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Feature

Metastability of Stochastic Partial Differential Equations and Fredholm Determinants

Alessio Figalli: Magic, Method, Mission

Discussion

Dynamics and Control of Covid-19

Society

Armenian Mathematical Union

Edinburgh, venue of the 30th Anniversary celebration of the EMS planned for October 2020. Photo: José Bidarra de Almeida

Editorial Team

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EMS Agenda

2020

29 October
EMS 30 Years Anniversary Celebration
Edinburgh, UK

30 October – 1 November
EMS Executive Committee Meeting
Edinburgh, UK

EMS Scientific Events

2020

Due to the Covid-19 measures most conferences were cancelled or postponed to 2021.

2021

20 – 26 June
8th European Congress of Mathematics
Portorož, Slovenia

21 – 23 September
The Unity of Mathematics: A Conference in Honour of
Sir Michael Atiyah
Isaac Newton Institute for Mathematical Sciences

A Message from the President

Volker Mehrmann, President of the EMS

Dear members of the EMS,

It is not easy to address you in these strange times. However, I am sure that science will find a way of dealing with the Covid-19 crisis. As in most countries, university life is still on hold or online, teachers and students have had to adapt to different ways of learning, and even the EMS is managing to run its organisation with modern tools. Due to the cancellation of the in-person Bled council meeting, it was held online on Saturday, 4 July 2020.

The council elected Jorge Buescu as new vice president and Jiří Rákosník as new secretary. New executive members are Frédéric Hélein, Barbara Kaltenbacher, Luis Narváez Macarro and Susanna Terracini. Beatrice Pelloni was reelected for a second term. I look forward to working with the new team.

My special thanks go to vice president Armen Sergeev, secretary Sjoerd Verduyn Lunel and the EC members Nicola Fusco, Stefan Jackowski and Vicente Muñoz, who leave their positions at the end of 2020, for their hard work for the society and their active participation in the executive committee. It was a pleasure to work with all of you.

My thanks also go to the editor of the Newsletter, Valentin Zagrebnov, whose term ended in July 2020, for his great work, and to his successor Fernando Manuel Pestana da Costa for his willingness to take over. Fernando is heading the transition of the Newsletter to the new format of an "EMS Magazine".

Another important decision by the council was the selection of the venue of the 9ECM. While the 8ECM in Portorož is still ahead of us and is planned for 20–26 June 2021, the council chose Sevilla for 2024, and we are looking forward to two great congresses in 2021 and 2024.

We plan to continue the discussions about the future of the EMS at a presidents meeting which hopefully (who knows these days) will take place as a personal meeting in Edinburgh on 30 October 2020. Topics will include the formation of activity groups, the organisation of specialised meetings and the start of a young academy.

On 29 October 2020 we are planning a celebration of the 30th anniversary of the EMS with a one-day meeting. The programme will be posted soon. In view of the anniversary, thanks to the great effort of vice president Betül Tanbay, a brochure was created that covers the past, the present and the future of the EMS.

I wish you all the best, in particular good health, and hope to meet many of you soon in Edinburgh.

A Message from the Editor-in-Chief

Fernando Pestana da Costa, Editor-in-Chief of the EMS Newsletter

Dear readers of the European Mathematical Society Newsletter,

It is my great honour, and an even greater responsibility, to accept the invitation of the Executive Committee of the EMS and of its president Volker Mehrmann to become, from 3 July 2020, the editor-in-chief of the Newsletter. The previous editor-in-chief, Valentin Zagrebnov, under whose guidance I had the privilege of working as an editor for the past three and a half years, has done a wonderful job in promoting the Newsletter and enhancing its interest for the European mathematics community. His wise words on how to efficiently run the journal and smoothly steer it in this transitional period are gratefully acknowledged.

These are indeed transitional times for the Newsletter: according to decisions taken by the Executive Committee of the EMS, our journal will soon undergo a deep restructuring, reflected in the change of its name from *Newsletter* to *Magazine*, the introduction of an online first publishing policy, the removal of news from the printed version, and the introduction of new subjects such as "Mathematics and the Arts", and "Mathematics and Industry", for which new editors have been added to the previously existing editorial team (see their names and biographical notes below), whose members have agreed to continue working with me as the new editor-in-chief.

All these changes will see the light in the first issue of 2021 and will be reflected in the introduction of a new layout and design, in line with the changes that are being prepared at EMS Press for other publications of our Society. Cooperation with the staff of EMS Press (CEO André Gaul, Editorial Director Apostolos Damialis, and Head of Production Sylvia Fellmann Lotrovsky) has been intense but very rewarding.

All of us, the editorial team and the EMS Press staff, hope the result will be an enhanced Newsletter/Magazine that will continue to keep the high standards set by the previous editors-in-chief, and will continue to serve the mathematically interested European community in progressively better ways.

I finish this editorial with the final words of the editorial by Valentin Zagrebnov in issue 101, when he became editor-in-chief: "We hope that all readers will feel free to contact the editorial board whenever they have ideas for future articles, comments, criticisms or suggestions."

New Editors Appointed

António Bandeira Araújo is an assistant professor at the Department of Sciences and Technology of the Universidade Aberta, Lisbon, and a researcher at CIAC, the Research Center for Arts and Communication. He works mainly on the connections between mathematics and the visual arts. He has developed geometrical methods for the construction of spherical perspectives and immersive anamorphoses, creating "hybrid models" that connect traditional sketching media with virtual reality visualisations. He graduated from the University of Lisbon with a physics degree and then went on to obtain a master's and a PhD in mathematics, specialising in geometry. He did research in contact geometry, studying the singularities of Legendrian varieties, before turning to his present area of research. He currently coordinates Aberta's branch of the CIAC research center and is a vice-coordinator of Aberta's PhD programme in Digital Media Arts (DMAD).

His homepage is http://www.univ-ab.pt/~aaraujo/

Krzysztof Burnecki is an associate professor at the Faculty of Pure and Applied Mathematics of the Wroclaw University of Science and Technology (WUST) and a vice-director of the Hugo Steinhaus Center, which specialises in modelling of random phenomena in natural sciences, economics and industry. He received his PhD degree in 1999 in Mathematics under the supervision of Aleksander Weron and his habilitation in 2013 in Control and Robotics from the Wrocław University of Technology. His research interests include self-similar processes, heavy-tailed models and time series analysis, industrial mathematics, insurance mathematics, mathematical physics and computational statistics.

He has written over 80 scientific publications, about 46 of which have appeared in peer-reviewed international journals such as *Nature*, *Physical Review Letters*, *Astrophysical Journal*, *Scientific Reports*, *IEEE Transactions on Signal Processing*, *Biophysical Journal*, and *Insurance: Mathematics and Economics*. He was a supervisor of two PhD theses at WUST and a co-supervisor of a PhD at Cape Town University. He has conducted several research projects with the industry, in particular constructed insurance and risk management strategies for energy sector companies. He is editor-in-chief of *Mathematica Applicanda*, associate editor of *Computational Statistics* (Springer) and ICIAM Dianoia. He is a member of the ECMI Council, Polish Mathematical Society and Polish Society of Actuaries. As a Newsletter editor he would like to pursue interesting success stories highlighting the interaction between Mathematics and Industry.

Ivan Oseledets graduated from the Moscow Institute of Physics and Technology in 2006, and defended his PhD in 2006 and habilitation in 2012, both at the Marchuk Institute of Numerical Mathematics of the Russian Academy of Sciences. He has been working at the Skolkovo Institute of Science and Technology in Moscow since 2013, and was promoted to full professor in 2019.

His research interests cover numerical linear algebra, numerical mathematics, data analysis, machine learning and high-dimensional data processing. He proposed the tensor-train decomposition of high-dimensional tensors and developed a series of methods and algorithms for solving different problems in physics, chemistry, biology and data analysis. He is an associate editor of the SIAM journal on Mathematics of Data Science, the SIAM journal on Scientific Computing, Advances in Computational Mathematics, and also area chair of ICLR 2020 and NeurIPS 2020.

Ivan's recent research focuses on fundamental questions in machine and deep learning, and their connections to other areas in mathematics such as algebraic geometry, topology and tensor methods.

2020 Jaroslav and Barbara Zemánek Prize

The Jaroslav and Barbara Zemánek Prize in functional analysis with emphasis on operator theory for 2020 is awarded to **Michael Hartz** (Universität des Saarlandes, Germany) for his work on operator-oriented function theory of several variables.

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The jury emphasized his recent breakthrough result showing that every complete Nevanlinna-Pick space has the column-row property with constant one, as well as several deep results on multipliers and functional calculi for tuples of commuting operators on spaces of holomorphic functions.

Following a generous donation of Zemánek's family, the annual Zemánek Prize was founded by the Institute of Mathematics of the Polish Academy of Sciences (IM PAN) in March 2018, in order to encourage the research in functional analysis, operator theory and related topics. The Prize is established to promote young mathematicians, under 35 years of age, who made important contributions to the field.

The awarding ceremony will take place at IM PAN, Warsaw, in Fall 2020.

A more detailed information about the Prize can be found on the webpage https://www.impan.pl/en/events/awards/b-and-j-zemanek-prize.

Metastability of Stochastic Partial Differential Equations and Fredholm Determinants

Nils Berglund (Université d'Orléans, France)

Metastability occurs when a thermodynamic system, such as supercooled water (which is liquid even below freezing point), lands on the "wrong" side of a phase transition, and remains in a state which differs from its equilibrium state for a considerable time. There are numerous mathematical models describing this phenomenon, including lattice models with stochastic dynamics. In this text, we will be interested in metastability in parabolic stochastic partial differential equations (SPDEs). Some of these equations are ill posed, and only thanks to very recent progress in the theory of so-called singular SPDEs does one know how to construct solutions via a renormalisation procedure. The study of metastability in these systems reveals unexpected links with the theory of spectral determinants, including Fredholm and Carleman–Fredholm determinants.

1 Introduction

Put a water bottle in your freezer. If the water is pure enough, when you take it out after a few hours you will find that the water is still in its liquid state although at a negative temperature. One says that the water is supercooled. Shake the bottle and you will see the water suddenly turn into ice.

Supercooled water is an example of a metastable state. In such a state, a thermodynamic potential of a physical system, such as its free energy, is minimised locally but not globally. The transition to its stable state requires the system to jump over an energy barrier, and this may take a long time if only fluctuations due to thermal agitation play a role. Thus, the transformation of supercooled water into ice is achieved by nucleation, that is, by the appearance of ice crystals that grow slowly,[1] although the presence of impurities or an outside energy source can speed up the solidification process.

There are many mathematical models describing metastability phenomena. The first to have been studied were lattice models such as the Ising model with a Metropolis-Hastings type stochastic dynamics. See for instance [8] for an overview of metastability results in lattice systems. Metastability also occurs in continuous systems such as stochastic (ordinary) differential equations, which we will consider in Section 2, and in stochastic partial differential equations that will be addressed in Section 3.

[1] A spherical crystal with radius r changes the system's energy in two ways: since the ice is more stable than the liquid water, the energy is decreased by an amount proportional to the crystal volume, r^3; however the interface between the crystal and the surrounding water increases the energy by a quantity proportional to the crystal's surface, r^2. For small values of r, the second contribution dominates the first one, whereas the opposite holds for large r. Therefore, the ice crystals grow rather slowly as long as their size is smaller than a critical value, for which the volume and the surface terms are comparable.

2 Reversible diffusions

The motion in $\mathbb{R}^n$ of a Brownian particle of mass m, subjected to a force arising from a potential V, a viscous damping force and thermal fluctuations can be described by *Langevin's equation*

$$m\frac{\mathrm{d}^2 x_t}{\mathrm{d}t^2} = -\nabla V(x_t) - \gamma\frac{\mathrm{d}x_t}{\mathrm{d}t} + \sigma\frac{\mathrm{d}W_t}{\mathrm{d}t}\;,$$

where W_t is a Brownian motion (see Appendix A), γ is a damping coefficient and the positive parameter σ is related to the temperature. We will assume in what follows that $V : \mathbb{R}^n \to \mathbb{R}$ is a confining potential (bounded below and converging to infinity quickly enough), and we are mainly interested in the case of small σ. In order to simplify a number of expressions we will write $\sigma = \sqrt{2\varepsilon}$.

When $\varepsilon = 0$, if the mass m is small compared with the damping coefficient γ, the particle converges without oscillating to a local minimum of V. In that case, the motion is said to be *overdamped*. For any ε and in the limit of very small m/γ, one can show that after a change of variables the dynamics is described by the simpler first-order equation

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = -\nabla V(x_t) + \sqrt{2\varepsilon}\,\frac{\mathrm{d}W_t}{\mathrm{d}t}\;, \tag{1}$$

called an overdamped Langevin equation. Mathematically speaking, this is an example of a *stochastic differential equation* (SDE) and its solution is also called a *diffusion*.

For instance, in dimension $n = 1$, if $V(x) = \frac{1}{2}x^2$ equation (1) becomes

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = -x_t + \sqrt{2\varepsilon}\,\frac{\mathrm{d}W_t}{\mathrm{d}t}\;, \tag{2}$$

and describes an overdamped harmonic oscillator subject to thermal noise. Its solution is called an *Ornstein–Uhlenbeck process*.

One way to describe the solutions of (1) is to determine their transition probabilities $p_t(x, y)$. These are such that if a particle starts from the point x at time 0, then the probability $\mathbb{P}^x\{x_t \in A\}$ of finding it in a region A at time $t > 0$ is given by

$$\mathbb{P}^x\{x_t \in A\} = \int_A p_t(x, y)\,\mathrm{d}y\;.$$

It is known that $p_t(x, y)$ satisfies the *Fokker–Planck equation*

$$\partial_t p_t = \nabla\cdot(\nabla V p_t) + \varepsilon\Delta p_t \tag{3}$$

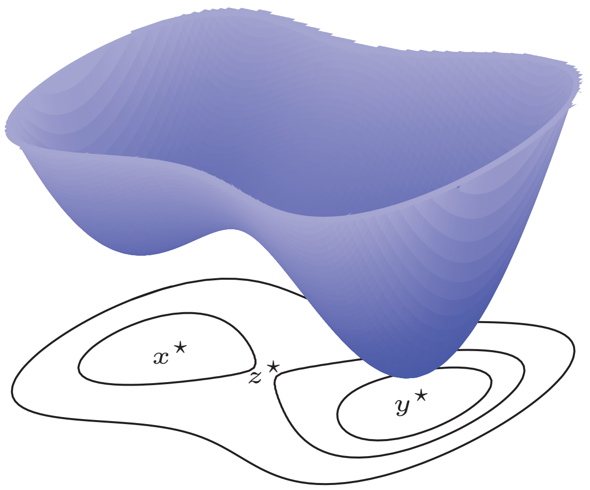

Figure 1. A double-well potential. The local minima $x^\star$ and $y^\star$ are separated by a saddle point at $z^\star$.

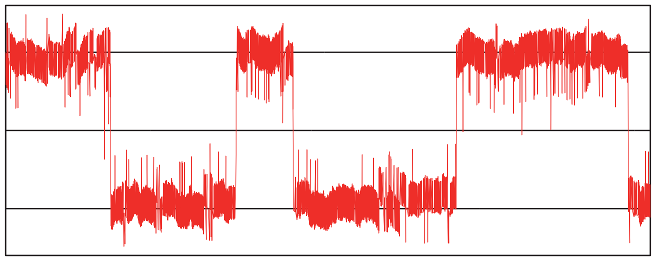

Figure 2. A trajectory x_t of the SDE (1) in dimension 1 and for the potential $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. For most of the time, the trajectory keeps fluctuating around the two local minima of V, $x^\star = -1$ and $y^\star = 1$, with occasional transitions from one minimum to the other. In this simulation, a relatively large ε was chosen to make transitions observable during the extent of the simulation.

(where the operators ∇ and Δ act on the variable y). The term $\nabla \cdot (\nabla V p_t)$ transports p_t by a distance proportional to $-\nabla V$, while $\varepsilon \Delta p_t$ is a diffusion term that tends to spread the distribution of x_t. In the case of the Ornstein–Uhlenbeck process (2) one can check that

$$p_t(x, y) = \frac{1}{\sqrt{2\pi\varepsilon(1 - e^{-2t})}} \exp\left\{-\frac{(y - x\,e^{-t})^2}{2\varepsilon(1 - e^{-2t})}\right\}, \qquad (4)$$

that is, x_t follows a normal law with expectation $x\,e^{-t}$ and variance $\varepsilon(1 - e^{-2t})$. Observe that when t tends to infinity this law converges to a centred normal law with variance ε: the smaller the temperature, the smaller the variance and the less important the fluctuations of x_t.

For general potentials V, one does not know how to solve the Fokker-Planck equation (3). However, it is known that the limit as $t \to \infty$ of $p_t(x, y)$ is always equal to

$$\pi(y) = \frac{1}{Z} e^{-V(y)/\varepsilon},$$

where Z is a normalising constant such that the integral of $\pi(y)$ is equal to 1. In fact,[2] $\pi(y)\,dy$ is also an invariant probability measure of the process, that is,

$$\int_{\mathbb{R}^n} \pi(x) p_t(x, y)\,dx = \pi(y) \qquad \forall y \in \mathbb{R}^n, \forall t > 0.$$

Furthermore, one can prove that the diffusion $(x_t)_{t\geqslant 0}$ is *reversible* with respect to π : its transition probabilities satisfy the *detailed balance condition*

$$\pi(x)p_t(x, y) = \pi(y)p_t(y, x) \qquad \forall x, y \in \mathbb{R}^n, \forall t > 0. \qquad (5)$$

This condition is easy to verify for the transition probabilities (4) of the Ornstein–Uhlenbeck process. From a physical point of view, it means that if we reverse the direction of time, the trajectories keep the same probabilities. Or, to put it another way, if we were to film the system, and then play the film backwards, we would be unable to tell the difference.

Metastability manifests itself in system (1) when V has more than one local minimum. Let us consider the simplest case when V is a double-well potential, meaning that V has exactly two local minima at $x^\star$ and $y^\star$, as well a saddle point at $z^\star$ (Figure 1). The two local minima represent two metastable states of the system, since the solutions of the SDE (1) remain in the neighbourhood of these points (Figure 2) for a long time.

The central question is then the following: let us suppose the diffusion starts in the first local minimum $x^\star$, and let $\mathscr{B}_\delta(y^\star)$ be a ball with small radius δ centered in the second minimum $y^\star$. For small ε, what is the behaviour of the first time $\tau = \inf\{t > 0 \mid x_t \in \mathscr{B}_\delta(y^\star)\}$ for which x_t visits $\mathscr{B}_\delta(y^\star)$?

Arrhenius' Law and large-deviation theory

A first answer to this question was given at the end of the 19th century by Jacobus van t'Hoff, then justified from a physical viewpoint by Svante Arrhenius [1]: the mean value of τ (its mathematical expectation) behaves like $e^{[V(z^\star)-V(x^\star)]/\varepsilon}$. Thus, it is exponentially large in the height of the potential barrier between the two local minima of V. When ε tends to 0, the mean transition time tends very quickly to infinity, reflecting the fact that no transition is possible in the absence of thermal fluctuations. Conversely, when ε increases, the mean transition time becomes shorter and shorter.

A rigorous version of this so-called *Arrhenius law* can be deduced from the *theory of large deviations*, developed in the SDE context by Mark Freidlin and Alexander Wentzell in the years 1960–1970 [11]. The idea is the following: fix a time interval $[0, T]$ and associate to every deterministic differentiable trajectory $\gamma : [0, T] \to \mathbb{R}^n$ the *rate function*

$$\mathscr{I}_{[0,T]}(\gamma) = \frac{1}{2} \int_0^T \left\| \frac{d\gamma}{dt}(t) + \nabla V(\gamma(t)) \right\|^2 dt. \qquad (6)$$

Observe that this function vanishes if and only if $\gamma(t)$ satisfies $\frac{d\gamma}{dt}(t) = -\nabla V(\gamma(t))$, which is Equation (1) with $\varepsilon = 0$. Otherwise, $\mathscr{I}_{[0,T]}(\gamma)$ is strictly positive and measures the "cost" of keeping x_t close to $\gamma(t)$. Indeed, the large-deviation principle for diffusions states that the probability that this happens is close (in a precise sense) to the exponential of $-\mathscr{I}_{[0,T]}(\gamma)/(2\varepsilon)$.

One can also estimate the probability $p(T) = \mathbb{P}^{x^\star}\{\tau \leqslant T\}$ of the diffusion starting from $x^\star$ to reach the ball $\mathscr{B}_\delta(y^\star)$ in time T at most. For that, observe that, for all $T_1 \in [0, T]$, the rate function is bounded below by $\mathscr{I}_{[0,T_1]}(\gamma)$, which can also

[2] The invariance of π follows from the fact that π is in the kernel of the Fokker-Planck operator on the right-hand side of equation (3), which is equivalent to the condition $\varepsilon\nabla\cdot(e^{-V/\varepsilon}\nabla(e^{V/\varepsilon}\pi)) = 0$. The detailed balance condition (5) follows from the fact that this operator is self-adjoint in L^2 with weight $e^{V/\varepsilon}$.

be written as

$$\mathscr{I}_{[0,T_1]}(\gamma) = \frac{1}{2}\int_0^{T_1} \left\| \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) - \nabla V(\gamma(t)) \right\|^2 \mathrm{d}t + 2\int_0^{T_1} \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \cdot \nabla V(\gamma(t))\,\mathrm{d}t \,.$$

The second term on the right-hand side can be integrated and is equal to $2[V(\gamma(T_1)) - V(\gamma(0))]$. Since the potential along any trajectory γ connecting $x^\star$ to $\mathscr{B}_\delta(y^\star)$ reaches at least the value $V(z^\star)$, the large-deviation principle shows that $p(T)$ is at most of order $\mathrm{e}^{-[V(z^\star)-V(x^\star)]/\varepsilon}$. Furthermore, one can construct a trajectory from $x^\star$ to $\mathscr{B}_\delta(y^\star)$ with cost $2[V(z^\star) - V(x^\star) + R(T)]$, where $R(T)$ is a term converging to 0 as $T \to \infty$.[3] The argument is concluded by comparing the system with a Bernoulli process performing independent attempts of reaching $\mathscr{B}_\delta(y^\star)$ in time intervals $[kT, (k+1)T]$, each with probability of success equal to $p(T)$, whose expectation is equal $1/p(T)$. The errors made in comparing the two processes become negligible when $\varepsilon \to 0$.[4]

The Eyring–Kramers Law and potential theory

The *Eyring–Kramers law*, proposed in the 1930s [9, 14], is more precise than Arrhenius' law[5], since it describes the prefactor of the mean transition time. Let us denote by Hess $V(x)$ the Hessian matrix of the potential V at x, which will always be assumed to be non-singular (i.e., with nonzero determinant). All eigenvalues of the matrix Hess $V(x^\star)$ are positive, whereas Hess $V(z^\star)$ has a single negative eigenvalue that will be denoted by $\lambda_-(z^\star)$.[6]

In this case, the Eyring–Kramers law states that

$$\mathbb{E}^{x^\star}[\tau] = \frac{2\pi}{|\lambda_-(z^\star)|}\sqrt{\frac{|\det \operatorname{Hess} V(z^\star)|}{\det \operatorname{Hess} V(x^\star)}}\, \mathrm{e}^{[V(z^\star)-V(x^\star)]/\varepsilon}[1 + R(\varepsilon)] \,, \qquad (7)$$

where $R(\varepsilon)$ is a remainder converging to zero when $\varepsilon \to 0$. There are at present several methods to prove this result. In the following we will explain the method based on potential theory, developed by Anton Bovier, Michael Eckhoff, Véronique Gayrard and Markus Klein in the early 2000s [6], which is generalisable to the case of stochastic PDEs (readers who are not interested in these technical details are invited to go directly to section 3).

Let us fix two disjoint sets, $A, B \subset \mathbb{R}^n$, with smooth boundaries — think of neighbourhoods of the minima $x^\star$ and $y^\star$ of the potential V. The basic observation is that *Dynkin's formula* (or Itô's formula for stopping times) allows us to express several probabilistic quantities of interest as solutions of partial differential equations. For example, the function $w_B(x) = \mathbb{E}^x[\tau_B]$, giving the expected value of the time to reach B starting from x, satisfies the *Poisson problem*

$$\begin{cases} (\mathscr{L} w_B)(x) = -1 & x \in B^c \,, \\ w_B(x) = 0 & x \in B \,, \end{cases} \qquad (8)$$

where $\mathscr{L}$ is the differential operator

$$\mathscr{L} = \varepsilon\Delta - \nabla V \cdot \nabla \,,$$

called the *generator* of the diffusion $(x_t)_{t\geqslant 0}$ (it is the adjoint in L^2 of the Fokker–Planck operator appearing in (3)).

The solution of the Poisson equation (8) can be represented in the form

$$w_B(x) = -\int_{B^c} G_{B^c}(x,y)\,\mathrm{d}y \,, \qquad (9)$$

where G_{B^c} is the *Green function* associated to B^c, solution of

$$\begin{cases} (\mathscr{L} G_{B^c})(x,y) = \delta(x-y) & x \in B^c \,, \\ G_{B^c}(x,y) = 0 & x \in B \,. \end{cases}$$

Reversibility implies that G_{B^c} satisfies the *detailed balance relation*

$$\mathrm{e}^{-V(x)/\varepsilon} G_{B^c}(x,y) = \mathrm{e}^{-V(y)/\varepsilon} G_{B^c}(y,x) \qquad \forall x,y \in B^c \,. \qquad (10)$$

In the case $V = 0$, Green's function has an electrostatic interpretation: $G_{B^c}(x,y)$ is the value at x of the electric potential generated by a unit electric charge at y when the region B is occupied by a conductor at zero potential.

A second important quantity is the *equilibrium potential* $h_{AB}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$, also called *committor*: it gives the probability, starting from x, of reaching the set A before reaching B. It is an $\mathscr{L}$-harmonic function that satisfies the *Dirichlet problem*

$$\begin{cases} (\mathscr{L} h_{AB})(x) = 0 & x \in (A \cup B)^c \,, \\ h_{AB}(x) = 1 & x \in A \,, \\ h_{AB}(x) = 0 & x \in B \,. \end{cases}$$

The equilibrium potential also admits an integral expression in terms of Green's function, namely

$$h_{AB}(x) = -\int_{\partial A} G_{B^c}(x,y) e_{AB}(\mathrm{d}y) \,, \qquad (11)$$

where e_{AB} is a measure concentrated in ∂A, called *equilibrium measure*, defined by

$$e_{AB}(\mathrm{d}x) = (-\mathscr{L} h_{AB})(\mathrm{d}x) \,.$$

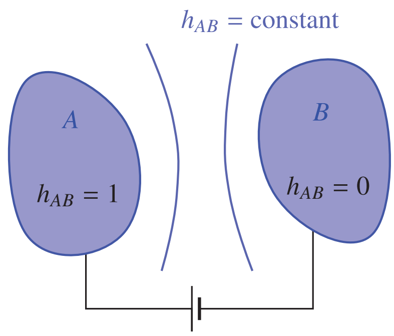

Figure 3. In the case $V = 0$, the equilibrium potential h_{AB} describes the electric potential of a capacitor formed by two conductors A and B at potentials 1 and 0, respectively.

3 For T sufficiently large, one connects points close to $x^\star$ and $z^\star$ in a time $(T-1)/2$ by a trajectory on which $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = +\nabla V(\gamma(t))$, with cost close to $2[V(z^\star) - V(x^\star)]$. Then, one connects a point close to $z^\star$ to $\mathscr{B}_\delta(y^\star)$ in a time $(T-1)/2$ by a zero-cost deterministic trajectory. Finally, one uses the remaining time 1 to connect these pieces of trajectory with straight line segments of negligible cost.

4 The precise statement of this result is that the mean transition time $\mathbb{E}^{x^\star}[\tau]$ satisfies $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}^{x^\star}[\tau] = V(z^\star) - V(x^\star)$.

5 The Eyring–Kramers law was indeed proposed about three decades before the first proof of Arrhenius' law.

6 Indeed, if Hess $V(z^\star)$ had more negative eigenvalues, one would be able to find a more economical path, in terms of maximal height, to go from $x^\star$ to $y^\star$. For example, in dimension 2, the stationary points of V at which the Hessian has two negative eigenvalues are local maxima of V, whereas we are interested in saddle points characterised by one positive and one negative eigenvalues.

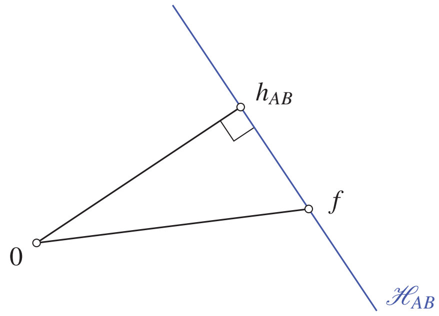

Figure 4. The Dirichlet principle states that the capacity minimises the distance to the origin, measured by the Dirichlet form, in the set $\mathscr{H}_{AB}$ of functions h with value 1 in A and 0 in B. This is due to the fact that $\langle f, -\mathscr{L} h_{AB}\rangle_\pi$ is constant for $f \in \mathscr{H}_{AB}$.

The electrostatic interpretation of h_{AB} is that it is the electric potential of a capacitor made of two conductors in A and B at potentials 1 and 0, respectively (Figure 3). Finally, the *capacity*

$$\operatorname{cap}(A, B) = \int_{\partial A} \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x)$$

is the normalising constant ensuring that

$$\nu_{AB}(\mathrm{d}x) = \frac{1}{\operatorname{cap}(A, B)}\, \mathrm{e}^{-V(x)/\varepsilon}\, e_{AB}(\mathrm{d}x)$$

is a probability measure on ∂A. In electrostatics, $\operatorname{cap}(A, B)$ is the total charge in the capacitor (which is equal to its capacity for a unit potential difference).

Combining the expressions (9) of w_B and (11) of h_{AB} with the detailed balance relation (10) of the Green function, we obtain

$$\int_{\partial A} \mathbb{E}^x[\tau_B]\nu_{AB}(\mathrm{d}x) = \frac{1}{\operatorname{cap}(A, B)} \int_{B^c} \mathrm{e}^{-V(x)/\varepsilon}\, h_{AB}(x)\,\mathrm{d}x\,, \quad (12)$$

which is a fundamental relation for the potential theoretic approach. Indeed, taking for A a small ball centred at $x^\star$, one can prove (either using Harnack inequalities, or by a coupling argument) that $\mathbb{E}^x[\tau_B]$ changes very little in ∂A. The left-hand side of (12) is, thus, close to the expected value $\mathbb{E}^{x^\star}[\tau_B]$. As for the right-hand side, we start by observing that if B is a small ball centred in $y^\star$, then h_{AB} is close to 1 in the basin of attraction of $y^\star$. Thus, Laplace's method allows us to show that

$$\int_{B^c} \mathrm{e}^{-V(x)/\varepsilon}\, h_{AB}(x)\,\mathrm{d}x \simeq \sqrt{\frac{(2\pi\varepsilon)^n}{\det \operatorname{Hess} V(x^\star)}}\, \mathrm{e}^{-V(x^\star)/\varepsilon}\,. \quad (13)$$

It remains to estimate the capacity, which can be done with the help of variational principles. The *Dirichlet form* is the quadratic form associated to the generator, and can be written using an integration by parts (Green's identity) as

$$\mathscr{E}(f, f) = \langle f, -\mathscr{L} f\rangle_\pi = \varepsilon \int_{\mathbb{R}^n} \mathrm{e}^{-V(x)/\varepsilon}\, \|\nabla f(x)\|^2\,\mathrm{d}x\,,$$

where $\langle f, g\rangle_\pi$ is the inner product with weight $\pi(x)$. The *Dirichlet principle* states that the capacity $\operatorname{cap}(A, B)$ is equal to the infimum of the Dirichlet form over all functions with value 1 in A, and 0 in B, and this infimum is attained at $f = h_{AB}$. This is a direct consequence of the fact that $\langle f, -\mathscr{L} h_{AB}\rangle_\pi = \operatorname{cap}(A, B)$ for all f satisfying these same boundary conditions, and the Cauchy–Schwarz inequality (see Figure 4). In electrostatics, the Dirichlet form is interpreted as the electrostatic energy of the capacitor, which is indeed minimal at the equilibrium state.

