NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

European
Mathematical
Society

Features

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Society

The Austrian Mathematical Society

December 2019
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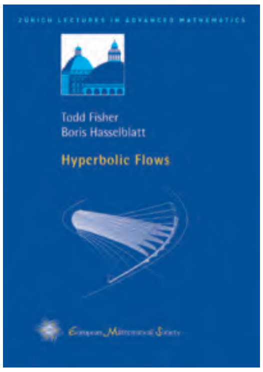
Yerevan, venue of the EMS
Executive Committee Meeting

New books published by the European Mathematical Society

Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.

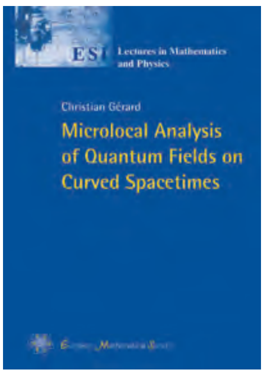


Todd Fisher (Brigham Young University, Provo, USA) and Boris Hasselblatt (Tufts University, Medford, USA)
Hyperbolic Flows (Zürich Lectures in Advanced Mathematics)
ISBN 978-3-03719-200-9. 2019. 737 pages. Softcover. 17 x 24 cm. 78.00 Euro

The origins of dynamical systems trace back to flows and differential equations, and this is a modern text and reference on dynamical systems in which continuous-time dynamics is primary. It addresses needs unmet by modern books on dynamical systems, which largely focus on discrete time. Students have lacked a useful introduction to flows, and researchers have difficulty finding references to cite for core results in the theory of flows. Even when these are known substantial diligence and consultation with experts is often needed to find them.
This book presents the theory of flows from the topological, smooth, and measurable points of view. The first part introduces the general topological and ergodic theory of flows, and the second part presents the core theory of hyperbolic flows as well as a range of recent developments. Therefore, the book can be used both as a textbook – for either courses or self-study – and as a reference for students and researchers.
There are a number of new results in the book, and many more are hard to locate elsewhere, often having appeared only in the original research literature. This book makes them all easily accessible and does so in the context of a comprehensive and coherent presentation of the theory of hyperbolic flows.

Frédéric Jaëck (Ecole Normale Supérieure, Paris, France), Laurent Mazliak (Université Pierre et Marie Curie, Paris, France), Emma Sallent Del Colombo (Universitat de Barcelona, Spain) and Rossana Tazzioli (Université Lille 1, Villeneuve-d'Ascq, France), Editors
Gösta Mittag-Leffler and Vito Volterra. 40 Years of Correspondence (Heritage of European Mathematics)
ISBN 978-3-03719-199-6. 2019. 438 pages. Hardcover. 17 x 24 cm. 88.00 Euro

This book contains the voluminous correspondence exchanged between the Swedish mathematician Gösta Mittag-Leffler and his younger Italian colleague Vito Volterra spanning a period of almost forty years at the end of the 19th and beginning of the 20th centuries. The relationship between the two men is remarkable for both personal and scientific reasons.
Mittag-Leffler met Volterra for the first time as a brilliant young student of Ulisse Dini in Pisa. He was soon captivated by the creativity and the skills of the young man, and eventually became his mentor. Being himself at the center of a major scientific network, Mittag-Leffler introduced Volterra to the major mathematicians of the time, especially the Germans (Weierstrass, Klein, Cantor…) and French (Darboux, Jordan…). In a few years, Volterra became the most prominent Italian mathematician and forged his own network of scientists all over Europe, and even in the United States which he was one of the first major European mathematicians to visit. Despite their difference in age, both men developed a deep and faithful friendship and their letters reflect the variety of themes of their exchanges. Of course, mathematics was the most prominent, and both men often used the letters as a first draft of their ideas and the addressee as a first judge of their soundness.
Besides mathematics, they also touched upon many aspects of both private and public life: matrimony, children, holidays, politics and so on. This vast set of letters affords the reader a general overview of mathematical life at the turn of the 19th century and an appreciation of the European intellectual spirit which came to an end, or at least suffered a drastic turn, when the Great War broke out. Volterra and Mittag-Leffler's exchanges illustrate how general analysis, especially functional analysis, gained a dramatic momentum during those years, and how Volterra became one of the major leaders of the topic, opening the path for several fundamental developments over the following decades.
The four editors are all specialists in the history of mathematics of the considered period. An extensive general introduction to the correspondence explains the context and the conditions in which it was developed. Moreover, the original letters are annotated with a large number of footnotes, which provide a broader cultural picture from these captivating documents.



