NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

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Systems of Points with Coulomb Interactions

Interviews
Alessio Figalli
Akshay Venkatesh

Societies
Georgian Mathematical Union

European Mathematical Society

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EMS Agenda

2019

22–23 March
EMS Executive Committee Meeting, Berlin, Germany

23–24 March
EMS Meeting of the Presidents, Berlin, Germany

EMS Scientific Events

2019

7–10 January
Variational Problems in Geometry and Mathematical Physics,
UK-Japan Winter School
University of Leeds, UK

1–5 April
Imaging and Machine Learning
Institut Henri Poincaré, Paris, France

9–13 July
SIAM Conference on Applied Algebraic Geometry
Bern, Switzerland

15–19 July
ICIAM 2019
Valencia, Spain

29 July–2 August
British Combinatorial Conference 2019
Glasgow, UK

26–29 August
Caucasian Mathematics Conference (CMC-III)
Rostov-on-Don, Russian Federation

2020

5–11 July
European Congress of Mathematics
Portorož, Slovenia

Editorial – From President to President

Pavel Exner, President of the EMS

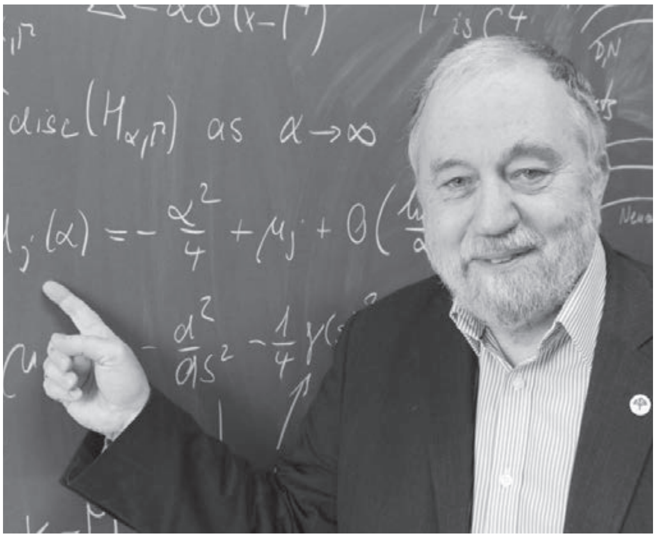
Dear colleagues,

This is my last message in this capacity: by the time this Newsletter issue reaches your hands, I will have passed the EMS leadership on to Volker Mehrmann. I wish him all the stamina necessary for the job. Having worked in close contact with Volker for the last four years, I have no doubt that the society will be in the best possible hands. This is also a fitting opportunity to thank him and the other members of the EMS Executive Committee, as well as the broader community involved in the full range of EMS activities, for the pleasure of collaboration, in the hope that it has brought some visible results.

I hope you will agree with me that our society is not in bad shape overall, although there is never a shortage of things to improve through our collective effort. We have a good number of active members, both corporate and individual, and our topical committees dealing with various aspects of mathematicians' lives are generally functioning very well. We have 10 of them, each broadly defined, which we regard as a better option than having over 100 as our American colleagues do, many with a rather narrow scope. This is not to say that our list could not be amended. One particularly important area where we could do better concerns young mathematicians. They face specific challenges at the start of their careers, which we, as older people, may sometimes not fully appreciate.

A related question concerns an even younger age group. The competition between different branches of science (and, more generally, different areas of intellectual endeavour) may be friendly but it is fierce. The future of mathematics stands or falls with the number of gifted children who choose it as their *métier*. Many factors may influence a choice of life path but the pecuniary aspect cannot be neglected. Very few get rich doing mathematics and we need to be able to show that intellectual excitement can outweigh more mundane drivers. The task is not easy, of course, especially when many among those who gain public attention, for whatever reason, are happy to trumpet that they succeeded despite being weak in mathematics.

Speaking of talent, another thing comes to mind. EMS life has its cycles. We have passed the mid-point between European Congresses and are heading for the eighth, which will convene, as we know, in Portorož (Slovenia) in July 2020. Now it is time to think about nominations for the prizes to be awarded there, starting with the 10 EMS Prizes for young mathematicians. In their not so long history, our prizes have achieved considerable renown, illustrated by the fact that of the 70 laureates so far, 12 have subsequently been awarded the Fields Medal. This is a sure sign of the strength of European mathematics – a tradition we certainly wish to continue. Hence, I encourage you to look for worthy nominees.



Some changes are less regular but no less important. The EMS is glad to have its own publishing house offering a palette of journals, several of them highly regarded, and beautiful books recognisable from afar by their blue covers. We are grateful to Thomas Hintermann, who built up this enterprise from scratch, bringing it to its current firm place on the global map of mathematical publishing. Nothing lasts forever, though, and Thomas is now approaching a well-deserved retirement. The future of the publishing house was always one of our main priorities and we are currently reforming it to ensure that it can continue to develop and serve, under a new director, the needs of the mathematical community even better.

This brings me to a larger theme. You will have certainly noticed – unless you arrived from Mars this morning – the debates about changes in models of academic publishing, in particular the obligatory gold Open Access scheme that the European Commission is planning to impose *de facto* from 2020 onward. It is presented coated in pleasing political sauce: nobody should be barred access from scientific literature by financial walls, especially if the work in question is supported by public money. It does not mention, however, that people may face the same walls when trying to publish their results. Older people with experience of living outside the Western world remember all too well the necessity of sending each submission with an accompanying beggar's letter, explaining that one has no way to pay the publication fee. That was an accepted game and I remember the editor-in-chief of a high-class journal telling us around 1980 that close to one half of the papers they published came with such a plea. In the brave new world of today, I am afraid, nobody would be spared.

The EMS has repeatedly said that we are aware that publication costs something and would be glad to see a serious discussion about the ways in which the present model could be transformed to better fit the needs of all the parties involved. Should such a dialogue be successful, though, it must be balanced. This is not the case currently, as the voice of big commercial publishers commands the most attention. The biggest was even chosen – in a legally crystal clear way, of course – to monitor the community's adherence to the aforementioned noble principles. This may open a can of worms. Strong and rich countries may strike a deal with the big publishers, shielding their researchers, but there is some doubt that all others would follow. The effect would be amplified in disciplines that are not rich and we know well that mathematics belongs to this category. The result may be a new wave of brain drain, with all the consequences that flow from that. Somebody recently called these plans "the new censorship of the rich" and I am afraid that this expression is painfully fitting.

It is vital to bear in mind that these problems don't just concern mathematicians. Finding a reasonable solution is in the interests of the whole academic community, in Europe and worldwide. We must seek a common language with colleagues in other areas, from those traditionally close, such as physics, to those far away, including the social sciences and humanities, remembering that there are values we all share. Of equal importance is the need to talk to politicians. This is not an easy task, especially noting that the negative consequences of major decisions may take much longer than one electoral term before fully manifesting. Nevertheless, one must keep reminding those elected to steer public affairs of their civic responsibility (a long shot, I admit).

Returning to the EMS itself, I can say with pleasure that it maintains active links with most parts of our continent's mathematics community, including places where the life of mathematicians is sadly more difficult. One example to celebrate is the revival of the Ukrainian Mathematical Society over the last two years, representing a large number of colleagues and a strong tradition. On the other hand, we have lost as corporate members two smaller societies that were not willing to accept even the bare minimum of duties that EMS membership entails. Still, I believe that sooner or later they will find their way back.

The world is a strongly connected space and mathematics is undoubtedly a universal language. One of our aims was to strengthen relations with mathematical communities beyond those we had already established. We have recently concluded cooperation agreements with mathematical societies in Japan, China and Australia. We exchange information and articles in our newsletters but we want to go further. We proposed the idea of common Euro-Pacific conferences, similar to what we have had (and will again) with the Americans; they could alternate between European and Far Eastern locations. Our partners found this a promising proposal and we have agreed to pursue it further.

To conclude, let me finally look at the activities of the EMS in the broadest context. The world's political tectonic plates are shifting ever faster; many of these developments influence the lives of our colleagues in various countries around the globe. We are a professional organisation, not a political one, but sometimes we must speak up. In addition, strengthening relations between mathematicians, especially in troubled regions, is itself a way to reinforce society's fabric. It may not be the strongest political medicine but its effect is not negligible either. We had a very positive experience with the Caucasian Mathematical Conferences and I believe that similar meetings "over troubled waters" could be organised elsewhere. As I said when opening the 2016 European Congress in Berlin, we remember the serious harm in previous generations when mathematicians bent to political pressures and ceased talking to each other. We are a closely interconnected community with common goals, culture and vision, and we must stick together, whatever political shocks the future may bring.

Having said all that, allow me to raise a toast to you on the occasion of the 2019 New Year, to wish you health and happiness and, above all, plenty of pleasure and satisfaction in mathematics.

Farewell within the Editorial Board of the EMS Newsletter

Vladimir R.Kostic ended his editorship of the Newsletter on 30 September 2018.
We express our gratitude for his work and for contributing to a friendly and productive atmosphere.

European Congress of Mathematics Call for Bids for 9ECM

Pavel Exner, President of the EMS and Sjoerd Verduyn Lunel, Secretary of the EMS

Outline bids from mathematicians to organise the 2024 congress are now invited and must reach the EMS Secretariat by 28 February 2019. The address of the Secretariat is Department of Mathematics & Statistics, P.O. Box 68 (Gustaf Hällströmink. 2b, 00014 University Helsinki, Finland) (Phone: +358 9 1915 1507, Fax: +358 9 1915 1400, Email: ems-office@helsinki.fi).

The information below may be helpful for possible organisers. Informal communication is welcome and may be addressed to the President of the EMS.

General information about ECMs

European Congresses of Mathematics are organised every four years. The first congress was held in Paris in 1992 and, since then, they have been held in Budapest, Barcelona, Stockholm, Amsterdam, Krakow and Berlin. In 2020, the congress will take place in Portorož (Slovenia).

Experience of previous congresses suggests that around a thousand or more people may attend. The duration of each congress has so far been five days. Ten EMS prizes are awarded to outstanding young European mathematicians at the opening ceremony, together with the Felix Klein and Neugebauer Prizes. The congress programme should aim to present various new aspects of pure and applied mathematics to a wide audience, to offer a forum for discussion of the relationship between mathematics and society in Europe, and to enhance cooperation between mathematicians of all European countries. The standard format of previous ECMs has been:

- About ten plenary lectures.
- Section lectures for a more specialised audience, normally with several held simultaneously.
- Mini-symposia.
- Film and mathematical software sessions.
- Poster sessions.
- Round tables.

An exhibition space for mathematical societies, booksellers and so on is required. No official language is specified and no interpretation is needed. Proceedings of the congress are published by the EMS Publishing House (http://www.ems-ph.org/).

Decision process for 9ECM

(i) Bids are invited via this notice in the EMS Newsletter and via letters to the EMS member societies; the deadline for bids is 28 February 2019. These bids need only be outline bids giving a clear idea of the proposal and possible sources of financial and local support.

(ii) The Executive Committee of the EMS will consider the bids received. It will invite one or more of the bids to be set out in greater detail so that it can decide which bids are sufficiently serious options to be considered further. The deadline for such "worked up" bids, which will include a draft budget and a commitment to follow the conditions set up by the Executive Committee, is 31 July 2019.

(iii) The Executive Committee will then create a shortlist of sites that appear to offer the best possibilities for a successful congress and appoint a Site Committee to visit the shortlisted sites between September and December 2019. They will check a range of items in connection with the development of the congress, for example:

- Size and number of auditoria (including locations and equipment).
- Room for exhibitors.
- Hotel rooms and dormitories (locations, prices, numbers in different categories and transportation to lectures).
- Restaurants close to congress site (number and prices).
- Accessibility and cost of travel from various parts of Europe.
- Financing of the congress and, in particular, support to participants from less favoured countries.
- Financing for the EMS Prizes.
- Experience in organising large conferences.
- Timing of the congress.
- Social events.
- Plans to use the occasion of the congress as publicity for mathematics.

The Executive Committee may or may not make a recommendation to the council based on the report of the Site Committee; the choice is at the discretion of the council delegates.

(iv) In 2020, at the EMS Council to be held in Bled prior to 8ECM, a decision will be reached.

Relations between the Executive Committee and the Organising Committee of 9ECM

The actual organisation of the congress is the responsibility of the local organisers but the EMS coordinates the setting of the scientific programme. Two main committees are appointed: the Scientific Committee and the Prize Committee.

The Scientific Committee is charged with the responsibility of conceiving the scientific programme and selecting the speakers. The Prize Committee is charged with the responsibility of nominating the EMS Prizewinners. For each of these committees, the chairs are appointed by the Executive Committee. There will be a call for suggestions for members of these two committees from the EMS member societies, after which the membership of each committee will be decided by the Executive Committee, in consultation with its chair. In addition, the Executive Committee will appoint the Otto Neugebauer Prize Committee, the Hirzebruch lecture committee (together with DMV), and the Felix Klein Prize Committee, the latter in co-operation with ECMI and the Fraunhofer Institute for Industrial Mathematics in Kaiserslautern. The Executive Committee wil also select the Abel Prize lecturer.

The Executive Committee must be kept informed by the chair of each committee about the progress of their work.

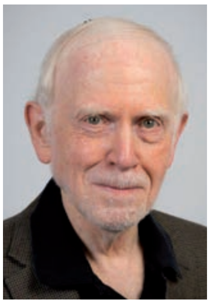
The local organisers are responsible for seeking financial support for the congress and for the meetings of its committees. However, the EMS will provide some financial support for mathematicians from less favoured countries in Europe and will also assist and advise in seeking sources of funding.

The level of the registration fee is of great importance to the success of an ECM and the Executive Committee must be involved before a final decision on the level of fees is made; members of the EMS normally receive a reduction of some 20% on the registration fees. The Executive Committee are pleased to offer advice to the local organisers on matters such as budget, registration, accommodation, publications, website, etc. In any case, the Executive Committee must be kept informed of progress at its regular meetings. Publicity for the ECM via the EMS Newsletter and the EMS website http://www.euro-math-soc.eu/ should appear regularly.

Prizes awarded by the IAMP

Robert Seiringer (IST Austria, Klosterneuburg, Austria)

The International Association of Mathematical Physics (IAMP) has awarded the *2018 Henri Poincaré Prize* for mathematical physics to: **Michael Aizenman**, Princeton University; **Percy Deift**, New York University; and **Giovanni Gallavotti**, Università di Roma La Sapienza.



Michael Aizenman was honoured "for his seminal contributions to quantum field theory, statistical mechanics and disordered systems, in which he pioneered innovative techniques that demonstrate the beautiful and effective interplay between physical ideas, mathematical analysis, geometric concepts and probability theory". Percy Deift was honoured "for his seminal contributions to Schrödinger operators, inverse scattering theory, nonlinear waves, asymptotic analysis of Fredholm and Toeplitz determinants, universality in random matrix theory and his deep analysis of integrable models". Giovanni Gallovotti was honoured "for his outstanding contributions to equilibrium and non-equilibrium statistical mechanics, quantum field theory, classical mechanics and chaotic systems, including, in particular, the renormalisation theory for interacting fermionic systems and the fluctuation relation for the large deviation functional of entropy production".

The *Henri Poincaré Prize,* which is sponsored by the Daniel Iagolnitzer Foundation, recognises outstanding contributions that lay the groundwork for novel developments in mathematical physics. It also recognises and supports young people of exceptional promise who have already made outstanding contributions to the field. The prize is awarded every three years at the International Congress on Mathematical Physics.

The IAMP has awarded the *2018 Early Career Award* to **Semyon Dyatlov** (UC Berkeley/Massachusetts Institute of Technology) "for the introduction and the proof of the fractal uncertainty principle, which has important applications to quantum chaos and to observability and control of quantum systems". The Early Career Award is given in recognition of a single achievement in mathematical physics and is reserved for scientists aged under 35. It is sponsored by Springer.

The prizes were awarded on 23 July 2018 in Montreal, Canada. For prior winners, selection committee members and laudations, see http://www.iamp.org.

André Lichnerowicz Prize in Poisson Geometry 2018

Eva Miranda (Universitat Politecnica de Catalunya, Barcelona, Spain)

The André Lichnerowicz Prize in Poisson geometry was established in 2008. It is awarded for notable contributions to Poisson geometry, every two years at the *International Conference on Poisson Geometry in Mathematics and Physics*, to researchers completing their doctorates at most eight years before the year of the conference.

The prize is named in memory of André Lichnerowicz (1915–1998), whose work was fundamental in establishing Poisson geometry as a branch of mathematics. It is awarded by a jury composed of members of the scientific/advisory committee of the conference.

The prize for the year 2018 was awarded to:

Brent Pym and Chelsea Walton

on 16 July 2018 at the Fields Institute, Toronto

Brent Pym

Brent Pym received his PhD at the University of Toronto in 2013, under the direction of Marco Gualtieri. He has held postdoctoral positions at McGill, Oxford and Edinburgh and recently accepted an assistant professorship in the Department of Mathematics and Statistics at McGill University. In his thesis work, Pym classified the noncommutative deformations of complex projective 3-space, proved the 4-dimensional case of the Bondal conjecture about Fano Poisson manifolds and, jointly with Gualtieri and Li, developed the theory of Stokes groupoids on Riemann surfaces. In recent work, Pym developed the notion of an elliptic singularity for a holomorphic Poisson structure and used it to obtain some of the only available classification results in dimension greater than three. He has also developed the notion of a holonomic Poisson manifold (jointly with Schedler), bringing the theory of perverse sheaves into the mainstream of Poisson geometry. In additional joint works, Pym has contributed to the enumerative geometry of noncommutative spaces and to the theory of Dirac structures and Courant algebroids as objects in shifted symplectic geometry.

Chelsea Walton Chelsea Walton completed her PhD in 2011 at the University of Michigan, under the direction of Toby Stafford and Karen Smith. Following postdoctoral stays at the University of Washington, MSRI and MIT, she took on an assistant professorship at Temple University in Philadelphia in 2015. In July 2018, she joined the Mathematics Department at the University of Illinois at Urbana-Champaign as an associate professor with tenure. Walton has written several important works in Poisson geometry, in addition to being a well-established expert in noncommutative algebra and quantum groups. Her work in Poisson geometry includes a deep investigation of the 3-dimensional and 4-dimensional Sklyanin algebras, especially those that are module-finite over their centre. Jointly with Wang and Yakimov, Walton showed that these are close analogues of Poisson algebras, namely Poisson Z-orders, which carry Poisson structures on the centre. Walton, in joint work with several collaborators, has written a deep series of works on actions of Hopf algebras on commutative and noncommutative domains, showing that semisimple Hopf actions generally factor through group algebra actions, also investigating the difficult non-semisimple case. She also gave a negative answer to the longstanding conjecture about whether the universal enveloping algebra of the Witt algebra is noetherian (jointly with Sierra).

Chelsea Walton (picture by Ryan Brandenburg)

Eva Miranda [eva.miranda@upc.edu] is an ICREA Academia Professor at Universitat Politècnica de Catalunya, an Honorary Doctor Vinculado at ICMAT-Madrid and a Chercheur Affilié at Observatoire de Paris. Eva Miranda is the Director of the Laboratory of Geometry and Dynamical Systems at UPC and a member of the Governing Board of the Barcelona Graduate School of Mathematics. In 2017, she was awarded a Chaire d'Excellence of the Fondation Sciences Mathématiques de Paris and an ICREA Academia Prize. Eva Miranda's research is at the crossroads of differential geometry, mathematical physics and dynamical systems. Her area of expertise is symplectic and Poisson geometry. She works with objects at the interface of geometry and physics such as integrable systems and group actions acquainting for symmetries of the systems. Her research deals with geometric and dynamic aspects of the singularities arising in symplectic and Poisson manifolds, as well as with mathematical models for their quantisation. During the period 2010–2018, she has been a member of the Editorial Board of the Newsletter of the EMS.

On the Solution of the Scalar-plus-compact Problem by Argyros and Haydon

András Zsák (University of Cambridge, UK)

Whether there is a Banach space on which every operator is a compact perturbation of a scalar multiple of the identity was one of the most famous, longstanding open problems in the theory of Banach spaces. It is known as the scalar-plus-compact problem and it was solved by Spiros Argyros and Richard Haydon some 10 years ago by constructing a Banach space that has the scalar-plus-compact property. In recognition of this major contribution to mathematics, Argyros and Haydon were invited speakers at the 2018 ICM in Rio de Janeiro. Their achievement was spectacular, particularly as at the time no one expected a breakthrough. The solution of the scalar-plus-compact problem caught not just the mathematical community but also Argyros and Haydon by surprise.

1 The origins

If X is an infinite-dimensional Banach space, what operators, *i.e.* continuous linear maps, on X are there? The Hahn–Banach theorem provides a rich supply of continuous linear functionals: for every non-zero vector x, there is a continuous linear map $f\colon X \to \mathbb{R}$ such that $f(x) = 1$. It follows that, for non-zero vectors x and y, there is a continuous rank-1 operator $T\colon X \to X$ with $T(x) = y$ given by $T(z) = f(z)y$. By taking sums, we obtain a large supply of finite-rank operators on X, and the limit of a sequence of finite-rank operators is compact. Recall that an operator T on X is *compact* if for every bounded sequence (x_n) in X, the sequence (Tx_n) has a convergent subsequence. Whether every compact operator is the limit of a sequence of finite-rank operators was another very famous open problem going back to Banach's book [9]. It was solved in the negative by Per Enflo in 1973 by constructing a Banach space that fails the so-called approximation property [12]. In this article, all Banach spaces under consideration have the approximation property, and so every compact operator can be expressed as the limit of a sequence of finite-rank operators. Assuming this for our space X, let us continue by asking if there are any non-compact operators on X. The answer is trivially 'yes': any non-zero multiple of the identity operator is non-compact since our space X is infinite-dimensional. The construction of operators in this generality stops here: there seems to be no general method of finding operators that are not of the form $\lambda \operatorname{Id} + K$, where K is a compact operator. In contrast, specific spaces, like Hilbert space, have very rich algebras of operators. In his famous list of open problems from 1976, Joram Lindenstrauss [26] began with the following question.

(Q.1) Does there exist an infinite-dimensional Banach space X so that every operator $T\colon X \to X$ is of the form $T = \lambda \operatorname{Id} + K$, where K is compact?

Lindenstrauss goes on and lists some of the peculiar properties such a space X would have to possess:

(i) X is not isomorphic to its subspaces of finite codimension.

(ii) X is *indecomposable*, *i.e.* every decomposition of X as a direct sum $X = Y \oplus Z$ is trivial: either Y or Z is finite-dimensional.

(iii) Every operator on X has a non-trivial, proper, closed invariant subspace, *i.e.* if T is an operator on X then there is a closed subspace Y of X such that $Y \neq \{0\}$, $Y \neq X$ and $T(Y) \subset Y$.

The first property follows since every operator on X is either compact or, being a compact perturbation of a non-zero multiple of the identity operator, Fredholm with index zero. The second property can be established as follows. If $T = \lambda \operatorname{Id} + K$ is a projection on X then λ must be 0 or 1; hence, either T or $\operatorname{Id} - T$ is a compact projection and thus either the range or the kernel of T is finite-dimensional. The third property is far from being straightforward and follows from a theorem of Aronszajn and Smith [8], which states that every compact operator on a (complex) Banach space has a non-trivial, proper, closed invariant subspace.

Independently of the scalar-plus-compact problem, the three properties above lead to three questions, each asking for the existence or otherwise of a Banach space satisfying the given property. These questions themselves became very important in their own right. This is hardly surprising; after all, these are fairly basic structure-theory questions of the kind that arise in most areas of mathematics. Indeed, in most fields of research, an important aim is to classify the central objects of study, in our case Banach spaces, and to decompose a general object into simpler building blocks. This usually goes hand-in-hand with a similar study of the morphisms, in our case operators. A very simple example is the question of diagonalisability of matrices. For general Banach space operators, a more basic question is concerned with the existence of invariant subspaces.

The first two of these three questions were already answered in the 1990s. Tim Gowers solved the so-called Banach hyperplane problem [18] by constructing a Banach space not isomorphic to its hyperplanes, *i.e.* its closed subspaces of codimension 1. His space also satisfied the stronger property (i) above. Another major breakthrough of the 1990s was the solution by Tim Gowers and Bernard Maurey of the unconditional basic sequence problem [21]: they constructed a space that contains no subspace with an unconditional basis. An *unconditional basis* is an infinite coordinate system with lots of symmetries. A Banach space that has an unconditional basis has many operators and is thus far from the scalar-plus-compact property. As pointed out by Bill Johnson at the time, the proof of Gowers and Maurey showed a much stronger property of their space: it is indecomposable (property (ii) above) and, in fact, even *hereditarily indecomposable*, meaning that no closed, infinite-dimensional subspace is decomposable. The study of hereditarily indecomposable (HI) spaces has since become a very important area in the field. HI constructions also happen to play a major role in the solution of the scalar-plus-compact problem.

Concerning property (iii), it was already known to John von Neumann that compact operators on Hilbert space have invariant subspaces. For general Banach spaces, this was proved by Aronszajn and Smith, as mentioned above. Whether every operator on an arbitrary Banach space has non-trivial, proper, closed invariant subspaces remained a major unsolved problem for some time. The first counterexamples were produced by Charles Read [28] and Per Enflo [13]. Both were highly complex examples that were later simplified, strengthened and generalised. It is still an open problem whether operators on Hilbert space have invariant subspaces. Indeed, it may even be possible that every dual operator on a dual Banach space has an invariant subspace. It is therefore significant that there is a Banach space on which it is true that every operator has an invariant subspace. The Argyros–Haydon space is the first such example.

2 History and ingredients

Recall that a Banach space is said to be hereditarily indecomposable (HI) if every subspace of X is indecomposable. The first example of an HI space is the space $\mathfrak{X}_{\mathrm{GM}}$ of Gowers and Maurey that solved the unconditional basic sequence problem. Studying the operators on $\mathfrak{X}_{\mathrm{GM}}$, Gowers and Maurey proved in [21] that every operator from a subspace of $\mathfrak{X}_{\mathrm{GM}}$ to the whole space is the sum of a scalar multiple of the inclusion operator and a strictly singular operator, which implies that $\mathfrak{X}_{\mathrm{GM}}$ is HI. (The reverse implication also holds for a complex HI space by a result of Valentin Ferenczi [14].) In particular, every operator on $\mathfrak{X}_{\mathrm{GM}}$ is a strictly singular perturbation of a scalar multiple of the identity. A strictly singular operator is one that is not an isomorphism on any (infinite-dimensional) subspace. Equivalently, an operator T on a Banach space X is strictly singular if every subspace contains for every $\varepsilon > 0$ an element x such that $\|Tx\| < \varepsilon\|x\|$. Strictly singular operators share many of the properties of compact operators. However, they form an operator ideal that is strictly larger than the ideal of compact operators. It was thus not clear whether $\mathfrak{X}_{\mathrm{GM}}$ was already a solution of the scalar-plus-compact problem.

Gowers was able to construct a strictly singular, non-compact operator on a subspace of $\mathfrak{X}_{\mathrm{GM}}$ [19]. Thus, $\mathfrak{X}_{\mathrm{GM}}$ could not be a solution of the stronger version of the scalar-plus-compact problem asking for all operators from a subspace to the whole space to be compact perturbations of scalar multiples of the inclusion map. Later, George Androulakis and Thomas Schlumprecht [1] showed that strictly singular, non-compact operators exist on the whole space $\mathfrak{X}_{\mathrm{GM}}$. A similar result was shown by Ioannis Gasparis [17] for Argyros-Deliyanni space (the first example of an asymptotic ℓ_1 HI space). For these reasons, it became widely accepted that the standard HI constructions would not solve the scalar-plus-compact problem and that a solution was some way away. This is why the landmark result of Argyros and Haydon came as such a surprise not just to the Banach space community but also to the authors. Indeed, Argyros and Haydon did not set out to solve the problem. They were trying to exhibit a hereditarily indecomposable predual of ℓ_1 as an extreme example of the phenomenon that the dual of an HI space need not be HI. They followed the classical ingredients for HI constructions: rapidly increasing sequences of ℓ_1-averages and special functionals using Maurey-Rosenthal coding. However, they did not follow the classical method of constructing HI spaces, which begins with the space of finite sequences and continues with the construction of a suitable exotic norm. Instead, they followed the method of Jean Bourgain and Freddy Delbaen [11] for constructing classes of $\mathscr{L}_\infty$-spaces. These are spaces that locally, *i.e.* at the level of finite-dimensional subspaces, look just like spaces of continuous functions on a compact space and have a priori nothing to do with HI spaces. The Bourgain–Delbaen method is very different from classical HI constructions in that they begin with a very familiar norm, the norm of the space ℓ_∞ of bounded sequences, and construct their space using carefully chosen exotic vectors in ℓ_∞. In the delicate construction of Argyros and Haydon, the classical HI ingredients are "woven into" the Bourgain–Delbaen method. Serendipitously, this extra $\mathscr{L}_\infty$-structure was exactly what was needed to show that strictly singular operators are compact. In order to explain this remarkable construction, we will have to delve into the constructions of HI spaces and their predecessors: Schlumprecht's space and Tsirelson space. We also need to take a close look at Bourgain–Delbaen spaces and the analysis of their subspace structure by Haydon some 20 years later.

3 Tsirelson space, Schlumprecht's space and HI constructions

Many constructions of Banach spaces, and in particular all the ones described in this section, begin with the space c_{00} of eventually zero scalar sequences. For an element $x = (x_i)$ of c_{00}, we let $\operatorname{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$ denote the *support of* x. For subsets A and B of $\mathbb{N}$, we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$; and for $x, y \in c_{00}$, we write $x < y$ if $\operatorname{supp} x < \operatorname{supp} y$, *i.e.* if the nonzero coordinates of x 'come before' the nonzero coordinates of y. Similarly, we write $n < x$ if $\{n\} < \operatorname{supp} x$, etc.

A norm on c_{00} is defined by means of a *norming set*, which is itself a subset of c_{00}. The action of an element $f = (f_i) \in c_{00}$ on $x = (x_i) \in c_{00}$ is defined as $\langle x, f\rangle = \sum_i x_i f_i$. Suppose that W is a subset of c_{00} containing the unit vector basis (e_n) (where e_n is the sequence whose n^{th} coordinate is 1 and all other coordinates are zero) and that W is contained in the unit ball of ℓ_∞. Then, the expression

$$\|x\| = \sup\{|\langle x, f\rangle| : f \in W\} \tag{1}$$

defines a norm on c_{00} satisfying $\|x\|_\infty \leqslant \|x\| \leqslant \|x\|_1$, where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ denote the ℓ_∞-norm and ℓ_1-norm respectively. The completion X of $(c_{00}, \|\cdot\|)$ is then a Banach space whose structure is determined by the choice of W. When considering the subspace structure of X, standard perturbation arguments show that it is sufficient to consider only a special type of subspace called a *block subspace*, which is one generated by a sequence of successive vectors $x_1 < x_2 < x_3 < \dots$ in c_{00}.

The first space we describe is the famous example of Boris Tsirelson [30]: a Banach space without a subspace isomorphic to one of the classical sequence spaces c_0 or ℓ_p, $1 \leqslant p < \infty$, which solved a longstanding open problem going back to Banach's book [9]. It is customary to use the term Tsirelson space for the dual space T of Tsirelson's original construction, which also solves the same problem. The description of T by Tadek Figiel and Bill Johnson [15] is an example of the general process described above for constructing Banach spaces. The norming set W is defined inductively: it is the smallest subset of c_{00} that contains $\pm e_n$ for all n and is closed under the following operation. If $f_1 < f_2 < \dots < f_n$ are in W and $n < f_1$ then $\frac{1}{2}(f_1+f_2+\dots+f_n)$ is also in W. Tsirelson space T is then defined to be the completion of $(c_{00}, \|\cdot\|)$, where the norm $\|\cdot\|$ is defined as in (1) above. Note that without the admissibility condition $n < f_1$ or without the factor $\frac{1}{2}$, the space T would be isomorphic to ℓ_1.

The proof that Tsirelson space contains no copy of any of the classical sequence spaces is surprisingly simple. First, it follows easily from the definition of the norming set W that T is an *asymptotic* ℓ_1 space, which roughly means that for any $n \in \mathbb{N}$, any n vectors 'deep enough' in the space behave like the unit vector basis of ℓ_1^n. More precisely, if $n < x_1 < \dots < x_n$ and $\|x_i\| = 1$ for all i then the norm of $\sum_{i=1}^n a_i x_i$ in T is the ℓ_1-norm $\sum_{i=1}^n |a_i|$ (up to a factor 2) for arbitrary scalars a_i. This immediately rules out the possibility of a subspace of T being isomorphic to ℓ_p for $1 < p < \infty$ or to c_0.

Proving that T contains no copy of ℓ_1 requires a bit more effort. This should not come as a surprise since, as we saw above, T is, in an asymptotic sense, close to ℓ_1. The proof is indirect: one demonstrates a hereditary property of T (*i.e.* one that passes to all subspaces) that ℓ_1 lacks. This property is *distortability*: an important property in its own right and one that is intertwined with the development of HI spaces. We say that a Banach space X with norm $\|\cdot\|$ is *distortable* if there exist $\lambda > 1$ and an equivalent norm $|||\cdot|||$ on X such that for every (infinite-dimensional) subspace Y of X, there are vectors $y, z \in Y$ with $\|y\| = \|z\| = 1$ and $|||y|||/|||z||| > \lambda$. In other words, $\|\cdot\|$ and $|||\cdot|||$ are not λ-equivalent, not even after passing to a subspace. In this case, we say that X is *λ-distortable*. A Banach space is *arbitrarily distortable* if it is λ-distortable for all $\lambda > 1$. An old result of Robert James [24] says that ℓ_1 and c_0 are not distortable, which means that a Banach space containing an isomorphic copy of one of these sequence spaces also contains almost isometric copies. Hence, a distortable space cannot contain a subspace isomorphic to ℓ_1 or c_0.