A lower bound for the capacity can be obtained with the help of *Thomson's principle*. Given a vector field $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, we define the quadratic form

$$\mathscr{D}(\varphi, \varphi) = \frac{1}{\varepsilon} \int_{(A\cup B)^c} \mathrm{e}^{V(x)/\varepsilon}\, \|\varphi(x)\|^2\,\mathrm{d}x\,.$$

Thomson's principle states that the inverse of the capacity is the infimum of $\mathscr{D}$ over all divergence-free vector fields, whose flux through ∂A is equal to 1.

Choosing appropriate test functions in both variational principles (which can be guessed from the explicitly solvable one-dimensional case), one obtains

$$\operatorname{cap}(A, B) \simeq \frac{|\lambda_-(z^\star)|}{2\pi} \sqrt{\frac{(2\pi\varepsilon)^n}{|\det \operatorname{Hess} V(z^\star)|}}\, \mathrm{e}^{-V(z^\star)/\varepsilon}\,.$$

Combining (13) and this last expression yields the Eyring–Kramers formula (7).

3 Metastability in the Allen–Cahn equation

We would now like to quantify, in a way similar to what was just done in the context of SDEs, the phenomenon of metastability in stochastic partial differential equations (SPDEs). We shall consider the Allen–Cahn equation

$$\partial_t \phi = \Delta\phi + \phi - \phi^3 + \sqrt{2\varepsilon}\xi\,, \quad (14)$$

which is a simple model for phase separation, for instance in a mixture of ice and liquid water, or in an alloy. It is also one of the simplest SPDEs displaying metastable behaviour.

The unknown $\phi(t, x)$ is a scalar field, where the space variable x lies in the torus $\mathbb{T}_L^d = (\mathbb{R}/L\mathbb{Z})^d$ of size L (we could work on the unit torus provided we introduce a viscosity parameter multiplying the Laplacian). The term ξ denotes a so-called *space-time white noise*. Intuitively, ξ represents a Brownian noise acting independently in each point of space, meaning that

$$\mathbb{E}[\xi(t, x)\xi(s, y)] = \delta(t - s)\delta(x - y)\,. \quad (15)$$

Mathematically, ξ is a centred Gaussian random distribution with covariance

$$\mathbb{E}[\langle\xi, \varphi_1\rangle\langle\xi, \varphi_2\rangle] = \langle\varphi_1, \varphi_2\rangle_{L^2}$$

for all pairs of test functions $\varphi_1, \varphi_2 \in L^2$. Indeed, formally replacing the test functions by Dirac distributions we recover relation (15). Furthermore, if $\varphi_T(t, x) = 1$ when $t \in [0, T]$ and x belongs to a set $A \subset \mathbb{T}_L^d$, and is otherwise equal to 0, then $W_T = \langle\xi, \varphi_T\rangle$ is a Brownian motion.

We can consider (14) as an infinite-dimensional analogue of the gradient diffusion (1) for the potential

$$V(\phi) = \int_{\mathbb{T}_L^d} \Bigl(\frac{1}{2}\|\nabla\phi(x)\|^2 - \frac{1}{2}\phi(x)^2 + \frac{1}{4}\phi(x)^4\Bigr)\mathrm{d}x\,. \quad (16)$$

Indeed, for all periodic functions ψ, the Gâteaux derivative of V in the direction ψ is

$$\lim_{h\to 0} \frac{V(\phi + h\psi) - V(\phi)}{h} = \int_{\mathbb{T}_L^d} (\nabla\phi(x)\cdot\nabla\psi(x) - \phi(x)\psi(x) + \phi(x)^3\psi(x))\,\mathrm{d}x\,,$$

and an integration by parts of the term $\nabla\phi \cdot \nabla\psi$ shows that this derivative is equal to $-\langle \Delta\phi + \phi - \phi^3, \psi\rangle_{L^2}$ which is, up to a change of sign, exactly the inner product of the right-hand side of (14) with ψ.

In the deterministic case $\varepsilon = 0$, the stationary solutions of (14) are the critical points of V, among which only two are local minima, and thus play the same role as $x^\star$ and $y^\star$ in the case of diffusions: these are the solutions identically equal to ± 1 that we shall denote by $\phi^\star_\pm$. If ϕ represents a mixture of ice and water, then $\phi^\star_-$ and $\phi^\star_+$ represent pure ice and pure water, respectively. Depending on the size L of the spatial domain, there can be additional critical points. In what follows, in order to simplify the presentation, we shall concentrate our attention on the case $L < 2\pi$. In this case there is only one other critical point: the identically zero function that we shall denote by $\phi^\star_{\text{trans}}$,since it represents the transition state when going from $\phi^\star_-$ to $\phi^\star_+$. It plays the same role as $z^\star$ in the case of diffusions.

Figure 5 illustrates the time evolution of a solution of (14) in dimension 2.[7] It presents a phenomenon of gradual phase separation (in solid state physics, for example in phase separation in an alloy, one speaks of *spinodal decomposition*), and corresponds to a rather slow convergence towards one of the equilibria $\phi^\star_\pm$. A difference with the one-dimensional case represented in Figure 2 is that here, one observes the coexistence of the two phases for a long time. Only after an extended time period (exceeding what is shown in Figure 5) does the system approach a single pure phase, be it blue or red. This is due to the fact that the initial condition, which is random and has zero mean, causes the field to first approach the saddle point $\phi^\star_{\text{trans}}$ (that also has zero mean) before being attracted by $\phi^\star_-$ or $\phi^\star_+$. Moreover, since we are in an infinite-dimensional situation, the system has plenty of "space" to evolve in before converging towards an equilibrium.

If, unlike what is shown in Figure 5, we were to start the simulation in one of the pure phases, say in $\phi^\star_-$, we would see the system stay close to that state for a very long time, before making a transition to the other state $\phi^\star_+$. Then, after another very long time period has ellapsed, we would see the system return to the initial state, and so on. The mean value of the field would then behave as illustrated in Figure 2. Therefore, we are indeed dealing with a metastability phenomenon. A natural question that arises when $\varepsilon > 0$ is the following: if we start with an initial condition close to $\phi^\star_-$, what is the precise asymptotics of the time needed to reach a small neighbourhood (in an appropriate norm) of the solution $\phi^\star_+$?

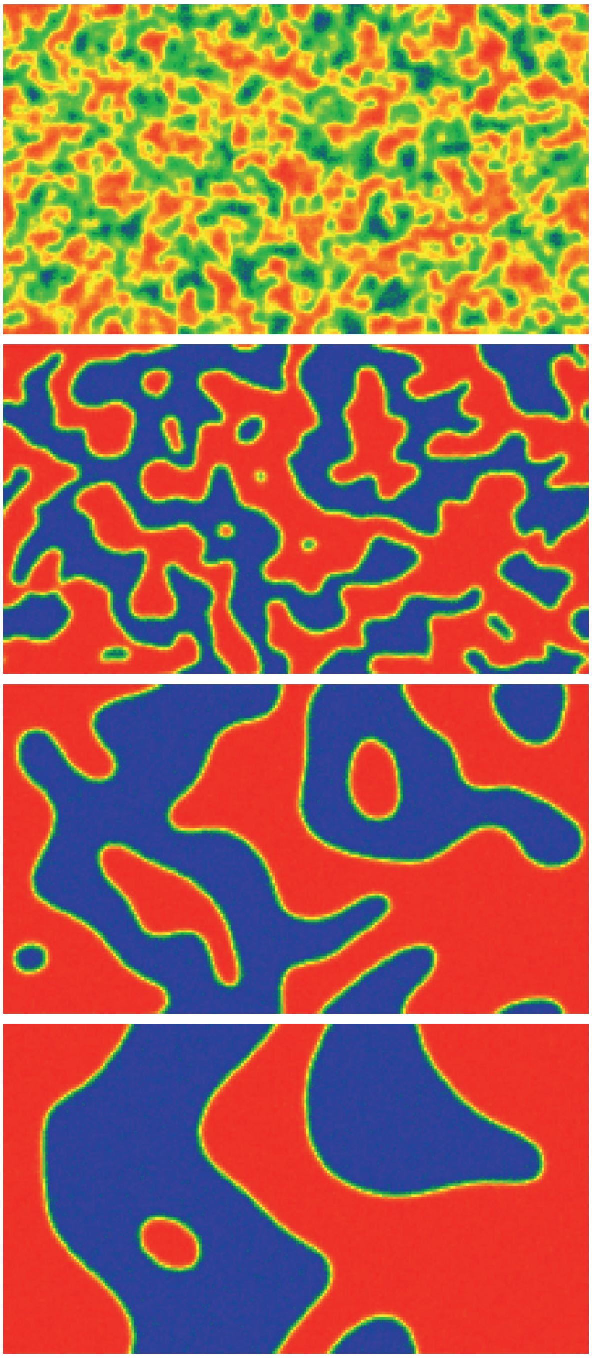

Figure 5. Time evolution of a solution of the stochastic Allen–Cahn equation on a two-dimensional torus, illustrating the phenomenon of spinodal decomposition, or slow phase separation. Red and blue colours represent regions where the field ϕ is close to 1 and to -1, respectively, whereas yellow corresponds to ϕ close to 0. The main effect of the noise in these simulations is to make the regions of red and blue phases slightly granular. The interfaces between the two phases remain relatively smooth due to the regularising effect of the Laplacian.

Dimension 1: Fredholm determinants

In the case of dimension $d = 1$, William Faris and Giovanni Jona-Lasinio proved in [10] a large-deviation principle with rate function (compare with the expression (6) for the rate function of a diffusion)

$$\mathscr{I}_{[0,T]}(\gamma) = \frac{1}{2}\int_0^T \int_{\mathbb{T}_L} \Big[\frac{\partial\gamma}{\partial t}(t,x) - \frac{\partial^2\gamma}{\partial x^2}(t,x) - \gamma(t,x) + \gamma(t,x)^3\Big]^2 \,\mathrm{d}x\,\mathrm{d}t\,. \qquad (17)$$

Let τ be the first-hitting time of a ball $B = \{\phi\colon \|\phi - \phi^\star_+\|_{L^\infty} < \delta\}$, with small radius $\delta > 0$ independent of ε. By a method similar to the one discussed in section 2, one obtains that τ satisfies

[7] Animations can be found on the web pages www.idpoisson.fr/berglund/simchain.html for dimension 1, and www.idpoisson.fr/berglund/simac.html for dimension 2. See also the YouTube page tinyurl.com/q43b6lf. Furthermore, one can find interactive simulations at the addresses experiences.math.cnrs.fr/Equation-aux-Derivees-Partielles.html and experiences.math.cnrs.fr/EQuation-aux-Derivees-Partielles-69.html.

Arrhenius' law

$$\mathbb{E}^{\phi_-^\star}[\tau] \simeq \mathrm{e}^{[V(\phi_{\text{trans}}^\star)-V(\phi_-^\star)]/\varepsilon} .$$

What about the Eyring–Kramers law? If we want to extend Expression (7), valid in finite dimension, to the present situation, we first need to determine what the analogues of the Hessian matrices of V at the critical points are. An integration by parts shows that the expansion up to order 2 of the potential around $\phi_{\text{trans}}^\star = 0$ is

$$V(\phi) = \frac{1}{2}\langle \phi, [-\Delta - 1]\phi\rangle_{L^2} + O(\phi^4) ,$$

and thus we can identify Hess $V(\phi_{\text{trans}}^\star)$ with the quadratic form $-\Delta - 1$. A similar argument applied to $\phi_-^\star$ shows that Hess $V(\phi_-^\star)$ can be identified with $-\Delta + 2$.[8] Taken separately, these two operators do not have a well-defined determinant. However, we can write their ratio as

$$\det((-\Delta + 2)(-\Delta - 1)^{-1}) = \det(\mathbb{1} + 3(-\Delta - 1)^{-1}) . \qquad (18)$$

This is a *Fredholm determinant*, an object generalising the characteristic polynomial of a matrix to infinite-dimensional operators[9]. To see that this determinant converges, let us observe that the eigenvalues λ_k of $3(-\Delta-1)^{-1}$ decrease like $1/k^2$ for large k. Hence, the logarithm of the determinant behaves like the sum of $\log(1 + \lambda_k)$, that is, like the sum of the λ_k, i.e., the trace of $3(-\Delta-1)^{-1}$. By Riemann's criterion this sum converges, and one says that $3(-\Delta - 1)^{-1}$ is *trace class*. In fact, using two of Euler's identities about infinite products, one can obtain the explicit value

$$\det(\mathbb{1} + 3(-\Delta - 1)^{-1}) = -\frac{\sinh^2(L/\sqrt{2})}{\sin^2(L/2)} .$$

The following theorem is a particular case of a result proved in [5] (and also of a result in [2] obtained by a different approach.)

Theorem 3.1. *For $L < 2\pi$, one has*

$$\mathbb{E}^{\phi_-^\star}[\tau] = \frac{2\pi}{|\lambda_-(\phi_{\text{trans}}^\star)|}\frac{\mathrm{e}^{[V(\phi_{\text{trans}}^\star)-V(\phi_-^*)]/\varepsilon}}{\sqrt{\left|\det(\mathbb{1} + 3(-\Delta - 1)^{-1})\right|}}[1 + R(\varepsilon,\delta)] , \qquad (19)$$

where $\lambda_-(\phi_{\text{trans}}^\star) = -1$ is the smallest eigenvalue of $-\Delta - 1$, and $R(\varepsilon, \delta)$ converges to 0 as $\varepsilon \to 0$. (The speed of this convergence depends on L, it becomes slower as L gets closer to 2π.)

Let us give an idea of the proof of this theorem. The first step consists of a *spectral Galerkin approximation*. Let $\{e_k\}_{k\in\mathbb{Z}}$ be a Fourier basis of $L^2(\mathbb{T}_L)$, and for a positive integer N (called the *ultraviolet cutoff parameter*), let P_N be the projection on the space $\mathscr{H}_N$ generated by $\{e_k\}_{|k|\leqslant N}$. The projected equation

$$\partial_t\phi_N = \Delta\phi_N + \phi_N - P_N(\phi_N^3) + \sqrt{2\varepsilon}P_N\xi$$

is equivalent to a finite-dimensional SDE of type (1), with V the potential (16) restricted to $\mathscr{H}_N$. We can then apply the potential theoretic approach discussed in section 2 above, carefully controlling the dependence of the error terms upon the cutoff parameter N and then letting $N \to \infty$.

A major difficulty is thus to get an estimate similar to (19) for the Galerkin approximation, with an error term $R(\varepsilon, \delta)$ independent of N. A key idea of the proof consists in decomposing the potential V into a quadratic part and a higher-order part. This allows for the interpretation of the capacity and of the integral on the right-hand side of Relation (12) as expectations, under a Gaussian measure, of certain random variables that can then be estimated with the help of probabilistic arguments. Details of these computations can be found in [3, Section 2.7].

Dimension 2: Carleman–Fredholm determinants

We will now consider the Allen–Cahn equation (14) on the torus of dimension $d = 2$. It turns out that, unlike in the case $d = 1$, the equation is no longer well posed! This is due to the fact that space-time white noise is more singular in dimension 2 than in dimension 1. In [7], Giuseppe Da Prato and Arnaud Debussche solved this problem by a *renormalisation* procedure inspired by Quantum Field Theory. Instead of (14) they considered, for $\delta > 0$, the regularised equation

$$\partial_t\phi = \Delta\phi + \phi + 3\varepsilon C_\delta\phi - \phi^3 + \sqrt{2\varepsilon}\xi^\delta . \qquad (20)$$

Here ξ^δ is a regularisation of space-time white noise defined as the convolution $\varrho^\delta * \xi$, where

$$\varrho^\delta(t, x) = \frac{1}{\delta^4}\varrho\Big(\frac{t}{\delta^2}, \frac{x}{\delta}\Big) ,$$

for a test function ϱ with integral 1. Consequently, ϱ^δ converges to the Dirac distribution when δ converges to 0. Furthermore, C_δ is a *renormalisation constant* that diverges like $\log(\delta^{-1})$ as δ tends to 0. Since ξ^δ is a function, and not a distribution, the so-called *renormalised* equation (20) admits solutions for all values of $\delta > 0$. Da Prato and Debussche then showed that these solutions converge to a well-defined limit when δ goes to 0.

At first sight, one might think that the stable equilibrium states of equation (20) are located at $\pm\sqrt{1 + 3\varepsilon C_\delta}$, and thus go to infinity as δ tends to 0 with ε fixed. In fact, this is not the case – a first indication of this was the proof by Martin Hairer and Hendrik Weber in [13] of a large-deviation principle with rate function analogous to that of the one dimensional case (see (17)). The key observation is that, as in dimension 1, this rate function does not include any renormalisation counterterm. This implies the Arrhenius law

$$\mathbb{E}^{\phi_-^\star}[\tau] \simeq \mathrm{e}^{[V(\phi_{\text{trans}}^\star)-V(\phi_-^\star)]/\varepsilon} ,$$

where V is the potential (16) without renormalisation term. As before, τ is the transition time between the equilibria $\phi_-^\star$ and $\phi_+^\star$, located at ±1, respectively. We can interpret this result as indicating that the only role of the *counterterm* $3\varepsilon C_\delta\phi$ is to make the nonlinearity ϕ^3 well defined.

What about the Eyring–Kramers law? It turns out that the Fredholm determinant (18) does not converge. In fact, $3(-\Delta - 1)^{-1}$ is no longer of trace class in dimension 2, since its eigenvalues are proportional to $1/(k_1^2 + k_2^2)$ with k_1 and k_2

[8] The values -1 et 2 are the second derivatives of the function $\phi \mapsto \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2$ at 0 and -1, respectively.

[9] The nonzero roots of the characteristic polynomial $c_M(t) = \det(t\mathbb{1} - M)$ of a matrix M are the inverses of the roots of $\bar{c}_M(s) = \det(\mathbb{1} - sM)$. The Fredholm determinant of $-sM$ is the analogue of $\bar{c}_M(s)$ when M is an infinite-dimensional linear operator.

two nonzero integers, and hence the sum of these eigenvalues diverges like the harmonic series!

The solution to this problem consists, first of all, in working, as in dimension 1, with a spectral Galerkin approximation with ultraviolet cutoff N. Instead of regularising the space-time white noise by convolution, one can again consider its spectral Galerkin projection $\xi_N = P_N\xi$, with a counterterm

$$3\varepsilon C_N = \frac{3\varepsilon}{L^2}\operatorname{Tr}(P_N(-\Delta-1)^{-1})$$

that diverges as $\log(N)$ (the constant C_N is the variance of the truncated *Gaussian free field*[10]). The renormalised potential can thus be written as

$$V_N(\phi) = \int_{\mathbb{T}_L^2}\Big[\frac{1}{2}\,\|\nabla\phi(x)\|^2 + \frac{1}{4}\phi(x)^4 - \frac{1}{2}(1+3\varepsilon C_N)\phi(x)^2\Big]\,\mathrm{d}x\,.$$

The crucial point is to observe that

$$V_N(\phi^\star_{\mathrm{trans}}) - V_N(\phi^\star_-) = \frac{L^2}{4} + \frac{3}{2}L^2\varepsilon C_N\,.$$

The new term $\frac{3}{2}L^2\varepsilon C_N$ is exactly the one that will make the prefactor converge. Indeed, the Eyring–Kramers formula involves the factor

$$\det(\mathbb{1} + 3P_N(-\Delta-1)^{-1})\,\mathrm{e}^{-3\operatorname{Tr}(P_N(-\Delta-1)^{-1})}\,,$$

which does have a limit as $N \to \infty$ (this follows from the fact that its logarithm behaves like the sum of $1/(k_1^2+k_2^2)^2$). This is, in fact, a known regularisation of the Fredholm determinant, also called the *Carleman–Fredholm* determinant, sometimes denoted $\det_2(\mathbb{1}+3(-\Delta-1)^{-1})$. Unlike Fredholm's determinant, this modified determinant is well defined for operators whose square is trace class, the so-called *Hilbert–Schmidt operators*, which include $3(-\Delta-1)^{-1}$.

The following theorem combines results of [4] and [15].

Theorem 3.2. *Let τ be the first-hitting time of a ball (in the Sobolev norm H^s for some $s < 0$), centred at $\phi^\star_+$. For $L < 2\pi$, we have*

$$\mathbb{E}^{\phi^\star_-}[\tau] = \frac{2\pi}{|\lambda_-(\phi^\star_{\mathrm{trans}})|}\,\frac{\mathrm{e}^{[V(\phi^\star_{\mathrm{trans}})-V(\phi^*_-)]/\varepsilon}}{\sqrt{\big|\det_2(\mathbb{1}+3(-\Delta-1)^{-1})\big|}}\,[1+R(\varepsilon,\delta)]\,, \tag{21}$$

where $\lambda_-(\phi^\star_{\mathrm{trans}}) = -1$ is the smallest eigenvalue of $-\Delta-1$, and $R(\varepsilon,\delta)$ is an error term converging to 0 as $\varepsilon \to 0$ (at a convergence rate depending on L.)

This result confirms that the renormalisation procedure does not displace the stationary states, since the theorem applies to the states $\phi^\star_\pm$ located at ± 1. However, the renormalisation procedure is necessary to get a finite prefactor for the transition time, since the ratio of the spectral determinants and the counterterm $\frac{3}{2}L^2\varepsilon C_N$ in the potential compensate each other exactly.

4 Some open problems

A natural question to ask is whether an Eyring–Kramers law exists for the Allen–Cahn equation in dimension $d = 3$ (in dimension $d = 4$ one does not expect the existence of non-trivial solutions to this equation). As shown by Martin Hairer in the widely noted paper [12][11], which earned him the Fields Medal in 2014, the form of the renormalised equation is

$$\partial_t\phi = \Delta\phi + \phi + \Big[3\varepsilon C_\delta^{(1)} - 9\varepsilon^2 C_\delta^{(2)}\Big]\phi - \phi^3 + \sqrt{2\varepsilon}\,\xi^\delta\,,$$

where $C_\delta^{(1)}$ and $C_\delta^{(2)}$ diverge like δ^{-1} and $\log(\delta^{-1})$, respectively. The first counterterm comes from the same renormalisation procedure as in dimension 2 (called *Wick renormalisation*), and does not introduce any new difficulties. On the other hand, the second counterterm is specific to dimension 3 and is at the origin of numerous problems. In particular, contrary to what happens in the case $d = 2$, the invariant measure of the Allen–Cahn equation is singular with respect to the Gaussian free field.

However, we can note that $(-\Delta-1)^{-1}$ is Hilbert–Schmidt in dimension 3 as well. As the second counterterm occurs with a factor ε^2, we expect that an Eyring–Kramers formula analogous to (21) is still valid. With Ajay Chandra, Giacomo Di Gesù and Hendrik Weber we managed to establish some of the estimates needed to prove that result. However, so far the lower bound on the capacity still resists our efforts.

Of course, it would be desirable to obtain Eyring–Kramers formulas not just for the Allen–Cahn equation but also for other SPDEs. An example is the Cahn–Hilliard equation describing phase separation in cases where the total volume of each phase is conserved, like in mixtures of water and oil. However, as in most mathematical models of metastable systems, these SPDEs remain based on a lattice dynamics: each lattice point is characterised by its state, but remains fixed in the same place. This is a good model for certain alloys or for ferromagnetic materials, which have a crystalline structure with different types of atoms or spins attached to each site. However, for a mixture of ice and liquid water there is no underlying lattice. One of the great challenges in the theory of metastability is to analyse models, taking into account the fact that ice crystals can move through liquid water to form larger crystals by agglomeration.

A Appendix: Brownian motion

Brownian motion is a mathematical model for the erratic movement of a particle immersed in a fluid, under the effect of collisions with the fluid's molecules. It was first observed by the naturalist Robert Brown in 1827, while studying pollen grains under a microscope.

The first mathematical descriptions of Brownian motion were proposed by the French mathematician Louis Bachelier in 1901, for applications in finance, and by Albert Einstein in 1905. Variants of their approaches were developed by Marian Smoluchowski in 1906 and by Paul Langevin in 1908. Einstein's computations allowed Jean Perrin to experimentally estimate Avogadro's number in 1909, a feat that earned him the Nobel Prize in 1926.

[10] For more information about the Gaussian free field see the article of Rémi Rhodes in the July 2018 issue of *La Gazette des Mathématiciens*.

[11] For further details of the theory introduced by Martin Hairer, called *theory of regularity structures*, the reader may consult the paper by François Delarue in the January 2015 issue of *La Gazette des Mathématiciens* and the paper of Bruned, Hairer and Zambotti published in the March 2020 issue of this Newsletter.

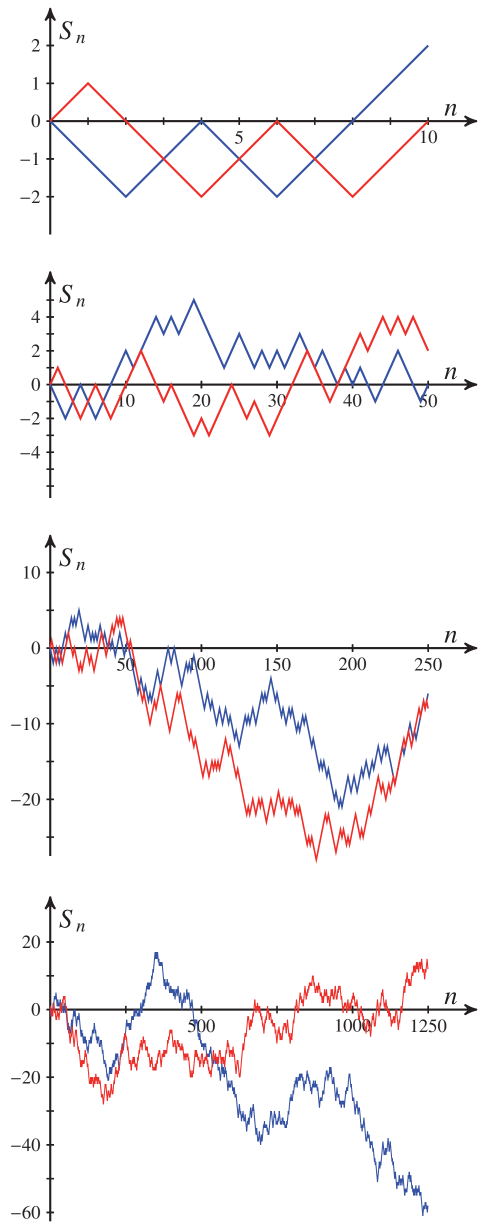

Figure 6. Two realisations (one in red and one in blue) of a symmetric random walk in $\mathbb{Z}$, seen at different scales. From one plot to the next, the horizontal scale is compressed by a factor of 5, while the vertical scale is compressed by a factor of $\sqrt{5}$.

Let us consider the case of dimension 1 and assume that the particle undergoes regular collisions, at time intervals Δt. Between two successive collisions, the particle travels a distance Δx with equal probability $\frac{1}{2}$ to the left or to the right. Its position at time $n\Delta t$ is thus given by $S_n \Delta x$, where S_n is a sequence of integers such that the increments $S_{n+1} - S_n$ have value 1 or -1, each with probability $\frac{1}{2}$. Furthermore, we assume that each increment is independent from all previous ones. The sequence S_n is called a symmetric random walk in $\mathbb{Z}$ (see Figure 6).

Since in practice the space and time intervals Δx and Δt are very small, it seems relevant to let them converge to zero in order to obtain a universal object. It turns out that this limit is only interesting if Δt is proportional Δx^2 (this is a consequence of the Central Limit Theorem). This amounts to setting

$$W_t := \lim_{n\to\infty} \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} .$$

This definition turns out to be equivalent to requiring that for all $t > s \geqslant 0$ the increment $W_t - W_s$ follows a centred normal law, with variance $t - s$, and independent of the values of the process up to time s.

Norbert Wiener showed in 1923 that the trajectories $t \mapsto W_t$ are continuous (W_t is also known today as the *Wiener process*). Other properties of W_t were established by several mathematicians, including Raymond Paley, Antoni Zygmund and Paul Lévy. In particular, we know that trajectories of Brownian motion are nowhere differentiable. This poses a problem for the definition of the SDE (1), which is solved by defining its solutions to be those of the integral equation

$$x_t = x_0 - \int_0^t \nabla V(x_s)\,\mathrm{d}s + \sqrt{2\varepsilon} W_t ,$$

that can be studied using a fixed-point argument. The theory was generalised by Kiyoshi Itô in the 1940s. His stochastic calculus allows one to solve variants of (1) in which the noise term is multiplied by a function of x. Some ideas at the basis of stochastic calculus were discovered independently by Wolfgang Döblin and sent to the French Academy of Sciences in a sealed envelope that was only opened in 2000.

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Acknowledgments. This is an English translation by Fernando Pestana da Costa of the French article *Métastabilité d'EDP stochastiques et déterminants de Fredholm* by Nils Berglund, published in *La Gazette des Mathématiciens* (N. 163, January 2020). The EMS Newsletter is grateful to the author, to *La Gazette des Mathématiciens*, and to the *Société Mathématique de France* for authorisation to republish this article. The author would like to thank Damien Gayet, editor in chief of *La Gazette des Mathématiciens*, for his critical remarks on early French versions of this text, which greatly improved its readability.

Alessio Figalli: Magic, Method, Mission

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This paper is based on [49], which chronicled for the Catalan mathematical community the Doctorate Honoris Causa conferred to Alessio Figalli by the UPC on 22nd November 2019, and also on [48], which focussed on the aspects of Figalli's scientific biography that seemed more appropriate for a society of applied mathematicians. Part of the Catalan notes were adapted to Spanish in [10]. It is a pleasure to acknowledge with gratitude the courtesy of the Societat Catalana de Matemàtiques (SCM), the Sociedad Española de Matemática Aplicada (SEMA) and the Real Sociedad Matemática Española (RSME) their permission to freely draw from those pieces for assembling this paper.

Origins, childhood, youth, plenitude

Alessio Figalli was born in Rome on 2 April 1984. His father Gennaro Figalli, now retired, was a professor of engineering, and his mother, Giuseppina Carola, is a teacher of Latin and Greek at a classical high school (*liceo classico*) in Rome. From Alessio's early years, in [45] Kevin Hartnett selects the following impressions:

> *As a kid Figalli liked to play soccer, watch cartoons, and hang out with his friends – and, he recalls, he always made the rational decision to get his homework done first, so that he could fully enjoy himself. "For me it was always a balance between how good a grade I could get and how much time I had to spend to get such a grade," he said. "I was always an optimizer, I wanted the best for the least effort."*

In the interview [45], conducted by Helga Rietz, Figalli talks of his first experiences in relation to mathematics:

> *As a child I liked math because it was easy for me. I was thinking of being an engineer. Then I had to decide, at age 13, which high school I would like to attend. There are many types in Italy, but the main ones that prepare for the university are scientific and classical lyceums. In the latter you learn Greek and Latin, philosophy, etc., and I chose this option. In Italy there is always the idea that the classical high school offers the broadest education and that with this training you can later study whatever you want…*

In his third year at high school, Antonio Corbo, a mathematician who worked at the same university as his father, suggested that Alessio participate in the Mathematical Olympiad. This led him to realise that there were mathematical problems whose solutions required inventiveness, and his aptitude for solving them, as well as the joy such magic insights brought him, were a truly revelatory experience. In the aforementioned interview [45], Figalli describes this experience:

> *At the Mathematical Olympiad, I met other teens who loved math. All of them dreamed of studying at the Scuola Normale Superiore in Pisa (SNSP), which offers a high level of education. Those who get one of the coveted scholarships do not have to pay anything. Living, eating and studying are free. I also wanted that.*
>
> *I concentrated on mathematics and physics on my own and managed to pass the entrance exam. The first year at Scuola Normale (SN) was tough. I didn't even know how to calculate a derivative, while my colleagues were much more advanced than me, since they came from the scientific lyceum.*

He undertook a methodical study plan that allowed him to catch up with his peers within a year. How far he went is revealed by the following episode. At the beginning of his second year, he began to read a highly technical work that Luigi Ambrosio and Xavier Cabré had recently written [4]. Ambrosio expected that the novice student would have to persist for quite some time to make some progress. But the surprise for Ambrosio was, as stated in [42], that "Alessio came to see me less than a week later and I realised that he understood everything". This step marked Figalli's course in mathematical research: one year later he completed the bachelor's degree [22] and in the following three years he obtained the master's degree [23] and the doctorate [25], always with full honours. His doctoral thesis was supervised by Luigi Ambrosio and Cédric Villani (Fields Medal 2010). In the words of David Jerison (MIT), "Alessio is incredibly fast. Quick on the essentials and quick to isolate the important points".