Christian Gérard (Université de Paris 11, Orsay, France)
Microlocal Analysis of Quantum Fields on Curved Spacetimes (ESI Lectures in Mathematics and Physics)
ISBN 978-3-03719-094-4. 2019. 228 pages. Softcover. 17 x 24 cm. 48.00 Euro

We focus on free fields and the corresponding quasi-free states and more precisely on Klein–Gordon fields and Dirac fields. The first chapters are devoted to preliminary material on CCR*-algebras, quasi-free states, wave equations on Lorentzian manifolds, microlocal analysis and to the important Hadamard condition, characterizing physically acceptable quantum states on curved spacetimes. In the later chapters more advanced tools of microlocal analysis, like the global pseudo-differential calculus on non-compact manifolds, are used to construct and study Hadamard states for Klein–Gordon fields by various methods, in particular by scattering theory and by Wick rotation arguments. In the last chapter the fermionic theory of free Dirac quantum fields on Lorentzian manifolds is described in some detail.
This monograph is addressed to both mathematicians and mathematical physicists. The first will be able to use it as a rigorous exposition of free quantum fields on curved spacetimes and as an introduction to some interesting and physically important problems arising in this domain. The second may find this text a useful introduction and motivation to the use of more advanced tools of microlocal analysis in this area of research.

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EMS Agenda

2020

13–15 March
EMS Executive Committee Meeting, CIRM Marseille, France

3–4 July
EMS Executive Committee Meeting, Bled, Slovenia

4–5 July
EMS Council Meeting, Bled, Slovenia

EMS Scientific Events

2020

2–5 February
Geometry of Complex Webs – GeoCoW2020
Les Diablerets, Switzerland

3 February–6 March
Thematic Month on Mathematical Issues in Biology
CIRM Marseille, France

18–21 February
Catania Set Theory and Topology Conference 2020
University of Catania, Italy

23–27 March
Mathematical aspects of the physics with non-self-adjoint operators
CIRM Marseille, France

6–10 April
Periods, motives and differential equations: between arithmetic and geometry
Institut Henri Poincaré, Paris, France

11–22 May
Recent trends in cryptology and cyber security
Kiev, Ukraine

25–29 May
Sub-Riemannian Geometry and Beyond, III.
Centro di Ricerca Matematica, Pisa, Italy

8–12 June
Conference on complex analysis and geometry
Institut de Mathématiques de Toulouse, France

29 June–3 July
Reductive Groups 2020
Bestwig, Germany

6 July–10 July
28th International Conference on Operator Theory
Timisoara, Romania

5–11 July
8th European Congress of Mathematics
Portorož, Slovenia

FAIR Research Data

Volker Mehrmann (TU Berlin, Germany), President of the EMS

In many European countries, e.g. Germany, there is a strong movement for all research data to be freely available according to the FAIR principles (findable, accessible, interoperable and reusable). See https://libereurope.eu/wp-content/uploads/2017/12/LIBER-FAIR-Data.pdf

This is a major challenge for scientists who produce massive data, e.g. from numerical simulations, but also for mathematical research as a whole.

How and in which form can we standardise the way to find mathematical formulas or mathematical theorems, when different communities adopt other terminology for the same objects while using the same formulas for different objects?