The proof that T is distortable features an ingredient that is crucial in all HI constructions: *rapidly increasing sequences of ℓ_1-averages*. We say that $x \in c_{00}$ is an *ℓ_1^n-vector of constant C* if there is a decomposition $x = x_1+x_2+\dots+x_n$ of x such that $x_1 < x_2 < \dots < x_n$ and $\|x_i\| \leqslant \frac{C}{n}\|x\|$ for all i. If, in addition, we have $\|x\| = 1$ then x is called an *ℓ_1^n-average of constant C* or an *ℓ_1-average of constant C and length n*. (Here $\|\cdot\|$ is the norm of T but the definitions make sense for any norm on c_{00}.) Note that, by the triangle inequality, C is necessarily at least 1. We say that a (finite or infinite) sequence $x_1 < x_2 < \dots$ is a *rapidly increasing sequence* (RIS) if x_i is an $\ell_1^{n_i}$-average of constant very close to 1 and $n_1 < n_2 < \dots$ is a very fast-growing sequence of integers. (These conditions are of course made precise.) If $x = x_1 + x_2 + \dots + x_n$, where $x_1 < x_2 < \dots < x_n$ is an RIS of length n, then we call x an *RIS vector (of length n)* and $x/\|x\|$ a *normalised RIS vector (of length n)*.

The earlier observation that T is asymptotic ℓ_1 shows that T, and indeed every block subspace of T, contains ℓ_1^n-averages of constant 2 for any n. James' argument showing that ℓ_1 is not distortable can be generalised to prove the following: if a space contains, for *some* $C > 1$ and for all n, ℓ_1^n-averages of constant C then it also contains, for *all* $C > 1$ and for all n, ℓ_1^n-averages of constant C. In particular, this holds in Tsirelson space and in all its block subspaces. In turn, this also implies that every block subspace of T contains RIS vectors of length n for all n. Let us now define a new norm $|||\cdot|||$ on T as follows. Fix some large $m \in \mathbb{N}$ and for $x \in T$, define

$$|||x||| = \sup \sum_{i=1}^{m} \|y_i\|, \tag{2}$$

where the supremum is over all decompositions $x = y_1 + \dots + y_m$ of x with $y_1 < \dots < y_m$. This is clearly m-equivalent to the original norm. By definition, an ℓ_1^n-average x of constant C comes with a specific decomposition $x = x_1 + \dots + x_n$, where $x_1 < \dots < x_n$ and $\|x_i\| \leqslant \frac{C}{n}$ for all i. These x_i add up in an ℓ_1-fashion in the sense that $\sum\|x_i\|$ is at most C. It turns out that the same is true for *any* decomposition of x; more precisely, if $x = y_1 + \dots + y_m$, where $y_1 < \dots < y_m$, then $\sum\|y_i\|$ is at most $C(1 + \frac{m}{n})$. Thus, if x is an ℓ_1^n-average of constant C with C very close to 1 and with n much larger than m then $|||x||| \approx 1 = \|x\|$. RIS vectors, on the other hand, behave differently. Let $z = z_1 + \dots + z_m$ be an RIS vector of length m, where $z_1 < \dots < z_m$ and z_i is an $\ell_1^{n_i}$-average of constant close to 1 and with $n_1 < \dots < n_m$ growing rapidly as in the definition. Then, $|||z||| \geqslant \sum_{i=1}^m \|z_i\| = m$. Since $\|z_i\| = 1$ for all i, there exist $f_1 < \dots < f_m$ in W such that $\langle z_i, f_i\rangle \approx 1$ for all i. Assuming, for argument's sake, that $m < f_1$, we have $f = \frac{1}{2}(f_1 + \dots + f_m) \in W$ and hence $\|z\| \geqslant \langle z, f\rangle \approx m/2$. The key combinatorial lemma for T shows that this lower bound is essentially the best possible, *i.e.* that $\|z\| \approx m/2$. The proof makes important use of the rapid growth of $n_1 < n_2 < \dots < n_m$. Thus, every block subspace of T contains a normalised vector whose $|||\cdot|||$-norm is at least approximately 2. This demonstrates that T is λ-distortable for any $\lambda < 2$.

We remark that if we replace the factor $\frac{1}{2}$ in the definition of the norming set W of T by an arbitrary $\theta \in (0, 1)$, we obtain a space that is λ-distortable for all $\lambda < \theta^{-1}$. Thus, for all $\lambda > 1$, there exists a λ-distortable Banach space. It is a famous open problem whether T is arbitrarily distortable. For some while, it was not known if such a space exists at all. A remarkable space constructed by Schlumprecht [29] provided the first example, as well as a key step toward subsequent constructions of HI spaces.

Schlumprecht's space S is defined in a similar way to Tsirelson space. We introduce the function $\varphi(x) = \log_2(1 + x)$

defined on $[1,\infty)$ and take our norming set to be the smallest subset W of c_{00} containing $\pm e_n$ for all n and closed under the following operation: for any $n \in \mathbb{N}$, if $f_1 < f_2 < \cdots < f_n$ belong to W then so does $f = \varphi(n)^{-1}(f_1 + \cdots + f_n)$. So, we no longer require the f_i to be 'far out' in c_{00}; instead, there is a higher price to be paid for a large value of n, namely the factor $\varphi(n)^{-1}$ replacing the constant factor $1/2$ of T. The space S is the completion of $(c_{00}, \|\cdot\|)$, where the norm $\|\cdot\|$ is defined as in (1) above.

The proof that S is arbitrarily distortable is similar to the argument used in T: the different behaviour of ℓ_1-averages and RIS vectors is key. We fix a large $m \in \mathbb{N}$ and define an equivalent norm $|\!|\!|\cdot|\!|\!|$ for S by (2). Again using James' argument, it is not hard to show that every block subspace of S contains ℓ_1^n-averages of constant C for all $C > 1$ and for all n, and hence RIS vectors of all lengths. Again, if x is an ℓ_1^n-average of constant C with C very close to 1 and with n much larger than m then $|\!|\!|x|\!|\!| \approx 1$. On the other hand, RIS vectors behave very differently. Let $z = z_1 + \cdots + z_m$ be an RIS vector of length m, where $z_1 < \cdots < z_m$ and z_i is an $\ell_1^{n_i}$-average of constant close to 1 and with $n_1 < \cdots < n_m$ growing rapidly as in the definition. Then, $|\!|\!|z|\!|\!| \geqslant \sum_{i=1}^m \|z_i\| = m$. Since $\|z_i\| = 1$ for all i, there exist $f_1 < \cdots < f_m$ in W such that $\langle z_i, f_i\rangle \approx 1$ for all i. Then, $f = \varphi(m)^{-1}(f_1 + \cdots + f_m) \in W$ and hence $\|z\| \geqslant \langle z, f\rangle \approx m/\varphi(m)$. The key combinatorial lemma, as in T but this time much harder to prove, says that this lower bound is essentially the best possible, *i.e.* that $\|z\| \approx m/\varphi(m)$. Thus, every block subspace of T contains a normalised vector whose $|\!|\!|\cdot|\!|\!|$-norm is at least approximately $\varphi(m)$. This shows that S is arbitrarily distortable.

The space $\mathfrak{X}_{GM}$ of Gowers and Maurey [21] was the first example of an hereditarily indecomposable Banach space and was a development of Schlumprecht's space. In addition to rapidly increasing sequences of ℓ_1-averages, the construction also made use of Maurey-Rosenthal coding. This was introduced by Bernard Maurey and Haskell Rosenthal in their construction of a normalised weakly null sequence without an unconditional subsequence [27]. As with the spaces T and S, the Gowers-Maurey space $\mathfrak{X}_{GM}$ is the completion of $(c_{00}, \|\cdot\|)$, where the norm $\|\cdot\|$ is defined by (1) with a suitably chosen $W \subset c_{00}$. The norming set W is defined as the smallest subset of c_{00} containing $\pm e_n$ for all n and satisfying certain closure properties. These include the one already used in the construction of S: if $f_1 < \cdots < f_n$ are in W then so is $\varphi(n)^{-1}(f_1 + \cdots + f_n)$. In addition, a crucial role is played by functionals of the form $\varphi(k)^{-1/2}(g_1 + \cdots + g_k)$ for certain *special sequences* $g_1, \ldots, g_k$ of elements of W. These have a certain tree-like structure: if $g_1, g_2, \ldots$ and $h_1, h_2, \ldots$ are special sequences and i is minimal such that $g_i \neq h_i$ then, for all $j, l > i$, the functionals g_j and h_l behave very differently. This is achieved by an injective function σ called the Maurey-Rosenthal coding: $\sigma(g_1, \ldots, g_{j-1})$ defines a certain parameter of g_j. These so-called *special functionals* are in some sense rare and, in particular, their length k must come from some very lacunary subset of $\mathbb{N}$.

It is again fairly straightforward to establish that every block subspace of $\mathfrak{X}_{GM}$ contains ℓ_1-averages of arbitrary length and constant arbitrarily close to 1, and hence RIS vectors of arbitrary length. The key inequality of S also holds in $\mathfrak{X}_{GM}$ with some restriction: if x is an RIS vector in $\mathfrak{X}_{GM}$ of length m then $\|x\| \approx m/\varphi(m)$, provided x is roughly speaking not aligned with a special functional that would force it to have a much bigger norm of at least $m/\sqrt{\varphi(m)}$. In particular, if m is far from the lacunary set of possible lengths of special sequences then $\|x\| \approx m/\varphi(m)$ always holds.

The key properties of T and S come from the different behaviour of ℓ_1-averages and RIS vectors. If we think of an RIS vector built from ℓ_1-averages as having complexity 1 then, in showing that $\mathfrak{X}_{GM}$ is hereditarily indecomposable, we shall need rapidly increasing sequences of complexity 2. By this, we mean sequences $x_1 < \cdots < x_m$, where each x_i is a normalised RIS vector of some length n_i and $n_1 < \cdots < n_m$ is a fast-growing sequence. The difference in behaviour comes from the alignment or otherwise of this complexity-2 RIS with a special sequence of functionals.

An easy equivalent definition of HI goes as follows. A Banach space X is hereditarily indecomposable if and only if, for any two (infinite-dimensional) subspaces Y and Z of X, the ratio $\|y + z\|/\|y - z\|$, $y \in Y$ and $z \in Z$, can get arbitrarily large. Informally, the 'angle' between any two subspaces is zero. Assume then that Y and Z are subspaces of X, which can be taken to be block subspaces after perturbation. It is then possible to build a complexity-2 RIS $x_1 < \cdots < x_k$ together with a special sequence $g_1 < \cdots < g_k$ of functionals such that $x_i \in Y$ for all odd values of i, $x_i \in Z$ for all even values of i and $\langle x_i, g_i\rangle \approx 1$. Set $y = x_1 + x_3 + \ldots$, $z = x_2 + x_4 + \ldots$ and $g = \varphi(k)^{-1/2}(g_1 + \cdots + g_k)$. Then, $y \in Y$, $z \in Z$ and $g \in W$ is a special functional. It follows that

$$\|y + z\| \geqslant \langle y + z, g\rangle = \varphi(k)^{-1/2} \sum \langle x_i, g_i\rangle \approx k/\sqrt{\varphi(k)}\,.$$

On the other hand, $\langle y - z, g\rangle = \varphi(k)^{-1/2} \sum (-1)^{i-1}\langle x_i, g_i\rangle \approx 0$ because of cancellations. Due to the tree-like structure of special functionals, the same holds for any special functional of length k. This is one of the situations when the key inequality of S applies, and we obtain $\|y - z\| \approx k/\varphi(k)$. The ratio $\|y + z\|/\|y - z\| \approx \sqrt{\varphi(k)}$ gets arbitrarily large and so $\mathfrak{X}_{GM}$ is hereditarily indecomposable.

As mentioned earlier, on the space $\mathfrak{X}_{GM}$, every operator is a strictly singular perturbation of a scalar multiple of the identity. This can be seen as follows. Given a bounded linear operator T on $\mathfrak{X}_{GM}$, we first find $\lambda \in \mathbb{R}$ such that $(T - \lambda)x_i \to 0$ for every infinite rapidly increasing sequence (x_i). This is done in two steps. First one shows that $d(Tx_i, \mathbb{R}x_i) \to 0$ as $i \to \infty$. If that were not the case then we could build an RIS vector x of complexity 2 (as in the proof that $\mathfrak{X}_{GM}$ is HI) such that $\|Tx\|/\|x\| > \|T\|$. It then follows that for some $\lambda \in \mathbb{R}$, we have $(T - \lambda)x_i \to 0$ as $i \to \infty$. By "merging" two infinite rapidly increasing sequences, it is clear that λ does not depend on (x_i). Finally, since every block subspace contains a rapidly increasing sequence, it follows that $T - \lambda$ is strictly singular. It is worth pointing out that for the stronger property of compactness, we would need $(T - \lambda)x_i \to 0$ for every bounded sequence $x_1 < x_2 < \ldots$.

Following the first construction of an HI space by Gowers and Maurey, the class of HI spaces has been studied extensively by many people, notably by Argyros and his coauthors. The motivation for this was Gowers' famous dichotomy theorem: for every Banach space X, either X is unconditionally saturated (every subspace of X contains a further subspace with an unconditional basis) or X has an HI subspace. It was therefore of interest to better understand the structure of HI spaces: their duality, quotients and operator algebras. In [2], Argyros showed that if a separable Banach space Z contains isomorphic copies of every separable, reflexive HI space then Z must, in fact, be universal, *i.e.* it must contain isomorphic copies of all separable Banach spaces. Thus, HI spaces are in some sense ubiquitous despite being so difficult to come by. Extending earlier work of Argyros and Vaggelis Felouzis [3], Argyros and Andreas Tolias [7] showed that every separable Banach space either contains ℓ_1 or is the quotient of an HI space, which implies that the class of HI spaces and their duals is very rich. A remarkable result of Argyros and Tolias [6] shows that there is a dual pair of Banach spaces: one is HI and the other is unconditionally saturated, *i.e.* they are on opposite sides of Gowers' Dichotomy. All these examples are Tsirelson-like constructions: they begin with c_{00} and a norming set W closed under certain operations that are defined by a sequence $(\theta_j, \mathcal{A}_j)$, where $\theta_j \in (0,1)$ and $\mathcal{A}_j$ is a family of finite subsets of $\mathbb{N}$ for each $j \in \mathbb{N}$. The norming set W is defined to be the smallest subset of c_{00} containing $\pm e_n$ for all n such that if $f_1 < \cdots < f_n$ are in W and $\{\min \operatorname{supp} f_i : 1 \leqslant i \leqslant n\} \in \mathcal{A}_j$ then $\theta_j(f_1 + \cdots + f_n) \in W$. Note that if $\theta_j = \frac{1}{2}$ and $\mathcal{A}_j = \{A \subset \mathbb{N} : |A| \leqslant \min A\}$ for all j then we obtain Tsirelson space T. If $\theta_j = \log_2(1 + j)^{-1}$ and $\mathcal{A}_j = \{A \subset \mathbb{N} : |A| \leqslant j\}$ for all j then we recover Schlumprecht's space S. It turns out that the classical sequence spaces ℓ_p can also be obtained by a suitable choice of $(\theta_j, \mathcal{A}_j)$. This was shown by Steven Bellenot in [10]. There is an alternative Tsirelson-like construction, which yields a norm equivalent to the ℓ_p-norm and which has a place in the story of the solution to the scalar-plus-compact problem. Fix real numbers $a, b \in (0,1)$ with $a + b > 1$ and let W be the smallest subset of c_{00} containing $\pm e_n$ for all n, such that if $f < g$ are in W then so is $af + bg$. Let us denote by $U_{a,b}$ the completion of $(c_{00}, \|\cdot\|)$, where $\|\cdot\|$ is defined by (1) for this choice of W. It was shown by Haydon in his work [22] on Bourgain–Delbaen spaces that $U_{a,b}$ is isomorphic to ℓ_p, where p is determined by the equations $1/p + 1/p' = 1 = a^{p'} + b^{p'}$.

4 The Bourgain–Delbaen construction

We now turn to the other line of research that paved the way to the solution of the scalar-plus-compact problem. Recall that a Banach space X is a $\mathscr{L}_\infty$-space if for some $\lambda \geqslant 1$ and for every finite-dimensional subspace E of X, there is a finite-dimensional subspace F of X such that $E \subset F$ and the Banach-Mazur distance $d(F, \ell_\infty^n) \leqslant \lambda$, where $n = \dim F$. Thus, locally, $\mathscr{L}_\infty$-spaces behave like $C(K)$ spaces, *i.e.* spaces of continuous functions on compact Hausdorff spaces. For this reason, it was believed that, in some sense, a $\mathscr{L}_\infty$-space cannot be too far from a $C(K)$ space. It therefore came as a surprise when Bourgain and Delbaen [11] constructed new examples that showed a great diversity of $\mathscr{L}_\infty$-spaces and settled several open problems in the theory of Banach spaces. In particular, they constructed separable $\mathscr{L}_\infty$-spaces not containing c_0 whose dual is ℓ_1. More precisely, for given real numbers a, b satisfying $0 < b < 1/2 < a < 1$ and $a + b > 1$, they constructed a separable $\mathscr{L}_\infty$-space $X_{a,b}$ that is somewhat reflexive (every subspace contains a further subspace that is reflexive) and whose dual is isomorphic to ℓ_1. The proof that $X_{a,b}$ is somewhat reflexive uses norm estimates involving the space $U_{a,b}$ defined above. Much later, Haydon carried out a careful analysis of the subspace structure of $X_{a,b}$. In [22], he proved that every block subspace of $X_{a,b}$ contains a subspace isomorphic to $U_{a,b}$ and that $U_{a,b}$ is isomorphic to some ℓ_p space.

All constructions of HI spaces before Argyros and Haydon begin with c_{00} and equip it with some exotic norm. The Bourgain–Delbaen construction is very different: they construct their space inside ℓ_∞ by choosing exotic vectors. We now describe this in a bit more detail, following [4]. The vectors constructed inside ℓ_∞ will be the biorthogonal functionals of a basis of ℓ_1. Denoting by X the subspace of ℓ_∞ generated by these vectors, ℓ_1 embeds into X^* (and will sometimes be the whole of X^*). For this reason, we think of elements of ℓ_1 as functionals. Accordingly, we denote by (e_n^*) the unit vector basis of ℓ_1, whereas we use the notation (e_n) for the same vectors when inside ℓ_∞. From now on, we shall also think of elements of ℓ_∞ as functions $x\colon \mathbb{N} \to \mathbb{R}$ and write $x(\gamma)$ instead of x_γ for $\gamma \in \mathbb{N}$. The support $\operatorname{supp} x$ of x now has the usual meaning for functions: $\operatorname{supp} x = \{\gamma \in \mathbb{N} : x(\gamma) \neq 0\}$.

A basis (d_n^*) of ℓ_1 is called *triangular* if $d_n^* = \sum_{j=1}^n a_{n,j} e_j^*$ with $a_{n,n} \neq 0$. We shall always take $a_{n,n} = 1$. In that case, the biorthogonal vectors (d_n) in ℓ_∞ form a basis of their closed linear span X. Moreover, for each $n \in \mathbb{N}$, the space $\operatorname{span}\{d_1, \ldots, d_n\}$ is isomorphic to ℓ_∞^n via the map sending x to its restriction to $\{1, \ldots, n\}$. To put it in another way, each $u \in \ell_\infty^n$ has a unique extension x into $\operatorname{span}\{d_1, \ldots, d_n\}$ with $x(\gamma) = u(\gamma)$ for all $\gamma \leqslant n$. This property will be crucial for the scalar-plus-compact property of the Argyros–Haydon space. The isomorphism constant between $\operatorname{span}\{d_1, \ldots, d_n\}$ and ℓ_∞^n turns out to be bounded by the basis constant of (d_i^*), which implies that X is a $\mathscr{L}_\infty$-space. The basis (d_n^*) is constructed recursively: at each step, an element $c_n^* \in \operatorname{span}\{e_i^* : i < n\}$ is chosen and d_n^* is defined to be $d_n^* = e_n^* - c_n^*$. (The recursion is actually done in batches and, at each step, several functionals are chosen at once.) Of course, the clever part of the Bourgain–Delbaen construction lies in the choice of the vectors (c_i^*). Since the norm on the space X is simply the ℓ_∞-norm, one needs estimates on how the evaluation functionals e_n^* act on certain vectors. Observing that $e_n^* = d_n^* + c_n^*$ and using induction, one can express these evaluation functionals as sums of successive functionals, which is called *the evaluation analysis* by Argyros and Haydon. In the Bourgain–Delbaen spaces $X_{a,b}$, the evaluation analysis features the functionals that define the spaces $U_{a,b}$. This makes the way in which the Tsirelson-like construction of $U_{a,b}$ is "woven" into the construction of $X_{a,b}$ more transparent.

5 The Argyros–Haydon space

We now finally turn our attention to the space $\mathfrak{X}_{AH}$ of Argyros and Haydon. As mentioned already, their initial aim was to construct a separable $\mathscr{L}_\infty$-space that is hereditarily indecomposable. Their idea was that since it was possible to recursively build into the dual of the Bourgain–Delbaen space $X_{a,b}$ the Tsirelson-like norming set of $U_{a,b}$, the same can perhaps be done with much more sophisticated Tsirelson-like constructions. Indeed, this was already done by Haydon for Tsirelson space T in [23]. The remarkable feat of Argyros and Haydon was to take this much further and build a Bourgain–Delbaen space modelled on a Tsirelson-like HI space. The construction begins with fixing two very fast-growing sequences (m_i) and (n_i) of positive integers. For $j \in \mathbb{N}$, let $\theta_j = m_j^{-1}$ and $\mathcal{A}_j = \{A \subset \mathbb{N} : |A| \leqslant 4n_j\}$. The sequence $(\theta_j, \mathcal{A}_j)$ then defines a Tsirelson-like space $T[(\theta_j, \mathcal{A}_j)]$ and this forms the backbone of $\mathfrak{X}_{AH}$, together with a Maurey-Rosenthal coding used in the construction of $\mathfrak{X}_{GM}$. As for the original Bourgain–Delbaen spaces, a triangular basis (d_n^*) of ℓ_1 is constructed recursively in batches. This leads to a partitioning of $\mathbb{N}$ into intervals $\Delta_1 < \Delta_2 < \dots$, where by an *interval* we mean a set of the form $\{m, m+1, \dots, n\}$ for some $m \leqslant n$ in $\mathbb{N}$. We begin with setting $\Delta_1 = \{1\}$ and $d_1^* = e_1^*$. The construction then proceeds in stages. At the n^{th} stage, intervals $\Delta_1 < \dots < \Delta_n$ and triangular basis vectors d_γ^*, $\gamma \in \Gamma_n = \bigcup_{i=1}^n \Delta_i$ for $\ell_1(\Gamma_n)$ are assumed to have been constructed. We then fix a certain, carefully chosen finite subset of $\ell_1(\Gamma_n)$ enumerated as b_γ^*, $\gamma \in \Delta_{n+1}$, which then defines the next interval Δ_{n+1}. The n^{th} stage of the construction is completed by setting $d_\gamma^* = e_\gamma^* - b_\gamma^*$ for each $\gamma \in \Delta_{n+1}$. This process then produces a basis $(d_\gamma^*)_{\gamma \in \mathbb{N}}$ of ℓ_1. As before, we now denote by (d_n) the biorthogonal sequence in ℓ_∞. The space $\mathfrak{X}_{AH}$ is defined to be the closed linear span of (d_n) in ℓ_∞. It follows that $\mathfrak{X}_{AH}$ is a $\mathscr{L}_\infty$-space with basis (d_n).

The key to the construction is of course the choice of the functionals b_γ^* in the recursive construction of the basis of ℓ_1. This is done in such a way that, for $\gamma \in \mathbb{N}$, the evaluation analysis of e_γ^* has the following form:

$$e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + m_j^{-1} \sum_{r=1}^a x_r^*,$$

where $1 \leqslant \xi_1 < \xi_2 < \dots < \xi_a = \gamma$ and the x_i^* are functionals so that the second term $m_j^{-1} \sum_{r=1}^a x_r^*$ is one of the norming functionals of the Tsirelson-like space $T[(\theta_j, \mathcal{A}_j)]$. The factor m_j^{-1} here is called the *weight* of γ, denoted $w(\gamma)$. The recursive construction of the basis (d_γ^*) of ℓ_1 is carried out in a way that ensures that there is an abundance of possible evaluation analyses produced by those γ that have *even weight* (meaning that j is even). On the other hand, there is a restriction on evaluation functionals of odd weight governed by a certain injective function $\sigma\colon \mathbb{N} \to \mathbb{N}$. This coding function is constructed alongside the triangular basis in the recursive process at the start. Now, if $\gamma \in \mathbb{N}$ has odd weight, say $w(\gamma) = m_{2j-1}^{-1}$, then e_γ^* has evaluation analysis

$$e_\gamma^* = \sum_{r=1}^a d_{\xi_r}^* + m_{2j-1}^{-1} \sum_{r=1}^a y_r^*,$$

where each y_r^* is some projection of an evaluation functional $e_{\eta_r}^*$, with $e_{\eta_r}^*$ having even weight $w(\eta_r) = m_{4\sigma(\xi_{r-1})}^{-1}$ for $1 < r \leqslant a$. Clearly, these evaluation functionals of odd weight correspond to the special functionals in Gowers-Maurey space. Because of the injectivity of σ, these evaluation functionals also have a tree-like structure.

Let us now turn to the properties of $\mathfrak{X}_{AH}$. As in any Tsirelson-like space, from the abundance of functionals of even weight, one can deduce that, for every $C > 1$, every block subspace of $\mathfrak{X}_{AH}$ contains arbitrarily long ℓ_1-averages of constant C. Recall that an ℓ_1^n-average of constant C comes with a decomposition as a sum of n vectors that add in an ℓ_1-fashion up to a factor C, and that the same holds for any decomposition into m successive vectors as long as m is much smaller than n. The analogous result in $\mathfrak{X}_{AH}$ has a similar proof and is as follows. If x is an $\ell_1^{n_j}$-average of constant C and γ is of weight m_i^{-1} with $i < j$ then $|x(\gamma)| \leqslant 2Cm_i^{-1}$. The next step is to define rapidly increasing sequences. Rather than defining this as a sequence of ℓ_1-averages, we instead use the property of ℓ_1-averages just observed. So, a (finite or infinite) sequence $x_1 < x_2 < \dots$ is a rapidly increasing sequence of constant C (or C-RIS) if:

(i) $\|x_k\| \leqslant C$ for all k

and there exists a fast-growing sequence $j_1 < j_2 < \dots$ such that, for all k,

(ii) $|x_k(\gamma)| \leqslant Cm_i^{-1}$ whenever $w(\gamma) = m_i^{-1}$ and $i < j_k$.

It follows from above that, for every $C > 2$, every block subspace of $\mathfrak{X}_{AH}$ contains a C-RIS.

Recall that the key inequality in Schlumprecht's space says that an RIS vector of length m has norm at most approximately $m/\varphi(m)$. There is an analogue in $\mathfrak{X}_{AH}$ with a similar proof and it is, in turn, a special case of what Argyros and Haydon call *the basic inequality*. It says that if $(x_k)_{k=1}^{n_{j_0}}$ is a C-RIS then

$$\Big\| \sum_{k=1}^{n_{j_0}} x_k \Big\| \leqslant 10C \frac{n_{j_0}}{m_{j_0}} . \tag{3}$$

(Note that in S one has $n_{j_0} = j_0$ and $m_{j_0} = \varphi(j_0)$.) Using terminology introduced earlier, the RIS above might be called a complexity-1 RIS. For a complexity-2 RIS, which can be built as in $\mathfrak{X}_{GM}$, there is a better norm estimate. Argyros and Haydon introduce the notion of an *exact pair*, examples of which include normalised RIS vectors. One can then recursively build sequences $(x_k)_{k=1}^{n_{2j_0-1}}$ of exact pairs, called *dependent sequences*, together with functionals that will end up being the evaluation analysis of some e_γ^* of odd weight $m_{2j_0-1}^{-1}$. This is analogous to building complexity-2 RIS vectors and special functionals in the proof that $\mathfrak{X}_{GM}$ is hereditarily indecomposable.

There are, in fact, two types of exact pairs (and hence dependent sequences): *0-pairs* and *1-pairs*. (This is not the exact terminology of [4].) As the name suggests, an exact pair is not just a vector x but a pair (x, η), where $\eta \in \mathbb{N}$ and $x(\eta)$ is 0 or 1. When the x_k are type-1 exact pairs taken from given block subspaces Y and Z by alternating between them, the resulting dependent sequence will satisfy the norm estimate

$$\Big\| \sum_{k=1}^{n_{2j_0-1}} (-1)^k x_k \Big\| \leqslant 40C \frac{n_{j_0}}{m_{j_0}^2},$$

which is an improvement on (3). This roughly follows from the fact that evaluating the above sum at γ would give something small and so, by the tree-like structure of odd-weight evaluation functionals, every other $e_{\gamma'}^*$ of weight $m_{2j_0-1}^{-1}$ is small on the sum. On the other hand, evaluation at γ gives

$$\Big\| \sum_{k=1}^{n_{2j_0-1}} x_k \Big\| \geqslant \frac{n_{j_0}}{m_{j_0}} .$$

This already implies that $\mathfrak{X}_{AH}$ is hereditarily indecomposable.

We now turn to operators. Assume that T is a bounded linear operator on $\mathfrak{X}_{AH}$. Assume that (x_i) is an infinite C-RIS such that $d(Tx_i, \mathbb{R}x_i) \not\to 0$. Using Hahn–Banach, one can then find $\delta > 0$ and build exact 0-pairs (z, η) with $Tz(\eta) > \delta$. One can then recursively construct a dependent sequence $(z_k)_{k=1}^{n_{2j_0-1}}$ of 0-pairs that forms an RIS, and for which one gets the improved estimate

$$\Big\| \sum_{k=1}^{n_{2j_0-1}} z_k \Big\| \leqslant C' \frac{n_{j_0}}{m_{j_0}^2},$$

where C' is a constant depending on C. On the other hand, the recursive construction produces an evaluation functional e_γ^* of weight $m_{2j_0-1}^{-1}$ for which

$$\sum_{k=1}^{n_{2j_0-1}} Tz_k(\gamma) \geqslant \delta' \frac{n_{j_0}}{m_{j_0}},$$

where δ' depends only on δ. Putting $z = \sum_{k=1}^{n_{2j_0-1}} z_k$, we obtain $\|Tz\|/\|z\| \geqslant \delta' m_{j_0}/C'$, which is a contradiction for large j_0. It follows that $d(Tx_i, \mathbb{R}x_i) \to 0$. It is not difficult to deduce that there is a $\lambda \in \mathbb{R}$ such that $(T - \lambda)x_i \to 0$ for *every* RIS (x_i). This already implies that $T - \lambda$ is strictly singular.

We now come to the most important property of $\mathfrak{X}_{AH}$ and show that the above operator $T - \lambda$ is in fact compact. At this point the "special functionals", *i.e.* evaluations of odd weight, play no role whatsoever; instead, the $\mathscr{L}_\infty$-structure becomes important. Suppose we are given $x \in \mathfrak{X}_{AH}$, which we write as $x = \sum_{\gamma\in\mathbb{N}} a_\gamma d_\gamma$ in terms of the basis (d_γ). Assume x has finite support and n is minimal so that $\{\gamma \in \mathbb{N} : a_\gamma \neq 0\} \subset \Gamma_n = \bigcup_{i=1}^n \Delta_i$. Call this n the *maximum range of* x, denoted max ran$x$. Let $u \in \ell_\infty(\Gamma_n)$ be the restriction of x to Γ_n. Recall that x is the unique element of span$\{d_\gamma : \gamma \in \Gamma_n\}$ that extends u, *i.e.* for which $x(\gamma) = u(\gamma)$ for all $\gamma \in \Gamma_n$ and, moreover, the map $u \mapsto x$ is an isomorphism. The set $\{\gamma \in \Gamma_n : x(\gamma) \neq 0\}$ is called the *local support of* x, which contains lots of information, in particular about rapidly increasing sequences. Suppose that (x_k) is a bounded block sequence. We say that (x_k) has *bounded local weight* if there exists $j \in \mathbb{N}$ so that each γ in the local support of x_k has weight at least m_j^{-1}. We say (x_k) has *rapidly decreasing local weight* if each γ in the local support of x_{k+1} has weight less than $m_{i_k}^{-1}$, where $i_k = \max\operatorname{ran} x_k$. In either case, the sequence (x_k) is an RIS. This allows one to split any bounded block sequence into the sum of rapidly increasing sequences (after passing to subsequence) by splitting the local support. Indeed, given a bounded block sequence (x_k), let u_k be the restriction of x_k to Γ_{i_k}, where $i_k = \max\operatorname{ran} x_k$. Write $u_k = v_k + w_k$ inside $\ell_\infty(\Gamma_{i_k})$ as a sum of disjointly supported vectors by splitting the support of u_k, *i.e.* the local support of x_k, according to the weight of the coordinates by choosing some suitable threshold. Let y_k and z_k be the unique extensions into span$\{d_\gamma : \gamma \in \Gamma_{i_k}\}$ of v_k and w_k respectively. Then, $x_k = y_k + z_k$ and (y_k) and (z_k) are bounded block sequences. After passing to a subsequence, it is possible to arrange that (y_k) has bounded local weight and (z_k) has rapidly decreasing local weight, and hence both are rapidly increasing sequences. Returning to our operator $T - \lambda$, since it converges to zero on any RIS, it follows that $T - \lambda$ also converges to zero on (x_k), which was an arbitrary bounded block sequence. In particular, $T - \lambda$ is compact.

6 Other developments and open problems

We already observed the peculiar properties of $\mathfrak{X}_{AH}$ that follow from the space having very few operators. In particular, it is the first Banach space on which every operator has a non-trivial invariant subspace. Let us now mention another application. In his 1972 memoir, Barry Johnson [25] introduced the notion of amenability of Banach algebras and posed the question of whether the algebra $\mathcal{L}(X)$ of all operators on a Banach space X can ever be amenable. It was pointed out by Garth Dales that the space $\mathfrak{X}_{AH}$ provides the first example simply because $\mathcal{L}(\mathfrak{X}_{AH}) = \mathcal{K}(\mathfrak{X}_{AH}) \oplus \mathbb{R}\,\mathrm{Id}$ is the unitisation of $\mathcal{K}(\mathfrak{X}_{AH})$, and it was already known that $\mathcal{K}(X)$ is amenable for a $\mathscr{L}_\infty$-space X.

