the time). Among such scientists doing their military service were, in 1985, some of our (then juvenile) colleagues like Julia and Golse.

Diffusion approximation

Dautray and J.-L. Lions entrusted the redaction of a volume on the transport equation to C. B. In particular, Dautray insisted on the rigorous formulation of the diffusion approximation.

The evolution of neutrons in the presence of reactive nuclei (uranium or plutonium) is described by a kinetic equation of Lorentz type (4). However, as with every kinetic equation, it depends on the $2d$ variables of the phase space and on the time variable. A direct calculation is thus not possible. Also in the 1940s-60s, Metropolis and Ulam at Los Alamos, Khasminski in Russia and Benoist at the CEA used, in order to calculate the "macroscopic" density of particles

$$\rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)dv \tag{13}$$

at point x and time t, an approximation called diffusion approximation, which is defined by the solution of the equation:

$$\partial_t \rho - \kappa\Delta\rho = c\rho\,. \tag{14}$$

This approach (valid near the critical regime) was based on a scale estimate together with a probabilistic interpretation. This is, by the way, the procedure on which is based the so-called Monte Carlo method for calculating integrals and solutions of equations with partial derivatives. While the method was well explained from the angle of physics in the book by Weinberg and Wigner in 1958 [22], one was far from having a precise mathematical formulation allowing its justification.

In 1974, Larsen and Keller were inspired by the role of the Knudsen and the Strouhal number in the Boltzmann equation to produce a direct proof based on functional analysis. Following Dautray's suggestion, this proof was reproduced in the book. For better understanding, we also reproduce it here, in the framework of the following simplified model

$$\partial_t f_\epsilon + \frac{1}{\epsilon}\omega\cdot\nabla_x f_\epsilon + \frac{1}{\epsilon^2}\Big(f_\epsilon - \frac{1}{4\pi}\int_{\mathbb{S}^2} f_\epsilon(x,\omega')d\omega'\Big) = c(x) f_\epsilon, \tag{15}$$

where $\omega \in \mathbb{S}^2$, $x \in \Omega \subset \mathbb{R}^3$ and $x \mapsto c(x)$ is a function depending only on x, positive or negative. Of course, it is convenient to add conditions on the limits. In neutron physics, the standard boundary condition is the absorbing condition (any particle that leaves the environment is absorbed by the surroundings and thus does not return). By introducing the external normal $\vec{n}$ to the boundary $\partial\Omega$, this can be expressed by the following:

$$\forall (x,\omega) \in \partial\Omega\times\mathbb{S}^2, \quad \omega\cdot\vec{n}(x) < 0 \Rightarrow f_\epsilon(x,\omega,t) = 0\,. \tag{16}$$

Applying the Green formula, multiplication by f_ϵ and integration on $\partial\Omega\times\mathbb{S}^2$ leads to the inequality

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_{\Omega\times\mathbb{S}^2} |f_\epsilon(x,\omega,t)|^2 d\omega dx \\ &\quad + \frac{1}{\epsilon^2}\int_\Omega dx \int_{\mathbb{S}^2}\Big(f_\epsilon - \frac{1}{4\pi}\int_{\mathbb{S}^2} f_\epsilon(x,\omega)d\omega\Big)^2 d\omega \\ &\qquad \le \int_{\Omega\times\mathbb{S}^2} c(x)|f_\epsilon(x,\omega,t)|^2 d\omega dx\,. \end{aligned} \tag{17}$$

It turns out that the solutions are, for $0 \le t \le T$, uniformly bounded with respect to ϵ in $L^2(\Omega\times\mathbb{S}^2)$ and that, for $\epsilon \to 0$, any value of adherence of the sequence f_ϵ in L^2 is a function $\rho(x,t)$ that is independent of ω, as will be the case for other weak limits $\overline{f_\epsilon}$. The integration of the relation (15) with respect to ω yields the conservation law:

$$\begin{aligned} \partial_t \frac{1}{4\pi}\int_{\mathbb{S}^2} f_\epsilon(x,\omega)d\omega + \nabla_x\cdot\Big(\frac{1}{4\pi\epsilon}\int_{\mathbb{S}^2}\omega f_\epsilon(x,\omega)d\omega\Big) \\ = c(x)\frac{1}{4\pi}\int_{\mathbb{S}^2} f_\epsilon(x,\omega)d\omega\,. \end{aligned} \tag{18}$$

With $\int_{\mathbb{S}^2}\omega f_\epsilon(x,\omega)d\omega$ converging weakly to 0, it is convenient (as has already been mentioned) to raise the indetermination in the second term of the first member of (18). To do this, we multiply the equation (15) a second time by $\epsilon\omega$ and obtain, after integration:

$$\begin{aligned} &\frac{1}{4\pi\epsilon}\int_{\mathbb{S}^2}\omega f_\epsilon(x,\omega)d\omega \\ &\quad = -\nabla_x\frac{1}{4\pi}\int_{\mathbb{S}^2}\omega\otimes\omega f_\epsilon(x,\omega,t)d\omega \\ &\qquad + \epsilon\frac{1}{4\pi}\int_{\mathbb{S}^2}\omega(c(x)f_\epsilon(x,\omega,t) \\ &\qquad - \epsilon\partial_t f_\epsilon(x,\omega,t)d\omega \\ &\to \nabla_x\frac{1}{4\pi}\int_{\mathbb{S}^2}\omega\otimes\omega d\omega : \nabla_x\overline{f(x,t)} = -\frac{1}{3}\nabla_x\rho(x,t)\,. \end{aligned} \tag{19}$$

By putting this into equation (18) and taking the limit (it is a linear problem and at this point the nature of the convergence does not need to be given), we obtain:

$$\partial_t\rho(x,t) - \frac{1}{3}\Delta\rho(x,t) = c(x)\rho(x,t)\,. \tag{20}$$

Now, the problem with the boundary condition on $\partial\Omega$ has to be solved. Taking (16) into account, the Dirichlet condition

$$x \in \partial\Omega \Rightarrow \rho(x,t) = 0$$

seems to be the most natural. However, in the 1950s, physicists (see page 198 of [22]) observed that the approximation was much better if one replaces the Dirichlet condition by a condition of the Robin type:

$$x \in \partial\Omega \Rightarrow \rho(x,t) + \lambda\partial_{\vec{n}}\rho(x,t) = 0\,.$$

The term λ with the dimension of length is called extrapolation length. Its evaluation is inspired by the observation of stellar radiation.

While writing the Dautray-Lions volume, with the scales that Larsen and Keller proposed, the direct (quantitative) demonstration for the calculation of this λ was found, by analysing the transport problem in a half-space named Milne space (after the astrophysician).

It turns out that this problem appears in an analogous manner in the relation between the Boltzmann equation and the compressible Navier-Stokes equation (11). The adaptation of the results obtained in Milne space to the Boltzmann equation was considered in 1986 in an article by Bardos, Caflish and Nicolaenko [1]. Also, this type of research was applied in the space shuttle project HERMES, planned in 1975 but then abandoned in 1992. Being a European project, it required regular collaborations between industry and university researchers, amongst them, in particular, (together with the French scientists) Neunzert from Kaiserlautern and Cercignani from the Politecnico de Milan (who had worked continuously and very successfully on the Boltzmann equation since 1962).

This collaboration also continued beyond the European borders, with Desphande (Indian Institute of Science, Bangalore), for example, and especially with Sone and his group at the Laboratory of Aeronautical Engineering in Kyoto.

Approximation of the critical size

In a volume dedicated to the transport equation and edited by the CEA, it was natural to evoke the critical size problem, which, in the kinetic regime, is expressed by the principal eigenvalue of the (unbounded) operator, defined on $L^2(\Omega\times\mathbb{R}^d_v)$ by

$$T(f) = f \mapsto -v\cdot\nabla_x f + \mathcal{L}f, \qquad (21)$$

with a convenient boundary condition (for example, the absorbing one), while $\mathcal{L}$ is a linear operator that acts on the variables v and represents the effects of the environment on the particles (absorption and re-emission). The Lorentz equations (4) and their simplification (5) give the most significant prototypes. The spectral analysis of the operator $f \mapsto T(f)$ is not simple because it is neither selfadjoint nor anti-selfadjoint and its spectrum may contain, at the same time, a continuous spectrum and eigenvalues with finite multiplicity.

Nevertheless, mathematicians like Albertoni- Montagnini in 1966 and Ghidouche-Point-Ukai in 1976 have demonstrated the existence of a real and simple principal eigenvalue. It is thus natural to expect that the eigenvalue obtained by the diffusion approximation would deliver a "good approximation" of the principal eigenvalue Λ_ϵ of the transport operator and therefore contribute to determining the critical character of the material.

In the case of the simplified model (5), using the scale change by Larsen and Keller, we are led to consider the pair $(\Lambda_\epsilon, f_\epsilon(x,v)\geq 0)$ as a solution of the equation (with absorbing boundary conditions):

$$-\frac{1}{\epsilon}\omega\cdot\nabla_x f_\epsilon - \frac{1}{\epsilon^2}\Big(f_\epsilon - \frac{1}{4\pi}\int_{\mathbb{S}^2} f_\epsilon(x,\omega')d\omega'\Big) + c(x)f_\epsilon = \Lambda_\epsilon f_\epsilon,$$
$$f_\epsilon(x,v)\geq 0\,,\quad \int_{\Omega\times\mathbb{S}^2} |f_\epsilon(x,\omega)|^2 dx d\omega = 1\,. \qquad (22)$$

In a paper [18], which was part of his "Thèse d'Etat" defended in 1981, Sentis proved that the pair $(\Lambda_\epsilon, f_\epsilon)$ converges to (Λ, u), a solution of the diffusion equation with Dirichlet boundary conditions:

$$\frac{1}{3}\Delta u + c(x)u = \Lambda u, \quad u = 0 \text{ on } \partial\Omega\,. \qquad (23)$$

The aim was also to demonstrate that, with the introduction of the extrapolation length λ given by Milne's problem, the principal eigenvalue corresponding to the same operator with the Robin condition

$$u_\epsilon + \epsilon\lambda\partial_{\vec{n}}u_\epsilon = 0 \quad \text{on} \quad \partial\Omega \qquad (24)$$

should give an approximation of higher order.

These results were then included in the last chapter of Dautray-Lions' book on transport. But, meanwhile, toward the end of 1984, Cessenat, entrusted by Dautray with the final proofreading of the different contributions, had discovered a "gap" in the proof. In the proof of the main result [18], it was not really established that $\int_{\mathbb{S}^2} f_\epsilon(x,\omega)d\omega$ converges to a non-zero function. He asked Sentis to solve the problem urgently (as the proofs needed to be sent to the editor). Sentis asked Golse, by that time working with him in the CEA as part of his military duties, and Perthame, also by that time a consultant at the CEA and sharing his office, to help him correct the proof.

It is clear that if $f\in L^2(\Omega\times\mathbb{S}^2)$ and $\omega\cdot\nabla_x f\in L^2(\Omega\times\mathbb{S}^2)$ then the function $x\mapsto f(x,\omega)$ possesses a supplementary regularity in the direction ω but that does not help further. In fact, they proved in a note to the CRAS, published at the beginning of 1985, that if the functions f are bounded in $L^2(\Omega\times\mathbb{S}^2)$ and are such that $\|\omega\cdot\nabla_x f\|_{L^2(\Omega\times\mathbb{S}^2)}$ are bounded then, for every function $\phi\in L^\infty(\mathbb{S}^2)$, the averages $\int_{\mathbb{S}^2} f(x,\omega)\phi(\omega)d\omega$ form a relatively compact set in $L^2(\Omega)$. That lemma allowed the correction of the proof of the announced result. Later on, Golse, Perthame and Sentis showed, together with P.-L. Lions, that these averages belong to the Sobolev space $H_x^{\frac{1}{2}}$, that is, one gains half a notch of regularity in x. More precisely, as an example, we have [12]:

Theorem 3.1. *For all test functions $\phi\in L^\infty(\mathbb{R}^d_v)$ with compact support, there exists a constant $C(\phi)$ such that if $f\in L^2(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^d_v)$ and*

$$\epsilon\partial_t f + v\cdot\nabla_x f = h \in L^2(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^d_v)$$

then

$$\Big\|\int f(x,v,t)\phi(v)dv\Big\|_{L^2(\mathbb{R}_t,H^{\frac{1}{2}}(\mathbb{R}^d_x))} \leq C(\phi)\|h\|_{L^2(\mathbb{R}_t\times\mathbb{R}^d_x\times\mathbb{R}^d_v)}.$$

Proof. We denote by $\hat{f}(\xi, v, \tau)$ the Fourier transform with respect to x, t of the function $f(x, v, t)$ and similarly for the function $\hat{h}$. We would like to bound

$$|\xi| \left| \int \hat{f}(\xi, v, \tau)\phi(v)dv \right|^2.$$

With this aim, we introduce the parameter α and decompose the integral into two parts respectively to the sign of $|\epsilon\tau + \xi \cdot v| - \alpha$. We bound

$$\left| \int 1_{|\epsilon\tau+\xi\cdot v|\le\alpha} \hat{f}(\xi, v, \tau)\phi(v)dv \right|^2$$

by

$$\Big(\int |\hat{f}(\xi, v, \tau)|^2 dv \Big)\Big(\int 1_{|\epsilon\tau+\xi\cdot v|\le\alpha} |\phi(v)|^2 dv \Big).$$

Next, we note that $|(\epsilon\tau + \xi \cdot v)\hat{f}(\xi, v, \tau)| = |\hat{h}(\xi, v, \tau)|$ to bound

$$\left| \int 1_{|\epsilon\tau+\xi\cdot v|>\alpha} \hat{f}(\xi, v, \tau)\phi(v)dv \right|^2$$

by

$$\Big(\int |\hat{h}(\xi, v, \tau)|^2 dv \Big)\Big(\int \frac{1_{|\epsilon\tau+\xi\cdot v|>\alpha}}{|\epsilon\tau + \xi \cdot v|^2} |\phi(v)|^2 dv \Big).$$

Both integrals depending on ϕ can directly be estimated. We conclude the proof with an optimal choice of α. □

Dautray later discovered that Theorem 3.1 had a forerunner. In 1984, Agoshkov established a result of that type. He used it for trace theorems that were very useful in numerical analysis. The broader significance of his formulation, however, he seemed to overlook.

4 Application of averaging lemmas

One of the challenges of the 1980s was the proof of the existence of solutions (possibly in a weak sense) for the Boltzmann equation for any natural initial datum, using only mass and energy conservation:

$$\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} \Big(1 + \frac{|v|^2}{2}\Big) f(x, v, t)dxdv = \int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} \Big(1 + \frac{|v|^2}{2}\Big) f(x, v, 0)dxdv,$$

as well as the decrease of entropy:

$$\int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} f(x, v, t) \log f(x, v, t)dxdv \le \int_{\mathbb{R}^d_x\times\mathbb{R}^d_v} f(x, v, 0) \log f(x, 0)dxdv,$$

leading to the idea of adapting Theorem 3.1 to other functions spaces. So, in 1988, Golse, P.-L. Lions, Perthame and Sentis [12] demonstrated by interpolation that the relations

$$f(x, v) \quad \text{and} \quad v \cdot \nabla_x f \in L^p(\mathbb{R}^d_x \times \mathbb{R}^d_v) \tag{25}$$

yield, for $1 < p < \infty$ (with $\phi \in L^\infty(\mathbb{R}^3_v)$ of compact support) and for $0 < s < \inf(1/p, 1 - 1/p)$, the estimate

$$\Big\| \int_{\mathbb{R}^d_v} f(x, v)\phi(v)dv \Big\|_{W^{s,p}} \le C(\phi)\|f\|^{1-s}_{L^p(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \|v \cdot \nabla_x f\|^s_{L^p(\mathbb{R}^d_x\times\mathbb{R}^d_v)}.$$

Using a basic example, they observed that this estimate does not extend to the case $p = 1$. In an attempt to overcome that obstacle, DiPerna and P.-L. Lions used, together with the averaging lemma, the a priori estimates of energy and especially entropy. This is how they came to prove the existence of solutions for the Boltzmann equation (in a relatively weak sense) called renormalised but global in time and depending only on the natural properties of the initial data (see [6]).

The analogy between this proof and the one given by Leray for the Navier-Stokes equations is striking, both for the results and the methods. It therefore becomes intuitive to have, at the macroscopic limit of the Boltzmann equation, "turbulent" Leray solutions of incompressible Navier-Stokes equations ($\nabla \cdot u = 0$) with strictly positive viscosity.

In 1991, Bayly, Levermore and Passot [3] observed that, if we introduce into the equations (11) a speed as well as fluctuations of density and temperature of order ϵ : $(u, \rho, \theta) = (\epsilon\tilde{u}, 1 + \epsilon\tilde{\rho}, 1 + \epsilon\tilde{\theta})$, we obtain formally, within the limit $\epsilon \to 0$, the incompressible Navier-Stokes equations with, in particular, strictly positive viscosity ν^* and thermal diffusiveness κ^*.

In order to link the solutions found by DiPerna–Lions to those given by Leray, it is convenient to consider functions

$$f_\epsilon = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}} (1 + \epsilon g_\epsilon(x, v, t)), \tag{26}$$

solutions of re-scaled Boltzmann equations:

$$\epsilon\partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon = \frac{1}{\epsilon} C(f_\epsilon). \tag{27}$$

This approach was suggested in 1990-91, firstly for the formal calculations in the stationary regime by Sone, then for regular regimes and finite times by Marra, Esposito and Lebowitz and finally by Bardos, Golse and Levermore (see [2]), inspired by the derivation of the diffusion approximation for Equation (15) by the method of moments. This calculation was, at the time, formal and it was justified only for very regular and small initial data in relation to the viscosity by Bardos and Ukai in 1991.

The final goal, which was the proof of convergence for any time and any "natural" initial data, was the object of various contributions and it was only completely achieved in 2004 by Golse and Saint-Raymond [13].

In [13], the authors use, in addition to the deduced estimates of energy and entropy, a control of the acoustic waves (given by P.-L. Lions and Masmoudi [14]), as well as a refined version of the averaging lemmas. They demonstrate that, for a family of solutions of the free transport equation, the properties of equi-integrability in the variables v can be transposed into equi-integrability in the variables (x, v) in a way that, with the Dunford-Pettis theorem, obtains strong convergence. This results from the following proposition.

Proposition 4.1. *For all $1 \le p \le q \le \infty$, every solution f of the equation*

$$\partial_t f + v \cdot \nabla_x f = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^d_x \times \mathbb{R}^d_v \tag{28}$$

satisfies the estimate

$$\|f(t)\|_{L^q_x(L^p_v)} \le |t|^{-d(\frac{1}{p}-\frac{1}{q})} \|f(0)\|_{L^\infty_x(L^p_v)}. \tag{29}$$

5 Averaging lemmas and the Wigner transform

In Proposition 4.1, the reader will note an astonishing similarity with the properties of regularisation of the free Schrödinger equation (and its Strichartz inequalities) that are obtained by representing, for $t > 0$, the solution of the free Schrödinger equation in $\mathbb{R}^d$

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = 0$$

by the formula

$$\psi(x,t) = \frac{e^{-i\frac{d\pi}{4}}}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2it}} \psi_0(y)dy, \tag{30}$$

from which can be deduced, on the one hand, the regularising effect (for example, for every distribution with compact support, the function $\psi(x,t)$ is analytic) and on the other hand, the effect of dispersion:

$$\|\psi(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |\psi_0(y)|dy. \tag{31}$$

Taking into account the relation

$$\frac{d}{dt}\|\psi(\cdot,t)\|_{L^2(\mathbb{R}^d)} = 0, \tag{32}$$

one can deduce the classical Strichartz estimates.

The relation between the dispersion effect for the transport equation and the Schrödinger equation was described in detail for the first time by Castella and Perthame [5]. It is naturally explained by considering the Wigner transform $w(x,v,t)$ [11], which allows the definition of a "quantum kinetic theory" in the "phase space" $(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ by introducing the Wigner function:

$$w(x,v,t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d_y} \psi\Big(x + \frac{y}{2}\Big)\psi^*\Big(x - \frac{y}{2}\Big)e^{-iv\cdot y}dy\,, \tag{33}$$

which transforms the complex-valued wave function into a real-valued function on the phase space, with the inconvenience that w may be negative. It is not a probability distribution in the strict sense like f in the classical kinetic equations. Apart from that, the Wigner function has all the good properties, with, for example, the moments in v giving macroscopic densities like densities of "position"

$$\int_{\mathbb{R}^d_v} w(x,v,t)dv = \big|\psi(x,t)\big|^2 = \rho(x,t)$$

and "current"

$$\int_{\mathbb{R}^d_v} vw(x,v,t)dv = \Im\mathrm{m}(\nabla\psi(x,t)\psi^*(x,t)) = J(x,t). \tag{34}$$

The free Schrödinger equation (30) transforms directly into the free transport equation

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = 0 \quad \Rightarrow \quad \partial_t w(x,v,t) + v\cdot\nabla_x w(x,v,t) = 0,$$

from which we deduce again (32), noting that

$$\begin{aligned} |\psi(x,t)|^2 &= \int_{\mathbb{R}^d_v} w(x,v,t)dv \\ &= \int_{\mathbb{R}^d_v} w_0(x - vt, v, t)dv. \end{aligned}$$

In the same way, in the papers by Perthame, Gasser and Markowich (see [17]), there is a systematisation from the kinetic point of view, in order to find the dispersion estimates, as well as a generalisation of that method for other PDEs.

More generally, with a real potential V and Planck constant $\hbar$ (the limit $\hbar \to 0$ of which represents the "(semi) classical" limit), $H_\hbar\psi = -\frac{\hbar^2}{2}\Delta\psi + V\psi$ defines a Hamiltonian operator (unbounded but selfadjoint) on the space $L^2(\mathbb{R}^d)$. Then, the solution of the Schrödinger equation

$$i\hbar\partial_t\psi_\hbar = -\frac{\hbar^2}{2}\Delta\psi_\hbar + V\psi_\hbar \tag{35}$$

is given by $\psi_0 \mapsto \psi_\hbar(t) = e^{-i\frac{t}{\hbar}H_\hbar}\psi_0$, with $e^{-itH_\hbar}$ being a unitary group.

Now, the time evolution of the Wigner function is given by the Wigner equation consisting of the classical free transport operator and a pseudodifferential operator in V (clearly non-local due to the Fourier transform in definition (33)):

$$\partial_t w_\hbar(x,v,t) + v\cdot\nabla_x w_\hbar(x,v,t) - [\Theta(V_\hbar)w_\hbar](x,v,t) = 0,$$

where $\Theta(V_\hbar)w_\hbar](x,v,t)$ is given by the integral in (y,v') of the product of

$$\frac{1}{(2\pi)^d}\frac{V(x+\frac{\hbar y}{2}) - V(x-\frac{\hbar y}{2})}{i\hbar}e^{-iv\cdot y}$$

and

$$e^{i\frac{v'\cdot(x-y)}{\hbar}} w_\hbar\Big(\frac{x+y}{2}, v', t\Big).$$

For a sufficiently regular potential V, one has

$$[\Theta(V_\hbar)w_\hbar](x,v,t) = \nabla_x V(x)\nabla_v w_\hbar(x,v,t) + O(\hbar)\,.$$

So, we recover, at least formally, the Liouville equation, in the limit $\hbar \to 0$. It can be demonstrated that, under certain hypotheses, the "Wigner function" $w_\hbar(x,v,t)$ converges to a non-negative measure $w_0(x,v,t)$, called "the Wigner measure" by P.-L. Lions and Paul [15], and, for this $w_0(x,v,t) \geq 0$, we recover, at least formally, classical kinetic theory, i.e., the Liouville equation

$$\partial_t w_0(x,v,t) + v\cdot\nabla_x w_0(x,v,t) - \nabla_x V\cdot\nabla_v w_0(x,v,t) = 0\,. \tag{36}$$

The above derivations can be rigorously justified: the general theory for the linear case is elaborated in [11] and the special nonlinear case for the limit of the "Schrödinger–Poisson" system toward "Vlasov–Poisson" is given in [15] and [25].

We would like to close this brief presentation of "quantum kinetics" with the following comments.

(1) The Wigner function is a reformulation of the "density matrix" $K_\hbar(x,y,t)$, defined for a "pure state" ψ like $K_\hbar(x,y,t) = \psi_\hbar(x,t)\otimes\psi^*_\hbar(y,t)$. This density matrix, which is the kernel of an integral operator in $L^2(\mathbb{R}^d)$ named the "density operator" $\hat{K}$, is a key object of statistical quantum mechanics. In the general case of a "mixed state", where the system is found with a probability λ_j in the state $\psi_j(x,t)$, with $\lambda_j \geq 0$ and $\sum_{j=1}^\infty \lambda_j = 1$, one has

$$K_\hbar(x,y,t) = \sum_{j=1}^{\infty} \lambda_j(\psi_j)_\hbar(x,t)\otimes(\psi_j)^*_\hbar(y,t)\,. \tag{37}$$

While the operator $\hat{K}_\hbar$ is a solution of the Heisenberg-von Neumann equation

$$i\hbar\frac{d}{dt}\hat{K}_\hbar + [H_\hbar, \hat{K}_\hbar] = 0\,, \tag{38}$$

the relation between the density matrix $K_\hbar(x,y,t)$ and the Wigner function $w_\hbar(x,v,t)$ is given by

$$K_\hbar(x,y,t) = \int_{\mathbb{R}^d_v} e^{i\frac{v\cdot(x-y)}{\hbar}} w_\hbar\Big(\frac{x+y}{2}, v, t\Big)dv \tag{39}$$

and

$$w_\hbar(x,v,t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d_y} e^{-iv\cdot y} K_\hbar\Big(x+\frac{\hbar y}{2}, x-\frac{\hbar y}{2}, t\Big)dy.$$

With $\hbar \to 0$, the formal calculation above yields a correspondence between the evolution of $\hat{K}_\hbar$ according to the Heisenberg–von Neumann equation (38) and the evolution of $w = \lim_{\hbar\to 0} w_\hbar$ according to the (kinetic) Liouville equation:

$$\partial_t w + \{H, w\} = 0. \tag{40}$$

We see how, within the "(semi)classical limit", the commutator of the operators becomes the Poisson bracket (2) of the functions.

(2) Under the hypotheses of convenient regularity of w, we find, as part of Weyl calculus, the theorem obtained by Egorov in 1970 [9], using the integral Fourier operators introduced by Hörmander. This theorem precisely states that, in this setting, the operator $e^{-i\frac{t}{\hbar}\hat{H}_\hbar}K_\hbar(0)e^{i\frac{t}{\hbar}\hat{H}_\hbar}$ is, up to a computable rest, described by Formula (39), with w being the solution of Equation (40).

The main lines of the computation above remain formally valid in the nonlinear case, where the potential V depends on the solution as in the Heisenberg–von Neumann–Poisson case:

$$i\frac{d}{dt}\hat{K}_\hbar + \Big[-\frac{\hbar^2}{2} + V_\hbar, \hat{K}_\hbar\Big] = 0\,,$$
$$V_\hbar(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3_x}\frac{1}{|x-y|}K_h(y,y)dy,$$

that is,

$$-\Delta V_\hbar = K_h(x,x) = \mathrm{Trace}\,(\hat{\mathrm{K}}_\hbar).$$

We thus obtain, in the limit $\hbar \to 0$, the Vlasov–Poisson equation

$$\partial_t w + w\cdot\nabla_x w - \nabla_x V\cdot\nabla_v w = 0,$$
$$\rho(x,v,t) = \int_{\mathbb{R}^3_v} w(x,v,t)dv,$$
$$-\,\Delta V(x,t) = \rho(x,t).$$

Because of the problem's nonlinearity, the justification, in the existing proofs of taking the limit $\hbar \to 0$, requires that the Wigner transform of the initial datum $K_h(0)$ is uniformly bounded in L^2 . In one space dimension, this was realised for weak (non-unique) solutions of Vlasov-Poisson with measure valued initial datum by Zheng, Zhang and Mauser [25]. But, in higher dimensions, as stated in the papers by P. L. Lions-Paul and Markowich–Mauser, it seems unavoidable to consider mixed states (37), with a very restrictive condition on the λ_j, which must depend on the Planck constant $\hbar$ in a way that

$$\sum_{j=1}^{\infty}(\lambda_j^\hbar)^2 \le C\hbar^3. \tag{41}$$

Interestingly, we see that this idea already appears in 1946 in a notice by Yvon [24]. Actually, we find Jacques Yvon's presence throughout the whole history of this subject!

Taking into account the different fields of application, many mathematical variants of the averaging lemmas have appeared; for example, in [12], the following theorem underlines the role of a transversal hypothesis.

Theorem 5.1. *Let μ be a bounded positive measure on $\mathbb{R}^d_v$ such that*

$$\sup_{v\in S^{d-1}} \mu\Big(\{v\in\mathbb{R}^d_v / |v\cdot e| \le \epsilon\}\Big) \le C\epsilon, \quad \forall\epsilon > 0\,. \tag{42}$$

Then, with u a solution of the equation $u + v\cdot\nabla_x u = f$, the map $f \mapsto \int u(x,v)d\mu(v)$ is continuous from $L^2(dx\otimes d\mu(v))$ to $H^{\frac{1}{2}}(\mathbb{R}^d_x)$.

This result has been generalised by Gérard and Golse [10] for averages with respect to y of solutions to pseudo-differential equations. In order to handle problems of Vlasov–Maxwell type, DiPerna and P. L. Lions [7] have, in the theory of kinetic equations, included averaging lemmas with differential operators in the variable v, and this point of view has been systematically generalised in the paper by Tadmor and Tao [20].

6 Conclusions: kinetic equations and statistical limits

The contributions mentioned above on the subject of averaging lemmas solely concerned the relations between kinetic and macroscopic equations. But, of course, their history also involves relations between the dynamics of N particles (molecules, atoms, ions and electrons) with N large or tending to infinity. One then enters into the realm of (classical or quantum) statistical mechanics, which could easily provide subjects for a lot of further articles.