Figalli's scientific output after his doctorate is surprising in all respects. Its leading role in the global mathematical landscape has steadily increased over the years. From the point of view of his academic positions, three phases can be distinguished. The first comprises of two terms: 2007–08, as a researcher at the University of Nice, and 2008–09, as *Professeur Hadamard* at the *École Polytéchnique de Palaiseau*. The second phase consists of the seven terms at the *University of Texas at Austin* (from 2009–10 to 2015–16): the first two as an associate professor (and *Harrington Faculty Fellow* in the first) and as professor in the following five (*R. L. Moore Chair* in the last three). Finally, since October 2016 he has been a professor at the ETH (Eidgenössische Technische Hochschule Zürich) and director of the Forschungsinstitut für Mathematik (FIM) since September 2019.

Rio de Janeiro, 1 August 2018

It is the first time that the International Congress of Mathematicians (ICM) has been hosted in the southern hemisphere of the American continent. It was a milestone for Brazilian mathematics, in general, and for the Instituto Nacional de Matemática Pura e Aplicada (IMPA), in particular. In fact, its director, Marcelo Viana, was the president of the Congress.

It is also appropriate to remember that Jacob Palis had been president (1999–2002) of the IMU (International Mathematical Union) and that Artur Avila was awarded the Fields Medal at the Seoul ICM (2014) for his "profound contributions to the theory of dynamic systems". The highlight of the day, the most looked forward to, was the announcement of the Fields Medal recipients:

Caucher Birkar (Cambridge, UK), Alessio Figalli (ETH, Zürich), Peter Scholze (Universität Bonn) and Akshay Venkatesh (IAS, Princeton).

In the case of Alessio Figalli, the mention was for "contributions to the theory of optimal transport, and its applications to partial differential equations, to metric geometry and to probability theory". Luis Caffarelli, in charge of the *laudatio*, stated that "his work is of the highest quality in terms of originality, innovation and impact, both in mathematics itself and in its applications", and that he was destined "to be one of the most influential mathematicians of his generation".

Figalli's plenary lecture as a medalist was titled "Regularity of interfaces in phase transitions via obstacle problems". Scheduled for Saturday, 4 August, it was a memorable occasion for its high quality, and also for the fact that at the end he mentioned recent results obtained in collaboration with Xavier Ros-Oton and Joaquim Serra. Both completed their studies in mathematics (bachelor and master) at UPC's Facultat de Matemàtiques i Estadística (FME), and earned a PhD from the UPC in June 2014 under the supervision of Xavier Cabré. They currently hold positions at the Universität Zürich (UZH) and the ETH, respectively. As stated on his website, Ros-Oton will hold positions at the "Institució Catalana de Recerca i Estudis Avançats" (ICREA), as a research professor, and at the Department of Mathematics of the Universitat de Barcelona, as a full professor, starting September 2020. It should also be mentioned that Serra has recently been awarded the EMS Prize.

Scientific opus

To begin with, it must be stated that an essential source for first-hand information about Alessio Figalli's life and work is his website at the ETH. Continuously updated, there one can find his curricula, in short and full versions; listings of his many awards, distinctions, honours and invited lectures; links to all his publications, classified according to different criteria; his commitments as editor of mathematical journals; lists of his doctoral students and postdocs; the many courses and seminars he has delivered; conference videos and links to interviews; and a special page for his ERC project.[1] It is a huge and amazing work on all counts, particularly because at the moment of writing these notes (July 2020) the thirteenth anniversary of his doctoral thesis was still three months into the future.

In this section the main focus will be the section

https://people.math.ethz.ch/~afigalli/.

"Research" of that web page, which is presented by declaring that his work falls within the areas of calculus of variations and partial differential equations (PDE),

> *with special emphasis on optimal transportation, Monge-Ampère equations, functional and geometric inequalities, elliptic PDEs of local and non-local type, free boundary problems, Hamilton-Jacobi equations, transport equations with rough vector-fields and random matrix theory.*

The high quality of Figalli's research was masterfully outlined by Luis Caffarelli in his *laudatio* [18]:

> *A recurring theme in Figalli's research is the interaction between the theory of optimal transport and other areas of mathematics. Optimal transport theory, although pioneered by G. Monge back in 1781, has only transpired as being of paramount importance in many areas of mathematics in the past two decades. One of the most valuable features of this theory is its unifying power, in the sense that key ideas and constructions in optimal transport have turned out to be useful in the most unexpected contexts. Figalli is currently one of the most authoritative experts in optimal transport and its many applications.*
>
> *Figalli's work is of the highest quality in terms of originality, innovation and impact both on mathematics per se as well as on its applications. He is clearly a driving force in the global mathematics community today. His approach to research is lively, dynamic and effective, and without a doubt will lead him to achieve many more stunning discoveries in years to come.*

This last forecast is accurate if we are to judge by the quality and quantity of work produced in the last two years, and there is no doubt that it will continue at this or even a higher pace for years to come.

[1] For a synopsis of a good part of this information, see https://fme.upc.edu/ca/la-facultat/activitats/2019-2020/arxius/expo_alessiofigalli-dhc.pdf, pp. 5 and 6.

Early productions

We have already touched upon the beginnings of Figalli's mathematical career at the SNSP. Actually, the episode in Ambrosio's office, related to Figallis's reading of [4] (see also [1]), suggests a crystallisation of Figalli's mission with research in mathematical analysis at its core. The successive steps amply confirm this. At the end of the second year, he defends his bachelor thesis, which deals with topics related to the cited articles, and more specifically on the relations between the **Bernstein problem** and the **De Giorgi conjecture** [The terms or phrases in this font refer to the glossary at the end of this article; when they appear in quoted text, this emphasis is not on the original]. Two years later, at the end of his fourth term at SNSP, he defends his master's thesis. With surprising conceptual and technical maturity, he obtains, in particular, a comprehensive theory of **optimal transport** for non-compact Riemannian manifolds in this work. And a year later, at the age of twenty-three, he earns his PhD.

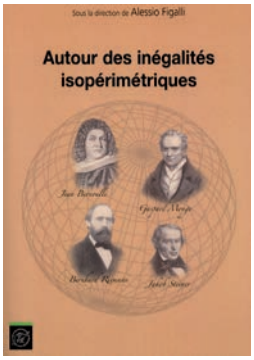
Let us take a moment to highlight some aspects of his doctoral thesis. As already stated, it was coadvised by Luigi Ambrosio and Cédric Villani. With more than 250 pages, each of its five chapters is based on one or more works already published or, at that time, pending publication. The titles of the chapters are quite vivid, but we find the description of their content in the thesis Introduction to be a bit more informative: 1) Optimal transport on manifolds with **geometric costs** (part of the results in collaboration with A. Fathi and C. Villani); 2) Optimal irrigation (with M. Bernot), a subject in which he quotes two pioneering works by Vicent Caselles [6, 7]; 3) Brenier's variational theory of incompressible fluids (with L. Ambrosio); 4) Aubry–Mather theory and solutions of the Hamilton–Jacobi equations (with A. Fathi and L. Rifford); and 5) DiPerna–Lions theory on stochastic differential equation solutions (based on [26]). The thesis ends with an appendix with two sections, one dedicated to describing semi-concave functions and their properties with great generality, and another dedicated to Tonelli's Lagrangian functions, a theory that links the calculus of variations to the optimal transport. Finally, let us note that the paragraph that precedes these descriptions summarises the research philosophy of its author in two brushstrokes:

> *In the development of the theory of optimal transportation, as well as in the development of other theories, it is important on the one hand to explore new variants of the original problem, on the other hand to figure out, in this emerging variety of problems, some common (and sometimes unexpected) features. This kind of analysis is the main scope of our thesis.*

By what has already been said, we notice Figalli's special predisposition towards sharing projects and research tasks with other people, generally leading the initiative. This observation is confirmed on verifying the large number of collaborators who participate in his publications, people who invariably have an outstanding profile. This is why it seems appropriate to document this aspect of his scientific record here. Basically, we will follow the order of the first research collaboration, as indicated by the year of the first article reviewed in Mathematical Reviews (MR), but grouped, for the convenience of the textual and graphic composition, in small successive groups (as in [40], poster 2).

Luigi Ambrosio (SNSP. Fermat Prize 2003. Balzan Prize 2019); **Cédric Villani** (Institut Henri Poincaré, Université de la Sorbone, Université de Lyon, Institut Camille-Jordan. EMS Prize 2008. Fermat Prize 2009. Fields Medal 2010. Doob Prize 2014); **Luis A. Caffarelli** (University of Texas at Austin. Steele Prize 2009. Wolf Prize 2012. Shaw Prize 2018); **Jean Bourgain** (1954–2018. Fields Medal 1994. Shaw Prize 2010. Crafoord Prize 2012. Steele Prize 2018); **Haïm Brezis** (Université Pierre et Marie Curie. Distinguished Visiting Professor at Rutgers University. Author of the treatise [12] (English translation of his Analyse fonctionelle, 1983, it has been a constant reference textbook in many universities). According to ISI, he is a highly cited researcher).

This first group is headed by his thesis advisors (an exception to the ordering rule) and includes Luis A. Caffarelli (profiled at "ArbolMat", AM[2]), Jean Bourgain and Haïm Brezis. It should be noted, to avoid confusion, that our scheme does not take into account the number of collaborations, nor, in this group, the time at which they occurred. Thus, Bourgain and Brezis appear for a specific collaboration [3] (2016), announced in [2] (2014), which establishes a significant isoperimetric inequality related to a previous result of both authors (together with P. Mironescu: *A new function space and applications*). Luis Caffarelli is co-author of five papers, the first one in 2012 and the last one in 2020. The collaborations with Cédric Villani, six in total, occurred in the period 2007–2012, and with Luigi Ambrosio, fourteen, in the period 2008–2017.



During his French period, he promotes the wonderful booklet *Autour des inégalités isopérimétriques* (Éditions de l'École Polytechnique, Palaiseau, 124 pages), which illustrates a productive model of collaboration, the interconnection of a diversity of subjects in the light of a common theme, and the nice coexistence of the research and didactic spirits. Figalli is the editor, and also the author of the preface. It is the outcome of the coordination, in the 2008–09 academic year, of a group of seven second-year students of the "École Politechnique" (W. Bench, C. De Franchis, L. Deproit, S. Gilles, B. Oh, A. Tenne, and K. Webster, who appear as authors) that "showed a particular interest in mathematics, and more specifically in the subject of isoperimetric inequalities,

[2] RSME portal of Mathematics, Science and Technology, colloquially 'ArbolMat', or AM.

which they wished to study both from a purely theoretical mathematical point of view and for their applications to other disciplines" (from the Preface). It contains seven chapters prepared jointly by the authors: 1) Relationship with the calculus of variations; 2) Generalization to any finite dimension; 3) Physico-chemical applications [shape of crystals, optimal structures; can you hear the shape of a drum?]; 4) Optimal transport; 5) Algebraic generalization [contains a section on applications to graph theory]; 6) Generalization to manifolds; and 7) Conclusion [in which "some famous inequalities that could have led us to the isoperimetric inequality are mentioned"]. On the back cover we find a good description of the book's scope:

> *One of the oldest optimization problems is connected with* ***isoperimetric inequalities****. They are linked to many theories, the frameworks of which clarify and enrich each other. This work synthesizes these different theories, highlighting their interrelationships, and presents different applications of each of them in various fields. From the most abstract areas to questions of everyday life, this book shows how an apparently so simple and so specific inequality actually extends to entire branches of mathematics and other sciences.*

In the second year of his stay in France, he lectures at the Bourbaki Seminar 2008–2009 on "Regularity of optimal transport maps (after Ma–Trudinger–Wang and Loeper)" [SB in what follows], which is published in *Astérisque* 332, 341–368, Société Mathématique de France (SMF). In addition, Villani's monumental work [47] on optimal transport appears and Figalli writes a detailed review [28].

These two works (and also [32]) are valuable indicators of the research context during his French stage and the degree of maturity and prestige that its author achieved in just seven years since entering the SNSP. In the acknowledgments, Villani writes that "Alessio Figalli took on the formidable challenge of reviewing the entire manuscript, from the first page to the last", with the connotation that he was the only one to accomplish such a feat. Naturally, this reading impacts on his review, whose purpose he sees as to "briefly examine the exciting and very active field of optimal transport, with an emphasis on the content and characteristics of the book being reviewed". In the review, Figalli also mentions [46], an interesting volume that in a way is a precursor to the book being reviewed, but which in fact has, by its approach, a complementary character.

The activity related to optimal transport is described authoritatively in the introductory paragraph of the review, as it appropriately corresponds to the leading status achieved by the reviewer since his doctoral thesis:

> *The problem of optimal transport has received the attention of many researchers in the past two decades, and its popularity continues to rise. This is mainly motivated by the discovery of unexpected connections between optimal transport and problems in physics, geometry, partial differential equations, etc.*

He explains the **optimal transport problem** (§2) and the relationship of the **optimal transport** with the **Monge–Ampère equation** (§3). This section ends with the following comment:

> *In chapter 12, the author offers a very good introduction to the theory of regularity of optimal transport. But since the book was completed in 2008, some of the more recent developments linking the Ma–Trudinger–Wang tensor with the manifolds's geometry are not covered,*

and refers to the SB article for a "recent presentation of these results". From the bibliography included in SB, it appears that among the results to which he refers, the following articles, of which he is co-author, should be counted: [33] (with Loeper), [35] (with Rifford), and [36–38] (with Rifford and Villani, announced as "in preparation").

The article SB is also an introduction to the historical evolution of the subject, as it highlights some of the milestones achieved by the most outstanding researchers: Monge (1781), Brenier (1987, 1991), Caffarelli (1990, 1991, 1992, 1996), Urbas (1997), McCann (2001, 2007, 2008), Nirenberg (2005), Ma–Trudinger–Wang (2005), Liu (2009), Loeper (2009 – article in *Acta Mathematica* [43] –, 2011), Trudinger–Wang (2009), Loeper–Villani (2010), etc.

From the beginning of the term 2009–2010, Figalli's destination is the University of Texas in Austin, but in the academic year 2011–2012 he is once again connected with France on the occasion of being awarded the Peccot-Vimont Prize and in charge of the corresponding "Cours Peccot" at the "Collège de France": *Stabilité dans les inégalités fonctionnelles, transport optimal et equations aux dérivées partielles*. With this distinction he is in the company of illustrious predecessors, such as Laurent Schwartz (FM 1950), Jean-Pierre Serre (FM 1954), Alexander Grothendieck (FM 1966), Pierre Deligne (FM 1978), Alain Connes (FM 1982), Pierre-Louis Lions (FM 1994), Cédric Villani (FM 2010), or Artur Avila (FM 2014).

To these credentials we must add the Prize of the European Mathematical Society, which was awarded at the VI European Congress of Mathematics (2–6 July 2012, Kraków, Poland). The corresponding plenary lecture, entitled *Stability in geometric and functional inequalities* [29], describes, after honouring its title in detail, applications to the asymptotic behaviour of some evolution equations.

Let us now consider some of the most relevant results obtained in the 2009–10 to 2015–16 academic years, apart from those already mentioned. In addition to the original articles, the main references for consideration that follow are [18] (the *Laudatio* of Alessio Figalli by Luis Caffarelli at ICM2018, designated as [L] in what follows), and the database MathSciNet, designated as [MR].

Although Figalli's specialties are the PDE and the calculus of variations, it can also be asserted, with what we have already written, that in his research the devel-

David Jerison (Massachusetts Institute of Technology. Bergman Prize 2012); **Juan L. Vázquez** (AM. Universidad Autónoma de Madrid. Premio Nacional de Investigación Julio Rey Pastor 2003. Plenary spekaer ICM-2006. Medal of the RSME 2018); **Albert Fathi** (Georgia Institute of Technology. Sophie Germain Prize 2013. Professeur émérite à l'ENS de Lyon); **Thierry Paul** (Directeur de Recherche, Centre de Mathématiques Laurent Schwartz, École Polytechnique, Paris, France); **Henrik Shahgholian** (Royal Institute of Technology (KTH), Stockholm, Sweden).

Francesco Maggi (University of Texas at Austin. Miranda Prize 2008); **José A. Carrillo de la Plata** (Imperial College London. Von Mises Prize 2006. Wolfson Prize 2012); **Robert J. McCann** (University of Toronto. Coxeter-James Prize 2005. Jeery-Williams Prize 2017); **Alice Guionnet** (École Normale Supérieure de Lyon. Loève Prize 2009. Pascal Medal 2018); **Kim Young-Heon** (University of British Columbia).

Ludovic Rifford (Université Nice Sophia Antipolis. Directeur du Centre International de Mathématiques Pures et Appliquées. Eisenbud Professor at MSRI 2013); **Enrico Valdinoci** (University of Western Australia. Most cited mathematician according to his graduation year (in all subjects and in Analysis). ERC Starting Grant 2011–2016); **Aldo Pratelli** (Università di Pisa. Medal of the President of the Italian Republic for young researchers 2004. Iapichino Prize 2005. ERC Starting Grant 2010–2015. Miranda Prize 2011. De Giorgi's Prize 2015); **Matteo Bonforte** (Universidad Autónoma de Madrid); **Álvaro Pelayo** (University of California at San Diego. Rubio de Francia Prize of the RSME 2009. NSF Faculty CAREER Award 2016).

Marco Di Francesco (Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila); **Dejan Slepčev** (Department of Mathematical Sciences, Carnegie Mellon University); **Giulio Ciraolo** (Departimento di Matematica Federigo Enriques, Università degli Studi di Milano); **Nicola Gigli** (Scuola Internazionale Superiore de Studi Avanzati. De Giorgi's Prize 2019); **Laura Valentina Spinolo** (IMATI-CNR, Pavia, Italy).

opment of the theory of **optimal transport** is central for the study of a multiplicity of spaces and cost functions, as well as for their application to a surprising variety of areas of mathematics. The power of this unifying capacity is especially manifest in cases where problems in one area can be fruitfully investigated with techniques developed in others. We have previously seen examples of this transfer, as in his doctoral thesis, in the articles [27] and [28], or in the book [21]. The following articles offer other paradigmatic cases (the names of the co-authors are included in parentheses after the year of publication):

2010 (F. Maggi and A. Pratelli) [34]

> *"[…] a sharp quantitative stability theorem for the **Wulff inequality** […] a most important mathematical result in our understanding of surface tension driven phase transitions, as it relates the microscopic structure of a given surface tension energy density to the macroscopic shape of the liquid/crystal observed at equilibrium. […] its sharp stability version allows one to describe […] every low energy state. This is a result of clear physical importance, whose proof required several original and innovative mathematical ideas.} [L]*

2012 "Stability in geometric and functional inequalities. *Proceedings of the 6th ECM* (Kraków, Poland, 2012), 585–599.
In thic congress, Alessio Figalli was awarded the EMS Prize,

> *For his outstanding contributions to the regularity theory of optimal transport maps, to quantitative geometric and functional inequalities and to partial solutions of the Mather and Mañé conjectures in the theory of dynamical systems.*

2013 (G. De Philippis) "$W^{2,1}$ regularity for solutions of the Monge–Ampère equation". *Inventiones mathematicae* 192.1, 55–69.

> *"…a fundamental, groundbreaking result on the second order Sobolev regularity of **optimal transport** maps and their link with the **Monge–Ampère equation**" [L].* This result allowed the authors to understand the existence and regularity of solutions to the **semigeostrophic equations**, a classical system of partial differential equations used in meteorology.

2016 (A. Guionnet) "Universality in several-matrix models via approximate transport maps". *Acta Mathematica* 217.1, 81–176.

> *"Large random matrices arise as a natural model in diverse fields such as quantum mechanics, quantum chaos, telecommunications, finance and statistics. The central mathematical question in this area is how much their asymptotic properties depend on the fine details of the model. […] Little was known about the universality of the fluctuations of the eigenvalues in several-matrix models, except in very particular situations. Figalli […] developed a new approach to these questions by introducing particular approximate transport maps. […] This is a real breakthrough, which in addition gives a firm mathematical grounding to the widely held belief coming from physics that a universality of local fluctuations holds, at least until some phase transition occurs." [L]*

2017 *The Monge–Ampère equation and its applications* [30]
By the year of its publication, this work would correspond to the Zürich stage, but the author tells us that

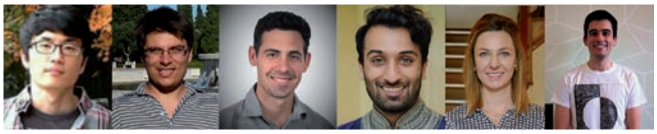
its origin is in "a series of lectures given by the author at ETH during the fall of 2014" with the same title. The purpose of the book is to provide a "a comprehensive introduction to the existence and regularity theory of the **Monge–Ampère equation**" and "to show some selected applications". His motivation, in contrast with the "classical" text [41], is "to include recent developments in the theory", fundamentally (Chapter 4) the study of the "interior regularity of weak solutions", "the geometry of solutions, mostly investigated by Caffarelli in the 1990s", and a proof of "interior $C^{1,\alpha}$, $W^{2,p}$ and $C^{2,\alpha}$ estimates". In addition, Chapter 5 presents "some extensions and generalisations of the results described in the previous chapters". The rest of the book is an extensive appendix (32 pages) in which the reader can find "precise statements of all the results used", and even, "whenever possible […] proofs of such results". Topics covered are linear algebra, convex geometry, measure theory, nonlinear analysis and partial differential equations. The bibliography contains 125 entries, each with a link to the Zentralblatt review in all cases where it exists.

Now, before turning to Figalli's work in the ongoing ETH period (since 2016), let us display the two most recent rows in the gallery of Alessio Figalli's collaborators.

Guido De Philippis (Courant Institute, New York University. EMS Prize 2016. Stampacchia Medal 2018. Doctorate from the SNSP, supervised by L. Ambrosio and L. Caarelli); **Maria Colombo** (EPFL Lausanne. Iapichino Prize 2016. Miranda Prize 2018. Doctorate from the SNSP, supervised by L. Ambrosio and A. Figalli); **Begoña Barrios** (Universidad de La Laguna, Tenerife, Spain. Doctorate from the UAM, supervised by Ireneo Peral Alonso and Fernando Soria); **Xavier Ros-Oton** (Universität Zürich. Rubio de Francia Prize of the RSME 2016. Antonio Valle Prize of the SEMA 2017. ERC Starting Grant 2018. Fundación Princesa de Girona Prize 2019. Doctorate from the UPC, supervised by Xavier Cabré); **Joaquim Serra** (ETH Zürich. Évariste Galois Prize of the SCM 2011. Josep Teixidó Prize of the SCM 2016. Antonio Valle Prize of the SEMA 2019. EMS Prize 2020. Rubio de Francia Prize of the RSME 2019. Doctorate from the UPC, supervised by Xavier Cabré).

2017 (D. Jerison) "Quantitative stability for the **Brunn–Minkowski inequality**". *Advances in Mathematics* 314, 1–47.

> *"[...] succeeded in the quite challenging task of combining tools from additive combinatorics, affine geometry, harmonic analysis and optimal transport to obtain the first quantitative stability result for the BM inequality in arbitrary dimension and on generic sets. [...] quite impressive, both for the technical complexity, the wealth of original ideas involved, and the mathematical beauty of the question under consideration. [L]*



Moon-Jin Kang (SookmyungWomen's University, Seoul, South Korea. Doctorate from the Seoul National University, supervised by Seung Yeal Ha); **Emanuel Indrei** (Department of Mathematics, Purdue University. Doctorate by the University of Texas at Austin, supervised by A. Figalli); **Connor Mooney** (University of California at Irvine. Firestone Medal for Excellence in Undergraduate Research, Stanford University, 2011. Doctorate from Columbia University, supervised by Ovidiu Savin); **Yash Jhaveri** (Princeton Institute for Advanced Study. Doctorate from the ETH Zürich, supervised by A. Figalli); **Robin Neumayer** (Northwestern University. Doctorate from the University of Texas at Austin, supervised by A. Figalli and F. Maggi); **Xavier Fernández-Real** (PhD student, supervised by A. Figalli at ETH Zürich. Whelan Prize in Mathematics from Christ's College, University of Cambridge, 2015. Évariste Galois Prize of the SCM, 2016. Ranked First in the Spanish National Award for Excellence in Academic Record 2013-2014, 2018. Coauthor, with X. Ros-Oton, of the memoir *Regularity Theory for Elliptic PDE* (2020).[3]

2017 (M. Cozzi) "Regularity Theory for Local and Nonlocal Minimal Surfaces: An Overview". *Nonlocal and Nonlinear Diffusions and Interactions. New Methods and Directions* (Cetraro, 4–8 julio, 2016), Lecture Notes in Mathematics 2186, 117–158. CIME Foundation Subseries, Springer (edited by Matteo Bonforte and Gabrielle Grillo).

> *This article is a basic reference on the subject of minimal surfaces. The volume also contains articles by José Antonio Carrillo (with Vincent Calvez and Franca Hoffmann), Manuel del Pino, Giuseppe Mingione and Juan Luis Vázquez (whose seventieth anniversary was celebrated with a special dinner in which his "big involvement in Italian mathematics" was highlighted).*

2019 (J. Serra) "On the fine structure of the free boundary for the classical obstacle problem". *Inventiones mathematicae* 215.1, 311–366.

> *"[…] great acclaim greeted the work of Figalli, together with his co-author Joaquim Serra, who in 2017 gave a complete and definitive description of the free boundary. […] The new methods introduced in this work are having a wide impact.* (A. Jackson, *The Work of Alessio Figalli,* IMU, 2018).

2020 (J. Serra) "On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4 + 1". *Inventiones mathematicae* 219.1, 153–177.

2020 (X. Cabré, X. Ros-Oton, J. Serra) [16]

> *"[…] we prove the following long-standing conjecture: stable solutions to semi-linear elliptic equations are bounded (and thus smooth) in dimension $n \leq 9$. This result, that was only known for $n \leq 4$, is optimal: $\log(1/|x|^2)$ is a $W^{1,2}$ singular stable solution for $n \geq 10$. […] As one can see by a series of classical examples, all our results are sharp. Furthermore, as a corollary,*

[3] https://people.math.ethz.ch/~xavierfe/Documents/Fernandez-Real_Ros-Oton_Book.pdf

we obtain that extremal solutions of Gelfand problems are $W^{1,2}$ in every dimension and they are smooth in dimension $n \leq 9$. This answers two famous open problems posed by H. Brezis and Brezis–Vázquez (from the Abstract; cf. [14], [13] and [11]).

2020 (Ros-Oton, J. Serra) [39]

Jaume Franch, Xavier Ros-Oton, Joaquim Serra, Alessio Figalli, Xavier Cabré, Juan Luis Vázquez, Matteo Bonforte.

UPC Doctorate Honoris Causa

Alessio Figalli was conferred a Doctorate Honoris Causa by the UPC on Friday, 22 November 2019, with Xavier Cabré in charge of the *laudatio*.[4]

The day before, on 21 November 2019, the FME organised a workshop in homage of the honoree. The invited speakers are outstanding collaborators of Figalli's: Juan Luis Vázquez and Matteo Bonforte, from the UAM; Ros-Oton and Serra, already mentioned, of the younger generation; and Cabré, who is responsible for closing the day.[5]

The panel is scheduled as the first activity in the afternoon, just before the lectures by Serra and Cabré. The panel members are Figalli, the five lecturers of the day, and the Dean (Jaume Franch), who chaired the activity. The game is to ask a round of questions to Figalli with the purpose of discovering aspects of his trajectory, experience and thinking that can hardly be elicited from his mathematical writings.

The first move is by Juan Luis Vázquez. After acknowledging the great popularity of mathematics, largely derived from its many and attractive applications, he expresses some concern about the possible relaxation of content that this phenomenon could entail, and concludes by asking Figalli's opinion on these perceptions, and in particular about what balance should be achieved between the study of fundamental and applied mathematics in the bachelor's curricula.

In his response, Figalli states that he does not see a real dichotomy and provides several considerations in favour of this point of view. He argues that in some countries, especially the United States of America and the United Kingdom, it is possible to pursue very intensive studies of mathematics, very theoretical, which give one the option of deciding later whether to continue in the academic sector or move to other areas in which their profiles are highly appreciated. This approach provides attractive opportunities for students who would otherwise go directly to careers where it is initially easier to find a job. In fact, this also favours that more people have the opportunity to dedicate themselves to mathematics.

Xavier Cabré asks Figalli if he can briefly describe the role of the SNSP in the Italian university system, also how important it was for his own career, and finally whether he can share any suggestions on how Spain could maximize the chances of getting a Fields Medal in the future.

"Good question! Yes, it is fortunate for Italians to have the SNSP, a French-like institution founded by Napoleon", says Figalli, adding that it was excellent for him. The entrance is competitive, open to all, and during the studies the system covers all expenses. He values as very positive that the state promotes equal opportunities and in that line welcomes the more recent creation of other research institutions, such as the Scuola Internazionale Superiore di Studi Avanzati (SISSA), in Trieste, and the Istituto Universitario di Studi Superiori (IUSS), in Pavia. Regarding the access to important distinctions and awards, Figalli thinks that the possibilities are correlated with the number of people in a community who acquire a robust mathematical training, and that the visits to prestigious centres around the world are very important, both for the Fields Medal and for many other awards.

Xavier Cabré asks what differences there are between studying at the SNSP or at another Italian university. The answer is that the study regime at the SNSP has a dual character, since it is based on being enrolled at the University of Pisa, like any other student of this institution, and supplementing it with additional courses at the SNSP, such as analysis and physics in the first year; measure theory, probability and electromagnetism in the second; and algebraic geometry and quantum mechanics in the third. For the fourth and fifth years, he does not specify any discipline, since they can vary from one year to another. Another distinctive feature he mentions is the problem list assigned every week. Indeed, each student is responsible for presenting a detailed written solution to one of the problems and sharing it with the class, a setup

Xavier Cabré has been an ICREA research professor since 2003 at the UPC's Department of Mathematics (DMAT), and full professor of applied mathematics since 2008. He earned his doctorate in 1994 at the Courant Institute in New York under the supervision of Louis Nirenberg (1925–2020) with the thesis *Estimates for Solutions of Elliptic and Parabolic Equations*. New York University Kurt Friedrichs Award, 1995. IAS Member, 1994–95. *Habilitation à diriger des recherches*, Université Pierre et Marie Curie-Paris VI, 1998. *Harrington Faculty Fellow*, The University of Texas at Austin, 2001–02. *Fellow of the American Mathematical Society, inaugural class*, 2012. Co-author, with Caffarelli, of the memoir [17]. As we have seen, the papers [4, 1] were among the first research articles to influence Figalli. Moreover, his former doctoral students Xavier Ros-Oton and Joaquim Serra, and Cabré himself, count among Figalli's most important collaborators in recent years. His profile is included in AM.

4 Cabré's *laudatio* and Figalli's lectio honoris causa can be accessed at the UPC website: see [15, 31].

5 The corresponding presentations, including their videos, can be found at https://fme.upc.edu/ca/la-facultat/activitats/2019-2020/Workshop-Figalli-FME-2019-ponencies-fotos-gravacio

that favours the development of the spirit and skills that a researcher needs. He considers that their solutions help to understand more fully the meaning of mathematics beyond the abstract theorems learned in theory.

In the next intervention, Matteo Bonforte is concerned with the proportion of research time to be devoted to problems which are considered difficult, and also with whether to worry, at any level (from students to established researchers), if the solution is not found immediately.

From the outset, Figalli thinks that it would be fallacious to give an answer for everyone, since the differences between people are considerable. There are some that are very effective at concentrating on a single problem for a long time, such as Andrew Wiles with Fermat's theorem. Others work with a more functional orientation, in the sense that they focus on a certain group of problems, which favours a more distributed and staggered treatment of difficulties. In collaborative work, he suggests bearing in mind that progress is often reinforced by differences in style rather than coincidences.

In the case of students, particularly doctoral students, he thinks that the distribution of effort is an issue in which supervision should be essential. Once reasonable research autonomy has been achieved, each person must find out what best suits them. As an illustration, he explains his work on the Monge–Ampère equations, in collaboration with De Philippis, which started in 2005 and was not completed until 2012. Initial attempts lasted months, but were unsuccessful. When they finally resumed the research, they found that the key ideas did not emerge until quite late. An important final consideration is that in his case he learned many things by struggling with problems in which repeated attempts were unsuccessful.

Xavier Ros-Oton puts forward the question of contrasting the merits and demerits of two ways of evaluating research: that of countries such as Spain and Italy, in which published articles or citations are counted, and that of countries such as the United States, Germany or Switzerland, in which the main evidences stem from the recommendation letters by experts and the quality of the best research carried out.