Since the solution of the scalar-plus-compact problem, a large number of articles have appeared, inspired by the landmark paper of Argyros and Haydon. We briefly mention a few of these. Around the same time that the Argyros–Haydon paper was published, Dan Freeman, Ted Odell and Thomas Schlumprecht [16] obtained another very important result that also uses the Bourgain–Delbaen method of constructing $\mathscr{L}_\infty$-spaces. They show the universality of ℓ_1 as a dual space: if X is a Banach space with separable dual then X embeds into a $\mathscr{L}_\infty$-space Y whose dual Y^* is isomorphic to ℓ_1. Combining and extending the ideas and techniques in this paper and in the Argyros–Haydon paper led to the following beautiful result of Argyros, Freeman, Haydon, Odell, Raikoftsalis, Schlumprecht and Zisimopoulou: every separable, uniformly convex space is isomorphic to a subspace of a Banach space X with the scalar-plus-compact property and such that X^* is isomorphic to ℓ_1. Note that being hereditarily indecomposable, $\mathfrak{X}_{AH}$ has a very limited subspace structure, so this result shows the richness of the class of spaces with the scalar-plus-compact property.

One important open problem mentioned in [4] is whether there is a reflexive Banach space with the scalar-plus-compact property. Such a space would also be the first example of a reflexive space with the *invariant subspace property (ISP)*, *i.e.* on which every operator has a non-trivial, proper, closed invariant subspace. Although this problem is still open, in a major breakthrough, Argyros and Pavlos Motakis [5] were able to construct the first example of a reflexive space $\mathfrak{X}_{ISP}$ with the invariant subspace property. Moreover, $\mathfrak{X}_{ISP}$ is the first space for which every subspace has the ISP.

We conclude with two further important open problems. As mentioned earlier, for every subspace Y of the Gowers-Maurey space $\mathfrak{X}_{GM}$, every operator $Y \to \mathfrak{X}_{GM}$ is a strictly singular perturbation of a scalar multiple of the inclusion operator. It is not known whether there is a space X such that for every subspace Y of X, every operator $Y \to X$ is a *compact* perturbation of a scalar multiple of the inclusion operator.

We began with a discussion of the most general operator one can construct on a Banach space with the approximation property. For a general Banach space, the Hahn–Banach theorem only guarantees the existence of *nuclear operators*, which can be written as an absolutely convergent sum of rank-1 operators. The question thus arises whether there is a Banach space X on which every operator is of the form $\lambda\,\mathrm{Id} + N$, where N is nuclear. As Tim Gowers commented in one of his blogs [20], the Argyros–Haydon space is already the ‘ultimate space’, and this space X would be ‘beyond ultimate’.

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Systems of Points with Coulomb Interactions

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Large ensembles of points with Coulomb interactions arise in various settings of condensed matter physics, classical and quantum mechanics, and even approximation theory, and give rise to a variety of questions pertaining to calculus of variations, partial differential equations and probability. We will review motivations from these fields as well as "mean-field limit" results that allow us to derive effective models and equations describing those systems at the macroscopic scale. We then explain how to analyse the next-order beyond the mean-field limit, obtaining information about systems at the microscopic level. In the setting of statistical mechanics, this allows, for instance, the observation of the effect of temperature and a connection with crystallisation questions.

1 General setup

The 18th century physicist Charles-Augustin de Coulomb was the first to postulate that electrically charged particles interact with one another by a force proportional to the inverse square of their distance apart, in a way similar to Newton's gravitational force. In this paper, we are interested in large systems of points (or particles) interacting by such forces, having as motivation, besides the case of classical mechanics, numerous other situations that we will detail below.

Recalling that force is the gradient of energy, we consider a system of N particles with energy of the form

$$\mathcal{H}_N(x_1,\ldots,x_N) = \frac{1}{2}\sum_{1\le i\ne j\le N} \mathsf{g}(x_i-x_j) + N\sum_{i=1}^N V(x_i). \quad (1.1)$$

Here, the points x_i belong to the Euclidean space $\mathbb{R}^\mathsf{d}$, although it is also interesting to consider points on manifolds. The interaction kernel $\mathsf{g}(x)$ is taken to be

$$\mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d}=2, \quad (1.2)$$

$$\mathsf{g}(x) = \frac{1}{|x|^{\mathsf{d}-2}}, \quad \text{in dimension } \mathsf{d}\ge 3. \quad (1.3)$$

Up to a multiplicative constant, this is the Coulomb kernel in dimension $\mathsf{d}\ge 2$, i.e. the fundamental solution to the Laplace operator, solving

$$-\Delta \mathsf{g} = \mathsf{c}_\mathsf{d}\delta_0, \quad (1.4)$$

where δ_0 is the Dirac mass at the origin and c_d is an explicit constant depending only on the dimension.

It is also interesting to broaden the study to the one-dimensional logarithmic case

$$\mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d}=1, \quad (1.5)$$

which is not Coulombian, and to more general Riesz interaction kernels of the form

$$\mathsf{g}(x) = \frac{1}{|x|^\mathsf{s}}, \quad \mathsf{s}>0. \quad (1.6)$$

The one-dimensional Coulomb interaction with kernel $-|x|$ is also of interest but has been extensively studied and is well understood.

We also include a possible external field or confining potential V, which is assumed to be sufficiently smooth and tending to infinity fast enough at infinity. The factor N in front of V makes the total confinement energy of the same order as the total repulsion energy, effectively balancing them and confining the system to a subset of $\mathbb{R}^d$ of fixed size.

The Coulomb interaction and the Laplace operator are obviously extremely important and ubiquitous in physics, as the fundamental interactions of nature (gravitation and electromagnetic) are Coulombic. Below, we will further review the reasons for studying this type of system.

There are several mathematical problems that are interesting to study, all in the asymptotic limit $N\to\infty$:

(1) Understand the minimisers and possibly critical points of (1.1).

(2) Understand the statistical mechanics of systems with energy $\mathcal{H}_N$ and inverse temperature $\beta>0$, governed by the so-called Gibbs measure

$$d\mathbb{P}_{N,\beta}(x_1,\ldots,x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta\mathcal{H}_N(x_1,\ldots,x_N)} dx_1\ldots dx_N. \quad (1.7)$$

Here, $\mathbb{P}_{N,\beta}$ is the probability density of observing the system in the configuration $(x_1,\ldots,x_N)$ if the inverse of the temperature is $1/\beta$. The constant $Z_{N,\beta}$, which is called the "partition function" in physics, is the normalisation constant that makes $\mathbb{P}_{N,\beta}$ a probability measure,[1] i.e.,

$$Z_{N,\beta} = \int_{(\mathbb{R}^\mathsf{d})^N} e^{-\beta\mathcal{H}_N(x_1,\ldots,x_N)} dx_1\ldots dx_N. \quad (1.8)$$

(3) Understand the dynamic evolutions associated to (1.1), such as the gradient flow of $\mathcal{H}_N$ given by the system of coupled ODEs

$$\dot{x}_i = -\frac{1}{N}\nabla_i\mathcal{H}_N(x_1,\ldots,x_N), \quad (1.9)$$

the conservative dynamics given by the system of ODEs

$$\dot{x}_i = \frac{1}{N}\mathbb{J}\nabla_i\mathcal{H}_N(x_1,\ldots,x_N), \quad (1.10)$$

where $\mathbb{J}$ is an antisymmetric matrix, or the Hamiltonian dynamics given by Newton's law

$$\ddot{x}_i = -\frac{1}{N}\nabla_i\mathcal{H}_N(x_1,\ldots,x_N). \quad (1.11)$$

We can also be interested in these dynamics with added noise.

1 One does not know how to explicitly compute the integrals (1.8) except in the particular case of (1.5) for specific cases of V where they are called Selberg integrals (see [Fo]).

From a mathematical point of view, the study of such systems touches on several fields of mathematical analysis (partial differential equations, calculus of variations, approximation theory), probability theory, mathematical physics and even geometry (when one considers such systems on manifolds). Some of the crystallisation questions they lead to also overlap with number theory, as we will see below.

2 Motivations

There is a large number of motivations for the study of the above questions. We briefly describe some of them:

(1) In superconductors, superfluids and Bose–Einstein condensates, one observes the occurrence of quantised "vortices", which behave mathematically like interacting particles with two-dimensional Coulomb interactions. In these systems, the vortices repel each other logarithmically, while being confined together by the effect of the magnetic field or rotation, and the result of the competition between these two effects is that, as predicted by Abrikosov, the vortices arrange themselves in a perfect triangular lattice pattern, called an *Abrikosov lattice* (see Figure 1; for more pictures, see www.fys.uio.no/super/vortex/).
These systems are, in fact, described by an energy (the Ginzburg–Landau energy) and associated PDEs but we can show rigorously (in a study started by Bethuel–Brezis–Hélein and continued by [SS]; see also [Se1]) that, in the case (1.2), the analysis of the vortices is reduced to the discrete problems described above.
Another motivation is the analysis of vortices in classical fluids, such as that initiated by Onsager (see [MP]) or in plasma physics.

(2) Fekete points in approximation theory: these points arise in interpolation theory as the points minimising interpolation errors for numerical integration. They are defined as those points maximising the quantity

$$\prod_{i \neq j} |x_i - x_j|$$

or, equivalently, minimising

$$-\sum_{i \neq j} \log |x_i - x_j|.$$

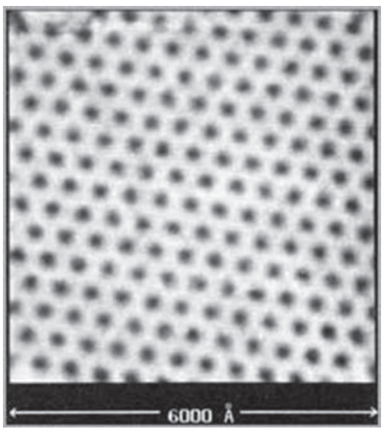

Figure 1. Vortices (in black) forming an Abrikosov lattice. H. F. Hess et al., Bell Labs, *Phys. Rev. Lett.* 62, 214 (1989).

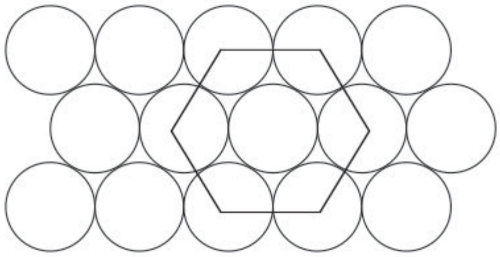

Figure 2. Solution of the sphere packing problem in dimension 2

They are often studied on the sphere or on other manifolds. In approximation theory [SK], we are also interested in the minimisation of Riesz energies

$$\sum_{i \neq j} \frac{1}{|x_i - x_j|^{\mathsf{s}}} \tag{2.1}$$

for all values of s. One can show that, by letting $\mathsf{s} \to 0$, the minimisers of Riesz energies converge to those of the logarithmic energy, whereas when $\mathsf{s} \to \infty$, they converge to the minimisers of the optimal sphere packing problem (whose solution in dimension 2 is known, from Fejes Tóth, to be the triangular lattice represented in Figure 2). It has been proven by Hales that the solution of the same packing problem in dimension 3 is an FCC (face-centred cubic) lattice, as was conjectured by Kepler. In higher dimensions, the solution is only known in dimensions 8 and 24, due to a recent breakthrough by Viazovska (see the presentation in [Coh] and the review [Sl]). In high dimensions, where the problem is important for error correcting codes, the solution is expected *not* to be a lattice.

(3) Statistical mechanics and quantum mechanics: in physics, the ensemble given by (1.7) in the Coulomb case is called a two-dimensional Coulomb gas or a one-component plasma and is a classic ensemble of statistical mechanics whose analysis is considered difficult due to the long range of the interactions. The study of the two-dimensional Coulomb gas, as well as the one-dimensional log gas, is also motivated by the analysis of certain quantum wave-functions (fractional quantum Hall effect, free fermions in a magnetic field, etc.), as well as by several stochastic models in probability (see [Fo]). The variant of the two-dimensional Coulomb case with coexisting positive and negative charges is interesting in certain theoretical physics models (XY-model, sine-Gordon), which exhibit a Kosterlitz–Thouless phase transition (see [Spe]).

(4) Random matrices (see [Fo]): in the particular cases (1.5) and (1.2), the Gibbs measure (1.7) corresponds, in certain instances, to the distribution law of the eigenvalues of certain well known ensembles:
   - The law of the complex eigenvalues of an $N \times N$ matrix where the entries are Gaussian i.i.d. is (1.7) with (1.2), $\beta = 2$ and $V(x) = |x|^2$. This is called the Ginibre ensemble.
   - The law of the real eigenvalues of an $N \times N$ Hermitian matrix with complex Gaussian i.i.d. entries is (1.7) with (1.5), $\beta = 2$ and $V(x) = x^2/2$. This is called the GUE (unitary Gaussian) ensemble.
   - The law of the real eigenvalues of an $N \times N$ symmetric matrix with Gaussian i.i.d. entries is (1.7) with (1.5),

$\beta = 1$ and $V(x) = x^2/2$. This is called the GOE (orthogonal Gaussian) ensemble.

(5) Complex geometry provides other examples of motivations. See, for instance, the works of Robert Berman and co-authors.

3 The mean-field limit and theoretical physics

Questions

The first question that naturally arises is to understand the limit as $N \to \infty$ of the *empirical measure* defined by

$$\mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \tag{3.1}$$

for configurations of points that minimise the energy (1.1), critical points, solutions of the evolution problems presented above and typical configurations under the Gibbs measure (1.7), thus hoping to derive effective equations or minimisation problems that describe the average or mean-field behaviour of the system. The term mean-field refers to the fact that, from a physics point of view, each particle feels the collective (mean) field $\mathsf{g} * \mu_N$ generated by all the other particles. Convergence in the mean-field is thus equivalent, in some sense, to the "propagation of molecular chaos" (see [Go]). From the statistical mechanics point of view, we also try to understand the temperature dependence of the behaviour of the system and the eventual occurrence of phase transitions.

The equilibrium measure

The energy (1.1) can be written as

$$\begin{aligned}\mathcal{H}_N(x_1,\dots,x_N) &= N^2\bigg(\frac{1}{2}\iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}\setminus\triangle}\mathsf{g}(x-y)d\mu_N(x)d\mu_N(y) \\ &\qquad + \int_{\mathbb{R}^\mathsf{d}} V(x)d\mu_N(x)\bigg),\end{aligned}$$

where $\triangle$ denotes the diagonal of $\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}$. Thus, it is natural to consider the "continuum version" of the energy, namely,

$$\mathcal{I}_V(\mu) := \frac{1}{2}\iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}}\mathsf{g}(x-y)d\mu(x)d\mu(y) + \int_{\mathbb{R}^\mathsf{d}} V(x)d\mu(x).$$

It is well known from potential theory that, in the space of probability measures, $\mathcal{I}_V$ admits a unique minimiser, μ_V, which is called the *equilibrium measure* and is characterised by the fact that there exists a constant c such that

$$\begin{cases} h^{\mu_V} + V \geq c & \text{in } \mathbb{R}^\mathsf{d} \\ h^{\mu_V} + V = c & \text{in the support of } \mu_V, \end{cases} \tag{3.2}$$

where

$$h^{\mu_V}(x) := \int_{\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d\mu_V(y) \tag{3.3}$$

is the (electric) potential generated by μ_V. This is true for Coulomb and for Riesz interactions, as well as for more general kernels. In the Coulomb case, the equilibrium measure can be interpreted with the help of an obstacle problem, and in the Riesz case with a fractional obstacle problem (see [Se1, Chap. 2]). An example is provided by Coulomb interaction (in any dimension) with confinement potential $V = c|x|^2$. In this case, we can verify that the equilibrium measure is always a (multiple of the) characteristic function of a ball. In the context of the Ginibre ensemble in Random Matrix Theory, this is known as the "circle law". Another important example is that of the logarithm interaction in dimension 1 with quadratic potential V: the equilibrium measure has density $\sqrt{x^2 - a^2}\mathbf{1}_{|x|<a}$, known in Random Matrix Theory as the (Wigner) *semi-circle law* for the ensembles GOE and GUE.

The energy $\mathcal{I}_V$ is the "mean-field limit" of the energy $\mathcal{H}_N$ and one can show, without much difficulty, that for the minimisers of $\mathcal{H}_N$, the empirical measure converges to μ_V and $\frac{1}{N^2}\min\mathcal{H}_N$ converges to $\mathcal{I}_V(\mu_V)$.

We can interpret $\nabla(h^\mu + V)$ as the total mean-force felt by a distribution with density μ. Therefore, in view of (3.2), it is null for the minimisers. More generally, we expect that the critical points of $\mathcal{H}_N$ have a limiting empirical distribution satisfying

$$\nabla(h^\mu + V)\mu = 0. \tag{3.4}$$

For the dynamics **(3)**, the formal limit of (1.9) or (1.10) is

$$\partial_t\mu = -\mathrm{div}\,(\nabla(h^\mu + V)\mu) \tag{3.5}$$

or

$$\partial_t\mu = -\mathrm{div}\,(\mathbb{J}\nabla(h^\mu + V)\mu), \tag{3.6}$$

again with $h^\mu = \mathsf{g} * \mu$. In the case (1.2), (3.6) with $V = 0$ is also well known as the vorticity form of Euler's equation.

The difficulty in rigorously proving the convergence toward solutions of these equations (whose well-posedness also needs to be proved) consists of passing to the limit in products of the type $\nabla h^\mu\mu$, which are nonlinear and a priori ill-defined in energy space. In the case of (1.2), we can overcome these difficulties by a reformulation of the terms introduced by Delort in the context of his works in fluid mechanics but this approach does not work in higher dimensions.

Until recently, all convergence results were limited to sub-Coulomb singularities ($\mathsf{s} < \mathsf{d} - 2$) or to dimension 1. Recently, a modulated energy method developed in [Se2] for the mean-field limit of the Ginzburg-Landau equations, based on the stability of solutions of the limiting equations for the "Coulomb norm" (or "Riesz norm")

$$\|\mu\|^2 = \iint \mathsf{g}(x-y)d\mu(x)d\mu(y),$$

allowed for the treatment of Coulomb interactions and for more singular Riesz cases.

Theorem 1 ([Se3]). *For the dynamics* (1.9) *and* (1.10)*, for all* d*, and all* $\mathsf{s} \in [\mathsf{d}-2, \mathsf{d})$ *in* (1.6)*, or* (1.5) *or* (1.2)*, the empirical measures converge to the solutions of* (3.5) *or* (3.6)*, when* $N \to +\infty$*, provided these are sufficiently smooth and the initial data energies converge to those of their limits.*

This result was preceded by one by Duerinckx in dimension 1 and 2 for $\mathsf{s} < 1$.

As far as (1.11) is concerned, the limiting equation is formally found to be the Vlasov-Poisson equation

$$\partial_t\rho + v\cdot\nabla_x\rho + \nabla(h^\mu + V)\cdot\nabla_v\rho = 0, \tag{3.7}$$

where $\rho(t,x,v)$ is the density of particles at time t with position x and velocity v, and $\mu(t,x) = \int\rho(t,x,v)dv$ is the density of particles. Notwithstanding recent progress, we do not yet know how to prove convergence of (1.11) to (3.7) when the interaction is Coulomb or has a stronger singularity. About this topic, one can consult the reviews [Jab, Go].

With temperature: statistical mechanics

Let us now consider (1.8) and turn our attention to problem (2). It is known that even with temperature the behaviour of the system is still governed by the equilibrium measure. The result can be phrased using the language of Large Deviations Principles and states, essentially, that if E is a subset of the space of probability measures, after identifying the configurations $(x_1, \dots, x_N)$ in $(\mathbb{R}^{\mathsf{d}})^N$ with their empirical measures, we have

$$\mathbb{P}_{N,\beta}(E) \approx e^{-\beta N^2(\min_E \mathcal{I}_V - \min \mathcal{I}_V)}, \tag{3.8}$$

which implies, due to the uniqueness of the minimiser μ_V of $\mathcal{I}_V$, that the configurations for which the empirical measure do not converge to μ_V have a very small probability. For example, in the case of matrices in GOE or GUE, for which the equilibrium measure is the semi-circle law, we deduce as an application a corollary of a result by Ben Arous and Guionnet: the probability that a GOE or GUE matrix is definite positive (and, thus, that all their eigenvalues are positive, which is incompatible with the semi-circle law because it is symmetric relative to 0) decreases like e^{-cN^2}.

In other words, at this leading order, temperature does not affect the mean-field behaviour of the system. (This is not what happens if we replace β by β/N: in this case, we have a modified equilibrium measure that spreads out with the temperature, minimising $\beta\mathcal{I}_V(\mu) + \int \mu \log \mu$.)

4 Beyond mean-field

In order to observe, for example, the effect of temperature (see Figure 3), it is interesting to go beyond the mean-field limit: expanding the energy $\mathcal{H}_N$ to next order we have, at the same time, access to information about the typical *microscopic* behaviour of the configurations. Observe that, at the microscopic scale, the typical distance between nearest neighbours is $N^{-1/\mathsf{d}}$.

Rigidity and Gaussian fluctuations

For minimisers of the energy $\mathcal{H}_N$ or of typical configurations under (1.7), since one already knows that $\sum_{i=1}^N \delta_{x_i} - N\mu_V$ is small, one knows, for instance, that the so-called discrepancy in balls $B_r(x)$, defined by

$$D(x, r) := \int_{B_r(x)} \sum_{i=1}^N \delta_{x_i} - N\, d\mu_V,$$

is of order $o(r^{\mathsf{d}}N)$ for fixed $r > 0$. It can be asked whether this estimation can be refined and if it remains true at mesoscopic scales, i.e., for r of order $N^{-\alpha}$ with $\alpha < 1/\mathsf{d}$, and for all temperatures. This would correspond to a *rigidity result*. We do get such a result for the energy minimisers. For configurations with temperature, in the context of bi-dimensional Coulomb interactions, we can prove a slightly different form of such a result: it is true when we integrate $\sum_{i=1}^N \delta_{x_i} - N\mu_V$ not over a ball but against a sufficiently smooth test function. In this way, we get an even more precise result, since we can prove that these quantities converge to a Gaussian with explicitly known mean and variance.

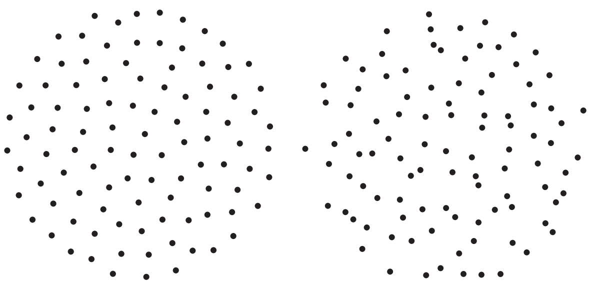

Figure 3. Case (1.2) **with** $N = 100$ **and** $V(x) = |x|^2$**, for** $\beta = 400$ **(left) and** $\beta = 5$ **(right)**

Theorem 2 ([LS2])**.** *In case* (1.2)*, let us assume that* $V \in C^4$ *and* μ_V *has connected support* Σ *with a regular boundary. Let* $f \in C_c^3(\Sigma)$*. Then,*

$$\sum_{i=1}^N f(x_i) - N \int_\Sigma f\, d\mu_V$$

converges in law to a Gaussian with

$$\textit{mean} = \frac{1}{2\pi}\left(\frac{1}{\beta} - \frac{1}{4}\right)\int_{\mathbb{R}^2} \Delta f \log \Delta V,$$

$$\textit{variance} = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2} |\nabla f|^2.$$

This result can be localised with test-functions f *supported on any mesoscale* $N^{-\alpha}$*,* $\alpha < \frac{1}{2}$*. It is also true for energy minimisers, formally taking* $\beta = \infty$*.*

For an idea of the proof, we suggest the lecture notes [Se4].

This result can be interpreted in terms of the convergence to a suitable *Gaussian free field*, a sort of two-dimensional analogue of Brownian motion. Note that a similar result was obtained by Bauerschmidt-Bourgade-Nikula-Yau and it was previously known for $\beta = 2$, and in the uni-dimensional logarithm case for all values of β.

If f is sufficiently smooth, the associated fluctuations are typically of order 1, i.e., much smaller than we could expect, for example comparing with the standard Central Limit Theorem, where the fluctuation of the sum of N i.i.d. random variables is typically of order $\sqrt{N}$. Proving this result in higher dimension or for more general interactions remains an open problem.

Next order in the energy

As we pointed out above, the approach we employ (initiated with Etienne Sandier and continued with Nicolas Rougerie, Mircea Petrache and Thomas Leblé) consists of studying the next order of the expansion of the energy about the measure $N\mu_V$, which is formally the minimiser. Expanding and using the characterisation (3.2), the "order 1" terms in $\sum_{i=1}^N \delta_{x_i} - N\mu_V$ vanish and we obtain

$$\mathcal{H}_N(x_1, \dots, x_N) = N^2\mathcal{I}_V(\mu_V) + F_N^{\mu_V}(x_1, \dots, x_N), \tag{4.1}$$

where

$$F_N^{\mu_V}(x_1, \dots, x_N) = \frac{1}{2}\iint_{\triangle^c} \mathsf{g}(x-y)\, d\left(\sum_{i=1}^N \delta_{x_i} - N\mu_V\right)(x)\, d\left(\sum_{i=1}^N \delta_{x_i} - N\mu_V\right)(y) \tag{4.2}$$

and again $\triangle$ denotes the diagonal $\mathbb{R}^d \times \mathbb{R}^d$. This is a next-order expansion of $\mathcal{H}_N$ valid for arbitrary configurations.

The "next-order energy" $F_N^{\mu_V}$ can be seen as the total Coulomb energy of the neutral system formed by N positive point charges at points x_i and a diffuse negative charge $-N\mu_V$ with the same mass. The goal is now to define the limit of this energy when $N \to \infty$, which will be the total Coulomb energy (per unit volume) of an infinite system of positive charges and a (let us say) uniformly distributed negative charge. In physics, such a system is called a *jellium*. The precise definition of this limiting energy is a bit complex but it uses, in a crucial way, the Coulomb nature of the interaction. In fact, since g is the kernel of the Laplacian, we observe that if $h^\mu = \mathsf{g} * \mu$ is the electrostatic potential generated by a charge distribution μ (with zero integral) then h^μ solves the Poisson equation

$$-\Delta h^\mu = \mathsf{c_d}\mu,$$

which is a local elliptic PDE. Additionally, using the Gauss-Green formula, we can write

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathsf{g}(x-y)d\mu(x)d\mu(y) &= -\frac{1}{\mathsf{c_d}}\int_{\mathbb{R}^d} h^\mu \Delta h^\mu \\ &= \frac{1}{\mathsf{c_d}}\int_{\mathbb{R}^d} |\nabla h^\mu|^2.\end{aligned}$$

In another way, we can rewrite the interaction energy (which involves a double integral) in the form of a single integral of a local function of the electrostatic (or Coulomb) potential generated by this distribution, itself a solution of a local equation. In Riesz's case, these manipulations can be replaced by similar ones using the fact that g is the kernel of an elliptic operator in divergence form, which is still local.

With the help of this observation, we succeed in defining an infinite volume energy for an infinite configuration of points C neutralised by a distributed charge (let us say -1) via the solutions of

$$-\Delta H = \mathsf{c_d}\left(\sum_{p\in C}\delta_p - 1\right).$$

We shall denote this energy by $\mathbb{W}(C)$. When the configuration of points C is periodic with respect to a lattice Λ, the energy $\mathbb{W}(C)$ has an explicit form: if there are M points a_i in the fundamental cell, we have (up to constants)

$$\mathbb{W}(C) = \sum_{1\le i\ne j\le M} G_{\mathbb{T}}(a_i, a_j),$$

where $G_{\mathbb{T}}$ is the Green function of the torus $\mathbb{T} := \mathbb{R}^d/\Lambda$.

We can show that $\mathbb{W}$ can be obtained as the limit (in a certain sense) of the functional $F_N^{\mu_V}$ in (4.1). It also follows from an expansion to the next order of the minimum of the energy $\mathcal{H}_N$ and from the fact that, after dilation, the minimisers of $\mathcal{H}_N$ must converge (almost everywhere with respect to the origin of the dilation) to a minimiser of $\mathbb{W}$ (see, for example, [Se1]).

We are therefore led to try to determine the minimisers of $\mathbb{W}$. This problem is extremely difficult, with the exception of the one-dimensional case, where we can prove that the minimum of $\mathbb{W}$ is attained by the lattice $\mathbb{Z}$. In dimension 2 and higher, the problem remains open and the only positive result is the following.

Theorem 3. *The minimum of $\mathbb{W}$ over lattices of volume 1 in dimension 2 is achieved uniquely by the triangular lattice.*

Here, the triangular lattice means $\mathbb{Z} + \mathbb{Z}e^{i\pi/3}$ properly scaled, i.e., what is called the Abrikosov lattice in the context of superconductivity. This partial result is, in fact, a result from number theory, known since the 1950s, about the minimisation of Epstein's zeta function (see [Mont] and references therein). It corresponds to minimising the height of a flat torus in Arakelov geometry.

Since the triangular lattice is observed in experiments with superconductors and since we have proved that the minimisation of the Ginzburg-Landau energy of the superconductor reduces to that of $\mathbb{W}$ [SS], it is natural to conjecture that the triangular lattice is a global minimiser of the energy.

According to a conjecture of Cohn-Kumar, the triangular lattice should be a universal minimizer in dimension 2 (i.e., should minimise a large class of interaction energies). An analogous role is played in dimension 8 by the lattice E_8 and in dimension 24 by the Leech lattice, which are solutions of the optimal packing problem, as was recently proved [Coh]. In these dimensions, the proof of the universal minimisation is near at hand.

In dimensions $\mathsf{d} \ge 3$ (except for $\mathsf{d} = 8$ and $\mathsf{d} = 24$), the minimisation of $\mathbb{W}$, even among lattices, is an open problem. As before, we can think that this relative minimum is global but we expect this to be true only in low dimensions since computer simulations provide clear indications that in dimensions $\mathsf{d} \ge 9$ the minimisers are not lattices.

These questions belong to the more general family of crystallisation problems for which very few positive results are known once the dimension is larger than or equal to 2 (see the review [BLe]).

Next order with temperature

In order to observe interesting temperature effects, as well as for applications to random matrices, we must consider $\beta_N = \beta N^{\frac{2}{\mathsf{d}}-1}$.

As we saw above, the macroscopic (or "mean-field") behaviour of the system does not depend upon the temperature and is given by the equilibrium measure. On the other hand, one can show that the microscopic behaviour depends on the temperature and is governed by a weighted sum of the energy $\mathbb{W}$ (from the previous paragraph) and a relative entropy. To formulate the result, one needs to dilate the configurations by $N^{-1/\mathsf{d}}$, as in the previous paragraph, and consider the limiting point process P^x obtained by averaging near each point x. Here, a point process is a law on infinite configurations of points. For instance, the Ginibre process is obtained by passing to the limit $N \to \infty$ (after dilation) of the Ginibre ensemble; the Poisson process Π with intensity 1 corresponds to points thrown independently of each other in such a way that the probability of having $N(B)$ points in a set B is

$$\Pi(N(B) = n) = \frac{|B|^n}{n!}e^{-|B|}.$$

Thus, one defines a "specific relative entropy" with respect to the Poisson process, denoted by $\mathsf{ent}[\cdot|\Pi]$, that we can think of as measuring how close the process P is to Poisson.

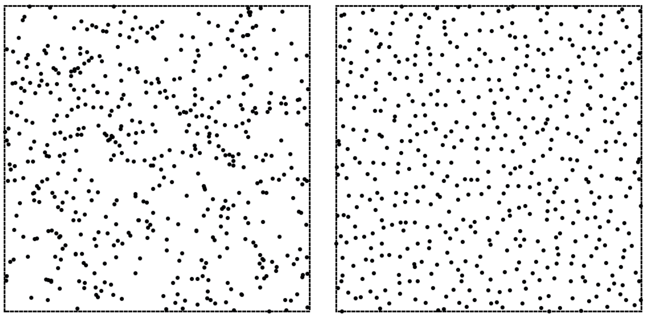

Figure 4. Simulation of the Poisson point process with intensity 1 (left) and the Ginibre process (right)

For all $\beta > 0$, we define the functional $\mathcal{F}_\beta$

$$\mathcal{F}_\beta(P) := \int_\Sigma \frac{\beta}{2}\mathbb{W}(P^x) + \mathsf{ent}[P^x|\Pi]dx, \qquad (4.3)$$

with $P = \int_\Sigma P^x dx$. We can now formulate a large deviations result.

Theorem 4 ([LS1])**.** *For all cases* (1.5)*,* (1.2) *and* (1.3) *with* $\mathsf{d} - 2 \leq \mathsf{s} < \mathsf{d}$*, with smooth assumptions on V and* μ_V*, and for all* $\beta > 0$*, we have a Large Deviations Principle at speed N with rate function* $\mathcal{F}_\beta - \inf \mathcal{F}_\beta$*, in the sense that*

$$\mathbb{P}_{N,\beta}(P_N \simeq P) \simeq e^{-N(\mathcal{F}_\beta(P) - \inf \mathcal{F}_\beta)}.$$

In this way, the Gibbs measure $\mathbb{P}_{N,\beta}$ concentrates on microscopic point processes that minimise $\mathcal{F}_\beta$. This minimisation is due to a competition between energy and entropy. When $\beta \to 0$, the entropy dominates and we can prove that the limit processes converge to a Poisson process. When $\beta \to \infty$, the energy $\mathbb{W}$ dominates, which, heuristically, forces the configurations to be more "ordered" and to converge to the minimisers of $\mathbb{W}$. Between these two extremes, we have intermediate situations and to know if there is a critical β corresponding to a crystallisation, or to a liquid-solid phase transition (which is conjectured to take place for (1.2) in some physics papers), is a problem that remains open. In dimension 1, on the other hand, due to the fact that we can identify the minimisers of $\mathbb{W}$, it can be concluded that a true crystallisation result holds when the temperature tends to 0.

One consequence of this result is to provide a variational interpretation for the few known limiting processes: the so-called "sine-β" process, the limit in the uni-dimensional case (1.5) and Ginibre's process, i.e., they must minimise $\beta\mathbb{W}+\mathsf{ent}$.

We would like to obtain more information about the limiting point processes, namely, the behaviour and decay of the "two-points correlation functions", which would shed light on the existence of phase transitions and crystallisation. Unfortunately, this theorem does not seem to provide much help for these problems.