The German Science foundation DFG has just started a large call for building research data infrastructures to deal with this, see e.g. https://www.dfg.de/en/service/press/press_releases/2018/press_release_no_58/index.html

Most people in the mathematical community seem to ignore these developments, but this may lead to real threats for the community if we do not join the movement right from the beginning.

Examples of such threats could be that standards are fixed which are incompatible with our current way of producing mathematical articles in LATEX and PDF, or that formulas are only stored graphically. Another problem may be that standards for model generation, mathematical software or simulation data are not to our taste. It is clear that commercial code providers are heavily lobbying governments to set standards that suit their particular needs, and that IT companies and data analytics people have their own views of how data should be addressed.

The mathematical community must unite in a common quest to be on board right away with the developments (the German maths community has already decided to do this and is participating in a joint consortium proposal), to make these principles realistic for mathematics and the neighbouring sciences and to preserve and improve established publishing standards to be able to deal with future developments. This may also require the construction of new and uniform concepts, such as semantic annotation of formulas or theorems.

Open Access Publishing in Mathematics

Volker Mehrmann (TU Berlin, Germany), President of the EMS, André Gaul and Laura Simonite (both EMS Publishing House, Berlin, Germany)

The last few years have borne witness to a number of important changes within the scholarly communication sphere that have the potential to radically disrupt research at large, and mathematics in particular.

Researcher and institutional dissatisfaction with traditional journal subscriptions and so-called “Big Deals” have taken the Open Access movement from grassroots activism to politically mandated regulation. The announcement in 2018 of Plan S, an initiative launched by an international consortium of research funders (including the European Commission and the European Research Council) with the explicit aim of making all publicly funded research Open Access, has rapidly accelerated publisher plans to embrace Open Access models of publication.

The European Mathematical Society, in association with a number of other learned societies in the field of mathematics, supports this transition to Open Research. Indeed, mathematics as a discipline has a long-standing culture of liberal Green Open Access policies. We are committed to developing sustainable models for Open Access publishing, as we believe these will come to dominate publishing models in the coming years.

However, this is a transition that should not be taken without due caution and consideration. In many countries in the Northern hemisphere there is a tendency to focus on Gold Open Access as the publishing model of choice - a model that requires authors to pay an article processing charge (APC) for publication. This may result in a number of unintended consequences, for example:

- Researchers without funding may be denied access to publish in their journal of choice. This may include researchers from developing economies, or those publishing in underfunded or niche areas of research.
- Journal profitability may become tied to published output, resulting in an inflation in the number of published articles, accompanied by a reduction in the quality of said research.
- APCs are often presented without a breakdown of where costs are incurred in the publication process, which can leave authors and their institutions unsure of the value a publishing house adds to the finished article.

Society-based publishing houses are not immune to these concerns, nor is the discipline of mathematics as a whole. For this reason it is important that, as the publishing houses of learned societies, we outline a set of guiding principles for our Open Access publishing models to ensure that quality and fairness remain at the heart of our publication programmes. At the same time, publishing houses play an important role in the curation and dissemination of research, so any publishing model must allow for the long-term sustainability of the organisation. We therefore propose the following criteria:

- The quality of publications is paramount and beyond compromise.
- Publications shall be accessible and available in perpetuity.
- Pricing models shall be transparent and fair.
- The publishing house serves the mathematics community, and commits surplus funds to community initiatives.
- The publishing house commits to collaborative relationships with other stakeholders within the mathematics community.

It is with these principles in mind that the publishing house of the European Mathematical Society has begun to investigate sustainable Open Access models for its journal and book portfolio. Over the coming months we will be presenting our findings and inviting feedback from our community of editors, as well as from librarians and researchers in the field.

Volker Mehrmann
President
European Mathematical Society

André Gaul
Managing Director
European Mathematical Society Publishing House

Laura Simonite
Head of Business Development
European Mathematical Society Publishing House
Press information: for enquiries contact Laura Simonite at simonite@ems.press.