We should mention that, in an intuitive and formal way, those relations could already be found in the spirit of Maxwell, Boltzmann and Lorentz: one of the most important tools was the BBGKY hierarchy, as introduced in this context by Yvon in 1935 [24], and the first rigorous works are due to Grad and Lanford.

It should also be noted that there are important similarities between the derivations of kinetic models from classical statistical mechanics and from quantum statistical mechanics. This can be explained by using the Wigner transformation. It is in this same context that the Vlasov equation mentioned above appears naturally.

We observe that Vlasov presented his equation in 1938, 66 years after the Boltzmann equations. This is explained by the fact that the Vlasov equation applies to modern domains of physics, while the Boltzmann equation was related to 19th century physics. However, because of its really nonlinear character, it contains the theory's essential difficulties.

After the pioneering works of Neunzert and Spohn (see [19]), the subject developed within the same community, also using the tools referred to above (for example, the version of the averaging lemmas given in [7]). Moreover, Vlasov's approach has morphed into new shapes motivated by contemporary physics, in particular in the modelling of plasmas, for instance in the context of nuclear fusion (the international project ITER, between Japan and Cadarache). A relativistic version of Vlasov's equation, where the force F is the Lorentz force containing the magnetic field, and where the speed v is bounded by the speed of light c and given by $v(\xi)$ with $\xi\in\mathbb{R}^d$, yields the nonlinear "Vlasov–Maxwell" system. Furthermore, there are "semi-non-relativistic" approaches of order $1/c$ or $O(1/c^2)$ (for a concise presentation of this hierarchy of nonlinear kinetic equations, see, for example, [4]). We notice that, for this kind of kinetic equation, a lot of problems are still unsolved in mathematical and numerical analyses.

Finally, as the Liouville equation (1) generates a Hamiltonian flow that preserves the measure on the phase space, there have been recent works (Brenier et al.) on the relations between Vlasov-Monge-Ampère equations and the optimal transport equation (see [21]).

With these observations in mind, it is clear that, since the decade of 2000, works on this subject multiplied with, amongst others, the work by Mouhot and Villani in 2011 on Landau damping [16], which contributed to a second Fields Medal for the French kinetic community. The European network HYKE (HYperbolic and Kinetic Equations, 2002-2005), with its 350 researchers in 16 European countries and the USA, amongst them nearly all the mathematicians cited in this article (and some young fellow researchers that left us too soon, like N. BenAbdallah and F. Poupaud), acted as an important catalyst in this evolution.

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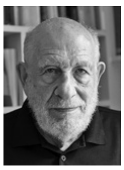

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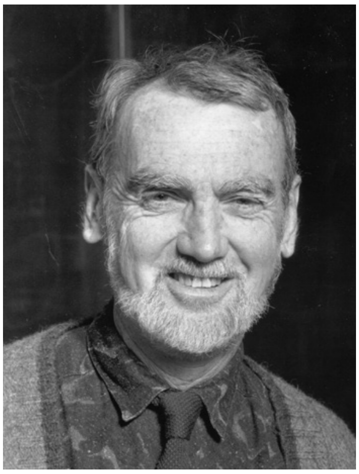
Interview with Abel Laureate Robert P. Langlands

Bjørn Ian Dundas (University of Bergen. Norway) and Christian Skau (Norwegian University of Science and Technology, Trondheim, Norway)



Robert P. Langlands, The Abel Prize Laureate 2018. © Randall Hagadorn/ Institute for Advanced Study.

Professor Langlands, firstly we want to congratulate you on being awarded the Abel Prize for 2018. You will receive the prize tomorrow from His Majesty the King of Norway.

We would to like to start by asking you a question about aesthetics and beauty in mathematics. You gave a talk in 2010 at the University of Notre Dame in the US with the intriguing title:* Is there beauty in mathematical theories? *The audience consisted mainly of philosophers – so non-mathematicians. The question can be expanded upon: Does one have to be a mathematician to appreciate the beauty of the proof of a major theorem or to admire the edifice erected by mathematicians over thousands of years? What are your thoughts on this?

Well, that's a difficult question. At the level of Euclid, why not? A non-mathematician could appreciate that.

I should say that the article was in a collection of essays on beauty. You will notice that I avoided that word; in the very first line, I said: "Basically, I do not know what beauty is!" I went on to other topics and I discussed the difference between theories and theorems.

I think my response to this is the same today. Beauty is not so clear for me; it is not so clear when you speak about beauty and mathematics at the same time. Mathematics is an attraction. If you want to call it beauty, that's fine. Even if you say you want to compare with the beauty of architecture. I think that architectural beauty is different from mathematical beauty. Unfortunately, as I said, I just avoided the question in the article and, if you forgive me, I will avoid it today.

One other reason we ask this question is that, as you are well aware, Edward Frenkel, who you have worked with and who is going to give one of the Abel Lectures later this week about aspects of the Langlands programme, wrote a best-seller with the title* Love and Mathematics *and the subtitle* The Heart of Hidden Symmetry. *The Langlands programme features prominently in that book. He makes a valiant effort to try to explain to the layman what the Langlands programme is all about. We were very intrigued by the preface, where Frenkel writes: "There is a secret world out there, a hidden parallel universe of beauty and elegance, intricately intertwined with ours. It is the world of mathematics. And it is invisible to most of us." You have probably read the book. Do you have any comments?

I have skimmed through the book but I have never read it. I am going to say something that is probably not relevant to your question. We are scientists: we ask about, we think about, we listen, at least, to what scientists say, in particular about the history of the Earth, the history of the creatures on it and the history of the Universe. And we even discuss sometimes the beginning of the Universe. Then, the question arises, something that puzzles me although I've seldom thought about it, except perhaps when I am taking a walk – how did it get started at all? It doesn't make any sense. Either something came out of nothing or there always was something. It seems to me that if I were a philosopher or you were a philosopher, we'd have to ask ourselves: how is it that something can be there? It's complicated; it's not irrelevant that the world is very complicated but the enigma is simply the fact that it is there. How did something come out of nothing? You may say with numbers it can happen but beyond that I don't know.

You have your creative moments, where all of a sudden you have a revelation. Hasn't that been a feeling of intense beauty for you?

You presumably mean when suddenly things fit together? This is not quite like looking at clouds or looking at the sea, or looking at a child. It is something else; it just works! It works and it didn't work before; it is very pleasant. The theories have to be structural and there has to be some sort of appealing structure in the theory.

But, you know, beauty… women are beautiful, men are beautiful, children are beautiful, dogs are beautiful, forests are beautiful and skies are beautiful; but numbers on the page or diagrams on the page? Beauty is not quite the right word. It is satisfying – it is intellectually satisfying – that things fit together, but beauty? I say it's a pleasure when things fit together.

As I said in the article, I avoided the word beauty because I don't know what it means to say that a mathematical theorem is beautiful. It is elegant, it is great, it is surprising – that I can understand, but beauty?!

But we can at least agree that Frenkel's book was a valiant effort to explain to the layman what beauty in

mathematics is and, in particular, that the Langlands programme is a beautiful thing.
Well, yes, I would wish that Frenkel were here so I could present my views and he could present his. I have studied Frenkel because he explained the geometric theory but I wasn't interested that much in the beauty. I wanted to read his description of the geometric theory and I got quite a bit from it but I also had the feeling that it wasn't quite right. So, if I wanted to say more, I would want to say it in front of him so he could contradict me.

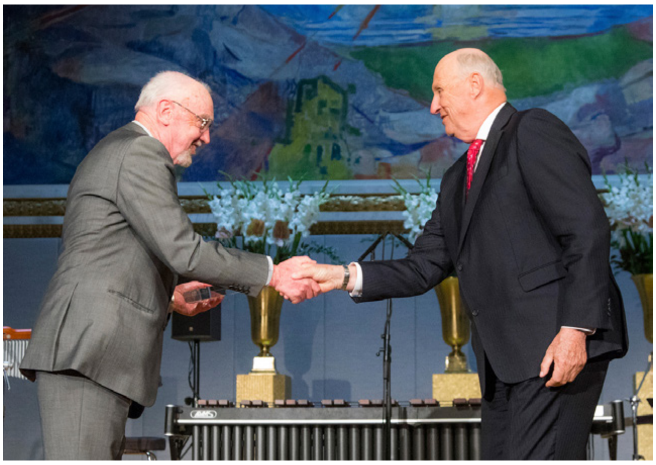
You have an intriguing background from British Columbia in Canada. As we understand it, at school you had an almost total lack of academic ambition – at least, you say so. Unlike very many other Abel Laureates, mathematics meant* nothing *to you as a child?
Well, except for the fact that I could add, subtract and multiply very quickly. There was an interview in Vancouver – actually, I was in New Jersey but the interviewer, *he* was in Vancouver – and he asked me a question along those lines and I answered rather frivolously. All the experience I had with mathematics was with arithmetic, apart from elementary school and so on, and I liked to count.

I worked in my father's lumberyard and those were the days when you piled everything on truck by hand and tallied it. And you counted the number of two by fours – is that a concept here? Two by fours: 10 feet, 12 feet, 8 feet, 16 feet… and then you multiply that and add it up with the number of 10s and multiply by 10, plus the number of 12-foot-lengths and multiply by 12, and so on and so forth and you get the number; convert it to board feet and you know how much it is worth. I would be loading the truck with some elderly carpenter or some elderly farmer from the vicinity. He would have one of these small carpenter pencils and he would very painfully be marking one, two, three, four, five; one, two, three, four, five; and so on. And then you would have to add it all up. And me, I was 12, 13 or 14 and I could have told him the answer even before he started. But I waited patiently when he did that.

So, that was my only experience with mathematics except for one or two things, one or two tricks my father used when building window frames to guarantee that the angles are right angles and so on, but that was just a trick, right? The diagonals have to be of equal length if the rectangle is going to be right-angled.

Then, why did you move toward mathematics? Why not languages or other things that you studied?
Actually, when I went to university in the almost immediate post-World War II period, it was still regarded as necessary for mathematicians to learn several languages: French, English, Russian or maybe even Italian. Now, that fascinated me. The instruction of French in English-speaking Canada was rather formal; nobody paid too much attention to it. But learning languages rather fascinated me and the fascination has been with me all my life (but that was incidental to mathematics).

Why did you start at university at all?
Why did I start…? Here is my conjecture. There are two things (I will come back to the second thing in just a minute). I went to high school. There were children from the neighbourhood and from the surrounding countryside, and they tested us. I was indifferent, you know. I didn't pay too much attention but they also used IQ tests and my conjecture has always been that I probably had an unusually high IQ – quite an unusually high IQ – I don't know. It didn't mean much to me then but that is my conjecture in retrospect. Many of our teachers were just former members of the army in World War II, who were given positions as teachers more as a gratitude for their service in the army. This fellow – he was young, he probably had a university degree and he took an hour of class time to say that I absolutely must go to university. So, I noticed that.


Robert Langlands receives the Abel Prize from H.M. King Harald. Photographer: Thomas Brun/NTB.

And there was another reason: I had acquired a mild interest in science because I had a book or, rather, my future father-in-law had a book (it was rather a leftist book about eminent scientists; of course, Marx was included, Darwin was included, Einstein was included and so on – various scientists from the 1600s, 1700s, 1800s, etc.) and he gave it to me. He himself had a childhood with basically no education and he learned to read aged about 37, during the Depression when the Labour parties were recruiting unemployed people. So, he learned to read but never very well and I think he never really could write. He always had a good memory so he remembered a number of things and he also had a library and, in particular, he had this book, which was very popular in the pre-war period. So, I began to read this book. My wife – my *future* wife – had a better idea of what one might do as an adult than I did and she influenced me. And I had this book, where I read about outstanding people like Darwin and so on, and that influenced me a little in the sense that it gave some ideas of what one might do.

And there was also the accident that I always wanted to leave school and hitch-hike across the country but when I turned 15, which was the legal age when you can stop going to school – I had only one year left – my mother made a great effort and persuaded me to stay another year. During that last year, things were changing for various reasons, e.g. the lecture of that teacher and an introduction to one or two books, so I decided to go to university.

You go on to obtain a Master's thesis at the University of British Columbia, you marry and then you go to Yale and start a PhD in mathematics. It is quite a journey that you were on there. How did you choose the thesis topic for your PhD at Yale?

First of all, Hille had this book – you may know it – on semi-groups and I was an avid reader of that book, and I took a course from Felix Browder on differential equations. You may not know but Felix Browder was an abysmal lecturer and so you had to spend about two or three hours after each lecture sorting things out. He knew what he was talking about but it took him a long time to get to the point or to remember this or that detail of a proof. I went home and I wrote out everything he had talked about.

So, I had this background in partial differential equations from his course and I had read all of Hille's book on semi-groups and I just put the two together. I really liked to think about these things.

In other words, you found your own PhD topic?

Yes, I found my own PhD topic.

But from there on, after your thesis, we have what we like to think of as a journey toward a discovery. Your work on Eisenstein series and your study of the theory of Harish-Chandra are crucial ingredients here. Would you care to explain to us what the background was that led to the Langlands programme?

There was a Hungarian fellow, S. Gaal, who had immigrated to the US after the difficulties in Hungary and that was in the middle of the 1950s. The Norwegian mathematician Atle Selberg was a member of the Institute of Advanced Study (IAS) in Princeton. Selberg's wife was Romanian and spoke Hungarian and I think Gaal was invited to the IAS by Selberg (he and his wife and maybe their children too). He had come to the US sponsored more or less by Selberg and he was giving a graduate course at Yale, where he talked about Selberg's paper, basically at the time of Selberg's second so-called "Indian paper", a Tata publication from 1960. Selberg didn't write that many papers at the time but I think there were two and Gaal talked about that. Also, I have to mention that there was an important seminar on convexity in the theory of functions of several complex variables.

So, you hear about Selberg and you hear about Eisenstein series, and this theory about convexity, and then you want to prove things and you move more or less instantly to an analytic continuation of Eisenstein series in several variables. So, I had already thought about that but I thought about them in a rather restricted context – no algebraic numbers, for example.

And then I got a position at Princeton University, not because of anything I had done about Eisenstein series but because of my work on one-parameter semi-groups. So, I gave a lecture in one seminar; Bochner didn't run it but he kept an eye on it. I think he was impressed simply because I was talking about something that wasn't in my thesis. I talked about this work with Eisenstein series and I think he was impressed by me. Now, Bochner's family was from Berlin. He wasn't born there but he lived there as a child. He went to German universities and he had connections with Emmy Noether and Hasse, for example. So, he took an interest in anything that had to do with algebraic number theory and he encouraged me to think about Eisenstein theory in a more general context, not just for groups over rational numbers but also for groups over algebraic number fields.

So Bochner was almost like a mentor for you for a while?

Not a mentor but he was like a foster father, if you like. He encouraged me – more than an encouragement; he *pushed* me. Bochner encouraged me to work over algebraic number fields rather than just over the rational number field. Algebraic number fields I basically learned from Hecke and I read papers by Carl Ludwig Siegel (because there are ways to handle analytic continuation of series, which you can take from Siegel's papers). I started to read a little in the literature of these two, Hecke and Siegel, and I wrote about Eisenstein series basically using their very classical methods.

In any case, one year – just about a week before the classes were to start – I was going to give a course in class field theory. Emil Artin had been in Princeton and was the expert on class field theory; he had gone back to Germany in 1958 and there were one or two disappointed students who had come to Princeton to learn a little bit of class field theory. There was no real information on class field theory to be obtained from the courses offered. I had attended a seminar that was arranged by these disappointed students but it wasn't such a good seminar, so I was quite ignorant. But Bochner said: "You are to give a course in class field theory." And I said: "How can I do it? I don't know anything about it and there is only one week left." But he insisted so I gave a course on class field theory from Chevalley's paper, which is the more modern view, and I got through it. There were three or four students, who said they learned something from it.

So, with that, I began to think about the fact that there was no non-abelian class field theory yet. Some people, like Artin, didn't expect there to be any. So, I was just aware of it, that's all. We are now in August of 1963 or something.

You already had a position at Princeton University at the time?

I had a comfortable position at the university and I went up the ladder reasonably rapidly. I think by 1967, I was an associate professor or something like that. Thanks to Selberg, I was at the IAS for a year, and I was at Berkeley, California, for a year. So, I was away two times.

And all this while you were contemplating the trace formula, is that correct?

Well, let me go back. I have forgotten something. I was concerned with the trace formula and I wanted to apply it. The obvious thing you want the trace formula for is to calculate the dimension of the space of automorphic forms; that is the simplest thing. So I wanted to do it. And, so, you plug in a matrix coefficient – as I understood it; it doesn't look like a matrix coefficient – of an infinite dimensional representation into the trace formula and you calculate.

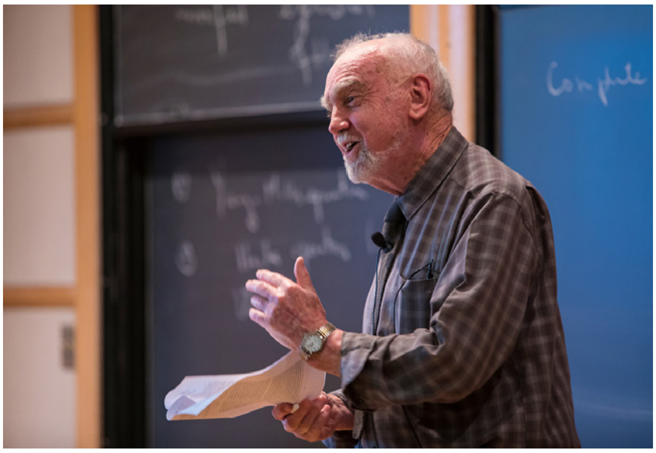
I didn't quite know what to do with this and then I spoke to David Lowdenslager – he died very young – and he said: "Well, people are saying that this is really something you can find in Harish-Chandra." So, I started to read Harish-Chandra and what I observed very quickly, *because* of reading Harish-Chandra, was that the integral that was appearing in the trace formula was an orbital integral of a matrix coefficient. And that orbital integral of a matrix coefficient, we know from representations of finite groups, is a character and, basically, you learn from Harish-Chandra's paper that this is indeed the case. So, that meant that I had to start to read Harish-Chandra – as I did.

And once you start to read Harish-Chandra, of course, it goes on and on; but that was the crucial stage: this observation of Lowdenslager that people were beginning to think that Harish-Chandra was relevant. So, there we are, we have it all. And then I began to think about these things, slowly; and sometimes it worked out, and sometimes it didn't. I could actually apply the trace formula successfully.

In 1962, Gelfand gave a talk at the ICM in Stockholm and a year later his talk was circulating. Now, Gelfand gave his views of the matter. The point was that he introduced the notion of *cusp forms* explicitly. The cusp form is a critical notion and it is a notion that I think appears in rather obscure papers by Harish-Chandra and Godement. But it is hard; you have to look for it. But with Gelfand it was clear why that was so fundamental. Now, an incidental question: I don't think Selberg ever really grasped the notion of a cusp form. Selberg, of course, didn't read other people's papers and I don't think he ever grasped the notion of a cusp form. I think that was an obstacle that he never overcame.

But as soon as you read Gelfand, you can do it – you can prove the general theory about Eisenstein series. You have to know something. In other words, you have to be someone who knows something about unbounded operators on Hilbert spaces. You have to be someone with this background or it doesn't mean anything to you. But if you had that background then you saw immediately what was to be done: take what Selberg had done in rank one to the general case.

Let me go back a little. I only talked mathematics with Selberg once in my life. That was in 1961, before I came to the Institute (IAS). It was at Bochner's instigation, I am sure. Selberg invited me over and he explained to me the proof of the analytic continuation in rank one. Now, of course, the proof of the analytic continuation in rank one is like Hermann Weyl's theory on differential equations of second-order on a half-line. I had read Coddington and Levinson's book *Theory of ordinary differential equations* not too long before, so I could just sit there and listen to Selberg – listen to the kind of things he knew very well – and he explained it to me. Whether he regretted that afterwards I cannot say but he explained to me how it works in rank one.



Robert P. Langlands giving a lecture.
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Were you impressed by his presentation?
I had never spoken mathematics with a mathematician on that level before in my life. I had really never spoken mathematics with Bochner and he is the one that came closest.

Even so, you didn't have further conversations with Selberg afterwards.
No, he wasn't a talkative man. Well, I did have occasional conversations because then I was still continuing to try to prove the fundamental analytic continuation of Eisenstein series. I approached him when I had done it in this or that case and he'd say: "Well, we don't care about this or that case. We want to do the general theory." So he didn't listen to me. While we were colleagues and our offices were basically side by side, we'd say hello but that's about it.

So you spent many years together in virtually adjacent offices at the IAS and you never really talked mathematics?
No. Selberg, you must know, didn't speak with very many people about mathematics. He spoke with one or two, I think, but not many. I am not sure how much he thought about mathematics in his later years. I just don't know.

But even so, your work on Eisenstein series really had some consequences in hindsight, didn't it?
Yes, it was critical in hindsight, right? So, that took me about a whole year and I think I was exhausted after that – it was one of the cases where you think you have it and then it slips away. There was, for example, an induction proof. In induction proofs, you have to know what to as-

sume: if you assume too much it is not true and if you assume too little it doesn't work.

In fact, there was a problem; things were happening that I didn't recognise. In other words, it could be a second-order pole where you naturally assume that there is only a first-order pole. And it took me a long time to reach that stage. Specifically, it is the exceptional group G_2 of the Cartan classification. You think this is going to work; and you try and it doesn't work – it doesn't work in general. Then you think about what and where it can really go wrong and it turns out that it *only* goes wrong for G_2. Then you make a calculation with G_2 and what do you see? You see this second-order pole or a new kind of first-order pole and that changes the game: you have a different kind of automorphic form.

It eventually worked; it was an exhausting year but it did eventually work.

And in the Autumn of 1964, you went to Berkeley, is that correct?

And then I went to Berkeley, pretty much exhausted by that particular adventure.

Were you really so exhausted that you thought about quitting mathematics?

Well, look, quitting mathematics is a rather strong statement. But I did decide to spend a year in Berkeley and got some things done in retrospect. I did more than I thought I had done. I was too demanding, you know. When you are younger, you are a little more demanding than when you are older. So, the next year I was really trying, I think, to do something with class field theory and I didn't see anything. I had a whole year where I don't feel I got anything done. In retrospect, in Berkeley, I *did* something but the year afterwards I didn't at first do anything and I was growing discouraged.

So, I decided on a little bit of foreign adventure. I pretty much decided that the time was right. I should just go away and maybe think of doing something else. I had a Turkish friend and he explained to me the possibility of going to Turkey. So, I decided to do that and, once I had decided to go to Turkey, there were various things to do; I wanted to learn some Turkish and then I went back to studying Russian. I had a very nice teacher. But I still had a little time to spare and I didn't know quite what to do and I began to calculate the constant terms of Eisenstein series.

Just for the fun of it?

Just for having something to do. And so I calculated them. I just calculated it for various groups and then I noticed that it was basically always of the form $f(x)/f(x+1)$ or something like that. But if you could continue the Eisenstein series you could continue the constant term and instead of $f(x)/f(x+1)$, you could continue $f(x)$. And these things are Euler products, so you have new Euler products. Of course, analytic number theorists just love Euler products. So you had it! You had something brand new: they had an analytic continuation and a functional equation. And you could basically do it for a lot of groups.

You could even do it for reductive groups?

You basically did it for split groups, i.e. those reductive groups with a split maximal torus, and then you have the classification. So, you had a whole bunch and, if you looked at them, you could see that somehow they were related to representations of Eisenstein series associated to parabolic groups of rank one. And they were somehow related to a representation; you have a parabolic group and you take the reductive subgroup – it is of rank one and you throw away the rank one part so you basically have some kind of L-function associated to the automorphic form on this reductive subgroup.

All right, so you have Euler products that are attached to a representation of a group. Euler products are Dirichlet series that number theorists love – and that is what you want. You have a large list of groups. And that already suggests something. You can formulate this – you can see somehow where this is coming from. You can see how to formulate it as a representation associated to an automorphic form and a particular representation of what I call the L-group, for L-series.

And there you are: you start to make a guess and you have this in general! For a particular reductive group, you have an Euler product with an analytic continuation, associated to a representation. But you think it works in general. So, once you have that – once you have something that might work in general – you have to think of how you are going to prove it in general.

This must have been extremely exciting?

Well, it was!

Incidentally, did you continue with your classes in Russian or Turkish?

I gave up both, even the Russian class where the teacher was this sweet woman; I think she liked me since I was an industrious student. She was very angry and wouldn't talk to me.

Is it fair to say, then, that your discovery comes out of… well, you were extremely exhausted, you let your shoulders down, you play, you have some evidence and you make a major discovery?

I think that's an apt description.

When did you have this epiphany, if you like, where you saw the connection with the Artin conjecture about the analytic continuation to the whole complex plane of the Artin L-functions?

It was during the Christmas vacation of 1966. Although I have forgotten the date the idea came to me, I still have a vivid recollection of the place. In the old Fine Hall at Princeton University, there was a small seminar room on the ground floor directly to the east of the entrance. The building itself, I recall, was of a Gothic style with leaded casement windows. I was looking through them into the ivy and the pines and across to the fence surrounding the gardens of the President's residency when I realised that the conjecture I was in the course of formulating implied, on taking $G=\{1\}$, the Artin conjecture. It was one of the major moments in my mathematical career.[1]

Was this a so-called* Poincaré moment *for you? You know the story about Poincaré getting on a bus when all of a sudden he saw the solution to a problem he had been thinking about for months and then put aside.
Except that somehow I was not searching. I had no idea I would stumble across a non-abelian class field theory.

And this was right before you sent the 17-page, handwritten letter to André Weil outlining your theory?
Yes. The letter to André Weil is somehow an accident. The point is, I went to a lecture by Chern. Weil went to the same lecture and we both arrived early. I knew him but not particularly well. We both arrived early and the door was closed so we couldn't go in. So, he was standing there in front of the door and I was standing there in front of the door. He wasn't saying anything so I thought I should say something. I started to talk about this business. And then he didn't understand anything, of course, and he probably behaved as you'd behave under those circumstances; I was this fellow talking to him and I just assumed he would walk away but he said "write me a letter". I wrote him a letter. He never read the letter so far as I know.

He had your letter typed and distributed, didn't he?
Yes, that's right.

This is not the only moment you describe where you are making a discovery while* not *sitting behind your desk and working. On another occasion, you tell of how you are walking from here to there and suddenly you see something. Is that a pattern of yours? Is that how you find things?
I have certainly seen these things very seldom in my life so I don't think one can speak about a *pattern*.

Perhaps it is time that you actually tell us about what the Langlands programme is all about? Just in broad brush strokes.
Okay, we sort of know what the quadratic reciprocity law is, right? There, two things that appear to be quite different are the same. Now, we also know that, after Weil, we can define zeta-functions (or L-functions would probably be better). You can define them over finite fields and you can also define them if you have a global field and you take the product of the ones over finite fields and you get some kind of an L-function associated to a variety or even, if you like, to a particular degree of the cohomology of that variety.

A basic problem in arithmetic for any kind of estimation of the number of solutions of Diophantine equations is reflected in the L-functions that you can formally associate – and you are in the half-plane – to the cohomology of the given degree of any kind of curve over a number field. They are there.

Presumably, if you can deal with these then you can, somehow or other, do more things about the estimation of the number of solutions and the nature of solutions. I think no one has a clear idea about this, except in very specific cases, i.e. what you can do with the knowledge of these global L-functions. But they are there, and you want to prove that they have analytic continuation. The only reasonable way, on the basis of evidence, is that they will be equal to automorphic L-functions.

Now, from the point of view of the variety and the cohomology of the variety, you have the Grothendieck formula. I don't know to what extent he actually had a complete theory – I don't think he had – but he had the notion of a motive, and a motive has certain multiplicative properties. So, you had a whole family of functions that behaved in a natural functorial manner. And you wanted to prove that they could be analytically continued. But he managed to associate a group; in other words, these motives were associated to representations of a group, whose nature had to be established. On the other hand, the group is *there*: you may never know its nature but you should be able to find out its relations to other groups.

Now, on the other hand, what you would like, normally, in order to establish the analytic properties of *these* things that are defined algebraically/geometrically is to associate them to something that is defined analytically because automorphic L-functions basically have analytic continuations. There are some questions about it, right, because you can do it if they are associated to GL_n and the standard representations of GL_n (that is the theorem by Jacquet and Godement from 1972). But, in the end, you need to do two things that are more or less mixed, namely, for an automorphic form associated with a general group, you need to show that that automorphic form really sits on GL_n; you push it toward GL_n and then you define the L-function. So, it is not just an automorphic form but it is an automorphic form that can be pushed toward GL_n.

Now, that will make you think that somehow an automorphic form is associated to a representation of a group, which has to be defined. In other words, there is a structure in the connection of all automorphic forms. You can pass it from one associated with G. (It is not true that if you associate it with G, you can pass it to another group G' if G goes to G'.)

This is the so-called L-group and you have to push it forward. If you have this motion and you can push, you could say you have the automorphic form here equal to one over there, and so the L-function is the same.

If the one you take over here is GL_n then you know, by Jacquet-Godement, that you can handle it. So, if you have a way of passing – whenever you have the form on one group – to other groups in the appropriate formalism then you can handle analytic continuation.

This is what you call* functoriality*?

[1] Langlands (2005): *The genesis and gestation of functoriality*. http://publications.ias.edu/sites/default/files/TheGenesis.pdf

Yes, this passing like that. So, this means that you can describe it by representations of a group. So, this is the *same* thing; something similar is happening over on the algebraic/geometric side. And there it is another group; it is the group defined in a similar way and that is the group of Grothendieck and its motive. And when you have the two, you can do all the analytic continuation you want and what you get is, of course, something for your great-grandchildren to discover.

It seems like a very naive question, and it is, but let's ask it anyway. Why is it so crucial to analytically or meromorphically continue the L-functions?
Why is that so crucial? That is a good question. Why is it so crucial to know anything about the zeta-function? Where do you go? In other words, you go for an estimate of the number of solutions and things like that. What do you do with the information you have about the zeta-function? And what would you do if you have all the possible information? Do you have an answer?