As we have seen, many questions remain open, notably those of crystallisation, identification of minimisers and minima of $\mathbb{W}$ and $\mathcal{F}_\beta$, and the generalisation of Theorem 2 to dimensions $\mathsf{d} \geq 3$, to Riesz interactions and even to more general interactions.

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Sylvia Serfaty [serfaty@cims.nyu.edu] is interested in developing analysis tools to understand problems from physics. She has extensively studied the Ginzburg–Landau model of super-conductivity and this has led her to questions on the statistical mechanics of Coulomb-type systems. She has been splitting her career between Paris and New York.

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Interview with Alessio Figalli

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter

UP: Let me start with a standard and somewhat stupid question. Were you surprised getting the Fields Medal?

AF: (*smiling*) How could I answer anything but yes to that question? In fact, I even worried that it was a joke or a mix-up of some sort. But, to be serious though, in recent years, I tried not to think about it too much, lest I would be disappointed?

In other words, you ended up not being disappointed?

I certainly am not.

Looking at your CV, so to speak, one gets the idea that you were not a child prodigy in mathematics because you entered university at a rather normal age of 18.

That is true. I really did not have any serious exposure to mathematics until I started university. I did not even know what a derivative was. In my last year of high school, I attended a local mathematics Olympiad and it was that which made me decide to apply to Scuola Normale. I attended the humanistic branch of high school, where you focus on the classics: Greek and Roman.

This is not too uncommon; many of my continental colleagues have the same background. Did you enjoy the classical education?

Some parts of it were nice of course (that is inevitable) but it was also a bit boring. Studying was not really fun; sports were my great interest as a teenager and as a child.

So you do not consider it a disadvantage coming to mathematics relatively late?

It was all fresh to me and exciting. Fellow students who had more exposure than me thought they knew everything and skipped classes and, in that way, they also missed something (in fact, much more than they realised). I, on the other hand, never missed a class and I did all the exercises.

You were a good student in other words – exemplary, in fact.

I was a good student.

How come you ended up doing analysis?

In fact, I was first fascinated by algebra. It struck me as so elegant. But I gained no real intuition; it all seemed like magic. It was actually when I took a course in functional analysis that I began really to be interested in analysis. There were so many neat proofs and simple but general arguments.

It is for this reason that functional analysis is referred to as soft analysis.

I do not agree.

And from functional analysis you move to harder and more technical analysis?

I was initially attracted by calculus of variations, which led me to PDEs. What I like about the subject is that you can draw pictures, that you can develop a visual intuition.

No magic.

No magic in the sense that you can understand what is going on, see things coming rather than have things thrown at you seemingly from nowhere. Of course, this was my very personal perspective.

Your career was fast. You got your PhD at 23, five years out of high school and, since then, it has been a straight road through French CNRS, a sojourn in the States and now at ETH. I guess Italian and French are your languages, in addition to ubiquitous English, but what about German?

I don't speak German yet but this is not really a problem. ETH is a very international environment so, in fact, not much German is spoken. The students are very good and the institution gives you very good support. It is close to an ideal environment.

Do you read a lot and, if so, how do you read?

I read, of course, a lot as a student. I was a good student and had to catch up quickly. I already started to read articles in my second year. I do not read articles linearly; I look for the good parts where something is actually "happening". Often, I do not bother with proofs but prefer to work it out for myself if I find something interesting. I am a fast reader and I go through a huge amount of articles, for my own work but also for evaluation – that is part of your duty.

It is often remarked that the way mathematics is written is not conducive to conveying mathematics, that precise statements and details tend to obscure the underlying ideas. The thoughts of a mathematician have to be encoded in strict, logical language, which therefore have to be decoded by the reader. A mathematical conversation is supposed to be the most efficient way.

Of course but it is a luxury. With mathematicians physically spread all over the world, you cannot have personal conversations with them all; you have to make do with papers and emails.

How do you write your papers?
I never use paper and pen, except perhaps for doodling when I am thinking of mathematics. I type directly into the computer. Of course, I do not put down the definitive version right away; my proofs go through several revisions, during which I try to simplify arguments and rearrange material more naturally and logically. I can do it up to 50 times.

All on the computer? But I guess you grew up with computers.
I did. I am very comfortable with them.

But do you not need paper and pen when you manipulate formulas?
Yes, I do need paper and pen for that but I am also quite adept at doing complicated manipulations in my head. I do them all the time, also when typing.

Do you think that too many papers are being published?
Definitely. There are too many bad papers or, rather, insignificant variations upon variations being written.

Most maths papers are never read, except possibly by the referee. They are not written to make a contribution to mathematics but to further the career of the authors to prove that they are active.
That could well be true and of course for students it is a very good exercise to write papers.

But they should not be published?
The screening process should definitely be more discriminatory. In fact, things are going much too fast nowadays. So much is produced and so much progress is being made. Things should cool down a bit; not so much should be thrown at you.

There are so many mathematicians nowadays so, unlike in the past, there are not enough good people to judge and form informed and personal opinions. So good young people may drown in a flood of mediocrity, while in the past they would have been spotted earlier – thus this emphasis on number of papers, modified by citations and prestige of journals in which they appear. This is good for bureaucrats by supplying a formal objective mechanism for evaluation, not taking into account contents. One particular measure I was told by younger colleagues was the h-index – in my generation we never had to worry about such things.
I know, of course, about the h-index. It is all very bad, I agree with you, and those indices can easily be doctored. On the other hand, it is something we just have to accept; it is unavoidable. We have to justify our existence; we cannot very well hope to be funded just because we are clever and beautiful. And those indices say something at least.

The problem is that those who pay us do not really care about what we are actually doing, as long as we are doing something.
It would be really bad for us if mathematicians had to justify themselves with practical applications. We should be grateful for this not being the case.

You said things go too fast.
Yes I did but this is exciting – so much progress. I would not like to have lived in the past because the foundations and techniques we now take for granted were not available to mathematicians then, and they had to struggle much more.

This reminds me of modern technology: people all the time using gadgets when they have no idea how they work. This makes for alienation. People in the past often manufactured their own implements and thus had a more intimate relation to them. A similar thing goes on in mathematics. To make progress and stay ahead, you need to use results you may not understand and, as a consequence, your relation to mathematics becomes much less intimate. Are you using results that you do not understand?
I guess I have to plead guilty on some accounts but I would say that it is minimal; once I am serious about something, even if I am only interested in citing a result, I do check all the details. It is part of the instinct of self-preservation after all. No self-respecting mathematician would like to be caught making mistakes.

Especially not a Fields Medallist.
Especially not. I want to understand everything. And besides, the exact formulation of a theorem may not be precisely what you need. If you have grasped the ideas, you are in a position to be able to modify.

Who are your mathematical heroes?
That is easy: Caffarelli and de Giorgi. Caffarelli I met during my sojourn in the States and he has had a very deep impact on me, personally and scientifically. Many of the problems I work on were inspired by him. De Giorgi died relatively young so I only got to meet him through his papers, although I of course heard a lot about him at Scuola Normale, where he was a legend. He was of the opinion that mathematicians should only write in their own native language because that is the only language you can express yourself very well in…

…but there is a problem of terminology…
…I know. When it comes to mathematics, English is my native language. So when I give a talk to Italians, I hope that there will be some non-Italian in the audience so I can proceed in English. If not, you feel stupid lecturing in English. But back to de Giorgi. His papers are very different from the papers of today. There is much more text and verbal explanations and not so many formulas. Modern papers are much more technical.

So a paper by de Giorgi you would not want to skim?
Ideally not but there is only so much time. Modern papers are more efficient. But he writes so beautifully and they are such a pleasure to read.

You do not have any heroes of the past like Euler or Gauss? Did you ever read Bell's "Men of Mathematics"?
Frankly, I do not know that much of history. I am not that interested either. I am very happy working as a mathematician in the present, when there has been so much progress – maybe too much, as I said before. And as to the book you mention, I never read it but I suppose I should someday.

Frankly, I fear it is too late. It is, in a sense, a children's book, which, in many cases, has inspired young people to become mathematicians. But you are already a mathematician. So let us change tack – what is your position on the philosophy of mathematics, Platonism for example?
To be honest, I do not have any. It does not interest me or, at least, I have never given it a thought, which I guess amounts to the same thing. But I should say that one thing that attracts me very much about mathematics is that it is either true or false. So precise. Someone may not like my theorem but he cannot argue about its correctness. Whether he likes it or not, it is correct and that is something you do not have outside mathematics.

That is a philosophical statement if any. What about physics?
It is interesting but physical intuition I believe is different from mathematical.

Harder and maybe more subtle than in mathematics. People make elementary mistakes in reasoning about physics they would never do in mathematics.
No, I would not say that. It is just different.

And some of the intuition of those physicists has had remarkable impact on mathematics. But that is another story. Finally, what do you think this celebrity status will do to you?
Hopefully not too much. It is somewhat unreal at the moment. I went up very early the other morning and went through 600 emails – incidentally, it is thanks to this that I was able to respond to your email. This attention might be the most tangible effect of my celebrity status…

…it will pass soon. As a consequence, you must be very busy and I should not keep you any longer. Thanks for your time and a very nice conversation.
You are welcome.

Interview with Akshay Venkatesh

Ulf Persson (Chalmers University of Technology, Göteborg, Sweden), Editor of the EMS Newsletter


Photo by Rod Searcy.

UP: As usual, my first stupid question is whether you were surprised to receive the Fields Medal?
AV: Honestly, I was.

Are you sure? How did you find out about it?
I got an email from Mori asking me to get in touch with him on Skype.

Do you think that being a Fields Medallist will change your life? You are now a kind of movie-star celebrity but that status may well evaporate after the congress is over.
It is pleasant while it lasts but I hope that I will not suffer from additional pressure of expectation and will be able to carry on in my normal, carefree way.

You were a prodigy of sorts, participating in all kinds of mathematical Olympiads even before you were in your teens. By the way, when did you go to university?
At 13.

This is young – exceptionally so, I would say. So you were a real prodigy, which leads me to the question of competition in mathematics. When you first encounter mathematics at school, this is the subject most adapted for excelling in and showing that you are smart. But when did you discover that mathematics was a subject interesting in itself?
As to your original question, let me put it this way: I like the social aspect of mathematics, working and interacting with other people. On the other hand, I would hate to be in the position of being one of two people working on the same problem and see who gets there first. I want to be in a more relaxed situation. As to your second question, I guess it was at university when I first encountered the richness of mathematics…

…One should not forget that you were only 13 at the time. Please go ahead!
I learned a lot at the time by reading a huge number of books and articles.

How do you read an article?
I never actually recall reading an article from cover to cover. I tend only to skim through them and look out for what is interesting.

This is a common experience among the Fields Medallists that I have interviewed. In a way, writers of articles waste their time constructing carefully structured

narratives. It is the same thing with lectures: you let it just pass and then once or twice you catch something of interest and decide to take it home with you. But, of course, even if the structured narrative of a lecture appears superfluous in retrospect, you would never enjoy just random mutterings by the lecturer throwing out unrelated titbits of facts and ideas at you (although I have `suffered' through such lectures). Even if you just pick a few of the fruits, you want them nicely presented. By the way, your talk was really very excellent.
(*beaming*) You think so?

Definitely. You dispensed with tedious details and elaborate reasoning and precise statements without being vague and vacuous (as such lectures otherwise often turn out to be). I particularly liked your way of suggesting what a reciprocity theorem is really about, namely connecting primes with similar behaviour in specific situations. Your remarks on Hecke operators were also instructive. Such remarks are usually omitted in standard presentations but may be conveyed in private conversations. Speaking a bit metaphorically, in a lecture you want to know why a certain computation is being made but not see it performed. When you give lectures to students in a lecture course, going through computations may be justified but never to colleagues. You did it very well.
Thank you. I actually worried a bit. I have never given a talk to such a large and varied audience before.

It went very well. But, of course, you need to know some mathematics. I do not think that my wife would have been as enraptured. If you are not familiar with primes and have never heard about the Langlands programme, you would not get much out of it, if you do not mind me saying.
Of course not.

You mentioned in the video that you liked manipulating numbers when you were a child.
Not only as a child; I still do as an adult.

It is hard to be a mathematician, especially a number theorist, and not know that 2 is a prime but, in principle, you could go through life as a mathematician without caring whether 23 is a prime or not. We have the famous Grothendieck prime 27. I guess you care and, like Ramanujan, individual numbers as such are friends to you.
I might not go as far as that and I certainly would not pretend to rival Ramanujan. But it is true, if you devote your life to numbers, you better love them as individuals; that is only human.

Would you care to elaborate more concretely what it means to be friends with numbers and how it may manifest itself not just in mathematics but in everyday life?
I would rather point out that for a number theorist, especially nowadays, there are richer objects than mere numbers to be friends with. As examples, I can mention the cubic number field of discriminant –23 or the elliptic curve of conductor 11. Both are very pleasant company and it is also very useful to have them at your fingertips. But, I have to admit that many other such examples I have outsourced to my laptop. There is a limit to how many intimate friends you can have.

Speaking about laptops, I guess you came into contact with computers very early.
No. In fact, it was not until eight years after my PhD that I discovered computers.

Really? Yet, being so young you must have grown up with them, unlike my generation.
It is true but I did not, strange as it may appear, connect computers with mathematics. Number theory is taught and conducted in a very abstract manner and you do not get your hands dirty caring about whether 23 is a prime – to do so is looked down upon. But computers really meant a revolution to me; it fundamentally changed the way I think of mathematics – especially number theory.

And ties up with your childhood pleasure of manipulating individual numbers.
Of course. Making experimental calculations makes the subject so much more tangible. It is a great experience to have a theorem of yours numerically verified. It makes matter almost uncannily real.

It does confirm the Platonic nature of mathematics: that although mathematics is done by humans, the facts and results are independent of us; we just throw some light on them and they do not care an iota about us. Also, a numerical test tends to be more persuasive than just going through the arguments. It is indeed mechanical and unsentimental, impervious to wishful thinking. I want to come back to that later. Sorry for the interruption.
More seriously, doing computer experiments really guides you and allows you to abandon unfruitful avenues of research.

In geometry and also in analysis, to some extent, you can acquire a visual intuition; this is not the case with numbers, hence you lack the same immediate overview that the visual sense supplies. Thus, such experiments are invaluable; people like Gauss and Euler and even Riemann calculated a lot…
…And by hand to boot. This is really impressive. Once my laptop crashed and I was forced to do a long computation by hand. Isn't it tedious, and how many mistakes do you make? In fact, when you program, you make a lot of mistakes, which are, of course, pitilessly shown up by the compiler. It makes me wonder how many mistakes there must be in a run-of-the-mill paper.

This has struck me too. I know I am being cynical but that is the privilege of my age, maybe the only one; in most cases, it does not make too much of a difference – most papers are not read.
But it is different. Mistakes in programming are seldom conceptual and can, in most cases, be easily fixed but of course are catastrophic to the running of the program (a

misplaced comma causes havoc). A mathematical proof is much more stable, unless of course it is very formal.

Programming is a very relaxing activity, in my opinion; you never get stuck as you do in mathematics. You are not alone; through the compiler and test runs, you can actually converse with the computer and, if things do not work out, you always feel that you are within an epsilon of resolving the problem and you can keep on for hours. It is very seductive.

But when it comes to mathematics, I can always rely on co-workers with new ideas when I get stuck. Anyway, I repeat: actual computations are essential to number theory.

May I be so bold as to presume that the discovery of computer-based computations gave you a new lease on mathematics, making it even more exciting because of that connection?

Yes, indeed.

Talking about formal proofs and reducing them to some kind of super computation invites the issue of AI, which has not so far made any major intrusions into mathematics as it has in games like chess and go. What is your opinion on computer-assisted proofs (not just proof checking), which should in principle be within reach soon (although I guess it is a labour-intensive task to translate actual proofs into formal strings amenable to manipulation by computers, and I am personally a bit suspicious, as there are many subtle forms of reasoning we employ in proofs and it is not clear to me that we can exhaust them in advance), and also the construction of actual proofs. I am sorry for being long-winded but if we think of the ultimate, it would mean that we could type in a conjecture and then the computer decides whether it is true or not. Would that not kill mathematics; there would be no fun at all. It is the road that matters and not the destination. But there are certain philosophers of AI who seriously claim that mathematicians are wasting their time trying to prove theorems; instead, they should devise theorem-proving programs. This reveals a deep misunderstanding of the nature of mathematics and what makes mathematicians tick.

If computers come up with long proofs that we humans cannot understand, just as they come up with opaque moves in chess and go, it is not much good to me.

Yes, what would be the point?

I would not say that it would be pointless. That would be too categorical a statement.

Still, much of mathematical activity consists of overcoming technical difficulties; this is where the professional shows his mettle. Without those struggles, there would be no appreciation of new concepts, nor would they develop in the first place because nothing comes out of nothing.

There you have a point of course. And I could elaborate on what I really mean but I am sorry – I have a lunch date coming up soon, which I do not want to miss, so I have not enough time. Fire again!

Ideas are the most important in mathematics but, unlike theorems, which can be formulated precisely and provide a stepping stone, an idea cannot be formulated; it has to be inferred, often by an example in which it resides, hidden and to be unfolded by the reader. A specific theorem is just one of many ways an idea can be expressed. To use a theorem without appreciating the guiding principle is cheating; usually what you need is not necessarily encapsulated in the formulation of the theorem but can be accessible through the idea. My question is whether you have been cheating, taking results on trust.

I have to admit that I have done so at times. For example, I make use of results from p-adic Hodge theory and unfortunately I have little understanding of the proofs. I try my best to avoid this situation. But I see your point. As mathematics becomes more and more "big science" where there are teams working together, such things will become inevitable and also a bit sad.

Yes. This is a development I see coming and it is not welcome. Would you have preferred to be a mathematician of the past?

There would have been many advantages, one being that you could be much more general and not as specialised as you are forced to be today (and even more so I fear in the future). I have studied some of the mathematicians of the 18th and 19th centuries and have been very impressed with what they were able to do with much less technology and such primitive notions. But now I really have to leave and meet Harald.

We are not finished yet!

Okay. I can give you some more time but I am sorry – it cannot be much.

One speaks about problem solvers and those who like Grothendieck, wanting to build theories and understand things in some "functorial" way. Not only should a theorem be proved, it should also be proved in a natural and inevitable way out of the theory in which it dwelt. Grothendieck hated tricks.

I must say that I am neither. I do not aim for a general understanding – that is too ambitious for me. I am more modest; I am just happy to come across objects that are congenial to me, with a lot of rich structure to explore. But now I really must be going.

Do you mind if I tag along?

Not all the way to lunch though; I have some things I want to discuss with Harald.

(approaching) But that is Helfgott. I did one on him a few years ago.

He survived.

Yes, and is still smiling. Hello Harald…

See you. It has been fun talking to you.

Mathematics in the Historical Collections of École Polytechnique Part I

Frédéric Brechenmacher (École Polytechnique, Palaiseau, France)

A product of the French Revolution and the Age of Enlightenment, École Polytechnique has a rich history that spans over 220 years.[1] The École Polytechnique library was created, along with the school, in 1794 [Thooris, 1999]. Issued from revolutionary deposits, collections have been enriched through regular acquisitions and outstanding contributions. Since its creation, the library has been encyclopaedic, with a very strong dominance in the basic sciences. Its historic, scientific and artistic collections form a rich heritage ensemble.

This diverse ensemble, preserved within the library by its Historical Resources Centre[2] offers the possibility of consulting archives, ancient instruments, successive editions of scientific works and any form of iconography relating to the polytechnique or its students. The Historical Resource Centre also offers online resources (databases, digitalisations, a virtual museum and exhibitions)[3] and organises events throughout the year, including exhibitions, colloquia and loans of objects.

École Polytechnique has recently given birth to a museum, the mus'x, which unveils to the public a large selection of the collections.[4] A full interpretative offering is set up around these works; this includes not only visitor circuits punctuated with multimedia devices but also workshops, lectures and film screenings. The mus'x also offers seasonal exhibitions; for 2018, the inaugural exhibition is devoted to the mathematician Gaspard Monge.

The patrimonial collections of École Polytechnique offer a unique testimony to the evolution of interactions between science and society. They consist of collections of scientific instruments (measurement, hydrostatics, heat, electricity, acoustics, optics and chemistry), archives (institutional archives of the school, as well as collections transferred by individuals documenting scientific developments, techniques, education, economics, military, business, politics, etc.), as well as iconographic collections (paintings, drawings, engravings and photographs) and museography (sculptures, medals, furniture, etc.) and an historical library of works published from the 15th century onwards. These collections are therefore far from being limited to mathematics; they actually provide crucial historical resources for investigating the intertwined developments of mathematics, science, technology, pedagogy, economy, industry and politics. The patrimonial collections of Ecole Polytechnique are organised into three main section:

Gaspard Monge by Alexandre Colin, Paris, 1863. © Collections École polytechnique.

- The reserve of ancient books, which is encyclopaedic but with a dominance of scientific books.
- The diverse but coherent museum collection: scientific objects, medals, busts, uniforms, iconography, etc.
- The historical, administrative and scientific archives of École Polytechnique.

This article is divided into two parts. In this part, we shall first discuss the role of mathematics in the reserve of ancient books and in the museum collection, while a second paper, to be published in the next issue of the EMS Newsletter, will be devoted to the archives of École Polytechnique.

The reserve of ancient books

The reserve of ancient books contains about 17,000 volumes dating from 1456 to 1850. The historical collections of École Polytechnique were created alongside the school itself. In 1793, the war that set revolutionary France in opposition to a coalition of European nations disorganised the schools of instruction and teaching.[5] In 1794, Jacques-Élie Lamblardie, Director of the École des Ponts et Chaussées, who lost a great number of his pupils,

[1] For the history of École polytechnique, see [Belhoste, 2003] and [Belhoste et al., 1994].

[2] The mission of the Historical Resource Center is the preservation and enhancement of the archives, the historical library and the heritage of the School. See https://www.polytechnique.edu/bibliotheque/en/historicalresource-center.

[3] The École Polytechnique heritage numerical portal presents a selection of the historical collections, as well as studies and works developed by students in the course of teaching the history of science or technology (https://www.polytechnique.edu/bibliotheque/en/heritage-portal). For accessing the (large) section of the collections that has been inventoried, see the database: https://bibli-aleph.polytechnique.fr.

[4] See https://www.polytechnique.edu/bibliotheque/fr/le-musx-un-musée-à-lécole-polytechnique.

[5] Among other references, see [Dhombres and Dhombres, 1989], [Gillispie, 2004].

thought of creating a preparatory school for bridges and roads, and then for all engineers. Gaspard Monge, a former professor at the School of Military Engineering at Mezieres, was enthusiastic about this idea and convinced several members of the Comité de Salut Public (French Public Welfare Committee) and the Convention. Under the support of people such as the chemist François Fourcroy, a decree of 11 March 1794 created the Central School of Public Works, which would be renamed École Polytechnique one year later, on 1 September 1795. Its mission was to provide its students with a well-rounded scientific education with a strong emphasis in mathematics, physics and chemistry, and to prepare them upon graduation to enter the national institutes of public works, such as École d'Application de l'Artillerie et du Génie (School of Artillery and Engineering Applications), École des Mines and École Nationale des Ponts et Chaussées (National School of Bridges and Roadways). The Comité de Salut Public entrusted Monge, Lazare Carnot and several other scholars with enlisting, by means of a competitive recruitment process, the best minds of their era and teaching them science for the benefit of the French Republic.

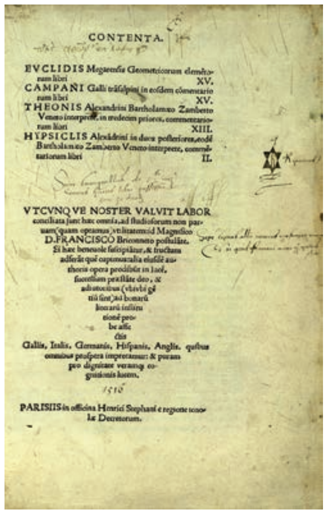
Mathematics in the reserve of ancient books. © Collections École polytechnique.

The content of the first plan of instruction was elaborated by Monge on two axes: the mathematics and the physics acquired by experiment in laboratories. This plan required both a library and a collection of scientific instruments. Both were initially constituted from property seizures that had been taken under the exigencies of revolution in three waves from 1789 to 1793, especially from private library collections of the aristocracy and clergy.

The property seizures started in 1789 when all the possessions of monastic communities were "put at the disposal of the Nation". As a result, the École Polytechnique library acquired in 1794 Vatani Florimondo Puteano's Latin edition of Euclid's *Elements*, printed in Paris in 1612 and belonging to the Collège of Bernardins, a Cistercian college founded in the 13th century that had served until the French Revolution as the residence for Cistercian monks and students at the University of Paris. It also acquired an earlier Latin edition of the *Elements* by Campanus of Novara, dating from 1537 and seized from the English Jesuit College at Leyden, as well as a first edition of François Viete's *Opera mathematica*, edited by Frans van Schooten in Leiden in 1646 and belonging to the English Jesuit College at Liège.

In 1792, a second wave of seizures confiscated the possessions of individuals, both laymen and clerics, who had left France as "emigrés" or "déportés". Amongst other volumes, Polytechnique's library acquired a first edition of Thomas Fincke's 1583 *Geometria rotundi* and

Euclidis, *Elementorum geometricorum lib. XV, cum expositione Theonis in priore XIII a Bartholomaeø [Zamberto], Veneto, latinitate donata, Campani in omnes,…*, Basileae : apud Johannem Hervagium, 1537. © Collections École polytechnique.



The 1516 edition of Euclid's *Elements* by the French humanist Jacques Lefèvre d'Étaples (also known as Jacobus Faber). © Collections École polytechnique.

Luca Valerio's 1606 edition of Archimedes' *Quadrature of the Parabola*, both gorgeously bound with the arms of Jacques Auguste de Thou, later belonging to the Earl Étienne Bourgevin Vialart de Saint Maurice who emigrated during the Revolution.

The white vellum gilded bounding of Fincke's 1583 *Geometria rotundi*, with the arms of Jacques Auguste de Thou. © Collections École polytechnique

The citrus veal gilded bounding of a 1606 edition of Archimedes' *Quadrature of the Parabola*, with the arms of Jacques Auguste de Thou and of his second wife Gasparde de La Chastre. © Collections École polytechnique.

The third wave of property seizures in 1793 targeted the possessions of the abolished royal institutions, such as universities, academies and corporations. The École Polytechnique library acquired about 500 volumes from École Royale du Génie de Mézières, a royal military engineering school that was suppressed in 1794 but nevertheless played a model role for the plan of instruction of the new École Polytechnique, established by Monge, a former professor of the Mézière school.

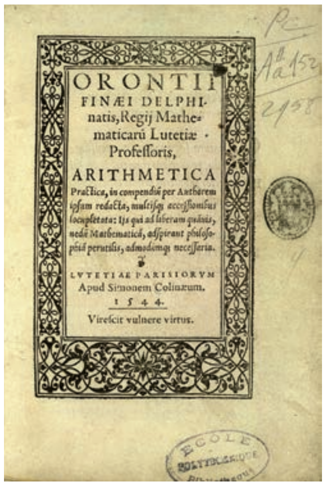
These various waves of seizures eventually resulted in the creation of the first library of École Polytechnique, with 564 volumes, including 76 books of mathematics.[6] The first book mentioned in the very first inventory is Leonhard Euler's 1768 *Institutiones calculi differentialis*. The second catalogue, established in 1799, amounted to 7555 volumes dating from the 15th century to the 18th century, including five incunabla printed during the earliest period of typographic printing in Europe, as well as a collection of periodical journals from the 17th and 18th centuries, among them the complete collections of the *Philosophical transactions of the London Royal Society* and the *Mémoires de l'Académie des Sciences* since their creation in 1664 and 1677 respectively. The first librarian was Pierre Jacotot, a physicist, assisted by François Peyrard, a professor of mathematics, who is especially remembered for his translations and commentaries of Euclid and Archimedes.[7]

This encyclopaedic collection covers all fields of knowledge in the sciences, the humanities and the arts. Its selection of mathematical volumes is therefore a testimony to a place attributed to mathematics in a classical culture rooted in the heritage of Greek antiquity and Renaissance humanism. The collection especially contains a great number of ancient editions of classical Greek works in which philosophy, poetry, dramaturgy and history sit alongside several Latin and French editions of Euclid's *Elements*, as well as works of Archimedes, Appolonius and Pappus ranging from the 16th century to the 18th century.

Christopher Clavius' edition of the 15th book of Euclid's *Elements*, printed in Cologne in 1591. © Collections École polytechnique.

The collections from the Renaissance contain the first 1543 edition of Copernicus' *De revolutionibus*, accompanied by the contemporary 1541 edition of Ptolemy's *Opera omnia*, along with first editions of several works that have played a key role in the development of algebra in Italy and in Europe: Luca Pacioli's *Divina proportione*, printed in Venice in 1509 with engravings of polyhedra by Leonardo da Vinci, Girolamo Cardano's *Opera omnia*, printed in 1663, Niccolo Fontana Tartaglia's 1556 *General trattato di numeri e misure*, Raphaël Bombelli's 1579 *Algebra opera*, Petrus Ramus' 1569 *arithmeticae*, Jacques Peletier's 1578 *Arithmeticae praticae*, Guillaume Gosselin's 1578 *L'arithmétique de Nicolas Tartaglia Brescia*, Christophorus Clavius' 1608 *Algebra*, and several works on arithmetic and geometry published from 1556 to 1585 by Orontius Finaeus, the first professor of mathematics of the Collège Royal (which would become the Collège de France) and one of the originators of Renaissance mathematics in France. These mathematical publications are part of a more general collection of works from Renaissance humanism, including treatises on religion, grammar, rhetoric, poetry, history, moral philosophy and the arts. Publications on the art of engineering include several treatises on mathematics, such as Jacob Köbel's 1551 *Astrolabii* or Jacques Besson's 1594 *Théâtre des instrumens mathématiques & méchaniques*. Historical treatises include works on the history of mathematics, such as Pierio Valeriano's 1556 investigation on the representation and symbolism of numbers in Ancient Egypt and Johannes Meursius' 1598 *De vita Pythagorae*.

Orontius Finaeus, *Quadrans astrolabicus, omnibus Europae regionibus inserviens…*, Parisiis: apud S. Colinaeum, 1534. © Collections École polytechnique.



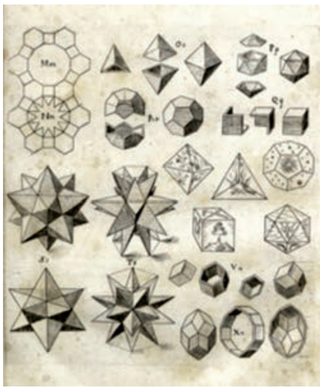
Orontius Finaeus, *Arithmetica practica, libris quatuor absoluta… Ex novissima authoris recognitione… emendatior facta. Aeditio tertia*, Parisiis: ex officina Simonis Colinae, 1542. © Collections École polytechnique.

The collection of books published in the 17th and 18th centuries presents quite a comprehensive panorama of the evolution of mathematics and especially the emergence of calculus, with first editions of the works of Joannes Kepler, Galileo Galilei, François Viète, William Oughtred, Pierre de Fermat, René Descartes, Marin Mersenne, Blaise Pascal, Frans van Schooten, John Neper, Christiaan Huygens, Guillaume de L'Hospital, Isaac Newton, Gottfried Wilhelm Leibniz, Johann Bernoulli, Leonhard Euler, Jean-Étienne Montucla, Étienne Bézout, Joseph-Louis Lagrange, Nicolas de Condorcet and Gaspard Monge. This collection also includes books devoted to what would nowadays be designated "applied mathematics", such as the Jesuit Georges Fournier's 1667 hydrographical investigations into practical and theoretical navigation, Nicolas Bion's 1752 treatise on the mathematical instruments used for navigation, artillery, fortifications, etc., as well as various tables of logarithms, such as the ones published by John Neper and Henry Briggs in 1628, and the decimal tables of logarithms published in the context of the definition of the metric system in 1800

[6] The inventory of the first collection of the École polytechnique library was established by Pierre Jacotot on 19 january 1795 (30 nivôse AN III), see [Rochas d'Aiglun, 1890].

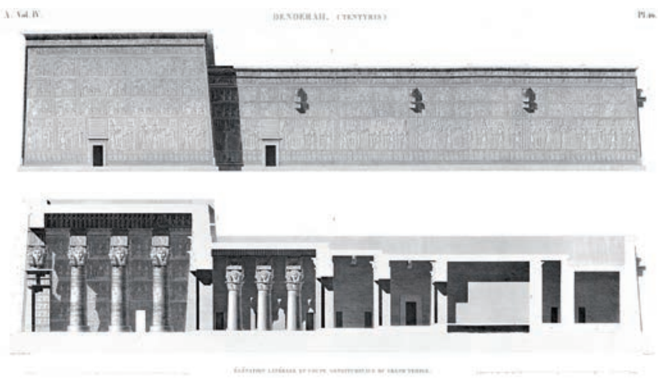
[7] For information on the works of Peyrard at the Polytechnique library, see [Langins, 1989].

by Jean-Charles Borda, Gaspard Prony, Joseph-Louis Lagrange, Pierre-Simon Laplace and Joseph Delambre.



Joannis Keppleri, *Harmonices mundi*, libri V, Lincii Austriae: sumptibus Godofredi Tampachii, 1619. © Collections École polytechnique.

From 1795 to 1802, about 10,000 new volumes were selected in the revolutionary deposits that had been constituted through the property seizures. The library collections also benefited from Napoléon's military campaigns. During the Italian campaigns of the French Revolutionary Wars, Monge was sent to Rome in 1798 and returned to France with about 100 books on architecture, including an edition of Vitruve printed in 1511 with a superb green leather binding made for Thomas Mahieu, a counsellor of Catherine de Médicis.



Lateral elevation and longitudinal cut of the great temple of Denderah by Édouard de Villiers du Terrage, an early student of École polytechnique in 1794. © Collections École polytechnique.