Announcement of the Next Meeting of the EMS Council in Bled, 4 and 5 July 2020

Sjoerd Verduyn Lunel (Utrecht University, The Netherlands), Secretary of the EMS

The EMS Council meets every second year. The next meeting will be held in Bled on 4 and 5 July 2020 at Hotel Kompas Bled. The registration starts on 4 July at 12:00 and ends at 13:45. The Council meeting starts at 14:00 on 4 July and ends at 12:00 on 5 July.

Delegates

Delegates to the Council shall be elected for a period of four years. A delegate may be re-elected provided that consecutive service in the same capacity does not exceed eight years. Delegates will be elected by the following categories of members.

(a) Full Members

Full Members are national mathematical societies, which elect 1, 2, 3 or 4 delegates according to their membership class. The Council decides the membership class and societies are invited to apply for theclass upgrade. However, the current membership class of the society determines the number of delegates for the 2020 Council.

Each society is responsible for the election of its delegates. To be eligible to nominate its delegates the society must have paid the corporate membership fee for the year 2019 and / or 2020. It is not compulsory but is highly appreciated that the full member delegates join the EMS as individual members.

The online nomination form for delegates of full members is below. The deadline for nominations for delegates of full members is **12 April 2020**.

(b) Associate Members

Delegates representing associate members shall be elected by a ballot organised by the ExecutiveCommittee from a list of candidates who have been nominated and seconded by associate members, and have agreed to serve. In October 2019, there were three associate members and, according to our statutes, (up to) one delegate may represent these members. Associate members delegates must themselves be members of the EMS and have paid the individual membership fees for the year 2019 and / or 2020.

The delegate whose term includes 2020 is Susanne Ditlevsen.

(c) Institutional Members

Delegates representing institutional members shall be elected by a ballot organised by the ExecutiveCommittee from a list of candidates who have been nominated and seconded by institutional members, and have agreed to serve. In October 2019, there were 48 institutional members and, according to our statutes, (up to) four delegates may represent these members. Institutional member delegates must themselves be individual members of the EMS and have paid the individual membership fees for the year 2019 and / or 2020.

The delegates whose term includes 2020 are David Abrahams, Alexander Mielke and Luis Vega. The delegate who can be re-elected is Klavdija Kutnar.

The online nomination form for delegates of institutional members (including the ones eligible for re-election) is below. The deadline for nominations for delegates of institutional members is **8 March 2020**.

(d) Individual Members

Delegates representing individual members shall be elected by a ballot organised by the Executive Committee from a list of candidates who have been nominated and seconded, and have agreed to serve. These delegates must themselves be individual members of the European Mathematical Society and have paid the individual membership fees for the year 2019 and / or 2020.

In October 2019 there were 2847 individual members and, according to our statutes, these members may be represented by (up to) 28 delegates.

Here is a list of the current delegates of individual members whose terms include 2020:

Jean-Marc Deshouillers
Maria Esteban
Vincenzo Ferone
Alice Fialowski
Luis Narvaáez Macarro
Jan Pospíšil
Primoz Potocnik
Jiří Rákosník
Muhammed Uludag

Here is a list of the delegates of individual members who could be re-elected for the 2020 Council:

Antonio Campillo
Piermarco Cannarsa
Jose Antonio Carrillo
Carles Casacuberta i Vergés
Bostjan Kuzman
Marta Mazzocco
Vicente Muñoz
Marta Sanz-Solé

The online nomination form for delegates of individual members (including the ones eligible for re-election) is below. The deadline for nominations for delegates of individual members is **8 March 2020**.

Agenda

The Executive Committee is responsible for preparing the matters to be discussed at Council meetings. Items for the agenda of this meeting of the Council should be sent as soon as possible, and no later than 30 April 2020, to the EMS Secretariat in Helsinki (ems-office@helsinki.fi).