No, we don't.
Neither do I but I think, in both cases, it is that we haven't worked with it in the right area.

Of course, we know that the classical zeta-function tells us something about prime numbers and their distribution. And Dirichlet's L-functions tell us something about prime numbers in arithmetic progressions.
So, you get that kind of information but…

It is clear that it is what people are hoping for. But you can ask: why do they want it? Only God knows. So, you're pushed by preconceptions and you're trapped in the way you think mathematics should work.

In 2009, the so-called Fundamental Lemma, conjectured by you in 1983, was proved by Ngô. He was awarded the Fields Medal in 2010 for this. Time Magazine selected Ngô's proof as one of the Top Ten Scientific Discoveries of 2009.
You can cancel your subscription to Time Magazine!

In a joint paper from 2010 titled "Formule des traces et fonctorialité", the authors being you, Ngô and Frenkel, the very first sentence – translated into English – reads: "One of us, Langlands, encouraged by the work of one of us, Ngô, on the Fundamental Lemma, whose lack of proof during more than two decades was an obstacle for a number of reasons for making serious progress on the analytic theory of automorphic forms, has sketched a programme to establish functoriality – one of the two principal objects of this theory." Any comments?
The Fundamental Lemma is needed to deal with a specific kind of technical question. Let's see if I can make it clear. This is not a good example but I'll try to explain something. Say you have something such as the group SL_n and you have SU_n. You know by Weyl's theory about the representations of SU_n. Those are basically the standard finite dimensional representations of this group. Now, look at the SL_n situation; SL_n has more representations than SU_n. SL_n is a non-compact group; it has a lot of representations. But, in particular, it has some things that are very much like those of SU_n; the characters are basically the same. For example, you know the characters of SU_2.

Now, lets go to SL_2. By Harish-Chandra's theory – actually, SL_2 is prior to Harish-Chandra – you have corresponding representations. In this whole theory of representations of semi-simple groups or reductive groups, and therefore the theory of automorphic forms, and therefore the whole theory, what happens for SL_2, for example – those things where there is only one? I mean, you know SU_2, where there is only one representation in each dimension. Each one has basically something corresponding for SL_2, the so-called discrete series, and at each end, it has two. It is just this one place where this unitary group becomes two for SL_2.

These two are, for all practical purposes, the same; they're just two pieces. Now, considering the Fundamental Lemma and what you have to do if you are worrying about the trace formula: you want some part that is really useful for, say, SL_2 and that's the part where you put these two together so they *look* like SU_2. Then, there is a supplementary part where you have to take into account the fact that they don't occur with the same multiplicity so you have this extra stuff. So, if you want to handle the trace formula, you have to see what you want to compare. You have to say that SU_2 is more or less like SL_2. So, you can compare the trace formula of the two but the extra bit over here is causing you trouble. And the reason is that somehow the one representation here breaks up into two representations there, and some of it doesn't have much to do with things and it is just there. You just take the difference of the characters rather than the sum.

If you are going to use the trace formula, you have to understand the part you don't really want. And there is some mysterious endoscopy. What is the so-called Fundamental Lemma? It is a fundamental lemma in the context of the specialised theory that was introduced for this special feature where things that should be the same could sometimes differ. What you do is that you treat them all as if they were the same and put them together and then you take the difference. You have to treat those differences separately so they look like something coming from the torus itself, the circle group that is sitting in there. So, it is a technical necessity; if you want to compare the representations of two groups you use the trace formula, but this stuff, this extra stuff, you have to get it out, put it aside and treat it separately, so you can compare what is left. And then what matters is just to understand what you can compare on its own. That means that you have to understand the differences – you have to look at just the circle group, which is all that matters, and for that you need the Fundamental Lemma, and that's all. The Fundamental Lemma is the fundamental lemma for these technical properties. It's a whole theory for this; it's rather complex but it takes care of that.

Functoriality is the most important part of the Langlands programme. And to make progress on functoriality you have said you think that the crucial tool is going

to be the Selberg–Arthur trace formula. Why is the trace formula going to be so important?
Well, what do you want to show? You want to show that you can transfer everything to GL_n, basically. Let's put this somewhat differently. You want to show that you can move automorphic forms from one group to another. This is something you want to use the trace formula for: you compare the two trace formulae, right?

You want to move things from the group G to the group G'. You want to be able, in particular, to handle the L-function, so you want to be able to move to GL_n. These things work at the level of the L-group but let's just work with GL_n, so we don't have to worry about that. So, how are we going to do it?

You say here is this group; every time I have a homomorphism of the group – really of the L-group – from one to the other then I have a transfer representation. This means that every representation is obtained by transfer; it is a natural transfer. You can see this if you see the distribution of conjugacy classes. So, what would you do? There is, so to speak, a smallest place, a smallest group where it sits and then it propagates to the other groups.

For example, say, you have one group G that you want to understand. So, you say here is the smaller group, so it has to be the contribution of those things that sort of sit inside the bigger groups in that smaller group, so one-away, one-away you do it all along, moving from the larger to the smaller. You look at the trace formula here and you look at the trace formula there, and they cancel. In other words, you come from one place and you look to see what it cancels – it cancels something – and you go along and along and along and you know you understand it. Ultimately, the real building blocks are those things in the big group that come from the trivial group. So, the last stage is to analyse those. I take the small group and I want to send it to the big group and I just have to look: I take the trace formula up here and it cancels everything I know from this. It just cancels everything; I said it should be made up by pieces and each should come from smaller groups and this just comes from the smaller group, and this comes from the smaller group, and this comes from the smaller group, and then I have to be careful because it can come from a bigger group and from a smaller group, and I have to be careful so I don't count it twice. So, I say they should be equal. I have to have a clear view of the combinatorics. Everything comes from a smaller group and some of it comes from two smaller groups and some is coming from three and so on. This depends on the image group. So, to show that this is really true, I just show that somehow the trace formula gives the same up here as it does for something in the selection of the various groups. This is pretty vague but in principle it is not so bad. And this is how it works but up until now at a very low level.

So, that is at the forefront of your investigation?
I mean, that is at the forefront of *Arthur*'s investigation. I think if you want to hear what is available along these lines, you have to ask Arthur.

From left to right: Bjørn Ian Dundas, Christian Skau and Robert P. Langlands. © Anne-Marie Astad / The Norwegian Academy of Science and Letters.

We understand you are currently thinking in more differential geometric terms?
I was thinking about the geometric theory and the geometric theory is not the trace formula, right? The geometric theory is basically Yang– Mills theory.

There are two papers – a brief one in English was premature and not entirely reliable. The other, which is longer and – so far as I know – reliable, is in Russian. This is already an obstacle but it is also very difficult to understand, in part because very few people, perhaps no one, understands the connection with Yang–Mills as in the paper of Atiyah–Bott. I might be able to help you with further questions but I have had difficulties with one Russian speaker who, in spite of encouragement, still does not understand the basic idea of the paper. He is a well-regarded mathematician. So it appears that the paper is difficult. I am nonetheless confident that it is correct. You might ask around!

That is a recent paper of yours that we can read?
It can be found on the web.[2]

In 1872, Felix Klein launched his famous Erlangen programme. To every geometry he associated an underlying group of symmetries. Klein stated in his autobiography that the Erlangen programme remained the greatest guiding principle, or "leitmotiv", for his subsequent research. Do you see any analogy with the Langlands programme?
I would hesitate to use the word programme but I think probably that leitmotiv is right. In other words, you have these two somewhat surprising structures on both sides: groups moving from one side to another. You have one side that is arithmetic and the other side that is analytic (or geometric, depending upon your view). So, you move around and you know that everything can go to GL_n, and with GL_n you have this one example of an Euler-product that you can analytically continue. These things give you a very clear focus – or leitmotiv if you like – on what one should try to achieve.

[2] http://publications.ias.edu/rpl/section/2659.

In 2016, we interviewed Andrew Wiles, who was awarded the Abel Prize for his proof of the modularity theorem for semistable elliptic curves, from which the Fermat theorem follows. The modularity theorem fits into the Langlands programme and Wiles expressed the sentiment that its central importance in mathematics lent him courage: one simply could not ignore it – it would have to be solved!

You propose a theory of mathematics that is rather encompassing: it is not a particular thing; it is a structural thing. What are your comments on this?
I think what one is looking for is a structural thing. All of the particular instances are of interest. Or something like that. There is so much you just can't do that I hesitate to answer really. But, if you like, you have this one structure on the one side, the Diophantine equation, which is sort of embedded in one of the automorphic forms. Automorphic forms have a lot of intricate structure on their own, so you have a lot of information about the L-functions there that moves back here, i.e. to the Diophantine side, and that is usually what you want. But I am not a specialist in those things.

Both Harish-Chandra and Grothendieck – two mathematicians we know you admired – were engaged in constructing theories, not being satisfied with partial insights and partial solutions. Do you feel a strong affinity with their attitude?
I greatly admired both of them and, incidentally, do not feel that I am at their level. Their impulses were, however, different. Grothendieck himself has described his own impulses. Harish-Chandra never did. He just went where the material led him. He abandoned the mathematics of his youth, as a student in India, on which he wrote many papers, and turned to the topic of his thesis with Dirac: representation theory, a theory that was gaining in popularity and depth when he came to the IAS with his advisor Dirac. He just went where it led him. In retrospect, he just went where his strength and ambition took him. Incidentally, his thesis was, in contrast to what followed, not very impressive.

To what extent has it been important to you to be around people and in an environment where new ideas circulate?
There were two people who made an absolute difference to my mathematical life. The first was Edward Nelson, whom I met basically by accident as a graduate student – I had come as a graduate student with a friend, who was an instructor at Yale, to the IAS to visit some of my friend's friends from his graduate-student days at Chicago, one of whom was Nelson. An incidental consequence of an informal conversation that day, during which we discussed mathematical matters of common interest, was that Nelson suggested to the Princeton mathematics department, where he was to begin teaching the following year, that I be offered a position as an instructor – no application, no documents, nothing.

The second is Salomon Bochner, who, after hearing me talk in an informal Princeton seminar, urged me to move from the rational number field to arbitrary number fields and to study the work of Hecke. He also recommended me to Selberg. As a consequence, I had my one and only mathematical conversation with Selberg. It was, of course, he who talked.

Harish-Chandra, too, made an enormous difference, principally because of his papers (these I read on my own initiative, many years before meeting him) but also because my appointment to the IAS was made – I suspect – at his initiative. I should also observe that it was a young Princeton colleague (although they were older than me) who directed me to Harish-Chandra's papers. So the answer to your question is certainly 'yes'. I owe a great deal to my education at UBC, where a very innocent young man, a boy if you like, was introduced to intellectual possibilities to which he has been attached all his life, and to Yale, where for two years he followed his own whims and where there were mathematicians who supported his independence. Whatever reservations I have about Princeton and its two academic establishments, it is clear from the preceding remarks that I am indebted in a serious way to specific individuals who were attached to them.

Perhaps before we conclude the interview, it might be interesting to hear whether you have private, non-mathematical passions or interests of some sort, e.g. music, literature, language or poetry?
Passions? I don't have any passions. But, you know, it is true that you want to take a look at other things, you know. History is fascinating: modern history, ancient history, the Earth's history, the Universe's history – these things are all fascinating. It is a shame to go through life and not have spent some time contemplating on that – certainly not everything of course but just to think about it a little bit.

On behalf of the Norwegian Mathematical Society and the European Mathematical Society, and the two of us, we would like to thank you for this very interesting interview, and again congratulations on the Abel Prize.
Thanks for inviting me.

Bjørn Ian Dundas is a professor of mathematics at the University of Bergen. His research interests are within algebraic K-theory and algebraic topology.

Christian Skau is a professor emeritus of mathematics at the Norwegian University of Science and Technology (NTNU) at Trondheim. His research interests are within C^*-algebras and their interplay with symbolic dynamical systems. He is also keenly interested in Abel's mathematical works, having published several papers on this subject.

A Glimpse of Sources for Historical Studies at the ETH Archive in Zürich

Nicola Oswald and Klaus Volkert (both Bergische Universität Wuppertal, Germany)[1]

An institution and its archive

The *Eidgenössische Technische Hochschule*[2] in Zürich was founded in 1855 as the *Eidgenössische Polytechnische Schule*, a name often abbreviated to *Eidgenössisches Polytechnikum*. It was one of the first institutions created and run in Switzerland by the *Eidgenossenschaft* (Swiss Federation) after the political reorganisation in 1848. This is important in order to understand its function and that of its archives. The polytechnic was headed by the *Schweizerische Schulrat* (Swiss School Board), a board with four members plus its president, elected by the *Bundesrat* in Bern. The president of the board was a full-time position with an office at the school; the other members met only a few times a year. There was also a director of the polytechnic but, for a long time, he did not have significant influence in the school.

Nearly all administrative correspondence was in the school itself and this is conserved in its archive. Quite a bit of it is accessible online. Important parts from the administrative archive are the *Schulratsprotokolle*, including the reports on the meetings of the *Schulrat*, the *Präsidialverfügungen* (the orders given by the *Schulratspräsident*) and the *Anhänge*, the calendars of the Polytechnic, also called the *Polyprogramme*. In the archive, there are many more documents concerning the *Schulrat*, including its *Missiven*, which are letters and documents sent by the *Schulratspräsident*, and the *Schulratsakten*, documents, and in particular letters, received by the *Schulrat*. At the archive of the ETH, one can find, for example, letters written by R. Dedekind and by B. Riemann (see Figure 1) applying for the vacant position of professor of mathematics at the polytechnic (1858).[3]

So, in respect to administrative information, the situation at the ETH is very convenient for the user – in particular, because many of these documents are now digitised and therefore easily accessible.

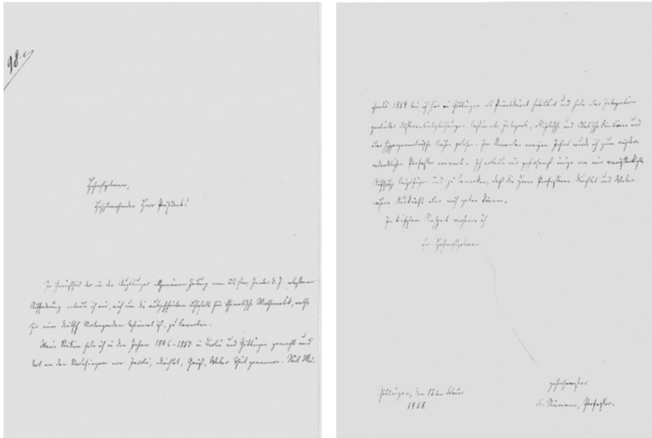


Fig. 1. Riemann applies for the position of professor of pure mathematics at the polytechnic in Zürich (13 June 1858).[4]

Transliteration:
"Hochwohlgeborener
Hochzuverehrender Herr Präsident!
In Anbetracht der in der Augsburger Allgemeinen Zeitung vom 22ten d. J. erlassenen Aufforderung erlaube ich mir, mich auf die ausgeschriebene Lehrstelle für reine Mathematik, welche für einen deutsch Vortragenden bestimmt ist, zu bewerben.
Mein Studium habe ich in den Jahren 1846–50 in Berlin und Göttingen gemacht und dort an den Vorlesungen von Jacobi, Dirichlet, Gauß, Weber theilgenommen. Seit Michaelis 1854 bin ich hier in Göttingen als Privatdocent habilitiert und habe über Integration partieller Differentialgleichungen, bestimmte Integrale und elliptische und Abel'sche Functionen und über hypergeometrische Reihen gelesen. Im November vorigen Jahres wurde ich zum außerordentlichen Professor ernannt. Ich erlaube mir gehorsamst, einige von mir veröffentlichte Aufsätze beizufügen und zu bemerken, daß die Professoren Dirichlet und Weber ihnen Auskunft über mich geben können.
In höflichem Respekt verbleibe ich
Ew. Hochwohlgeboren
Göttingen, den 13. Februar
1858
Gehorsamst
Professor B. Riemann"

[1] We want to thank Evelyn Boesch, Monica Bussmann and Wiebke Kolbmann from the archives and collections of the ETH and Norbert Hungerbühler as well as Urs Stammbach (Department of Mathematics of the ETH) for their kind assistance. Volker Remmert (Wuppertal) and David Rowe (Mainz) read the manuscript and made valuable suggestions. Peter Morley and Nadine Benstein (both at Wuppertal) helped us with the English language.

[2] This name was officially given to the school in 1911.

[3] See Volkert 2017 for more details on that nomination.

[4] ETH-Bibliothek, Hochschularchiv, Anmeldungen für theoretische Mathematik 1858, Hs 1230: 181. Riemann's application had no success and his friend Dedekind was appointed (see below).

Riemann's letter was addressed to Karl Kappeler (1816–1888), the president of the school board (1857–1888), and handed to the director of the school (W. von Deschwanden). Kappeler was perhaps the most famous president of the polytechnic; he had the idea to promote pure mathematics at the polytechnic in order to strengthen its scientific standing. As is demonstrated by the list of mathematicians below, he was very successful in doing so. Kappeler visited the professorship candidates in order to get an impression of their teaching and their personalities.[5] At Göttingen, Kappeler was impressed by Dedekind's teaching whereas Riemann's did not convince the president. Of course, Kappeler also asked the opinion of experts and in the case of Dedekind and Riemann, this was Dirichlet.

For a long time, there was no real self-administration at the polytechnic. The director was not elected but appointed and the professors had only rare meetings. Mathematics was part of the so-called sixth department (VI. Abteilung[6]), responsible for the training of prospective teachers and providing teaching to other students (for example, future engineers) in basic disciplines such as mathematics and physics. The training of future teachers was a special feature of the polytechnic in Zürich; in almost all of the other polytechnic schools in German speaking regions, it was introduced only later. It was important for the mathematicians of the school because it offered them the opportunity to teach mathematics at a high level. In other words, the professors of mathematics at the Polytechnikum were not forced to provide highly standardised courses for future engineers. Whereas their studies were heavily regulated, the students of the Fachlehrerabteilung had a relatively free hand in choosing their courses, and the professors in choosing the subjects of their courses.

Mathematics and mathematicians at the ETH

Almost from the beginning of the Polytechnikum, there were two chairs for mathematics (often called pure mathematics), as well as one chair for descriptive geometry and geometry of position[7] in German, and another chair for mathematics in French. Later in the 19th century, a chair for descriptive geometry in French was created as well. Thus, there was teaching in both German and French – another consequence of the fact that the school was run by the Swiss Federation. The list of German speaking mathematicians is highly impressive:

Bruno Elwin Christoffel (1862–1869), Friedrich Prym (1865–1869), Hermann Amandus Schwarz (1869–1875), Georg Frobenius (1875–1892), Friedrich Schottky (1882–1892), Heinrich Weber (1870–1875), Adolf Hurwitz (1892–1919), Hermann Minkowski (1898–1902), Hermann Weyl (1913–1930) and Heinz Hopf (1931–1965).[8]

There were also other professors (without a chair) like Carl Friedrich Geiser (1873–1913) and Ferdinand Rudio (1889–1929). The German-speaking chair for descriptive geometry was occupied by Wolfgang von Deschwanden (1865–1866) then Wilhelm Fiedler (1867–1907) and then Marcel Grossmann (1907–1927). The French-speaking chairs were occupied by Edouard Armand Méquet (1860–1886) and then Jerôme Franel (1886–1929). Marius Lacombe (1894–1908) was responsible for the teaching of descriptive geometry.

Between 1859 and 1881, there were mathematicians at the so-called Vorschule providing preliminary courses to future students in order to improve their mathematical knowledge. Among them were Johannes Orelli and Gustav Stocker, the latter also serving as secretary to the *Schulrat*.

Many of the mathematicians listed above came to Zürich quite early in their careers and left Zürich again some years later, therefore leaving few traces in the archive. Others, like Hurwitz and Hopf, spent a long time in Zürich until their retirement and their estates, or parts of them, were passed to the archive. Of particular interest are the inventories named *Handschriften* and *Autographen der ETH-Bibliothek*. These are inventories of the estates, or parts of estates, given to the ETH-archive. Such inventories exist for the following mathematicians:[9] Alexandroff [copies of his correspondence, in particular letters by E. Noether], Bernays, Burckhardt, Bützenberger, Deschwanden,[10] Wilhelm Fiedler, Ernst Fiedler, Finsler, Grossmann, Herzog,[11] Hopf, Kollross, Pólya, Jakob Steiner,[12] van der Waerden[13] and Weyl.

In Hurwitz' case, the inventory has the title "Die mathematischen Tagebücher und der übrige handschriftliche Nachlass von Adolf Hurwitz (1859–1919): Katalog". This catalogue was prepared by Hurwitz' friend George Pólya (1887–1985). Through it, you get a lot of information of where to find what kind of document as well as information concerning the origin of selected documents.

[5] This procedure is nicely described in a letter (1866) from C. Culmann, a famous professor of construction at Zürich Polytechnic, to K.M. Bauernfeind (see Maurer 1988, 281–282).

[6] Other names were *Fachlehrer* or *Allgemeine Abteilung*. Of course, the structure of the polytechnic underwent changes. The structure described above was introduced in 1865 by the first revision after the polytechnic was founded in 1855. It lasted, with some slight modifications (e.g. the VI. department was divided into two departments later – one for future teachers and one for the disciplines of general interest), for the rest of the century. Our presentation is mostly focused on the second half of the 19th century.

[7] Today, we would say projective geometry. The name *Geometrie der Lage* was a reference to von Staudt's book (1847).

[8] For more information, see Frei/Stammbach 2007. The dates indicate the period when the people served as professors at the polytechnic.

[9] Here, only some names are mentioned, and, of course, there are others. Note that these mathematicians were not all professors at the ETH.

[10] Because Deschwanden was the first director of the polytechnic, there is a lot of administrative correspondence by him in the archive.

[11] Herzog also served as director of the polytechnic.

[12] This is only a part of his estate – mainly manuscripts for publications and lecture notes taken by his students. The material in Zürich comes from Kollros, who was an active member of the Steiner committee. Bützberger was also a great expert in Steiner studies and, of course, Geiser was his grandnephew.

[13] This is a collection of van der Waerden's correspondence.

In particular, you can find Hurwitz' *Mathematische Tagebücher* (Mathematical Diary) in 29 booklets, an important source on his mathematical ideas.[14]

Turning the pages of Hurwitz' diary, amongst the numerous advanced entries on geometry, number theory, analysis, etc., probably the most surprising entries deal with topics from recreational mathematics, e.g. on mathematical origami and puzzles circulating at his time in the mathematical community. This glimpse of the private Hurwitz indicates the broad interest he had in mathematics and not only at the high level of his excellent research. Another peculiar entry in his last diary, bearing the date 20 May 1918, deals with Arthur Cayley's work on counting the isomers of hydrocarbons C_nH_{2n+2} (so-called alkanes or paraffins). Extending Cayley's approach, Hurwitz employed generating functions to find an explicit formula for the number of these chemical compounds with exactly three primary carbons, namely

$$[(n(n-2)+4)/12],$$

where [x] denotes the largest integer less than or equal to x. Since carbons have valency four, they may be classified with respect to the number of adjacent carbons, so Hurwitz was counting those isomers having exactly three carbons each of which bound to only one further carbon (see Figure 2).[15] In view of the later work by George Pólya [Pólya 1937], another prominent mathematician of the ETH and curator of Hurwitz' estate, it might be surprising that such an explicit formula is possible. Pólya succeeded in providing asymptotic formulae for chemical compounds (such as, for example, isomers of alkanes, alcohols, etc.) of exponential growth (with respect to the number of carbons). However, his enumeration method applies to the set of all isomers of hydrocarbons and the very special isomers Hurwitz was studying and counting constitutes only a tiny subclass. We may speculate whether Pólya was inspired for his research by this entry in Hurwitz' diary.[16]

Furthermore, there is a huge number of Hurwitz' meticulously handwritten manuscripts of lectures throughout his academic life.

Many of the aforementioned inventories have been digitised and there are more to follow. The *Findbuch* of the Hopf estate, for instance, contains 14 pages with dense information on Hopf in three parts: I. manuscripts, II. correspondence, III. documents concerning Hopf's biography. Of course, not all estates are as rich as Hopf's. His correspondence is very interesting, not least because he lived in neutral Switzerland and thus could serve as an intermediary between mathematicians in countries at war with each other (see Figure 3 for an example).

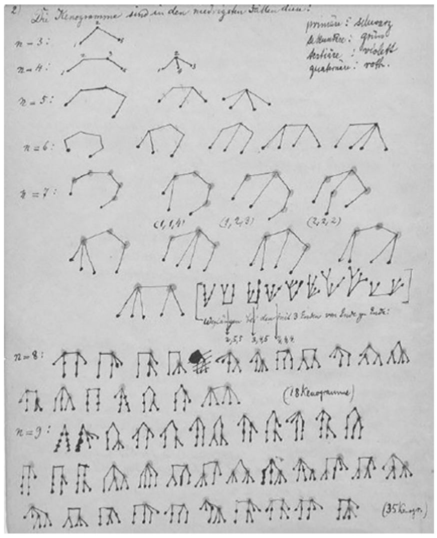
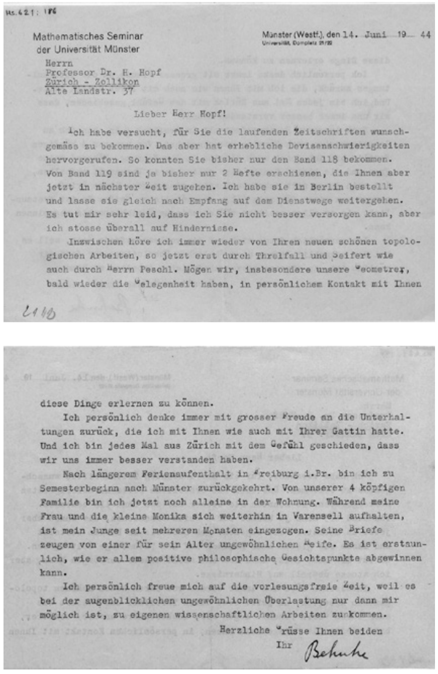


Fig. 2. Hurwitz counting alkanes.[17]



Hs.621: 186

Mathematisches Seminar der Universität Münster

Münster (Westf.), den 14. Juni 1944
Universität, Domplatz 20/22

Herrn
Professor Dr. H. Hopf
Zürich - Zollikon
Alte Landstr. 37

Lieber Herr Hopf!

Ich habe versucht, für Sie die laufenden Zeitschriften wunschgemäss zu bekommen. Das aber hat erhebliche Devisenschwierigkeiten hervorgerufen. So konnten Sie bisher nur den Band 118 bekommen. Von Band 119 sind ja bisher nur 2 Hefte erschienen, die Ihnen aber jetzt in nächster Zeit zugehen. Ich habe sie in Berlin bestellt und lasse sie gleich nach Empfang auf dem Dienstwege weitergehen. Es tut mir sehr leid, dass ich Sie nicht besser versorgen kann, aber ich stosse überall auf Hindernisse.

Inzwischen höre ich immer wieder von Ihren neuen schönen topologischen Arbeiten, so jetzt erst durch Threlfall und Seifert wie auch durch Herrn Peschl. Mögen wir, insbesondere unsere Geometrer, bald wieder die Gelegenheit haben, in persönlichem Kontakt mit Ihnen diese Dinge erlernen zu können.

Ich persönlich denke immer mit grosser Freude an die Unterhaltungen zurück, die ich mit Ihnen wie auch mit Ihrer Gattin hatte. Und ich bin jedes Mal aus Zürich mit dem Gefühl geschieden, dass wir uns immer besser verstanden haben.

Nach längerem Ferienaufenthalt in Freiburg i.Br. bin ich zu Semesterbeginn nach Münster zurückgekehrt. Von unserer 4 köpfigen Familie bin ich jetzt noch alleine in der Wohnung. Während meine Frau und die kleine Monika sich weiterhin in Varensell aufhalten, ist mein Junge seit mehreren Monaten eingezogen. Seine Briefe zeugen von einer für sein Alter ungewöhnlichen Reife. Es ist erstaunlich, wie er allem positive philosophische Gesichtspunkte abgewinnen kann.

Ich persönlich freue mich auf die vorlesungsfreie Zeit, weil es bei der augenblicklichen ungewöhnlichen Überlastung nur dann mir möglich ist, zu eigenen wissenschaftlichen Arbeiten zu kommen.

Herzliche Grüsse Ihnen beiden
Ihr Behnke

Fig. 3. Communication in wartime (Behnke to Hopf, Münster, 14 June 1944).[18]

Behnke describes the situation of his family and the difficulties of sending the last issues of the journal *Mathematische Annalen* to Hopf. He remembers warmly the mathematical discussions he had with Hopf in Zürich during his last visit there.

Soon after the war, Hopf was involved in the rebuilding of mathematics in Germany.[19] Hopf was a moral and scientific authority with important influence. To some of his German colleagues, it seemed important to justify themselves before that authority. In the ETH-archive, there is a surprising document in which L. Bieberbach (1886–1982) did so (see Figure 4). During the Nazi regime, Bieberbach was the most prominent proponent of the so-called *Deutsche Mathematik*. Together with Th. Vahlen,

[14] Cf. Oswald 2015, Schmidt 2017. For another interesting aspect of Hurwitz' work cf. Helmstetter/Oswald 2016. The Hurwitz' correspondence is preserved today at Göttingen in the *Universitäts- und Staatsbibiothek*. The diary is digitalized by e-manuscripta.ch – the digital platform for manuscript material from Swiss libraries and archives: http://dx.doi.org/10.7891/e-manuscripta-12833.

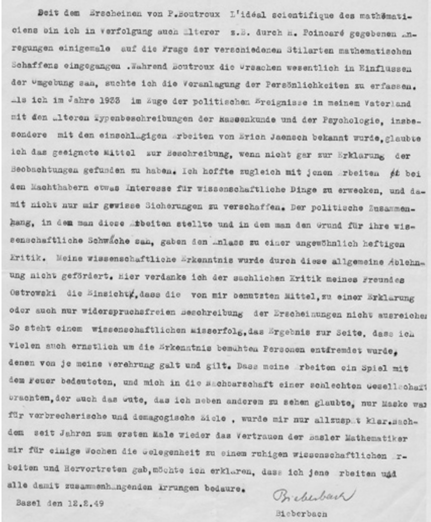
[15] For more information cf. Helmstetter, Oswald 2016.