The French Campaign in Egypt and Syria (1798–1801) included an enormous contingent of scientists and scholars ("savants") assigned to the invading French force (167 in total). These scholars included several founding members of École Polytechnique such as Monge and the chemist Claude Louis Berthollet, as well as professors of the school, such as the mathematician Joseph Fourier, and many early students, such as the mathematician Étienne Malus and Pierre-François-Xavier Bouchard, who discovered the Rosetta Stone in July 1799. They founded the Institut d'Égypte with the aim of propagating Enlightenment values in Egypt through interdisciplinary work. The young polytechnicians applied Monge's descriptive geometry for establishing fortifications in Cairo, mapping the cartography of Egypt and describing the monuments of Ancient Egypt, which gave rise to a fascination of Ancient Egyptian culture in Europe and the birth of Egyptology. In addition to the many sections and elevations of ancient monuments the polytechnicians produced, François Edme Jomard established a correspondence between the various ancient measurement system and the new metric system. Even though many of the antiquities collected by the French in Egypt were seized by the British Navy and ended up in the British Museum, the scholars' research in Egypt gave rise to the monumental *Description de l'Égypte*, published on Napoleon's orders between 1809 and 1821. The first volumes were given by Napoléon himself to École Polytechnique in 1815.

Napoléon donating the *Description de l'Égypte* to École polytechnique in 1815. Drawn and engraved by Peronard. © Collections École polytechnique.

At the beginning of the First French Empire in 1804, Napoleon Bonaparte had granted École Polytechnique its military status and had also given the school its motto: "Pour la Patrie, les Sciences et la Gloire" (For the Nation, for Sciences and for Glory). The school was relocated from Palais Bourbon to the site of the former College of Navarre on Mount Sainte-Geneviève in Paris, providing enough space for the students to be housed on campus.

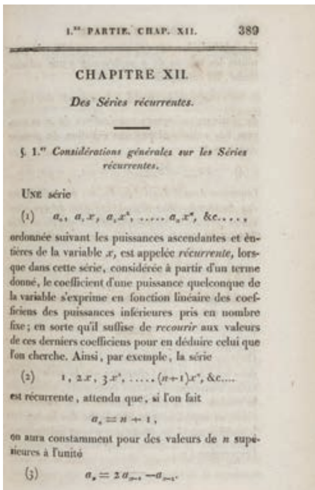
During the Bourbon restoration, the Duke of Angoulême became the protector of the school (from September 1816). Thanks to the friendship between the duke and Ambroise Fourcy (the director of the library from 1818 to 1842), École Polytechnique received a great many donations, including rare maps and medals edited by the Monnaie de Paris, as well as precious bindings for the library's ancient books and for the publication of the school's lectures, decorated with the arms of École Polytechnique.[8]

The front gate of École polytechnique on Mount Sainte-Geneviève: the encyclopaedic vocation of the school is represented on the right with the symbols of geometry, mechanics, navigation, cartography, astronomy, the humanities, the arts, etc. The mathematician Joseph-Louis Lagrange is represented on the first medallion in the top right corner.

The museum collection

This collection has the same origin as the reserve of ancient books.[9] The creation of the school in 1794 was marked by strong ideals that had developed during the Enlightenment on the role that the teaching of science should play in education. Mathematics was placed at the core of the instruction programme of École Polytechnique because it was considered to provide results

8 See [Thooris, 1997] and [Thooris, 2006].
9 For the history of this collection, see [Thooris, 1997].



1.re PARTIE. CHAP. XII. 389

CHAPITRE XII.

Des Séries récurrentes.

§. 1.er *Considérations générales sur les Séries récurrentes.*

Une série

(1) $a_0, a_1 x, a_2 x^2, \ldots\ldots a_n x^n,$ &c.....,

ordonnée suivant les puissances ascendantes et entières de la variable x, est appelée *récurrente*, lorsque dans cette série, considérée à partir d'un terme donné, le coefficient d'une puissance quelconque de la variable s'exprime en fonction linéaire des coefficiens des puissances inférieures pris en nombre fixe; en sorte qu'il suffise de *recourir* aux valeurs de ces derniers coefficiens pour en déduire celui que l'on cherche. Ainsi, par exemple, la série

(2) $1, 2x, 3x^2, \ldots\ldots (n+1)x^n,$ &c....

est récurrente, attendu que, si l'on fait

$$a_n = n + 1,$$

on aura constamment pour des valeurs de n supérieures à l'unité

(3) $a_n = 2a_{n-1} - a_{n-2}.$

Augustin Louis Cauchy, *Cours d'analyse de l'École royale polytechnique*, 1822. © Collections École polytechnique.

closer to the truth than any other science. Education in mathematics therefore had a moral value: practising such a science aimed to ensure the continuation of progress not only in science and technology but also in the morality of the younger generations. Furthermore, the ideal of universality associated with mathematics was at the core of the creation of the system of competitive exams for entering the "grandes écoles" such as École Polytechnique and École Normale. This system aimed to replace hereditary privileges with individual value, to be proved by solving mathematical problems. These strong ideals went along with the goal to create a new pedagogy that would promote both theoretical and practical knowledge. On the model of the recent pedagogical innovations that had been made in mining schools such as Schemnitz's school in Hungary, the founders of École Polytechnique aimed to promote science activities and experiments. This ambition required the creation not only of a library of theoretical treatises but also a collection of scientific instruments that could be used for practical experiments.

The first scientific objects were taken from the revolutionary deposits. This collection quickly expanded with the new apparatus that was invented by the professors (often alumni of the polytechnique) for their teaching and for their research. The Historical Resources Centre retains more than 400 ancient scientific objects, of which 87 have been the subject of a procedure for classifying historic monuments. These objects are classified in categories: measurement, hydrostatics, heat, electricity, acoustics, optics and chemistry.

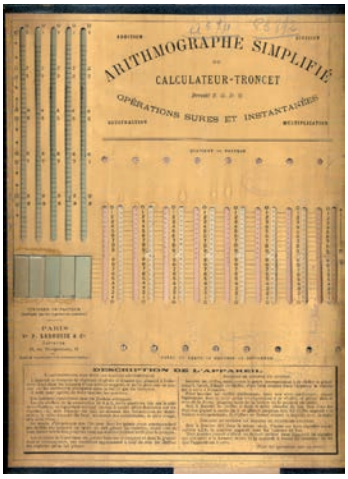
The measurement section includes the first prototype metre bar. Several of the founders and first professors of École Polytechnique actively participated in the meridional definition of the metre and, more generally, in the decimal-based metric system for length, mass and time. Lagrange, in particular, was a strong advocate for extending the metric system to decimal time, which was officially used for about a year from the beginning of the Republican Year III (22 September 1794) to 18 Germinal of Year III (7 April 1795).

The prototypes of the metre bar and the decalitre. © Collections École polytechnique.

The measurement collection also includes several calculating devices such as Thomas' arithmometer, Troncet's arithmograph and a Curta calculator. Thomas' arithmometer was the first digital mechanical calculator reliable enough to be used daily. It could add and subtract two numbers and could perform long multiplications and divisions effectively by using a movable accumulator for the result. This calculator was patented in France by Thomas de Colmar in 1820 and manufactured from 1851 to 1915. The device in the collection of École Polytechnique is not the original model conceived by Thomas de Colmar but a model improved by his son Thomas de Bojano in 1876.

Thomas' arithmometer. © Collections École polytechnique.

The arithmograph introduced by Louis Troncet in 1889 was a simple, cheap and popular mechanical add/subtract calculator. It is composed of sheet-cardboard sliders inside a cardboard envelope and is manipulated by a stylus, with an innovative carry mechanism, doing subtract ten, carry one with a simple stylus movement. This mechanism was later improved into metallic apparatuses, including the Addiators manufactured in Berlin by Addiator Gesellschaft from 1920 to 1982. A model of a small pocket Addiator from the 1960s is also retained in the collections of École Polytechnique.

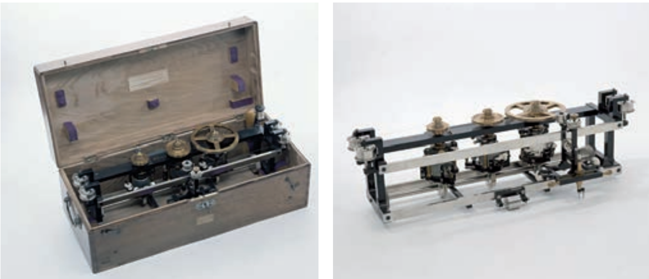


Troncet's arithmograph. © Collections École polytechnique

The Curta is a small mechanical calculator developed by Curt Herzstark. The Curta's design is a descendant of Gottfried Leibniz's Stepped Reckoner and Thomas' arithmometer, accumulating values on cogs, which are added or complemented by a stepped drum mechanism. It has an extremely compact design: a small cylinder that fits in the palm of the hand. About 140,000 models of the Curta were produced from 1948 to 1972.

The Curta calculator. © Collections École polytechnique.

The acoustic section includes an harmonic analyser from 1933. This analyser was designed by Olaus Henrici of London in 1894 for determining the fundamental and harmonic components of complex sound waves. It was then developed and mechanically constructed by G. Coradi of Zurich. The analyser consists of multiple pulleys and glass spheres, called rolling-sphere integrators, connected to measuring dials. The image of a curve (for example, a tracing of a sound wave) is placed under the device. The user moves a mechanical stylus along the curve's path, tracing out the wave form. The resulting readings on the dials give the phase and amplitudes of up to ten Fourier harmonic components.

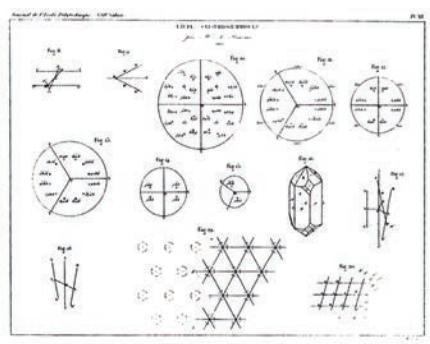


The Coradi analyser. © Collections École polytechnique.

In addition to the objects used for mathematical operations or investigations, several other scientific instruments retained in the collection of École Polytechnique shed light on the changing interactions between mathematics and other sciences and are therefore very relevant for investigations in the history of mathematics. To give just one example, the collections of minerals, polyhedra cardboard models and goniometers highlight intimate connections between crystallography, geometry, mechanics, optics (especially Augustin Fresnel's theory of light) and algebra with the emergence of group theory in the works of the polytechnicians Louis Poinsot, Augustin Louis Cauchy, Auguste Bravais, Joseph Bertrand, Camille Jordan and Henri Poincaré.

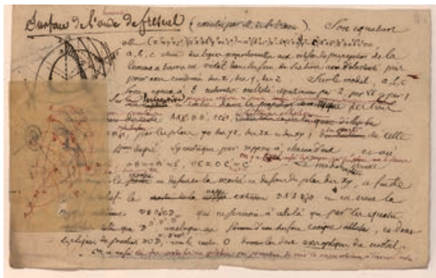
The collection of scientific objects is still expanding today thanks to donations of alumni, professors and researchers, and objects purchased by alumni associations.

Cardboard cristal model. © Collections École polytechnique.



Bravais' cristallographic investigations in the *Journal de l'École polytechnique* in 1850.

Fresnel polyprism. © Collections École polytechnique.



A manuscript by Adhémar Barré de Saint-Venant on the Fresnel wave surface, 1855. © Collections École polytechnique.

Very recently, in June 2018, the collection received a specimen of a herpolhodographer that was used for experimental works in mechanics to trace herpolhodes, i.e. surfaces created by the rotation of a rigid body around its centre of gravity.[10] This instrument was conceived by the mathematicians Gaston Darboux and Gabriel Koenigs in the early 1890s; it was constructed by Château Père et Fils in Paris in 1900 and presented to the Exposition Universelle in Paris, the world fair that hosted the International Congress of Mathematicians in which David Hilbert presented his famous list of problems.

Herpolhodographer of Gaston Darboux and Gabriel Koenigs. © Collections École polytechnique.

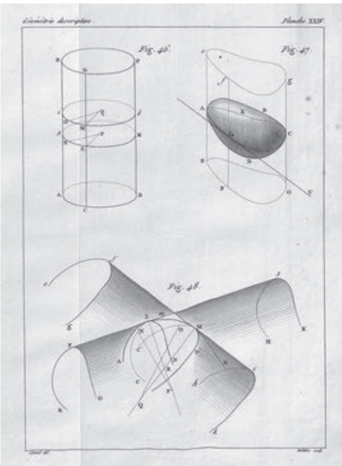
In addition to scientific instruments, the museum collection of École Polytechnique also contains a variety of other types of objects, such as drawings, engravings, paintings, busts, medals, uniforms, stained glasses and photographs. When the school was founded, a number of master drawings from the 18th century were taken from revolutionary deposits for the teaching of drawing by imitation. For instance, the *Bélisaire* drawn in 1779 by Jacques-Louis David was previously in the possession of the Marquis de Clermont d'Amboise, Ambassador of France in Napoli.

Jacques-Louis David, *Bélisaire*, 1779. © Collections École polytechnique.

At the foundation of the school, the teaching of drawing was very close to that of geometry. Monge's descriptive geometry provided a mathematical formulation for the diversity of graphical techniques

[10] See [Résal, 1895, p.249], and [Appel, 1953, p.192]

that had been developed by engineers for the arts of stone cutting, mapping, building, fortifying, etc. Monge's own pedagogical ideals gave an important role to action-learning through experimentation and drawing. It is known from the archives of École Polytechnique that Monge had created a cabinet of mathematical models that were used as drawing models for the teaching of descriptive geometry, along with other types of models, such as crystallography models and mechanical devices.[11] Yet, because engineers did not always have the adequate conditions to make geometrical drawings rigorous and precise, they also had to master more classical forms of drawing, which they learned by imitating master drawings such as David's *Bélisaire*. Several students of Monge later advocated the use of drawing models in the teaching of descriptive geometry. Among them, Théodore Olivier and Libre Bardin – who both taught descriptive geometry and mathematical drawing – were pioneers in constructing mathematical models in the 1830s and 1840s. In the late 1860s, Olivier's string and metal models and Bardin's plaster models (as edited by Charles Muret, one of Bardin's students) made great impressions on mathematicians such as Félix Klein and Gaston Darboux, who, in turn, played a key role in the emergence of large collections of mathematical models, such as the ones of the Göttingen Mathematisches Institut and the Institut Henri Poincaré in Paris [Brechenmacher, 2017].



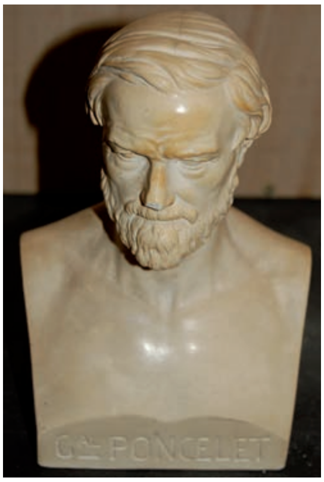
Manuscripts of Monge's lectures on descriptive geometry. © Collections École polytechnique.

The École Polytechnique museum collection also contains a very large iconographic collection that documents the evolution of mathematician figures, including engravings of the young Lagrange in the mid-18th century, plaster medallions made by David d'Angers and representing Poinsot, Cauchy and Pierre-Simon Laplace, 19th centuries paintings of Lagrange, Laplace and Joseph Bertrand, and busts of Monge, Laplace, Jean-Victor Poncelet, Bertrand, Michel Chasles, Ossian Bonnet, Georges Halphen and Henri Poincaré, as well as 20th century photographs of mathematicians such as Jacques Hadamard, Paul Lévy and Laurent Schwartz.

Monge.

Cauchy.

Poinsot.



A bust of Jean-Victor Poncelet. © Collections École polytechnique.

Henri Poincaré in the picture album of the 1873 promotion of École polytechnique. © Collections École polytechnique.

Joseph Bertrand painted by Léon Bonnat in 1896. © Collections École polytechnique.

A photograph of Jacques Hadamard from Studio Harcourt. © Collections École polytechnique.

[11] For the application of Monge's teaching of descriptive geometry to the drawing of machines, especially by Hachette, see [Dupont, 2000].

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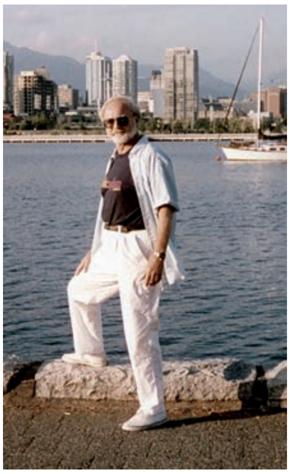
In Memoriam of Edwin F. Beschler (1931–2018)

Thomas Hintermann (Director, EMS Publishing House)



Edwin Beschler looking cool in front of the Vancouver skyline, 1993.

For almost 40 years, Edwin Beschler was one of the great figures of mathematical publishing. From 1961 until his move to Springer in 1989, he played a major role in the turbulent history of Academic Press, in varying positions from acquisitions editor in mathematics and related fields to president of the company. He himself wrote a highly interesting article on the origins and early days of this enterprise, arguably the most influential American publisher at that time [1]. Also noteworthy is his account of the foundation of the *Journal of Combinatorial Theory*, giving a lively and often amusing glimpse of the publishing processes in the 1960s [2]. From 1989 until his retirement in 1999, Edwin Beschler served as Executive Vice President and Editorial Director of Birkhäuser Boston, a subsidiary of Springer New York.

The first time I met Edwin was in Basel, in early 1990, shortly after becoming Mathematics Editor of Birkhäuser. At the time, Edwin Beschler was head of Birkhäuser Boston and he made the trip to Basel twice each year to attend the board of directors meetings. For the next 28 years, through our changing jobs and lives, we maintained a friendship even if our personal encounters remained rare. Professionally, Edwin was a great inspiration. While I was just a fledgling editor, he was looking back on a long and distinguished publishing career. His rich supply of anecdotes, insights and reminiscences provided an essential part of my education as a publisher. For years I collected letters written by Edwin which not only gave testimony to his superior linguistic style, but also his inimitable ways of putting even the most delicate and unpleasant messages into winsome words.

Edwin and T.H. on the road to the AMS-DMV Meeting in Heidelberg, 1993.

Fern and Edwin Beschler and T.H. during the ICM in Zürich, 1994. (All photos private collection Thomas Hintermann)

With Edwin a great figure of mathematical publishing, and an extraordinary man, has left us. A short obituary by Fern Beschler and Ina Mette appeared in the *Notices of the AMS*, Volume 65 (2018), number 6, p. 716.

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On How to Climb the Highest Mountains – Experiences During a Week in the Heidelberg Laureate Forum

Demian Nahuel Goos (Universidad Nacional de Rosario, Rosario, Argentina)

The author shares his thoughts and experiences of the 5th Heidelberg Laureate Forum in 2017, during which he conversed with numerous mathematics and computer science laureates.

Introduction

A gale is blowing. Whiteness everywhere. A heavy blizzard has reigned over the landscape for two days, with no end in sight. The snow does not fall but rather hits you in the face, and the wind is as loud as the engines of a Boeing 747-400. It's cold. It's freezing. However, you no longer feel it. You lose your senses: up is down and down is up. What can you see? Nothing. Only wilderness. Both the peak that was recently left behind and home feel as far away as far can get. Surely, even though people think that getting to the peak is the goal, they easily overlook the fact that this is the most dangerous moment. Exhausted, stunned, overwhelmed and alone. It's you and the mountain – and nothing more. You have to get back. Only then have you completed the adventure. Getting back is the goal – nothing less, nothing more.

Using the words of Hans Kammerlander, an Italian mountaineer who specialised in fast climbing (setting a series of world records in this area):

> *We own a mountain only when back in the vale.*
> *Until then, we are the ones who belong to the mountain.*

As a beautiful and poetic way to describe this feeling, this phrase acquires a deeper significance when one learns that Kammerlander himself had lost friends during an expedition.

However, what does this have to do with maths? How could the life of a mathematician be remotely comparable to the struggle for life in extreme circumstances? This is certainly possible. I like to compare the mountains to mathematical problems, conjectures or – if a proof is completed – theorems:

> *We own a theorem only when a proof is completed.*
> *Until then, we are the ones who belong to the problem.*

This describes in many ways what mathematicians deal with. The solution to a problem can be compared to the peak of the mountain: it is of no value to us if we do not prove that this conjecture is true. Without the proof, it is comparable to a mountaineer who decides to stay back and remain on the mountain. An unfinished quest. Finding the solution is not the goal but a first sense of achievement, and it is followed by the fundamental task of proving its trueness.

I frequently hear that people compare the work of a mathematician to playing a game. After all, you set up some rules – our precious axioms – and start working with them. This visualises very well the fact that most, if not all, mathematicians love what they do and truly do enjoy it. But a game is no serious business – and maths truly is! Mathematicians do not play with toys but they face important and challenging questions. Mathematicians are adventurers, the mountaineers of truth. Similar to a mountaineer, to face the difficult problems, you first need to gain some experience with many small mountains or you will not succeed. The big mountains are left for only the greatest adventurers among us. With each mountain, you become more experienced. Sometimes you fail and do not succeed, and other times, it takes different attempts until the quest is completed. Just like mountain climbing, it is best done in small groups but not alone.

However, this metaphor has more facets. It also describes what goes through a mathematician's mind. A mountaineer enjoys climbing every mountain he comes across. Big mountains, small mountains – he enjoys every challenge, regardless of whether he already knows that he will succeed. Honestly, what mathematician does not enjoy the simple joy of solving puzzles in the newspaper or of facing some problems that undergrads deal with? Further, once we start a problem, there is no escape. Our mind will not let off until it is solved. We cannot help but think of our work while showering, while walking down the street or before we go to bed. They catch us and won't let us go. Until it is solved, we are the ones who belong to the problem.

This is the tale of those who got to the peak, saw the world from above and came back to tell the story, those adventurers who faced the highest mountains, the trickiest quests, those who know the *Aconcagua* from within, survived the *Himalayas* or even built a bridge from one *Kilimanjaro* peak to the other together with Arthur Wilson.

First, let me briefly introduce you to the young climber who is addressing you.

The author

My name is Demian Nahuel Goos Bosco. I am half-German and half-Argentinian, and I originally wanted to become an archaeologist. The reason for this desire is probably hidden in the Roman background of my hometown *Mainz* in Germany, which, owing to the constant discoveries of historic nature, can also be described as a playground for archaeologists, and in the palaeontological riches of *Patagonia*, the southern region of Argentina, where I spent time while growing up.

However, when I read *Secret Codes* by Simon Singh at the age of 14, this desire suddenly had to move aside for a new one. By reading this book, I got to know Whitfield Diffie and Martin Hellman, who together developed the *Diffie–Hellman key exchange method*, which was published in their paper "On new directions in cryptography". I was fascinated not only by their story – the background of their friendship, the difficulties and conflicts that arose because of their work and how they faced them – but also by the cryptographic method itself. I was amazed by the simplicity and power of their algorithm. For instance, modern digital signatures are based on their method. It was a true game changer! And all that was obtained with simple computations!

I wanted to do that. I became enthusiastic, started creating my own cryptosystems and wanted to know what I had to do to work on these kinds of problems; after some research, I found that maths would be a good choice. Currently, I actually am a mathematician and although I do not exactly work on cryptography or anything somehow related to it, after many years of hard work, I got to the amazing moment of getting to know both of them (Whitfield Diffie and Martin Hellman) in person. An absolute highlight of my career! Pure excitement!

But how did this happen? What led to this incredible moment? To answer this question, I would like to go some steps back to the origins of this event.

The idea

Behind every big idea, there is a visionary, and the Heidelberg Laureate Forum is no exception. Born in December 1940, Klaus Tschira was what in Germany is called a *Kriegskind*, someone who was born during the barbaric Second World War, a term that calls to mind the traumatic circumstances under which these children were raised. However, traumatised is not an appropriate description for this man – quite the contrary!

If you look him up, you will probably find that he was one of the cofounders of SAP – one of the market leaders in software developing (but he would not appreciate this description). Being the forward-looking and passionate adventurer he was, a probable reaction would have been: "That happened so many years ago. I have done much more interesting things since I left SAP!" And indeed he did!

Once an entrepreneur, always an entrepreneur. He created the *Heidelberger Institut für Theoretische Studien*, an institution devoted to the research of natural sciences, mathematics and computer sciences. He also thought up the *Tschira Jugend Akademie*, whose mission is to connect young teenagers with biology and natural sciences. Art exhibitions and film festivals in which these dots are connected with mathematics and computer sciences are also part of his legacy.

In all of his projects and activities, there is always a fundamental idea: to create awareness of maths and computer science and their key role in everyday life. He understood that it is the scientist who must take the first step to do so, and this is what guided him throughout his work as a science patron. Two other beautiful examples are the *KlarText* Prize for communication in science, which is awarded to those young professionals who manage to describe their PhD thesis in a summary with such clarity that even schoolchildren can understand it, and his promotion of and collaboration with the *MS Wissenschaft*, a ship that, in 2010, sailed from city to city in Germany, bringing an interactive exhibition about the natural sciences to the people, particularly covering the environment, the dangers it faces and how science can contribute to avert this danger.

Last but not least, there is the *Heidelberg Laureate Forum Foundation*.

The Heidelberg Laureate Forum

As the name already suggests, the Heidelberg Laureate Forum takes places in this beautiful, charming, medieval city in the Bundesland of *Baden-Württemberg*. Heidelberg could not be a better host city for this event. It is not only a centre of research, knowledge and entrepreneurship but is also the hometown of the eldest university in Germany, a university that is connected to 56 Nobel Laureates and whose motto – *Semper Apertus* – seems to anticipate the nature of the forum and the spirit of its attendants. Inspiration is omnipresent during HLF and the city plays a vital role in it. Each morning, walking down the route from your hotel through the *Altstadt* and to the university, you feel like you are in an old village of a Grimm's tale, surrounded by picturesque medieval houses, the *Fachwerkhäuser*. At the end of the day, when it is getting dark, ambitious students from around the globe meet their friends on the streets to spend time in one of the many *Kneipen* and pubs in the old town.

The concept of the Heidelberg Laureate Forum is simple: bring together the laureates of mathematics and computer science – the brilliant minds who were awarded the *Abel Prize*, the *Fields Medal*, the *ACM Turing Award*, the *ACM Prize in Computing* or the *Nevanlinna Prize* – and the next generation of outstanding young researchers and students in these two disciplines from around the world. In fact, more than 60 countries were represented in 2017, which also reflects the multicultural essence of the forum. While the young researchers are mainly PhD students and postdocs, undergrads can also apply to participate. These young researchers are selected by the award-winning organisations together with the *Heidelberg Institute for Theoretical Studies*, the *Mathematisches Forschungszentrum Oberwolfach*, the *Schloss Dagstuhl* and the *Heidelberg Laureate Forum Foundation*, whose Scientific Board makes the final decision. Nevertheless, these are not the only participants. Local researchers, scientific bloggers and representatives of science-related companies are invited to participate, and different media from around the world cover the event.

The forum can be partitioned into three groups of activities. In the first group, the plenary lectures, the laureates play the key role. Here, they can share their work, ideas and life stories through what I would call amazing and encouraging talks. In 2017, sixteen lectures were given and their contents could not have been more diverse. Manuel Blum shared his current research on consciousness in machines, Leslie Lamport introduced a modern structure of mathematical proofs and Vinton Cerf presented his (not so) futuristic work about an interplanetary internet. In the second group, which we could call scientific activities, the young researchers come to the fore. Postdocs can apply to organise a workshop in collaboration with a laureate, PhD students are invited to present their research during a poster session, and all young researchers can visit local scientific institutions and companies and, during a hot-topic discussion, interact with experts on a selected topic (this was quantum computing in 2017). Nevertheless, the most relevant part of the entire forum, which makes this event such a special and unforgettable moment in every attendant's career, is the third group: the social programme. Each activity is conceptualised with a particular love for detail to promote enriching, encouraging and inspiring conversations not only with the laureates but also with the other young researchers, journalists and others. On one day, you are taken on a long, relaxing boat trip on the *Neckar* among forests and castles and on the next day, you celebrate a private *Oktoberfest* with *Bier*, *Brezeln*, *Lederhosen* and a dancing *Volkstanzgruppe*. While one dinner is held within the *Speyer Museum of Technology* – chatting and dining with rockets, airplanes and even Da Vinci's inventions flying above you – the farewell dinner is held in the famous Heidelberg castle. What amazing scenery to conclude this spectacular week!

The conversations

It is very hard to describe the feelings that arise when talking to these renowned scientists. As computational complexity is one of my most significant and lively interests in mathematics, talking to Stephen Cook about the $\mathcal{P}$ vs. $\mathcal{NP}$ problem and discussing the motivation and inquisitiveness that led him to formalise the concepts (which laid the cornerstone for one of the most relevant problems in theoretical computer science) was by a long way one of the milestones of this week for me. Not every day do you get first-hand advice from Jeff Dean, the lead of Google's Artificial Intelligence Division, to start working on machine learning. Nor do you always get to discuss results in group theory with Efim Zelmanov – a mathematician who describes and explains his work in such a natural, intuitive and didactic way that it makes you feel that you actually understood each and every one of the explanations about his extremely abstract and intricate research.

Of course, it would be tempting to straightforwardly ask the laureates how they managed to be awarded these prizes so I could learn which path to trace but, obviously, there is no recipe for success. I actually overheard some comments such as: "If you want to be awarded such a prize, forget about its existence! None of us ever intended to get it." We could say that every path is like a *One–Time–Pad*: reuse is doomed to fail. Nonetheless, I certainly wanted to know what made these people stand out, how they think, how they faced their career and what drove them during their life. Basically, what do they have that others do not? What do they have in common?

To tap the full potential of this opportunity, I previously wrote a small booklet in which a spread is dedicated to each attending laureate, summarising the prizes they won and some biographic items I found interesting. The idea behind this was to check my booklet whenever I was talking with a laureate so I could ask more directed questions and avoid missing the chance to ask something I was curious about. The *book of the laureates* and the work put into it aroused much attention but, more importantly, it undoubtedly helped me achieve my goals.

Here, I put into writing the many observations I made, the most significant and valuable ideas the laureates shared with us and the most remarkable quotes I noted. I have made a particular effort to share these concepts as faithfully as possible and to properly reproduce the intention behind these words. I hope that I have succeeded.

Inspiring minds

The first thing that struck the eye was their humility. I was originally not quite sure how these conversations were going to go – formal and uptight or natural and casual – but the latter was clearly the case. The laureates were, as it turned out, very easy to approach and we changed from one topic to another as naturally as old friends. Not only in this sense could we notice their simplicity but also in many of the answers that they gave. To explain how the award changed his life, Alexei Efros, who generally always has a humorous reply ready, answered that he is still "that stupid, normal guy" that he was before he won the prize. When asked how it felt like to work on an idea he had many years ago when he was a young student – Manuel Blum chose his career because he wished to create *mechanical brains* and half a century later, he is working on consciousness in machines – he gave an unexpected answer: "It's scary. And very difficult." This was no less unexpected than Shigefumi Mori's answer to why he chose to study maths: "It was a mistake." This was not quite the answer you expect from someone who seems to have been predestined to do maths. He explained that he was a terribly shy kid, blushing whenever someone talked to him, and that he originally thought that maths was a way to avoid interacting that much with people. The current president of the *International Mathematical Union*, who attended this forum with the sole purpose of talking and interacting with people, concluded his anecdote with "I was wrong" and a likeable smile.

As a mathematician, I am aware of the fact that many deep-rooted prejudices people have about mathematicians are false. What I learned during HLF was something deeper. Part of the success of these laureates was precisely their ability to find a balance between their work and their lives. What I concluded is that each and every one of them has a very human aspect, and I got to know them as people and not as mathematicians. They did not allow their work to consume their lives; instead, they made their work part of their lives. Religion is a vital part of Frederick Brooks' life and he explained to me that being religious did not contradict his scientific career because he found his way to religion through critical reasoning. Martin Hellman says that meeting Whitfield Diffie was like "walking in a desert and coming across an oasis", and Diffie describes his wife as a "source of inspiration", without whom he would not have achieved anything. In particular, Hellman and Diffie's lives are a beautiful example of how important it is to get support in difficult times not only from your friends and family but also from the institution you work for and your colleagues. Religion, friendship, love – everyone found an equation of happiness in which their work is part of a whole, a variable in a multidimensional problem. They

emphasised this throughout the week and underlined the importance of striving for happiness in all aspects of our lives. But what precisely makes us happy? Well, it is naturally something personal. Nobody can decide what fills our hearts with warmth, nor can anybody impose upon us what we are passionate about – not even ourselves! This is also valid in our research: we do not really decide which mountains we desire to conquer or which mountains have already conquered us. Madhu Sudan summed it up poetically with a Harry Potter reference: "The wand chooses the mathematician, not the other way around!" Vinton Cerf, whose hat seems to be a sympathetic homage to Hilbert himself, agreed: "We did not plan our careers. We did not plan the stuff we do!"

However, once we know the wand, the instruction is clear. "Do what you are passionate about!" Ivan Sutherland euphorically advised. Of course, I must say that this was not the only thing Sutherland told me with enthusiasm; I would say that in every word he shares, there is a conviction, certainty and passion. Without exception, they all work on those things that they love and we must do the same to succeed. We should not let anybody tell us otherwise – neither advisers nor colleagues – as Sutherland continued with a mathematical contradiction: "Do what you love – what makes you happy. And don't listen to anybody else. Not even to me!" After a brief reasoning, Bertrand Russell would probably have smiled.

The next tip was to be a detailed observer and to be open-minded when facing problems. What does this mean? I think a good example to visualise this concept is a real historical problem that the *United States Military* faced during World War II. They wanted to improve the performance of their air force and consulted the mathematician Abraham Wald. He observed the bombers returning from combat and noticed that some of their sections were far more damaged than others, which should intuitively induce us to improve the protection of these damaged sections. However, Wald did something different. He suggested that the areas that he observed to be undamaged should be protected, reasoning that if these sections were damaged by enemy fire, it would inevitably lead to the destruction of the aircraft, whereas any other damage would make it possible for the pilot to fly back to their base. To reach such a conclusion, it is important to think carefully, observe in detail, distrust your first intuition and reason about the implied consequences of each possible decision.

There is a beautiful painting of René Magritte from 1936 called *La clairvoyance*, which visualises precisely this idea of observing more than the eye can see. It shows an artist observing an egg but painting an already born bird on the canvas.