Executive Committee

The Council is responsible for electing the President, Vice-Presidents, Secretary, Treasurer and other members of the Executive Committee. The present membership of the Executive Committee, together with their individual terms of office, is as follows:

President:	Volker Mehrmann (2019–2022)
Vice-Presidents:	Betül Tanbay (2019–2022)
	Armen Sergeev (2017–2020)
Secretary:	Sjoerd Verduyn Lunel (2019–2022)
Treasurer:	Mats Gyllenberg (2019–2022)
Members:	Jorge Buescu (2019–2022)
	Nicola Fusco (2017–2020)
	Stefan Jackowski (2017–2020)
	Vicente Muñoz (2017–2020)
	Beatrice Pelloni (2017–2020)

Members of the Executive Committee are elected for a period of four years. The President can only serve one term. Committee members may be re-elected, provided that consecutive service shall not exceed eight years.

The Council may, at its meeting, add to the nominations received and set up a Nominations Committee, disjoint from the Executive Committee, to consider all candidates. After hearing the report by the Chair of the Nominations Committee (if one has been set up), the Council will proceed to the elections to the Executive Committee posts.

All these arrangements are as required in the Statutes and By-Laws, which can be found here together with the web page for the Council: http://www.euro-math-soc.eu

The online nomination form for *full member delegates*:
https://elomake.helsinki.fi/lomakkeet/90751/lomake.html
The deadline for nominations is **12 April 2020**.

The nomination form for *institutional, associate and individual member delegates*:
https://elomake.helsinki.fi/lomakkeet/90750/lomake.html
The deadline for nominations is **8 March 2020**

Secretary: Sjoerd Verduyn Lunel (s.m.verduynlunel@uu.nl)
Secretariat: ems-office@helsinki.fi

Call for Applications Emerging Regional Centre of Excellence – ERCE 2020–2024

Sophie Dabo (University of Lille, Villeneuve d'Ascq, France), Giulia Di Nunno (University of Oslo, Norway) and Francesco Pappalardi (University Roma Tre, Rome, Italy)

ERCE (Emerging Regional Centre of Excellence) is a label awarded by the EMS-CDC (European Mathematical Society-Committee of Developing Countries) to centres which have achieved an outstanding level in their area of influence in research and education, thus attracting students from other regions and countries. Indeed, the education of master and PhD students is an asset for raising the quality and diffusion of mathematics worldwide, and it has particular value in the developing world. ERCE centres are among those which play an important role in training students in their region, particularly those students coming from less developed areas.

With the global proliferation of emerging economies worldwide, there are varying degrees of development among developing countries, just as there are within the developed world. Very good centres exist in emerging economies where students from the least developed regions can be trained to the master level and beyond. Indeed, the most talented students may wish to pursue further education after the master's degree and be eligible for a PhD. A higher number of masters and PhDs is an enrichment for any country in terms of human resources with specialised high competence. A full education obtained in one of the outstanding centres in the region is a better guarantee for returning to the original country and as such is an effective way of fighting brain drain, whilst also being cost effective.

In this spirit, the first ERCE centre was awarded in 2011. Since then, 9 institutes have been awarded. Please visit the webpage of the EMS Committee for Developing Countries, under the page ERCE, for a complete list:

https://nickpgill.github.io/emscdc/erce

With the success of this scheme the committee is now opening a new call for applications. Here are some details.

About ERCE

The ERCE label is a recognition of excellence awarded for a period of four years. It provides a number of advantages to the awarded institution:

- The label adds prestige and visibility to the centre, which will then likely attract more and better students..
- Often, in turn, this will secure funding from local and regional sources.

Moreover, the CDC maintains a particular relationship with these centres:

- The CDC will give advice, whenever needed, related to scientific training and education programmes.
- The CDC will make grants available to support ERCE centres activities. Allocation of limited funding will be done via calls restricted to ERCE centres.
- The CDC will facilitate experienced lecturers to give short or medium-length courses, e.g. by involving the Voluntary Lecturers Scheme run by the IMU.
- The CDC will facilitate European hosts for researchers from these centres for visits and/or collaborations.