After considering that the issue is very delicate, Figalli first describes how the Italian system works, then assesses the role of the letters of recommendation and, finally, defends a supervised system, based on clear rules of procedure, that can guarantee the highest objectivity and transparency. As for the Italian system, the averaging filter stands out. To be considered for a habilitation, for example, one must have a number of publications not less than that reached by 60% of the researchers in the area, and similarly with the number of citations and with the so-called index h. Among the problems of this system, he points out that it can leave out important papers, also the absence of a minimum quality control of those that exceed the indicators, and the fact that the definition of the averages implies that it is increasingly difficult to reach them.

The justification for the letters of recommendation is that there comes a time when you have to enter the contents and their significance, assess their real impact regardless of the journal in which they have been published, since no magazine can be an indisputable reference for all areas, and for that purpose an appropriate selection of experts can be decisive. But it also warns that it cannot be abused, since it involves a lot of effort on everyone's part, so its use is only recommended after some type of prior filter.

Finally, Figalli expresses that he likes the rules-based systems, for example to deal with conflicts of interest, as commonly used in the USA. A system of clear, impersonal rules simplifies decision-making, especially if the process is supervised, as is the case in Switzerland, by someone who knows how to guide its unfolding regardless of the people involved. A supervisor can ask a person to leave the room, after hearing what he or she wants to say, if that person has a conflict of interest in the matter to be resolved. Such a decision would hardly work spontaneously if the request had to come from one of the committee members.

During his turn, Joaquim Serra introduces the topic of rating the professional destinations of the doctorates in mathematics. While in countries like the USA and Switzerland the majority go into industry, and that these destinations are well regarded, in others, such as Italy and Spain, it is perceived as a certain devaluation of the academic career, although this impression may have begun to change already. How could the university benefit from the prestige of mathematics in society if it increased the number of excellent doctorates in non-academic positions and, therefore, the university's interaction with society? In his answer, Figalli ponders that the difficulty lies, at least in the case of Italy, in the fact that for companies the doctoral degree does not usually represent any added value compared to a master's degree. It is natural, therefore, that potential doctoral students do not see, from the point of view of jobs in the business world, any tangible benefit in the effort to obtain a doctorate. In the USA, on the other hand, a doctorate is well considered, a fact that is reflected, as a general rule, in better wages. However, the social benefit of the doctorate lies in the people trained to think critically, to solve problems, skills that are relevant for many occupations. As the acceptance of this view by society is an upward value, so is the benefit obtained by the university. Achieving a mature university-business interaction has in its favour the fact that the number of higher graduates in mathematics is very small compared to the population of all graduates of similar levels. Figalli also considers some of the elements of the situation at ETH that enhance this dynamic. The polytechnic nature of ETH implies that the Department of Mathematics teaches in all other departments. This guarantees a good mathematical foundation for all, leading to a better acceptance of the role of mathematicians in industry, which in turn allows the Department of Mathematics to access more resources, particularly for research. This circle indicates that the industry can be an ally, and as an illustration mentions the joint doctorates with companies such as Google in the case of ETH. This is a good time for maths!

The last question from the panel is asked by Jaume Franch, in two parts: For a student who has just received a doctorate, which strategy is better: to stay in the same institution and try to get a job at it or venture to go abroad? In this sense, how important has your stay in Austin been for your fulfillment?

The predictable answer is clear: Figalli is personally against staying in the same place, but it seems natural to him that there are people who return later. In fact, he thinks that a sojourn of a few years abroad is a very positive experience for everyone, more than anything because of the gain in knowledge and experience. His own training as a mathematician began in Italy, where he learned the techniques of the De Giorgi school, including calculus of variations and geometric measure theory. He then spent three years in France, where he became familiar with the interrelationship between optimal transport and isoperimetric inequalities, and finally came his stay in Austin, hosted by Luis Caffarelli, and where he solved, after three years, the problem of regularity for the Monge–Ampère equation (solution already alluded to above, see p. 22). He underlines that the climate of Austin was very important for the kind of research he was working on in that period. In fact, he stresses that he considers himself a product of his travels and thinks that this recipe may be valuable for other researchers.

Taking advantage of the opportunity given by the Dean to the assistants, a doctoral student states that he very much values research collaborations, as he sees in them many advantages, but with the disadvantage, according to people in his environment, that the credit for the work is diluted.

In Figalli's comment, he first points out that there are many differences between people in this regard, and that the views of the thesis advisor can be decisive. He also considers that some publication should be about a problem solved from beginning to end by the student, since this autonomy is part of what the access to a doctor's degree should insure. On the other hand, what mainly counts in the publications are the results, whether in collaboration or not. To illustrate this, he comments on his first paper. At the beginning of his doctorate, his supervisor, Luigi Ambrosio, sent him to Lyon for a semester. There he met Albert Fathi, who gave him a recent article on optimal transport on compact manifolds, and suggested that it would be nice to extend it for non-compact manifolds. Figalli reacted by saying that it seemed very easy, which led Fathi to say that precisely because of that it was an interesting problem to start with. Figalli went to work and realised that the generalisation had more difficulties than he had originally foreseen. Filling in all the details, and writing them in an orderly and complete manner, was a decisive learning process at that point in his career [24].

Xavier Cabré supplements the previous comments with the idea that one encounters difficult problems that can probably only be solved by collaborating with other researchers, and states his belief that this awareness has a positive effect on the research dynamics.

Glossary

If $X \subset \mathbb{R}^n$ is measurable, $|X|$ denotes the Lebesgue measure of X.

Bernstein problem. A function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ whose graph in $\mathbb{R}^n$ is a minimal hypersurface (that is, such that it satisfies $\operatorname{div}\left(\nabla f / \sqrt{1+|\nabla f|^2}\right) = 0$, which is the Euler–Lagrange equation deduced from the area functional $A(f) = \int \sqrt{1+|\nabla f|^2}$), is it necessarily linear? In 1914, Sergei N. Bernstein (1880–1968) solved it affirmatively for $n = 3$. See [9], in which the question is answered affirmatively for $n \leq 8$ and negatively for $n \geq 9$. For a report on the successive advances on this matter, including its close relationship with the *De Giorgi conjecture*, see [1].

Brenier theorem. Let X and X' be open sets in $\mathbb{R}^n$ with probability distributions μ and μ', respectively. Let $c : X \times X' \to \mathbb{R}$ be the cost function $c(x, x') = |x - x'|^2$ and assume that that μ is absolutely continuous with respect to the Lebesgue measure. Then there exists a unique *optimal transport* T from μ to μ'. Moreover, there is a convex function $u : X \to \mathbb{R}$ such that $T(x) = \nabla u(x)$ for all $x \in X$. If in addition $\mu = f dx$, $\mu' = f' dx$ and T is a diffeomorphism, then u satisfies $\det(D^2u) = f/f'(\nabla u)$, which is a *Monge–Ampère equation*. See [5].

Brunn–Minkowski inequality. Given two compact sets X and Y in $\mathbb{R}^n$, and setting $X + Y = \{x + y : x \in X, y \in Y\}$ (the Minkowski sum of X and Y), then $|X + Y|^{1/n} \geq |X|^{1/n} + |Y|^{1/n}$. See [46, §6.1.2] for an elementary proof using mass transportation theory. Intuitively, the "linear size" of $X + Y$ is at least the sum of the "linear sizes" of X and Y.

De Giorgi conjecture. Let $u : \mathbb{R}^n \to (-1, 1)$ be a C^2 solution of the semi-linear equation $\Delta u = u^3 - u$ such that $\partial_{x_n} u > 0$ in $\mathbb{R}^n$. Then all level sets $\{u = s\}$ are hyperplanes, at least for $n \leq 8$. See [1].

geometric costs. They are costs $c(x, x')$ defined on a manifold X given by expressions of the form

$$c(x, x') = \inf_{\gamma} \int_0^1 L(\gamma, \dot{\gamma}) dt,$$

where L is a Tonelli Lagrangian (see [24, §6.2]) and the infimum is taken with respect to all absolutely continuous curves on X going from x to x'.

isoperimetric inequality. If $X \subset \mathbb{R}^n$ is regular and bounded, $|\partial X| \geq n|B|^{1/n}|X|^{(n-1)/n}$, where $B \subset \mathbb{R}^n$ denotes the ball of radius 1. See [30, §4.6.2] for a brief proof using *optimal transport*. This inequality is *stable*, in the sense that if X almost reaches equality, then X is quantitatively close to a ball: See [34].

Minkowski problem. Let $X \subset \mathbb{R}^{n+1}$ be the graph of a function $u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$ open, and let $K : \Omega \to \mathbb{R}$. The condition for X to have curvature $K(x)$ at $(x, u(x))$ turns out to be a *Monge–Ampère equation*: $\det(D^2u) = K(1 + |\nabla u|^2)^{(n+2)/2}$.

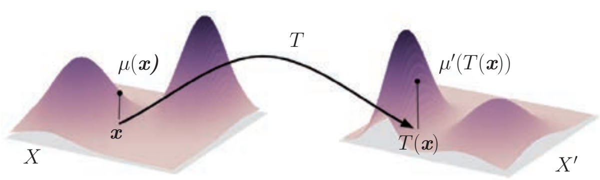
Monge–Ampère equation. Given a function $u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$ open, it is an equation of the form $\det(D^2u) = f$, where D^2u the Hessian of u and $f = f(x, u, \nabla u) > 0$ in Ω. It is a nonlinear second order PDE.

optimal transport, ~map, ~problem. The optimal transportation problem is finding the cheapest way to transport a mass distribution from one place to another given some measure of cost. Seen in this way, it can be considered as a problem in economics, but it actually turns out to be a useful technique in PDE, fluid mechanics, geometry, probabilities or functional analysis. The variability of this scheme is unlimited in terms of the geometry of the ambient space and the shape of the cost function, and in each case the basic questions are to ensure the existence, uniqueness and regularity of optimal transport.

Formally, and with the notations introduced in *transport map*, let $c : X \times X' \to \mathbb{R}^+$ (*cost function*). A transport $T : X \to X'$ from μ to μ' is *optimal* if the total cost of T, which by definition is $\int_X c(x, T(x)) d\mu(x)$, is minimal with respect to all transports from μ to μ'. In the case where $X' = X$ and d is a distance on X, then $d^p(x, x')$ $(p > 0)$ is an example of a cost function. The case $p = 2$ is of special importance, particularly when X is an open set of $\mathbb{R}^n$, $\mu = \rho dx$ and $\mu' = \rho' dx$ (ρ and ρ' positive functions with compact support) and d the Euclidean distance, since these conditions guarantee the existence of a single optimal transport and that it has the form $T = \nabla u$, u a convex function [5]. Cf. *Brenier theorem*.

semigeostrophic equations. In the case of the atmosphere, the geostrophic flow assumes that the pressure gradient is in equilibrium with the Coriolis force (*geostrophic balance*), in which case the wind direction is parallel to the isobars. In semi-geostrophic flow, this balance occurs only approximately and other forces are taken into account. One of the important equations of this flow is in the form of a *Monge–Ampère equation*, which points out to the role of this equation in such dynamics.

transport map. Let μ and μ' be probability distributions defined over measurable spaces X and X', respectively. A measurable map $T : X \to X'$ is said to transport μ in μ' if $T_\sharp\mu = \mu'$, that is, if $\mu(T^{-1}A') = \mu'(A')$ for all measurable $A' \subseteq X'$.



Wulff inequality. It appears in the study of crystals and phase transitions. On convex sets, it is equivalent to *Brunn–Minlowski inequality*. In [34] the *optimal transport* technique is used to establish an improved inequality for convex sets $X, Y \subset \mathbb{R}^n$: $|X + Y|^{1/n} \geq (|X|^{1/n} + |Y|^{1/n})(1 + c_n(A(X, Y)))$, with c_n a constant that only depends on n and $A(X, Y)$, a function that measures the "relative skewness" between X and Y.

Wulff shape, Wulff construction. It is the equilibrium surface for a crystal or a drop that has minimal anisotropic surface energy for a given volume. In the isotropic case it is the sphere. See [20, Ch. 5], [44, Ch. 7], [8]. The *Wulff construction* is a method for finding the equilibrium shape (*Wulff shape*) of a drop or crystal of a given volume.

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Gotthold Eisenstein and Philosopher John

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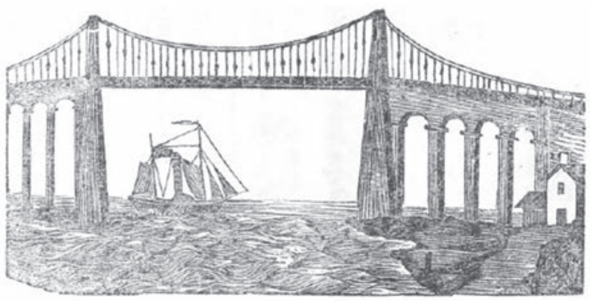
Before the recent publication of the correspondence between Gauss and Encke, nothing was known about the role that John Taylor, a cotton merchant from Liverpool, had played in the life of Gotthold Eisenstein. In this article, we will bring together what we have discovered about John Taylor's life.

Eisenstein's journey to England

Gotthold Eisenstein belonged, together with Dirichlet, Jacobi and Kummer, to the generation after Gauss that shaped the theory of numbers in the mid-19th century, and like Galois, Abel, Riemann, Roch and Clebsch, Eisenstein died young. Today, Eisenstein's name can be found in the Eisenstein series, Eisenstein sums, the Eisenstein ideal, Eisenstein's reciprocity law and in his irreducibility criterion, and he is perhaps best known for his ingenious proofs of the quadratic, cubic and biquadratic reciprocity laws. Eisenstein's father Johann Constantin Eisenstein emigrated to England in 1840; Eisenstein and his mother followed in June 1842, although Eisenstein's few remarks on this episode in his autobiography [3] belie the dramatic events that he experienced in England. On their journey to England, the Eisensteins passed through Hamburg; during the Great Fire in May 1842 about a third of the houses in the Altstadt had burned down. What we learn from Eisenstein's account is that he was impressed by the sight of railroad lines running right under the foundations of houses (in London?) and by the Menai suspension bridge in Wales: Eisenstein mentions that he undertook six sea voyages, and that on one of them they sailed under the tremendous suspension bridge in Anglesey, which was so high that the Berlin Palace would easily have fitted under its main arch.

Eisenstein also writes that he later made the acquaintance of William Hamilton in Dublin (who would discover the quaternions in October 1843), as well as that of the mayor of Dublin, Daniel O'Connell (who died in 1847 in Genoa, where he wanted to organise help for the Irish during the Great Famine).



Menai suspension bridge (from [1]).

Eisenstein does not mention what happened after his arrival in England; the correspondence between Encke and Gauss reveals that he fell seriously ill and was saved by the assistance of a certain Mr. Taylor from Liverpool. Encke first mentions this story in his letter from 10 June 1844 (see [10], p. 1141):

> *I have the honour, most esteemed Privy Counsellor, to send you the second volume of the Berlin Observations, and I have enclosed a collection of papers by a young local mathematician Eisenstein, whose assessment seems to me to be of importance in the interest of science, and for which no better judge could ever be found than you, the true head of this part of higher mathematics. I would not have concerned myself with this matter, but would have left it to Prof. Dirichlet, if the latter had not already been away for a year.*
>
> *Young Eisenstein went to the same Gymnasium as my sons, where he did not excel at other subjects, but surprised his teacher in mathematics[1] with a curious derivation of a series first developed by Lagrange, without having known this paper. He was then introduced to Prof. Dirichlet, who believed him to be an extraordinary genius. Owing to his family being in special financial circumstances, he followed his parents to Liverpool two years ago, where illness and family relations (his father seems to have speculated unsuccessfully) reduced him to such a harassed state that his local acquaintances thought it necessary to send him back home. Dirichlet told me of these circumstances and asked me to try to aid his return with the help of my English friends. Although I did not have any acquaintances in Liverpool, it turned out that an important cotton merchant, Taylor, had submitted a book on ancient Roman festivities and chronology to the Academy, and I learned that he published articles in journals on our comet and in particular on the short-period comet.[2] Based on this, I turned to this gentleman, who was otherwise completely unknown to me, and asked for his assistance. With a generosity that can perhaps only be found in Englishmen, he sought out young Eisenstein, arranged for a doctor and medicine (Eisenstein was suffering from typhoid fever), provided him with the means for travelling to Dublin (where Hamilton received him very obligingly) and also gave*

1 This might have been Karl Heinrich Schellbach.

2 Encke had computed the elements of the comet now named after him; it has an orbital period of only 3.3 years.

him and his mother the necessary funds to return to Berlin. Here, Eisenstein occupied himself with investigations of the kind you will find among the enclosed papers and has obtained, since other avenues were unsuccessful, a yearly sum of money from the King (through the efforts of Mr. v. Humboldt, as he told me), with which this young man is completely content (in his own words). He would like to visit Göttingen in the near future in order to present himself to you and, if the circumstances allow, profit from your teaching.

Because of his young age (he cannot yet be or must only just be 20) and his talent, which he certainly must possess, although I admittedly cannot tell whether it is as great as Dirichlet's remarks would have one believe, your judgement would be of such importance for his future position that I entreat you with some urgency not to deny my request. He seems to me to be pleasant company, and his experiences do not appear to have weighed him down, but rather shown him that he must pull himself together, and I believe I may hope that he will not be a burden to you. The manner in which I am involved in these matters is not entirely comfortable to me, since I feel indebted to Mr. Taylor and must see how I can thank him.

Gauss answers Encke's letter on 23 June and writes:

I would be pleased to make the acquaintance of such a promising young man, and I would be delighted […] if he would spend some time in G[öttingen]. It would be my pleasure if I could be of any assistance to him, if not by actual teaching, since he clearly has by far surpassed this stage.

Gauss's judgement on Eisenstein's skills was based on the articles that Eisenstein published in *Crelle's Journal* in early 1844. Gauss had only studied one of these articles in detail, namely Eisenstein's proof of the quadratic reciprocity law using multiple Jacobi sums (see [3], vol. I, pp. 100–107), but this was sufficient to convince Gauss of Eisentein's talent.

On 13 August 1844, Encke writes to Gauss, asking him to reply to a letter from Taylor and reminding him what Taylor has done for Eisenstein:

Young Eisenstein, in whom you have shown so much interest, went to Liverpool in early 1843 (unless I'm mistaken) with his (rather worthless) father; there his family got into a serious plight, and in addition young Eisenstein became gravely ill with typhoid fever. Prof. Dirichlet told me about it and asked if I happened to have an acquaintance in Liverpool who could take an interest in the young man, for otherwise he would perish.

Although I did not know Mr. Taylor (I had only heard from Prof. Mitzscherlich that he was a wealthy cotton merchant and an amateur astronomer who published news about comets and was familiar with my name), I took the chance of approaching him about the matter. He immediately did a lot more than I had hoped, sending a doctor to young Eisenstein, providing for him by a subscription, supplying him with money for travelling to Dublin in order to meet Hamilton, and facilitating his return to Berlin.

John Taylor

Who was this Mr. Taylor, cotton merchant from Liverpool, who saved Eisenstein's life and paid for his travels to Ireland and back to Germany? Encke writes in a letter to Gauss from 15 August 1846:

I must confess that I think very highly of his actions, for which he received no compensation, and so I am sorry that he got into a bitter dispute with Sheepshanks concerning the building of the observatory, in which he (probably deservedly) drew the short straw, since he does not know much about modern astronomy. He knows more about ancient astronomy, since he has translated the first four books of Ovid's fastis and published them with explanations about the knowledge of the skies at that time.

Should you thus feel inclined to answer his letter, I would sincerely request that you make friendly mention of his truly noble behaviour towards young Eisenstein, which he would value highly.

The information that we can glean from Encke's letter suffices to identify Taylor as the cotton merchant John Taylor from Liverpool; both his *Poems and Translations*, including the English translation of Ovid's *fastis* [9], and his heated exchanges with Reverend Sheepshanks in [7] can be found online.

Richard Sheepshanks (1794–1855) had been a Fellow of the Royal Society in London since 1830. The list of people he quarreled with is long; his dispute with James South, which Charles Babbage and his difference engine got drawn into later, is described in Hoskin's article *Astronomers at War* [5].

The dispute with Taylor was about the best position for the future obervatory in Liverpool. In a letter to the Liverpool Mercury, Taylor heaps scorn upon Sheepshanks' choice:

according to the writers of this Royal Astronomical Report, the proper situation for our Observatory is in the lowest point of land that can be found, surrounded by hills that cut off the true horizon, and where, in fact, there is no horizon at all, although, to be sure, when the trough of the river is filled with smoke and fog, which is commonly the case, there may seem to be a horizon at a few yards distance […] where nor sun, nor moon, nor star is ever seen, or was ever seen, to rise or set, and where no meridian line can ever be drawn or determined […].

Sheepshanks prevailed, but 20 years later the observatory was closed due to an extension of the harbour, and

[3] Eilhard Mitzscherlich (1799–1863) was a chemist and mineralogist from Berlin.

in 1868 the new observatory was erected on Bidston Hill, the position that had been suggested by Taylor.

Another source of information on John Taylor comes from the diary of the American astronomer Maria Mitchell. In 1857 she travelled through Europe; on 3 August 1857 (see [6], p. 86) she delivered a letter to John Taylor and observed that he must have been around 80:

> *I brought a letter from Professor Silliman to Mr. John Taylor, cotton merchant and astronomer; and to-day I have taken tea with him. He is an old man, nearly eighty I should think, but full of life, and talks by the hour on heathen mythology. He was the principal agent in the establishment of the Liverpool Observatory, but disclaims the honor, because it was established on so small a scale, compared with his own gigantic plan. Mr. Taylor has invented a little machine, for showing the approximate position of a comet, having the elements. […] He struck me as being a man of taste, but of no great profundity.*

If Taylor was about 80 years old in 1857, then he must have been born around 1777. His cometarium was studied by Beech [2]. The information that Taylor died in 1857 can be found in [4], p.191:

> *One of Mr. William Ewart's strongest political supporters during the stirring times which preceded the passage of the first Reform Bill (of which Mr. Ewart became an energetic advocate) was Mr. John Taylor, who, along with his brother Richard, had commenced business as cotton broker in 1821, but who since 1826 had been by himself. He was commonly called "Philosopher John", for besides being an active politician, both as writer and speaker, he was also noted as a poet and as an astronomer. He was the first to propose the erection of an Observatory in Liverpool, and out of this suggestion originated the present establishment at Bidston Hill. He was one of the original members of the Cotton Brokers' Association, and continued in business until his death, which occurred in 1857.*

William Ewart (1798–1869) was a liberal politician from Liverpool fighting for the abolition of capital punishment; he voted for the legalisation of the metric system in England in 1864.

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Stokes at 200

For thirty years, Sir George Gabriel Stokes was an inimitable Secretary of the Royal Society and its President from 1885 to 1890. Two hundred years since his birth, Stokes is a towering figure in physics and applied mathematics; influencing fluids, asymptotics, optics, and acoustics among many other fields.

At the Stokes200 meeting, held at Pembroke College, Cambridge, from 15–18th September 2019, an invited audience of about 100 discussed the state of the art in all the modern research fields that have sprung from his work in physics and mathematics, along with the history of how we have got from Stokes' contributions to where we are now.

Royal Society Publishing has recently published a special double issue of *Philosophical Transactions A* entitled *Stokes at 200* (Parts 1 & 2), compiled and edited by Silvana Cardoso, Julyan Cartwright, Herbert Huppert and Christopher Ness, based on work presented at the Stokes200 meeting. The articles can be accessed at https://bit.ly/TransA2174 and https://bit.ly/TransA2179 Both issues are currently freely available online until 1 September 2020.

In bringing together people whose work today is based upon Stokes' own, the aim was to emphasize his influence and legacy at 200 to the community as a whole.

Dynamics and Control of Covid-19: Comments by Two Mathematicians

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Since the Covid-19 pandemic is not yet over, it may appear to be premature to draw some conclusions. However, we may also be just in time to recapitulate some lessons we as mathematicians should have learned and are urged to apply now. Thus, we pose the question: Why are the dynamics and control of Covid-19 most interesting for mathematicians and why are mathematicians urgently needed for controlling the pandemic?

We shall first present our comments in a bottom-up approach, i.e., following the events from their beginning as they evolved through time. They happened differently in different countries, and the main objective of this first part is to *compare* these evolutions in a few selected countries with each other.

Still, there are some general features, which we present separately as we are used to doing in mathematics. They include the history of certain epidemics which have influenced the reactions of people in many countries, and some basic mathematical tools. In addition, there is a common factor, which one of the authors (KK) defined on 12 March 2020 in an e-mail to a German health office:

> *"The extension and evolution of Covid-19 in various countries and regions reflects the state of their health systems. This was for instance already very obvious in the case of Ebola."*

It is in fact the *public health* component of the health system that plays a crucial role.

The second part of the article is not "country-oriented" but "problem-oriented". From a given problem we go "top-down" to its solutions and their applications in concrete situations. We have organised this part by the mathematical methods that play a role in their solution. Here is an example where specially much mathematics is needed: to develop a vaccine and the strategy for applying it without losing sight of basic ethical principles.

Bottom-up

1. Prehistory

In the following, the gentle reader may consult when necessary the book [KPP] for the basic concepts of epidemiology.

Demography as a mathematical subject area was already developed centuries ago well beyond its elementary beginnings. For a long time, it remained the only mathematical tool in the study of the evolution of infectious diseases. Here is a famous early example. In China, India and Europe one tried to confer immunity against smallpox by infecting individuals slightly so they would contract a mild form of the disease and be immune afterwards. Some of them died by this procedure, but in 1766 the Swiss mathematician Daniel Bernoulli showed by a demographical approach that the procedure would increase life expectancy if applied to everybody [DI1]. Nowadays, evaluating the cost-effectiveness of a public health measure is done widely; it is based on methods of mathematical economy.

The 19th century saw the discovery of microorganisms as pathogens of many diseases and their study by mainly microbiological methods. The mathematical tools for following up an epidemic remained essentially demographical well into the 20th century. A few physicians suggested that every epidemic ends because there are finally not enough people left to be infected, which is a naïve predecessor to the mathematical-epidemiologic concept of herd immunity (see Sect. 8). Nevertheless, even the abundant literature on the influenza pandemic of 1918-19, wrongly called Spanish flu, only discusses two possible ways for it to end: better clinical treatment and mutations of the pathogen.

Seen from a virological viewpoint, the Spanish flu was an extreme form of the so-called seasonal influenza. The virus which causes them can be one of a large variety, its genus being denoted by A, B, C or D, where some of them include several species. A is the most serious one; it has subtypes $A(H_xN_y)$, $x = 1,\ldots,18$ and $y = 1,\ldots,11$, where x and y represent proteins on the surface of the virus. The strategy for controlling the "normal" seasonal influenza epidemic is widely known even among laymen: identify the strain of the virus in the autumn, develop a vaccine as fast as possible, and vaccinate people thought to be at risk. Nevertheless, the number of infections and deaths by a seasonal influenza can be as high as those caused by some of the pandemics to be described now.

The Spanish flu was due to $A(H_1N_1)$. Pictures from that time show people wearing masks that resembled those used now. In the years 1957–58 another "digression" from seasonal influenza occurred, called the Asian flu and caused by $A(H_2N_2)$. It started in China and then became a pandemic, passing from neighbouring states through the UK and the USA. Estimations of the number of cases vary around 500 million and of the number of deaths around 3 million. Its beginnings looked much like those of the Spanish flu, but towards the end a vaccine became available, a predecessor to the ones being routinely used now against the seasonal flu.

The Hong Kong influenza of 1968–69, generated by the virus $A(H_3N_2)$, had similar characteristics and will not be described further.

Parallel to these and other epidemics entering the scene, and partly motivated by them, basically new mathematical tools of public health emerged in the first part of the 20th century, preceded by a few studies in the late 19th. They were twofold. The first tool was called a "statistical-mathematical model". Its aim is the study of the influence of *factors*, also called determinants, on the health of the population. Such factors may, for instance, be a lack of hygiene or a polluted environment. A factor can also be a preventive or curative treatment by an immunisation or a drug, respectively; in this case the main objective of a study is to estimate the *efficacy* of the treatment. Sampling plans are statistical-mathematical models of a different but related kind. They form the basis of sample surveys, which are likewise being done in profusion about Covid-19, and are not always very illuminating.

The second tool is called "mathematical modelling of the evolution of an epidemic", or briefly "mathematical modelling". There are two kinds of it. First, one may aim for the epidemic curve, which is the cumulated number of cases up to a moment t as a function of t . In this case, mathematical modelling serves to estimate or predict this curve under various assumptions on the infectivity of infected subjects. Early predecessors are presented in [FIN], see Figure 1; the question whether the infectivity remains constant or decreases already played a role. Refined versions are still being used, in particular for Covid-19 (Sect. 7).

Second, one may build so-called compartmental models (Sect. 8). The first one, for measles, was published in 1889 by P.D. En'ko; see [DI2]. Around the year 1900, compartmental models for malaria appeared. Then in the 1920s, new models for the evolution of measles in closed populations were defined and intensively studied. They became very influential because they already displayed many basic features that reappeared later in mathematical models of epidemics in other and more complex settings.

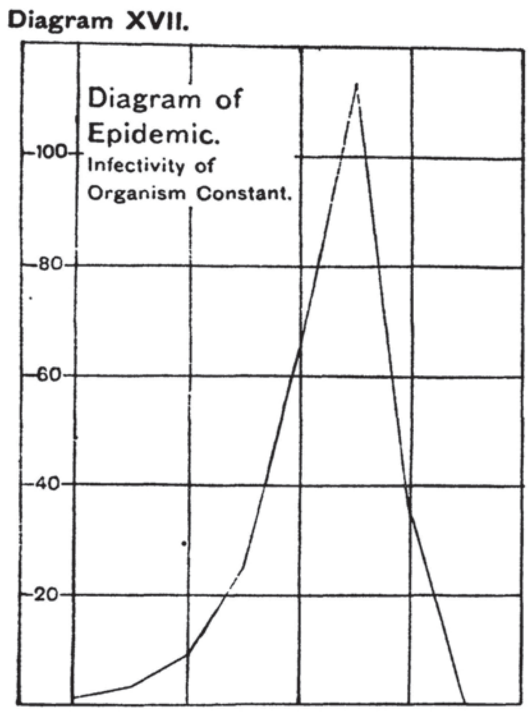

Fig. 1 Early numerical simulation of an epidemic curve by J. Brownlee, 1907, discussed in [FIN].

Such tools found many applications. Dealing with large epidemics mathematically was no longer a matter of demography alone, although that continued to be the main tool for estimating the number of cases and deaths. Statistical-mathematical models were employed to estimate the efficacy of antiviral drugs, for instance against HIV-infections, and the efficacy of various immunisations, including those against forms of influenza. Mathematical modelling of epidemics was used in planning strategies to eradicate smallpox, poliomyelitis, measles and maybe others. The first articles on modelling influenza epidemics appeared in the scientific literature. Planning a vaccination strategy involves both statistical-mathematical and mathematical models [HAl].

These roads to progress may have produced a general feeling of success in dealing with epidemics. Then in the period from 2002 to 2019, a few events occurred that evoked memories of previous pandemics and undermined such beliefs.

2. Unexpected Events 2002–2018

In November 2002 the first SARS (Severe Acute Respiratory Syndrome) epidemic broke out. It was a zoonosis generated by the virus SARS-CoV-1, a strain of the species SARS-CoV. It was first identified in China and never spread much beyond the surrounding countries and Canada. In July 2003 it was declared eradicated after having caused 8,096 cases and 774 deaths.

Similarly, the Middle East Respiratory Syndrome (MERS), due to the corona virus MERS-CoV, lead to around 2,500 cases and 870 deaths between 2012 and 2020. It was essentially concentrated in Saudi Arabia and, to a minor degree, in South Korea, with most infections happening around the years 2014 to 2015. Being a zoonosis carried largely by camels, it is also called the camel flu.

Moreover, a pandemic influenza invaded the world that resembled the Spanish flu in several respects. Its pathogen was a new strain called $A(H_1N_1)09$ of the H_1N_1 influenza virus. Its origin is being debated; a likely hypothesis says that, being a zoonosis carried by pigs, it infected a human on a Mexican pig farm around January 2009. It was therefore called swine flu or Mexican flu. It spread from North America to the whole world and was declared "extinguished" in August 2010. Estimations of the number of infections and deaths fluctuate enormously, but there were apparently more cases and fewer deaths than by the Spanish flu. Accusations against the WHO were raised about its handling of vaccines against the swine flu.