Moreover, when the omnipresent Michael Atiyah, who, even at 88 years of age, was one of the most active, dynamic and compelling laureates, suggested that "we need to have quantum minds", he somehow told me to consider all possible solutions and not to discard anything until I find the best – until our mind has collapsed into one optimal observed state. There is an experiment that I usually do with my undergrads to show them how quickly we focus on one solution and discard a huge amount of alternate ideas. I ask them to tell me a prime number. After some answers, I ask them to tell me something that is not a prime number, and they intuitively answer with even numbers. At this point, I repeat my question and note that the Renaissance or even Lionel Messi are not prime numbers either and are indeed valid solutions to my question. Many students humorously complain that I cheated because I did not define a universal set or that I tricked them and that my first question directed their second answer – but then again, that is what I wanted to show them.

The laureates constantly encouraged us to work hard, to be persistent and to believe in our ideas. Michael Atiyah summed up this message very well when he said: "You don't need a million dollars, but energy, brains and perseverance." Stephen Smale gave a personal and very impressive illustration of this idea when he told me that he worked on the Poincaré conjecture but did not succeed at first. Later, when working on PDEs on manifolds, he thought that this could be a successful approach, which it eventually was. The idea of the necessity of failing to succeed was, in fact, hidden in many conversations. Manuel Blum commented with a bit of self-irony that he is "very good at getting his work rejected" and Robert Tarjan admitted that, even today, he frequently faces ups and downs in his work. Failure was explained not only to be an inevitable part of research and life in general but also a vital part of growth and success. But wait, there is more! Not only does failure seem to be quite necessary for success but so does a good bit of luck. At least, when presenting his intense way of working, Daniel Spielman gave me his real-life example that supports this idea. He told me that he came across the problem that eventually got him his award by pure chance: "What I did, I did it by accident."

This sounds disappointing at first glance, for what hope is left if we are subject to the capriciousness of chance? However, this is precisely a mistaken line of reasoning because luck is no random variable; we can shape our own luck and force it to go in certain directions. Efros shared with me two concepts in this context. The first is that luck is a consequence of attitude, which he illustrated by introducing two kinds of people. The first group consists of people who feel chased by bad luck, i.e. those who miss the bus, spill the coffee and forget their umbrellas on rainy days. Group two consists of those among us who seem to attract luck in an unbelievable way, i.e. those who find money precisely when they need it and catch the falling porcelain at the very last second. I am sure everybody knows someone who fits in each group. Clearly, both groups have a certain attitude toward everyday life, with the first group overlooking lucky moments and the latter not perceiving certain situations in which there is no luck at all. Efros then told me about a scientific experiment he read about, in which people from both groups were given the simple task of counting the number of triangles in a geometric figure in a magazine. The surprising result that the "lucky" ones were indeed quite faster had a simple explanation. Within the magazine, there was a page in which the solution to the problem was visualised and the people in the second group saw this and gave the printed number as their answer. Thus, people with a more positive attitude toward life are more likely to solve problems in a more effective way, perceiving alternate solutions and taking risks if need be. The second concept that Efros presented was luck as a consequence of hard work: "Luck helps the prepared mind. Great findings are never what you are looking for. But they are the result of hard work." Cerf also followed this direction when he motivated us to work hard to shape our luck: "You have to take your opportunities. Notice them and work. Work hard." How can you not follow this advice if it is given by someone who partially created the internet? Do not wait for the grass to grow.

Until now, everything I have summed up seems to be a pretty good recipe for how to achieve greatness in what you do, no matter what it is. Nevertheless, there is still something missing. After all, these guys have not solved just "something" but the big questions in mathematics and computer science.

Thus, how do you solve these big questions? How do you prove that $\mathcal{P}$ is strictly included in $\mathcal{NP}$ or that the Riemann Zeta-function has its zeros on the line $Re(z) = 0.5$? Well, there is one first step, which is quite simple: work on those problems and face them or, as Atiyah said: "Don't play with toys. Go for the big things!" In an unconnected conversation with Spielman, he unknowingly gave his own research as an example, when he told me that he started working on the Riemann hypothesis after finding a research article that somehow connected this problem to random matrices. Of course, facing these conjectures appears to be foolish. If so many brilliant minds have not found a solution, why would I? What could a fool like me achieve in comparison to them? Then again, as Hellman said: "Don't be afraid of being a fool. Only a fool goes where no one else goes." He is precisely a per-

son who achieved what he achieved because of being a "fool". Only a fool would question the intuitive notion that every statement within a consistent system of axioms can be proven to be either true or false, but Kurt Gödel did and thereby caused the foundations of mathematics to totter. Only a fool would question the natural idea that the position and momentum of a particle can be known with as much precision as desired, but Heisenberg's uncertainty principle puts a limit to precisely that precision.

I would like to finish with something Efros told me, which I found to be beautiful and inspiring as a strong, deep message:

> *"Look for ideas that change the world. Maybe you won't succeed. But you will have done a whole lot more than those who don't even try."*

Epilogue

After this week in Heidelberg, I needed some time to process the experience I had just had. All these lectures (where I barely talked at all), all these activities and conversations, all these thoughts and emotions – and all in just seven days! Back in Argentina, I wished to share my experiences with my friends, my peers and my undergrads; I wanted to make them part of this experience and really hoped to motivate someone to apply to participate in the HLF. With this in mind, I gave a talk at my university at the celebrations of the 50th anniversary of the Degree in Mathematics in *Rosario* (my hometown), which was the basis for this manuscript.

Now, the main purpose of this manuscript and the hope connected to it is to encourage more people to participate in this event. Whether you are a young scientist eligible for participation or a not-so-young scientist, I hope that the message is clear.

Hopefully, this work also shows new aspects on how mathematicians are and how they think and communicate. People frequently have a false image of mathematicians and I hope that this work provides a more realistic portrayal.

I would also like to honour the memory of Vladimir Voevodsky. He was one of the laureates who confirmed his participation but was missed during the event. One day after the forum was finished, he passed away. Whenever I hear someone talk about him, it is always with affection, respect for his achievements and admiration for his passionate way of working and communicating.

Acknowledgements

Finally, this work is also a way to express my gratitude to all those who somehow participated in this amazing experience:

- The laureates who participated in HLF 2017: all of those mentioned in this manuscript but of course also those laureates whom I did not mention, namely Martin Hairer, Tony Hoare, John Hopcroft, William Kahan, Joseph Sifakis, Richard Stearns and Leslie Valiant. The main reason for this is that during this week I did not have the opportunity to talk with you but I honestly wish to do so at another opportunity.
- The young researchers, with whom I also shared uncountably many incredible and encouraging conversations.
- The local scientists, journalists, bloggers and photographers.
- A special thanks to the organisers of the HLF for making this event even possible and letting me be part of it.
- And last but not least, my family and friends, who always support me in life.

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Problem Solving and IMC (International Mathematics Competition) Activities in Göttingen

Chenchang Zhu (Universität Göttingen, Germany)

This article is about some interesting mathematics activities I do with some of my students and some of my colleagues in Göttingen – once the home of Gauss, Riemann, Hilbert, Noether and other beloved and respected mathematicians.

Since 2011, Dima Dudko and I open a course or seminar on mathematics problem solving every Summer semester for university students. At the beginning of the course, we give lectures on various problems in analysis, number theory, algebra and geometry. We propose problems and ask the students to solve them during the lecture. Then we give hints and summarise the cool points of the problems. The students may also propose their own solutions. Later, it becomes a show for the students themselves. They are in charge of looking for problems and presenting them, and asking their fellow students to solve them during the seminar. We usually gather 10–20 people each year, which makes it manageable. We especially want to emphasise the beauty and fun in mathematics, to challenge the rejecting and fearful feelings most of our students have at the beginning of university. We also try to establish a collaborative environment instead of competitive one, and promote discussion and sharing.

After two classes, we conduct two exams in the seminar, each consisting of six problems in four hours. This is a selection exam. The best three or four students will represent Göttingen in attending the International Mathematics Competition for University Students (IMC)[1] held in Blagoevgrad, Bulgaria. Our faculty and university (Unibond) give us funding to cover travel and registration for the IMC.

Although these three or four students are selected, everyone still attends the seminar and presentations go on. For these three or four students, we offer more exams to prepare them for the IMC test. After years of effort, we have gradually achieved good results in the IMC and are able to attract new maths talents to join our maths institute in Göttingen. This is also connected to MO in Niedersachsen, which is organised by my students for high school and elementary schoolchildren. More will appear on this event in a later article.

The IMC is a competition now held every year during the last week of July or the first week of August. There are now 300–400 participants from across the entire world, from around 60–70 countries. In 2016, the team of Göttingen ranked sixth in the world. In 2018, we even received a Grand First Prize (almost perfect score) for Christian Bernert. This is granted to only two students amongst the 351 students from all over the world. Recalling his experience, Christian said: "There was one moment during the closing ceremony of this year's IMC where I finally didn't know the answer. It was the granting of the Grand First Prizes and the two winners, Daniel Klyuev from St. Petersburg and me, were on stage in front of the other 350 students. For Daniel, this was already the third time winning this prize but for me, it was a completely new and utterly unexpected experience. I had competed at the competition already for the third time, with some success in the previous years. But I would have never expected to perform that well.

It was at this moment where John Jayne, the longstanding organiser of this great event, told the students in the crowd to ask us, the winners, for our secret at the final dinner. Fortunately, no one actually followed this advice, for the only thing I would have been able to tell them is to continue to do math, to have fun doing it, to feel no pressure and to hope for quite a bit of luck."

Team photo with John Jayne.

As a female mathematician, and since Göttingen is the home university of the famous female mathematician Emmy Noether, I also try to encourage female students to join us in mathematics, or at least to overcome their fear of mathematics. We have always had several female students in our seminar of problem solving. I have to admit that we have not yet had any girls on our IMC team. For this, we also do something especially for high school girls, to promote their interest in mathematics, in the spirit of starting early. In our view, encouraging girls at an early stage to do mathematics is vital in order to compensate for classical stereotypes, which are unfortu-

[1] The website of IMC is http://imc-math.org.uk/.

Girls day.

nately still present in society and can be quantitatively assessed in self-assessment studies of high school children. That is why, within the framework of the German "Zukunftstag" (future-day) and "girls-day", together with my colleagues Anna-Lena Martins and Robin Richter in the equal opportunity office, we organise a day that should give girls aged from 10 to 15 a peek into the study of mathematics.. The day has two main targets that we try to achieve via group exercises and riddles, library and mathematical exhibition exploration, a little introduction to logic and the sharing of experiences by faculty members and students. First of all, we want to show that there exists a very rich world outside of "high school mathematics" that is worth exploring. Logical exercises and riddles are especially well suited for this, since they are completely different from German "school mathematics" whilst being challenging and intuitive at the same time. Secondly, we want to emphasise the teamworking character of mathematics. From the first years in Göttingen, exercise sheets have often been solved in interacting groups of students who organise themselves, an aspect that differs greatly from their experience of school and often from the expectations with which new students come to university. During the day, female (PhD) students and professors engage to give insight into their routes through life, thus establishing important role models that the girls might want to follow. We hope events like this will encourage girls to participate in mathematics at university and avoid lost talent because of traditional gender bias.

[2] There is also a "boys-day" event, and some event which can be included in both frameworks.

Chenchang Zhu has been tenured at the University of Gottingen since 2013 and, before moving to Gottingen in 2008, she was Maitre de conferences (assistant professor) at the Fourier Institute, Grenoble (from 2006), a post-doc at ETH Zurich (2004–2006) with Giovanni Felder, a PhD student at U.C. Berkeley with advisor Alan Weinstein (until 2004) and an undergraduate at Bei Jing University. Since 2011, she has been in charge of the selection of the IMC team and training in Göttingen.

The 100th Anniversary of the Norwegian Mathematical Society and the Viggo Brun Prize

Petter Andreas Bergh, President of the Norwegian Mathematical Society

The Norwegian Mathematical Society was founded in Oslo on 2 November 1918. During the first decades after 1900, the mathematics community in Norway grew substantially and the society was established with the intention of fostering contact between the nation's mathematicians. The society is amongst the smaller ones in Europe, with a little over 300 members as of 2018. It serves as the main professional organisation for mathematicians in Norway and works to promote research, cooperation and recruitment. Its monthly newsletter, Infomat, is freely available online.

The society's activities are aimed at mathematicians in all stages of their careers, as illustrated by the following three examples. Every year, it organises a national competition for high school students, the Abel Competition, which also serves as the national qualification for the International Mathematical Olympiad. This is a very popular competition and the winners usually receive their awards from the minister of education at a prize ceremony. The society also awards Abel Scholarship Grants. These are grants awarded annually to some of the best students on Master's programmes in Norway, supporting shorter and longer visits to universities abroad. Finally, each year, the society organises the Abel Symposium, a conference with distinguished international speakers. These three activities are funded by the Abel Prize.

From left to right: Kristian Seip, Finn Faye Knudsen, Rune Haugseng (recipient of the Viggo Brun Prize) and Ragni Piene. The photo is courtesy of Helge Skodvin.

In September 2018, the society celebrated its 100th anniversary at a national meeting of mathematicians held in Bergen. At the meeting, the society announced a new national prize, the *Viggo Brun Prize* (see https://web.matematikkforeningen.no/viggo-brun-prize/):

The prize is named after the noted number theorist Viggo Brun, who became an honorary member of the Norwegian Mathematical Society in 1974. It is to be awarded biannually to a young mathematician, and comes with a cash reward of NOK 50.000. The 2018 laureate is *Rune Haugseng*, who receives the prize for his fundamental contributions to the theory of higher categories, with applications to quantum field theory, representation theory, algebraic geometry, and geometric topology, and for the development of higher Morita theory and enriched ∞-categories.

Petter Andreas Bergh is a professor of mathematics at the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway. His research interests are in homological algebra, triangulated categories and representation theory of algebras. He is currently the President of the Norwegian Mathematical Society.

Georgian Mathematical Union – History and Activity

Roland Duduchava (University of Georgia, Tbilisi, Georgia), Guram Gogishvili (St. Andrew the First Called Georgian University, Tbilisi, Georgia) and David Natroshvili (Georgian Technical University, Tbilisi, Georgia)

The Georgian Mathematical Union (GMU), formerly the Georgian Mathematical Society, was founded on 21 February 1923. One of its founders and the first president was the distinguished mathematician Andrea Razmadze. Active co-founders and members were Nikolz Muskhelishvili, Giorgi Nikoladze and Archil Kharadze. These Georgian mathematicians, educated in Moscow and Saint Petersburg, founded the Georgian Mathematical School and launched mathematical education in Georgia at the first university in Caucasus, opened in Tbilisi in 1918. A. Razmadze represented Georgia at the International Congress of Mathematicians in Toronto, Canada, in 1924. At the same congress in Toronto, the Georgian Mathematical Society was accepted as a member of the International Mathematical Union.

After the death of A. Razmadze in 1929, the Georgian Mathematical Society became inactive until 1962, when it was re-established due to the efforts of Viktor Kupradze, Boris Khvedelidze, Levan Gokieli and Archil Kharadze. Mathematicians from Georgian universities, scientific research institutes, institutions of higher education and secondary schools consolidated in its ranks. Viktor Kupradze became the president of the newly

Founders of the Georgian Mathematical School and Georgian Mathematical Union: A. Kharadze, A. Razmadze, G. Nikoladze and N. Muskhelishvili.

revived society from 1962 to 1966. Later presidents of the GMU were Levan Gokieli (1966–1970), Archil Kharadze (1970–1974), Levan Magnaradze (1974–1994), David

Natroshvili (1994–1997), Roland Duduchava (1997–2001), Teimuraz Vepkhvadze (2001–2005), Jondo Sharikadze (2005–2009) and Roland Duduchava (2009–2017). The current president of the GMU is David Natroshvili (since 2017).

In 1990, the Georgian Mathematical Society became one of the founders of the European Mathematical Society. In 1991, it became a member of the International Mathematical Union. In 1994, it was renamed the Georgian Mathematical Union.

The mission of the Georgian Mathematical Union is to promote mathematical sciences, especially among the young generation, and to establish and strengthen contacts with colleagues from abroad and with international professional organisations such as the International Mathematical Union and the European Mathematical Society. To this end, the GMU organises public meetings where novel scientific results are reported and discussed. The GMU often organises memorial meetings dedicated to outstanding mathematicians, which essentially promote mathematics popularisation.

Special workshops are devoted to contemporary problems related to the teaching of mathematics at schools and universities. The GMU actively collaborates with schoolteachers. Members of the GMU intensively participate in consideration of educational standards and in the preparation of Georgian mathematical textbooks for secondary schools and universities. The GMU organises competitions of young scientists and gives awards to authors of the best mathematical papers.

Every four years, before the International Congress of Mathematicians, the GMU organises a congress, where the President and Vice-Presidents of the GMU, the Chair of the Georgian National Committee for Mathematics, the Chairs of different commissions and Members of the Council of the GMU are elected. At the congress, the Niko Nikolkadze Prize is awarded to a young mathematicians under the age of 40.

Since 2010, at the beginning of September, the GMU organises Annual International Conferences in Batumi, a wonderful Black Sea resort of Georgia. The 9th conference took place on 3–7 September 2018 and was dedicated to the centenary of the Ivane Javakhishvili Tbilisi State University.

In 2014, the GMU hosted the First Caucasian Mathematics Conference (CMC) in Tbilisi.

The idea of CMCs was initiated by the Turkish Mathematical Union (President B. Tanbay), the Moscow Mathematical Society (Vice-President A. Sergeev) and the Georgian Mathematical Union (President R. Duduchava) at their meeting in Istanbul at the beginning of 2014. The initiative was supported by the European Mathematical Society. Later, the Mathematical Societies of Armenia, Azerbaijan and Iran joined the project. It was decided to organise the CMC bi-annually under the auspices of the European Mathematical Society and the cooperation of the aforementioned mathematical societies. The scope of the CMC is to bring together mathematicians from Caucasian and neighbouring countries to promote scientific collaboration in the region.

In CMC I in Tbilisi, about 150 mathematicians participated from Caucasian and other countries. CMC II was held in Van, Turkey, and members of the GMU actively participated in the work of the conference.

Among other international meetings organised by the GMU is the 26th International Workshop on Operator Theory and Applications (IWOTA), which took place in Tbilisi, Georgia, 6–10 July 2015. It brought together mathematicians and engineers working in operator theory and its applications to related fields, ranging from classical analysis, differential and integral equations, complex and harmonic analysis to mathematical physics.

The GMU was a co-founder, alongside 11 other societies, of the Silkroad Mathematical Centre of the Chinese Mathematical Society (SMC-CMS) in Beijing, China, in 2016 (http://www.cms.org.cn/en/mathcenterintro_332.html). Professor R. Duduchava is a member of the Steering Committee of the SMC-CMS. One representative and three young mathematicians under 35 from all the member countries of the centre are invited to the annual conference in Beijing.

In 1995, at the I. Vekua Institute of Applied Mathematics (VIAM) of Ivane Javakhishvili Tbilisi State University, the *Tbilisi International Centre of Mathematics and Informatics* (TICMI) was founded. The TICMI is supported by the GMU and the EMS. The aims of the centre are to help young scientists of the Black Sea Basin, to improve their professional skills and to promote the exchange of scientific information worldwide. This is achieved through lecture series and Summer schools devoted to topics determined by the current interests of the centre.

The work of the centre is guided by an international scientific committee, which consists of seven members. Four members are elected for a period of four years by the Scientific Council of the VIAM and are approved by the Presidium of the GMU. They represent different interests and at least one amongst them is a member of the VIAM. The three other members are appointed for the same period by the Executive Committee of the EMS.

The current members of the international scientific committee of the TICMI are:

Lucian Beznea (Bucharest, Simion Stoilow Institute of Mathematics of the Romanian Academy).
Alice Fialowski (Budapest, Institute of Mathematics, Pazmany Peter setany 1/C).
George Jaiani, Chairman (Tbilisi, I. Vekua Institute of Applied Mathematics, Tbilisi State University).
Vaxtang Kvaratskhelia (Tbilisi, N. Muskhelishvili Institute of Computational Mathematics).
Alexander Meskhi (Tbilisi, A. Razmadze Mathematical Institute, Tbilisi State University).
David Natroshvili (Tbilisi, Georgian Technical University).
Eugene Shargorodsky (London, King's College London).

The centre presents an annual report to the GMU and the EMS.

Nikoloz (Niko) Muskhelishvili

Ilia Vekua

Viktor Kupradze

In conclusion, we would like to mention the three most outstanding members who stood at the origins of the GMU.

The most outstanding Georgian mathematician and founder of the well-known Georgian Mathematical School Academician *Nikoloz (Niko) Muskhelishvili* conducted fundamental research on the theories of mathematical elasticity, integral equations and boundary value problems. He was one of the first to apply the theory of functions of complex variables to the problems of elasticity theories, proposing a number of methods that have been successfully implemented in numerous areas of mathematics, theoretical physics and mechanics. In his works, almost all major problems of the plane elasticity theory are solved. He is also well-known for his contributions to the theory of linear boundary value problems for analytic functions and one-dimensional singular integral equations. N. Muskhelishvili is the author of various scientific articles, monographs and textbooks on mathematics, which are still widely used today. His highest regarded monographs are *Singular Integral Equations*, Groningen, P. Noordhoff, 1953 (67 editions published between 1946 and 2011 in five languages and held by 849 libraries worldwide), and *Some Basic Problems of the Mathematical Theory of Elasticity*, Groningen, P. Noordhoff, 1963 (78 editions published between 1933 and 1977 in three languages and held by 846 libraries worldwide).

The distinguished Georgian mathematician Academician *Ilia Vekua* is well-known worldwide for his fundamental contributions to the theory of partial differential equations, singular integral equations, generalised analytic functions and the mathematical theory of elastic shells. Fundamental monographs of I. Vekua are: *New Methods of Solution of Elliptic Equations*, North-Holland Publ. Co., Amsterdam, 1967; *Generalized Analytic Functions*, Oxford University press, Oxford-London-New York-Paris, 1962; and *Shell Theory: General Methods of Construction*. Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

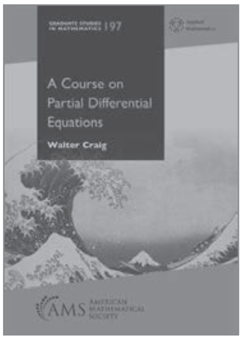
Mathematicians and scientists working in the fields of continuum mechanics throughout the world are familiar with the name of Academician *Viktor Kupradze*, who made essential contributions in mathematical physics and the 3-dimensional theory of elasticity. The following monographs are the most popular works of V. Kupradze: *Fundamental Problems in the Mathematical Theory of Diffraction*. Los Angeles, 1952; *Potential Methods in the Theory of Elasticity*. Israel Program of Scientific Translations, Jerusalem, 1965; and *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*. North-Holland Publ. Comp. Amsterdam, 1979.

Roland Duduchava [RolDud@gmail.com] is Chairman of the National Committee of Georgian Mathematicians and Head of the Department of Mathematics at the University of Georgia (Tbilisi, Georgia). His research interests are in partial differential equations, integral and pseudodifferential equations, solid mechanics, wave scattering and mathematical physics.

Guram Gogishvili [guramgog@gmail.com] is Head of the Commission of Mathematical Education and History of Mathematics at the GMU, and is Chief of the Educational Centre of St. Andrew the First-Called Georgian University. His research interests are in arithmetical aspects of the theory of quadratic forms, representation of numbers by forms, estimation of corresponding arithmetical functions and class numbers of quadratic forms.

David Natroshvili [natrosh@hotmail.com] is President of the Georgian Mathematical Union and Head of the Department of Mathematics at the Georgian Technical University (Tbilisi, Georgia). His research interests are in partial differential equations, integral and pseudodifferential equations, solid mechanics, generalised thermo-electro-magneto elasticity, wave scattering and mathematical physics.

AMERICAN MATHEMATICAL SOCIETY



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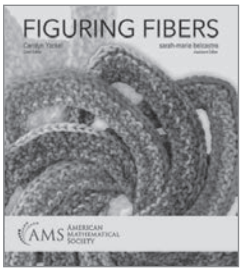

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Leelavati Prize 2018 to Ali Nesin[1]

Gert-Martin Greuel (University of Kaiserslautern, Germany)

Citation

Ali Nesin has been awarded the 2018 Leelavati Prize[2] in recognition of his outstanding contributions and great achievements in increasing public awareness of mathematics in Turkey, especially his tireless work in creating and developing the "Mathematics Village" as an exceptional, peaceful place for education, research and the exploration of mathematics for a wide range of people.

Fig. 1. Ali Nesin at the Award Ceremony on 9 August 2018.

Short scientific CV (Research)

Hüseyin Ali Nesin was born in Istanbul in 1957; he is a Turkish citizen. After junior high school in Turkey, he went to high school in Switzerland and studied mathematics at Paris VII in France. In 1981, he went to the USA and studied at Yale University, where he received his PhD in 1985. He held positions at several major universities in the United States, namely Berkeley, Notre Dame and Irvine, where he was an associate professor from 1991 to 1996. During this period, his research focused on algebra, in particular on the interplay of mathematical logic, model theory and group theory. In 1996, he became a professor at the newly established Bilgi University in Istanbul, where he founded the Department of Mathematics, which he has chaired until today.

He has published 37 research articles in leading mathematical journals and the highly cited research monograph *Groups of Finite Morley Rank* together with Alexandre Borovik.

Return to Turkey

Ali Nesin's career took a significant turn in 1995 due to the death of his father, Aziz Nesin, a legendary Turkish writer of over 100 humorous and satirical books. Aziz Nesin had assigned the earnings from all rights from his published works to a charitable foundation, the Nesin Foundation, which he had established for the education of orphans and children from destitute families.

Ali Nesin gave up an academic career in the US and returned to Turkey to ensure the continuation of the Nesin Foundation.

He changed his life in order to realise his vision: the enhancement of understanding of mathematics among the youth as an essential force for the economic, social and cultural development of his country.

Fig. 2. Ali Nesin working with elementary school students.

Nesin's outreach programme

Ali Nesin's activities toward the awareness of mathematics in Turkish society are numerous. He initiated several outreach activities directed at the Turkish public in general as well as students of mathematics at the high school level and beyond.

From 2003 to 2013, he was editor-in-chief of *Matematik Dünyası* ("The World of Mathematics"), a monthly magazine for the popularisation of mathematics, which presented many fundamental notions of mathematics to the public (e.g. foundations of mathematics, theory of limits, graph theory, theory of groups and p-adic numbers). Each issue sold around 20,000 copies, which is very rare for such a journal.

He established his own publishing house, publishing popular mathematical texts, including nine of his own works of popularisation, as well as an ongoing series representing curriculum development at the Mathematics Village.

His materials for an open source courseware in Turkish were made available online through the Turkish Mathematical Society. He also authored many popular mathematical articles in Turkish, which appeared in periodicals aimed at a national audience.

In addition, he developed and taught a supplementary mathematical instruction programme for undergrad-

[1] Modified version of my laudation on Ali Nesin at the Opening Ceremony of the ICM 2018 in Rio de Janeiro.
[2] See https://www.mathunion.org/imu-awards/leelavati-prize.

uates outside the regular school term. It was designed to bring the students quickly to a competitive international level and developed into the nucleus of the Mathematics Village after 2007.

Approximately 7,000 videos related to the activities of the Mathematics Village have been posted on YouTube, often lectures by Ali Nesin to students in the Mathematics Village, as well as a public lecture given at Gezi Park in central Istanbul. In total, these have been viewed several million times.

Ali Nesin's open source courseware received a prize from the Turkish Mathematical Society for expository excellence in 2010. He has also won prizes for several of his textbooks.

The idea of a Mathematics Village

However, what makes the work of Nesin unique, and what goes beyond all envisaged activities for the Leelavati Prize, is his creation, organisation and development of the Mathematics Village, in the face of considerable financial, bureaucratic and practical difficulties, and against ideologically motivated resistance.

Fig. 3. The Nesin Mathematics Village – a panoramic view.

The Nesin Mathematics Village is located on a physical site south of Izmir, near the village of Şirince in the region of the ancient Greek town of Ephesus. This site is owned by the Nesin Foundation and is dedicated entirely to teaching and learning of mathematics on a non-profit basis. It is fully devoted to the enhancement of understanding of mathematics of gifted students at all levels.

However, it is not in competition with the official education system but a supplement, giving students access to knowledge free from examinations and fear of failure, in the inspiring environment and stimulating atmosphere of a camp.

Construction of the Village

When Nesin first returned to Turkey, he quickly realised that the students in Turkey needed additional preparation for an education in line with international standards. He came to the conclusion that a new model of education was a necessity for the proper development of Turkish mathematics. He first organised additional lectures in his own house and in tents in various rural locations in Turkey, which soon attracted talented undergraduates from all over Turkey. Mathematically, these lectures were very successful but the organisation of the workshops at varying venues was very difficult, both financially and organisationally. He decided that the programme needed a permanent home.

In 2007, Ali Nesin, together with a close friend, the self-educated architect (and also a prominent Turkish linguist) Sevan Nisanyan, started the construction of the village.

Fig. 4. The Sevan Nişanyan Library, today.

Ali undertook the construction of the village, initially on a small scale. He had bought the land in a nice but remote area adjacent to the small village Şirince near Selçuk, Izmir. It was a great idea but also a daring adventure to start such an endeavour. The architectural challenges were to build a village nestled organically in the landscape of a deserted area and to transform the site into an inspiring environment aligned with the spirit of a new way of conveying mathematical understanding to Turkish youth.

Bureaucratic difficulties

Not counting the numerous practical problems, the bureaucratic difficulties to create the village were even greater.

The authorities refused an official building permit because the village had no officially registered "street". Ali tried very hard to register the already existing path into an official street, spending a lot of money, but he failed. The building permit was never denied but it was also not granted. After some time, he gave up the idea of getting the permission and started the construction.

He was accused of having founded an illegal educational institution and of "teaching without permission", contrary to the freedom of teaching of sciences guaranteed by Turkish constitution. However, the Mathematics Village is not an educational institution. Young people come there only for a week or two. There are no exams or grades and no diplomas or transcripts of any kind are issued. Nevertheless, the Mathematics Village was raided by Gendarmes and sealed. Fortunately, the case was dropped and Ali was saved from prison, and the village was reopened.

Part of the resistance to the Mathematics Village stemmed from antipathy to the name Nesin: Ali's father was a well known leftist and an avowed atheist, a very controversial issue in the religious-conservative part of Turkish society. In 2014, Sevan Nişanyan (also a prominent atheist) was imprisoned on politically motivated made-up charges. He fled the prison in 2017 and now lives in Greece.

Ali just promotes sharing and learning of mathematics, and all his actions are non-ideological. The difficulties still continue, although less than in the beginning.

Support for the Village

The above mentioned difficulties could not stop Ali from realising the dream of a place where mathematical sharing and learning is possible in an informal environment, free from all constraints and only dedicated to enhancing mathematical understanding.

He received support from his colleagues and students, not only within Turkey but also within the international community.

Lectures were initially given in the so-called "Langlands Shed," named in honour of Robert Langlands, who had generously donated a significant portion of his Shaw Prize and recently also his Abel Prize.

Fig. 5. The Langlands Shed, 2007.

The continuing support of the international mathematics community in giving advanced, intensive lectures on a voluntary basis during the graduate Summer programmes also played an important role in the success of the project.

The operation of the Mathematics Village is now self-sustaining, with public donations used only for the purpose of expansion and development.

Developing the Village

In the first year, only about 100 students participated in the Summer programme at an undergraduate and graduate level, taught by Ali and a few enthusiastic mathematicians.

The building complex expanded and more and more students could be admitted. Turkish graduate students also volunteered to give courses. Moreover, the village has become an important venue for international conferences.

In addition to university students, high school children from the age of 14 have also been admitted to the two-week camps during Summer holidays. The fees for the high school programme are $25 per day, reduced or waived whenever there are financial difficulties. The project enjoyed such a good reputation that it was considered an honour to be admitted. Many of the lectures are now viewable as videos on YouTube.

While normal high school education in Turkey is focused on university admission, with the typical focus on memorising, the focus in the Mathematics Village is on communicating, understanding and independent thinking.

Teaching and Learning

All teaching in the village is voluntary and unpaid; in return, the accommodation and meals for the lecturers are free. Most of the courses are in Turkish but some are also in English. The two-week cycles begin and end on Sundays with the departure and arrival of hundreds of students.

The university level courses are organised according to topics. The range of subjects is large. The programme for 2018 lists the following events on the English webpage: Week of Theoretical Physics, Antalya Algebra Days XX, Workshop on Integer Partitions, Randomness in Complex Geometry and Complex Analysis, Categories and Toposes and Non-Commutative Geometry, TMD Undergraduate and Graduate Summer School, High School Philosophy Summer School, International Aegean School of Human Rights.

Moreover, the Turkish webpage lists another 46(!) events, such as computer programming courses and physics schools but also completely different topics like seminars on cinema, architecture, arts, history, politics and a freedom philosophy Summer school.

The village attracts the best Turkish teachers and promoters of mathematics, as well as mathematicians from all over the world. Many lecturers are former students of previous Summer programmes in the village.

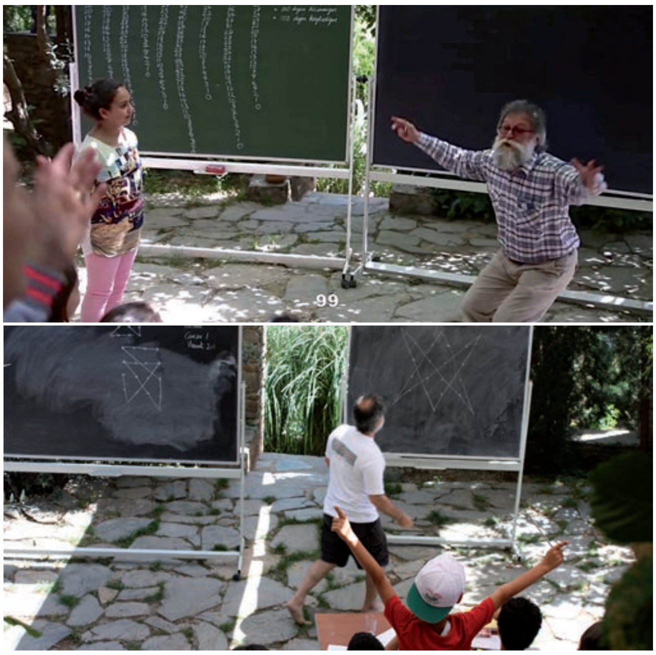

Fig. 6. Ali's enthusiasm during open-air lectures is contagious.