How to apply

The application must be sent by e-mail to the address:

EMS Office <ems-office@helsinki.fi>

Deadline for this application is: **February 15th, 2020**

The application must be written in English and should be as comprehensive as possible. Here is a list of important pieces of information that should be contained in the applications:

1. ERCE centre Director
   - Name, title, place of the institution, academic affiliation, address, email and other relevant contact information.
   - CV.
2. Vision statement of the ERCE centre
   - An aspirational description of the centre and how it will play a leading role in the region.
   - Plans (covering 4 years) for how to implement such a vision.
   - Within these plans, a motivated list of feasible activities that could be supported internationally, if funding were available.
3. Background material about the applying institute
   a. Webpage address of the institute (in English).
   b. General presentation of the centre: location, facilities. If a multi-university centre, clarify the institution's position in the academic structure.
   c. List of academic staff (professors, lecturers, PhD students) that will be associated with the ERCE centre, with details of the subject area of interest and possibly homepages.

d. List of significant publications from the last 10 years by the academic staff involved in the ERCE centre.
e. List of master programmes (description).
f. List of PhD programmes (description), if applicable.
g. Number of students (national and international) subscribed to the master/PhD programmes (number to the last 4 years). For international students, add information about the country of origin.
h. Information about the policy of enrollment of international students (qualification requested, subscription fees, grants, etc.).

Pay particular notice to the following point:
The application **must** elaborate in detail on **all the criteria** for participation.

Sophie Dabo
Chair of EMS Committee of Developing Countries
https://euro-math-soc.eu/committee/developing-countries

A Problem for the 21st/22nd Century

Sylvain Crovisier (CNRS and Université Paris-Sud, France) and Samuel Senti (Universidade Federal do Rio de Janeiro, Brazil)

Sylvain: *At the end of 1997, Jean-Christophe welcomed me in his huge office at the Collège de France. I had already met him two years earlier at an oral "concours". This time I came to ask him to supervise my DEA thesis. On the board he talked at length about dynamical systems, which he presented with a diagram.*

QUASI-PERIODIC SYSTEMS	*HYPERBOLIC SYSTEMS*
small-divisors & KAM theory	*uniform/non-uniform*

He had already done a lot of work on quasi-periodical systems (circle diffeomorphisms, rational maps), but in recent years he was more interested in (non-uniformly) hyperbolic systems; a very hot topic. I left full of enthusiasm, with his lecture notes on Jakobson's Theorem.

Samuel: *My experience is quite similar to yours. Two years earlier, the inaugural course at the Collège de France on Jakobson's theorem had not been held yet. Still the same discussion took place, with the same supporting diagram, a well-known schema to all those who had the chance to see Jean-Christophe expose his programme: a short, medium and long-term programme, and even a very long-term programme since he dared not hope to fulfill it during his career.*

We had the opportunity to follow a little bit of Jean-Christophe's path in the non-uniformly hyperbolic world, and to hear him share his dream of one day understanding conservative systems. In these few pages we try to give an overview of that dream.

Hyperbolic dynamics

The study of the dynamics of a map f from a space X to itself consists in the description of its orbits, i.e., of sequences of the form $x, f(x), f(f(x)), f(f(f(x))), \ldots$ We denote by $f^n(x)$ the n^{th} iterate of x.

Uniform hyperbolicity in dimension one

Let us start by describing the dynamics of the quadratic map $f_a : x \mapsto x^2 + a$ on the real line for parameter values $a < -2$. We observe two kinds of orbits: some go to infinity; the others belong to an invariant Cantor set K. Moreover, for all points $x \in K$ the sequence of iterated derivatives $Df_a^n := Df_a \circ \cdots \circ Df_a(x)$ grows exponentially at a given uniform rate which is independent of x.