Finally, another zoonotic influenza appeared, popularly called bird flu and in scientific language Highly Pathogenic Avian Influenza (HPAI). The main pathogen was an $A(H_5N_1)$ influenza virus. It had been known long ago but reached a peak in the years 2013–2017. Whether there was airborne transmission from poultry to humans was a hotly debated question with obvious economic consequences. The bird flu spread widely over the whole world, but the number of known human cases remained small, at just over 70.

In addition to various forms of influenza and the epidemics generated by the corona virus SARS-CoV-1, SARS-CoV-2 or MERS-CoV, other epidemics occurred. It is instructive to compare them with those just mentioned, also applying in addition mathematical yardsticks. We shall restrict ourselves to Ebola epidemics. Their most widespread outbreak was the Western African Ebola virus epidemic from 2013 to 2016, which caused 28,646 cases and 11,323 deaths. There is a fundamental difference in the evolution of a case of influenza or SARS-CoV-1 or SARS-CoV-2 on the one hand and of an Ebola case on the other, which leads to a basic difference in their mathematical modelling (Sects. 4 and 8). A carrier of an influenza or corona virus can transmit it to others well before the first symptoms appear, that is well before the end of the incubation period. A subject infected by Ebola will only become infectious around the end of the incubation period. They could then be immediately isolated together with their latest contacts in order to avoid further transmission of the infection, provided that there is a health service nearby to do it. Therefore, Ebola did not spread to countries that have a sufficiently dense primary healthcare network, but it caused much suffering in countries that do not have it. The strategy of the WHO to control the epidemic was wrong. It insisted on drugs and the search for a vaccine (which only became available in December 2019), but neglected primary healthcare. For the present purpose it would even have been most useful to rapidly train village health workers and "barefoot doctors" as it had been done decades ago.

3. Looking at some countries

Only very few countries profited from the experiences of these premonitory 18 years to prepare much in advance for a possible, and probable, new outbreak of an epidemic. Some others only took appropriate measures at the first signs of Covid-19, and many started planning when the epidemic had almost reached its zenith.

We shall sketch some examples. For simplicity we will always describe the result of the strategy of a country by indicating its cumulated numbers of confirmed cases and deaths around the 1 June 2020. Regarding the reliability of these data, see Sects. 5 and 6.

We begin with those that had planned early.

Taiwan: Already in 2004, the year after the SARS-epidemic outbreak, the government established the National Health Command Center (NHCC), which was to prepare the country for a possible new epidemic. From 2017 on it was headed by the popular Minister of Health, Chen Shih-chung, who had studied dentistry at the Taipei Medical College. The Vice-President of Taiwan from 2016 to 2020, Chen Chien-jen, had been Minister of Health from 2003 to 2005 after having studied human genetics, public health, and epidemiology at the National Taiwan University and the Johns Hopkins University in the USA, followed by research. Thus, decisions about the control of Covid-19 were taken by politicians competent in matters of health, including public health.

Taiwan counts 23 million inhabitants, and many of them travel to and from China. From the 31 December 2019 onwards, when the WHO was notified of the epidemic in Wuhan, all incoming flights from there were checked, followed by controls of passengers arriving from anywhere else. An "Action Table" was produced in the period of 20 January to 24 February 2020, which listed 124 measures to be taken. The public obtained daily revised clear information by all existing means. "Contact tracing", which means repeated follow-up of symptomatic persons, of confirmed cases and of all of their contacts, was rapidly established on the basis of the electronic health insurance card that everybody has. The virological PCR-tests used (Sect. 4) were already available and quarantines well organised. In late January, rules about the wearing of masks were edited; a sufficient supply existed already.

As a result, 442 confirmed cases had been found and 7 deaths recorded up to the 1 June.

Vietnam: The Vietnamese strategy resembles the Taiwanese one in almost all aspects, with the exception of contact tracing. A Steering Committee to deal with new epidemics existed in the Ministry of Health. It put into effect its plan right after the 23 January, when the first infected individuals arrived at Vietnamese airports, among them a Vietnamese returning from the UK. All schools were closed on the 25 January, and since the 1 February everybody entering Vietnam had to spend two weeks in quarantine.

Other measures were imposed or relieved in accordance with the evolution of the epidemic, for instance a limited confinement or the wearing of masks. The Ministry of Health issued regular precise and clear information for the entire population by all available means, including smartphones. In addition, there is a personalised information system of so-called "survival guides" given to everybody. Every survival guide defines three categories of persons: F0: a confirmed case; F1: suspected to be infected or having had contact with an infected person; F2: having had contact with a person in F1. Each person is expected to find the category to which it belongs. The survival guide then provides printed information about what it must do depending on its category, for example submit to a test. Only PCR-tests are used.

In contrast to Taiwan, contact tracing does not use electronic tools. It is done by the population itself, aided by the survival guides, together with a large number of well-trained members of the health services, for example university lecturers.

At the end of 2019, Vietnam had 98,257,747 inhabitants. On the 1 June there had been 328 confirmed cases and 0 deaths. These data are based on a strong demographic section of the "General Statistical Office" and on several Health Information Systems [KKR] and can hardly be contested.

The preceding sketch of control measures in Taiwan and Vietnam has shown us the three main components of their epidemiologic side: contact tracing; lock-down, that is physical or social distancing in the wide sense including quarantine and border controls; and wearing of masks. We may call this the "surveillance-containment strategy". In addition, there is the medical-clinical side, from primary healthcare such as general practitioners up to large hospitals. Its state is crucial to the number of deaths caused by the virus SARS-CoV-2.

In contrast to Taiwan and Vietnam, it seems that all other countries in the world were unprepared at the end of December 2019. A few of them took fairly systematic and strict measures that covered the entire population as soon as the first cases had declared themselves. For a quick overview see Figure 2 on page 33. This was for example true for **China** at the end of January 2020, for **Slovakia** and **Greece** on the 27 and 28 February, for **Austria** on the 10 March and for **Denmark** on the 12 March. An alternative Danish strategy, based on rigorous contact tracing and quarantine, but not implemented until now was argued for in [SIA].

Regarding the results, the turbulent evolution in China is well known. In Denmark, with a population of 5.806 million, about 12,000 cases had been confirmed and 593 deaths recorded, and the corresponding figures for Austria were 8.86 million people, 16,979 cases and 672 deaths.

The comparison of Slovakia, a country of around 5.5 million inhabitants, with Greece, which counts 10.72 million people, is particularly striking because it makes visible the role of their physicians and hospitals. In Slovakia there were 1,528 confirmed cases and 28 deaths. The corresponding data for Greece were 3,058 and 183. The relatively much higher number of fatalities in Greece, in spite of equally early reaction and almost the same number of cases per number of inhabitants, is no doubt due to the catastrophic state of its medical-clinical system, caused mainly by the debt crisis from 2010 onwards.

Next, we move to a group of countries that reacted late and not systematically, applying the various measures in a haphazard way and only to part of the population. Here are some of them with their numbers of inhabitants in millions, cumulated numbers of confirmed cases and numbers of fatalities:

Belgium: 11.46; 59,348; 9,606.
Spain: 46.94; 289,046; 27,136.
Italy: 60.36; 235,561; 34,043.
France: 66.99; 154,591; 29,296.
Germany: 83.02; 187,000; 8,831.

The relatively low number of deaths in Germany mainly reflects a sufficient medical-clinical system that could readily adapt itself to the epidemic. The opposite was true in France. There, about 100,000 hospital beds had been eliminated in the period between 1993 and 2018. An arbitrary strict "confinement", not determined by epidemiologic reasoning, was imposed on the 17 March.

Finally, there are countries that decided to do nothing, at least for a long while. Their motivation, or pretext, was above all a belief in herd immunity (Sects. 1 and 8), according to which the epidemic would stop by itself. This was the strategy of **Sweden**, a country counting 10.23 million people, which resulted in 37,814 cases and 4,403 deaths. In the **United Kingdom** there were, among 66.65 million inhabitants, around 290,000 cases and 41,128 fatalities, and in the USA these data were 328.2 million, 2.04 million, 115,000 deaths.

This overview of strategies confirms that, as said in the introduction, the results indeed depend heavily on the state of public health. Note that nowadays in every language of the world the concept "public health" is designated by a literal translation or a slight modification of this expression. For instance, in Danish it is "folkesundhed", that is, "health of the people".

Top-down

In this second part, we will sketch the scientific and in particular mathematical principles involved in the study of successive stages of the pandemics. In short: Sect. 4: Discovery of the new virus, basic properties, testing for its presence in a person. 5 and 6: Data on the evolution of Covid-19 in a population. 7: Attempts at analysing mathematically and predicting such an evolution by representing it by an epidemic curve. 8: The analogous for a representation by a compartmental model. 9: Trying to stop the epidemic by a vaccine. 10: What to learn and what to do?

4. The New Virus SARS-CoV-2

After the often-depicted outbreak in late December 2019 of cases of pneumonia of unknown aetiology around Wuhan, in the course of January 2020 Chinese scientists identified a new virus as the pathogen. They followed the usual procedures, i.e., they determined the load of 26 common respiratory pathogens in the patients. They found none of them in abundance. They suspected SARS-CoV, but could not find it either. Then they investigated all kinds of viral load that had a slight similarity (coincidence in a number of genomes) with SARS-CoV and detected a novel virus which displayed abundant virions in respiratory specimens from patients. Electron microscopy and mathematical pattern analysis [MUM, PEV] showed that it belongs to the same species as SARS-CoV-1 and MERS-CoV (Sect. 2); hence the name SARS-CoV-2.

Starting with this work in China, a large number of publications about the peculiar properties of the pathogen and the ways it is acting have appeared. On the virological side, its genetic sequence was determined. The new virus is believed to have zoonotic origins, but human to human infection was rapidly established. The combination of SARS and influenza features, that is intensive respiratory inhibition of patients and rapid transmission, make Covid-19, the disease caused by SARS-CoV-2, particularly dangerous. For further work see [AND].

In the clinical context, several periods in the evolution of a case were determined (see their definition in [KPP, Sect. 5.2]): The median *incubation period* is 5.2 days; the mean *latency period* is 4.6 days, i.e., in general the *infectious period* indeed starts before the *prodromal phase*. We have discussed the implications in Sect. 2 in comparison with Ebola. The mean length of the infectious period is 6 days for mild and asymptomatic cases; for severe and critical cases this period lasts on average 22 days and ends only by recovery or death.

The manifold applications for the control of the pandemic of both their virological and their clinical characteristics will appear in Sects. 7, 8, 9 and 10. Their study is still active and may even reverse former results; this happened recently for example about so-called cross-immunities. However, in this article we will only treat applications to the basic element of well-designed control strategies, namely testing for infections.

The first step of a test programme is to define the target population. Who will be tested? Subjects who had contact with infected people? Or those who complain about symptoms? Or everybody coming from a region where cases exist? See the example of Vietnam in Sect. 3.

Next, what will be the objective? To discover the presence of the virus or that of some kind of antibodies? Depending on the objective there are virological and serological tests. The usual virological test is called the PCR-(Polymerase Chain Reaction) Test. Dozens of serologic tests of varying quality have been and still are being developed and even offered to the general public in some countries. Recall that the characterisation of a test with a given target population and a given objective is a classical subject of clinical epidemiology [KPP, Sect. 19.2].

Coming back to the fundamental role of testing in control strategies, we only remark that in poor countries or in rich countries with inattentive public health officials, the target population was often determined by the shortage of test kits and by the influence of institutions that required them for themselves.

5. Demography: Descriptive Epidemiology

This is classical *medical statistics*, which gives for a specific disease the number of cases and deaths together with the when and where and a few additional data such as sex, age and sometimes profession of the subjects. In the present context we are dealing with Covid-19 as the disease due to an infection by the virus SARS-CoV-2.

In principle, the methods for finding the number of confirmed cases and fatalities by Covid-19 are the same as for any other disease. They fluctuate widely between countries. Both the diagnosis of a case of a disease and the description of the cause of a death may be relatively correct or most unreliable. In particular, finding a correct diagnosis for somebody who complains about acute health problems depends very much on the local contact tracing methods and on the state of the clinical-medical system. An additional difficulty arises from the existence of asymptomatic forms of the disease, that is, subjects infected by SARS-CoV-2 who display no symptoms.

In Sect. 3 we have mentioned Vietnam, which uses its normal demographic and health information systems [KKR]. It includes in its statistics asymptomatic cases found by contact tracing. Other countries obtain their morbidity and mortality data from a "health reporting system". Such a system is partly based on sampling methods from various sources, for example hospitals and local health offices. In Germany, the Robert Koch Institute, a central institute mainly devoted to infectious diseases, reports on the results for Covid-19. In the USA, the Johns Hopkins University plays a similar role. Still other countries use data from health insurance offices.

However, many countries have neither a health information system nor a health reporting system, or they do not use it for Covid-19. A host of alternative methods is being employed. For example, France only counts hospitalised confirmed cases and only deaths which happen in a hospital or in a retirement home that is connected with a medical structure.

Summing up, we may say that morbidity data, and to a lesser degree mortality data, for Covid-19 that one finds in various periodic publications are fairly unreliable, with very few exceptions. The sources are not always clearly indicated.

An important alternative idea is to compare the present situation with that in years past. Speaking naïvely again, we assume that the present higher case frequencies and death toll, and only these, are the result of Covid-19. Given the diagnostic difficulties mentioned above, this idea is mainly applied to fatalities and hardly to nonlethal cases. Thus, in the method of "excess mortality", we only measure how many more deaths by any cause happened this year than in the corresponding period in the past. For the UK we have for instance quoted in Sect. 3 the figure of 41,128 deaths up to the 1 June as supplied by the National Health Service. By contrast, the National Statistical Office advanced about 62,000 deaths as excess mortality!

Finally, here is an interesting idea based on the most classical form of a statistical-mathematical model. A graphic in the paper [FIT] (see Figure 2) shows in double logarithmic scale for every one of 16 selected countries the point in the plane whose coordinates are, respectively, the estimated number of infections per million inhabitants on lock-down day, and the excess mortality. A short glance convinces us that they are positively correlated. A

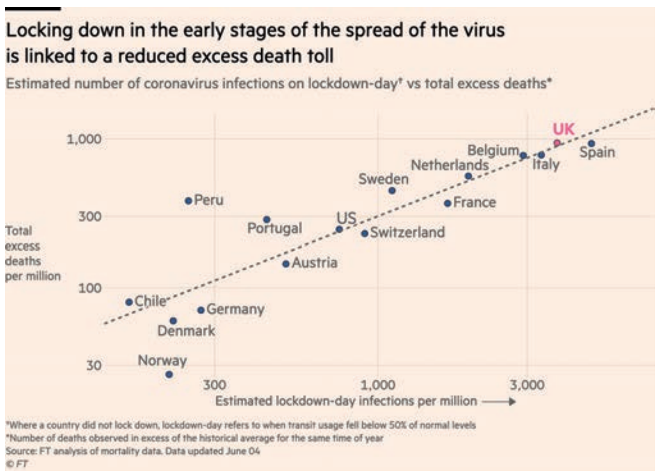

Fig. 2. Estimated number of infections on lock-down day and excess mortality for 16 selected countries. Reproduced from [FIT]. Used under licence from the Financial Times. All Rights Reserved.

simple regression analysis based on this graphic would also allow us to estimate one of these values by the other one for any other country.

6. Advanced Demography

It goes in several directions beyond classical health statistics, all of them likewise relevant to Covid-19. Firstly, *sample surveys* are conducted instead of using the data from the entire "target population". They have for example been used to study the influence of *social factors* on the evolution of various aspects of the disease. In particular the factor "to be an immigrant or to descend from them" was thoroughly investigated in some countries. Secondly, more types of data about cases and deaths are collected, for example about morbidity and mortality by age groups. Thirdly, data sets are not only being registered and perhaps published, but also transformed and interpreted in various ways. Here, standardisation is the best-known procedure. A fictitious example would be the number of fatalities by Covid-19 in Denmark, if Denmark had the same age structure as Vietnam and in each age group it had the same Covid-19 mortality as in the same age group in Vietnam.

In Sect. 9 we shall meet statistical-mathematical models as a basic mathematical tool in developing a preventive treatment of Covid-19. With their help, one studies the influence of various factors on some outcome variable E of interest in a clinical trial. Here the idea of "controlling" for the influence of another factor, which might be a "confounder" in the study of the action of E, plays a role. It looks as if most demographers on the one hand, and most clinical epidemiologists on the other, ignore that the mathematical procedure of standardising is the same as that of controlling for a confounder [KPP, Lesson 21]. A mathematician will not be astonished, though!

7. Modelling the Epidemic Curve

We have mentioned this classical concept in Sect. 1; see [KPP, Sect. 4.6]. Let C be an epidemic, V a geographical region, t_0 a moment of time which may be that of the first case of C in V, and $f(t)$ for $t \geq t_0$ the number of observed and reported cases of C that had declared themselves in V before or at the instant t. Then f is called the *epidemic curve* of C in V. In particular, it needs to be said whether unconfirmed cases are included or not. Measuring $f(t)$ as the time t goes along is the task of the relevant demographic services (Sects. 5 and 6). This process is therefore subject to all the deficiencies listed there.

To get some knowledge about f for various regions V is of course one of the main concerns of the population of a country invaded by C. Such knowledge is equally vital for health authorities who attempt to control C. However, much more knowledge is desirable. What can we learn about the mechanism of C by observing $f(t)$? This was already the subject of the papers described in [FIN]; see Sect. 1. In particular, is there a way to predict aspects of the future evolution of f, having observed the values $f(t)$ for a while?

Answers to these questions are generally given by *modelling f*, that is by making certain assumptions about its shape and by estimating certain parameters in it. A very large number of papers have been published about this issue. Some of them use extrapolation methods known from mathematical economy. A recent survey on various basic ideas and techniques can be found in [KRM], where a model is described in terms of an integro-differential equation.

We will restrict ourselves to a discussion of an application, namely a so-called basic reproduction number R_0. It appears constantly in popular publications. To define it, let us look at a subject s that is infected at a time $t^* \geq t_0$. Let $\mu(s, t^*)$ be the number of all subjects infected by s after t^* in the form of secondary, tertiary etc. infections. Then R_0 is the average of $\mu(s, t^*)$ over all s. Thus, it depends on t^*. It is precisely this dependence in which people are interested: a value less than 1 is looked upon as predictor of the extinction of C after t^*. In the case $C = $ Covid-19, values as high as 5.7 had been estimated in the beginning, that is, for t^* close to the time of the first outbreak of C. The article [SIA] presents an interesting factorisation of R_0 in order to compare different approaches to control the size of it.

8. Compartmental Models

We have sketched their historical origin in Sect. 1. We distinguished between two ways of mathematically modelling the evolution of an epidemic. Models of the first kind (Sect. 7) represent the temporal evolution of the number of subjects in a certain state, for instance the state "to be infected". By contrast, compartmental models also represent *changes* of this state at some moments in the form of transitions of a subject from one compartment to another one.

The SIR-model, which we designated in Sect. 1 as "intensively studied in the 1920s", is particularly simple and has served as a paragon for many others, in particular for those applied to Covid-19. It involves three compartments: S are the susceptible, not yet infected subjects, I the infected ones, and R consists of subjects removed by recovery with immunity or death. The transitions between compartments are described by differential equations for the numbers $S(t)$, $I(t)$ and $R(t)$ of subjects in the compartments as a function of time t. They involve certain parameters such as transition probabilities from one compartment to another one. Under various assumptions, the resulting system of differential equations for S, I and R can be solved explicitly or numerically.

A first important application is to estimate a basic reproduction number R_0 as defined in Sect. 7. It can be expressed by the basic parameters.

Secondly, it turns out that the limit S_∞ of $S(t)$ for $t \to \infty$ is strictly positive, which means that a certain part of the population will never be infected. This led to the concept of *herd immunity*, which, however, gave rise to much confusion among people who thought they had something to say about the matter.

After the outbreak of Covid-19, many more involved compartmental models were defined and analysed. Their parameters represented among other features the underlying control strategy to be used. There was for instance the "do nothing" strategy and also the "mitigation" strategy, which consisted of the less stringent components of the "surveillance-containment strategy" defined in Sect. 3. In the much-discussed paper [FER] Neil Ferguson and collaborators described the shape of the function I, that is the number of infected subjects, for the "do nothing" strategy. From the value 0 on it increases, reaches a maximum, decreases and finally reaches 0 at a certain moment t_{happy}. This had apparently motivated the countries UK, USA, Sweden and Brazil to adopt this strategy for too long, ignoring that Ferguson predicted (see Figure 3) about 500,000 deaths caused by the epidemic in the UK and 2.2 million in the USA before extinction at the moment t_{happy}.

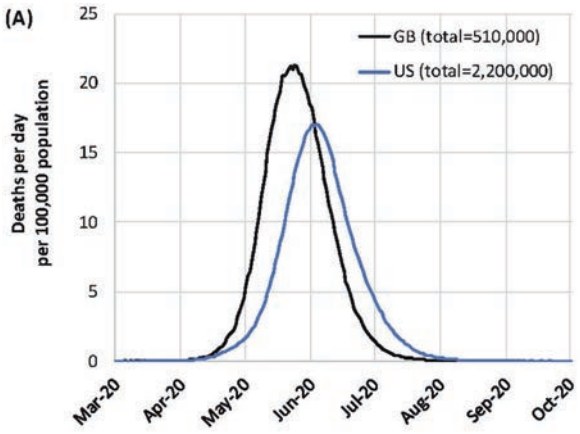

Fig. 3 Expected deaths caused by the epidemic for the do-nothing strategy, reproduced from [FER] with permission of School of Public Health, Imperial College London.

At present, compartmental models play hardly any practical role, mainly because they contain too many unknown parameters. Some parameters such as infectivity are first estimated with the help of a model of the epidemic curve, which does not seem to be a very successful detour.

9. Preventive and Curative Treatments

It will hardly be a surprise that several pharmacological companies have started a run for developing curative and preventive treatments of various ailments which SARS-CoV-2 may inflict on a person. Up until now, no curative treatment was found. There are only the well-known methods to be used in the treatment of non-specific aspects of a case, such as reducing pain, facilitating breathing or shortening the recovery time by an antiviral drug. We shall therefore restrict ourselves to *preventive* treatments, that is, to immunisations.

The objective of an immunisation by a vaccine against a Covid-19 connected health deficiency needs to be defined in the same way as for any other infectious disease. First the target population needs to be determined: Whom do we intend to protect? Next, what are the health deficiencies we want to prevent? For how long is the preventive effect to last? This is a particularly important aspect of the vaccine, but is usually suppressed when a new one is announced. For instance, the measles vaccination remains active in most subjects for their entire lives. For Covid-19, the company which is trying to develop the vaccine may be satisfied with a few months, hoping that SARS-CoV-2 will have disappeared after that. Finally, the *efficacy* needs to be found, which represents the part of the target population actually protected. It may also be defined in epidemiologic terms by regarding as "exposed" all subjects that have not obtained the treatment. Then the efficacy is the "aetiological fraction among the exposed subjects", see also [KPP, Sects. 15.1, 16.1, 17.1].

Nowadays, there is general agreement that the process of developing a vaccine against an infectious disease needs to run along a well-defined common line [KPP, Lesson 18, and HAL]. This ought to hold for Covid-19, too, and we shall therefore recall it here.

First, one or several substances are selected, which, for some reasons, usually virological ones, look like possible candidates for a vaccine. Each of them needs to be submitted to a "clinical trial" in order to explore its most important properties. Such a clinical trial consists of three "phases" I, II and III. Phase I deals with various mainly pharmacologic aspects such as side effects for various possible dosages.

Statistical-mathematical models are the essential tools of the phases II and III. Phase II aims at providing a first idea of the efficacy of the selected vaccine. Thus, a relatively small target population is built artificially. Here two basic problems arise. The first is the definition of the outcome variable of interest. Often only the "immunogenicity" is being studied, which means the formation of antibodies, but not protection against the disease. It is a particularly complex and manifold problem in the case of Covid-19. Secondly, the target population needs to include among the vaccinated subjects a sufficient number of people who would attract the disease when not vaccinated. Since Covid-19 morbidity in the entire population of a country is small, such a group must be constructed by "challenge", that is, by infecting its members artificially. They are usually volunteers and their risk of dying is small except in the age groups where the lethality by the disease is high, that is, in the case of Covid-19, for elderly people. Faced with this ethical problem, the USA used, for various previous infectious diseases, prison inmates whose terms were shortened as a reward. There was a time when Vietnam, while developing a certain vaccine, sent its samples for the Phase II trial to the USA to be tested in this way because Vietnamese ethical standards forbade all kinds of "challenge".

There are usually several phase II trials in order to select the potential vaccine to finally be studied in a phase III trial. This is a field trial in the sense that a sample of subjects is drawn from the entire population of interest, for instance from among all inhabitants of a country within a certain age group. The outcome variable is not immunogenicity but protection against the disease in the sense of the desired efficacy. The size of the sample is determined beforehand by the precision of the intended estimate of the efficacy. As noted above, the decision about the duration of the trial is a crucial element. If high efficacy during the first two weeks after vaccination is considered sufficient, the trial may be stopped after two weeks; this philosophy underlies the vaccinations against the seasonal influenza. If we are interested in its efficacy during the first ten years after vaccination, it must last ten years. This has, in addition to other problems, caused the long delay in developing an Ebola vaccine (end of Sect. 2). We hope that it will not be glossed over by those who are trying to sell a Covid-19 vaccine very soon.

10. Outlook

The pandemic has functioned like a magnifying glass. In some places, it has shown a basically well-functioning society. In other places it has revealed scandals and intolerable social inequalities. In particular, it has reflected the state of a country's public health system.

The present article aimed at describing the role of mathematics in the pandemic. As said above, there are two parts to this "outlook". Let us take up the first one, namely: What can be learnt from the epidemic? In Sect. 1 we gave an overview of the main branches of mathematics that play a role. Then the Sects. 4–9 sketched the most frequent applications; their titles and their order correspond vaguely to the branches of mathematics concerned. Thus, there were mathematical pattern analysis in laboratory work and statistical-mathematical models in judging the quality of tests; demographic methods in the collection of data; different ways to model the evolution of the pandemic mathematically; and clinical epidemiology in attempts to develop a vaccine.

In this way, the article aimed to clarify the potential role of mathematics in making decisions. On the one hand, it turned out that in practise the role of the epidemic curve or compartmental models is much more restricted than advertised in many publications. Decisions based on them may even have disastrous consequences, for instance those based on the mathematical concept of herd immunity. Thus, blind trust in mathematical arguments is unjustified.

On the other hand, denying the existence of a valid mathematical-scientific foundation for a control strategy is just as detrimental. It was done in Denmark with the "tracing and lock-down" strategy by a report of an "expert group" of health academics and officials, which reflected the interests of medical, industrial and governmental circles.

This comment leads us to the second part of our "outlook", namely: what to do in the future? The authors of this article started it in early May with the line "Since the Covid-19 pandemic is not over…". While we are finally finishing our work in the middle of July, it is still not over! It is still even very active but has taken a largely different form. Hence it seems natural to analyse its present characteristics in the light of the facts we have described in the Sections 4–9 above and to ask ourselves: Which lessons can we draw regarding the control strategies to be applied now?

Covid-19 no longer surges from a single source. It reappears in large or small regions of many parts of the world, which may be of various forms and extensions: a single home for the elderly in France, two districts in Germany, a large city like Beijing, an entire province in Spain, or a whole country like New Zealand. We shall call them "nests" to distinguish them from "clusters", which denote certain discrete sets of people. A precise follow-up of the evolution of cases in these nests meets with the manifold difficulties explained in Sects. 5 and 6 and will not be repeated here.

A first natural question to ask is, then: Why do "active" nests persist and reappear? Sect. 3 presented three components of successful control strategies: contact tracing; lock-down and masks. While contact tracing continues reluctantly, lock-down and wearing masks were widely abandoned, often as a result of governmental policies seeking popularity.

Next, what should be done? In the Sections 7, 8 and 9 we have explained, using in particular mathematical arguments, to what extent the strategies of control treated there suffer from serious drawbacks. This leaves us with the combination of two measures: within a nest a rigorous lock-down such as social distancing and preventing larger assemblies of people; at its borders: closing them or only allowing passage when combined with quarantine. For example, New Zealand, regarded as a single nest, has taken such rigorous measures. As a result, there are now no new cases, except two cases around the 14 July in "managed isolation facilities". Other nests will act similarly, we hope.

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Working from Home. ~~2 Months~~ 4 Months and Still Counting…

Allessandra Frabetti (Université de Lyon, Villeurbanne, France), Vladimir Salnikov (La Rochelle Université, France) and Laura Schaposnik (University of Illinois, Chicago, USA)

Dear friends and colleagues,

These days most of us are working from home and this seems to be a unique experience. We thought of writing an article about this topic for the Newsletter of the European Mathematical Society.

So a couple of questions for you:

One way or another, some of us replace school teachers for our kids from kindergarten through to university (tough job, right?) You have probably invented some nice recipes for how to explain mathematics and science in general to non-specialists that you are willing to share. Or you have found some resources that you have used successfully and would like to advertise. Please do!

The other way around: your nearest and dearest can now see you working. Maybe you have finally taken the time to explain what your job really is? What kind of science you do and why. Maybe even convinced someone that watching a seminar on your TV in the living room is more interesting than cartoons? Or just some interesting discussion happened unexpectedly? Or even better, a funny episode? Please share!

Impatiently waiting for your stories.
Yours, Vladimir

My letter, 23/04/2020 (Vladimir Salnikov)

This is the e-mail that I wrote to a rather large list of people to whom I usually make scientific announcements. It sort of speaks for itself: I realised that we were (and still are) crossing some historic period, and I was curious about how other people are handling it.

I received several replies, from which we chose some that people were ready to share with a broader audience, hence this article.

Actually, I received a lot of material, so we will continue in the next issue of the EMS Newsletter. And since we now have no idea how the situation will evolve, I will be attaching dates to be able to fit the stories into the context. In this issue we will address mostly the family aspect of the story, and the second part will be about everything "online". I do hope it will not become a regular section, though. And I will start with my own short anecdote.

My son Miron has just turned 4 years old, and he actually motivated me to write the above letter by showing me how I work. He glued together two sheets of A4 paper, drew some sort of rectangle on one of them and small rectangles on the other one. You have guessed it: he said it was his laptop. Then the following happened (I just watched, with no interference). He sat on the sofa in the living room, opened his "laptop" and said, "I am going to work now". He pressed the "buttons" for a couple of minutes, then said "No, it is too noisy here", he closed the "laptop" and went to his bedroom. He closed the door and stayed there for about 3 minutes (I do not know what he was doing, but from the sound I guess he was still typing). He came back, his "laptop" under his arm and said, "I was working, but now I am finished", and put it aside. Well, I recognised myself … did you?

Laura Schaposnik, letter from 28/04/2020

Being pregnant and thus at high risk, we started self-isolating before any official advice was given here in Chicago. Hence, now being in week 6 of house confinement, we've developed some favourite ways in which we're teaching our 20 month-old son about numbers and letters.

Our day starts at 7 am, and after some light reading we offer little Nikolay some "pasas de uva": that's when he runs to the table, knowing there will be 5 raisins ready for him, displayed in a line which will get shorter as he eats. Each time one is gone, we count backwards… by now, the little habit has grown into him tidying all his food into lines and making as if he were counting the objects. Maybe he's counting them in his head – the sounds he makes, albeit not English or Spanish numbers, are certainly precise and get repeated every time he is counting!

Whilst most of our books are not about counting – in fact, we don't have any about it! – we have turned most of them into letters and counting games… this is how

we end up in an "arándanos" party, counting backwards from 8 most lunchtimes! The newest problem on this front is that Nikolay loves hearing that "no hay ningún arándano más!" and thus munches 8 blueberries within seconds.

We have also decided to make certain actions be accompanied by counting every time they occur – when waiting to cross a street, heating something up in the microwave, or preparing a piece of toast, we'll always count to 10 (in Spanish) and this has by now led to Nikolay "counting" by himself (and moving his hands adding random fingers to the count) when these events happen.

One of the things we love the most from these last weeks is that he knows what "one", "two" and "many" are (with one finger, two fingers, or ten fingers) and you can now ask him how many of something he wants… which is usually either none, or many many more!

Alessandra Frabetti, letters from 27/04/2020 and 29/04/2020

I submit three contributions :-)

1) My son is seventeen, he is finishing the Lycée[1] and fortunately he does not need my help anymore to do his homework (if he ever needed it), he is fully autonomous. But I do have a funny story to tell about replacing cartoons with online math seminars. The funny side is that I didn't force my son to, didn't even suggest it, rather the opposite!