[16] Pólya's enumeration method was anticipated by John Redfield in 1927, however, it was not noticed by the community for a long time.

[17] A. Hurwitz, Mathematisches Tagebuch 30, ETH-Bibliothek, Hochschularchiv, Hs 582:30, page 2 (20. Mai 1918).

[18] ETH-Bibliothek, Hochschularchiv, Hs 621: 186.

[19] See Volkert 2018 for an example.


Seit dem Erscheinen von P.Boutroux L'idéal scientifique des mathématiciens bin ich in Verfolgung auch älterer z.B. durch H. Poincaré gegebenen Anregungen einigemale auf die Frage der verschiedenen Stilarten mathematischen Schaffens eingegangen .Während Boutroux die Ursachen wesentlich in Einflüssen der Umgebung sah, suchte ich die Veranlagung der Persönlichkeiten zu erfassen. Als ich im Jahre 1933 im Zuge der politischen Ereignisse in meinem Vaterland mit den älteren Typenbeschreibungen der Rassenkunde und der Psychologie, insbesondere mit den einschlägigen Arbeiten von Erich Jaensch bekannt wurde,glaubte ich das geeignete Mittel zur Beschreibung, wenn nicht gar zur Erklärung der Beobachtungen gefunden zu haben. Ich hoffte zugleich mit jenen Arbeiten bei den Machthabern etwas Interesse für wissenschaftliche Dinge zu erwecken, und damit nicht nur mir gewisse Sicherungen zu verschaffen. Der politische Zusammenhang, in dem man diese Arbeiten stellte und in dem man den Grund für ihre wissenschaftliche Schwäche sah, gaben den Anlass zu einer ungewöhnlich heftigen Kritik. Meine wissenschaftliche Erkenntnis wurde durch diese allgemeine Ablehnung nicht gefördert. Hier verdanke ich der sachlichen Kritik meines Freundes Ostrowski die Einsicht,dass die von mir benutzten Mittel,zu einer Erklärung oder auch nur widerspruchsfreien Beschreibung der Erscheinungen nicht ausreichen. So steht einem wissenschaftlichen Misserfolg,das Ergebnis zur Seite, dass ich vielen auch ernstlich um die Erkenntnis bemühten Personen entfremdet wurde, denen von je meine Verehrung galt und gilt. Dass meine Arbeiten ein Spiel mit dem Feuer bedeuteten, und mich in die Nachbarschaft einer schlechten Gesellschaft brachten,der auch das Gute, das ich neben anderem zu sehen glaubte, nur Maske war für verbrecherische und demagogische Ziele , wurde mir nur allzuspät klar.Nachdem seit Jahren zum ersten Male wieder das Vertrauen der Basler Mathematiker mir für einige Wochen die Gelegenheit zu einem ruhigen wissenschaftlichen Arbeiten und Hervortreten gab,möchte ich erklären, dass ich jene Arbeiten und alle damit zusammenhängenden Irrungen bedaure.

Basel den 12.2.49

Bieberbach

Bieberbach

Fig. 4. Letter from Bieberbach to Hopf.[22]

he edited the journal with this title. Bieberbach tried to get influence in several contexts like the *Deutsche Mathematiker-Vereinigung* and the *Mathematische Annalen*. In Berlin, he created an influential centre of mathematics and he held important positions at the university like dean.[20] After the war, Bieberbach lost his position; only in 1949 was he invited to Basel by A. Ostrowski to lecture there as a guest. At that point, Bieberbach could hope for some rehabilitation. In his letter to Hopf,[21] he speaks about his racist theory of types of mathematical thinking. He explains that Poincaré and Klein had similar ideas before him and that he regrets everything.

In case there is no inventory, other documents can be of help: often you can find a *Biographisches Dossier* on a mathematician created by the archive; *Personalakten* (personal files) of the professors are conserved at Bern by the *Bundesarchiv* (archive of the Swiss Federation); and there is a *Professorenbuch* at the ETH-archive with some information on the professors and their careers. It may be interesting to have a look at the *Missiven* – because the *Schulrat* sent appraisals of candidates proposed for the nomination of professor to the *Bundesrat*, and to the *Schulratsakten* because sometimes other mathematicians were asked for their opinions on a candidate. The nominations were proposed by the *Schulrat* but made by the *Bundesrat* and the nomination procedure was very different to that of the universities. Habilitation was also decided by the *Schulrat* when demanded by the candidate. The right to assign a doctorate was only given to the polytechnic in 1908. Before that, the candidates had to go to a university – usually in Zürich – to get their degree. The documents on these procedures are conserved in the *Staatsarchiv*[23] in Zürich, as are many other administrative documents of the university. From the very beginning of the polytechnic, it was possible to get a habilitation there and so become *Privatdozent*. The candidate demanded their habilitation at the *Schulrat* and, when the candidate was viewed positively, they were promoted to *Privatdozent*. Some names here are Theodor Reye, Heinrich Durège, Carl Friedrich Geiser, Ludwig Stickelberger and Ernst Fiedler.

One can find a lot of lecture notes in the ETH archive. Very nice and legible examples are Marcel Grossmann's lecture notes of courses delivered by Hurwitz, Geiser, Wilhelm Fiedler and others. It was a lucky thing that Marcel Grossmann was very engaged with mathematics and a diligent student – he was able to help his friend Albert Einstein with mathematics. There are also very nice booklets written by one of Grossmann's pupils at the *Kantonsschule*. It seems characteristic of the situation at the polytechnic that there were strong links to schools like the *Kantonsschule*: the training of future teachers led to the creation of a network connecting the polytechnic with its surroundings. This is especially significant in the case of Wilhelm Fiedler, who taught future teachers for 40 years in the field of geometry. His son Ernst Fiedler became director of the *Industrieschule* (later named *Oberrealschule*) at Zürich and has left some very nice lecture notes on courses given by his father and other professors like Weierstrass. The only difficulty with them is that they were written in shorthand.[24] Of course, there are further, more specific sources and documents in the archive. You may search for them with the help of the *Wissensportal* (Knowledge Portal).

Naturally, other interesting people near to mathematics were in Zürich like Einstein, Pauli and Schrödinger. Many documents on them are in the ETH-archive.

From the point of view of the history of mathematics, it may also be interesting to know that there is a small

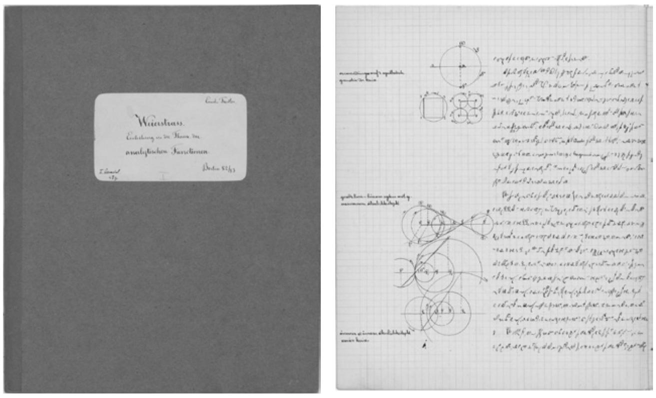


Fig. 5. Pages of Ernst Fiedler's notebooks.[25]

[20] See Mehrtens 1987 and Schappacher 1998.

[21] Hopf was of Jewish origin. After a visit to Germany in 1938, he was arrested and had some trouble with the Nazi regime (see Stammbach 2009).

[22] ETH-Bibliothek, Hochschularchiv, 641: 253a.

[23] This is the archive of the *Kanton Zürich* and it is located in Zürich on the new university campus called Milchbuck. The archive of the Swiss Federation is called *Bundesarchiv* and is located in Bern.

[24] Hermann Weyl also liked shorthand, as did his friend Fritz Medicus, professor of philosophy at the polytechnic (see Eggert 1957).

[25] ETH-Bibliothek, Hochschularchiv, Hs 109 and 110. Title page of a course on analytic functions delivered by Weierstrass at Berlin and a page from lecture notes taken by Ernst Fiedler of a course on Zyklographie delivered by his father Wilhelm Fiedler.

collection of mathematical models bought by Wilhelm Fiedler in order to support the teaching of descriptive geometry and geometry in general (see Figures 6 and 7).[26] A project on these models is in progress.

Fig. 6. Front page of the inventory of the collection of models for the teaching of descriptive geometry.[27]

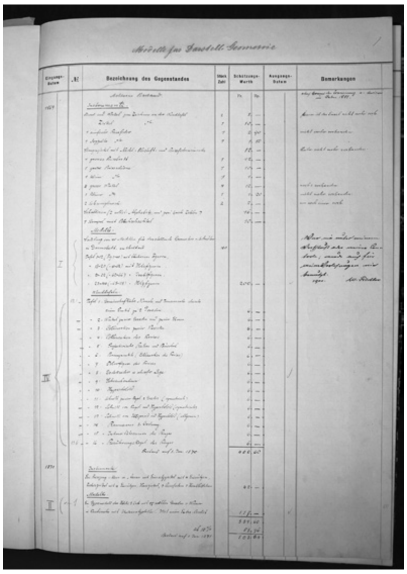


Fig. 7. A page of Fiedler's inventory: buying models for teaching descriptive geometry in 1869.[28]

The library of the ETH owns many interesting books and thesises. An inventory with useful comments was published by P. Richter some years ago.[29]

Sometimes you can find surprising and unexpected documents. We end our survey with such a document: a letter by Jacques Feldbau (1914–1945) written in 1942 to Werner Gysin (see Figure 8).[30] Feldbau was a young Jewish mathematician from Strasbourg (Alsace) who worked on his thesis first at Strasbourg and then – after the transfer of the Strasbourg University to Clermont-Ferrand in 1940 – at Clermont-Ferrand under the supervision of Charles Ehresmann. The year after he wrote the letter below, he was arrested at Clermont-Ferrand and transported to Auschwitz. He died some days before the end of the war on a *Todesmarsch* in Bavaria.[31]

[26] Besides well-known models sold by Brill and later by Schilling, there are also nice examples produced by Jakob Schröder's firm and self-made objects in this collection.
[27] ETH-Bibliothek, Hochschularchiv, Hs 1196: 30.
[28] ETH-Bibliothek, Hochschularchiv, Hs 1196: 50.
[29] See Richter 2015.
[30] ETH-Bibliothek, Hochschularchiv, Hs 646: 1. Gysin was a graduate student of H. Hopf (like B. Eckmann) working in topology.
[31] See Audin 2012.

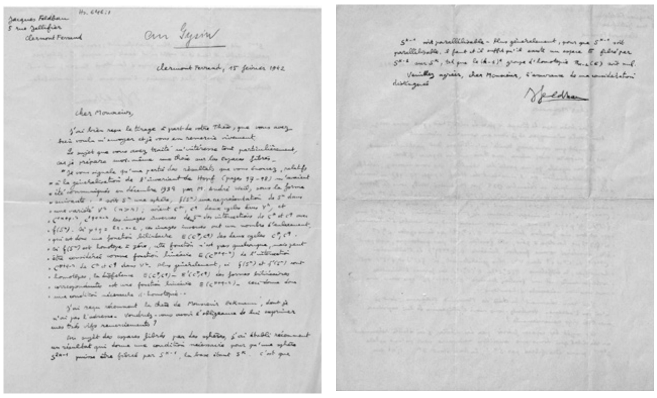


Fig. 8. Letter from Jacques Feldbau to Werner Gysin.[32]

Feldbau thanks Gysin for sending him his thesis. He discusses some results of that thesis and explains that they were communicated to him by André Weil in 1939. Feldbau also thanks B. Eckmann for sending him his thesis and announces some results on fiber spaces that he had discovered recently.

The archive of the ETH is a rich source for interesting documents on the history of mathematics since the middle of the 19th century. Many mathematicians, among them some of the most important ones, worked here, leaving a lot of documents in the archive. It is certainly worth of a visit, not least because you are warmly welcomed here and you get kind and competent assistance from the persons working in the archive. And if you are tired of mathematics, you can go upstairs and have a look at the Max Frisch archive, a famous student of the ETH.

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[32] ETH-Bibliothek, Hochschularchiv, Hs 641: 1.

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Nicola Oswald graduated from the university at Würzburg in number theory. She is now working in the history of modern mathematics, in particular on Adolf Hurwitz, at Wuppertal.

Klaus Volkert is also working at Wuppertal. His main interest being the history of geometry and topology since 1800, in particular that of descriptive geometry. Both are members of the "AG Didaktik und Geschichte der Mathematik" and the "Interdisziplinäres Zentrum für Wissenschafts- und Technikforschung (IZWT)" of the Bergische Universität Wuppertal.

G. I. Barenblatt in Memoriam (1927–2018)

Michiel Bertsch (University of Rome Tor Vergata, Italy), Alexandre J. Chorin (University of California at Berkeley, USA), Nigel Goldenfeld (University of Illinois at Urbana-Champaign, USA) and Juan Luis Vázquez (Universidad Autónoma de Madrid, Spain)

Grigory Isaakovich Barenblatt, December 2005.

The great mathematician and physicist Grigory Isaakovich Barenblatt died in Moscow on 22 June at almost 91 years of age, in the city where he was born on 10 July 1927. We say goodbye with great sorrow to the Master who taught us the beauty of self-similarity applied to flows and fractures and to whom some of us have dedicated so many mathematical solutions that bear his name.

Barenblatt (Grisha to his friends) was the son of the virologist Nadezhda Veniaminovna Kagan, who developed a vaccine against encephalitis and who became infected and died in a laboratory accident, and the Moscow endocrinologist Isaak Grigorievich Barenblatt. His grandfather was the mathematician Veniamin Kagan. He was the son-in-law of Pelagueya Yakovlevna Polubarinova-Kochina, illustrious pioneer of Soviet applied mathematics, who reached her centenary without stopping her work. His younger brother on his mother's side is the well-known mathematician Yakov Grigorievich Sinai, who received the Abel Prize in 2014. These data introduce us to the great Moscovite intelligentsia of the last century and they also speak to us of women with a strong imprint on science.

The early years

Grigory Isaakovich graduated in 1950 from Lomonosov Moscow State University, in the famous Department of Mechanics and Mathematics where he studied under Boris Moiseevich Levitan and Andrei Nikolaevich Kolmogorov. He obtained his first doctorate degree (*Kandidat Nauk*) in 1953 from that university under the supervision of Kolmogorov, one of the greatest mathematicians of the 20th century and someone who had a deep influence on him. Kolmogorov and his collaborators created the cornerstone of all turbulence research – the Kolmogorov-Obukhov analysis of the inertial range of scales, the intermediate range between the energy-containing scales and the scales where dissipation occurs – and also produced the first practical mathematical models of turbulent flows of engineering interest. Barenblatt maintained a lifelong passionate interest in these topics, as well as in dimensional analysis and scaling, tools Kolmogorov used in his analyses and whose offshoots have acquired an enormous role in contemporary mathematical physics. He was also greatly influenced by a long collaboration with Yakov Borisovich Zel'dovich, which later led to a seminal generalisation of the approach that Kolmogorov had used, uncovering the relationship between similarity and asymptotics. This work would be the cornerstone of many contributions to continuum mechanics, including fluid flow in porous media or elasto-plastic porous media, turbulence and fracture.

Grigory Isaakovich received the title of *Doktor Nauk* (second doctorate, a post-doctoral degree equivalent to a British DSc or a French *habilitation à diriger des recherches*) in 1957. In 1962, he became a professor, also at Moscow State University.

He has held the following positions in the USSR:

1953–61 Research Scientist, Senior Research Scientist, Institute of Petroleum, USSR Academy of Sciences.
1961–75 Head, Department of Plasticity, Institute of Mechanics, Moscow University.
1975–92 Head, Theoretical Department, Institute of Oceanology, USSR Academy of Sciences.

Barenblatt's main contributions to science in the Soviet Union are noted below.

Fracture mechanics:
- Fundamental mathematical model of elastic body with cracks based on the explicit introduction of cohesion forces and solutions of basic problems.
- Introduction of one of the basic characteristics of fracture toughness: cohesion modulus.
- Basic model of the kinetics of crack propagation.
- Applications to fracture problems in metals, rocks and polymers.
- Similarity laws for brittle and quasi-brittle fracture.
- Scaling laws for fatigue cracks and multiple fractures and model of small fatigue cracks.
- Mathematical model of non-local damage accumulation.
- Mathematical model of self-oscillation and self-similar phenomena in fatigue fracture.

Theory of fluid and gas flows in porous media:
- Fundamental model of flow in fissurised porous rocks and solutions of basic problems.
- Asymptotic solutions of basic problems of unsteady groundwater and gas flow in porous media.
- Fundamental model of fluid flow in elasto-plastic porous media and solutions of basic problems.
- Non-equilibrium two-phase flow in porous media (capillary imbibition, water-oil displacement and solid phase precipitation): basic mathematical model and fundamental solutions.
- Mathematical model of gas-condensate flow in fissurised porous media.
- Mathematical model of very intense pulse in groundwater flow in porous and fissurised porous rocks.

Mechanics of non-classical deformable solids:
- Mathematical models of neck propagation in polymers (with an analogue to flame propagation) and of thermal vibro-creep in polymers.
- Mathematical model of the impact of a viscoplastic body on a rigid obstacle.

Turbulence:
- Turbulence in stratified fluids.
- Mathematical models of the transport of heavy particles in turbulent flow.
- Basic model of turbulent patch dynamics in stably stratified fluids and self-similar asymptotic laws; relations to oceanic microstructure.
- Mathematical model of non-steady heat and mass transfer in stably stratified turbulent flows.
- Model of turbulent drag reduction by polymeric additives.
- Mathematical models of turbulent bursts and turbulent shearless wake evolution.
- Mathematical model of laminar-turbulent transition taking into account the evolution of pre-existing turbulence.
- Mathematical model of temperature step formation in stably stratified turbulent flows.
- Scaling laws for developed turbulent shear flows, in particular for pipe and boundary-layer flows and wall-jets.
- Mathematical models of dust storms and tropical hurricanes.

Self-similarity:
- Nonlinear waves and intermediate asymptotics (long-term work generally performed in close collaboration with Ya. B. Zel'dovich).
- Concepts of intermediate asymptotics and self-similar asymptotics of the first and second kinds.
- Nonlinear eigenvalue problems.
- Relations between intermediate asymptotics and renormalisation groups.
- Basic model of the stability of self-similar solutions and travelling waves.
- Contributions to the theory of combustion and thermal explosion.
- New model of surface-tension-driven thin films.

Barenblatt abroad

With the *glasnost* movement, the doors were opened in the USSR and Grigory Isaakovich arrived in the West in 1990, visiting the Université de Paris VI. In 1991, he spent the Spring at the IMA Institute of the University of Minnesota, where two of the authors of this article (MB, JLV) first met him. He also met an old friend, Shoshana Kamin, a professor in Tel Aviv, who received her mathematical education in Oleinik's PDE group in Moscow. It was a stellar moment in which Barenblatt presented his ideas and posed multiple mathematical problems that would occupy researchers in our countries for years. He came from the "World on the Other Side", with his applicable equations, a permanent smile and an endless flow of stories. He sought to unite in science the best of both worlds and his life is an example that shows that it is possible. A number of the attendees reoriented their research to welcome his ideas and mathematical problems and have been solving some of those problems and raising new ones ever since. Since that Spring in Minnesota in 1991, they never lost scientific and human contact.

Then, Barenblatt became G.I. Taylor Professor of Fluid Mechanics at the University of Cambridge from 1992 to 1994 and has been G.I. Taylor Professor Emeritus of Fluid Mechanics since then. He held this to be his highest honour and the stay in Cambridge affected him deeply. Nobody understood better than he the importance of the longstanding British tradition in fluid mechanics and, with several of its representatives, such as Batchelor, Crighton and Lighthill, he had a deep scientific and human relationship. As an active Professorial Fellow of the Gonville & Caius College in Cambridge, he enjoyed British academic traditions in general. He was a visiting professor at the University of Rome Tor Vergata (1992) and at the University of Minnesota (1994). In 1993, he visited the Department of Mathematics of the Universidad Autónoma of Madrid as a BBVA Visiting Professor. He returned to Spain in 1996 as Iberdrola Visiting Professor at the Universidad Autónoma de Madrid.

Grigory Isaakovich arrived in Berkeley in February 1996 as a visiting professor, after a stay at the University of Illinois. This was when another of the article's authors (AJC) met him; this was the beginning of a long collaboration and friendship.

It soon became clear that the Mathematics Department at Berkeley would be delighted to have Grigory Isaakovich for a much longer stay as Professor in Residence, a highly honoured position that does not require a heavy teaching load, with a concurrent appointment as a mathematician at the Lawrence Berkeley National Laboratory. In particular, many of the applied mathematicians at Berkeley were oriented towards computing and were enthusiastic to collaborate with a great master who had a different perspective. When he came to Berkeley, Grigory Isaakovich was already committed to a semester-long visit to Stanford, so he went there for a few months and then returned to Berkeley.

The mathematics group at the Berkeley Lab had a suite of offices where the doors were open and it was easy for faculty, postdocs and students to talk and collaborate. Grigory Isaakovich fit wonderfully into this environment. He provided advice, information, perspective and leadership. He used to invite the young mathematicians to afternoon tea in his office where he talked to them about great scientific problems, about his career and life and about science in the Soviet Union. They loved it, stayed for hours and asked questions; he gave them a perspective on the joys and possibilities of a great scientific career and a model to emulate. He taught some extraordinary courses, on topics such as fluid mechanics, fracture, turbulence and porous media; a large part of his audience was made up of faculty, from a variety of departments. He became an essential part of the applied mathematics seminar; some adaptation was needed because traditionally scientific seminars in Russia tend to be more confrontational than the ones in the United States and Grigory Isaakovich could be quite critical of poor presentations. Under his impact, the seminars became more lively, more interesting and more instructive.

Grigory Isaakovich excelled at linking different worlds, such as Russian and Western science. His major pedagogical impact at Berkeley was to link computing people to asymptotics and scaling. This produced better scientists and its impact is growing.

Grigory Isaakovich returned to live full-time with his family in Moscow in his last years. At 90, he still went to work in his Oceanology Laboratory every day he could.

With his collaborator Prostokishin on his 86th birthday in Moscow.

Honours

Barenblatt held foreign memberships at the US National Academy of Sciences, the US National Academy of Engineering, the American Academy of Arts and Sciences and the Royal Society of London, as well as a long list of scientific societies in multiple countries. The long list of his honours and awards includes the G. I. Taylor Medal of the US Society of Engineering Science, the J. C. Maxwell Medal and Prize of the International Congress for Industrial and Applied Mathematics and the Timoshenko Medal of the American Society of Mechanical Engineers.[1]

Writings

Barenblatt was a superb and dedicated writer. His books include the following:

- *Flow, Deformation and Fracture*. Cambridge University Press, Cambridge, UK, 2014.
- *Scaling*. Cambridge University Press, Cambridge, UK, 2003.
- *Dimensional Analysis*. Gordon and Breach NY, USA, 1987.
- (With V.M. Entov and V.M. Ryzhik), *The Motion of Fluids and Gases in Natural Rocks*. (in Russian) Nedra, Moscow, 1984; (in English) Kluwer, 1990.
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- (With A.P. Lisitzin) *Hydrodynamics and Sedimentation*. (in Russian) Nauka, Moscow, 1983.
- *Scaling, Self-Similarity, and Intermediate Asymptotics*. (with a foreword by Ya.B. Zel'dovich) (in Russian), Gidrometeoizdat, Leningrad, USSR, 1978, 1982; (in English) Plenum Press, NY, USA 1979; Cambridge University Press, Cambridge, UK, 1996.
- *Dimensional Analysis and Self-Similar Solutions* (in Russian), USSR Academy of Sciences, Moscow, 1975.

1 See full listing in http://math.lbl.gov/barenblatt/barenblatt_paper_mono.html

- (With V.M. Entov and V.M. Ryzhik) *Theory of Unsteady Filtration of Fluids and Gases*, (in Russian) Nedra, Moscow, 1972.

He was also co-editor of a number of books and author of numerous scientific publications dating from 1952 to his late years.

Comments on his scientific work and legacy

One of the major themes of Grigory Isaakovich's work over the years has been the development and application of scaling ideas to a variety of problems. Much of this work grew out of his seminal work with Ya.B. Zel'dovich on self-similarity as intermediate asymptotics. This was summarised in a remarkable review article (G.I. Barenblatt and Ya.B. Zel'dovich, "Self-similar solutions as intermediate asymptotics", *Annual Review of Fluid Mechanics*, 4 (1) (1972), pp. 285–312), which brought to a Western audience three under-appreciated ideas. Firstly, the conventional use of what is called "dimensional analysis" to simplify problems when one dimensionless group is negligible, makes an assumption about regularity that is frequently unjustified. When this assumption is removed, paradoxes can be resolved and new scaling laws emerge with non-rational exponents. In these problems, there is "scale-interference": the equation retains the memory of initial conditions, for example, even at asymptotically large time, in contrast to naive expectations. Secondly, these examples of so-called incomplete similarity can be dynamical attractors for a wide range of initial conditions and are associated with degenerate initial conditions that are generalised solutions (but not delta-functions). These attractors describe the dynamics for intermediate times between when initial transients have decayed and when the system reaches its final state, and hence are called intermediate asymptotics. Thirdly, these similarity solutions are, in fact, special cases of rather general Lie group symmetries. An important example is the analysis of travelling waves, where the front interpolates between a stable and an unstable fixed point. The speed can then be mapped by a simple change of variable into an exponent associated to a solution with incomplete similarity. This insight pertained especially to the famous waves analysed by Kolmogorov, Piskunov and Petrovsky in 1937. On the wall of his home office in Berkeley, Grigory Isaakovich had photos of the first and last authors of this work, along with several other individuals who meant a lot to him.

Barenblatt also speculated that there could be some connection between incomplete similarity and the problems of critical exponents at second-order phase transitions. The latter problem exercised the theoretical physicists around the world during the 1960s and 1970s, perhaps nowhere more so than in Moscow. This link was indeed later established directly by one of the article's authors (NG) and co-workers, showing *inter alia* how the elasto-plastic porous medium equation bearing Barenblatt's name could be solved by renormalisation group methods and providing analytic series expansion formulae for the exponents arising in its intermediate asymptotics. Grigory Isaakovich singled out these developments as a partial motivation for the updated revision of his classic book *Similarity, Self-similarity and Intermediate Asymptotics*, which proved to be highly influential in the West.

In Berkeley, he focused on the possibility that incomplete similarity would provide a more faithful description of the "intermediate" region in bounded shear flows, i.e. of the region not immediately adjoining the wall but not yet far from the wall, as well as a coherent theory of corrections to Kolmogorov's scaling in the inertial range of turbulence. The shear flow problem is of great practical importance – the scaling law for the intermediate layer provides a crucial input for a wide variety of computational models, for example, in the tracking of hurricanes and in the design of combustion engines. Grigory Isaakovich and his associate V.M. Prostokishin had deduced the coefficients in the scaling by processing experimental data and comparing the resulting laws to the data. This showed that the incomplete similarity solution fit the data to within experimental error in all cases, unlike the older, simpler scaling law (known as "the law of the wall"). Also, unlike in previous scaling laws, the parameters assumed to be constant remained constant in all cases, unlike the "constants" in the previous law, which varied by as much as 30% from one flow to the other. This left theoretical work to be done, showing that the resulting intermediate layer is asymptotically consistent with the rest of the flow and providing examples where the new scaling can be checked analytically. This was done during Grigory Isaakovich's stay in Berkeley in collaboration with AJC.

Kolmogorov's scaling of the intermediate ("inertial") scales of turbulence far from boundaries is the foundation of all theories of turbulence, yet experiments show that it is not exact and its derivation has been challenged many times, in particular by L.D. Landau. Kolmogorov himself eventually offered a "corrected" theory that has not been widely accepted. Barenblatt showed that incomplete similarity provides a self-consistent viscosity-dependent correction that converges to the original Kolmogorov solution in the limit of vanishing viscosity. This is a novel idea, which is well supported by the available experimental data.

A recurrent theme of his research was the use of similarity methods in wide classes of nonlinear heat equations known in the Russian literature as filtration equations. A simple nonlinear version of the heat equation that first comes to mind is the following equation: $\partial_t u = \Delta(u^m)$, with $m>1$, which is usually called the Porous Medium Equation (PME). It is relevant since it is nonlinear and, moreover, it is degenerate parabolic at the level $u = 0$ so that the diffusion coefficient is just mu^{m-1}. Hence, it can serve as a model of restricted propagation that leads to free boundaries, sharp fronts with finite speed of propagation. It has been studied in d-dimensional Euclidean space, with interest in the cases $d = 1, 2, 3$ for the applied scientist and with no dimension restriction for the mathematician. Δ represents the Laplace operator acting on the space variables. This equation appears in the description of different natural phenomena and its theory and properties depart strongly from the classical heat equation, $\partial_t u = \Delta u$, its most famous relative – hence the interest in its study, both for the pure mathematician and the applied scientist. There are a number of physical applications where this simple model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Maybe the best known of them is the description of the flow of an isentropic gas through a porous medium, modelled independently by the engineers L.S. Leibenzon in the USSR and M. Muskat in the USA around 1930. An earlier application is found in the study of groundwater infiltration by Boussisnesq in 1903. Another important application refers to heat radiation in plasmas, developed by Zel'dovich and co-workers around 1950. Indeed, this application was at the base of the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, the spread of viscous fluids, boundary layer theory and other fields.