Despite the system's apparent simplicity, describing the dynamics turns out to be a surprisingly complex task for parameter values $-2 < a < 0$. Indeed, denoting by β_a the positive fixed point of f_a, the interval $[-\beta_a, \beta_a]$ turns out to be invariant and the orbit of the critical point 0 remains bounded. For some values of the parameter (for instance for $a = -0.8$), one can decompose the system as follows:

- The orbit of any point outside $[-\beta_a, \beta_a]$ goes to infinity.
- There is an invariant compact set K which does not contain 0.

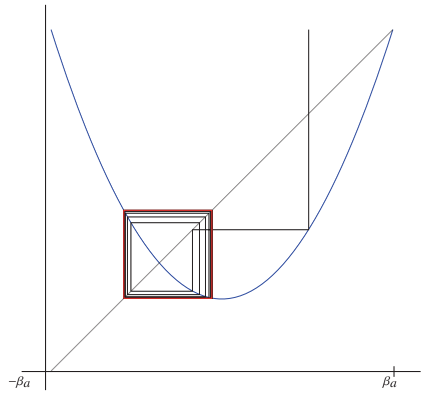


Figure 1. Orbit of a uniformly hyperbolic map

- There is an attractive periodic orbit which attracts the orbit of any point in $]-\beta_a, \beta_a[\setminus K$.

This dynamic is *uniformly hyperbolic*: for orbits contained in K, the sequence of iterated derivatives $|Df_a^n| := |Df_a \circ \ldots \circ Df_a|$ grows exponentially at a uniform rate; on the other hand, for any other bounded orbit, it decreases to 0 exponentially.

Hyperbolicity is a fundamental property: it implies that the orbits of any two points in K, however close, always end up being distinct from one another. We can thus associate to each orbit a sequence in $\{-1, 1\}$ which encodes the sign of each iterate. Furthermore, hyperbolicity implies the system's stability: for any mapping C^1-close to f_a, the structure of the space of orbits is unchanged.

Non uniform hyperbolicity

In 1981 Jakobson [6] showed that there exists another typical dynamical behavior:

Theorem (Jakobson)**.** *There is a set of parameters of non-zero measure $\mathcal{P} \subset [-2, 0]$, such that for $a \in \mathcal{P}$ and almost all $x \in [-\beta_a, \beta_a]$,*

- *the orbit of x for the quadratic map f_a is dense in the interval $[a, a(a + 1)]$,*
- *the asymptotic growth rate of the sequence of iterated derivatives $|Df_a^n(x)|$ is exponential.*

The dynamics of the maps f_a is then rather well understood: the orbit of almost every point of the interval $[-\beta_c, \beta_c]$ equidistributes to the same measure (which has support in $[a, a(a + 1)]$); outside that interval all orbits escape to infinity. The dynamics is non-uniformly hyperbolic: we observe an increase of the derivatives along the orbit of almost all points x, but that exponential growth is no longer uniform in x, since some orbits can land more or less close to the critical point 0 where the derivative is zero.

The proof of Jakobson's theorem is delicate: although the behaviour described has non-zero measure in $[-2, 0]$, its complement is dense: this is Fatou's conjecture, proved by Graczyk and Świątek, as well as by Lyubich in the 1990s.

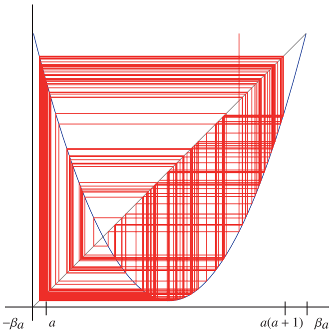


Figure 2. Orbit of a non-uniformly hyperbolic map

Afterwards, Rees [11] extended this result to rational maps on the Riemann sphere.

Uniform hyperbolicity on surfaces

For surface diffeomorphisms, the horseshoe and Plykin's attractor are typical examples of hyperbolic dynamics. Now the following property (H) holds at every point z:

- ∘ $(Df^n(z))_{n\geq 0}$ exponentially expands vectors *outside* a subspace $E^s(z)$.