Fig. 7. Lecture in an open-air theater.

Fig. 9. Sign my math book please!

A few numbers

The village now comprises more than 35,000 square metres, approximately half of it consisting of olive groves. The complex of buildings, at the moment, consists of:

- Sixteen bedrooms, two amphitheatres and four closed and four open-air lecture halls.
- Two Turkish baths, 29 single or double rooms and a fully functional kitchen.
- A cafeteria, a small shop and a wonderful two-storey library.

The village has the capacity to accommodate 150 people, with the option of pitching tents if more capacity is needed. Sometimes the place is overflowing with about 400 students. In 2017, there were 10,379 visits by children of the age 13 or older as part of organised groups.

The holdings of the Sevan Nişanyan Library at the Mathematics Village are currently 15.000 books at a wide range of levels.

Fig. 8. Library conference hall and joint dining area.

Basic principles

In 2014, Ali Nesin expressed his ideas to expand the Mathematics Village by constructing adjacent Philosophy and Art Villages with the words:

> *"The whole valley should be dedicated to education – not a standard one but a 'pirate' one".*

This idea has been put into action. An independent Theatre School has been established on an adjacent site, also with the assistance of Sevan Nişanyan.

The Nesin Mathematics Village is now a cultural magnet and even a tourist attraction whose architectural principles have become fashionable in Turkey.

The governing aim is: access to knowledge, education and freedom, based on the principles of *safety – independence – responsibility*, derived from his experiences as Director of the Nesin Foundation. Quoting Ali:

> *"It is not possible to have a proper education in an environment without freedom. You can give an average education in an environment with restricted freedom, but not a proper one."*

Conclusion

According to the statutes of the Leelavati Prize, the prize is awarded to a person *in recognition of outstanding contributions for increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavours*".

It is hard to imagine that someone else has earned this award more than Ali Nesin.

Fig. 10. Sevan Nişanyan and Ali Nesin at the Award Ceremony.

Acknowledgement

I would like to thank Alexandre Borovik and Gregory Cherlin for sharing their knowledge about Ali Nesin and the Mathematics Village. I also want to thank the administration of the Nesin Mathematics Village for providing recent financial and operational data.

Furthermore, I have made use of the articles [1] and [2].

Photo credits

Fig. 1: Photo by Alexandre Campbell, ICM 2018. https://www.flickr.com/photos/icm_2018/.

Figs. 2, 6-1, 9: Screenshots from YouTube Video *Matematik Köyü* (Mathematics Village) by Ayser Sude Gök (2014). https://www.youtube.com/watch?v=YUd3HvlQMEc.

Figs. 3, 4, 8-1, 8-2: Photos by Burak Barutcu, courtesy Aslı Can Korkmaz.

Figs. 5, 6-2, 7: Photos by Alexandre Borovik.

Fig. 10: Photo by Alexandre Campbell, ICM 2018. https://www.flickr.com/photos/icm_2018/.

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[1] Krishnaswami Alladi, Gabriela Asli\ Rino Nesin: The Nesin Mathematics Village in Turkey. *Notices of the AMS}*, 62(6), 2015.

[2] Gizem Karaali: Nesin Math Village: Mathematics as a Revolutionary Act. *The Mathematical Intelligencer*, Springer Science+Business Media, 36(2), 2014.

Gert-Martin Greuel is a professor emeritus of mathematics at the University of Kaiserslautern. He has been director of the mathematics research institute Oberwolfach (2002–2013), Editor-in-Chief of Zentralblatt Math (zbMATH) (2012–2015), and founder and scientific advisor of IMAGINARY, the platform for public awareness of mathematics.

ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

ICME 14 in 2020*

The first international programme committee (IPC) meeting of ICME 14 was held in Shanghai, 11–17 September 2017. Twenty-one IPC members participated in the meeting.

As a result of friendly but extensive discussions and negotiations during the meeting, the overall scientific structure of ICME 14 was determined, speakers of plenary lectures and invited lectures were nominated, and themes and teams of plenary panels, survey teams and TSGs were proposed. In contrast to previous ICMEs, it has been decided that TSGs in ICME 14 will be grouped into two classes, Class A and Class B, to be arranged into two different time slots so that more TSGs can be accommodated and participants can be more flexible in attending TSG activities. Details of the main academic activities of ICME 14 can be found at http://icme14.org (the official website of ICME 14).

The first announcement of ICME 14 was published on the official website of ICME 14 and can be downloaded from http://icme14.org/images/icme/announcemen/FirstAnnouncement.pdf.

Important information, such as submissions of proposals and papers, registration and the ICME 14 solidarity fund, has been provided in the announcement. Moreover, calls for national presentations at ICME 14 have been announced on the official website of ICME 14 http://icme14.org/ and the ICMI website https://www.mathunion.org/icmi/news-and-events/2018-08-01/call-national-presentations-icme-14-2020. Any intention to organise a national presentation at ICME 14 is warmly welcomed.

The IPC members will meet for a second time in March 2019 in order to finalise the programme and discuss issues related to the conference website system (including the registration system, the submission system and the review system), the proceedings and the venues, etc.

* With the permission of Binyan Xiu who published it first in the November 2018 ICMI Newsletter.

ICMI-25 Study Briefing – June 2018
Mathematics teachers working and learning in collaborative groups

The set of ICMI Studies was launched in the mid-1980s and has acquired a growing importance and influence in the field. It contributes to a better understanding and resolution of the challenges that face multidisciplinary and culturally diverse research and development in mathematics education. Each study focuses on a topic or issue of prominent current interest in mathematics education. Built around an international conference, it is directed toward the preparation of a published volume intended to promote and assist discussion and action at the international, regional or institutional level.

The 25th ICMI Study has just been launched; it will focus on "Mathematics teachers working and learning in collaborative groups".

The idea of mathematics teachers working and learning through collaboration is gaining increasing attention in educational research and practice, particularly since the report on Lesson Study in Japan from the TIMSS classroom video study (Stigler, Gonzales, Kawanaka, Knoll & Serrano, 1999). Across education systems and at all educational levels, mathematics teachers work and learn through various forms of collaboration, which may contribute to their learning and development in different ways. Efforts to understand what teachers do as they work in collaborative groups, and how this leads to improvement in their practice and expertise, has led to increasing interest in examining the different activities, processes and contexts for teacher collaboration around the world. The work completed by the ICME-13 Survey Team on this theme is evidence of the considerable international interest in research on teachers working and learning through collaboration. However, the ICME-13 Survey also identified several gaps and limitations, not only in the existing research base but also in the survey's coverage of relevant topics within the theme, which highlights the need for the proposed ICMI Study.

The study's theme of teachers working and learning in collaborative groups implies a focus on teachers as they work within teams, communities, schools or other educational institutions, teacher education classes, professional development courses, and local and national networks – that is, in any formal or informal grouping. Teachers' collaborative work might also include those who support their learning and development, such as trainers or coaches, mentors and university academics. Collaboration can extend over different periods of time and can take place in face-to-face settings or at a distance. The role of online platforms and technology-enabled social networks is an additional focus in supporting "virtual" collaboration.

Because there are different ways of understanding teacher collaboration and its characteristics, enablers and consequences, the study will include multiple theoretical, methodological and contextual perspectives. It will be particularly important to solicit contributions from teachers as well as researchers, so that teachers' voices are given equal prominence in accounts of their learning. Likewise, the study will acknowledge that learning is mutual, that is, that those who work collaboratively with teachers to develop their practice are also learning from these interactions.

Some of the areas and questions that the study will investigate are:

- Conceptualising and enacting collaboration.
- Supporting and researching teachers' work and learning through collaboration.
- Goals of collaboration.
- Resources for teacher collaboration.
- Cultural and political contexts for teacher collaboration.
- Cross-cutting issues in studying and supporting teacher collaboration.

The international programme committee appointed by the ICMI executive committee is formed of the following researchers and mathematics educators with experience in this area:

- Hilda Borko (Co-Chair, Stanford University, USA).
- Despina Potari (Co-Chair, Athens State University, Greece).
- Joao Pedro da Ponte (University of Lisbon, Portugal).
- Shelley Dole (University of Sunshine Coast, Australia).
- Cristina Esteley (National University of Cordoba, Argentina).
- Rongjin Huang (Middle Tennessee State University, USA).
- Ronnie Karsenty (Weizmann Institute of Science, Israel).
- Takeshi Miyakawa (Joetsu University, Japan).
- Ornella Robutti (University of Turin, Italy).
- Luc Trouche (Ecole Normale Supérieure de Lyon, France).
- Ex Officio members: Jill Adler (ICMI President) and Abraham Arcavi (ICMI Secretary General).

The first meeting of the international programme committee will take place in Berlin, 11–14 February 2019, and a discussion document with a call for papers for the study conference will be distributed soon thereafter. The study conference is planned for January 2020.

Call For Nominations for the 2019 Felix Klein and Hans Freudenthal Awards

Since 2003, the International Commission on Mathematical Instruction (ICMI) has awarded biannually two medals to recognise outstanding accomplishments in mathematics education research:

- The Felix Klein Award, for lifelong achievement in mathematics education research.
- The Hans Freudenthal Award, for a major programme of research on mathematics education.

The Felix Klein medal is awarded for lifetime achievement in mathematics education research. This award is aimed at acknowledging those excellent senior scholars who have made a field-defining contribution over their professional life. Past candidates have been influential and have had an impact both at the national level, within their own countries, and at the international level. We have honoured in the past those candidates who have not only made substantial research contributions but have also introduced new issues, ideas, perspectives and critical reflections. Additional considerations have included leadership roles, mentoring and peer recognition, as well as the actual or potential relationship between the research carried out and improvement of mathematics education at large, through connections between research and practice.

The Hans Freudenthal medal is aimed at acknowledging the outstanding contributions of an individual's theoretically robust and highly coherent research programme. It honours a scholar who has initiated a new research programme and has brought it to maturation over the past 10 years. The research programme will be one that has had an impact on our community. Freudenthal awardees should also be researchers whose work is ongoing and who can be expected to continue contributing to the field. In brief, the criteria for this award are depth, novelty, sustainability and impact of the research programme.

See http://www.mathunion.org/icmi/activities/awards/the-klein-and-freudenthal-medals/ for further information about the awards and for the names of past awardees (to date, eight Freudenthal Medals and eight Klein Medals).

The ICMI Klein and Freudenthal Awards Committee consists of a Chair (Professor Anna Sfard) nominated by the President of the ICMI and five other members, who remain anonymous until their terms have come to an end. The ICMI Klein and Freudenthal Awards Committee is now entering the 2019 cycle of selecting awardees and is currently welcoming nominations for the two awards from individuals or groups in the mathematics education community.

Nominations for the Felix Klein Award should include the following:

1) A document (of a maximum of eight pages) describing the achievements of the nominee (e.g. their theoretical contribution and/or empirical research, leadership roles, graduate supervision and mentoring, and peer recognition) and reasons for the nomination (including a description of the nominee's impact on the field).
2) A one-page summarising statement.
3) A curriculum vitae of the nominee (of a maximum of 20 pages).

4) Electronic copies of three of the nominee's key publications.
5) Three letters of support (preferably from different countries).
6) The names and email addresses of two persons, other than the nominee themselves, who could provide further information, if needed.

Nominations for the Hans Freudenthal Award should include the following:

1) A document (of a maximum of five pages) describing the nominee's research programme and reasons for the nomination (including a description of the nominee's impact on the field).
2) A one-page summarising statement.
3) A curriculum vitae of the nominee (of a maximum of 10 pages).
4) Electronic copies of three of the nominee's key publications.
5) Three letters of support (from different countries, if possible).
6) The names and email addresses of two persons, other than the nominee themselves, who could provide further information, if needed.

All nominations must be sent by email to the Chair of the Committee (annasd@edu.haifa.ac.il, sfard@netvision.net.il) no later than 31 March 2019.

Professor Anna Sfard
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Israel

ERME Column

Susanne Prediger (President of ERME, TU Dortmund University, Germany) & Elmar Cohors-Fresenborg (Founding Vice-President of ERME, University of Osnabrück, Germany), Jason Cooper (Weizmann Institute of Science, Rehovot, Israel)

ERME's 20th Anniversary

In 2018, the European Society for Research in Mathematics Education (ERME) celebrated its 20th anniversary. This article includes a recollection of the first conference in 1998, written by Elmar Cohors-Fresenborg, who was the host of CERME 1 and the founding Vice-President of ERME, and a presentation of a rich anthology, written by the ERME community on the occasion of the anniversary, highlighting 20 years of research.

CERME1 – The First Conference of the European Society for Research in Mathematics Education was held 27–31 August 1998

Twenty years ago, it was a very exciting time for mathematics education in Europe. During the weekend of 2–4 May 1997, mathematics educators from 16 countries met to discuss what a European society in mathematics education research might look like. The meeting took place in Haus Ohrbeck, near Osnabrück in Germany. Representatives from these 16 countries formed the initial constitutive committee of ERME. It was agreed that the founding philosophy of the society should be one involving *Communication*, *Cooperation* and *Collaboration* throughout Europe.

The formal constitution of ERME took place in a conference scheduled for August 1998. To this *First Conference of the European Society for Research in Mathematics Education (CERME1)* participants from all over Europe would be invited. CERME1 was organised by the following programme committee: Elmar Cohors-Fresenborg (Germany) as coordinator, Milan Hejny (Czech Republic), Barbara Jaworski (United Kingdom), Joao Pedro da Ponte (Portugal) and André Rouchier (France).

CERME1 was held 27–31 Aug 1998 in Haus Ohrbeck, near Osnabrück, with 120 participants from 24 countries. For the coordinator, it was very important to find substantial support for scholarships to let participants from Middle and East European Countries attend the conference. A generous grant of the "Deutsche Forschungsgemeinschaft" (German National Science Foundation) enabled 22 participants from Middle and East European Countries to attend.

Three sessions at the conference were devoted to considering the new society, its ongoing work and its constitution.

The scientific work was organised in seven thematic groups. Its members worked together on common research. They shared their individual work in order to develop a joint programme of work for the future. It was the intention that each group would engage in scientific debate with the purpose of deepening mutual knowledge about topics, challenges and methods of research in the field. In these ways, the style of the conference deliberately and distinctively moved away from research presentations by individuals toward collaborative group work.

The results of CERME1 have been published in three volumes of more than 1400 pages. Inge Schwank did the editing work and she also translated the papers into a common layout, which was an incredible amount of work in those days.

From CERME 2 to CERME 11

Since 1998, ERME kept this spirit of *Communication*, *Cooperation* and *Collaboration* across Europe. The community has grown immensely. CERME11, to be held February 2019 in Utrecht, has attracted a huge number of researchers, and we will be obliged – for the first time – to limit the number of participants to 850.

During these years since 1998 the exchange about mathematics education research has fuelled substantial developments in the field. These developments are documented in the following book:

Anthology presenting the 20 years of ERME research

After three years of preparation, an anthology documenting 20 years of ERME research has appeared: Dreyfus, T., Artigue, M., Potari, D., Prediger, S. & Ruthven, K. (Eds.) (2018), *Developing Research in Mathematics Education – Twenty Years of Communication, Cooperation and Collaboration in Europe*", Oxon: Routledge (https://www.routledge.com/Developing-Research-in-Mathematics-Education-Twenty-Years-of-Communication/Dreyfus-Artigue-Potari-Prediger-Ruthven/p/book/9781138080294).

The book chapters were written by experienced ERME researchers and were discussed by all CERME participants in 2017. This procedure reflects the ERME spirit of *Communication*, *Cooperation* and *Collaboration*. The ERME community thanks all authors and editors, especially Tommy Dreyfus for taking the lead on a huge, joint enterprise!

Developing Research in Mathematics Education is the first book in the series *New Perspectives on Research in Mathematics Education*, to be produced in association with ERME. This inaugural volume sets out broad advances in research in mathematics education that have accumulated over the last 20 years through the sustained exchange of ideas and collaboration between researchers in the field.

The chapters provide perspectives on major areas of research in the field on topics that include:

- The content domains of arithmetic, geometry, algebra, statistics and probability.
- The mathematical processes of proving and modelling.
- Teaching and learning at specific age levels from early years to university.
- Teacher education, teaching and classroom practices.
- Special aspects of teaching and learning mathematics, such as creativity, affect, diversity, technology and history.
- Theoretical perspectives and comparative approaches in mathematics education research.

Current news from CERME

The upcoming CERME11 (https://cerme11.org/) comprises 26 Thematic Working Groups (TWGs), reflecting the growing diversity of the interests in our community. Since the September 2017 issue of the newsletter, we have been introducing these working groups to readers from the mathematics community. To date, we have introduced six of the groups: TWG 14 (University Mathematics Education, September 2017 – Issue 105), TWG 15-16 (Teaching and Learning Mathematics with Technology and Other Resources, December 2017 – Issue 106), TWG 1 (Argumentation and Proof, March 2017 – Issue 107), TWG 12 (History in Mathematics Education, June 2018 – Issue 108) and TWG 9 (Mathematics and Language, September 2018 – Issue 109).

Susanne Prediger has been a full professor for mathematics education research since 2006 at IEEM, the Institute for Development and Research in Mathematics Education at TU Dortmund University. Since 2017, she has been President of ERME (the European Society for Research in Mathematics Education). Her fields of expertise concern design research, conceptual understanding in mathematics, language diversity and mathematics teacher education research.

Elmar Cohors-Fresenborg was full professor for mathematics education research, from 1975 until 2013, at the Institute for Cognitive Mathematics (IKM) at the University of Osnabrueck. From 2001 until 2005 he was vice president of ERME, the European Society for Research in Mathematics Education. His fields of expertise concern a cognitive approach to research in mathematics education, especially the role of pupils' and teachers' metacognitive and discursive activities as an indicator of instructional quality in the learning and teaching mathematics.

Jason Cooper is a researcher at the Weizmann Institute's Department of Science Teaching. His research concerns various aspects of mathematics teacher knowledge, including roles of advanced mathematical knowledge in teaching, and contributions of research mathematicians to the professional development of teachers. He has been a member of the ERME board since 2015.

The Mathematical Space in the Portuguese Language (EMeLP)

Fernando Pestana da Costa (Univ. Aberta, Lisboa, Portugal), João Semedo (Univ. de Cabo Verde, Praia, Cape Verde), Marcos Cherinda (UNESCO Office, Maputo, Mozambique), Mário Carneiro (Univ. Federal de Minas Gerais, Belo Horizonte, Brazil), Yuriko Baldin (Univ. Federal de São Carlos, São Carlos, Brazil)

The Mathematical Space in the Portuguese Language (EMeLP – Espaço Matemático em Língua Portuguesa) is an international partnership, the members of which are scientific societies and scientific and academic institutions whose main interest is the promotion and development of mathematics in its broadest sense in countries that have Portuguese as one of their official languages. These countries, located in Africa (Angola, Equatorial Guinea, Cape Verde, Guinea-Bissau, Mozambique, S. Tomé and Principe), America (Brazil), Asia (East Timor) and Europe (Portugal), form a community of more than 260 million, with a broad cultural, historical and social diversity, diverse stages of economic and scientific development, and various needs concerning mathematics teaching and research.

Although the idea to create EMeLP (inspired by the already existing Espace Mathématique Francophone) goes back to the first decade of the 21st century, when it was suggested by Jaime Carvalho e Silva, then the representative of Portugal at the ICMI, real efforts to put the idea on a firm footing only started in January 2014. The founding act of EMeLP took place shortly afterwards, on 7 June 2014, when the then representatives of Brazil (Victor Giraldo), Mozambique (Marcos Cherinda) and Portugal (José Francisco Rodrigues) at the ICMI signed the creation act of EMeLP at IMPA in Rio de Janeiro, Brazil (Figure 1). At present, EMeLP has the active participation of the founding countries and also Cape Verde, which joined the "Space" shortly after. One of our main goals is to invigorate the mathematical community of the remaining Portuguese speaking countries to adhere and get actively involved in EMeLP initiatives.

The objective of EMeLP is to develop and help disseminate, among Portuguese speaking countries, joint initiatives in mathematics in its widest sense, namely in: mathematics research; applications of mathematics in other sciences, technology, arts and society at large; education; teacher training; communication and popularisation of mathematics for the general public; and dissemination of educational resources and materials.

Fig. 1. Signing the creation act of EMeLP in January 2014 in Rio de Janeiro. From left to right: Marcos Cherinda, Victor Giraldo and José Francisco Rodrigues.

Currently, EMeLP publicises its own activities, as well as other mathematics related activities in Portuguese speaking countries, through its Facebook page (https://www.facebook.com/emelportuguesa); a dedicated webpage is planned for creation soon. However, the first widely visible initiative of EMeLP was the promotion of its 1st international congress (CiEMeLP 2015), which took place in Coimbra, Portugal, 28–31 October 2015, with the moto "The Multiple Ways to Produce and Communicate Mathematical Culture in the Portuguese Language" (Figure 2).

Fig. 2. Webpage of the first international congress of EMeLP, which took place in Coimbra (Portugal), 28–31 October 2015.

This meeting had more than 260 participants from 10 countries (including a few from three countries outside EMeLP – Paraguay, Spain and the UK). There were seven invited plenary talks, two special sessions about the life and work of the late ethnomathematician Paulus Gerdes and the mathematician José Sebastião e Silva, 12 parallel talks and more than 120 contributed talks in the five Discussion Groups ("Mathematics, culture and society", "Relations between mathematics in schools, universities and in other social practices", "Mathematics communication inside and outside schools", "Mathematics' teacher formation" and "Technologies in the teaching and communication of mathematics"), in addition to a significant number of poster presentations. The conference was home to many well-attended and lively discussion sessions, with the participation of mathematicians, mathematical educators and teachers (Figure 3).

Fig. 3. A panel discussion at the first international congress of EMeLP.

According to the by-laws of EMeLP submitted to the ICMI for approval, these international congresses are to be a regular feature of EMeLP, where discussions of past activities and planning for future initiatives will take place, as well as the election of new direction boards.

The second international congress of EMeLP is scheduled to take place in Maputo, Mozambique, in November 2019.

Although different Portuguese speaking countries have different needs in mathematics education, research and public awareness, and have different cultural contexts in which mathematical practices and reasoning impinge, the need for: public recognition of the importance and ubiquity of mathematics; an improved mathematics education at all levels; a stronger mathematical background for teachers; and improved quality and enhancement of international collaboration and the impact of mathematical research are recognised as important goals in all of the countries involved and, thus, are aspects to which EMeLP will certainly be paying attention in the years to come.

Fernando Pestana da Costa [fcosta@uab.pt] teaches at the Department of Sciences and Technology of Universidade Aberta, Lisbon, and is a researcher at the Centre for Mathematical Analysis, Geometry and Dynamical Systems, Instituto Superior Técnico, University of Lisbon, Portugal. His research interests are in mathematical analysis and differential equations, with a particular emphasis on dynamical aspects. He received his PhD at Heriot-Watt University, Edinburgh, in 1993. He has been Vice-President (2012-2014) and President (2014–2016) of the Portuguese Mathematical Society. Since 2017, he has been the representative of Portugal at the ICMI.

João Felisberto Semedo [joao.semedo@docente.unicv.edu.cv] is currently the President of the Scientific Council at the University of Cabo Verde. He graduated in mathematics from Universidade Federal Fluminense, Rio de Janeiro (1998), and has an MSc in didactics of mathematics (2004) and a PhD in education (didactics of mathematics), both from Universidade de Lisboa (2011). He is a professor at the University of Cabo Verde and was President of the Pedagogical Council of the Department of Science and Technology and President of its Board of Directors from September 2012 to June 2015. He is a founding member of AMATCV (Mathematical Association of Cape Verde) and is the representative of Cabo Verde at the ICMI.

Marcos Cherinda [mcherinda@gmail.com] received the degree of Teacher of Mathematics and Physics (1980–1981) for Secondary Education at the Faculty of Education of Universidade Eduardo Mondlane in Maputo, Mozambique. In 1989, he completed his Diplom Lehrer in mathematics in Germany. In 2002, he received a PhD in mathematics education from the University of the Witwatersrand, Johannesburg, South Africa. Until 2017, he worked as a professor of mathematics education at the Pedagogical University in Mozambique, where he was Director of the Faculty of Natural Sciences and Mathematics. Since 2012, he has been the national representative of Mozambique at the ICMI. He is currently an official of UNESCO at the Maputo office.

Mário Jorge Dias Carneiro [carneiro@ufmg.br] is an emeritus professor at Universidade Federal de Minas Gerais, Brazil. His main research area is dynamical systems, with a special interest in conservative systems. He received his PhD from Princeton University in 1980 and has held post-doctoral positions at IMPA, Brazil, and Princeton University. He has been Vice-President of the Brazilian Mathematical Society (1993–1996) and, in 2005, was the coordinator of a curriculum reform in mathematics for the State of Minas Gerais, Brazil. He is the representative of Brazil at the ICMI.

Yuriko Yamamoto Baldin [yuriko@dm.ufscar.br] received a PhD in mathematics from the Universidade Estadual de Campinas, Brazil, in 1984, with a thesis in differential geometry. She has done postdoctorate studies in the USA, made scientific visits to Japan and has been a visiting professor at the University of Tsukuba (2014). She is a senior professor of mathematics at the Department of Mathematics of the Universidade Federal de São Carlos, where she has worked since 1977. Her research interests include teacher education and development of didactical materials. She has represented Brazil at the ICMI (2008–2012) and is currently a member at large of the Executive Committee of the ICMI for 2017–2020, after also holding this position for 2013–2016. She is the appointed liaison of EMeLP to the ICMI.

Green, Gold, Platinum, Nickel: On the Status of Open Access in Mathematics

Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

The various facets of Open Access (OA) have been a topic in the EMS Newsletter for many years, including articles on such diverse topics as a global description of the publishing ecosystem with its political ramifications [1] and the concrete circumstances and policies leading to the flip of a single journal [2]. Recently, the announcement of the plan of cOAlition S [3] and the choice of Elsevier as a subcontractor to monitor the progress of OA by the European Commission triggered many reactions, including a statement by the EMS [4]. The aim of this note is to give a status report on the various OA facets in the mathematical corpus, which may also provide some quantitative information helpful to estimate the impact of envisioned changes.

The ongoing progress of green OA in mathematics

Almost three years ago, we discussed the progress of green OA in mathematics [5]. While much more data are now available, it is still true that the arXiv accounts for most of the green OA deposits in mathematics. In addition, institutional repositories seem to contribute a growing share. One caveat here is that an arXiv deposit may not count as full green OA since frequently only preliminary versions are available; funders' policies may help to encourage mathematicians to include updates with equal math content to the published version (even though the layout may differ).

Figure 1 shows the share of arXiv versions matched to publications indexed in zbMATH for the main MSC subjects.[1] We can see that for several core areas in mathematics, almost half (or more) of recent articles are available. Quite remarkably, the share of arXiv submissions is sometimes now even higher than in areas of mathematical physics (MSC 81–83), which traditionally formed arXiv's nucleus. Given the relatively stable share of around 40% in the latter areas over the last few years, propelled by large journals like JHEP, which have a 100% arXiv overlap, one might wonder which figures would form a possible limit of saturation; but, so far, the overall trend still points upward.

This strong growth is less obvious from the total figures, which are still slightly below 10% of recent publications; this is, however, due to the effect of large applied areas being less present on the arXiv. However, with the achieved share and the ample amount of reference data available, we are in a position to test the effectiveness of the arXiv with respect to early dissemination.

To do so, we employ citations as a proxy for awareness. Naturally, there is the obstacle that citation numbers are influenced by many factors (not related to the quality of a paper) [6] and that the presence on the arXiv may come along with an inherent bias. Therefore, we select several larger, mostly general, mathematics journals (for our approach to this term, see a previous newsletter article [7]), which have reached a share of around 50% of arXiv overlap. Altogether, they account for about 12% of the published maths arXiv. This aggregation aims at minimising selection or subject bias. The following table and picture show publication vs. citation year for nine jour-

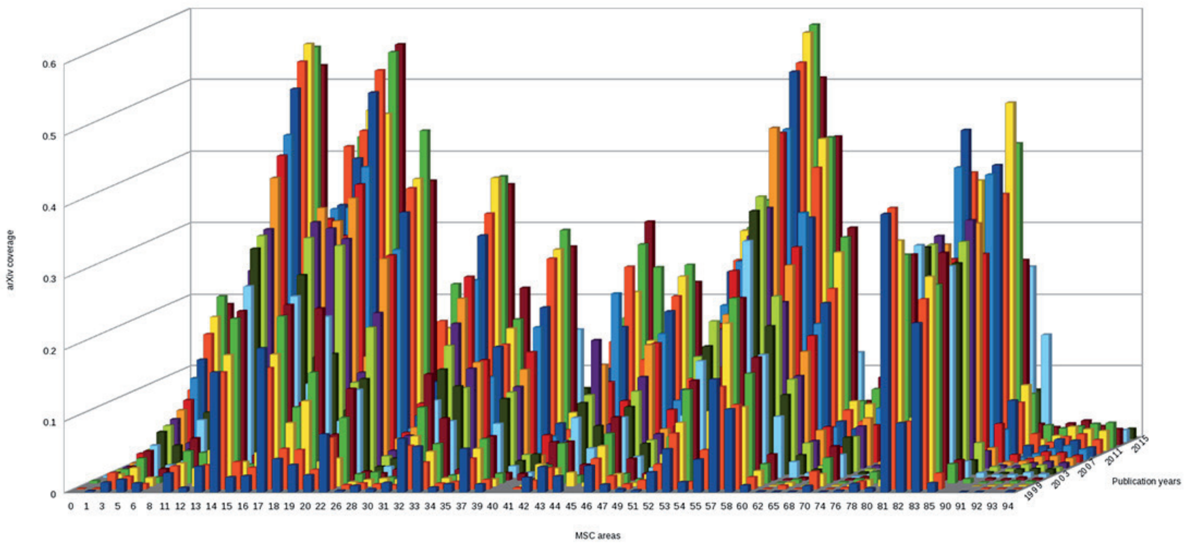

Figure 1: Progress of the arXiv share for the main MSC classes for publication years 1999–2017.

[1] Recent figures can be retrieved via zbMATH by adding "en:arXiv" to queries; in particular, the filters for MSC, years, authors or journals provide detailed information.

Table 1: Citations per year for nonarXiv/arXiv articles published.

Publication year	2011		2012		2013		2014		2015	
Type	nonarXiv	arXiv	nonarXiv	arXiv	nonarXiv	arXiv	nonarXiv	arXiv	nonarXiv	arXiv
# of publications	2006	1137	1877	1302	1616	1238	1752	1509	1885	1768
Citations in 2011	448	519	106	192	47	96	7	24	5	15
Citations in 2012	719	1098	462	452	119	186	40	86	15	22
Citations in 2013	740	1202	834	660	289	383	92	161	37	75
Citations in 2014	902	1520	1361	1031	900	767	434	630	109	258
Citations in 2015	987	1511	1483	1083	1234	1034	1060	1079	493	725
Citations in 2016	927	1498	1363	1098	1245	990	1260	1188	1057	1161
Citations in 2017	905	1408	1363	1031	1151	988	1300	1253	1321	1301
Citations in 2018	764	1055	1020	846	864	815	994	1008	1064	1144

nals (*Adv. Math, J. Algebra*, *J. Funct. Anal.*, *J. Math. Anal. Appl.*, *Int. Math. Res. Not.*, *Math. Z.*, *Pac. J. Math.*, *Proc. AMS* and *Trans. AMS*), which have the advantage of a rough balance recently of nonarXiv/arXiv articles, as well as presenting a reasonably broad variety of subjects.

Figure 2 illustrates the ratio of citations adjusted by article numbers in relation to the timeframe.

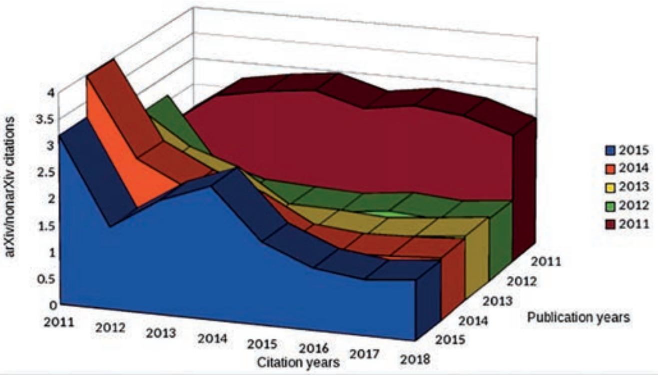

Figure 2: Adjusted citation ratio ((arXiv cit)*(nonarXiv pub)/(non-arXiv cit)*(arXiv pub))

As a first observation, we see that the ratio is always greater than one. That it approaches one in the long term seems to indicate that there is indeed no inherent selection bias. However, prior to publication, this ratio is usually larger than two, confirming (unsurprisingly) that green OA via the arXiv is indeed an effective way of dissemination (and not just preservation, as emphasised by plan S).

OA in mathematics journals: moving walls, (non-)APCs and hybrids

While this indicates that green OA serves well in supporting the dissemination and preservation role, the journal system is still (and perhaps, due to the growth of publications, more than ever) essential for ensuring scientific quality. Indeed, mathematicians consider the quality of peer review as by far the most important asset of a journal, much more than other aspects, such as access [8]. This seems completely sensible given the potentially enormous waste of resources connected to research based on false assumptions or incentives for duplicated or not genuinely new work. Hence, it is in the interest of the scientific community to maintain a reliable system of quality control and reputation management, without diverting too many resources in this process. In the past, this has been provided by the subscription model, with the revenues of society journals frequently used for grants, prizes, etc. Inherently tied to the print costs, subscriptions also came along with an incentive to create quality content. This aspect has faded in the age of digitisation and large bundles; indeed, the subscription model came under just criticism in connection with cases of monopolistic profits, frequently generated by the sale of bundles bloated by dubious content and lopsided incentives due to the problematic use of bibliometric measures by funders. The push for OA models was also a means to counteract these developments.