Serious progress in understanding this equation was reached around 1950 in Moscow by Zel'dovich, Kompaneets and Barenblatt, who found and analysed a solution representing heat release from a point source. This solution has an explicit formula in self-similar form $U(x, t) = t^{-\alpha}\mathrm{F}(|x|t^{-\beta})$, with similarity exponents determined by the algebra of the equation and mass conservation. They found that the profile F has the explicit form of an inverted parabola cut at the level $u = 0$ so that two important properties are manifested: firstly, there exists a free boundary located at a distance $|x| = ct^{\beta}$ and secondly, the solutions are not differentiable enough to be classical. Since β turns out to be less than 1/2 for $m > 1$, the propagation does not follow the Brownian scaling and the process can be considered as a relevant example of anomalous diffusion. Barenblatt not only did the rigorous analysis of these solutions (which now usually bear his name) in his 1952 paper but also found the self-similar source-type solution for the nonlinear heat equation with gradient diffusivity that we now call the evolution p-Laplacian equation (EPLE).

The PME equation has had enormous success with pure and applied mathematicians after Oleinik and collaborators supplied the first proof of existence, uniqueness and finite propagation of non-negative solutions of the PME in 1D in 1958. Gradually, the theory was developed with increasing generality in the last decades of the 20th century, much to Grigory Isaakovich's surprise and delight, and the results bear distinguished names: Kamin and Friedman supplied the first asymptotic proof; Bénilan, Brezis, Crandall, Evans and Pierre were busy with the construction of solutions with quite general data and the corresponding nonlinear semigroup theory; Angenent, Aronson, Caffarelli, DiBenedetto, Friedman, Sacks and others supplied the regularity theory for solutions and interfaces; and Peletier and collaborators supplied new self-similar solutions. Parallel progress occurred for the EPLE. Based on all this knowledge, DiBenedetto wrote a book on Degenerate Diffusions (1993) and one of the article's authors (JLV) wrote a book describing the state of the art for the PME (Oxford Univ. Press, 2007). This goes to show the enormous influence of the ideas originally coming from Zel'dovich and given mathematical treatment by Barenblatt.

In his office in the Shirshov Institute of Oceanology, Moscow 2013.

In the current century, Barenblatt's ideas have had an important impact on the theory of nonlinear PDEs in various directions. One was the growth of entropy methods, coming from statistical mechanics and used to prove the convergence of general solutions to stable profiles. The techniques of scaling, so dear to Grigory Isaakovich, made possible the use of entropy methods on nonlinear heat equations of the previous types to establish intermediate asymptotics to Barenblatt solutions with rates (we refer to work by Markowich, Toscani, Carrillo, del Pino, Dolbeault, Bonforte, Grillo, etc.). A second direction is the study of nonlocal diffusion, based mainly on the use of fractional Laplacian operators. The name "Barenblatt solutions" appears in a number of models of nonlinear, nonlocal models as self-similar profiles evolving from a point-mass initial datum, and asymptotic convergence proofs have been found for more general data. One of the article's authors (JLV) has been intensely involved in such directions and has been witness to the sense of wonder that this ongoing "Barenblatt activity" always produced in Grigory Isaakovich. A third direction derives from the mathematical treatment of heat and mass transfer in stably stratified turbulent shear flow, where the physically relevant regularisation of an ill-posed diffusion equation is still so degenerate that solutions may become discontinuous. This has led to a systematic study of discontinuous transient solutions of nonlinear PDEs, mainly by a PDE group in Rome that was strongly influenced by Barenblatt's continuous encouragement to develop PDE theories motivated by his models of turbulence, flow in porous media, thin liquid films and damage accumulation. In this sense, Grigory Isaakovich's enthusiasm had no limits: he asked such PDE questions even during his last years at the Oceanology Laboratory in Moscow, where Joost Hulshof, Grigory Isaakovich's collaborator and friend from Amsterdam, and one of the article's authors (MB) had the privilege to visit him more than once to discuss turbulent shear flows with suspended particles.

The writers of this article think that Grisha's life adventure will endure as a brilliant tale of the fruitful encounter of two worlds, a rare event that we were fortunate enough to witness and that was, in large part, due to his immense curiosity. Grisha held clear ideas about the need for a strong engineering-physics-mathematics interaction that includes pure mathematics, and also about the need for a strong connection between science and culture and for a better understanding between cultures. He will always remain in our memory as an example and a scientific hero.

Michiel Bertsch has been a professor of mathematical analysis at the University of Rome Tor Vergata since 1990. Between 1997 and 2014, he headed the Istituto per le Applicazioni del Calcolo "Mauro Picone" (CNR) in Rome. His research area is nonlinear PDEs and their applications.

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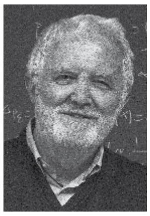


Juan Luis Vázquez has been a professor of applied mathematics at the University Autónoma de Madrid from 1986 to 2016 and is now an emeritus professor and member of the Spanish Royal Academy of Sciences. His research areas are nonlinear PDEs, free boundaries and asymptotic methods.

Flipping JACO

Mark C. Wilson (University of Auckland, New Zealand), Hendrik van Maldeghem (Ghent University, Belgium), Victor Reiner (University of Minnesota, USA), Christos Athanasiadis (University of Athens, Greece), Akihiro Munemasa (Tohuku University, Sendai, Japan) and Hugh R. Thomas (Université du Québec, Montréal, Canada)

This article is about the conversion of a subscription mathematics journal owned by a large commercial publisher into an independent, open access journal. Mark C. Wilson discusses the background to this and the main editors of the journal describe their impressions and experiences.

Introduction (Mark C. Wilson)

In July 2017, the four editors-in-chief and the whole editorial board of *Journal of Algebraic Combinatorics* (JACO) gave notice of their resignation to the publisher, Springer. A new journal with the same editors, *Algebraic Combinatorics* (ALCO), published by the Centre Mersenne based at the University of Grenoble, was announced almost immediately and began publication in January 2018. What was behind these moves?

MathOA (of which I am a board member) is a foundation (*Stichting*), set up in the Netherlands in late 2016, dedicated to achieving precisely this kind of outcome, that is, the "flipping" of established mathematics journals to a model known as "Fair Open Access". The five principles behind Fair Open Access are as follows (many more details are available at http://fairopenaccess.org):

1. The journal has a transparent ownership structure and is controlled by and responsive to the scholarly community.
2. Authors of articles in the journal retain copyright.
3. All articles are published open access and an explicit open access licence is used.
4. Submission and publication is not conditional in any way on the payment of a fee from the author or its employing institution, or on membership of an institution or society.
5. Any fees paid on behalf of the journal to publishers are low, transparent and in proportion to the work carried out.

Many existing journals satisfy (more or less!) these five conditions. Some of them are fairly well known (for example, *Electronic Journal of Combinatorics* and *Journal of Computational Geometry*). However, many established mathematics journals are owned by large commercial publishers such as Elsevier, Springer, Taylor & Francis and Wiley. MathOA is engaged in systematically contacting editorial boards of such journals and raising money to support flipped journals. The main obstacles in the minds of editors appear to be a fear of loss of quality and reputation, worries about financial stability of open access journals and concerns about the work involved in transitioning to a new publisher. MathOA was established precisely in order to alleviate these concerns.

Some prominent, commercially published journals are owned by learned societies: for example, *Communications in Pure and Applied Mathematics* (owned by the Courant Institute and published by Wiley), *Israel J. Math.* (owned by The Hebrew University Magnes Press and published by Springer) and *Publications Math. de l'IHES* (owned by IHES and published by Springer), and there is some chance of bringing market forces to bear. However, large commercial publishers own the titles of such well known journals as *Inventiones Mathematicae*, *Advances in Mathematics*, *Journal of Algebra*, *Journal of Functional Analysis*, *Journal of Number Theory*, *Journal of Combinatorial Theory A*, *Journal of Combinatorial Theory B* and *Discrete Mathematics.* They will not relinquish these names lightly, since trading on the reputation mostly created by others is the basis of their enormous profit margins. Thus, flipping a journal often requires a change of name. With appropriate community buy-in, the reputation of the old journal (which has almost nothing to do with the publisher and everything to do with the authors, reviewers and editors who have contributed) usually transfers quickly to the new (for example, see my basic analysis at https://mcw.blogs.auckland.ac.nz/2016/10/08/what-happens-to-journals-that-break-away/).

Hendrik Van Maldeghem

We were first contacted on 26 September 2016 by Mark Wilson as to whether we would consider a "flip". There was some discussion among the editors-in-chief, including Vic Reiner, a former EiC (Editor-in-Chief) who would help us a lot with the flip. We quickly converged to agree with the flip. Our initial concerns were the reputation of the journal and the amicable relationship with Springer that we would put on the line, not yet being that concerned either about funding or about the platform we would be using. On 6 October, we replied to Mark that we all agreed.

One thing that made things easier was that all our contracts, except Hugh's, ended at the same time (31 December 2017). So, canonically, we could start the new journal then. Hugh, who began his work with JACO on 1 January 2017, pulled off a one-year contract, so we could all resign at the same time. Up until April 2017, not much happened except that a major funder suddenly withdrew. However, we were assured that funding would still be no problem so we did not have to worry about this, and we did not. In March 2017, we started composing a letter to the full board to ask them to join us at the new journal and, from that moment on, things felt more serious. So serious, in fact, that I started to have doubts about the reliability of the project. In particular, when we showed the letter that we had composed to send to the editors to Mark, he reacted "not to be over-promising" to the editors. That made me nervous since suddenly and for the first time it seemed that we could not take things for granted. The four of us decided to contact Vic and chat with him. He persuaded us to continue with solid arguments and since then I have had no doubts anymore. It was natural for us to ask Vic to be the interim EiC of the new journal, which we would baptise "Algebraic Combinatorics". In fact, Vic has been a great help at every step of the transition; he was the one we could rely on and, for me, his presence always gave me a feeling of certainty. His determination and energy was very contagious.

Our letter was sent to the editors and then, for me, the work became less. As a non-native English speaker, I preferred not to be involved with the advertising, other PR jobs or helping compose a constitution. I could just focus on the rest of my editorship with JACO.

Around the end of June, we let Springer know that we would not renew our contracts and that was also a difficult moment, if only psychologically. Indeed, we had worked with people like Elizabeth Loew for several years and kept good contact, and so it felt like a betrayal. But we kept things professional and, after all, we were not doing anything illegal. Springer made some modest attempts to keep us on board and change our minds but this did not do any good.

The next thing to confront us was the installation of the new EiC of JACO, who began his duties in October. Ilias Kotsireas is his name. His point of view was completely opposite to ours. For instance, he wanted to send every submitted paper to referees, which, in our opinion, is very inefficient. Soon, one of us (Christos) could no longer deal with Ilias' way of working and quit the team early. I remember that moment was hard since, on the one hand, one wants to show solidarity with a co-editor but, on the other hand, we had a responsibility toward the authors and submitted papers. The three of us (Hugh, Akihiro and I) decided to stay and I wrote a letter of reconciliation to Elizabeth. She appreciated that and we could continue working. But it was very difficult. Ilias was granted the sole right to reject and accept papers. Although we were seriously doing our jobs, by suggesting immediate rejections of some manuscripts, Ilias never followed these suggestions. On the contrary, he sent all manuscripts to reviewers. So, in fact, we gave up after a few weeks in trying to educate the new EiC; after all, it was not our concern anymore. We were cut off from access to the electronic manager a few days earlier than we expected and so our time at JACO had come to a definite end. I was happy to be able to concentrate on the new journal, ALCO. I just hope that the manuscripts at JACO with a pending decision were treated seriously, which I doubt. I wrote a letter to Elizabeth to show my appreciation for her as a person and to let her know that our decisions were, of course, purely professional and that we could part with a positive feeling.

Victor Reiner

The notion of JACO leaving Springer was not totally new. I was a JACO EiC during 2000–2005 and, around 2004, we had several issues of dissatisfaction surrounding the journal's transition from ownership by Kluwer to Springer, and Springer's business practices (including high subscription fees and outrageously high fees for à la carte article purchases). This eventually led the four EiCs to a negotiation with the new Springer representative, Ann Kostant. We entered the negotiation with the support of our editorial board, leaving open the possibility of leaving Springer and following the lead of the successful open access *Electronic Journal of Combinatorics*. After that negotiation, the four EiCs had differing feelings about whether to stay or leave Springer but in the end we decided to stay.

Fast-forward 13 years to 2017 and the landscape had changed. One of the issues holding us back in 2004 was concern over publication of paper volumes – this no longer seems important. Another issue in 2004 was the service provided by a commercial publisher via its editorial management software – this issue has evaporated through the development of free systems like OJS (Open Journal Systems) or reasonably-priced systems like EditFlow. The nail-in-the-coffin for me was when I was approached by Mark Wilson of MathOA in early September 2016. He told me of MathOA's goal of flipping maths journals and their efforts to get financial support from library consortia to support such moves, along with the legal and technical experience of people at MathOA, such as Johan Rooryck, who had flipped the commercially-owned linguistics journal *Lingua* to the open access journal *Glossa*. I eventually suggested that JACO might be a candidate for such a flip and expressed my willingness to work toward it.

From there onward, things went as Hendrik has described. On my end, once I had discussed the possibility of flipping with the four JACO EiCs and they seemed willing, I then mustered the support of the editorial board of the journal. The board took it seriously and there was discussion of alternatives and concerns. In the end, they were willing to follow the will of the EiCs and resign at the same time, on 31 December 2017. Once the EiCs had presented their resignation letter to Springer in June 2017, I and the board presented a joint resignation letter to Springer in July.

Meanwhile, since the JACO EiCs were working for the old journal until the end of the year, I set about starting the new open access journal ALCO, as an interim EiC until the JACO EiCs could take over. Fortunately, Satoshi Murai was available to join me as the second interim EiC of ALCO, with the intention that he would take the place of Christos Athanasiadis as the fourth EiC after 1 January 2018. Satoshi turned out to be a fantastic partner and the two of us began handling the papers that came pouring in. During the journal set-up process, Mark Wilson and Benoît Kloeckner of MathOA were indispensable in helping us examine various options for service providers and publishing structure. Eventually, we decided to use the new Centre Mersenne in Grenoble as a service provider, which has worked out very well. They provide much of the infrastructure, including maintaining a version of the editorial software OJS, hosting the papers on their site, helping us choose a style file for the journal and providing DOIs for papers for almost no cost. Their main charge is for the copy-editing required to fix some authors' articles after they have prepared them in the journal style file, for which they charge the very reasonable price of 7 euros per page (many articles do not need these fixes). Centre Mersenne's service and responses to our questions and concerns have been excellent.

In terms of community support, almost everyone has been behind us – support has been particularly strong among younger mathematicians. In July 2017, I sent a mass email to the algebraic combinatorics community, announcing and explaining the flip and asking people to support ALCO and to stop supporting JACO. As Springer and the new JACO EiC Kotsireas began inviting new names to populate their various editorial and advisory boards, many people told me that they had refused. However, as a few new names appeared on the list of editorial boards posted on the JACO website, I tried to politely explain the situation to these new editors and ask them to consider removing themselves from these boards. Some responded, explaining their reasons, and others did not. Some withdrew from the boards. My hope is that, in the end, the old journal will fade away.

Christos Athanasiadis

Another of our initial concerns (as far as I remember) was the amount of our time and effort we would have to put into the flip. As it turned out, this was not so bad (mainly thanks to MathOA support) so other editors considering flipping in the future should not be discouraged by this matter.

I was a bit surprised about Springer appointing an EiC without consulting us about his suitability. This showed that they only cared about retaining the journal as their property and not at all about its content or quality. I could not possibly have cooperated and did the right thing for everyone (I believe) in quitting when the chance came up.

My impression is that the procedure was easier than we initially expected (and the publisher had no way to stop it).

Akihiro Munemasa

Since I became an editor-in-chief in 2000, I have received a lot of respectful words from mathematicians in this area, probably much more than I deserve. It became clear to me that, in addition to daily editorial work, editors are supposed to be decision-makers of what our research area should be aiming for. Let me explain why and what made us leave Springer. Our activity is almost entirely based on our own mathematical interest, at least in the beginning, and the publisher's role was minimal when I started. Around 2004, Springer bought the former publisher Kluwer, so we started working with Springer's editorial office. Also, around the same time, quantitative evaluation of research performance started to prevail; notably, the publisher and editors needed to care about Impact Factors. Springer, as a commercial publisher, worked hard to sell the journal to a larger number of customers. Trying to improve Impact Factors is one thing but packaging with some other journals to increase subscriptions is another. While this behaviour affected our work very little, I started to feel that I was not doing the highest priority work for mathematicians. We should have the right to say if a journal is not serving the mathematical community in the best way we could imagine.

However, with a commercial publisher, a journal runs according to their policy and editors are kind of "hired" to run the journal. Instead, editors should work voluntarily based on their own mathematical interest and should choose a publisher that does the job non-commercially.

I might be too optimistic. Increasing use of quantitative measures in research in all disciplines makes people ignore such opinions. Mathematicians might be in a minority by resisting commercial intervention in their research. But I was encouraged to hear that a journal in linguistics had already made a flip before us and this was one of the important reasons that made us think we were making the right move.

Hugh Thomas

I volunteered to be the contact person for the editors-in-chief in dealing with media inquiries. To get ready for this, we collectively wrote a set of responses to questions we anticipated being asked, which was a helpful exercise. I was a bit worried that being the contact person would mean journalists calling me at all hours of the day and night for months; this turned out not to be the case. In fact, one journalist got in touch, from Inside Higher Ed, which led to an article (https://www.insidehighered.com/news/2017/07/31/math-journal-editors-resign-start-rival-open-access-journal) giving a pretty clear and accurate account of what had happened. On the whole, I would say talking to the media was less of a big deal and less stressful than I had imagined.

Afterword (Mark C. Wilson)

In the case of *Algebraic Combinatorics*, it is already clear that it is a success and is the true successor to the journal founded in 1992 by Chris Godsil, Ian Goulden and David Jackson. More details behind the critical claims made above about the expertise of the current JACO boards can be found at http://mcw.blogs.auckland.ac.nz/2018/04/19/alco-vs-jaco-a-stark-comparison/.

Since the start of 2018, a new organisation – the Free Journal Network (FJN) https://freejournals.org – has been created. It is intended to promote and nurture Fair OA-compliant journals and now has 24 members in mathematics, including *Algebraic Combinatorics*. I am on the steering committee.

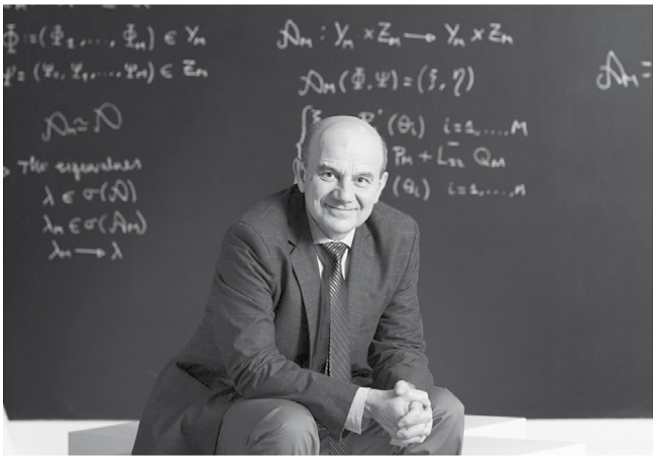
I urge all editors of subscription journals to think hard about their responsibilities to the research community and the wider society and to contact MathOA to discuss flipping their journals. Also, I urge all mathematicians to support journals run according to the Fair Open Access model (e.g. members of FJN) and not contribute their volunteer labour to journals that transfer public money to private corporations while providing inferior service. Finally, all mathematicians are welcome to contribute to the discussion/action forum https://publishing-reform.gitlab.io/.

BCAM, the Basque Center for Applied Mathematics

Jean-Bernard Bru and Carlos Pérez (both University of the Basque Country and Basque Center for Applied Mathematics, Bilbao, Spain)

The Basque Center for Applied Mathematics (BCAM) is a world-class research centre in the field of applied mathematics, located in Bilbao (Spain). Its main goal is promoting scientific and technological advances worldwide through interdisciplinary research in mathematics and training and attracting talented scientists.

Embedded in a multicultural environment, with more than 90 people from over 25 nationalities working there, BCAM is a young centre that provides the right atmosphere for research and promotes the creation of hard-working international and interdisciplinary teams.



Luis Vega, BCAM's Scientific Director.

10 years of research at the frontiers of mathematics

BCAM was founded 10 years ago, in September 2008, and Professor E. Zuazua was the founding scientific director. It was promoted by the Basque Government through Ikerbasque (Basque Foundation for Science) in the framework of the BERC (Basque Excellence Research Centers) network. The University of the Basque Country and Innobasque (Basque Innovation Agency) joined BCAM as founding members and the Biscay Regional Government joined later as an institutional member.

In 2013, BCAM was accredited for the first time as a "Severo Ochoa" centre of excellence by the State Research Agency, which has been part of the Spanish Ministry of Science, Innovation and Universities for four years. The accreditation was given to the centre for the second time during the 2017 call for proposals. This distinction, which recognises the international relevance of the scientific research carried out, as well as the global interest of the proposed work programme for the next four years, is given to the best research institutions in the world in their fields.

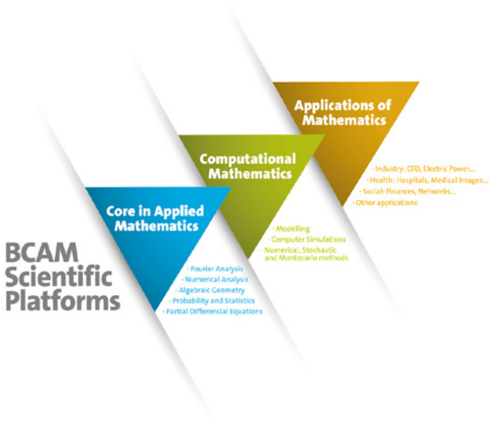
BCAM's team is led by Professor Luis Vega, a well renowned mathematician with an extensive professional career, who became BCAM's scientific director in 2013. Vega has been a full professor of mathematical analysis at the University of the Basque Country for 23 years and a visiting professor at several international universities. He is also a member of many international research institutes, a fellow of the American Mathematical Society, a member of the European Academy of Science and has recently been elected a member of the Spanish Royal Academy of Sciences (RAC). He received the Euskadi Research Prize 2012 and a European Research Council Advanced Grant in 2014 and was awarded with the 2015 Blaise Pascal Medal in Mathematics. Vega is also BCAM's representative at ERCOM, a committee under the EMS consisting of scientific directors of 26 European mathematical research centres.

According to him, making scientific progress and improving the social appreciation of mathematics will be BCAM's main goals on its 10th anniversary, as well as dissemination and training in this area. Regarding the day-to-day life at the centre, he claims that "it is essential that a research centre creates a healthy environment in which there is complicity among researchers and in which one can learn just by breathing" and he "strongly" believes that this is being achieved at BCAM.

Research areas and projects

The research carried out at the centre is oriented by three scientific platforms: Core in Applied Mathematics, Computational Mathematics and Applications of Mathematics.

The first platform, Core in Applied Mathematics, refers to research carried out on mathematical challenges at the frontier of knowledge. This corresponds to the purest mathematics and is somehow timeless because it faces the most important mathematical challenges of our time. Computational Mathematics, the centre's second platform, focuses on developing precise mathematical models and simulations, as well as numeric experiments, using powerful computational resources. Finally, the third platform, Applications of Mathematics, covers the techniques and methods that have real-life applications in industry and society in general.



BCAM's Scientific Platforms.

This structure makes BCAM a multidisciplinary centre in which mathematics and computer science occupy more than 50% of the scientific production. The team at BCAM is distributed into five research areas covering various relevant fields of applied mathematics.

1. Computational Mathematics (CM)

This area is divided into three research lines: Simulation of Wave Propagation, CFD Microfluids & Rheology and CFD Computational Technology. The researchers in this area work on new mathematical methods and robust numerical schemes and software to solve complex and large-scale, challenging, real-life problems on massively parallel computers. Some applications of their developments are the characterisation of the Earth's surface composition for CO2 sequestration and oil and gas extraction, computational fluid dynamics applied to medicine, meteorology, oceanography, aeronautics, naval architecture, acoustics and turbomachinery, and the tackling of several complex microflow problems in material, food and biomedical sciences.

2. Mathematical Modelling with Multidisciplinary Applications (M3A)

Many open challenges in life sciences modelling require efficient algorithms and robust supporting theories. The objective of the research lines included in the M3A group – Modelling and Simulation in Life and Materials Sciences, Mathematical Modelling in Biosciences, and Mathematical, Computational and Experimental Neuroscience – is the elaboration of novel theoretical and computational tools for efficient and detailed simulation of multi-scale complex systems describing real-life problems in biology, medicine, public health and society. In fact, BCAM is working on a new Neuroscience Laboratory that will develop new technologies based on mathematics and analyse large volumes of clinical data in collaboration with other research facilities. This laboratory will serve as an interface between neuroscientists and clinicians.

3. Mathematical Physics (MP)

At the interface between mathematics and physics is the so-called mathematical physics, which at BCAM is represented by the research lines of Quantum Mechanics, Statistical Physics and Singularity Theory & Algebraic Geometry. This group aims toward the mathematical understanding of theories of physics and the development of methods that could, in the future, be applied, for example, to quantum technologies, the forecast of wildland fire propagation to preserve natural heritage, cryptography and string theory.

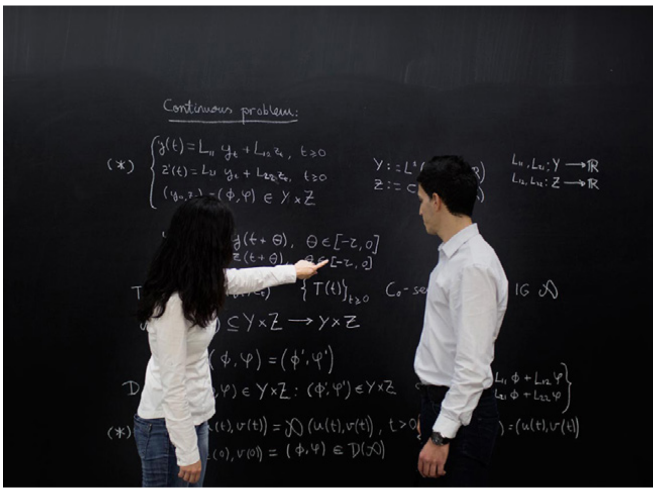
4. Analysis of Partial Differential Equations (APDE)

This group uses PDE models to describe real-life phenomena efficiently. They explore and exploit the deep connections between partial differential equations, harmonic analysis, inverse problems and applied mathematics so as to describe the most diverse phenomena. The understanding of the fundamental principles that control the relevant phenomena in physics and biology could eventually become of use for scientists working in those fields. The group consists of three research lines: Linear and Non-linear Waves, Harmonic Analysis and Applied Analysis.

5. Data Science (DS)

The increase in data generation (big data) and problem sizes has made indispensable the development of new statistical and machine learning methods and algorithms for knowledge extraction and optimisation. The Data Science group at BCAM, divided into the Heuristic Optimisation, Applied Statistics and Machine Learning research lines, works on massive data and optimisation problems in the following areas: financial and social media, cybersecurity, marketing, medical domains (diagnosis and prognosis), genetics, environmental modelling, demography and biostatistics, logistics, and scheduling and planning.

Although BCAM is a multidisciplinary research centre, in the last period, a special effort has been made to encourage collaboration between different lines to foster synergies. In addition to the common research interests that some groups share, there are new initiatives in which teams that belong to different research areas participate together.



Researchers at BCAM's facilities.

All the groups mentioned above work on highly competitive projects that range from top level projects, funded by the European Research Council (ERC) or the Marie Skłodowska-Curie Research and Innovation Staff Exchange, to research and development projects, funded by the Basque and Spanish Governments and even grants awarded by private companies.

Knowledge transfer and dissemination

Under the motto "Mathematics in the service of society", the Basque Center for Applied Mathematics aims to spread knowledge and technology in industry and society in general. It is critical for the centre to transfer the obtained research results to sectors such as biosciences, health, energy, advanced manufacturing, telecommunications and transport, including local, national and international entities.

Toward that goal, BCAM recently promoted its Knowledge Transfer Unit (KTU), a platform to develop mathematical solutions for scientific challenges based on real-life applications and collaborations with industry. These collaborations are developed in the form of strategic partnerships, R&D&I projects, joint positions, training courses, supervision of Master's and PhD students, organisation of dissemination activities, etc.

Some examples of the projects that have been developed by BCAM's KTU for private companies include the analysis of the oil price market, computational modelling for cardiac radiofrequency ablation and CFD simulation of the beam deposition process for a control laser device in additive manufacturing. Another interesting project that has just started is a collaboration with an athletics club football team to predict, prevent and manage injuries among the players through data science.

In line with its commitment to knowledge transfer, BCAM aims to promote the MSO & Data Analysis Laboratory. The main goal of this platform will be to use applied mathematics to promote cooperation with other agents (social, industrial, clinical, etc.) by supporting them through the design of experiments, the simulation of diverse phenomena and the analysis of big amounts of data.

At the same time, BCAM has created and developed a very broad programme of scientific activities, addressed to all sectors of society and, particularly, to young scientists. In the last four years, the centre has organised more than 600 seminars and working groups, 90 workshops in collaboration with the main R&D agents, 10 colloquiums and 100 courses. The aim of this comprehensive training programme is to transfer the specialised knowledge and results generated and nurtured at BCAM and communicate the importance of mathematical research and its applications to society.

Dissemination of mathematics for the general public and fostering scientific culture among citizens are also two important priorities of the centre. Researchers actively participate in dissemination activities in order to bring their research on mathematics closer to society.

Part of the team working at BCAM.

BCAM team

From the administrative staff to the highly qualified and specialised international researchers, people within BCAM are the driving force of the centre. The current team is formed of more than 90 people, with over 25 nationalities represented, and the average age of the researchers is less than 35.

Aware that young researchers are the heart of the centre, one of BCAM's main targets is to attract and provide them with competitive training so that they can develop their future careers in a successful way. With that purpose, BCAM has put into place several programmes to attract international researchers and students.