My son wants to study informatics next year, and he is a great fan of online lectures on his favorite topics, in particular those offered by MIT. A few evenings ago, at dinner, he explained to me the so-called samples representation of polynomials, that I had never come across, in particular its good behaviour with respect to the basic operations among polynomials. He was so excited that he finally proposed my husband and I immediately watch, on the laptop, the lecture "Divide & Conquer: FFT"[2] by Professor Erik Demaine of the Department of Electrical Engineering & Computer Science at MIT.

And that's how the whole family watched an MIT online course about the fast Fourier transform during supper!

An unusual, interesting and amusing supper, indeed!

2) I would like to suggest a trick for encouraging children to compute sums and products, and also for opening the wonderful world of abstraction up to them. I experienced it a long time ago (when my son was six years old, in 2008), but it adapts perfectly to the current confinement situation and I would surely do it again if I had young children around.

Every parent knows that answering "what is three times eight?" is of no interest to a child in primary school.

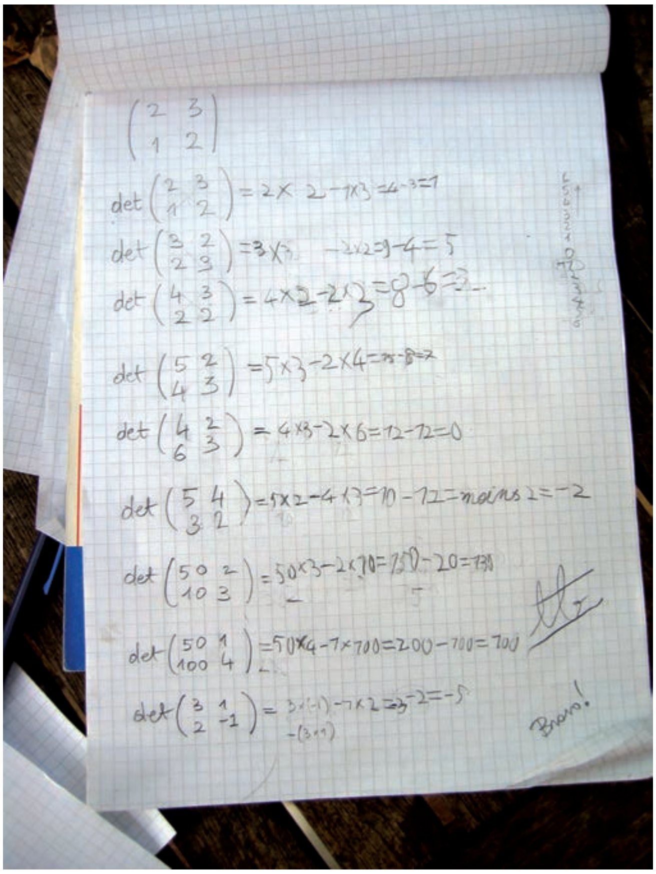
But if you first tell the child what a two-by-two matrix is a mysterious box with four numbers, from which one can extract so much information! – especially if you spice it up by saying that it is a topic of your lectures at the first year of university, and you tell them the rule to compute its determinant, I'm pretty sure that any child will play at "Let's Compute Two-by-two Determinants" for quite a while!

And if the confinement lasts longer, why not try computing the product of two-by-two matrices? And the inverse?

I think any operation with matrices is really fun!

With this trick, I once kept my son and a friend, both six years old, busy for half an afternoon. I started with matrices with small numbers (1 to 5), and after half an hour the children asked me for bigger numbers because it was "too simple". I had to insert some 50 and 100 to make it fun enough! I attach two pictures to show its success.

[1] High school in France, age from 15 to 18.
[2] https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-046j-design-and-analysis-of-algorithms-spring-2015/lecture-videos/lecture-3-divide-conquer-fft/



In the next days they asked me for more, and I moved to the product and to the inverse of matrices.
A short time after, I opened the new fantastic world of "Solving Equations in the Unknown x": find x knowing that $x+2=3$, or $2x=6$, or even $2x+1=5$. They truly adored it!

Can you predict the answer when I asked my son, during a long annoying trip by car, "Can you find all x such that x times x is equal to x?" He found three solutions, yet his answer was correct: "0, 1 and infinity"!

And can you predict the reaction when I proposed enlarging the set of numbers to some totally unreal, imaginary numbers (the complex numbers of course)? I think my son was around 8 when I dared.

I saw his full attention on this new trick, his ears wide open and his eyes shining :-) Try it!

3) Finally, in the same age range, my son simply fell in love with the representation of base-2 numbers on one's fingers. Finger up, finger down, he learned to count in base-2 at the speed of light! The game is then to be the quickest to count up to 31 without making any mistakes. And even higher, with the second hand. Of course he always won!

And now he knows the powers of two up to… well, I don't know, because I don't know so many powers of two ;-)

Vladimir Salnikov

Dear Alessandra!

Thanks for sharing the stories – I liked them a lot.
I think I'll give the determinants a try. Eventually explaining areas as well (we are geometers after all). Daria is precisely 6 now, and is obviously bored by the usual counting stuff. I guess that would bring some excitement to my kindergarten self-isolation :-)

Greetings to the family!
Ciao, Vladimir

Alessandra Frabetti

Ah, Daria is at the perfect age! :-)

About areas, sure! I had two more experiences with Adrien, inspired by areas and geometry, still around 6–8 years old.

The second one can be done with even younger children if you skip the explanations on the dimension and on fractions. I presented it to some *Fête de la Science* in Lyon and in Paris, for children of the Ecole Maternelle.

1) The game "What's the Dimension?"
First, I asked Adrien to draw segments of length 1, 2, 3, etc. and I asked him what the length was (rhetorical question).

Then I asked him to draw squares of side 1, 2, 3, and I asked him what the area was (the number of little squares).

Then I asked him to draw cubes of side 1, 2, 3, and I asked him what the volume was (the number of little cubes).

Finally, I drew a table

side	area	volume
1	1	1
2	4	8
3	9	27

and I asked him to guess the line with side N (any length): the exposant D of the power N^D is called the 'dimension' of the drawn object. The dimension of an object counts the number of important 'thicknesses' of the object.

One can remark that the 'boundary' or 'contour' of a square (of dim 2) is composed of segments (of dim 1), and that the 'boundary' of a cube (dim 3) is composed of squares (of dim 2)!

You can also push to the 'boundary' of a segment (of dim 1), which are two points: let's call them of dimension 0!

To go from dimension D to dimension D-1 is equivalent to slimming down one thickness until it becomes 'negligible', insignificant, with respect to the others.

Then we played "What's the Dimension?" everywhere!

For instance, at lunch: the tablecloth is clearly of dimension 2, and we can view a very thin plate as a 2-dim object, and a very thick one as a 3-dim one. What about an empty glass? What is the boundary of a glass?

What about the dimension of the water inside a glass full of water?

What is the dimension of an apple? That of a crêpe (a think pancake)? That of a single piece of spaghetti? What is the dimension of a cat?

A joke: a material is 'liquid' if it takes the form of its container. My brother showed me that a cat is liquid! In fact, as it is well known, it takes on the shape of any container :-)

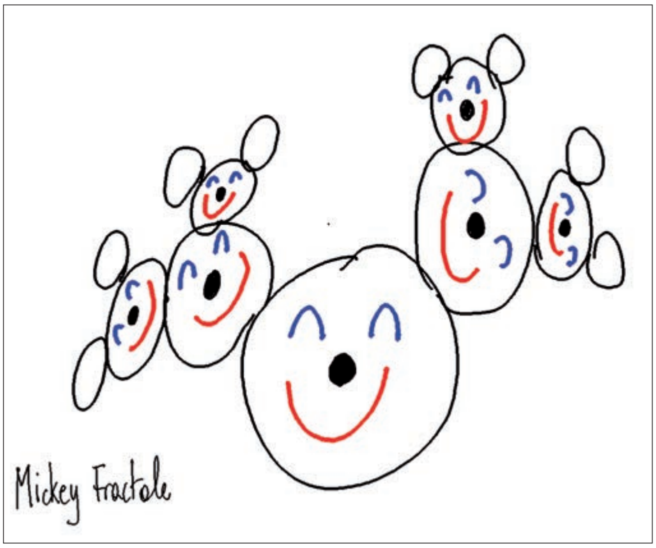
2) The drawing "Mickey Fractal"
Once the 'integer dimension' is well established in the child's mind, and the child is comfortable with fractions, one can go further. Usual objects have integral dimension, but there exist objects with non-integer dimension: fractals!

The trick of fractals is that they are plounged in a D-dimensional space in a crazy way, so that their dimension is not D (they are sensibly thinner than the whole space) but it is bigger than D-1 because they occupy too much space by zigzagging here and there at any point: none of the D-space's thicknesses can be neglected!

For instance, Norway's coast full of fjords is a fractal coast on the Earth surface.

In 2007 I invented a Christmas card called the Fractal Christmas Tree for Adrien's class, to be constructed with two sheets of paper with different colours, a pencil, scissors and glue. I then proposed it to some Festival of Science and it is now well known, you can find the instructions to build it online, cf. at https://eduscol.education.fr/maths/actualites/actualites/article/carte-de-voeu-fractale.html

I then invented another easy fractal for very small children, in order to understand the procedure of 'doing the same action in a smaller size': Mickey Fractal.



Start with Mickey's face: a big circle with a nose, two eyes and a (smiling) mouth, together with two big round ears.

Then imagine that each of Mickey's ears becomes a smaller Mickey: draw again the nose, the eyes, the mouth and the two ears, and so forth, cf. the joint drawing.

Of course you can imagine plenty of easy fractals of this type, and there are thousands of pictures online.

Alessandra Frabetti (frabetti@math.univ-lyon1.fr, http://math.univ-lyon1.fr/~frabetti/) is an Associate Professor at the Mathematics Department of the University Lyon 1. Her main research interests are algebraic methods in quantum field theory, renormalization Hopf algebras, combinatorial groups, operads.

Vladimir Salnikov [vladimir.salnikov@univ-lr.fr, https://www.vladimir-salnikov.org/] is a researcher at CNRS, La Rochelle University, France. His scientific interests are graded and generalized geometry, dynamical systems, applications to theoretical physics and mechanics. He is also involved in various science outreach activities, including the Young Mathematicians' Column of the EMS Newsletter.

Laura P. Schaposnik [schapos@uic.edu, https://schapos.people.uic.edu/] is an associate professor at the University of Illinois at Chicago, USA. Her scientific interests are integrable systems and decorated bundles, and their applications to mathematical physics. She is also involved in various science outreach activities for young women.

Faculty Position in Applied Mathematics

at the Ecole polytechnique fédérale de Lausanne (EPFL)

The School of Basic Sciences (Physics, Chemistry and Mathematics) at EPFL seeks to appoint a Tenure-Track Assistant Professor of Applied Mathematics. We have a particular interest in outstanding candidates whose research centers on the mathematical foundations of learning, with emphasis on: approximation theory, high dimensional statistics, applied analysis and probability.

We expect candidates to establish leadership and strengthen the EPFL's profile in the field. Priority will be given to the overall originality and promise of the candidate's work over any particular specialization area.

Candidates should hold a PhD and have an excellent record of scientific accomplishments in the field. In addition, commitment to teaching at the undergraduate, master and doctoral levels is expected. Proficiency in French teaching is not required, but willingness to learn the language expected.

EPFL, with its main campus located in Lausanne, Switzerland, on the shores of lake Geneva, is a dynamically growing and well-funded institution fostering excellence and diversity. It has a highly international campus with first-class infrastructure, including high performance computing.

As a technical university covering essentially the entire palette of engineering and science, EPFL offers a fertile environment for research cooperation between different disciplines. The EPFL environment is multi-lingual and multi-cultural, with English often serving as a common interface.

Applications should include a cover letter, a CV with a list of publications, a concise statement of research (maximum 3 pages) and teaching interests (one page), and the names and addresses (including e-mail) of at least three references.

Applications should be uploaded (as PDFs) by **September 30, 2020** to:

https://facultyrecruiting.epfl.ch/positiondetails/23691269

Enquiries may be addressed to:
Prof. Victor Panaretos
Chair of the Search Committee
E-mail: direction.math@epfl.ch

For additional information, please consult www.epfl.ch, sb.epfl.ch, math.epfl.ch

EPFL is an equal opportunity employer and family friendly university.
It is committed to increasing the diversity of its faculty.
It strongly encourages women to apply.

Faculty Position in Mathematics

at the Ecole polytechnique fédérale de Lausanne (EPFL)

The School of Basic Sciences (Physics, Chemistry and Mathematics) at EPFL seeks to appoint a Tenure-Track Assistant Professor of Mathematics. We seek outstanding candidates with research interests in the theory of dynamical systems, broadly construed. Indicative areas of interest include, but are not limited to: algebraic/analytical dynamics, dynamics and geometry, ergodic theory, locally homogeneous spaces.

We expect candidates to establish leadership and strengthen the EPFL's profile in the field. Priority will be given to the overall originality and promise of the candidate's work over any particular specialization area.

Candidates should hold a PhD and have an excellent record of scientific accomplishments in the field. In addition, commitment to teaching at the undergraduate, master and doctoral levels is expected. Proficiency in French teaching is not required, but willingness to learn the language expected.

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E-mail: direction.math@epfl.ch

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Armenian Mathematical Union – History and Activity

Yuri Movsisyan (Yerevan State University, Armenia), President of the AMU

The evolution of mathematics in Armenia dates back millennia. The first indications of this were the ancient Urartian cuneiform inscriptions, particularly rich in algebraic and geometric regularities.[1,2] For instance, Urartian construction workers used the golden ratio relationship between IX–VI BC before it was utilised for the facade of the Parthenon by the Greek architect and sculptor Phidias.

Interest in mathematics in Armenia grew significantly in the Early Middle Ages thanks to Anania Shirakatsi (7th c. AD), considered to be the father of exact and natural sciences in Armenia. His most valuable contribution is the textbook on Arithmetic, currently being held at the Museum of Ancient Manuscripts Matenadaran. Shirakatsi's 1400th anniversary entered the UNESCO list of important anniversaries in 2012. Other notable influencers of the time were Leo the Philosopher (9th c.), Grigor Magistros (11th c.), Hovhannes Sarkavag (11–12th cc.) and Nikoghayos Artavazd (14th c.).

In more recent history, Armenian mathematicians began forming an association during the short-lived First Republic in the year 1920. The organiser of the first mathematical association, and its first president, was a professor of Yerevan State University, Ervand Kogbetlianz.

Ervand Kogbetliantz (far left) at the International Congress of Mathematicians, Zürich 1932.

Koghbetlianz's emigration to France slowed the activity of the association, until it was reinvigorated in the 1940s and 1950s by the academics Artashes Shahinyan, Mkhitar Jrbashyan and Sergey Mergelyan.

During the 1960s to 70s, the list of academics grew and incorporated Alexander Talalyan, Rafayel Aleksandryan, Norayr Arakelian, Anry Nersessian, Ruben Ambartzumian and Rom Varshamov.

From left to right: Artashes Shahinyan, Mkhitar Jrbashyan and Sergey Mergelyan.

From left to right: Rafayel Aleksandryan, Norayr Arakelian and Alexander Talalyan.

The theory of functions of a complex variable, real analysis, differential equations, functional analysis and equations of mathematical physics were actively advancing in Armenia during these years. These were followed by the development of probability theory, integral geometry, algebra and mathematical logic, and discrete mathematics in the 1970s to 80s.

Following Armenia's independence from the Soviet Union in 1991, a group of young mathematicians formalised the association into the Armenian Mathematical Union (AMU), registering it at the Ministry of Justice with elected members of the Council and an elected president of the Union. The first elected Council was comprised of Sergey Antonyan, Samvel Apresyan, Norayr Arakelian, Victor Arzumanian, Bagrat Batikyan, Grigor Barsegian, Gegham Gevorgyan, Samvel Dalalyan, Norayr Yengibaryan, Vanik Zakaryan, Alexander Talalyan, Sargis Hakobyan, Hrachik Hayrapetyan, Tigran Harutyunyan, Felix Harutyunyan, Haik Ghazaryan, Valeriy Martirosyan, Vardan Martirosyan, Yuri Movsisyan, Karen Yagjyan, Vladimir Yavryan, Boris Nahapetyan, Romen Shahbaghyan, Faizo Shamoyan, Arthur Sahakyan, Shvayts Sahakyan, Sokrat Simonyan, Ashot Vagharshakyan and Victor Ohanyan. Many of these inaugural Council members shaped the leadership of the AMU in the years to come. The following is a complete list of the AMU's elected presidents since its formal registration:

[1] Schulz, F. E. (1840). Mémoire sur le lac de Van et ses environs. *JA* 9: 257–323

[2] Salvini, M. (2008). *Corpus dei testi urartei, 1: Le iscrizioni su pietra e roccia* (Documenta Asiana 8). CNR/Istituto di studi sulle civiltà dell'Egeo e del Vicino Oriente, Rome.

1991 – Alexander Talalyan
1994 – Norayr Arakelian
1997 – Ruben Ambartzumian
2001 – Norayr Yengibaryan
2004 – Hrachik Hayrapetyan
2011 – Tigran Harutunyan
2017 – Yuri Movsisyan

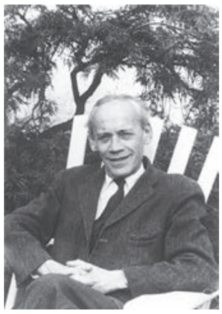
The mission of the Armenian Mathematical Union is to promote mathematical sciences, especially among the young generation, to establish and strengthen contacts with colleagues from around the world and with international professional organisations in the area of pure and applied mathematics and computer science. Various committees function under the AMU, the most important of them being the Education Committee, which focuses on teaching curricula, methods and the quality of textbooks.

The AMU has been a member of the International Mathematical Union since 1993 and a member of the European Mathematical Society since 2016. It is also one of the twelve members of the Silkroad Mathematics Center (SMC) that was established in 2016 under the auspices of the China Association for Science and Technology. The Union collaborates directly with a number of universities and research institutes in Armenia and abroad. It convenes general meetings, hosts discussions with visiting eminent scholars, and organises periodic thematic and general annual conferences.

The current Council of the AMU includes a number of international members, among them:

S. Adian (Steklov Mathematical Institute of RAS)
A. Arutunov (RUDN University)
S. Antonyan (Universidad Nacional Autónoma de México)
A. Basmajian (City University of New York, Hunter College)
L. Beklaryan (Central Economics and Mathematics Institute of RAS)
L. Budaghyan (University of Bergen)
S. Grigoryan (Kazan State Power Engineering University)
A. Hajian (Northeastern University, Boston)
S. Mkrtchyan (Rochester University)
R. Mnacakanov (West Virginia University)
A. Petrosyan (Purdue University)
V. Pambuccian (Arizona State University)
A. Sergeev (Steklov Mathematical Institute of RAS)
H. Shahgolyan (KTH Royal Institute of Technology)
M. Toomanian (University of Tabriz)

Since 2001, members of the AMU have presented the Emil Artin Junior Prize in Mathematics to former students of Armenian educational institutions for their outstanding contributions in algebra, geometry, topology, and number theory. The award is announced in the *Notices of the American Mathematical Society*, and carries a cash award of US$1,000. It is named after Emil Artin, a leading 20th century mathematician of Armenian descent, and covers the fields in which Artin made major contributions. The most recent 2020 recipient of the Prize was Levon Haykazyan of Oxford Asset Management. Haykazyan was chosen for his paper "Spaces of Types in Positive Model Theory", published in the *Journal of Symbolic Logic* in 2019.



Emil Artin.

Dedicated to Emil Artin's 120th anniversary, the AMU organised a large International Conference, celebrating his scientific heritage. The conference took place in May–June 2018 in Yerevan, Armenia, and was attended by renowned mathematicians from around the world. Concurrently, an International School in Mathematics was organised for students (mostly from around the region), featuring lectures by invited professors from the United States, Germany, Italy and the Netherlands.

Efim Zelmanov (right) speaking at the Emil Artin International Conference in Yerevan, Armenia, 2018.

Other recent conferences organised by the AMU were: the International Conference Dedicated to the 90th Anniversary of Sergey Mergelyan (May 2018); the International Conference Dedicated to the 100th Anniversary of Mkhitar Djrbashian (October 2018); the International Conference Dedicated to the 100th Anniversary of the Yerevan State University (October 2019); and the Scientific Conference Dedicated to the International Day of Mathematics (14th March 2020).

Prof. Yuri Movsisyan [movsisyan@ysu.am] is the current president of the AMU and the honorary head of the Department of Algebra and Geometry at the Yerevan State University. His research interests include algebra, mathematical logic and discrete mathematics.

ICMI Column

Jean-Luc Dorier (Université de Genève, Switzerland)

Renewal of the ICMI Executive Committee

Despite the fact that ICME14 has been postponed (see below), the General Assembly of ICMI, which was supposed to take place just before the opening of ICME14 in Shanghai, was organised electronically over a 24h period of time between the 13th and 14th of July. One of the main issues was the election of the new ICMI Executive Committee by the 84 ICMI country representatives.

This new committee will be in service from the 1st of January 2021 for 4 years. It is composed of:

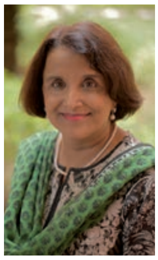
President: Fredrick K.S. LEUNG (Hong Kong, SAR, China)
Secretary general: Jean-Luc DORIER (Switzerland)
Vice-presidents: Merrylin GOOS (Australia/Ireland)
Anjum HALAI (Pakistan)
Members at large: Marta CIVIL (USA)
Patricio FELMER (Chile)
Mercy KAZIMA (Malawi)
Nùria PLANAS (Spain)
Susanne PREDIGER (Germany)

Moreover, the ex-president of ICMI, Jill ADLER (South Africa), as well as the president and the secretary general of IMU, Carlos E. KENIG (USA) and Helge HOLDEN (Norway), are members ex-officio.
Paolo PICCIONE (Brazil) is the liaison person for IMU.


Frederick K.S. Leung


Jean-Luc Dorier


Merrylin Goos


Anjum Halai


Marta Civil


Patricio Felmer


Mercy Kazima


Nùria Planas


Susanne Prediger


Jill Adler


Carlos E. Kenig


Helge Holden


Paolo Piccione

ICME-14 postponed to July 2021

Due to the global pandemic caused by COVID-19, the ICMI and the ICME14 organising committee have decided, after careful discussion and consultation, to postpone the original session and hold ICME14 from the 11th to the 18th of July 2021 at the East China Normal University (Putuo campus).

Relevant arrangements are summarised as follows:

1. All standard practices continue to be valid (including registration, contributions, payment, TSG paper and posters review and acceptance, etc.)
2. The registration system will remain open throughout this state of emergency.
   The address for registration is https://reg.icme14.org/login.
3. The payment channel will reopen on the 1st of September 2020.
   Fully paid before (inclusive) the 31st of March 2021, RMB 3500;
   Fully paid between the 1st of April and the 31st of May 2021, RMB 3800;
   Paid on and after the 1st of June 2021, RMB 4000;
   Accompanying person's registration at any time, RMB 1000.
4. The submission channel for TSG papers and posters will reopen on the 1st of August 2020. We consider that in one year's time, researchers are likely to refine or come up with new research advances and results, or want to fine-tune their paper to a higher quality. Therefore, a participant whose poster has been accepted is allowed to withdraw it and re-submit a paper. For detailed operation instructions, please see updates on our website.
5. New proposals for organising Discussion Groups and Workshops are also welcome. Please send emails to DG@icme14.org and WS@icme14.org respectively to submit a proposal.
6. Application for the Solidarity Fund will reopen on the 1st of November 2020. The new round of applications is only open to attendees who have never submitted an application before. For an application form, please see https://www.icme14.org/static/en/news/60.html?v=1590631349483.

Applicants who submitted application forms last year are requested not to submit them again.
However, if:

(1) the grade of your contribution has changed, e.g. from poster to paper; or
(2) your status has changed, e.g. from normal attendees to invited participants, including Invited Lecturers, team members of Plenary Panels, Survey Teams and Topic Study Groups, or you have become contributors or proponents of TSG, National Presentations, Discussion Groups, Workshops or Thematic Afternoons, please inform us by emailing to grant@icme14.org with your name and ID number included.

7. Registered participants are requested to send a confirmation letter of your attendance to ICME-14 in 2021 to reg@icme14.org before the 31st of October (instead of the 30th of June as in our earlier announcement).

For more details and latest news please stay tuned to the updates on www.icme14.org.

ERME Column

Peter Liljedahl (Simon Fraser University, Burnaby, Canada), Stanislaw Schukajlow (University of Münster, Germany) and Jason Cooper (Weizmann Institute of Science, Israel)

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME), holds a biennial conference (CERME), in which research is presented and discussed in Thematic Working Groups (TWG). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME Thematic Working Group 8 – Affect and Teaching and Learning of Mathematics

Thematic Working Group rationale

We would like to begin with a quote by G. Pólya, a Hungarian mathematician, who wrote *How to Solve It* (1945) – his seminal work on problem solving:

> *"Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. … If he [the teacher] fills his allotted time with drilling his students in routine operations, he kills their interest…"* (Pólya, 1945, p. V)

This quotation demonstrates that affective concepts such as emotions and interest were identified as important antecedents of mathematical thinking and problem solving even in the early days of mathematics education. Moreover, improving students' positive affective experience was suggested as essential for teaching mathematics. However, despite the importance of affective factors for learning and teaching mathematics, and for problem solving and mathematical thinking, it took a long time until mathematics educators began to focus on affect as a research domain. A considerable attempt at systematising the field was done by Douglas McLeod in the early 1990s (1992). He proposed important characteristics of affect, such as stability and intensity of affective variables, and made a comprehensive review of research results on affect, pointing out the lack of theories in research on affect. Since then, the research on affect has received more and more attention and has been featured in a number of keynotes at conferences and in high-impact journals in mathematics education.

Discussions about the importance of affect in TWG8 (founded in 2003) have contributed significantly to our understanding of affect and its role in teaching and learning mathematics. McLeod (1992) identified affect as including a wide range of concepts including beliefs, attitudes, values, goals, needs, motivation, identity, self-esteem and emotions. Discussions on similarities and differences between these affective concepts is an important part of work of TWG8 at each conference. Because of conceptual overlap between affective variables and origins of affective concepts in different theories, clarifying the relations of affective constructs to each other is demanding work. Stability (stable trait vs. unstable state), intensity (high vs. low), or valence (positive, negative, neutral) are the main characteristics that can be assigned to each affective variable. Another topic of high relevance in TWG8 is the evaluation of affective measures. We have explored, and continue to explore, the advantages and shortcomings of traditional approaches, such as questionnaires and student interviews, and innovative approaches, such as eye-tracking, measures of heart rate or skin conductance as indicators of students' affect.

Most relevant for the readers of this newsletter might be the relation between affect, mathematical thinking, and the learning and teaching of mathematics. Recent research has demonstrated that there is a feedback loop between affect and mathematical thinking, and has stressed the importance of teachers and parents in the development of students' emotions, motivation and other affective concepts.

Affect in early childhood and in schools

We do not know much about the development of affect in early childhood and in primary school. Findings from research on anxiety indicate that some young children report anxiety around mathematics even at the very start of formal schooling. A negative relationship between maths anxiety and mathematical performance has been identified in students as early as first grade. Children with early mathematical anxiety receive lower grades, and lower grades in turn increase their mathematical anxiety (Carey et al., 2017). This can follow students all the way to upper secondary school, where low self-confidence, low enjoyment and high levels of boredom have been identified as reasons for not continuing with mathematics in university.

One of the factors that has been shown to contribute to the development of anxiety is students' experience with teachers and teaching methods. Although early research on mathematical affect was more concerned with identifying the sources of such anxiety, more recent research has been looking at ways to mitigate anxiety. Such research has shown that teachers' positive affect toward mathematics increases students' positive affect towards mathematics. For example, teachers' enthusiasm and enjoyment of teaching mathematics affected students' interest and enjoyment of mathematics in lower secondary school. Likewise, prompting students to develop, compare, and contrast multiple solutions for a problem was shown to increase students' enjoyment, autonomy and perceived competency during mathematics classes, and improved their interest in mathematics.

Affect in universities

The role of affect in tertiary education is of great interest in TWG8. We all know that students struggle in their first-year mathematics courses and, as a result, many students drop out of mathematics. Many of these students are accustomed to being successful in mathematics, and to struggle is unexpected for them and creates an onset of negative affect – "*… for the first two years I was in a permanent state of anxiety and distress"; "I think that my experience at the Bachelor in mathematics left me with less confidence in my ability to study"; "I remember [of that period] just a lot of tears"* (Di Martino & Gregorio, 2017). One reason for these affective difficulties is a big difference between mathematics in upper secondary and tertiary education. Whereas in upper secondary school mathematics is a subject in which students apply mathematical procedures and solve word problems, in university mathematics is a subject of formal language, justification and proof. Consequently, many students do not recognise their favourite school subject and develop motivational problems.

In order to support students through this transition, many universities have started various intervention programmes. These programmes often begin with preparation courses that aim at refreshing students' prior knowledge and introducing them to the main proof techniques. Another intervention is the introduction of learning centres, which offer students the opportunity to share their thoughts while working on lectures and sample problems under the guidance of teaching assistants. Although these types of interventions reduce the failure and dropout rates of students in first year mathematics courses, we do not know much about the effects of such programmes on students' emotions and motivation. Research on instructional practices that support students' perceived competence and autonomy, increase students' motivational beliefs and improve their positive emotions, is still developing. However, an approach which has shown promise is to focus on the individual development of students' understanding ("I improved my knowledge in the last week") instead of their performance ("I got a higher score last week").

In this overview, we have offered a brief description of the development of affect and have focused on topics that have been discussed in TWG8 in recent years. We welcome in our group mathematicians, teacher educators and researchers to share our views on affect and its relation to the teaching and learning of mathematics at CERME 12 in 2021.

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Transforming Scanned zbMATH Volumes to LaTeX: Planning the Next Level Digitisation

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1 LaTeX conversion as the next essential step for math digitisation

Since the advent of the internet, mathematicians have pursued the vision of a comprehensive, open and accessible digital collection of mathematical resources. The International Mathematical Union (IMU) supports this goal under the brand of the Global Digital Mathematics Library, and the diverse activities are fostered by the International Mathematics Knowledge Trust (IMKT). Obviously, this aim is not likely to be achieved in the near future – several technical and legal obstacles need to be overcome. The extent of the mathematics literature alone has been estimated at > 120 million pages of very diverse status [3]. However, the first technical step of digitisation – namely, the scanning of the existing mathematics literature – has been mostly successful due to public and private efforts: about 60 % of the pages with mathematics literature are now available in some digital form [3]. However, scanned files have many limitations – although they can be read by most humans, their content is neither easily searchable nor machine-processable (e.g., for content analysis). Since mathematical content is intrinsically linked to formulæ, it would be highly desirable to have LaTeX sources available. Unfortunately, this is not even the case for most digitally born documents (with only a few exceptions, most notably the arXiv) – currently, less than 3 % of all maths pages are available as LaTeX [3]. Transforming scans to LaTeX is still challenging and costly. When the *Jahrbuch über die Fortschritte der Mathematik* was digitised, it required massive investments into typesetting formulæ (as well as correcting OCR errors). During the last years, MathOCR technology has made considerable advances, but it is still far from being seamlessly applicable in a scalable way. We will report here on the current state of the art for MathOCR with a view towards transforming the zbMATH volumes 1–529 (currently mostly just available as scans) into LaTeX. Due to the diversity of content and formulæ in zbMATH, as well as of the types through the decades, we believe that this could serve as a meaningful representative model for the full corpus of mathematical literature.

2 Hoards of scanned reviews in zbMATH

Currently, zbMATH, and the aforementioned volumes in particular, includes more than 800 000 reviews and abstracts that exist as scanned images alone. Those items are distributed over 250 000 pages. For today's zbMATH users, the usage experience for those items is not satisfying. For example, the fonts are hard to read, one cannot search for text, and copy and pasting of text is not possible. Moreover, the text is inaccessible for people with disabilities and also for information retrieval systems. Consequently, those reviews do not occur in recommendations and are not considered while scanning new articles for plagiarism. To improve this situation, we have manually transcribed about 15 000 abstracts over the past years. These reviews are now available to zbMATH users in the known digital form. Based on this experience, we estimate that the effort for manual retro-digitisation is immense. Outsourcing the LaTeX processing part would cost about half a million Euro, and will take several years, and is thus infeasible. However, recent advancements in computer science, in particular deep learning, might drastically reduce the effort. In this paper, we will discuss our plans to use modern deep learning approaches to digitise the rest of the scanned images with reduced manual effort and discuss the challenges we foresee in this context.