Due to the extreme longevity of mathematical research [9], by far the gravest issue with respect to making the maths corpus openly available is still to open up the knowledge assembled in the past. This is especially true since green OA simply doesn't work retroactively on a large scale [5]. The obvious solution is provided by wall ("embargo") policies, which require making the content available after a defined period of time. This approach has been pursued in various initiatives; notably, EuDML [10] has been very successful in assembling free resources, especially from European society journals. ProjectEuclid [11] and MathNet.Ru [12] have pursued a similar approach. The push from mathematicians has also motivated other publishers to adopt moving wall policies. Experience from societies (e.g. the AMS and the EMS) indicates that this hasn't harmed subscriptions; in addition, it perfectly complemented the arXiv corpus. Another important side effect of establishing sustainable moving walls is that they effectively undermine monopolistic approaches.

On the other hand, even with voluntary self-archiving and moving walls broadly available, a fraction of the corpus would remain at least temporarily behind paywalls. More importantly, both the mature status of green OA and the importance of historical content is quite specific for mathematics; other areas, along with funders, have therefore more eagerly ventured to establish business models allowing for immediate OA at the journal source. So far, the main distinction is the existence of article processing charges (APCs) (usually referred to as "gold OA") or the lack thereof ("platinum" or "diamond

OA"), in which case the infrastructure costs are usually covered by a third party (very often by an individual or local efforts but sometimes by larger infrastructure funds or libraries realigning subscription resources). APCs have been successfully established in several amply funded disciplines (e.g. medicine and biology) to an extent that both existing large commercial publishers consider them as the major tool for future revenue growth beyond the saturated subscription market (see the 2018 Springer Nature IPO prospect) and various new players have been attracted by this market. This development has been accompanied by intense lobbying both at European and national level, supported by the establishment of policies strongly in favour of APCs (e.g. the UK Finch report [13]). However, mathematicians overwhelmingly reject APCs for different reasons, perhaps the most prominent being that they install incentives to eradicate quality control (see N. Taubert [14] for a very detailed sociological study about the attitude of mathematicians toward the various aspects of OA). Below are some indications that this expectation has been mostly confirmed so far. A related development has been the installation of hybrid OA models, i.e. APCs for single articles in subscription journals. While some publishers adopted it as a means to allow submissions under constraints like those imposed by the Finch report, they are now widely considered to be problematic due to the option of double dipping when connected with the sale of large bundles of these journals, and limited benefits for science. While several measures have been proposed to adjust pricing with respect to APC-covered articles, the inherent lack of transparency of the bundle sales, and of a penalties mechanism in case of double charges, makes it hard to imagine them as a viable long-term option, although they may have at least the advantage that there are less incentives for lower quality, since this may destroy the journal's reputation and hence affect the demand for subscriptions. This would, however, only be true in the rare case of non-bundled sales. In any case, hybrid OA has failed to meet the demand for broad OA (in mathematics, its existence is extremely scarce), such that even APC advocates seem now to be in favour of its abolishment.

Finally, non-APC journals exist in a large variety. Some of them were established at the beginning of the digitisation era and flourished basically as one-person projects, with all the relevant advantages (reduced overhead) and disadvantages (lack of sustainability and scalability). Others are supported by institutions, foundations or donations. Some projects have established a scalable technical infrastructure, e.g. by creating arXiv overlay journals and employing open source software like OJS, but they must rely on third-party resources for the remainder of work (see, for example, M. C. Wilson et al. [2]). While the list of quality platinum journals in mathematics is considerable (see below), they are far from being already sufficient to ensure a full transformation to OA. That would require the allocation of considerable funds from libraries (or funders currently willing to cover APCs) to their maintenance. If the implementation of plan S would lead to such a support, this could lead to immense progress. However, experience advises that even in the case of existing funds, the transition of a journal to such a model requires at least a year; hence the process could be obstructed by too close timeframes.

However, there exists a long tail of less prominent journals without APCs (see the table below). Frequently, their number is cited as an argument that the majority of OA journals don't require APCs, making a swift and broad transition likely. However, a closer look reveals that they are of quite a diverse nature. For instance, a considerable number have been founded in countries that rely heavily on quantitative measures of research evaluation. Since the introduction of the Hirsch index, which takes both publication and citation figures into account, the incentives have grown there to create large clusters of publications that are often only small improvements or variations of each other. There are a number of OA journals aimed at bringing them to account. Notably, they have a thematic bias toward certain areas, e.g. fixed point theorems, fractional DE, elementary numerical methods, inequalities, fuzzy structures, classes of functions or sequence spaces, and combinatorial identities. In other cases, groups of researchers are engaged in a relatively small subject that finds obstacles to being published in traditional journals so, again, founding a specific OA journal is a welcome solution. In both examples, frequently (but not necessarily), there is a considerable overlap of the authors with the editorial board, making it harder to judge the quality of peer review. On the other hand, such overlap may also occur initially for very good OA journals, just to get the project started. Often, only time will tell in what direction the journal will evolve.[2]

Frequently, journals (or whole publishers) with dubious policies are referred to as predatory OA. This name is somewhat imprecise: similar bad practices can also occur in subscription journals (as in the case of Chaos, Solitons & Fractals, where the accumulation of bogus science also served commercial interests by inflating citation and publication figures). On the other hand, inappropriate quality control may not always derive from bad intentions. Perhaps the most clear-cut cases are APC publishers who limit the rejection of papers out of commercial interest, and even get rid of boards they consider to be too selective. Often, this is accompanied by practices like spam invitations (both for editors and authors).

Table 2 gives a rough breakdown of the various OA journal types in zbMATH[3] according to five categories. In addition, the number of total and recent (2016–2018) articles in these is given, indicating that sheer journal numbers may be misleading. Note that the categories are not a fully adequate proxy for quality – they are primarily internally employed for workflow priorities at zbMATH.

[2] The author would suggest naming the less convincing efforts of this type 'nickel OA' from the element of the platinum group famously named after a mountain spirit responsible for letting precious metals vanish from the ore. But one has to keep in mind that a large spectrum of alloys is possible.

[3] In the zbMATH serial search, results can be restricted to OA by adding the st:o option. Additional options are available under the recently introduced structured serial search.

Table 2: Categorisation of various journal types in zbMATH, with the number of journals, and their corresponding total and recent (since 2016) articles.

Category	non-APC OA			APC OA			Moving Wall (Eventually OA)			Subscription		
	Journal	Total#	Recent#	Journal	Total#	Recent#	Journal	Total#	Recent#	Journal	Total#	Recent#
FAST TRACK	13	9053	885	4	226	128	26	191721	16419	128	350425	28384
CAT 1	113	99279	13394	8	6786	922	36	116641	12386	336	512591	53734
CAT 2	201	73563	9316	39	21607	2513	23	39553	2878	435	545825	55961
CAT 3	114	14332	1785	52	10300	2359	2	1252	125	227	159908	17365
Under scrutiny[1]	85	20719	1370	31	7011	7720	1	2440	194	86	87452	6295

One caveat here is that these distinctions have been made partially heuristic and might be subject to discussion individually. As an example, looking at one of the smallest categories (the four FAST TRACK APC journals), both the Forum of Mathematics journals seem to very frequently waive APCs, even though they are much lower than average journal APCs, so they might almost be considered to be non-APC journals. The other two are also peculiar since they are the APC spinoffs of the Proceedings and Transactions of the AMS (titled "B"), which puts them close to the function of a hybrid journal (and therefore the subscription category). Both also offer a good measure for the acceptance of APCs in the mathematical community: since 2015, the subscription variants had almost a hundred times more articles than their "B" alternatives. The vast majority of APC journals indexed in zbMATH are currently published by Hindawi, SpringerOpen or de Gruyter Open.

On the other hand, a large majority of journals are still not immediately OA. In the top categories, this especially concerns journals by society publishers, while this column is overall dominated by Springer and Elsevier.

Some conclusions

Mathematics has made considerable progress toward an OA corpus, especially in the direction of green OA via the arXiv. There is certainly room for improvement, both concerning the coverage, the deposited versions and the licences (older arXiv content usually comes only with a distribution licence, while there are now diverse compliant licenses like CC BY 4.0, CC BY-SA 4.0 and CC BY-NC-SA 4.0). Diversity is important here; e.g., CC BY (as recommended by plan S) would, e.g., allow for rebundling genuinely OA content in commercial platforms even behind paywalls, in contrast with the NC (non-commercial) option. The SA (share alike) option would require making also derived work equally openly available. It should be noted that the arXiv also provides, as quite a unique feature, the distribution of LaTeX sources, which already facilitates formula search [15] and might be essential in the future to develop semantic features. This is complemented by a large ecosystem of journals that are platinum or moving wall OA, accounting for a considerable proportion of core mathematics. However, a swift change to immediate OA, like that envisioned by plan S, would pose considerable challenges in order to maintain the benefits of the ecosystem of quality control. Especially, APCs appear to be an inadequate alternative. While large commercial publishers have shown a lot of flexibility in generating revenues, society publishers would face imminent problems. An appropriate means to enable their OA transition could consist of funding models supporting platinum OA models, ideally via platforms that also allow for support in sustaining the diverse existing platinum projects, which too often rely on local efforts with rather scarce resources.

[4] The "under scrutiny" category comprises journals that are currently (re)considered, both from category and indexing viewpoints. Especially in the area of low-category APC journals, indexing has been discontinued in several cases due to questionable peer review. It indeed became quite a pattern that already suspicious journals eventually started publishing article of elementary proofs of the Riemann hypothesis or the Goldbach conjecture.

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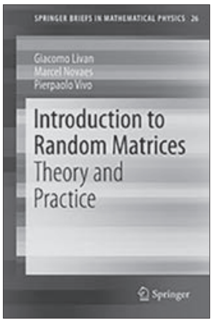
Photo and CV of the author can be found in previous Newsletter issues.

Book Reviews



Giacomo Livan, Marcel Novaes and Pierpaolo Vivo

Introduction to Random Matrices. Theory and Practice

Springer, 2018
ix, 124 p.
ISBN 978-3-319-70883-6

Reviewer: Rabe von Randow

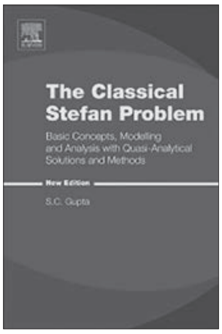
The Newsletter thanks zbMATH and Rabe von Randow for the permission to republish this review, originally appeared as Zbl 1386.15003.

In their preface, the authors state that 'this is a book for absolute beginners'. Well, that depends on the definition of a beginner! Here, (s)he should be thoroughly at home in probability theory and statistical methods and prepared to do a lot more than dot the i's and cross the t's, because further along the preface, the authors 'are sure … that any seriously interested reader … willing to dedicate some of their time to read and understand this book till the end, will next be able to … understand any other source … on RMT, …'. That's a tough assignment in 124 pages: from the beginning the reader is plunged in at the deep end.

The book covers standard material – classical ensembles, orthogonal polynomial methods, spectral densities and spacings, and also some advanced topics – the replica approach and free probability. There are 17 chapters (listing the titles doesn't reveal much) on topics like Wigner's surmise, joint pdf of eigenvalues of Gaussian matrices, Dyson index, classification of matrix models, Wigner matrices, Wigner's semicircle law for Gaussian matrices, the Coulomb gas technique, Wishart-Laguerre ensemble, Vandermonde determinant, resolvent, Sokhotski-Plemelj formula, inverse participation ratio, Porter-Thomas distribution, Dyson-Gaudin integration lemma, Andríef identity, Anti-Wishart ensemble, Marcenko-Pastur density, Edwards-Jones formula, the replica trick, and an overview of free probability theory.

The emphasis is on concepts, computations and tricks of the trade, and the style is modern and informal, which the reader will no doubt enjoy. There are many 'question boxes' for asides on the main text, and "To know more" sections at the end of most chapters. Moreover, every chapter ends with references relevant to that chapter, and many calculations are accompanied by references to their numerical verification in an online file provided by the authors. The book has, however, no index (but the reviewer's copy has 3 empty pages at the end where the reader can start their own). This monograph should prove to become a very welcome companion to the serious and enthusiastic reader setting out to get to know this vast and very useful subject.

Rabe von Randow studied mathematics, physics, and chemistry at Auckland University, New Zealand, graduating with an MSc in mathematics. He then did his PhD with Professor F. Hirzebruch in Bonn, Germany. After university teaching posts in Dunedin, New Zealand; Tucson, Arizona; Durham, England; and Cologne, Germany, he returned to the University of Bonn, where he remained until his retirement.



S.C. Gupta

The Classical Stefan Problem
Basic Concepts, Modelling and Analysis with Quasi-Analytical Solutions and Methods
2nd edition

Elsevier, 2017
xxiii, 726 p.
ISBN 978-0-444-63581-5

Reviewer: Aleksey Syromyasov

The Newsletter thanks zbMATH and Aleksey Syromyasov for the permission to republish this review, originally appeared as Zbl 1390.80001.

The problem this book deals with is very complicated from the physical and also from the mathematical point of view. The physical side of the question requires the description of heat transfer in continua with different properties, taking into account heat balance on the moving phase-change boundary. The mathematical side of the question includes the solution of a system of PDEs that are connected via conditions on a moving boundary (as a rule, they are heat transfer equations of some type).

The area of scientific and engineering applications of this problem is very wide: from ice melting and metal casting to oxygen diffusion. Accordingly, the papers on this topic are numerous and almost impossible to review. The author of the book under review, a well-known specialist in the area of free-boundary problems, set an ambitious goal to himself: to describe a wide variety of Stefan and Stefan-like problems and their solutions in a restricted book volume.

The main contents of the book is represented in 12 chapters which can be grouped into several groups.

Chapters 1–5 deal mainly with physical aspects of Stefan problems, including their classical formulation, types of boundary conditions, their meaning, multiphase problems, and so on. Chapters 6–9 are about mathematical peculiarities of Stefan problems. In particular, stationary, degenerate, hyperbolic and inverse problems are discussed. Chapters 10 and 11 form the "reasoning" group. They contain results on existence, uniqueness and regularity of Stefan problems' solutions. Finally, Chapter 12 contains an impressively wide review of analytical and semianalytical methods of Stefan problems' solutions: from "good lucks" when a solution can be found in closed form and from series expansions to implementation of conformal mappings, homotopies and perturbation methods. This chapter is very large: its extent approximately equals the total extent of all other chapters.

The exposition is illustrated by examples of certain problems' solutions. Different chapters are tightly connected with each other. Numerous links in the text, leading from one section to another, help to tackle the same problem from different points of view.

Besides a traditional detailed index and a list of symbols, the supplementary materials include a comprehensive bibliography (more than 800 books and papers) and four appendices, containing a brief introduction to function theory.

This book cannot be treated as easy reading. To fully understand its content, one must have enough knowledge in physics and must be some kind of expert in mathematics. Of course, it is not a disadvantage of the book, but an essential feature of the discussed topic. It is well known that actual engineering and scientific problems are very sophisticated; Sobolev spaces, integral transforms, variational inequalities, regularization techniques, etc., have become necessary instruments of modern science.

Summing up, it might be said that the book under review is an impressing monograph containing up-todate results in an important branch of mathematical physics.

Aleksey Syromyasov graduated from Ogarev Mordovia State University in 2004 as mathematician and received PhD in fluid and gas mechanics in 2007 from Kazan State University. Now he is associate professor of Department of Applied Mathematics, Differential Equations and Theoretical Mechanics in Ogarev Mordovia State University. His research interests are mechanics of viscous fluids and suspensions, mathematical and computer modelling in technics and science, teaching of mathematics and mechanics in higher school.

Albert Kubzdela

Selected Topics in Non-Archimedean Banach Spaces

Nicolaus Copernicus University, Torun, 2018
170 p.
ISBN 977-2082-433-80-9

Reviewer: Ştefan Cobzaş

The Newsletter thanks zbMATH and Ştefan Cobzaş for the permission to republish this review, originally appeared as Zbl 06871283.

The book contains some recent results on non-Archimedean (n-A) Banach spaces, including the author's contributions. Let $\mathbb{K}$ be a field with an n-A valuation $|\cdot|$, i.e., $|\lambda+\mu| \le \max\{|\lambda|,|\mu|\}$, $\lambda,\mu \in \mathbb{K}$. Typical examples of n-A valued fields are the p-adic field $\mathbb{Q}_p$ and $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$.

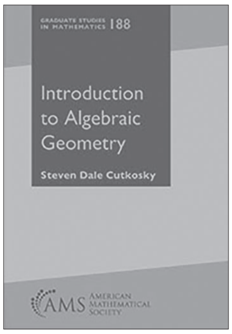
Let $(E,\|\cdot\|)$ be an n-A normed space over $\mathbb{K}$, i.e., the norm $\|\cdot\|$ satisfies $\|x+y\| \le \max\{\|x\|,\|y\|\}$ for all $x,y \in E$. The first chapter of the book contains some preliminary notions and results needed in the subsequent chapters. The second chapter is concerned with orthocomplemented subspaces of n-A Banach spaces. Two subspaces D_1,D_2 of E are called orthogonal if $\|x+y\| = \max\{\|x\|, \|y\|\}$ for all $x \in D_1$ and $y \in D_2$, and orthocomplemented if, in addition, $E = D_1 + D_2$. Among the results contained in this chapter, we mention: complete descriptions of the orthocomplemented subspaces of $c_0(I)$ and $\ell^\infty(I)$, and an example of an n-A Banach space over $\mathbb{C}_p$ containing a non-orthocomplemented weakly closed proper subspace with the Hahn-Banach extension property. The case of n-A Hilbertian spaces is also discussed. An n-A Hilbertian space is defined as an n-A normed space such that every finite-dimensional subspace has an orthogonal complement (the known characterization of Hilbert spaces as those Banach spaces in which every closed subspace is complemented does not work in the n-A case).

In the third chapter, several measures of noncompactness (including the n-A analogs of the Hausdorff measure of noncompactness and the de Blasi measure of weak noncompactness) are introduced and applied to obtain quantitative versions of Grothendieck's theorem (on pointwise compactness in $C(T,\mathbb{R})$) and Schauder's and Gantmacher's theorems on the compactness of conjugate operators.

In the fourth (and the last) chapter, one studies isometries of n-A Banach spaces. One shows that the Mazur-Ulam theorem, on the linearity of surjective isometries, holds only in the trivial case $E=\{0\}$. The Alexandrov problem – every mapping preserving unit distances is an isometry – and the surjectivity of isometries are also studied.

The book is well written, with clear proofs, motivations and historical references. It is of interest for researchers working in non-Archimedean analysis.

Ştefan Cobzaş is Emeritus Professor at Babeş-Bolyai University, Department of Mathematics, Cluj-Napoca-Romania. He graduated from the same University in 1968 with a thesis on non-Archimedean functional analysis and obtained a Ph. D. in 1979 with a thesis on best approximation with restrictions. His scientific interests concern mainly applied functional analysis – optimization and best approximation in Banach spaces (generic existence results for perturbed problems, the existence of bounded closed convex antiproximinal sets in Banach spaces). In the last years he worked on some problems in asymmetric functional analysis and published several papers and a book (in the series Frontiers in Mathematics, Birkhäuser-Springer, 2013) on this topic.



Steven Dale Cutkosky

Introduction to Algebraic Geometry

AMS, 2018
xii, 484 p.
ISBN 978-1-4704-3518-9

Reviewer: Werner Kleinert

The Newsletter thanks zbMATH and Werner Kleinert for the permission to republish this review, originally appeared as Zbl 1396.14001.

This book offers a three-semester introductory course in algebraic geometry with the goal to provide most of the fundamental classical results of the subject for graduate students. As the author points out in the preface to this comprehensive textbook, particular emphasis is put on developing the theory of quasi-projective varieties over an algebraically closed field of arbitrary characteristic, thereby stressing the connection between the geometry of algebraic varieties, on the one hand, and their regular functions via commutative algebra on the other. Also, the differences between the theory in characteristic zero and positive characteristic are especially emphasized, which may be seen as another outstanding feature of the current primer.

As for the precise contents, the entire text consists of twenty-two chapters. Roughly speaking, Chapters 2–10 serve as a one-semester introduction to algebraic geometry via affine and (quasi-) projective varieties, their regular and rational maps, and their properties.

Before that first part, Chapter 1 provides a crash course in commutative algebra, in which all the necessary definitions, concepts and results are briefly explained and stated, mostly with references to the standard books on commutative algebra as for proofs. Basically, this chapter is placed in front of the main text in order to keep the latter as self-contained as possible. The more detailed titles (and contents) of the first part (Chapters 2–10) are as follows:

Chapter 2: Affine varieties (including regular functions and maps, finite maps, dimensions, and rational maps),
Chapter 3: Projective varieties (including graded algebras, Grassmann varieties, regular functions and maps of quasi-projective varieties);
Chapter 4: Regular and rational maps of quasi-projective varieties (including rational maps like projec- tions and the Veronese embedding);
Chapter 5: Products of varieties (including graphs of maps and the Segre embedding); Chapter 6: The blow-up of an ideal in an affine or projective variety;
Chapter 7: Finite maps of quasi-projective varieties (including affine and finite maps as well as the concept of normalization);
Chapter 8: Dimension of quasi-projective varieties;
Chapter 9: Zariski's main theorem;
Chapter 10: Nonsingularity (including tangent spaces, singular loci, projective embeddings of smooth surfaces, and complex manifolds).

The following second part of the book comprises Chapters 11–20 and covers the material for a second- semester course. The topics treated here are clearly summarized by the titles of the subsequent chapters.

Chapter 11: Sheaves (including quasi-coherent and coherent sheaves);
Chapter 12: Applications to regular and rational maps (focussing on blow-ups of ideal sheaves, resolutions of singularities, valuations, factorizations of birational maps, and the author's theory of monomializations of maps);
Chapter 13: Divisors on quasi-projective varieties (including class groups, linear systems, invertible sheaves, and closed embeddings);
Chapter 14: Differential forms and the canonical divisor;
Chapter 15: Schemes (as subschemes of quasi-projective varieties, general schemes, abstract varieties, and varieties over non-closed fields);

Chapter 16: The degree of a projective variety (and Hilbert polynomials);
Chapter 17: Cohomology (containing sheaf cohomology, Čech cohomology, higher direct images of sheaves, local cohomology and regularity);
Chapter 18: Curves (including Serre duality, the Riemann-Roch theorem, the Hurwitz theorem, elliptic curves, abelian varieties, and Jacobians);
Chapter 19: An introduction to intersection theory (including intersection numbers, their properties, and applications to degree and multiplicity à la Bezout);
Chapter 20: Nonsingular projective surfaces (with focus on the Riemann-Roch theorem for surfaces, the Hodge index theorem, Castelnuovo's contraction theorem, and further examples of linear systems).

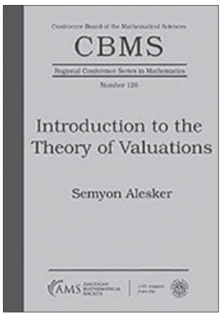
The last two chapters form the third part of the book, and could serve as a subject for a third-semester course on some special, more advanced topics.

Chapter 21 deals with the ramification of regular maps, Abhyankar's generalizations of some of Zariski's theorems, Galois theory of varieties, étale maps and uniformizing parameters, the purity of the branch locus of a map, and the classical Abhyankar-Jung theorem.
Chapter 22 is devoted to the behavior of the general fibers of dominant regular maps of varieties (in characteristic zero), on the one hand, and to the two classical Bertini theorems in O. Zariski's version on the other. In these two last chapters, the distinctions between characteristic 0 and positive characteristic $p > 0$ are carefully worked out, and instructively illustrated through guided exercises.

Generally, a set of related exercises is given at the end of many sections and chapters, and numerous examples throughout the text accompany the respective material.

All together, this textbook on algebraic geometry contains a wealth of fundamental topics, and that in an utmost clear, detailed and rigorous presentation. No doubt, this excellent primer is a highly useful and valuable companion to the comparable classical textbooks on algebraic geometry by D. Mumford, I. Shafarevich, R. Hartshorne, and others. Also, the great classical works of O. Zariski, S.S. Abhyankar, and J.-P. Serre are duly and faithfully reflected in the course of this text.

Werner Kleinert received his doctoral degree in commutative algebra in 1971. After his postdoctoral qualification (habilitation) in the field of algebraic geometry in 1979, he was promoted to university lecturer at Humboldt University in Berlin, an academic position that he held there until his retirement in 2010. His main research interests have always been the geometry of moduli spaces of algebraic curves and abelian varieties, together with related topics such as Riemann surfaces, theta functions and Teichmueller theory.



Semyon Alesker

Introduction to the Theory of Valuations

AMS, 2018
vi, 83 p.
ISBN 978-1-4704-4359-7

Reviewer: Rolf Schneider

The Newsletter thanks zbMATH and Rolf Schneider for the permission to republish this review, originally appeared as Zbl 06916392.

The theory of valuations on convex bodies had its first culmination in the work of Hugo Hadwiger in the 1950s. In the last two decades (roughly), this theory has seen a surprising expansion in depth, scope, and applications. An essential part of this development is due to the work of Semyon Alesker. The present book gives a brief introduction to the highlights of this work. It is based on lectures given at the NSF CBMS conference "Introduction to the Theory of Valuations on Convex Sets", held at Kent State University in August 2015. The first five chapters are in the classical style and present fundamentals and preparations. Treated are here, for example, basic extension theorems, the canonical simplex decomposition, McMullen's polynomiality and decomposition theorems, Hadwiger's characterization of n-homogeneous valuations, McMullen's description of $(n-1)$-homogeneous valuations, and the Klain–Schneider characterization of simple valuations. Then come some preparations for the main topic: generalized functions on manifolds, the Goodey–Weil embedding, a digression on vector bundles. After this, the central point is treated, Alesker's irreducibility theorem of 2001. It says that the natural representation of $GL(V)$ (V a finite-dimensional vector space) in the space of even, translation invariant, continuous and k-homogeneous valuations on V is irreducible; a similar result holds for odd valuations. One of the first applications, a proof of McMullen's conjecture on the approximation of translation invariant, continuous valuations by linear combinations of suitable mixed volumes, is presented. The final chapter gives a brief survey of the deep structural discoveries about valuations that Alesker (partly together with others) has made after his proof of the irreducibility theorem. Here one finds, after a presentation of smooth valuations, the product on valuations, the convolution of valuations, the Fourier type transform on valuations, pull-back and push-forward on valuations, valuations invariant under a subgroup of the rotation group transitive on the sphere, valuations related to Monge–Ampère type operators. Altogether, this book gives a brief introduction to the part of modern valuation theory that has been revolutionized under the hands of Semyon Alesker. 'Brief' implies that neither deeper proofs can be carried out nor the later influence, such as the development of an 'algebraic integral geometry', is mentioned. For the newcomer to the field, the book gives a first orientation, insights into the underlying structures, and valuable hints to the original literature.

Photo credit: Archives of the MFO Oberwolfach.

Rolf Schneider was born in 1940. He studied mathematics and physics at the University of Frankfurt a.M. In 1970 he became full Professor at the University of Berlin, and since 1974 he has been at the University of Freiburg i.Br. from where he retired in 2005. He is working in convex geometry and stochastic geometry.

Personal Column

Please send information on mathematical awards and deaths to newsletter@ems-ph.org.

Awards

Oxford mathematician **Robin Wilson** was awarded the **2017 Stanton Medal**. The medal is awarded every two years by the Institut of Combinatorics and its Applications for outreach activities in combinatorial mathematics.

Matthias Liero (WIAS Berlin, Germany) was awarded the **ISIMM Junior Prize 2018** and **Sir John Ball** (Oxford, UK) was awarded the **ISIMM Prize 2018** by the International Society for recognition of their many contributions in the Interaction of Mathematics and Mechanics.

The European Mathematical Society awards the **Gordin Prize 2018** to **Mateusz Kwaśnicki**, Wroclaw Technical University, for his outstanding contribution to the spectral analysis of Lévy processes.

The Austrian Mathematical Society awarded its **2018 Promotion Prize for Young Scientists** to **Vera Fischer** for her outstanding achievements in Mathematical Logics.

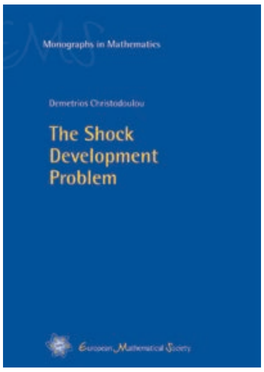
The Society for Industrial and Applied Mathematics (SIAM) has given awards to distinguished applied mathematicians for their research and contributions at the 2018 SIAM Annual Meeting. The **SIAM Prize** winners 2018 include the forthcoming EMS President **Volker Mehrmann**.

The **Jaroslav and Barbara Zemánek Prize 2018** in functional analysis with emphasis on operator theory is awarded to **Karl-Mikael Perfekt** (University of Reading, UK) for essential input in a variety of topics in operator theory.

Herbert Edelsbrunner, Institute of Science and Technology Austria, has received the **Wittgenstein Award 2018**, the most prestigious award for science and research in Austria.

Luis Caffarelli (University of Texas at Austin, USA) has been awarded the **2018 Shaw Prize** in the Mathematical Sciences for his ground breaking work on partial differential equations.

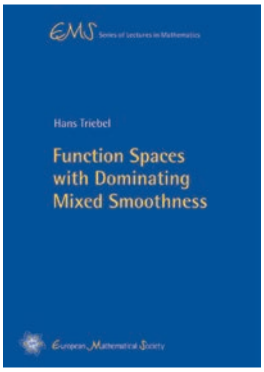
New books published by the European Mathematical Society



Demetrios Christodoulou (ETH Zürich, Switzerland)
The Shock Development Problem (EMS Monographs in Mathematics)

ISBN 978-3-03719-192-7. 2019. 932 pages. Hardcover. 16.5 x 23.5 cm. 128.00 Euro

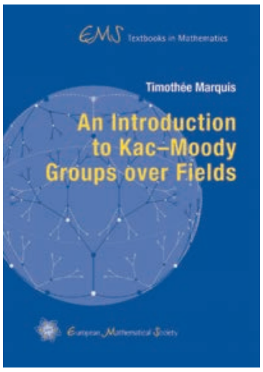
This monograph addresses the problem of the development of shocks in the context of the Eulerian equations of the mechanics of compressible fluids. The mathematical problem is that of an initial-boundary value problem for a nonlinear hyperbolic system of partial differential equations with a free boundary and singular initial conditions.
The free boundary is the shock hypersurface and the boundary conditions are jump conditions relative to a prior solution, conditions following from the integral form of the mass, momentum and energy conservation laws. The prior solution is provided by the author's previous work which studies the maximal classical development of smooth initial data. New geometric and analytic methods are introduced to solve the problem. Geometry enters as the acoustical structure, a Lorentzian metric structure defined on the spacetime manifold by the fluid. This acoustical structure interacts with the background spacetime structure. Reformulating the equations as two coupled first order systems, the characteristic system, which is fully nonlinear, and the wave system, which is quasilinear, a complete regularization of the problem is achieved.



Hans Triebel (University of Jena, Germany)
Function Spaces with Dominating Mixed Smoothness (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-195-8. 2019. 210 pages. Softcover. 17 x 24 cm. 36.00 Euro

The first part of this book is devoted to function spaces in Euclidean n-space with dominating mixed smoothness. Some new properties are derived and applied in the second part where weighted spaces with dominating mixed smoothness in arbitrary bounded domains in Euclidean n-space are introduced and studied. This includes wavelet frames, numerical integration and discrepancy, measuring the deviation of sets of points from uniformity.
These notes are addressed to graduate students and mathematicians having a working knowledge of basic elements of the theory of function spaces, especially of Besov–Sobolev type. In particular, it will be of interest for researchers dealing with approximation theory, numerical integration and discrepancy.

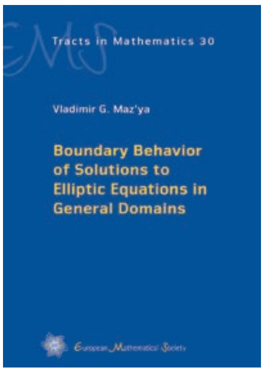


Timothée Marquis (Université Catholique de Louvain, Louvain-la-Neuve, Belgium)
An Introduction to Kac–Moody Groups over Fields (EMS Textbooks in Mathematics)

ISBN 978-3-03719-187-3. 2018. 343 pages. Hardcover. 16.5 x 23.5 cm. 48.00 Euro

The interest for Kac–Moody algebras and groups has grown exponentially in the past decades, both in the mathematical and physics communities, and with it also the need for an introductory textbook on the topic. The aims of this book are twofold:
- to offer an accessible, reader-friendly and self-contained introduction to Kac–Moody algebras and groups;
- to clean the foundations and to provide a unified treatment of the theory.

The book starts with an outline of the classical Lie theory, used to set the scene. Part II provides a self-contained introduction to Kac–Moody algebras. The heart of the book is Part III, which develops an intuitive approach to the construction and fundamental properties of Kac–Moody groups. It is complemented by two appendices, respectively offering introductions to affine group schemes and to the theory of buildings. Many exercises are included, accompanying the readers throughout their journey.
The book assumes only a minimal background in linear algebra and basic topology, and is addressed to anyone interested in learning about Kac–Moody algebras and/or groups, from graduate (master) students to specialists.



Vladimir Maz'ya (Linköping University, Sweden and University of Liverpool, UK)
Boundary Behavior of Solutions to Elliptic Equations in General Domains (EMS Tracts in Mathematics)

ISBN 978-3-03719-190-3. 2018. 441 pages. Hardcover. 17 x 24 cm. 78.00 Euro

The present book is a detailed exposition of the author and his collaborators' work on boundedness, continuity, and differentiability properties of solutions to elliptic equations in general domains, that is, in domains that are not a priori restricted by assumptions such as "piecewise smoothness" or being a "Lipschitz graph". The description of the boundary behavior of such solutions is one of the most difficult problems in the theory of partial differential equations. After the famous Wiener test, the main contributions to this area were made by the author. In particular, necessary and sufficient conditions for the validity of imbedding theorems are given, which provide criteria for the unique solvability of boundary value problems of second and higher order elliptic equations. Another striking result is a test for the regularity of a boundary point for polyharmonic equations.
The book will be interesting and useful for a wide audience. It is intended for specialists and graduate students working in the theory of partial differential equations.

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Aims and Scope

Bernoulli is published by the Bernoulli Society for Mathematical Statistics and Probability and disseminated by the Institute of Mathematical Statistics on behalf of the Bernoulli Society. The journal provides a comprehensive account of important developments in the fields of statistics and probability, offering an international forum for both theoretical and applied work.

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