- The Visiting Fellow Programme: Every year, BCAM offers research opportunities for outstanding mathematicians from all over the world, for short-term and long-term visits.
- The Visitor Programme: Internationally-leading scientists are invited for short-term visits to disseminate and convey their ideas and recent and ongoing research through seminars and collaboration with BCAM researchers.

BCAM's facilities located in Bilbao.

- The Internship Programme: Young people, appointed to BCAM in this programme, come from institutions where they are studying graduate or post-graduate courses and join BCAM as members of a specific team with a predefined area and tutor.

Finally, we must point out that BCAM signed its commitment to the European Researchers' Charter and the Code of Conduct for Researcher Recruitment in December 2008. In 2016, the centre was awarded the HR Logo related to "Human Resources Strategy For Researchers (HRS4R)", promoted by the European Commission.

Although the road ahead presents many challenges, such as seeking funding to consolidate research teams, developing new programmes and activities and strengthening links with international and local partners, BCAM will continue working to achieve outstanding scientific results to demonstrate that mathematics is a fundamental tool for the development of society.

If you would like to join us or find out more about BCAM's research, scientific activities and open positions, please visit www.bcamath.org, subscribe to our newsletter at https://bit.ly/2HshsXA or follow us on Twitter @BCAMBilbao.

Jean-Bernard Bru is an Ikerbasque Research Professor at the Mathematics department of the University of the Basque Country (UPV/EHU) and at the Basque Center for Applied Mathematics (BCAM). He started his career as an independent researcher in 1999 with a PhD in mathematical physics from the Aix-Marseille University (France). The bulk of his research covers a scope from mathematical analyzes of the many-body problem to operator algebras, stochastic processes, differential equations, convex and functional analysis. He has participated in more than 64 conferences, performed at least 94 seminars and accomplished many research visits to universities across Armenia, Brazil, Europe and the USA.

Carlos Pérez graduated in Mathematics at the Autonomous University of Madrid and obtained his PhD in Mathematics in 1989 at Washington University, Saint Louis (USA). He authored one book and more than 80 papers in International journals. He was full professor at the University of Seville until 2014 where he was appointed as Ikerbasque Research Professor at the University of the Basque Country and BCAM.

A Brief History of the French Statistical Society (SFdS)

Gérard Biau, President of the SFdS

The French Statistical Society was founded in 1997 through the merger of two associations of statisticians: the Statistical Society of Paris (SSP) and the Association for Statistics and its Applications (ASU).

Formed in 1860, the main goal of the SSP was to "popularise statistical research through its work and publications". From the beginning, the group evolved in a way that stayed close to economists, while also being attuned to demographers, actuaries and doctors, not to mention politicians. In order to underpin its projects, the SSP published – since the year it was created – the *Journal of the Statistical Society of Paris* and did so regularly for nearly 140 years. This journal was then succeeded by the *Journal of the French Statistical Society* in the context of the merger mentioned above.

The ASU originated in a meeting of around 30 statisticians in 1969 in Toulouse. They were mainly university professors, oriented toward applied statistics and practising their profession mainly outside Paris, who wished to establish a more formal relationship. It was also a question of reflecting on the content of statistics teaching, both in university curricula and secondary school education, and to study how to participate in programme reforms.

The first action of the ASU was to organise the *Statistics Days* conference, which is now held every year at the end of May or the beginning of June in a different town in France or abroad. From 1976 onward, a bulletin was also published, entitled *Statistics and Data Analysis*. A third initiative was the creation in October 1984 of the *Study Days in Statistics* (JES) venture, which, every two years, offers an in-depth course for one week on a statistical subject of interest, giving rise to a book written by the speakers and organisers.

In 1987, the ASU became the *Association for Statistics and its Uses* in order to emphasise a turn toward applications as well as a will to interact with non-university statisticians. A second important decision concerned the idea of specialised think tanks "to encourage certain aspects of statistics in line with the aims of the association". These quickly acquired a life of their own, organising seminars, courses, specialised meetings and so on. Today, these groups are named "Agro-Industry", "Banking-Finance-Insurance", "Biopharmacy and Health", "Chemometrics", "Surveys", Models and Applications", "Teaching Statistics", "Environment and Statistics", "Reliability and Uncertainty", "History of

Statistics, Probability and their Uses", "Young Statisticians", "Machine Learning and Artificial Intelligence", "Mathematical Statistics", "Statistics and Public Issues" and "Statistics and Sport".

Among the first activities of the SFdS was the birth of the *Journal of the French Statistical Society*, which aimed to be a tool for disseminating scientific information amongst statisticians. This journal acted as an extension of the *Journal of the Statistical Society of Paris* (in issue numbering also) and merged in 2007 with the previously established *Journal of Applied Statistics*. The journal is currently published in electronic form, as are three other journals formed since: *Statistics and Teaching*, *Statistics and Society and Case Studies in Business*, *Industry and Government Statistics* (CSBIGS).

The SFdS also supports collections of scholarly texts. In addition to those from the *Study Days in Statistics*, currently published by Technip, there are also the *Statistical Practice* collection, published with Rennes University Press, *A Fresh Look at Statistics* collection, published by Technip and aimed primarily at teachers and statistics users, and finally the *World of Data* collection, published by EDP Sciences for a broad audience.

Note also that the SFdS gives out several awards. The oldest, the *Dr Norbert Marx Award*, is given every two years for applied statistical methodology work in the fields of epidemiology, public health and health economics. Since 2004, there is also the *Marie-Jeanne Laurent-Duhamel* Award, rewarding doctoral theses in statistics defended in the preceding three years. The jury awards this prize once every three years to work in theoretical statistics and the following year to applied statistics. In the third year, the award is replaced by an homage – the *Pierre Simon de Laplace Prize* – to an experienced statistician whose contribution to the French statistics community has been particularly remarkable.

It should also be mentioned that the SFdS and the learned societies that preceded it were the originators of both the *European Courses in Advanced Statistics* (ECAS) in 1987 and the *Federation of European National Statistical Societies* (FENStatS) in 2014. The SFdS also organises training courses and debates, such as the *Statistical Workshops*, the *Statistics Café* and the *Young Statisticians Meeting*.

Gérard Biau is a full professor at the Probability, Statistics and Modelling Laboratory (LPSM) of Sorbonne University, Paris. He is Deputy Director of LPSM and has served as the President of the French Statistical Society (SFdS) since 2015. He is an elected member of the International Statistical Institute and was a member of the prestigious Institut Universitaire de France from 2012 to 2017. His research is mainly focused in developing new methodology and rigorous mathematical theory in statistical learning, artificial intelligence and massive and high-dimensional data, whilst trying to find connections between statistics and algorithms. Gérard Biau has co-authored two books and more than 60 articles and research notes in international, peer-reviewed journals and he has been the PhD advisor of 17 students.

Slovenian Discrete and Applied Mathematics Society Joins the EMS

Klavdija Kutnar and Tomaž Pisanski (both University of Primorska, Koper, Slovenia)

Slovenian Discrete and Applied Mathematics Society, founded in December 2016.

At the EMS council meeting in Prague, 23–24 June 2018, the young Slovenian Discrete and Applied Mathematics Society (SDAMS) was accepted as a full member of the EMS. There are now 56 full member societies from 44 countries. Slovenia joined the group of countries that have more than one full member society: Spain (4), Italy (3), UK (3), France (3), Germany (2) and Russia (2). It is the first European country with a member society that has the word *discrete* in its title and the first society from a country behind the former Iron Curtain that has the name *applied* in its title.

It seems to be a bit unusual for a country of two million inhabitants to have two mathematical societies. However, we want to show that this is a natural step in the development of mathematics in Slovenia.

Firstly, we present a brief overview of the historical development of Slovenian mathematics.

Historical development of mathematics in Slovenia

Habsburg rule until 1918

For most of its history, Slovenia was under Habsburg rule, which ended in 1918 after World War I. Slovenian mathematicians who were born in Austria, or later in the Austro-Hungarian empire, include Herman de Carinthia, also known as Herman Dalmata, who translated Euclid's *Elements* from Arabic to Latin, Andrej Perlach, who became a rector of the University of Vienna and who taught diverse subjects such as mathematics and medicine, Jurij Vega, known for his logarithm tables and calculations of digits of π, Franc Močnik, one of the most prolific writers of high school textbooks in the Austrian Empire (and translated into 12 languages), Franc Hočevar, the first Slovenian mathematician who published papers in the modern sense, Josip Plemelj, known for instance for the Sokhotski–Plemelj theorem from complex analysis, Ivo Lah, known for Lah numbers in combinatorics, and Ivan Vidav of the Vidav–Palmer theorem from functional analysis.

Kingdom of Yugoslavia, from 1918 until 1945

In 1919, the University of Ljubljana was founded and Plemelj became its first rector. He focused on producing a high quality curriculum, essentially covering algebra and number theory, differential equations and analytic functions. Unfortunately, he only published the corresponding three textbooks at the end of his career after World War II. Nearly all students of mathematics at that time became high school teachers of mathematics, with two exceptions. His first PhD student, Anton Vakselj (PhD in 1924), became a professor of mathematics at the Technical Faculty of the University of Ljubljana while his second PhD student Ivan Vidav (PhD in 1941) became his successor.

Socialist Federal Yugoslavia, from 1945 until 1991

In 1949, the Society of Mathematicians, Physicists and Astronomers of Slovenia was founded. At that time, less than 10 Slovenian mathematicians were involved in research in mathematics, with Plemelj and Vidav being the leading mathematicians focusing on various aspects of mathematical analysis. Although Plemelj only retired in 1957, it was Vidav who created the mathematical school in Slovenia. Between 1955 and 1985, he had 17 PhD students and currently has 105 academic descendants.[1] However, in the 1970s, a number of important things happened.

In 1960, the Institute of Mathematics, Physics and Mechanics was founded in Ljubljana. In the following years, most active mathematicians in Slovenia became partially employed there in order to conduct research in mathematics. This model of separating teaching at the university from research at the institute was not uncommon in Eastern Europe.

In 1972, the first graduate programme in mathematics started at the University of Ljubljana.

[1] Mathematics Genealogy Project.

In 1973, the University of Maribor (the second university in Slovenia) was established.

Since 1972, over 20 Slovenian research mathematicians got their PhDs abroad or had foreign advisors and returned to Slovenia to pursue their academic careers at home, bringing new research areas and reducing the risk of inbreeding in a small community. As shown in Table 1, seven of them have at least 10 academic descendants.

(a)	(b)	(c)	(d)	(e)	(f)	(g)
Ivan Vidav	1941	34	Josip Plemelj	Austria	17	105
Jože Vrabec	1972	57	James Cannon	USA	2	11
Boštjan Vilfan	1972	68	Albert deSilva Meyer	USA	4	10
Dragan Marušič	1981	05	C. St.J.A. Nash-Williams	UK	7	14
Tomaž Pisanski	1981	05	Torrence Parsons	USA	16	78
Franc Forstnerič	1985	32	Edgar Lee Stout	USA	9	14
Franc Solina	1987	68	Ruzena Bajcsy	USA	13	20

Table 1. Mathematicians having international education with at least 10 academic descendants in Slovenia. The data are collected mostly from the Mathematics Genealogy Project:
(a) Mathematician,
(b) Year when PhD was received,
(c) Math Subj Classification of the thesis,
(d) Advisor,
(e) Country,
(f) Number of doctoral students,
(g) Number of academic descendants.

These figures indicate that discrete mathematics in Slovenia, in particular combinatorics and graph theory, started in the last quarter of the 20th century.

Independent Slovenia after 1991

In an independent and democratic Slovenia, mathematics witnessed further development. Ease of travel has increased the number of mathematicians who study abroad. At the same time, the number of students from abroad is steadily increasing. Several accomplished mathematicians have also decided to continue their academic careers in Slovenia.

In 2003, the third public university, the University of Primorska, was established, followed by FAMNIT (the faculty where mathematics is taught) in 2006. Currently, all three public universities in Slovenia offer PhD programmes in mathematics.

In 2006, Slovenia very successfully hosted the 47th International Mathematical Olympiad (IMO), with participation of competitors and their team leaders totalling over 2000. It has a very efficient computer system for running mathematical competitions and it hosts the IMO homepage. The IMO's current secretary Gregor Dolinar also comes from Slovenia. Currently, high school competitions form a dominant activity of the Society of Mathematicians, Physicists and Astronomers of Slovenia.

In 2008, the first high-quality mathematical journal *Ars Mathematica Contemporanea* was established in Slovenia. It mainly covers discrete mathematics.

According to the Mathematics Genealogy Project, a little over 300 PhDs in mathematical sciences were awarded by Slovenian universities, almost one half in the area of discrete and applied mathematics or theoretical computer science.

All three universities grant honorary PhDs. They all recognise the importance of mathematics. At the University of Ljubljana, the following mathematicians have been granted this honour: Josip Plemelj (1963), Alojz Vadnal (1981) and Ivan Vidav (1997) from Slovenia, and Ruzena Bajcsy and Dana Scott (2003) from the US. In 2018, Cheryl Praeger (Australia) received an honorary doctorate from the University of Primorska and, in the same year, Wilfried Imrich (Austria) received an honorary doctorate from the University of Maribor. They both work mainly in discrete mathematics.

Slovenia has attracted accomplished mathematicians to spend sabbaticals at its universities and several international mathematicians have decided to pursue their academic careers in Slovenia: one at the University of Maribor, three at the University of Ljubljana and three at the University of Primorska.

One of the biggest achievements of mathematics in Slovenia is winning the bid for the 8th European Congress of Mathematics, to take place in Portorož, Slovenia, in July 2020.

A New Learned Society

The *Slovenian Discrete and Applied Mathematics Society* (SDAMS) was founded in Koper (Slovenia) on 14 December 2016.

The aim of this society is to promote the mathematical sciences, with a special emphasis on discrete and applied mathematics. The society is research-oriented and publishes scientific literature and organises scientific meetings. In particular, the SDAMS is involved in publishing two international mathematical journals: *Ars Mathematica Contemporanea* (https://amc-journal.eu/) and *The Art of Discrete and Applied Mathematics* (https://adam-journal.eu/).

The SDAMS is also involved in organising valuable scientific meetings. So far, it has co-organised five such meetings. The next one will be the “Discrete Biomathematics Afternoon on the Adriatic Coast”, 13–14 February 2019, Koper, Slovenia (https://conferences.famnit.upr.si/event/9/).

Historical centre of the coastal city of Koper in Slovenia.

In 2018, SDAMS became one of the co-publishers of AMC, founded in 2008, and the main publisher of ADAM, an electronic journal that was founded in 2018 (both journals are high-quality, peer-reviewed, of no charge for authors and are freely available to readers).

The SDAMS has members, fellows and honorary members. Currently it has about 50 members, mostly from Slovenia but also from Bosnia, Canada, Colombia, Hungary, Italy, Mexico, New Zealand and the USA.

A ‘member’ may be any individual actively engaged in mathematical research, as evidenced in practice by authorship of a paper covered by *MathSciNet* or *zbMATH* (formerly *Zentralblatt MATH* or by enrolment on a research degree (supported by a recommendation letter from the student’s supervisor). A ‘fellow’ is a member who has strong international visibility and has made a positive impact on mathematics in Slovenia. An ‘honorary member’ is an individual who has made outstanding contributions to the development of discrete or applied mathematics in Slovenia.

The SDAMS has a council to oversee its operations. It has a nomination committee for nominating candidates for fellowship and for considering candidates for honorary membership. Under the current rules, the council of the society will elect new fellows and a limited number of honorary members at its annual meeting each year.

The SDAMS seeks contact with other similar domestic and international societies worldwide. In 2018, it was admitted as a full member of the European Mathematical Society. It welcomes international members, in particular mathematicians interested in discrete and applied mathematics. The current annual membership fee is 20 euros. For more information, see http://sdams.si/en.

Klavdija Kutnar (Klavdija.Kutnar(at)upr.si) is a member of the executive committee of the SDAMS and the Dean of FAMNIT, the faculty of the University of Primorska, which will host 8ECM in 2020. She is the deputy chair of the organising committee of 8ECM. Her main research interests include algebraic graph theory.

Tomaž Pisanski (Tomaz.Pisanski(at)upr.si) is the President of the SDAMS and a professor of mathematics and computer science at the University of Primorska. He is the chair of the organising committee of 8ECM. His research interests include various aspects of discrete mathematics. He is the co-author of a book on configurations.

The New International Science Council – A Global Voice for Science

Maria J. Esteban (President of the International Council for Industrial and Applied Mathematics (ICIAM), affiliate member of the ISC) and Gabriella Puppo (member of the ICIAM ISC Committee)

On 3–5 July 2018, the founding general assembly of the new International Science Council (ISC) took place in Paris. The ISC is a new international body, representing more than 200 scientific organisations from 127 countries. This novel council results from the merger of two previous organisations: the ICSU (International Council for Science) and the ISSC (International Social Science Council).

Its mission is to represent science and scientists in a world that seems to need science more than ever and, at the same time, presents science with new challenges concerning its role in society, in economy and in politics.

This new body aims to represent all sciences, to be a global voice for science and to advance science as a global public good. As stated on the ISC website: "scientific knowledge, data and expertise must be universally accessible and its benefits universally shared. The practice of science must be inclusive and equitable, also in opportunities for scientific education and capacity development."

The council's goals are twofold. On one hand, it intends to stimulate and support international scientific research on major issues of global concern, such as global sustainability, poverty, disaster risk reduction, urban health or wellbeing, coordinating studies and efforts in these directions. On the other hand, the council aims to promote the rigour and the relevance of science in analysing complex situations, and speak for the value of all sciences and the need for evidence-informed decision-making.

A particularly strong effort is devoted to the defence of the free and responsible practice of science. This includes concerns for the way in which science is currently evaluated and the need for science and scientific education to be free and accessible worldwide. Furthermore, the ISC pours energy into filling the gender and economic gaps that prevent many potential researchers from participating in the quest for knowledge.

These topics are also of particular interest to the mathematics community and the ISC can amplify the public concerns of mathematicians. As public opinion struggles with fake news, distorted data and a widespread mistrust of science and quantitative data, a body such as the ISC can help to increase the impact of the scientific point of view in decision-making. Mathematicians, and scientists in general, should be more aware of the importance of the ISC as their global voice and also the opportunities that it provides to interact with other sciences and propose solutions to global issues concerning the future of our planet and our societies.

ICMI Column – Espace Mathématique Francophone 2018

Jean-Luc Dorier (University of Geneva, Switzerland)

Launched by the French Sub-Commission of the ICMI (CFEM – *Commission française de l'enseignement mathématique*) during the World Mathematical Year 2000, the series of *Espace Mathématique Francophone* conferences (occurring every three years) is built on the notion of "region" defined in linguistic rather than geographical terms, French being the common language amongst participants. It is recognised as a regional conference of the International Commission on Mathematical Instruction (ICMI) – https://www.mathunion.org/icmi/conferences/icmi-regional-conferences.

L'Espace Mathématique Francophone (EMF) was set up to promote reflection and exchanges within the French-speaking world on the vital issues of mathematics education in today's societies, at primary, secondary and higher levels, as well as on issues relating to initial and in-service teacher training. The EMF contributes to the development of a Francophone community rich in cultural diversity around mathematics education at the crossroads of continents, cultures and generations.

All information about the EMF, including all proceedings (with an internal browser), the composition of the executive bureau and the status of the conference can be found at www.emf.unige.ch, where you can also subscribe to the distribution list.

The seventh EMF (EMF-2018) will be held in Paris (France):

Mathematics on stage, bridges between disciplines
Paris, 22 to 26 October 2018
https://emf2018.sciencesconf.org/

Mathematics has been built and continues to be built in interaction with other disciplines. The historical testimonies that have come down to us show that several mathematical notions were born as answers to problems (concrete needs of individuals or groups) and that they evolved in a dynamic of producing tools or concepts and theoretical results. This dynamic was sometimes generated in other fields and sometimes in response to a need for generalisation and theorisation of mathematics itself.

Whilst remaining a means of expressing problems posed by other disciplines and a powerful set of tools for solving some of these problems, mathematics can also be seen as a set of tools for understanding the world and its evolution. It is in the development and use of models to understand or transform this world that mathematical activity tends today to get realised, often in collaboration with other scientific disciplines.

Moreover, the growing presence of technological tools in social, cultural and educational contexts amplifies the complexity of contemporary realities. Understanding the relationship between these tools and the models and algorithms they operationalise is a challenge that must be met if critical thinking and genuine citizen participation are to be exercised. Paradoxically, mathematics has never been so present but with so little visibility.

It becomes essential to understand the complexity of the world in a multidisciplinary, holistic and systematic approach. The educational system does not escape this approach and therefore thinking about mathematics education in relation to other subjects in today's changing world is a way of thinking about the tools needed by today's pupils to live in tomorrow's world.

From an institutional point of view, collaboration between disciplines is increasingly promoted through curriculum reforms in different countries, often as a solution chosen by the institution to help students understand complexity. Thus, various mechanisms promote the convocation of several disciplines: approaches to solving complex problems, investigative approaches, project pedagogy, etc. In addition, in some countries, there are new profiles/competitions for bivalent teachers, in addition to the installation, in some contexts, of multidisciplinary teaching teams. But do all these reforms imply a real exchange between disciplines or rather a minimal collaboration that is limited to juxtaposition?

The design of genuine multidisciplinary/interdisciplinary learning situations requires significant research work to build authentic bridges between disciplines. In addition, the collaboration of specialists in the various disciplines involved seems necessary, both to ensure the quality and authenticity of the situations proposed and to encourage a significant contribution to learning and avoid the great risk of betraying one or other of the disciplines.

The crucial question of teacher training in mathematics inevitably arises. The challenge here would be to open mathematics teachers to other modes of scientific thought, in particular to their transversal nature, but also to initiate them into epistemological reflection on the articulation of knowledge relating to the different disciplines.

These issues of multidisciplinary and interdisciplinary approaches in the teaching and learning of mathematics are at the heart of the theme of the *Espace Mathématique Francophone 2018* conference.

Several questions thus feed this theme:

- To which societal, research and teaching problems does the question of interdisciplinary apply?
- How can the participation of mathematics in interdisciplinary activities be qualified in relation to other disciplines from epistemological and teaching points of view?
- What challenges and opportunities does the articulation of concepts from different disciplines pose for teaching?
- How do teaching practices take into account institutional injunctions?
- What are the contributions to a student's learning and what are the difficulties?

Finally, a fundamental question involves the conditions that are conducive to a "true" collaboration between disciplines, which favours mutual insights between these disciplines. It is all these questions and challenges that the *Espace Mathématique Francophone 2018* conference proposes to address through plenary sessions, working groups and special projects.

ERME Column

Núria Planas (Universitat Autònoma de Barcelona, Catalonia-Spain), Marie Therese Farrugia (University of Malta, Malta), Kirstin Erath (TU Dortmund University, Germany) and Jason Cooper (Weizmann Institute of Science, Rehovot, Israel)

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME) holds a biennial conference (CERME) in which research is presented and discussed in Thematic Working Groups (TWGs). The initiative, which began in the September 2017 newsletter, of introducing the working groups continues here, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. The aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME's Thematic Working Group 9 – Mathematics and Language

Group of co-leaders: Núria Planas, Marie Therese Farrugia, Jenni Ingram, Kirstin Erath and Marcus Schütte

Perspectives on mathematics and language in Europe

Mathematics and language is not a new domain of knowledge. There has been research in this field for about 40 years (Austin & Howson, 1979) and our group has been contributing in this domain for the last two decades (Planas, Morgan & Schütte, 2018). Thanks to a well-established tradition, we have come to know that language in mathematics is more than the language of mathematics, and language in the mathematics classroom is diverse.

Language in mathematics is more than just the language of mathematics

Mathematicians have largely recognised mathematics as a language with specific notation, symbols, vocabulary, grammar, syntax, structures, etc. Nonetheless, the mathematics and language connection goes far beyond the production and use of a unique human language with its spoken, written and symbolic forms. Even if we agree to take a linguistic approach to what mathematics is (i.e. a language in many ways), languages other than the language of mathematics are involved and they matter in mathematical learning, teaching and thinking. We learn the language of mathematics through Catalan, Spanish, Maltese, English, German, etc., and we specifically come to learn how to speak and write mathematical Catalan, mathematical German…

In TWG9, we examine language in mathematical learning, teaching and thinking. This includes considering language in many roles: as a medium of instruction, as an epistemic tool and a pedagogic resource, as a learning goal and a learning condition, etc. People learn and think mathematics through one or more languages in interaction with each other, and through engagement with the "mathematics itself". Despite this being rather obvious, the myth of mathematics as an almost 'language-free' curricular area persists. There is also the myth that the more symbolisation involved in the mathematics, the less the dependence on the language of learners in teaching and learning. This belief runs through all levels of education and takes different forms at each level. At university level, for example, there is a strong thought that symbolisation (and visualisation) can supply verbalisation. In line with this belief, many school and university teachers view late arrival learners who are in the process of learning the language(s) of instruction as being ready for the mathematics lessons and their mathematical languages. Research in TWG9 shows, however, that mathematics learning and language learning are integral to each other. Some of the questions that interest us are: What is speaking and writing mathematically in the realm of educational practice? How are mathematical and everyday languages related? What are the connections between teaching language and teaching mathematics?

Language in the mathematics classroom is diverse

In the mathematics classroom, one expects to find ways of speaking and functioning mathematically. These ways never develop in a context of unicity of language and meaning. Let us take the example of the meaning of fraction, which is foundational to algebra, trigonometry and calculus. Learners, mathematics teachers and mathematicians require human languages other than mathematical language in order to make sense of the diversity of semantic meanings linked to, for example, the symbolic representation a/b or the phrase 'a parts of an object divided into b equal parts'. To interpret the sign we pose questions like 'what kind of whole is involved in a/b?' or 'is there a unit implicit in the situation of representation of this fraction?'. Here, English (or some other language, of course) is the language for posing the questions; it provides the context of culture that first suggests a meaning for whole, unit and the relationship unit-whole. In a lesson with learners who were asked to "cut 1/3 out of 1/2 of a pizza", some language issues emerged when the teacher wanted them to identify "the new whole after cutting the pizza piece out". One of the learners said that there was not a whole anymore because the pizza was not complete. The teacher addressed the polysemy of whole by bridging mathematical and everyday languages in the lesson. The misconception about the word 'whole' brings to the fore the need to integrate diverse languages in the process toward speaking and writing mathematically. The meaning of 'cutting' (a fraction out of another fraction of a pizza) as 'calculating' (the fraction resulting from an operation) is not language-free either. Furthermore, the meaning of a fraction as a number on a number line takes words such as 'distance', 'length', 'measurement', 'order' and 'position', while the phrase 'share equally' helps to express the quotient meaning – 3/4 as representing 3 pizzas divided among 4 people. Diversity exists in university classrooms as well, where learners also face the challenge of integrating mathematical and everyday languages and where some of them, if not many, are not fluent in the language of instruction. Given the myth that high symbolisation can supply verbalisation (and hence everyday languages), the challenge for university learners is even more transparent and more difficult.

Overall, we have that (i) *language in mathematics is more than the language of mathematics* and (ii) *language in the mathematics classroom is diverse*. The implications of this view of language for mathematics teaching and learning are enormous. By seeing language learning as integral to mathematics learning, we can interrogate misconceptions that are not necessarily grounded in difficulties with the mathematics but in the pedagogic and institutional lack of attention to the everyday languages through which mathematics is taught and learned. A line of concern in TWG9 is the recognition of the everyday and mathematical languages of learners in mathematics learning and teaching. Instead of thinking of some languages (and their speakers) as 'the problem', we see them as an asset and an opportunity for building richer mathematical practices.

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Jason Cooper is a research fellow at the University of Haifa's Faculty of Education. He is also a researcher at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of teacher knowledge, including roles of advanced mathematical knowledge in teaching and contributions of research mathematicians to the professional development of teachers. He has been a member of the ERME board since 2015.

Pseudonyms and Author Collectives in zbMATH

Octavio Paniagua Taboada, Nicolas Roy and Olaf Teschke (all FIZ Karlsruhe, Berlin, Germany)

Writers and authors in general – and also mathematicians – have often used pseudonyms for many different reasons. Sometimes an author wanted to avoid certain political, sexist, ethnic or religious discrimination, or even persecution. Others, perhaps just wanting to keep their private life away from their work, have pursued a career under an alias. Lastly, authors occasionally change their names for practical reasons,[1] which is, however, beyond the scope of this note. Frequently, the humour of mathematicians has given birth to fictitious persons – often, but not always, the creation of such characters is the idea of a group of authors. Of course, there are also practical reasons for creating collective authors, like the case of larger collaborations whose composition is subject to fluctuations. In this column, we describe some examples and how such information can be retrieved from zbMATH.[2]

Literary pseudonyms

Perhaps the most famous cases of pseudonyms used by mathematicians have their origin in parallel literary careers. *Lewis Carroll* (for Charles Lutwidge Dodgson) and *Paul Mongré* (for Felix Hausdorff) are the most prominent examples, to an extent that even works signed with their pen names have been indexed in the zbMATH database. Here, the zbMATH author profile will automatically display the pseudonyms once the contributions have been acknowledged. The same is true for Helga Bunke, whose literary writings, appearing under her maiden name Helga Königsdorf, arguably outshone her mathematical publications. On the other hand, in an otherwise similar case, the literary pseudonym *Irina Grekova* of E.S. Wenzel, who also wrote several popular Russian textbooks on probability theory (see, for example, [10]), will not be visible via her indexed publications. The same is true for the pen name *John Taine* of Eric Temple Bell.

Female mathematicians

Women's empowerment has been a long and difficult process, wherein they have fought for redefining roles and positions that were previously completely restricted or denied. Science is not an exception and women had to struggle for recognition of their scientific contributions. Sophie Germain (1776–1831) faced the prejudices of 19th-century society and used the alias *M. Le Blanc* in her early correspondence with Adrien-Marie Legendre and Carl Friedrich Gauss [7].