3 Ingredients for digitisation

In this section, we will elaborate on the building blocks to retrodigitise past reviews before we discuss an example in the next section.

What is digitisation about?

The exponentially increased options for storing, transmitting, processing, linking, interpreting and reproducing information through digitisation have also set in motion fundamental transformation processes in science. Digitisation enables an open culture of innovation, in which data, information and ideas can be freely introduced and exchanged. If scientific literature can be digitally accessed from anywhere, the question arises as to whether printed research literature can be retrospectively digitised and made usable. For example, publications can be recorded using a scanner so that its software generates an image file in which the image is displayed as a raster graphic. A disadvantage of this form of digitaliation is that the quality of the image file depends on the scanning hardware and the paper of the original document, and often due to the book form, the text lines are not displayed straight. On the other hand, the font size and line spacing of printed or scanned articles are usually minimised due to the number of pages, cf. Figure 1. Therefore, digitised publications are often associated with poor readability and very limited further processing. Research literature in digital form is computer-generated liter-

Da für jedes ganze Wertepaar (m, n) von einer gewissen Stelle an die Teilnenner größer sind als die Teilzähler, ist der Wert nach bekannten Sätzen irrational. Da aber $\operatorname{tg} \frac{1}{4}\pi = 1$ ist, muß π irrational sein. *O.-H. Keller.*

Erdös, P.: On the irrationality of certain series. Nederl. Akad. Wet., Proc., Ser. A **60**, 212—219 (1957).

The author shows that the series

$$\sum_{n=1}^{\infty} t^{-\varphi(n)} \text{ and } \sum_{n=1}^{\infty} t^{-\sigma(n)} \quad (t = 2, 3, 4, \ldots)$$

are irrational; here φ and σ are Euler's function, and the sum of divisors. The proof depends on a general Lemma 1 on irrational series, and on these properties: (1) There are only $O(x)$ integers n satisfying $\varphi(n) \leq x$ (or $\sigma(n) \leq x$). (2) There are only $o(x)$ integers $n \leq x$ for which $\varphi(k) = n$ (or $\sigma(k) = n$) has a solution k. — Lemma 1 is obtained as a special case of the more general Lemma 4: Let $\{a_k\}$ and $\{b_k\}$ be two infinite sequences of integers ≥ 0 such that $a_k \leq k^s$ and $b_k \leq k^s$ (s a constant). Let $f(n)$ and $g(n)$ denote the number of positive a_k and b_k with $1 \leq k \leq n$; assume that $f(n) \to \infty$ as $n \to \infty$. Let there exist an infinite sequence of integers $\{m_i\}$ such that

$$\sum_{k=1}^{m_i} (a_k + b_k) < c_1 m_i, \; f(m_i) = o(m_i), \quad g(m_i) = o\left(\frac{m_i}{\log m_i}\right).$$

Finally assume there exists a constant $c > 0$ as follows: When i_1 and i_2 are consecutive suffixes with $b_{i_1} b_{i_2} > 0$, and when $i_1 + c\,x < i_2$, then there is a k with $i_1 + x < k < i_2 + c\,x$ such that $a_k > 0$. Under these conditions all series $\sum_{k=1}^{\infty} \frac{a_k \mp b_k}{t^k}$ $(t = 2, 3, 4, \ldots)$ are irrational. — From Lemma 4, the author deduces Theorem 2: Let $\{n_k\}$ be a strictly increasing sequence of positive integers such that $\limsup_{n\to\infty} \frac{n_k}{k^l} = \infty$ ($l > 0$ a constant). If $t \geq 2$ is an integer, then the series $\sum_{k=1}^{\infty} t^{-n_k}$ cannot have as its sum an algebraic number of degree $\leq l$. *K. Mahler.*

Figure 1. Excerpt from Zentralblatt für Mathematik, Series 79, page 74. Zbl 0079.07401 [1]

ature which, in contrast to digitised literature, is existentially dependent on the digital medium [8, p. 15–28]. It is characterised by at least one of the specific features of digital media: interactivity, intermediality and staging [7, p. 3–21]. Accordingly, digital literature is described as hypertextual and multimedia rather than as a self-contained work. If we transfer this to zbMATH reviews and abstracts, this means that formulæ from the publication can be copied into Word/LaTeX for further processing. An additional advantage of digital research literature is the adaptation of page and font size as well as fonts to the respective end device so that the reader is provided with better usability. A manual full-text entry, i.e., typing the complete text and formulæ, is hardly conceivable due to the high personnel costs and the high error rate.

Associate image and metadata

A first problem that arises when digitising the scanned images of zbMATH reviews and abstracts is that they are not yet separated. Namely, there is usually more than one review on a scanned page and a review can start on one page and end on another. Thus, we first have to solve a special segmentation problem where we have to split the scanned pages into individual reviews and associate them to their corresponding zbMATH entry. We know that the review of an article starts with the author names and the title of the given article and ends with the name of the reviewer. The author names, title and name of the reviewer of an article are metadata in our database. Thus, all this information is already given in a digital form and known in advance. This is very different from other segmentation problems in the field of OCR, as usually the next section is unknown.

In recent years, machine learning approaches, in particular deep recurrent and convolutional neural networks were successfully applied to document segmentation problems, see for example [4]. We investigate how these methods can be applied to our setting and how we can make use of the special metadata information we have.

Convert images to LaTeX code

According to our research, there are two major commercial services that specialised in the conversion of images of mathematical documents to LaTeX code. The Infty project from Japan is a research-driven project that has been developing technology to convert scanned images to LaTeX code since 1998. A new alternative is the MathPix project, which provides an online API as well as a desktop app to convert images of mathematical formulæ to LaTeX source code. While the desktop app was originally designed to convert individual formulæ, a recent API addon was published to allow conversion of whole pages at once. In contrast to the Infty Reader, Mathpix does not link an extracted fragment to the position in its image. See Section 3 for an exemplary comparison of the two selected commercial services.

Assess conversion quality

After converting the scanned images to LaTeX code, we have to assess the quality of our results. A usual measure in OCR would be the character error rate (CER), which makes sense for plain text. However, in a mathematical context it is not clear how to measure the similarity of two given formulæ in LaTeX. We do not need exact character matches as long as the semantic meaning of a formula stays unchanged.

In the literature, several metrics are used to evaluate mathematical formula recognition algorithms. One possible metric is the recognition rate for complete expressions or individual symbols. This metric states whether two expressions or symbols are the same or different. There are also more refined metrics which look at different kinds of errors and weigh them or compare the symbol layout trees of mathematical expressions.

Sain, Dasgupta, Abhishek and Garain develop in [5] a method that compares the structure of two mathematical formulæ given in MathML. They convert the given formulæ into ordered trees and measure their similarity by the tree edit distance of the associate trees.

Another approach captures both the individual symbols of an expression and its structure as a bipartite graph [9]. Then different metrics are defined to measure the similarity of two bipartite graphs.

In our setting, we need metrics that directly compare LaTeX strings. We investigate the metrics mentioned above and compare them. If they turn out to be unsuitable for our purpose, we develop a new metric.

New search opportunities and TDM tools in copyright review

Our planned encompassing digital development of zbMATH opens up extensive research and use opportunities for scientists. The new conditions, introduced in 2019 and 2020, of the EU Copyright Directive and the German Copyright Law have been integrated into this development. In this respect, we want to conduct a legal investigation into the concrete meaning of new legal provisions for Text and Data Mining (TDM), deep learning techniques and digital research for the digital expansion of the directory of mathematical literature. With the investigation, we intend to make clear how the current approach to copyright law will impact open access transformation projects. In addition, editors, journal managers and

Listing 1: Infty

```
1   The author shows that the series
2
3   $\displaystyle \sum_{n\Rightarrow1}^{\infty}
       t^{-\varphi(n)}$ and $\displaystyle
       \sum_{n=1}^{\infty} t^{-\sigma(n)} (t=2,3,4,\ .\ .\
       .)$
4   are irrational; here $\varphi$ and $\sigma$ are
       Euler's function, and the sum of divisors.
5   The proof depends on a general Lemma 1 on irrational
       series, and on these properties:
6   (1) There are only $O(x)$ integers $n$ satisfying $
       \varphi(n)\leq x$ (or $\sigma(n)\leq x$).
7   (2) There are only $o(x)$ integers $n\leq x$ for which
       $\varphi(k)=n$ (or $\sigma(k)=n$) has a solution $
       k$. --
8   Lemma 1 is obtained as a special case of the more
       general Lemma 4:
9   Let $\{a_{\alpha}\}$ and $\{b_{k}\}$ be two infinite
       sequences of integers $\geq 0$ such that $a_{k}\leq
       k^{s}$ and $b_{k}\leq k^{s}$ ($s$ a constant).
10
11  Let $f(n)$ and $g(n)$ denote the number of positive $
       a_{k}$ and $b_{k}$ with $1\leq k\leq n$; assume
       that $ f(n)\rightarrow\infty$ as $
       n\rightarrow\infty$.
12  Let there exist an infimte sequence of integers $
       \{m_{i}\}$ such that
13  $$
14  \sum_{k=1}^{m_{i}}(a_{k}+b_{k})<\mathrm{q}m_{i},\
       f(m_{i})=o(m_{i}),\ g(m_{i})=o(\frac{m_{i}}{\log
       m_{i}})\ .
15  $$
16  Finally assume there exists a constant $c>0$ as
       follows:
17  When $i_{1}$ and $i_{2}$ are conse-cutive suffixes
       with $b_{i_{1}}b_{i_{-}}. >0$, and when $
       i_{1}+cx<i_{2}$, then there is a $k$ with $
       i_{1}+x<k<i_{2}+cx$ such that $a_{\gamma}>0$.
18  Under these conditions all series $\displaystyle
       \sum_{k=1}^{\infty}\frac{a_{k}\mp
       b_{\lambda}}{t^{1}}$ $(t=2,3,4,\ .\ .\ .)$ are
       irrational.
19  $-\Gamma \mathrm{r}\mathrm{o}\mathrm{m}$ Lemma 4, the
       author deduoes Theorem 2:
20  Let $\{n_{k}\}$ be a strictly increasing sequence of
       positive integers such that $\displaystyle
       \lim_{n\rightarrow}\sup_{\infty}\frac{n_{k}}{k}=\infty$
       ($1>0$ a constant).
21  If $t\geq 2$ is an integer, then the series $
       \displaystyle \sum_{k=1}^{\infty}t^{-n_{k}}$ cannot
       have asits sum an algebraic number of degree $\leq
       1$.
```

Listing 2: Manual transcript

```
1   The author shows that the series
2   \[
3   \sum_{u=1}^\infty t^{-\varphi (n)} \text{ and }
       \sum_{n=1}^\infty t^{-\sigma(n)}
    \]
    \((t=2,3,4,...)\)
4   are irrational; here \(\varphi\) and \(\sigma\) are
       Euler's function, and the sum of divisors.
5   The proof depends on a general Lemma 1 on irrational
       series, and on these properties:
6   (1) There are only \(O(x)\) integers \(n\) satisfying \(
       \varphi(n) \leq x\) (or \(\sigma(n) \leq x\)).
7   (2) There are only \(o(x)\) integers \(n \leq x\) for
       which \(\varphi(k) = n\) (or \(\sigma(k) = n\)) has a
       solution \(k\).
8   Lemma 1 is obtained as a special case of the more
       general Lemma 4:
9   Let \(\{a_k\}\) and \(\{b_k\}\) be two infinite sequences
       of integers \(\geq 0\) such that \(a_k \leq k^s\) and \(
       b_k \leq k^s\) (\(s\) a constant).
10
11  Let \(f(n)\) and \(g(n)\) denote the number of p0ositive \(
       a_k\) and \(b_k\) with \(1\leq k \leq n\); assume that \(
       f(n) \to \infty\) as \(n \to \infty\).

12  Let there exist an infinite sequence of integers \(
       \{m_i\}\) such that
13  \[
14  \sum_{k=1}^{m_i} (a_k+b_k) < c_1 m_i,\qquad f(m_i) =
       o(m_i),\qquad g(m_i) = o\left({m_i \over \log
       m_i}\right).
15  \]
16  Finally assume there exists a constant \(c>0\) as
       follows:
17  When \(i_1\) and \(i_2\) are consecutive suffixes with \(
       b_{i_1}b_{i_2} >0\), and when \(i_1+cx < i_2\), then
       there is a \(k\) with \(i_1+x < k < i_2+cx\) such that \(
       a_k > 0\).
18  Under these conditions all series \[
    \sum_{k=1}^\infty {a_k \mp b_k \over t^k}\qquad
       (t=2,3,4,...)
    \] are irrational.
19  From Lemma 4, the author deduces Theorem 2:

20  Let \(\{n_k\}\) be a strictly increasing sequence of
       positive integers such that \(\limsup_{n \to \infty}
       {n_k \over k^l} = \infty\) \((l>0\) a constant).

21  If \(t \geq 2\) is an integer, then the series \(
       \sum_{k=1}^\infty t^{-n_k}\) cannot have as its sum
       an algebraic number of degree \(\leq 1\).
```

Listing 3: Mathpix

```
1   The author shows that the seriess
2   \[
3   \sum_{n=1}^{\infty} t^{-\varphi(n)} \text { and }
       \sum_{n=1}^{\infty} t^{-\sigma(n)} \quad(t=2,3,4,
       \ldots)
    \]
4   are irrational; here \( \varphi \) and \( \sigma \) are
       Euler's function, and the sum of divisors.
5   The proof depends on a general Lemma 1 on irrational
       series, and on these properties:
6   (1) There are only \( O(x) \) integers \( n \) satisfying \(
        \varphi(n) \leq x(\text { or } \sigma(n) \leq x) \)
7   (2) There are only \( o(x) \) integers \( n \leq x \) for
       which \( \varphi(k)=n(\text { or } \sigma(k)=n) \)
       has a solution \( k .- \)
8   Lemma 1 is obtained as a special case of the more
       general Lemma 4:
9   Let \( \left\{a_{k}\right\} \) and \( \left\{b_{k}\right\} \)
       be two infinite sequences of integers \( \geq 0 \)
       such that \( a_{k} \leq k^{s} \) and \( b_{k} \leq
       k^{s} \) ( \( s \) a constant).
11  Let \( f(n) \) and \( g(n) \) denote the number of
       positive \( a_{k} \) and \( b_{k} \) with \( 1 \leq k
       \leq n \); assume that \( f(n) \rightarrow \infty \)
       as \( n \rightarrow \infty . \)
12  Let there exist an infinite sequence of integers \( \left
       \{m_{i}\right\} \) such that
13  \[
14  \sum_{k=1}^{m_{i}}\left(a_{k}+b_{k}\right)<c_{1} m_{i},
       f\left(m_{i}\right)=o\left(m_{i}\right), \quad
       g\left(m_{i}\right)=o\left(\frac{m_{i}}{\log
       m_{i}}\right)\]
16  Finally assume there exists a constant \( c>0 \) as
       follows:
17  When \( i_{1} \) and \( i_{2} \) are consecutive suffixes
       with \( b_{i_{1}} b_{i_{2}}>0, \) and when \( i_{1}+c
       x<i_{2}, \) then there is a \( k \) with \(
       i_{1}+x<k<i_{2}+c x \) such that \( a_{k}>0 . \)
18  Under these conditions all series \(
       \sum_{k=1}^{\infty} \frac{a_{k} \mp b_{k}}{t^{k}} \)
      \( (t=2,3,4, \ldots) \) are irrational. \( - \)

19  From Lemma \( 4, \) the author deduces Theorem 2:

20  Let \( \left\{n_{k}\right\} \) be a strictly increasing
       sequence of positive integers such that \( \lim _{n
       \rightarrow \infty} \sup \frac{n_{k}}{k^{l}}=\infty
      \)\( \left(l>0 \text { a constant).
21  If } t \geq 2 \text { is an integer, then the series }
       \underset{k=1}{\stackrel{\infty}{\Sigma}} t^{-n_{k}}
    \) cannot have asits \right. sum an algebraic number of
       degree \( \leq l \).
```

Figure 2. Comparison of different approaches to infer LaTeX code from a scanned example image

project coordinators will be provided with legal information and recommendations to be taken into account in the context of new usages such as TDM.

4 An example

After having introduced the basic ingredients in the last section, we will now discuss the conversion quality based on the example shown in Figure 1. Figure 2 compares the output of Infty project[1] (left), the manual conversion (middle), and Mathpix[2] (right) for the select example. In the image, mistakes are highlighted in red, whereas alternative formatting is highlighted in orange. The begin and start delimiter, such as `$,$$,\(,\),\[,\]` as well as irrelevant grouping brakes `{}`, and `\left(\right]` combinations of brackets are shown in gray. Unfortunately, the output of Mathpix did not compile for this example, as the left and right brackets were not balanced. In particular, the opening left bracket in line 20 ends with an invisible right bracket in line 21 which is not in maths mode. Other than that the more excessive use of the left-right version of brackets by Mathpix did not create a visual difference for the selected example. For the text recognition, Mathpix recognised an additional s in the word 'series' line 1, and missed a space in 'as its' in line 21. This was also incorrectly recognised by Infty. Infty has spelling issues for the word 'infinite sequence' in line 12 and 'deduces' line 19. Moreover, Infty recognised the word 'From' as formula Γrom (line 19).

Regarding the mathematical formulæ, the human reviewer made a typo in the summation in line 3 where u was used instead of the correct identifier n. While both MathOCR solutions recognised this correctly, Infty recognised an $\Rightarrow$ instead of an $=$ sign in line 3. Both Mathpix and Infty did not correctly recognise $\limsup_{n\to\infty}$ in line 20. They did split limit and sup as separate operators. Additionally, the very last inline formula $\leq\, l$ was correctly recognised by both systems, but was falsely manually transcribed as ≤ 1. In this example, Infty had several issues with sub and superscripts

$$a_k \to a_\alpha \qquad (1.9)$$

$$c_1 \to q \qquad (1.14)$$

$$b_{i_2} \to b_{i_-} \qquad (1.17)$$

$$a_k \to a_\gamma \qquad (1.17)$$

$$b_k \to b_\lambda \qquad (1.18)$$

$$t^k \to t^1. \qquad (1.18)$$

Besides these optical differences there are several differences in the LaTeX code that generate identical or very similar output. This is different delimiters in grey as discussed before, a different encoding of spaces, different ways of switching between maths and text mode and different encoding of dots, arrows and mathematical operators such as the sum sign (cf. line 21, Mathpix). To fully normalise these issues, either a LaTeX grammar parser (like texvcjs included in mathoid [6]) is required, or the LaTeX code needs to be converted to MathML to simplify comparison by using prebuild APIs such as the math tools [2].

1 http://www.inftyproject.org
2 https://mathpix.com

5 The road ahead

We are planning to apply for grant money to eliminate the dark spot of scanned but not fully digitised reviews in zbMATH. As a supplement to zbMATH Open, we are planning to investigate the capabilities of MathOCR tools further and combine the strength of them. With well-defined evaluation metrics, we will be able to continuously improve the conversion quality and involve the community in the final human judgment of the quality. We are committed to the FAIR and open principles, and therefore we will document share and open-source our developments. Thus not only the more than 250 000 zbMATH pages will become available as LaTeX code, but also the effort required to convert from scans to LaTeX code will decrease for follow-up projects. We will pave the road for more than 100 million pages of mathematical literature that is not available as LaTeX code to eventually become digital.

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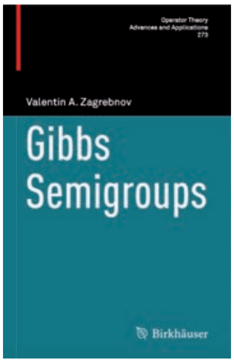
Book Reviews



Valentin A. Zagrebnov
Gibbs Semigroups

Birkhäuser, 2019
xv, 319 p.
ISBN 978-3-030-18876-4

Reviewer: Michael Th. Rassias

In the 1940s, it was understood that one-parameter semigroups of operators have direct applications to a broad spectrum of mathematical areas, including PDEs, random processes, infinite-dimensional control theory and mathematical physics, to name just a few. More specifically, in the vast majority of applications in PDEs with bounded domains, the corresponding solution semigroups turn out to be families of compact operators. This widespread applicability has elevated this topic to a really essential research area of contemporary functional analysis.

The book under review is mainly focused on strongly continuous semigroups with values in the trace-class ideal of bounded operators acting on a Hilbert space. The book presents an exhaustive and up-to-date overview of results on the Gibbs semigroup theory and on the operator semigroups in symmetrically-normed ideals. This book constitutes a really valuable and much needed contribution to the corresponding literature, as the better part of two decades have passed since the publication of the Leuven University Notes [V. A. Zagrebnov, *Topics in the Theory of Gibbs Semigroups*, Leuven Notes in Mathematical and Theoretical Physics, vol. 10 (Series A: Math. Phys.), Leuven University Press, 2003], devoted to this topic. The present book not only provides an update of previous publications, but also a vast expansion on the topics treated. The author extends the content in several directions, providing detailed information on the Trotter–Kato product formulae in the trace-norm topology as well as general results on semigroups in symmetrically-normed ideals, including the Dixmier ideal. The book succeeds in providing a complete and reader friendly introduction to various interesting aspects of the theory of Gibbs semigroups.

The book comprises of seven chapters and four useful appendices. It commences with a very didactic introduction providing the necessary background, namely providing a brief account of basic results of the theory of generators of strongly continuous operator semigroups. Subsequently, the second chapter studies norm ideals (classes) of compact operators on a Hilbert space. The third chapter is devoted to trace inequalities. This chapter plays a crucial preparatory role for the subsequent chapters and mainly treats topics regarding convergence of product formulae, useful to chapter 5. These first three chapters constitute introductory preparatory theory, which have the really important feature of allowing the reader to be initiated to the subject in a completely comprehensive way. This is a very nice aspect of this book.

The main part of the book is featured in chapters 4–7, in which the central topics are investigated. More specifically, the fourth chapter introduces Gibbs semigroups, presenting all necessary notations and definitions. The fifth chapter constitutes one of the principal chapters of this book. It is presented in a self-contained manner so that researchers working in the domain can follow its content independently of other chapters or sources. An important part of this chapter is the discussion of the so-called "lifting lemma" which is of central importance in order to establish the trace-norm estimate from the operator-norm estimate in the Trotter–Kato product formulae. Subsequently, in the sixth chapter of the book the main problem is still the proof and the estimate of the rate of convergence of the Trotter–Kato product formulae, but in the general setting of symmetrically-normed ideals of compact operators, where a particular case that is important for the Gibbs semigroups is the trace-class. The final chapter of the book constitutes an introduction to singular traces, with its main purpose being to present product formulae in the Dixmier ideal.

Overall, the book is masterfully composed to provide not only a complete and up-to-date overview of the problems treated, but also a very elegant introduction to this topic for the uninitiated reader. Being written by one of the world experts in this domain, the book is very well and rigorously structured, having the real attribute of being valuable to both experienced researchers in the domain, but also graduate students in mathematics and mathematical physics who wish to explore the problems treated in this versatile area of research. As such, I believe that this book would also be excellent for seminar use. Finally, this book certainly belongs to all libraries of universities and research institutes with mathematics or physics departments.

Michael Th. Rassias is on the Editorial Board of the EMS Newsletter. His photo and CV can be found in previous Newsletter issues.

Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

The present column is devoted to Algebra.

As for everything else, so for a mathematical theory:
beauty can be perceived but not explained.
Arthur Cayley (1821–1895)

I Six (plus one) new problems – solutions solicited

226. Let $\mathbb{C}^n$ stand for the space of complex *column* n-vectors, and let $\mathbb{M}_n$ stand for the space of complex $n \times n$ matrices.
The inner product $\langle x|y\rangle$ of $x, y \in \mathbb{C}^n$ is defined as

$$\langle x|y\rangle = x^* y \quad \text{(matrix product)}.$$

Therefore $\langle x|y\rangle$ is linear in y and conjugate linear in x.
Let A, B be $n \times m$ complex matrices. Write them as

$$A = [a_1, \ldots, a_m] \quad \text{and} \quad B = [b_1, \ldots, b_m]$$

with $a_j, b_j \in \mathbb{C}^n$ $(j = 1, 2, \ldots, m)$.
Then it is immediate from the definition of matrix multiplication that

$$A^* B = \left[\langle a_j|b_k\rangle\right]_{j,k=1}^{m} \in \mathbb{M}_m.$$

Show the following relation:

$$AB^* = \sum_{j=1}^{m} a_j b_j^* \in \mathbb{M}_n$$

where each product $a_j b_j^*$ $(j = 1, \ldots, n)$ is a rank-one matrix in $\mathbb{M}_n$.

(T. Ando, Hokkaido University, Sapporo, Japan)

227. Let p and q be two distinct primes with $q > p$ and G a group of exponent q for which the map $f_p : G \to G$ defined by $f_p(x) = x^p$, for all $x \in G$, is an endomorphism. Show that G is an abelian group.

(Dorin Andrica and George Cătălin Ţurcaş, Babeş-Bolyai University, Cluj-Napoca, Romania)

228. Let $(G, \cdot)$ be a group with the property that there is an integer $n \geq 1$ such that the map $f_n : G \to G$, $f_n(x) = x^n$ is injective and the map $f_{n+1} : G \to G$, $f_{n+1}(x) = x^{n+1}$ is a surjective endomorphism. Prove that G is an abelian group.

(Dorin Andrica and George Cătălin Ţurcaş, Babeş-Bolyai University, Cluj-Napoca, Romania)

229. Let A and $B \in \mathrm{Mat}_k(K)$ be two matrices over a field K. We say that A and B are *similar* if there exists an invertible matrix $C \in \mathrm{GL}_k(K)$ such that $B = C^{-1}AC$.
Let A and $B \in \mathrm{GL}_k(\mathbb{Q})$ be two similar invertible matrices over the field of rational numbers $\mathbb{Q}$. Assume that for some integer l, $A^{l+1}B = BA^l$. Then A and B are the identity matrices.

(Andrei Jaikin-Zapirain, Departamento de Matemáticas, Universidad Autónoma de Madrid & Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, Spain, and Dmitri Piontkovski, Faculty of Economic Sciences, Moscow Higher School of Economics, Russia)

230. We are trying to hang a picture on a wall. The picture has a piece of string attached to it forming a loop, and there are 3 nails in the wall that we can wrap the string around. We want to hang the picture so that it does not fall down, but it will upon removal of any of the 3 nails.

(Dawid Kielak, Mathematical Institute, University of Oxford, UK)

231. Given a natural number n and a field k, let $M_n(k)$ be the full $n \times n$ matrix algebra over k. A matrix $(a_{ij}) \in M_n(k)$ is said to be centrosymmetric if

$$a_{ij} = a_{n+1-i,n+1-j}$$

for $1 \leq i, j \leq n$. Let $C_n(k)$ denote the set of all centrosymmetric matrices in $M_n(k)$. Then $C_n(k)$ is a subalgebra of $M_n(k)$, called centrosymmetric matrix algebra over k of degree n. Centrosymmentric matrices have a long history (see [1, 5]) and applications in many areas, such as in Markov processes, engineering problems and quantum physics (see [2, 3, 4, 6]). In the representation theory of algebras, a fundamental problem for a finite-dimensional algebra is to know if it has finitely many nonisomorphic indecomposable modules (or in other terminology, representations). In our case, the concrete problem on $C_n(k)$ reads as follows.
Does $C_n(k)$ have finitely many nonisomorphic indecomposable modules? If yes, what is the number?

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An additional interesting problem (not intimately connected to algebra). Intervals of monotonic changes in the polynomial are located between the roots of its derivative. A derivative of a polynomial is also a polynomial, although of a lesser degree. Using these considerations, construct an algorithm for calculating the real roots of the quadratic equation. Improve it to calculate the real roots of the polynomial of the third, fourth and generally arbitrary degree.

(Igor Kostin, Moscow, Russian Federation)

II (A) Open problems, by Maxim Kontsevich (Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France)

Consider the group ring A of a free finitely generated group (i.e., noncommutative Laurent polynomials) with coefficients in a field $\mathbf{k}$ of characteristic zero. Denote by $\tau : A \to \mathbf{k}$ the "trace map" given by the coefficient of monomial 1 ().

Theorem For any $a \in A$ the following series is *algebraic*:

$$G = G(a) := \exp\Big(-\sum_{n\geq 1} \tau(a^n)\frac{t^n}{n}\Big) = 1 + \cdots \in \mathbf{k}[[t]]$$

The rationale for the minus sign is that if we replace algebra A by $Mat(N \times N, \mathbf{k})$ and τ by the usual trace, the resulting series $G = \det(1 - at)$ is polynomial.

Example For $a = x_1 + x_1^{-1} + x_2 + x_2^{-1} + \cdots + x_m + x_m^{-1}$ we have

$$G = ((f+1)/2)^m/((mf + m - 1)/(2m-1))^{m-1},$$
$$f := \sqrt{1 - 4(2m-1)t^2}.$$

In 2007 I found a ridiculous proof in three steps (basically no progress afterwards):

(1) It is known that $F := G'/G = -\sum_{n\geq 1} \tau(a^n)t^{n-1}$ is algebraic (see, e.g., Corollary 6.7.2 in [10], I learned about it from a paper [6] where it was rediscovered). This is a corollary of the theory of algebraic formal languages developed by N. Chomsky and M. Schützenberger in the 60s [4].
(2) Assume that all coefficients of a are integers (it is not a severe restriction), then it is easy to see that coefficients of G are integers (it follows solely from the fact that the free group is torsion-free).
(3) Then we have (a) $G \in \mathbb{Z}[[t]]$, (b) $dG/dt = FG$, i.e., G is a flat section of a line bundle with connection on an algebraic curve over $\mathbb{Q}$. Property (a) implies that the p-curvature of this connection is 0 for almost all primes p, thus fitting to the realm of general Grothendieck–Katz conjecture. By a result of Yves André [1], [2] based on ideas of D. V. and G. V. Chudnovsky [5], or by later results of J.-B. Bost [3], series G is also algebraic.

If we replace the free group by a finite group Γ, the resulting series G is again algebraic, a fractional power of a polynomial. This follows from the fact that after an extension of scalars $\mathbf{k}' \supset \mathbf{k}$, the group ring of the finite group is a direct sum of matrix algebras, and that the canonical trace on the group algebra is proportional to the matrix trace on each matrix algebra summand $Mat(N_i \times N_i, \mathbf{k}')$ rescaled by $N_i/|\Gamma| \in \mathbb{Q}_{>0}$.

For a finitely-generated free abelian group, the series F can be calculated by residue formula, and is holonomic, i.e., it satisfies a non-trivial algebraic linear differential equation. Neveretheless, the series G in this case is typically not holonomic, in particular not algebraic. For nonhyperbolic groups the situation is much more tricky, see [7] where it was shown that the analog of generating series F can already be nonholonomic for the arithmetic group $SL(4, \mathbb{Z})$.

232*. Find a purely combinatorial proof of algebraicity of G, not based on results from number theory. In particular, what is an adequate upper bound on the degree of the equation satisfied by G? Is it true that the function G considered as a multi-valued function is either $1 - ct$ for some constant $c \in \mathbf{k}$, or it does not attain zero value for $t \neq 0$? Also, is it true that the values of all branches of G at $t = 0$ belong to the set $\{0, 1\} \subset \mathbb{P}^1$?

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II (B) Open problems on Iwahori–Hecke algebras, by George Lusztig (Department of Mathematics, M.I.T., Cambridge, USA)

Below we state four open problems (see (233*)–(236*)) on Iwahori–Hecke algebras.

1. Let I be a finite set and let $(m_{ij})_{(i,j)\in I\times I}$ be a symmetric matrix whose diagonal entries are 1 and whose nondiagonal entries are integers ≥ 2 or ∞. Let W be the group with generators $\{s_i; i \in I\}$ and relations $(s_i s_j)^{m_{ij}} = 1$ for any i, j such that $m_{ij} < \infty$; this is a Coxeter group. (Examples of Coxeter groups are the Weyl groups of simple Lie algebras; these are finite groups. Other examples are the affine Weyl groups which are almost finite.) For $w \in W$ let $|w|$ be the smallest integer $n \geq 0$ such that w is a product of n generators $s_i, i \in I$. We assume that we are given a weight function $L : W \to \mathbf{N}$ that is a function such that $L(w) > 0$ for all $w \in W - \{1\}$ and

$$L(ww') = L(w) + L(w')$$

for any w, w' in W such that

$$|ww'| = |w| + |w'|.$$

(For example, $w \mapsto |w|$ is a weight function.) Let $A = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. Let H be the free A-module with basis $\{T_w; w \in W\}$. There is a unique structure of associative A-algebra on H for which

$$(T_{s_i} + v^{-L(s_i)})(T_{s_i} - v^{L(s_i)}) = 0$$

for $i \in I$ and

$$T_w T_{w'} = T_{ww'}$$

for any w, w' in W such that $|ww'| = |w| + |w'|$; this is the Iwahori–Hecke algebra associated to W, L.

For $c \in \mathbf{C} - \{0\}$ let $H_c = \mathbf{C} \otimes AH$ where $\mathbf{C}$ is viewed as an A-algebra via the ring homomorphism $A \to \mathbf{C}, v \mapsto c$. Now H_c is also referred to as an Iwahori–Hecke algebra.

233*. Show that the algebras associated in [10] to a supercuspidal representation of a parabolic subgroup of a p-adic reductive group are (up to extension by a group algebra of a small finite group) of the form H_q where q is a power of p, with H associated to an affine Weyl group W and with L in the collection Σ_W of weight functions on W described in [4, §17], [5], [6].

For example, if W is of affine type F_4, then Σ_W consists of all L whose values on $\{s_i; i \in I\}$ are $(1,1,1,1,1)$ or $(1,1,1,2,2)$ or $(2,2,2,1,1)$ or $(1,1,1,4,4)$; if W is of affine type G_2, then Σ_W consists of all L whose values on $\{s_i; i \in I\}$ are $(1,1,1)$ or $(1,1,3)$ or $(3,3,1)$ or $(1,1,9)$.

The statement analogous to (233*) for groups with connected centre over a finite field F_q instead of p-adic groups is known to hold, without the words in parenthesis; in that case, W is a Weyl group and Σ_W consists of the weight functions on W described in [3, p. 35].

2. There is a unique group homomorphism$^-$: $H \to H$ such that

$$\overline{v^n T_w T_{w^{-1}}} = v^{-n}$$

for $n \in \mathbf{Z}$, $w \in W$; it is a ring isomorphism. Let

$$H_{\leq 0} = \sum_{w \in W} \mathbf{Z}[v^{-1}] T_w \subset H.$$

For any $w \in W$ there is a unique element $c_w \in H_{\leq 0}$ such that

$$\bar{c}_w = c_w$$

and

$$c_w - T_w \in v^{-1} H_{\leq 0}$$

(see [2], [7]). Then $\{c_w; w \in W\}$ is an A-basis of H. For x, y, z in W we define $f_{x,y,z} \in A$, $h_{x,y,z} \in A$ by

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z,$$

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z.$$

234*. Show that there exists an integer $N \geq 0$ such that for any x, y, z in W we have $v^{-N} f_{x,y,z} \in \mathbf{Z}[v^{-1}]$.

(See [7, 13.4]) If W is finite this is obvious. If W is an affine Weyl group, this is known.

We will now assume that (234*) holds. With N as in (234*), we see that

$$v^{-N} h_{x,y,z} \in \mathbf{Z}[v^{-1}]$$

for any x, y, z in W. It follows that for any $z \in W$ there is a unique integer $a(z) \geq 0$ such that

$$h_{x,y,z} \in v^{a(z)} \mathbf{Z}[v^{-1}]$$

for all x, y in W and

$$h_{x,y,z} \notin v^{a(z)-1} \mathbf{Z}[v^{-1}]$$

for some x, y in W. Hence for x, y, z in W there is a well-defined integer $\gamma_{x,y,z^{-1}}$ such that

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{a(z)} \mod v^{a(z)-1} \mathbf{Z}[v^{-1}].$$

Let J be the free abelian group with basis $\{t_w; w \in W\}$. For x, y in W we set

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z.$$

(This is a finite sum.)

235*. Show that this defines an (associative) ring structure on J (without 1 in general).

Assume now that W is a Weyl group or an affine Weyl group and $L = ||$. In this case, (235*) is known to be true and the ring J does have a unit element.

More generally, assume that W is an affine Weyl group and $L \in \Sigma_W$ (see (233*)); in this case there is a (conjectural) geometric description [8, 3.11], of the elements c_w. From this one should be able to deduce (235*) as well as the well-definedness of the $\mathbf{C}$-algebra homomorphism $H_q \to \underline{J}$ in [7, 18.9], where H_q is as in (233*) and $\underline{J} = \mathbf{C} \otimes J$ is independent of q. One should expect that the irreducible (finite dimensional) $\underline{J}$-modules, when viewed as H_q-modules, form a basis of the Grothendieck group of H_q-modules. (This is indeed so if $L = ||$.) This should provide a construction of the "standard modules" of H_q which, unlike the construction in [5],[6], does not involve the geometry of the dual group.

3. Assume that W is finite and that $L = ||$. Let C be a conjugacy class in W; let C_{min} be the set of all $w \in C$ such that $|w|$ is minimal. For $w \in C$ let $N^w \in A$ be the trace of the A-linear map $H \to H$,

$$h \mapsto v^{2|w|} T_w h T_{w^{-1}}.$$

We have $N^w \in \mathbf{Z}[v^2]$. (Note that $N^w|_{v=1}$ is the order of the centralizer of w in W.) From [1] one can deduce that for $w \in C_{min}$, N^w depends only on C, not on w. We say that C is *positive* if $N^w \in \mathbf{N}[v^2]$. For example, if C is an elliptic regular conjugacy class (in the sense of [11]) then C is positive (see [9]). If W is of type A_n, the positive conjugacy classes are $\{1\}$ and the class of the Coxeter element. In the case where W is a Weyl group of exceptional type, a complete list of positive conjugacy classes in W is given in [9].

236*. Make a list of all positive conjugacy classes in W, assuming that W is a Weyl group of type B_n or D_n.

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III Solutions

218.
Determine the sum of the series
$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{2^n-1},$$
where φ is the Euler's totient function.

(Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania)

Solution by the proposer. Let $(a_n)n \geq 1$ be a sequence of real numbers. From the equality
$$\frac{x^n}{1-x^n} = x^n + x^{2n} + \cdots + x^{kn} + \cdots,\ |x| < 1$$
we derive
$$\sum_{n=1}^{\infty}\frac{a_n x^n}{1-x^n} = \sum_{n=1}^{\infty} A_n x^n, \tag{1}$$
where
$$A_n = \sum_{d|n} a_d$$
and assuming that the power series in the right-hand side of (1) is convergent for $|x| < 1$. This is the main idea of the so-called Lambert series with the related identities.

Now, using the well-known Gauss' identity $\sum_{d|n}\varphi(d) = n$, the formula (1) yields
$$\sum_{n=1}^{\infty}\frac{\varphi(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2},\ |x| < 1. \tag{2}$$
Setting $x = \frac{1}{2}$ implies $\sum_{n=1}^{\infty}\frac{\varphi(n)}{2^n-1} = 2$.

Also solved by George Miliakos (Sparta, Greece), Moubinool Omarjee (Paris, France), Richard G.E. Pinch (Gloucestershire, UK), Rudolf Rupp (Nuremberg, Bavaria, Germany), Muhammad Thoriq (Yogyakarta, Indonesia), Socratis Varelogiannis (France) and James J. Ward (Galway, Ireland).

219[a]. Let $\omega(n)$ denote the number of distinct prime factors of a non-zero natural number n.

(i) Prove that $\sum_{n\leq x}\omega(n) = x\log\log x + O(x)$.
(ii) Prove that $\sum_{n\leq x}\omega(n)^2 = x(\log\log x)^2 + O(x\log\log x)$.
(iii) Using (i) and (ii), prove that $\sum_{n\leq x}(\omega(n)-\log\log x)^2 = O(x\log\log x)$.
(iv) Using (iii), prove that $\sum_{n\leq x}(\omega(n)-\log\log n)^2 = O(x\log\log x)$.
(v) Using (iv), prove that $\omega(n)$ has normal order $\log\log n$, i.e., for every $\varepsilon > 0$,
$$\#\{n \leq x : (1-\varepsilon)\log\log n < \omega(n) < (1+\varepsilon)\log\log n\} \sim x \quad (\text{as } x \to \infty).$$

a. Parts (i)–(v) of Problem 219 are extracted from a proof by Paul Turán (1910–1976), published in 1934, of a theorem of G. H. Hardy (1877–1947) and S. Ramanujan (1887–1920), published in 1917; see references below:

[1] G. H. Hardy and S. Ramanujan, The normal number of prime factors of a number *n*. *Quart. J. Math.* **48** (1917), 76–92.
[2] P. Turán, On a Theorem of Hardy and Ramanujan. *J. London Math. Soc.* **9**(4) (1934), 274–276.

(Alina Carmen Cojocaru, Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, USA, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, Romania)

Solution by the proposer. In what follows, the letters p, p_1, p_2 denote primes.
(i) Using the definition of $\omega(n)$ and Mertens' theorem, we obtain
$$\sum_{n\leq x}\omega(n) = \sum_{p\leq x}\left[\frac{x}{p}\right] = x\sum_{p\leq x}\frac{1}{p} + O(x) = x\log\log x + O(x).$$
(ii) As above, using the definition of $\omega(n)$ and Mertens' theorem, we obtain
$$\begin{aligned}\sum_{n\leq x}\omega(n)^2 &= \sum_{n\leq x}\sum_{p_1|n}\sum_{p_2|n}1 = \sum_{p_1\leq x}\sum_{p_2\leq x}\sum_{\substack{n\leq x\\ p_1|n,\, p_2|n}}1\\ &= \sum_{\substack{p_1,p_2\leq x\\ p_1\neq p_2}}\left[\frac{x}{p_1p_2}\right] + \sum_{p\leq x}\left[\frac{x}{p}\right]\\ &= \sum_{\substack{p_1,p_2\\ p_1p_2\leq x}}\left[\frac{x}{p_1p_2}\right] + O(x\log\log x)\\ &= x\sum_{\substack{p_1,p_2\\ p_1p_2\leq x}}\frac{1}{p_1p_2} + O(x\log\log x).\end{aligned}$$
Now observe that
$$\left(\sum_{p\leq\sqrt{x}}\frac{1}{p}\right)^2 \leq \sum_{\substack{p_1,p_2\\ p_1p_2\leq x}}\frac{1}{p_1p_2} \leq \left(\sum_{p\leq x}\frac{1}{p}\right)^2$$
and, again by Mertens' theorem,
$$\sum_{p\leq\sqrt{x}}\frac{1}{p} = \log\log\sqrt{x} + O(1) = \log\log x + O(1).$$
Thus
$$\sum_{n\leq x}\omega(n)^2 = x(\log\log x)^2 + O(x\log\log x).$$
(iii) Expanding the square and using parts (i), (ii), we deduce that
$$\begin{aligned}&\sum_{n\leq x}(\omega(n)-\log\log x)^2\\ &\quad= \sum_{n\leq x}\omega(n)^2 - 2(\log\log x)\sum_{n\leq x}\omega(n) + (\log\log x)^2\sum_{n\leq x}1\\ &\quad= O(x\log\log x).\end{aligned}$$
(iv) First we relate the summand $(\omega(n)-\log\log n)^2$ to the summand $(\omega(n)-\log\log x)^2$ and then we use (iii):
$$\begin{aligned}&\sum_{n\leq x}(\omega(n)-\log\log n)^2\\ &\quad= \sum_{n\leq x}(\omega(n)-\log\log x+\log\log x-\log\log n)^2\\ &\quad\ll \sum_{n\leq x}(\omega(n)-\log\log x)^2 + \sum_{n\leq x}\left(\log\frac{\log x}{\log n}\right)^2\\ &\quad\ll x\log\log x + \sum_{n\leq x}\left(\log\frac{\log x}{\log n}\right)^2.\end{aligned}$$
We split the remaining sum over $n \leq x$ into sums over $n \leq \sqrt{x}$ and $\sqrt{x} < n \leq x$, which we bound trivially.
(v) Let $\varepsilon > 0$ and, for any $y < x$, denote by $N(y, x)$ the number of natural numbers $y < n \leq x$ for which $|\omega(n)-\log\log n| \geq \varepsilon\log\log n$. Using elementary observations and part (iv), we obtain
$$\begin{aligned}N(1,x) &\leq \sqrt{x} + N(\sqrt{x},x) \leq \sqrt{x} + \sum_{n\leq x}\left(\frac{\omega(n)-\log\log n}{\varepsilon\log\log n}\right)^2\\ &\ll \sqrt{x} + \frac{x\log\log x}{\varepsilon^2(\log\log x)^2} = o(x).\end{aligned}$$
We leave the proof of (vi) as a challenge to the reader.

Also solved by Mihaly Bencze (Romania) and Socratis Varelogiannis (France).

220. Using Chebyshev's Theorem, prove that for any integer M there exists an even integer $2k$ such that there are at least M primes p with $p+2k$ also prime. *Unfortunately $2k$ will depend on M. If it did not, we would have solved the Twin Prime Conjecture, namely, there are infinitely many primes p such that $p+2$ is also prime.*

(Steven J. Miller, Department of Mathematics & Statistics, Williams College, Massachusetts, USA)

Solution by the proposer. By Chebyshev's theorem, there exist explicit positive constants A and B such that, for $x > 30$:

$$\frac{Ax}{\log x} \le \pi(x) \le \frac{Bx}{\log x}.$$

Ignoring the lone even prime 2, the number of positive differences between the odd primes at most x is $\binom{\pi(x)-1}{2}$, or

$$(\pi(x)-1)(\pi(x)-2)/2\,.$$

Looking at the lower and upper bounds for $\pi(x)$, we get that the number of these differences is essentially at least $A^2x^2/\log^2 x$ and basically at most $B^2x^2/\log^2 x$; however, there are only about $x/2$ odd numbers which can be these differences.

Thus by the pigeonhole principle, at least one of these positive odd differences must occur at least the average number of times, and thus there is a difference that occurs essentially at least

$$(A^2x^2/\log^2 x)/(x/2) = 2A^2x/\log x\,.$$

The proof is completed by choosing x sufficiently large so that this exceeds M.

Also solved by Mihaly Bencze (Romania), Efstathios S. Louridas (Athens, Greece), George Miliakos (Sparta, Greece).

221. For any three integers a, b, c, with $\gcd(a,b,c) = 1$, prove that there exists an integer m such that

$$0 \le m \le 2^{2^{2002}} c^{\frac{1}{1000}} \text{ and } \gcd(a+mb, c) = 1.$$

(Abhishek Saha, School of Mathematical Sciences, Queen Mary University of London, UK)

Solution by the proposer. We begin with an elementary lemma.

Lemma 1 For positive integers N, k, we have

$$\prod_{p|N} 2 \le 2^{2^k} N^{\frac{1}{k}}.$$

Proof. We have

$$\frac{\prod_{p|N} 2}{N^{\frac{1}{k}}} \le \prod_{p|N} \frac{2}{p^{\frac{1}{k}}} \le \prod_{p|N, p\le 2^k} \frac{2}{p^{\frac{1}{k}}} \le \prod_{p|N, p\le 2^k} 2 \le 2^{2^k} \qquad \square$$

We now begin the solution proper. Let a, b, c as in the problem. *We may assume without loss of generality that* $\gcd(a,b) = 1$ *and that c is squarefree.* Indeed, if these conditions are not met, we can replace a by $\frac{a}{\gcd(a,b)}$, replace b by $\frac{b}{\gcd(a,b)}$ and replace c by its largest squarefree divisor, so that this modified setup does satisfy the conditions. Any integer m that satisfies the required conditions in this modified setup will automatically satisfy it in the original setup.

Next, define $Q = c/\gcd(b,c)$; note that Q is squarefree and hence $\prod_{p|Q} 2 = \sum_{d|Q} 1$. Let $X = 2^{2^{2002}} Q^{\frac{1}{1000}}$. Using well-known properties of the Mobius function μ and the Euler totient function ϕ, we have

$$\sum_{\substack{1\le m\le X\\ \gcd(a+mb,Q)=1}} 1 = \sum_{1\le m\le X} \sum_{\substack{d|Q\\ d|a+mb}} \mu(d) = \sum_{d|Q} \mu(d) \sum_{\substack{1\le m\le X\\ m\equiv -\bar{b}a \mod (d)}} 1$$
$$= \sum_{d|Q} \mu(d)\left(\frac{X}{d} + r_d\right),$$

where $|r_d| \le 1$ for all d, leading to

$$\sum_{\substack{1\le m\le X\\ \gcd(a+mb,Q)=1}} 1 \ge X\sum_{d|Q}\frac{\mu(d)}{d} - \sum_{d|Q} 1 = X\frac{\phi(Q)}{Q} - \prod_{p|Q} 2$$
$$\ge X\prod_{p|Q}\left(1-\frac{1}{p}\right) - \prod_{p|Q} 2.$$

Now, using $1 - \frac{1}{p} \ge \frac{1}{2}$, putting in the bounds from the lemma (with $k = 2000$), and substituting the value of X, we obtain that

$$\sum_{\substack{1\le m\le X\\ \gcd(a+mb,Q)=1}} 1 \ge \left(2^{3\cdot 2^{2000}} - 2^{2^{2000}}\right) Q^{\frac{1}{2000}} > 1.$$

It follows that there exists some m (between 1 and X), such that

$$\gcd(a+mb, c/\gcd(b,c)) = 1.$$

However, since $\gcd(a,b) = 1$, this implies that $\gcd(a+mb, c) = 1$. This completes the solution of the problem.

Also solved by Socratis Varelogiannis (France).

222. Show that

$$\sum_{n=1}^{\infty} \frac{\sin^2(\pi\delta n)}{n^2} = \tfrac{1}{2}\pi^2\delta(1-\delta) \quad \text{for } 0 \le \delta \le 1,$$
$$\sum_{n=1}^{\infty} \frac{\sin^3(\pi\delta n)}{n^3} = \tfrac{1}{2}\pi^3\delta^2(\tfrac{3}{4}-\delta) \quad \text{for } 0 \le \delta \le 1/2,$$
$$\sum_{n=1}^{\infty} \frac{\sin^4(\pi\delta n)}{n^4} = \tfrac{1}{2}\pi^4\delta^3(\tfrac{2}{3}-\delta) \quad \text{for } 0 \le \delta \le 1/2.$$

Setting $\delta = 1/2$, deduce the values of $\zeta(2)$ and $\zeta(4)$.

(Olof Sisask, Department of Mathematics, Stockholm University, Sweden)

Solution by the proposer. Let $f = 1_{[-\delta/2,\delta/2]}$ be the indicator function of the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$ in

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [-\tfrac{1}{2}, \tfrac{1}{2}].$$

We use Fourier analysis on $\mathbb{T}$; in particular, for $n \in \mathbb{Z}\setminus\{0\}$, by definition and computation we have

$$\widehat{f}(n) = \int_{\mathbb{T}} f(x)e^{2\pi i n x}\,dx = \int_{-\delta/2}^{\delta/2} e^{2\pi i n x}\,dx = \frac{1}{2\pi i n}\left(e^{\pi i n\delta} - e^{-\pi i n\delta}\right)$$
$$= \frac{\sin(\pi\delta n)}{\pi n}.$$

Parseval's identity $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 = \int_{\mathbb{T}} |f(x)|^2\,dx$, which is easily verified, then yields

$$\delta^2 + 2\sum_{n=1}^{\infty} \frac{\sin^2(\pi\delta n)}{\pi^2 n^2} = \delta,$$

which is a rearrangement of our first identity.

For the third identity, we use the convolution identity $\widehat{f*f}(n) = \widehat{f}(n)^2$ and Parseval to write

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^4 = \sum_{n\in\mathbb{Z}} |\widehat{f*f}(n)|^2 = \int_{\mathbb{T}} |(f*f)(x)|^2\,dx, \qquad (\star)$$

where, by definition of convolution followed by a simple computation (where we use that $\delta \le 1/2$),

$$f*f(x) = \int_{\mathbb{T}} f(y)f(x-y)\,dx = \begin{cases} \delta - |x| & \text{for } |x| \le \delta \\ 0 & \text{elsewhere} \end{cases}.$$

Thus

$$\int_{\mathbb{T}} |(f * f)(x)|^2\,dx = 2\int_0^{\delta} (\delta - x)^2\,dx = \tfrac{2}{3}\delta^3,$$

and so ($\star$) gives

$$\delta^4 + 2\sum_{n=1}^{\infty} \frac{\sin^4(\pi\delta n)}{\pi^4 n^4} = \tfrac{2}{3}\delta^3.$$

Rearranging, we have the third identity.

For the second identity, note similarly that, by the convolution identity and Fourier inversion formula $g(x) = \sum_n \widehat{g}(n)e^{-2\pi i n x}$,

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)^3 = f * f * f(0) = \int_{\mathbb{T}} (f*f)(x)f(x)\,dx = \int_{-\delta/2}^{\delta/2} (\delta - x)\,dx$$
$$= \tfrac{3}{4}\delta^2$$

for $0 \le \delta \le 1/2$, whence

$$\sum_{n=1}^{\infty} \frac{\sin^3(\pi\delta n)}{n^3} = \tfrac{1}{2}\pi^3\delta^2(\tfrac{3}{4} - \delta).$$

To deduce the ζ-values, we take $\delta = 1/2$ and note that

$$\sin\left(\tfrac{1}{2}\pi n\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

In particular, the first of our identities yields

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \tfrac{1}{8}\pi^2.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \tfrac{1}{4}\zeta(2),$$

we see that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\zeta(2)$$

and, rearranging,

$$\zeta(2) = \frac{\pi^2}{6}.$$

In exactly the same manner we obtain

$$\zeta(4) = \frac{\pi^4}{90}.$$

Comment. The integrals computed above all have an additive combinatorial interpretation: in the case of the fourth-power denominators, the integral measures how many solutions there are to

$$a_1 + a_2 = a_3 + a_4$$

in the interval $[-\frac{\delta}{2}, \frac{\delta}{2}]$. In the case of third-powers, the equation is

$$a_1 + a_2 + a_3 = 0,$$

and in the case of squares it is simply

$$a_1 = a_2.$$

The reason for restricting to $\delta \le 1/2$ in two of the cases is that these solutions are easy to measure provided there is no 'wrap-around' when adding two variables.

Also solved by Mihaly Bencze (Romania) and Rudolf Rupp (Nuremberg, Bavaria, Germany).

223. Fix a prime number p, and an integer $\beta \ge 2$. Consider the function defined on $x \in \mathbf{R}$ by $e(x) = \exp(2\pi i x)$. Given a coprime residue class $r \bmod p^\beta$, consider the additive character defined on integers $m \in \mathbf{Z}$ by $m \mapsto e\left(\frac{mr}{p^\beta}\right)$. Given a complex parameter $s \in \mathbf{C}$ with $\Re(s) > 1$, consider the Dirichlet series defined by

$$D(s, r, p^\beta) = \sum_{m\ge1} e\left(\frac{rm}{p^\beta}\right) m^{-s}.$$

Show that this series has an analytic continuation to all $s \in \mathbf{C}$, and moreover that it satisfies a functional equation relating values at s to $1 - s$.

(Jeanine Van Order, Fakultät für Mathematik,
Universität Bielefeld, Germany)

Solution by the proposer. The key idea is to express the additive character as a certain linear combination of multiplicative (Dirichlet) characters, to reduce to the well-known classical setting of Dirichlet L-series and their functional equations. To be more precise, let χ denote a primitive Dirichlet character modulo p^β, and let

$$\tau(\chi) = \sum_{\substack{x \bmod p^\beta \\ (x,p^\beta)=1}} \chi(x)e\left(\frac{x}{p^\beta}\right)$$

denote the corresponding Gauss sum. We claim that for any integer $n \ge 1$, we have

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)^n = \varphi(p^\beta)\,\mathrm{Kl}_n(r, p^\beta), \tag{1}$$

where the sum runs over primitive Dirichlet characters $\chi \bmod p^\beta$, $\varphi(p^\beta)$ denotes the Euler-phi function evaluated at p^β, and

$$\mathrm{Kl}_n(r, p^\beta) = \sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^\beta}} e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right)$$

denotes the hyper-Kloosterman sum of dimension n and modulus p^β evaluated at a coprime residue class $r \bmod p^\beta$. Indeed, let us write

$$\varphi^\star(p^\beta) = \varphi(p^\beta) - \varphi(p^{\beta-1})$$

to denote the number of primitive Dirichlet characters $\chi \bmod p^\beta$. Note that by the Möbius inversion formula, we have for any integer $m \ge 1$ prime to p the relation

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \chi(m) = \sum_{\substack{0\le x\le\beta \\ p^x \mid (m-1,p^\beta)}} \varphi(p^x)\mu\left(\frac{p^\beta}{p^x}\right),$$

from which it is easy to derive the corresponding orthogonality relation

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \chi(m) = \begin{cases} \varphi^\star(p^\beta) & \text{if } m \equiv 1 \bmod p^\beta, \\ -\varphi(p^{\beta-1}) & \text{if } m \equiv 1 \bmod p^{\beta-1} \text{ but } m \not\equiv 1 \bmod p^\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Now to show (1), we open up sums and switch the order of summation to find

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)^n = \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \sum_{x_1,\ldots,x_n \bmod p^\beta} \chi(\overline{r}x_1\cdots x_n)e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right)$$
$$= \varphi^\star(p^\beta) \sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^\beta}} e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right) - \varphi(p^{\beta-1})$$
$$\sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv r \bmod p^\beta}} e\left(\frac{x_1 + \cdots + x_n}{p^\beta}\right).$$

Here, we write $\overline{r}$ to denote the multiplicative inverse of the class $r \bmod p^\beta$, so that $r\overline{r} \equiv 1 \bmod p^\beta$. Let us now consider the second sum in this latter expression, which after putting $y := x_1 \cdots x_{n-1}\overline{r}$ is the same as

$$\sum_{\substack{x_1,\ldots,x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^{\beta-1} \\ x_1\cdots x_n \not\equiv r \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right)$$
$$= \sum_{x_1,\cdots,x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}}{p^\beta}\right) \sum_{\substack{x_n \bmod p^\beta \\ x_n \equiv y \bmod p^{\beta-1} \\ x_n \not\equiv y \bmod p^\beta}} e\left(\frac{x_n}{p^\beta}\right)$$

Notice that the inner sum in this latter expression can be parametrised equivalently by $x_n = y + lp^{\beta-1}$ for l ranging over integers $1 \leq l \leq p-1$, so that we have

$$\sum_{x_1,\cdots,x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}}{p^\beta}\right) \sum_{\substack{x_n \bmod p^\beta \\ x_n \equiv y \bmod p^{\beta-1} \\ x_n \not\equiv y \bmod p^\beta}} e\left(\frac{x_n}{p^\beta}\right)$$
$$= \sum_{x_1,\cdots x_{n-1} \bmod p^\beta} e\left(\frac{x_1+\cdots+x_{n-1}+y}{p^\beta}\right) \sum_{l=1}^{p-1} e\left(\frac{lp^{\beta-1}}{p^\beta}\right)$$
$$= \sum_{\substack{x_1\cdots x_n \bmod p^\beta \\ x_1\cdots x_n \equiv r \bmod p^\beta}} e\left(\frac{x_1+\cdots+x_n}{p^\beta}\right) \sum_{l=1}^{p-1} e\left(\frac{l}{p}\right) = -\operatorname{Kl}_n(r, p^\beta)$$

via the well-known elementary identity $\sum_{l=1}^{p-1} e\left(\frac{l}{p}\right) = -1$. Hence, see that

$$\sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)^n = \left(\varphi^\star(p^\beta) + \varphi(p^{\beta-1})\right)\operatorname{Kl}_n(r, p^\beta)$$
$$= \varphi(p^\beta)\operatorname{Kl}_n(r, p^\beta).$$

Let us now consider the series in question, which after using (1) is equivalent to a sum over Dirichlet series $L(s,\chi)$ attached to each Dirichlet character $\chi \bmod p^\beta$,

$$\sum_{m\geq 1} e\left(\frac{rm}{p^\beta}\right)m^{-s} = \sum_{m\geq 1}\operatorname{Kl}_1(rm, p^\beta)m^{-s}$$
$$= \varphi(p^\beta) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)\sum_{m\geq 1}\overline{\chi}(m)m^{-s}$$
$$= \varphi(p^\beta) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)L(s,\overline{\chi}).$$

Observe that we have for each $L(s,\chi)$ the asymmetric functional equation

$$L(s,\chi) = p^{-\beta s}\tau(\chi)\left(\frac{\Gamma_{\mathbf{R}}(\frac{1-s}{2})}{\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)}\right)L(1-s,\overline{\chi}), \tag{3}$$

i.e., after bringing gamma factors $\Gamma_{\mathbf{R}}(s) = \pi^{-s}\Gamma(s)$ over to the right-hand side. Substituting this functional equation (3) for each $L(s,\overline{\chi})$ then gives us

$$\sum_{m\geq 1} e\left(\frac{rm}{p^\beta}\right)m^{-s}$$
$$= \varphi(p^\beta)p^{-\beta s}\left(\frac{\Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right)}{\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)}\right) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)\tau(\chi)\tau(\overline{\chi})L(1-s,\chi)$$
$$= \varphi(p^\beta)p^{\beta(1-s)}\left(\frac{\Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right)}{\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)}\right) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} L(1-s,\chi).$$

Here, we use that

$$\tau(\chi)\tau(\overline{\chi}) = \tau(\chi)\overline{\tau(\chi)} = |\tau(\chi)|^2 = p^\beta$$

for each primitive Dirichlet character $\chi \bmod p^\beta$. Hence, we derive the relation

$$\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)D(s,r,p^\beta) = \varphi(p^\beta)p^{\beta(1-s)}\Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right) \sum_{\substack{\chi \bmod p^\beta \\ \chi \text{ primitive}}} \overline{\chi}(r)L(1-s,\chi).$$

Observe that when $\Re(s) < 0$, we can open up the absolutely convergent Dirichlet series on the right-hand side and use the orthogonality relation (2) to see that

$$\Gamma_{\mathbf{R}}\left(\frac{s}{2}\right)D(s,r,p^\beta)$$
$$= \varphi(p^\beta)p^{\beta(1-s)}\Gamma_{\mathbf{R}}\left(\frac{1-s}{2}\right)$$
$$\left(\varphi^\star(p^\beta)\sum_{\substack{m\geq 1 \\ m\equiv r \bmod p^\beta}} \frac{1}{m^{1-s}} - \varphi(p^{\beta-1})\sum_{\substack{m\geq 1 \\ m\equiv r \bmod p^{\beta-1} \\ m\not\equiv r \bmod p^\beta}} \frac{1}{m^{1-s}}\right).$$

Note that a similar discussion carries over to the more general setting of Dirichlet series defined with respect to $\operatorname{Kl}_n(mr, p^\beta)$ instead of $\operatorname{Kl}_1(mr, p^\beta)$, as well as to additive twists of $\mathrm{GL}_n(\mathbf{A}_\mathbf{Q})$-automorphic L-functions (see [1] for more details).

Reference

[1] J. Van Order, Dirichlet twists of GL_n-automorphic L-functions and hyper-Kloosterman Dirichlet series preprint (2017). To appear in *Ann. Fac. Sci. Toulouse Math. (6)*.

We would like for you to submit solutions to the proposed problems and ideas on the open problems. Send your solutions by email to Michael Th. Rassias, Institute of Mathematics, University of Zürich, Switzerland, michail.rassias@math.uzh.ch.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Probability Theory*.

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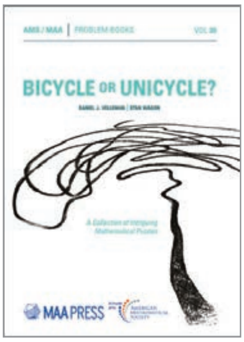

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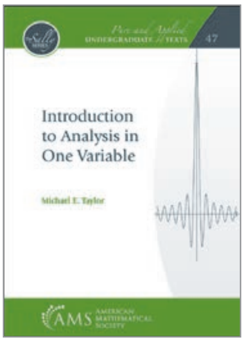

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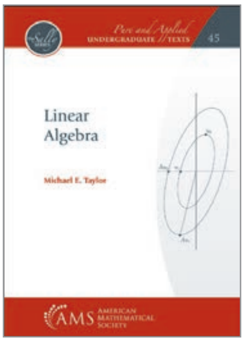

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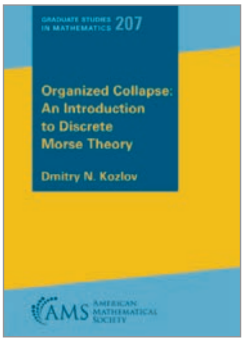

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