War, persecution and political pressure

In the darkest days of European history, before and during World War II, numerous mathematicians not only used pseudonyms but actually had to change their names permanently in order to hide their Jewish origin after the Nazi party seized power. One remarkable example is André Bloch, who was confined to a psychiatric institution and wrote under the pseudonyms *René Binaud* and *Marcel Segond* [3]. Jacques Feldbau was another victim of the Nazi regime, dying in Auschwitz; before being captured, he wrote several works incognito under *Jacques Laboureur* [4]. Stalinist persecution also led to pseudonyms: both Nikolay S. Koshlyakov and A.I. Lapin could only publish works under the pseudonyms *N.S. Sergeev* and *A.I. Ivanov*, in the 1940s and 1950s respectively [5]. Even in the West, Jean van Heijenoort preferred to publish a political article as *Jean Vannier* in the McCarthy era. While it no longer seems dangerous to make these identities public, there are also recent situations where this is not true. Purportedly, scientists in Iran frequently prefer to use pseudonyms when their work involves colleagues from Israel. For obvious reasons, zbMATH would not make these identities public even if they were known to us.

Other reasons for pseudonyms

It is quite interesting that there are mathematicians who wrote some articles under their real name and some under an alias (not an author collective). In this category, there is William Sealy Gosset, who wrote several papers under the pen name *Student*. From these papers came the famous t-test or Student test in statistics [9]. Fields Medallist Heisuke Hironaka once published a paper under the alias of *Hej Iss'sa* as an homage to a Japanese poet [6]. The mathematician R.P. Boas wrote a very famous article under the pseudonym *E.S. Pondiczery*; this name was later adopted in the Hewitt–Marczewski–Pondiczery theorem. Apparently, Boas was known for his remarkable sense of humour [1]. Humour also appears to be the origin of pseudonyms like *Sally Popkorn* (of Harold Simmons) and the names of many collectives (see below). The cheekiness of some authors sometimes leads them to sign their publications with multiple names, like, for example, Hermann Laurent, who published a paper in 1897 under the double name "*C.A. Laisant & É. Lemoine*" (two of his colleagues). The famous feline co-author *F.D.C. Willard* of the physicist and mathematician Jack H. Hetherington is another interesting example [11].

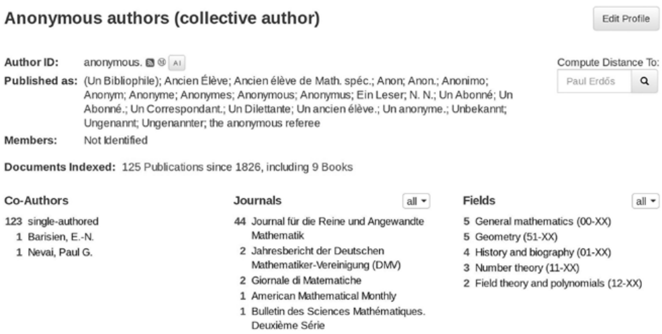
[1] E.g. Rabinowitsch/Rainich, Tajtelbaum/Tarski, Weiss/Fejér and Zaritsky/Zariski.

[2] The authors would like to thank the MathOverflow community, Walter Warmuth and Dirk Werner for many valuable hints.

Anonymous authors

Certain mathematicians wanted to be kept anonymous completely. Sometimes, they signed their papers with "X", "former student", "a student" or "anonymous". There is also the famous case of Joseph Diez Gergonne, who submitted several of his articles anonymously or under an alias. He subsequently added his name to his own copies, which were later donated to the Sorbonne Library [8].

Actually, you can check the profile "anonymous." (https://zbmath.org/authors/anonymous.) in zbMATH and you will see how many interesting variants were used by different authors:



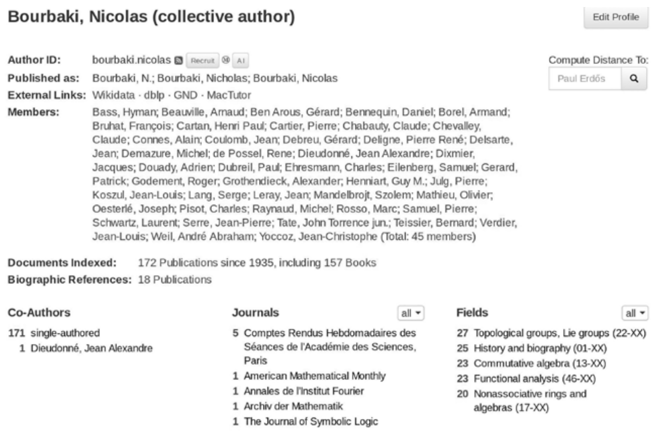
All these variants are collected into this "anonymous" group until further information is available. We invite the reader to read the very interesting discussion on MathOverflow concerning pseudonyms of mathematicians: https://mathoverflow.net/questions/45185/pseudonyms-of-famous-mathematicians.

There, you will also find some very interesting anecdotes (some of them apocryphal) and more pseudonyms.

Author collectives

Nicolas Bourbaki is arguably the most famous author collective in the mathematics literature. L. Banlieu, in her article [2], states that in the early years of Bourbaki (1934–1935), the following members formed the Bourbaki team: Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné, Szolem Mandelbrojt, René de Possel and André Weil. The physicist Jean Coulomb and Charles Ehresmann joined the team later. If you check the profile of the collective author in zbMATH, you can see that 45 members have worked at various points in this collective. Is this all of them? Maybe not – it would not be surprising if we were still missing some names.

Bourbaki is, in fact, the inspiration for other collective authors. Here, we mention the collective Arthur Lancelot Besse, which is a group of mathematicians publishing mainly in differential geometry. Other collectives include the famous John Rainwater, K. Blizzard, R.B. Honor, Peter Ørno, Y.T. Rhineghost, Boto (meaning Bochumer Topologen) von Querenburg and K.M.S. Humak,[3] which also involved Helga Bunke who was mentioned above. By design, the massive collaborative Polymath projects involve many mathematicians. While resulting achievements have been published only under the name of D.H.J. Polymath, it might make sense to distinguish it in the future by the project number. In any case, it remains challenging to maintain an exhaustive list of members. In recent times, Polymath has produced several deep results on bounded gaps between primes and bounded intervals containing primes.



A list of all collective authors currently identified in zbMATH can be retrieved in the author database via https://zbmath.org/authors/?q=st:o.

We invite our readers to check other collective authors in zbMATH and to send us your feedback on all these topics!

[3] Kollektiv Mathematische Statistik: Humboldt-Universität zu Berlin und Akademie der Wissenschaften der DDR.

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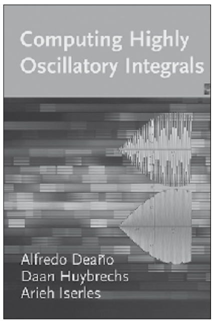
Photos and CVs of the authors can be found in previous Newsletter issues.

Book Reviews



Alfredo Deaño, Daan Huybrechs and Arieh Iserles

Computing Highly Oscillatory Integrals

SIAM, 2018
x, 180 p.
ISBN 978-1-61197-511-6

Reviewer: Manfred Tasche

The Newsletter thanks zbMATH and Manfred Tasche for the permission to republish this review, originally appeared as Zbl 06841753.

Highly oscillatory integrals occur in fluid dynamics, acoustic, and electromagnetic scattering. This important monograph presents efficient algorithms for computing highly oscillatory integrals, such as

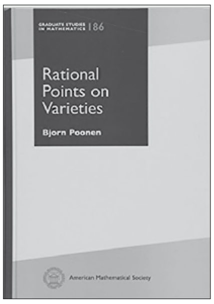
$$I_\omega[f] := \int_{-1}^{1} f(x)\,\mathrm{e}^{\mathrm{i}\,\omega\, g(x)}\,\mathrm{d}x$$

where f is a smooth function, $\omega \gg 1$, and $g(x) = x$ or $g(x) = x^2$. In contrast to $g(x) = x$, the function $g(x) = x^2$ has a stationary point at $x = 0$.

Chapter 1 has preliminary character. Since efficient numerical methods for oscillatory integrals use the asymptotic behavior of $I_\omega[f]$ for large ω, Chapter 2 presents an asymptotic theory of highly oscillatory integrals. Chapter 3 handles with the Filon quadrature and Levin-type methods. In the Filon method one calculates $I_\omega[p]$ with $g(x) = x$, where p is a polynomial p interpolating $p^{(j)}(-1) = f^{(j)}(-1)$ and $p^{(j)}(1) = f^{(j)}(1)$, $j = 0, \ldots, s$. Chapter 4 is devoted to extended Filon methods, such as Filon-Jacobi quadrature and Filon–Clenshaw–Curtis quadrature. Numerical methods based on steepest descent are discussed in Chapter 5. Complex-valued Gaussian quadrature for oscillatory integrals are presented in Chapter 6. In Chapter 7, the authors compare the various quadrature methods at several test functions. The final Chapter 8 contains some conclusions and further extensions. In an appendix, properties of orthogonal polynomials on $\mathbb{R}$ and orthogonal polynomials with complex weight function are sketched.

This well-written monograph is intended for graduate students in applied mathematics, scientists, and engineers who encounter highly oscillatory integrals. The authors consider mainly univariate oscillatory integrals and give some hints for the multivariate case. This book contains numerous examples and instructive figures. Doubtless, this excellent work will be stimulated the further research of computing highly oscillatory integrals.

Manfred Tasche is a retired Professor at the University of Rostock. He completed his Dissertation in 1969 and his Habilitation in 1976. His areas of research are Fourier analysis, wavelet theory, algorithms for discrete Fourier transform, interpolation and methods of approximation. He has published three books and more than 90 research papers.



Poonen, Bjorn

Rational Points on Varieties

AMS, 2017
xv, 337 p.
ISBN 978-1-4704-3773-2

Reviewer: Tamás Szamuely

The Newsletter thanks zbMATH and Tamás Szamuely for the permission to republish this review, originally appeared as Zbl 1387.14004.

Bjorn Poonen has written a textbook that takes readers to the frontiers of research in a very active field where the author himself is a prominent contributor. His main topic is the existence of rational points on algebraic varieties and local-global principles for them. These questions have a long history and the methods go back to such classics as Fermat's infinite descent and Hasse's local-global principle for quadratic forms. The modern development of the subject started with work of Manin in the early 1970's. Manin introduced a cohomological obstruction that explained the failure of the Hasse principle (i.e. the existence of varieties over a number field with points over every completion but no rational point) in all cases that were known at the time – it was only in 1998 that Skorobogatov found the first example where the failure of the Hasse principle was unaccounted for by Manin's obstruction.

These ideas are discussed in Chapter 8 of Poonen's book where the Manin obstruction to the Hasse principle and to weak approximation is presented together with its later refinements. The most general of these is the descent obstruction of Harari and Skorobogatov which is a non-commutative extension of fundamental earlier work by Colliot-Thélène and Sansuc. We owe to the latter authors the crucial discovery of a link between Manin's obstruction and the descent method in Diophantine geometry which until then had only been applied to finite Galois coverings. Poonen gives beautiful worked-out examples for both the Manin obstruction and the descent method, and presents in detail a counter-example to the Hasse principle that is not explained by the Manin obstruction nor, in fact, by the more general descent obstruction.

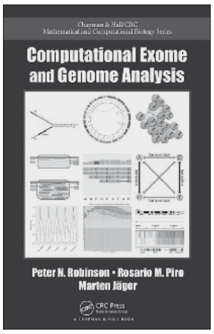
The core of Chapter 9 is devoted to del Pezzo surfaces that are a rich source of examples where the methods in the subject can be successfully applied. After a thorough survey of relevant algebro-geometric topics, Poonen discusses the validity of the Hasse principle and weak approximation for each possible degree.

The earlier chapters of the book are devoted to background material needed for the constructions of Chapter 8. Among the topics treated here we find descent theory, group schemes and torsors under them, Brauer groups and methods of étale cohomology. An important feature throughout these chapters is the attention paid to global fields of positive characteristic which are often left in the shade in favour of number fields. This is particularly noteworthy in the chapter on group schemes where recent work of Conrad, Gabber and Prasad gets its due share.

There is also a chapter devoted to basic scheme-theoretic constructions that are often used in arithmetic geometry, such as spreading out techniques. This part is very useful for non-experts as it gives a rigorous and detailed treatment of techniques that are usually labelled 'standard arguments' in research papers. More dispensable is the chapter surveying the Weil conjectures (proven by Grothendieck and Deligne) and the (still unproven) Tate conjecture. These subjects are a bit off-topic in the present book, and a reasonably thorough treatment would require a book of its own. Nevertheless, the reader is given a nice quick introduction.

Poonen's exposition is atypical for an introductory textbook: as he himself writes, it is closer in style to an extended survey. Thus the reader should not expect complete and self-contained proofs for most of the results; their inclusion would have at least tripled the book's current size. However, there is no handwaving either. Concepts are always introduced clearly and rigorously, followed by key examples and counterexamples. The more accessible statements are then given with proof, but the reader is often directed to some of the best available references in the literature instead. It should be emphasized that all concepts are presented in the way they occur in present-day research, even in cases where more elementary approaches would have been possible. This entails that the prerequisites for reading the book are rather on the high side: the reader should have a solid working knowledge of basic algebraic number theory and algebraic geometry including scheme theory, but some familiarity with more advanced topics such as Galois cohomology is also required. In return, the author always presents what is nowadays considered the 'right point of view', and he is never lost in arid technicalities. In so doing, he has done a great service to the community and his book will be much appreciated by those wishing to enter this fascinating field of research in arithmetic.

Tamás Szamuely is a researcher in arithmetic and algebraic geometry at the Alfred Renyi Institute for Mathematics, Budapest. He is the author of Central Simple Algebras and Galois Cohomology *(with Ph. Gille, 2nd edition 2017) and* Galois Groups and Fundamental Groups *(2009), both with Cambridge University Press.*



Peter N. Robinson, Rosario M. Piro and Marten Jäger

Computational Exome and Genome Analysis

Chapman & Hall/CRC Mathematical and Computational Biology, 2018
xxi, 552 p.
ISBN 978-1-4987-7598-4

Reviewer: Irina Ioana Mohorianu

The Newsletter thanks zbMATH and Irina Ioana Mohorianu for the permission to republish this review, originally appeared as Zbl 1384.92004.

This book is an excellent example of a hybrid between a textbook and an up-to-date research reference on the latest bioinformatics tools available in this field. Its rigorous and thorough approach makes it a reliable starting point for bioinformaticians and biologists. By including details on methodological aspects of some of the algorithms used for various components of the data analysis and coupling these with fully-commented examples and exercises, this book presents itself as a must-have for novices and experts alike. Given the fast pace of the field, no book can be exhaustive, however, the wide variety of tools presented here recommend it to a wide audience, both as expertise and focused research interests.

The book consists of seven parts focused on investigating (and data mining) the human genome for scientific and medicine related questions; Mendelian diseases and the use of precision medicine are a recurrent theme throughout the chapters; the framework of the book is described in the first chapter. The authors start with an overview of sequencing history from Sanger sequencing to Next Generation Sequencing and Illumina technologies all in the light of Moore's law. The third chapter is built as a detailed description of the Illumina sequencing and includes elements of the library preparation with its particular steps: fragmentation, repair adenylation and adapter ligation. The flow cell preparation and the individual steps for the sequencing by synthesis are thoroughly presented. In the fourth chapter, the whole genome and whole exome sequencing (WGS and WES, respectively) are introduced using as example the Corpasome, i.e., genomic data from the Corpas family, publicly available since 2012. The step-by-step WES/WGS analysis is presented in detail including the commands for downloading and processing the data.

The second part of the book is dedicated to raw data processing; it starts with a detailed overview of the fastQ format, including the description of phred scores. Next, the authors present some quality checks such as base quality, nucleotide distribution, GC content distribution, duplication rate and contamination with the sequencing adapter; the interpretation of the *k*-mer content and the per-tile sequence quality is also included. Chapter 6 is built as a description of the fastQC tool developed at the Babraham Institute. The last chapter in this section revolves around trimming, i.e., removing of sequencing artefacts, namely sequencing adapters, before the data analysis. The tool presented for this task is trimmomatic (Java-based). A discussion on the usefulness of trimming and on the usage of other tools such as trimadapt and SAMtools is included.

The third part of the book focuses on alignment tools; the SAM and BAM formats are introduced and approaches for the quality control of the alignment data are discussed. In Chapter 8, the authors describe the mapping of reads to a reference genome or transcriptome. The examples make use of the BWA-MEM. An overview of the human genome reference with details on the availability of sequences is presented next to the Burrows–Wheeler transform used for the mapping. In the ninth chapter, the sequence alignment map (SAM) and the binary alignment map (BAM) are introduced. A full description of the output for single-end and paired-end reads is included; the cigar string is also presented next to the interpretation of the mapping quality output. The ninth chapter describes the post-processing of alignments using Picard tools; methods for realigning of reads and for base quality score recalibration are also presented. The last chapter in this section describes the quality-control of alignment data on depth and coverage. A detailed description of coverage analysis using the browser extensible data (BED) is presented; a script to create a coverage plots in R is included.

Part 4 is built on approaches for variant calling. Chapter 12 focuses on variant calling using the GATK tool, more specifically the Haplotype caller module which is suitable for both single and multiple sample analysis and the BCFtools. The hard filtering option as well as the variant quality score recalibration (VQSR) are discussed with examples. The chapter concludes with an analysis on the concordance of variant callers. The output of variant calling tools, the VCF (variant calling format) is presented at large in Chapter 13. The features and approaches for a variant normalisation are also included. The next chapter presents Jannovar, a stand-alone Java application for the identification of transcripts affected by a given variant. The tool is applicable for variants in either coding or non-coding transcripts and can be used to perform pedigree analyses for the identification of Mendelian disorders. In Chapter 15, the authors present the standards set by the Human Genome Variation Society (HGVS), including the numbering conventions, the annotation of files and the variant categories. Chapter 16 focuses on the quality control of variant calling; it includes a description of the transition-transvertion ratio and the proportion of other variants. Chapter 17 presents a Java-based integrative genomics viewer (IGV) for visualising alignments and variants with approaches for recognising poor quality alignments described using examples. In Chapter 18, a method for the identification of de novo variants is discussed which is based on single sample calling or joint calling. In the last chapter, the authors focus on structural variation including causes for structural variation and known categories copy number variants, inversions, and translocations. Tools like conifer, cnvator and DELLY analysis are presented as examples.

Part 5 focuses on variant filtering. In Chapter 20, the authors present pedigree and linkage analyses, starting with an overview of locations sets, of pedigree symbols and types of files. Analyses of homozygous and heterozygous variants (examples of X chromosomal recessive pedigrees) are presented coupled with the annotation of vcf files using Jannovar. In the next chapter, the authors present to some rare variant association studies (RVAS), followed by the variant frequency analysis and its integration with Jannovar presented in Chapter 22. In Chapter 23, the authors discuss the prediction of variant pathogenicity starting with criteria for deleteriousness of a variant. The effects on proteins and on RNA and DNA are examined. The tool MutationTester is presented as an example for pathogenicity prediction.

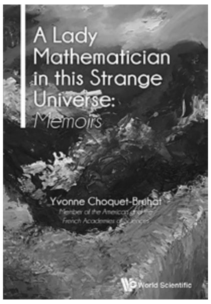
Part 6 is built on approaches for gene prioritization based on random walks methods for phenotype analyses. Chapter 24 describes variant prioritization, an algorithm to determine the likelihood that a disease gene is found. The authors integrate functional variation, gene expression and pathway annotation to evaluate further the priority of genes and for determining a diagnosis. In Chapter 25, the random walk analysis for the prioritization of genes is introduced. The effect of direct protein-protein interactions for determining disease gene families and the advantage of selecting the shortest path between interacting proteins is also examined. Next, the authors present phenotype analyses starting with an overview of the human phenotype ontology (HPO), approaches for the interpretation of annotations, and the integration with other disease databases. The semantic similarity of items annotated by ontology terms and the statistical significance of semantic similarity scores is also discussed; as an example, the phenogramviz is included. In Chapter 27, the authors present two software suites, exomiser and genomiser, that enable an all-inclusive phenotype driven analysis whole exome and whole genome sequencing data. The phive algorithms are presented and a full tutorial is also included, coupled with the integration with

Exome-Walker, described in Chapter 25. The last chapter of this part focuses on the medical interpretation of the results using examples to highlight the effect of single exon deletions, of mutations in enhancers, of repeat expansions or structural variations.

Part 7 focuses on cancer studies and it commences with a short introduction to cancer characteristics, somatic variants in the light of tumour evolution, sample purity, driver mutations and mutational signatures. In Chapter 29, an overview of the basics of tumour biology and its integration with hereditary cancers syndromes is presented; databases frequently used in cancer bioinformatics are also included. Chapter 30 focuses on the analysis of somatic variants in cancer and exemplifies, on glioblastoma data, VarScan2, a tool for variant calling based on pileup files resulting from SAMtools. In Chapter 31, the authors present approaches for the estimation of tumour purity and clonality using the PurityEst algorithm. Chapter 32 discusses driver mutations and mutational signatures coupled with their integration into recurrently mutated pathways. The mathematical description is presented in tandem with the SomaticSignatures library in R.

The book is written for biologists, bioinformaticians, computational biologists or computer scientists who would like to either initialize their study on the computational analysis of human whole-exome and whole genome sequencing or be exposed to alternative analysis approaches. The examples assume a comfortable use of the command line and of compilation and execution of scripts.

Irina Mohorianu has a BSc in Computer Science from University "Al I Cuza" Iasi, Romania and a PhD in Computer Science (Bioinformatics) from University of East Anglia, Norwich, UK. She is currently the Bioinformatics Lead at the Oxford Vaccine Group (Department of Medical Sciences) and Lecturer in Computer Science (Department of Computing Sciences) at the University of Oxford, UK. Current interests span from using machine learning (deep learning) approaches for the analysis of high throughput medical datasets and the identification and characterisation of regulatory networks modelled as weighted graphs..



Yvonne Choquet-Bruhat

A Lady Mathematician in this Strange Universe: Memoirs

World Scientific, 2018
x, 351 p.
ISBN 978-981-3231-62-7

Reviewer: Hans-Jürgen Schmidt

The Newsletter thanks zbMATH and Hans-Jürgen Schmidt for the permission to republish this review, originally appeared as Zbl 1387.83002.

This book represents a great contribution to the history of general relativity theory. It really closes several gaps in the literature. First, the majority of texts on this topic concentrate on the physical, astronomical and philosophical issues of that theory, and here, as seen already in the title of the book, the mathematical issues of the development of general relativity theory are in the center of the consideration.

Second, Yvonne Choquet-Bruhat is one of the very few scientists in the world who had relevant scientific and personal contacts to Albert Einstein (who died in 1955) and are still active in relativity research today in 2018. This leads to a well-balanced presentation of the topics under discussion.

Example: For the ADM-formalism, which is named after R. Arnowitt et al. [*Gen. Relativ. Gravitation* 40, No. 9, 1997–2027 (1962/2008; Zbl 1152.83320)], being relevant both for the Cauchy problem of the Einstein field equation as well as to the mathematical details for the introduction of the notion of gravitational energy, she presents insightful comments to the background discussion of these topics including her own contributions to them, e.g. [*C.R. Acad. Sci.,* Paris 252, 3411–3413 (1961; Zbl 0100.40502)]. By the way, in 1970 she was one of the founders of that mentioned journal *General Relativity and Gravitation*.

Publisher's description: "In this book, the distinguished mathematician and physicist, Yvonne Choquet-Bruhat, at the urging of her children, recounts and reflects upon various key events and people from her life-first childhood memories of France, then schooling, followed by graduate studies, and finally her continuous research in the mathematics of General Relativity and other fundamental physical fields. She recalls conversations, collaborations and even arguments shared with many great scientists, including her experiences with Albert Einstein. She also describes some of her numerous trips around the world, spurred by a passion for travel, beauty and mathematics. At once reflective, enlightening and bittersweet, this book allows readers a look into the life and thought processes of an esteemed female academic."

For the French original edition see [Zbl 1391.83002].

Hans-Jürgen Schmidt has been a Lecturer at the mathematics department of Potsdam University since 1996. He received the diploma and doctor degree in mathematics from Greifswald University in the field of topology. His research interests are on relativistic theories of gravity and their application to cosmology.

Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

Probability theory is nothing but common sense reduced to calculation.

Pierre-Simon Laplace (1749–1827)

The column this month is devoted to probability theory. The proposed problems range from basic to fairly demanding so a wide range of our readers should be able to tackle them. As always, there is also a proposed open research problem. The open problem, along with the relevant discussion, is provided by Martin Hairer.

Probability theory traces back to the 16th century, when the Italian polymath Gerolamo Cardano attempted to mathematically analyse games of chance. More specifically, his book about games of chance, published in 1663 (written ca. 1564), contains the first systematic treatqment of probability. Probability theory also traces back to 17th century France, when Blaise Pascal and Pierre de Fermat corresponded about problems of games of chance. In modern mathematics, probability theory is an extremely applicable and versatile field, which is used in a surprisingly broad spectrum of areas, such as weather prediction, medicine/biology, equity trading, machine perception, music, etc.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

197. In a game, a player moves a counter on the integers according to the following rules. During each round, a fair die is thrown. If the die shows "5" or "6", the counter is moved up one position and if it shows "1" or "2", it is moved down one position. If the die shows "3" or "4", the counter is moved up one position if the current position is positive, down one position if the current position is negative and stays at the same position if the current position is 0. Let X_n denote the position of the player after n rounds when starting at $X_0 = 1$. Find the probability p that $\lim X_n = +\infty$ and show that $X_n/n \to 1/3$ with probability p and $X_n/n \to -1/3$ with probability $1 - p$.

(Andreas Eberle, Institute for Applied Mathematics, Probability Theory, Bonn, Germany)

198. Let $B := (B_t)_{t\geq 0}$ be Brownian motion in the complex plane. Suppose that $B_0 = 1$.

(a) Let T_1 be the first time that B hits the imaginary axis, T_2 be the first time after T_1 that B hits the real axis, T_3 be the first time after T_2 that B hits the imaginary axis, etc. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(b) More generally, let ℓ_n be lines through 0 for $n \geq 1$ such that $1 \notin \ell_1$. Let $T_1 := \inf\{t \geq 0\,;\ B_t \in \ell_1\}$ and recursively define $T_{n+1} := \inf\{t > T_n\,;\ B_t \in \ell_{n+1}\}$ for $n \geq 1$. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(c) In the context of part (b), let α_n be the smaller of the two angles between ℓ_n and ℓ_{n+1}. Show that $\sum_{n=1}^{\infty} \alpha_n = \infty$ iff, for all $\epsilon > 0$, the probability that $\epsilon \leq |B_{T_n}| \leq 1/\epsilon$ tends to 0 as $n \to \infty$.

(d) In the context of part (a), show that

$$\lim_{n\to\infty} \mathbf{P}\Big[\exp(-\delta_n \sqrt{n}\,) \leq |B_{T_n}| \leq \exp(\delta_n \sqrt{n}\,)\Big] = \int_{-2\delta/\pi}^{2\delta/\pi} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$$

if $\delta_n \geq 0$ tend to $\delta \in [0, \infty]$.

(Russell Lyons, Department of Mathematics, Indiana University, USA. [Partially supported by the National Science Foundation under grant DMS-1612363])

199. Suppose that each carioca (native of Rio de Janeiro) likes at least half of the other 2^{23} cariocas. Prove that there exists a set A of 1000 cariocas with the following property: for each pair of cariocas in A, there exists a *distinct* carioca who likes both of them.

(Rob Morris, IMPA, Rio de Janeiro, Brazil)

200. Let X, Y, Z be independent and uniformly distributed in $[0, 1]$. What is the probability that three sticks of length X, Y and Z can be assembled together to form a triangle?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

201. Suppose that each hour, one of the following four events may happen to a certain type of cell: it may die, it may split into two cells, it may split into three cells or it may remain a single cell. Suppose these four events are equally likely. Start with a population consisting of a single cell. What is the probability that the population eventually goes extinct?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

202. We flip a fair coin repeatedly and record the outcomes.

(1) How many coin flips do we need on average to see three tails in a row?

(2) Suppose that we stop when we first see heads, heads, tails (H, H, T) or tails, heads, tails (T, H, T) come up in this order on three consecutive flips. What is the probability that we stop at H, H, T?

(Benedek Valkó, Department of Mathematics, University of Wisconsin Madison, Madison, Wisconsin, USA)

II An Open Problem, by Martin Hairer (Mathematics Institute, Imperial College London, UK)

Before trying to formulate this open problem, I would like to start by introducing one of the most important objects in probability theory, namely Brownian motion. One way of viewing Brownian motion is as a random variable B taking values in the space $\mathbb{C}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and satisfying the following two properties.

Claim 1 (i) One has $B(0) = 0$ almost surely.

(ii) For any finite sequence of times $(t_1, \ldots, t_n)$, the $\mathbb{R}^n$-valued random variable $(B(t_1), \ldots, B(t_n))$ is a centred Gaussian random variable such that $\mathbf{E}(B(t_i) - B(t_j))^2 = |t_i - t_j|$ for any $i, j \in \{1, \ldots, n\}$.

Why is Brownian motion so important? One reason is that it appears in the following "functional" version of the central limit theorem. Consider a sequence $\{\xi_i\}_{i\in\mathbb{Z}}$ of independent and identically distributed (i.i.d.) random variables with vanishing expectation and unit variance. We use these to define a collection of random variables S_n for $n \in \mathbb{Z}$ by specifying that

$$S_0 = 0 , \qquad S_{n+1} - S_n = \xi_n .$$

The central limit theorem then tells us that, as $n \to \infty$, $S_n/\sqrt{n}$ converges in law to a standard Gaussian random variable. On the other hand, we can define a random continuous function $S(t)$ by setting $S(n) = S_n$ for $n \in \mathbb{Z}$ and by extending this to arguments in $\mathbb{R}$ by linear interpolation. If we rescale this random function appropriately by setting $S^{(N)}(t) = S(Nt)/\sqrt{N}$, we obtain the following result [3].

Theorem 1 As $N \to \infty$, the sequence of $\mathbb{C}$-valued random variables $S^{(N)}$ converges in law to a Brownian motion.

This way of obtaining Brownian motion immediately suggests a number of properties that are not completely obvious at first sight from the definition above, although they can easily be read off property (ii) above. First, since the ξ_n are i.i.d., the collection $\tilde\xi_n = \xi_{n+m}$ is equal in law to the original sequence for every fixed $m \in \mathbb{Z}$. At the level of S, this implies that if we define the translation operators

$$(\mathcal{T}_\tau S)(t) = S(t+\tau) - S(\tau) \quad \text{so that} \quad \mathcal{T}_\tau \mathcal{T}_\nu = \mathcal{T}_{\tau+\nu} , \tag{1}$$

$\mathcal{T}_\tau S \overset{\text{law}}{=} S$ for every $\tau \in \mathbb{Z}$. Similarly, we can define rescaling operators

$$(\mathcal{S}^\lambda_\alpha S)(t) = \lambda^{-\alpha} S(\lambda t) \quad \text{so that} \quad \mathcal{S}^\lambda_\alpha \mathcal{S}^\mu_\alpha = \mathcal{S}^{\lambda\mu}_\alpha , \tag{2}$$

as well as $S^{(N)} = \mathcal{S}^N_{1/2} S$. Finally, we note that since the ξ_i are independent, there exists $\delta > 0$ such that, conditional on the 'present' $\{S(t) : |t - t_0| \le \delta\}$, the 'future' $\{S(t) : t > t_0\}$ is independent of the 'past' $\{S(t) : t \le t_0\}$ for every $t_0 \in \mathbb{R}$. This suggests the following.

Proposition 1 Brownian motion satisfies the following properties.

Claim 2 *Translation invariance:* $\mathcal{T}_\tau B \overset{\text{law}}{=} B$ for all $\tau \in \mathbb{R}$.
Scale invariance: $\mathcal{S}^\lambda_\alpha B \overset{\text{law}}{=} B$ for $\alpha = 1/2$ and all $\lambda > 0$.
Markov property: For any $t_0 \in \mathbb{R}$, conditional on B_{t_0}, $\{B(t) : t > t_0\}$ is independent of $\{B(t) : t \le t_0\}$.

As a matter of fact, up to multiplication by a real number, Brownian motion is the *only* $\mathbb{C}$-valued random variable with these properties that also vanishes at the origin. Furthermore, even if we relax the second condition to allow for values $\alpha \neq 1/2$, it remains the case that Brownian motion is the only *continuous* stochastic process satisfying all of these properties. If we allow for discontinuous processes then we can find other processes satisfying these properties but there are still very "few" of them. More precisely, for each value $\alpha > 1/2$, there is a process L_α (the so-called 'spectrally positive $1/\alpha$-stable Lévy process') such that every process satisfying the properties of Proposition 1 is of the form $\kappa_+ L_\alpha - \kappa_- \tilde L_\alpha$, where $\kappa_\pm$ are two positive numbers and $\tilde L_\alpha$ is an independent copy of L_α.

Processes satisfying the three properties of Proposition 1 arise naturally (or rather, in many cases, are conjectured to arise) as scaling limits of various "toy models" of statistical mechanics. In these situations, however, one is typically interested in processes that do not depend on a time parameter but instead on two or more "spatial" parameters. Furthermore, in most known cases, the processes arising in this way are random Schwartz distributions, so that some care has to be taken with the formulation of the Markov property. One formulation is the following.

Definition 1 A random distribution η on $\mathbb{R}^d$ satisfies the germ Markov property if, for any smooth domain D and any neighbourhood U of ∂D, the laws of $\{\eta(\phi) : \operatorname{supp}\phi \subset D\}$ and $\{\eta(\phi) : \operatorname{supp}\phi \subset D^c\}$ are independent, conditional on $\{\eta(\phi) : \operatorname{supp}\phi \subset U\}$.

For $d > 2$, the "free field" is the analogue of Brownian motion and is defined as the random distribution η such that all random variables of the type $\eta(\phi)$ with $\phi \in \mathcal{C}^\infty_0$ are jointly centred Gaussians with covariance given by

$$\mathbf{E}\eta(\phi)\eta(\psi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\psi(y)|x-y|^{2-d}\,dx\,dy .$$

Again, this is translation invariant, has the germ Markov property and is scale invariant with exponent $\alpha = 1 - \frac{d}{2}$. (A similar object also exists for $d = 2$ and is of great interest but the associated notion of "translation invariance" is more involved.) An answer to the following question would be a gigantic breakthrough in probability theory and mathematical physics.

203* Open Problem. For $d \ge 2$ and $\alpha < 0$, characterise all (if any) random distributions that are invariant under the Euclidean transformations, scale invariant with exponent α and satisfy the germ Markov property.
Any partial result, including the description of any previously unknown non-Gaussian random distribution with these properties, would be very welcome. Besides the free field, one such random distribution was recently constructed in $d = 2$ with exponent $-1/8$ as the scaling limit of the Ising model at criticality [1, 2]. Conformal field theory provides a conjectured characterisation of a whole family of such objects for a range of exponents α in $d = 2$ but the case $d \ge 3$ is wide open, even at the conjectural level. Another breakthrough in this direction was the recent characterisation [4] of the "KPZ fixed point", a space-time random function H (in space dimension 1) that is translation invariant, has the germ Markov property and is scale invariant in the sense that $\lambda^{-1} H(\lambda^2 x, \lambda^3 t) \overset{\text{law}}{=} H(x,t)$.

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III Solutions

187. Let $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$ and $(c_n)_{n\ge0}$ be sequences such that $a_n > 0$, $b_n > 0$ and $c_n > 0$ for $n \ge 1$ and:
(G1) $c_0 = 0$ and c_n is increasing,
(G2) $c_{n+1} - c_n$ is decreasing for $n \ge 0$,
(G3) $c_k\left(\frac{a_{k+1}}{a_k} - 1\right) \ge c_n\left(\frac{b_{n+1}}{b_n} - 1\right)$ for $1 \le k < n$.
Given a function f, let

$$A_n = \frac{1}{c_{n-1}}\sum_{k=1}^{n-1} f\left(\frac{a_k}{b_n}\right), \qquad n \ge 2.$$

Then, if f is real, convex increasing and non-negative on an interval $[D, E]$ that includes all the points $\frac{a_k}{b_n}$ for $k < n$, prove that A_n increases with n.

(Shoshana Abramovich, University of Haifa, Israel)

Solution by the proposer. Similarly to [1, Theorem 5.1], we will show that under our conditions,

$$A_{n+1} - A_n \geq 0, \qquad n \geq 2. \tag{3}$$

By the definition of A_n, we get

$$\begin{aligned} A_{n+1} - A_n &= \frac{1}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big) - \frac{1}{c_{n-1}}\sum_{k=1}^{n-1} f\Big(\frac{a_k}{b_n}\Big) \\ &= \frac{1}{c_{n-1}}\Big[\frac{c_{n-1}}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big) - \sum_{k=1}^{n-1} f\Big(\frac{a_k}{b_n}\Big)\Big]. \end{aligned} \tag{4}$$

To enable proving (3), we rewrite

$$\frac{c_{n-1}}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big), \qquad n \geq 2. \tag{5}$$

By using (G2), we get that

$$c_{n-1} > c_n - c_k + c_{k-1}.$$

As f is non-negative and $c_n > 0$ when $n \geq 1$, we get from (5) that

$$\begin{aligned} \frac{c_{n-1}}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big) &\geq \sum_{k=1}^{n} \frac{c_{k-1} + c_n - c_k}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big) \\ &= \sum_{k=1}^{n} \frac{c_{k-1}}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big) + \sum_{k=1}^{n} \frac{c_n - c_k}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big). \end{aligned} \tag{6}$$

It is given that $c_0 = 0$, therefore (6) leads to

$$\begin{aligned} \frac{c_{n-1}}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big) &\geq \sum_{k=1}^{n} \frac{c_{k-1}}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big) + \sum_{k=1}^{n} \frac{c_n - c_k}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big) \\ &= \sum_{k=2}^{n} \frac{c_{k-1}}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big) + \sum_{k=1}^{n-1} \frac{c_n - c_k}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big) \\ &= \sum_{k=1}^{n-1} \Big(\frac{c_k}{c_n} f\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big)\Big). \end{aligned} \tag{7}$$

From (7), by using the convexity of f, we get that

$$\begin{aligned} \frac{c_{n-1}}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big) &\geq \sum_{k=1}^{n-1} \Big(\frac{c_k}{c_n} f\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n} f\Big(\frac{a_k}{b_{n+1}}\Big)\Big) \\ &\geq \sum_{k=1}^{n-1} f\Big(\frac{c_k}{c_n}\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n}\Big(\frac{a_k}{b_{n+1}}\Big)\Big). \end{aligned} \tag{8}$$

We see now by (G3), because $a_k > 0$, $b_k > 0$ and $c_k > 0$, $k \geq 1$, that

$$\begin{aligned} &\Big(\frac{c_k}{c_n}\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n}\Big(\frac{a_k}{b_{n+1}}\Big)\Big) - \frac{a_k}{b_n} \\ &\qquad = \frac{a_k}{c_n b_{n+1}}\Big(c_k\Big(\frac{a_{k+1}}{a_k} - 1\Big) - c_n\Big(\frac{b_{n+1}}{b_n} - 1\Big)\Big) \geq 0, \\ &\qquad\qquad 1 \leq k < n. \end{aligned} \tag{9}$$

Hence, from (9),

$$\Big(\frac{c_k}{c_n}\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n}\Big(\frac{a_k}{b_{n+1}}\Big)\Big) \geq \frac{a_k}{b_n}, \qquad 1 \leq k < n, \tag{10}$$

and, as f is increasing on the interval $[D, E]$, from (10):

$$f\Big(\frac{c_k}{c_n}\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n}\Big(\frac{a_k}{b_{n+1}}\Big)\Big) \geq f\Big(\frac{a_k}{b_n}\Big), \qquad 1 \leq k < n. \tag{11}$$

From (7) and (11), we get that

$$\begin{aligned} \frac{c_{n-1}}{c_n}\sum_{k=1}^{n} f\Big(\frac{a_k}{b_{n+1}}\Big) &\geq f\Big(\frac{c_k}{c_n}\Big(\frac{a_{k+1}}{b_{n+1}}\Big) + \frac{c_n - c_k}{c_n}\Big(\frac{a_k}{b_{n+1}}\Big)\Big) \\ &\geq f\Big(\frac{a_k}{b_n}\Big), \qquad 2 \leq k < n. \end{aligned} \tag{12}$$

From (12) and (4), we get that (3) holds, which means that A_n is increasing with n, $n \geq 2$. □

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Also solved by Mihaly Bencze (Romania), Socratis Varelogiannis (France), Alexander Vauth (Germany).

188. For a function $f : \mathbb{R} \to \mathbb{R}$ and a positive integer n, we denote by f^n the function defined by $f^n(x) = (f(x))^n$.

(a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a function that has an antiderivative then $f^n : \mathbb{R} \to \mathbb{R}$ satisfies the intermediate value property for any $n \geq 1$.

(b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that has an antiderivative and for which $f^n : \mathbb{R} \to \mathbb{R}$ has no antiderivatives for any $n \geq 2$.

(Dorin Andrica, Babeş Bolyai University, Cluj-Napoca, Romania, and Vlad Crişan, University of Göttingen, Germany)

Solution by the proposers.

(a) Since f has an antiderivative, f satisfies the intermediate value property (IVP). The compositum of two functions satisfying the IVP is again a function satisfying IVP. Note that

$$f^n = g \circ f,$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined as $g(x) = x^n$. It is easy to see that g has the IVP, hence $f^n = g \circ f$ must also satisfy the IVP.

(b) We use the following classical result.

Lemma 1 For $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, the function defined by

$$f_{a,b}(x) = \begin{cases} \cos\frac{a}{x} & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{cases}$$

has an antiderivative if and only if $b = 0$.

From Lemma 1, it follows that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \cos^2\frac{1}{x} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

has an antiderivative.

We now have that

$$f^n(x) = \begin{cases} \cos^{2n}\frac{1}{x} & \text{if } x \neq 0 \\ (\frac{1}{2})^n & \text{if } x = 0. \end{cases}$$

We shall prove that f^n has no antiderivative for $n \geq 2$. For this, one uses the following identity, whose proof is just a simple induction.

Lemma 2 For any reals $x_1, \dots, x_m$, one has

$$\cos x_1 \cos x_2 \cdots \cos x_m = \frac{1}{2^m}\sum \cos(\pm x_1 \pm x_2 \pm \cdots \pm x_m),$$

where the sum on the right side is over all 2^m possible choices of signs.

Using Lemma 2, we can write

$$f^n(x) = \begin{cases} \frac{1}{2^{2n}} \sum_{+,-} \cos(\underbrace{\pm\frac{1}{x} \pm \cdots \pm \frac{1}{x}}_{2n \text{ terms}}) & \text{if } x \neq 0 \\ \frac{1}{2^n} & \text{if } x = 0, \end{cases}$$

$$= \frac{1}{2^{2n}} \sum_{+,-\neq 0} f_{(\underbrace{\pm 1 \pm \cdots \pm 1}_{2n \text{ terms}},0)} + \begin{cases} \frac{1}{2^{2n}} S_n & \text{if } x \neq 0 \\ \frac{1}{2^n} & \text{if } x = 0, \end{cases}$$

where the last sum is over all possible choices of signs for which we have

$$\underbrace{\pm 1 \pm \cdots \pm 1}_{2n \text{ terms}} \neq 0$$

and S_n is the number of combinations for which

$$\underbrace{\pm 1 \pm \cdots \pm 1}_{2n \text{ terms}} = 0.$$

Using Lemma 1, we have that f^n has an antiderivative if and only if $\frac{1}{2^{2n}} S_n = \frac{1}{2^n}$, i.e., $S_n = 2^n$. On the other hand, it is clear that

$$S_n = \binom{2n}{n},$$

so the last condition becomes $\binom{2n}{n} = 2^n$. This equation has no solution for $n \geq 2$ since, for example, any prime p between n and $2n$ divides $\binom{2n}{n}$ but does not divide 2^n. □

Also solved by Mihaly Bencze (Romania), John N. Daras (Greece), Sotirios E. Louridas (Greece).

189.

(a) Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of continuous real-valued functions on a compact metric space X that converges pointwisely to a continuous function f. Show that the convergence must be uniform.

(b) Show by a counterexample that the compactness of X in (a) is necessary.

(c) Determine whether (a) remains valid if the sequence $\{f_n\}_{n=1}^{\infty}$ is not monotone.

(W. S. Cheung, University of Hong Kong, Pokfulam, Hong Kong)

Solution by the proposer.

(a) For any $n \in \mathbb{N}$, write $g_n := f - f_n$. As $f_n \uparrow f$ pointwisely on X, we have $g_n \downarrow 0$ pointwisely on X.
Let $\varepsilon > 0$ be given. For any $x \in X$, there exists $N_x \in \mathbb{N}$ such that

$$g_{N_x}(x) < \varepsilon .$$

Since g_{N_x} is continuous, there exists an open neighbourhood B_x of x in X such that

$$g_{N_x}(y) < \varepsilon \qquad \text{for all } y \in B_x .$$

Do this for every $x \in X$. The open cover $\{B_x : x \in X\}$ of X has a finite subcover, say $\{B_{x_1}, \ldots, B_{x_k}\}$. Write

$$N := \max\{N_{x_1}, \ldots, N_{x_k}\} .$$

For any $y \in X$, there is an $i \in \{1, \ldots, k\}$ such that $y \in B_{x_i}$. Hence, for any $n \geq N \geq N_{x_i}$, we have

$$0 \leq g_n(y) \leq g_N(y) \leq g_{N_{x_i}}(y) < \varepsilon .$$

Hence $\{g_n\} \to 0$ uniformly on X and so $\{f_n\} \to f$ uniformly on X.

(b) For any $n \in \mathbb{N}$, consider $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) := \Big(1 - \frac{1}{n}\Big)x .$$

Clearly, $f_n(x) \uparrow f(x) := x$ pointwisely on $\mathbb{R}$ but, as

$$\sup_{n\to\infty} \Big\{\big|f_n(x) - f(x)\big| : x \in \mathbb{R}\Big\} = \sup_{n\to\infty} \Big\{\frac{|x|}{n} : x \in \mathbb{R}\Big\} = \infty ,$$

the convergence is not uniform.

(c) Without the monotonicity, (a) will no longer be valid. For example, consider the sequence $f_n : X = [0, 1] \to \mathbb{R}$ given by

$$f_n(x) := \begin{cases} 0 & 0 \leq x \leq 1 - \frac{2}{n} \\ n^2x - n^2 + 2n & 1 - \frac{2}{n} \leq x \leq 1 - \frac{1}{n} \\ -n^2x + n^2 & 1 - \frac{1}{n} \leq x \leq 1 . \end{cases}$$

f_n is continuous on X for each n and $f_n \to f \equiv 0$ pointwisely on X but, as

$$\sup_{n\to\infty} \Big\{\big|f_n(x) - f(x)\big| : x \in X\Big\} = \sup_{n\to\infty} \Big\{f_n(x) : x \in X\Big\} = \sup_{n\to\infty} \Big\{n\Big\} = \infty ,$$

the convergence is not uniform. □

Also solved by Socratis Varelogiannis (France), Alexander Vauth (Germany), Jeff Webb (UK).

190. Let $\{a_n\}$ be a sequence of positive numbers. In the ratio test, we know that the condition

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$$

is not sufficient to determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent. For example, if $a_n = 1/n$ then

$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} = 1 - \frac{1}{n+1} = 1 - \frac{n+1}{(n+1)^2}$$

and if $a_n = 1/n^2$ then

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = 1 - \frac{2n+1}{(n+1)^2}.$$

Hence, the coefficient a in the expression $1 - \frac{an+1}{(n+1)^2}$ plays an important role in the convergence of $\sum a_n$. In this question, we would like to study it more closely.
Let a be a non-negative real number and let $\{a_n\}$ be a sequence with $a_n > 0$, satisfying

$$\frac{a_{n+1}}{a_n} \leq 1 - \frac{an+1}{(n+1)^2} \tag{13}$$

for all $n \geq n_0 := [|2 - a|] + 1$, where $[x]$ is the integral part of x.

(i) Show that if $a > 0$ then

$$\lim_{n\to\infty} a_n = 0.$$

If $a = 0$, for any $\lambda > 0$, find an example such that

$$\lim_{n\to\infty} a_n = \lambda.$$

(ii) Show that if $a > 1$ then

$$\sum_{n=1}^{\infty} a_n$$

is convergent. Is this still true when $a = 1$?

(Stephen Choi and Peter Lam, Simon Fraser University, Burnaby B.C., Canada)

Solution by the proposers. (i) In view of (13), the sequence $\{a_n\}$ is eventually monotonically decreasing and bounded below by 0. So, $\lim_{n\to\infty} a_n$ exists and is non-negative.

By (13), we have

$$a_{n+1} \le \left(\frac{n^2+(2-a)n}{(n+1)^2}\right)a_n \le \cdots \le a_{n_0}\prod_{i=n_0}^{n}\left(\frac{i^2+(2-a)i}{(i+1)^2}\right) \tag{14}$$

for all $n \ge n_0$. It follows that

$$\frac{1}{a_{n+1}} \ge \frac{1}{a_{n_0}}\prod_{i=n_0}^{n}\left(\frac{(i+1)^2}{i^2+(2-a)i}\right) = \frac{1}{a_{n_0}}\prod_{i=n_0}^{n}\left(1+\frac{ai+1}{i^2+(2-a)i}\right) \tag{15}$$

for all $n \ge n_0$.

If $a > 2$ then

$$\sum_{i=n_0}^{\infty}\frac{ai+1}{i^2+(2-a)i} \ge \sum_{i=n_0}^{\infty}\frac{a}{i} = \infty.$$

If $2 > a > 0$ then

$$\sum_{i=n_0}^{\infty}\frac{ai+1}{i^2+(2-a)i} \ge \sum_{i=n_0}^{\infty}\frac{a}{2i} = \infty$$

because $i > (2-a)$ for $i \ge n_0$. Since

$$\prod_{i=n_0}^{n}\left(1+\frac{ai+1}{i^2+(2-a)i}\right) \ge 1+\sum_{n=n_0}^{n}\frac{ai+1}{i^2+(2-a)i},$$

we have $\lim_{n\to\infty} a_n = 0$ by (15).

If $a = 0$, for any $\lambda > 0$, take $a_n = \lambda(n+1)/n$. Then,

$$\frac{a_{n+1}}{a_n} = 1-\frac{1}{(n+1)^2}$$

and $\lim_{n\to\infty} a_n = \lambda$.

(ii) If $a > 1$, in view of (14), for any $n \ge n_0$, we have

$$\begin{aligned} a_{n+1} &\le a_{n_0}\prod_{i=n_0}^{n}\left(\frac{i^2+(2-a)i}{(i+1)^2}\right) \\ &= a_{n_0}\left(\prod_{i=n_0}^{n}\frac{i}{i+1}\right)\prod_{i=n_0}^{n}\left(\frac{i+(2-a)}{i+1}\right) \\ &= a_{n_0}\frac{n_0}{n+1}\prod_{i=n_0}^{n}\left(1-\frac{a-1}{i+1}\right) \\ &= \frac{a_{n_0}n_0}{n+1}\exp\left\{\sum_{i=n_0}^{n}\log\left(1-\frac{a-1}{i+1}\right)\right\}. \end{aligned}$$

Now, using an elementary inequality $\log(1-x) \le -x$ for $0 \le x < 1$, we have

$$\begin{aligned} a_{n+1} &\le \frac{a_{n_0}n_0}{n+1}\exp\left\{-(a-1)\sum_{i=n_0}^{n}\frac{1}{i+1}\right\} \\ &\le \frac{a_{n_0}n_0}{n+1}\exp\left\{-(a-1)\int_{n_0+1}^{n+2}\frac{1}{x}dx\right\} \\ &\le \frac{a_{n_0}n_0}{n+1}\exp\left\{-(a-1)\log\left(\frac{n+2}{n_0+1}\right)\right\} \\ &= \frac{a_{n_0}n_0}{n+1}\left(\frac{n_0+1}{n+2}\right)^{(a-1)} \\ &\le \frac{a_{n_0}(n_0+1)^a}{(n+1)^a} \end{aligned}$$

for all $n \ge n_0$. Since the series $\sum_{n=1}^{\infty}\frac{1}{(n+1)^a}$ converges, so does $\sum_{n=1}^{\infty} a_n$.

If $a = 1$, the statement is false by considering the counterexample $a_n = 1/n$. □

Also solved by Mihaly Bencze (Romania), Panagiotis Krasopoulos (Greece).

191. Show that for any $a, b > 0$, we have

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

(Silvestru Sever Dragomir, Victoria University, Melbourne City, Australia)

Solution by the proposer. Integrating by parts, we have

$$\int_a^b \frac{b-t}{t^2}dt = \frac{b-a}{a} - \ln b + \ln a \tag{16}$$

for any $a, b > 0$.

If $b > a$ then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{17}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \ge \int_b^a \frac{t-b}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{18}$$

Therefore, by (17) and (18), we have for any $a, b > 0$ that

$$\int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (16), we then get the desired result. □

Also solved by Jim K. Kelesis (Greece), Panagiotis Krasopoulos (Greece), Alexander Vauth (Germany).

192. Let $a, b, c, d \in \mathbb{R}$ with $bc > 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix}\cos\frac{a}{n} & \sin\frac{b}{n} \\ \frac{c}{n} & \cos\frac{d}{n}\end{pmatrix}^n.$$

(Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania)

Solution by the proposer. The limit equals

$$\begin{pmatrix} \cosh\sqrt{bc} & \frac{b}{\sqrt{bc}}\sinh\sqrt{bc} \\ \frac{c}{\sqrt{bc}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} \cos\frac{a}{n} & \sin\frac{b}{n} \\ \frac{c}{n} & \cos\frac{d}{n} \end{pmatrix}.$$

The characteristic equation of A is given by

$$\lambda^2 - \left(\cos\frac{a}{n} + \cos\frac{d}{n}\right)\lambda + \cos\frac{a}{n}\cos\frac{d}{n} - \frac{c}{n}\sin\frac{b}{n} = 0.$$

The discriminant of this equation is $\Delta = \left(\cos\frac{a}{n} - \cos\frac{d}{n}\right)^2 + \frac{4c}{n}\sin\frac{b}{n}$. Since $bc > 0$, one has that either both b and c are positive or both are negative real numbers. If b and c are positive numbers, one has that for large n, $0 < \frac{b}{n} < \pi$ and hence $\Delta > 0$. If b and c are negative real numbers then, for large n, one has that $-\pi < \frac{b}{n} < 0$ and it follows, since $\sin\frac{b}{n} < 0$, that $\Delta > 0$. Therefore, there are two real distinct eigenvalues of A given by

$$\lambda_1 = \frac{1}{2}\left(\cos\frac{a}{n} + \cos\frac{d}{n}\right) + \frac{1}{2}\sqrt{\left(\cos\frac{a}{n} - \cos\frac{d}{n}\right)^2 + \frac{4c}{n}\sin\frac{b}{n}},$$

$$\lambda_2 = \frac{1}{2}\left(\cos\frac{a}{n} + \cos\frac{d}{n}\right) - \frac{1}{2}\sqrt{\left(\cos\frac{a}{n} - \cos\frac{d}{n}\right)^2 + \frac{4c}{n}\sin\frac{b}{n}}.$$

Now we need Theorem 4.7 on page 194 (see also Remark 3.1 on page 109) in [1], which states that if $n \in \mathbb{N}$, $A \in \mathcal{M}_2(\mathbb{C})$ and $\lambda_1 \neq \lambda_2$ are the eigenvalues of A then

$$A^n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}A + \frac{\lambda_1\lambda_2^n - \lambda_2\lambda_1^n}{\lambda_1 - \lambda_2}I_2.$$

Let $k \in \mathbb{N}$. An easy calculation, based on the previous formula, shows that

$$A^k = \frac{\lambda_1^k(1-\lambda_2) + \lambda_2^k(\lambda_1 - 1)}{\lambda_1 - \lambda_2}I_2 + \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}\begin{pmatrix} \cos\frac{a}{n} - 1 & \sin\frac{b}{n} \\ \frac{c}{n} & \cos\frac{d}{n} - 1 \end{pmatrix}.$$

When $k = n$, one has that

$$A^n = \frac{\lambda_1^n(1-\lambda_2) + \lambda_2^n(\lambda_1 - 1)}{\lambda_1 - \lambda_2}I_2 + \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\begin{pmatrix} \cos\frac{a}{n} - 1 & \sin\frac{b}{n} \\ \frac{c}{n} & \cos\frac{d}{n} - 1 \end{pmatrix}. \qquad (19)$$

We have that $\lim_{n\to\infty}\lambda_1^n = e^{\sqrt{bc}}$ and $\lim_{n\to\infty}\lambda_2^n = e^{-\sqrt{bc}}$. On the other hand, a calculation shows that $\lim_{n\to\infty}\frac{1-\lambda_2}{\lambda_1-\lambda_2} = \lim_{n\to\infty}\frac{\lambda_1-1}{\lambda_1-\lambda_2} = \frac{1}{2}$, $\lim_{n\to\infty}\frac{\cos\frac{a}{n}-1}{\lambda_1-\lambda_2} = \lim_{n\to\infty}\frac{\cos\frac{d}{n}-1}{\lambda_1-\lambda_2} = 0$, $\lim_{n\to\infty}\frac{\sin\frac{b}{n}}{\lambda_1-\lambda_2} = \frac{b}{2\sqrt{bc}}$ and $\lim_{n\to\infty}\frac{\frac{c}{n}}{\lambda_1-\lambda_2} = \frac{c}{2\sqrt{bc}}$.

Passing to the limit as $n \to \infty$ in (19) and using the previous limits we have that

$$\lim_{n\to\infty}A^n = \frac{e^{\sqrt{bc}} + e^{-\sqrt{bc}}}{2}I_2 + \left(e^{\sqrt{bc}} - e^{-\sqrt{bc}}\right)\begin{pmatrix} 0 & \frac{b}{2\sqrt{bc}} \\ \frac{c}{2\sqrt{bc}} & 0 \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{bc} & \frac{b}{\sqrt{bc}}\sinh\sqrt{bc} \\ \frac{c}{\sqrt{bc}}\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{pmatrix}$$

and the problem is solved. □

References

[1] Vasile Pop, Ovidiu Furdui, *Square Matrices of Order 2. Theory, Applications and Problems*. Cham, Springer, 2017.

Also solved by Mihaly Bencze (Romania), Jim K. Kelesis (Greece), Sotirios E. Louridas (Greece), Julio Cesar Mohnsam (Brazil), Socratis Varelogiannis (France).

We would like you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *topology*.

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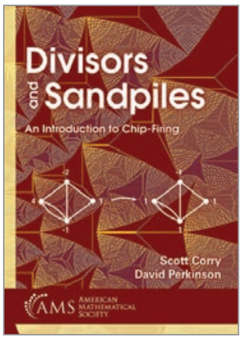

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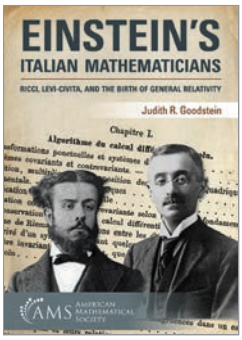

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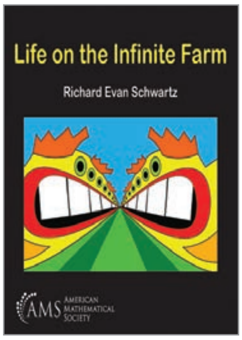

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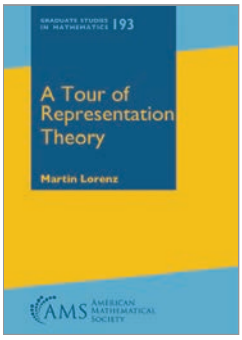

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