($\mathcal{H}$) ∘ $(Df^{-n}(z))_{n\geq 0}$ exponentially expands vectors *outside* a subspace $E^u(z)$.

- ∘ $E^s(z), E^u(z)$ span the tangent space at z.

And so, there is a *uniform* integer $N \geq 1$ such that for every z and every non zero vector v, the image $Df^N(z).v$ or $Df^{-N}(z).v$ has norm bigger than $2\|v\|$. Again, we can decompose the dynamics as follows: there is a finite collection of elementary invariant compact sets onto which past and future orbits accumulate. There are three different types of such sets:

– Subsets that attract all orbits originating from a neighbourhood are the *attractors*. They can be periodic attractive orbits (sinks), or laminated sets, expanded by the dynamics, of which Plykin's attractor is an example.
– Subsets attracting every past orbit originating from a neighbourhood are the *repellers*.
– There are also saddle-like subsets, in the neighbourhood of which some orbits escape by future and past iterations. The *horseshoe* is an example of such a set.

Figure 3. A Plykin attractor. It carries a lamination by curves. The dynamics expands along the leaves and contracts in the transversal direction.

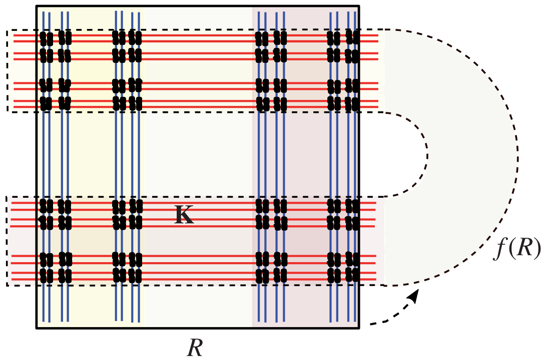


Figure 4. The horseshoe: A Cantor set K having a product structure. The dynamics stretches and folds a rectangle R, contracting in the vertical directions and expanding in the horizontal ones.

To learn more about uniformly hyperbolic dynamics we strongly recommend the introductory text written by Jean-Christophe [16].

Non uniform hyperbolicity on surfaces

Hyperbolic diffeomorphisms were extensively studied in the 1960–70s, but it soon became clear that they were not sufficient to describe most dynamics. In a 1991 paper, Benedicks and Carleson [1] generalised Jakobson's theorem to surface diffeomorphisms. They studied the Hénon family, i.e. diffeomorphisms of the plane of the form:

$$f_{a,b}\colon (x, y) \mapsto (x^2 + a - y, bx).$$

Theorem (Benedicks-Carleson)**.** *There is an open region U of the plane and a set of parameters $\mathcal{P} \subset \mathbb{R}^2$ of non-zero measure, such that for every $(a, b) \in \mathcal{P}$ there is a compact set $\Lambda_{a,b} \subset U$ satisfying:*

– *$\Lambda_{a,b}$ is an attractor of the Hénon map $f_{a,b}$: the future orbit of every point in U accumulates on a part of $\Lambda_{a,b}$,*
– *$\Lambda_{a,b}$ is transitive: it contains a point z whose orbit is dense in $\Lambda_{a,b}$,*
– *$\Lambda_{a,b}$ is not a periodic orbit.*

Subsequent work by Benedicks–Young [3] and Benedicks–Viana [2] showed that orbits of almost all points of U distribute towards the same probability measure $\mu_{a,b}$, supported on the attractor $\Lambda_{a,b}$. Property ($\mathcal{H}$) stated previously is satisfied, asymptotically, $\mu_{a,b}$-almost everywhere. It is not satisfied everywhere because U's topology is an obstruction to uniform hyperbolicity.

Benedicks and Carleson's attractor is a "non-uniform version" of Plykin's attractor. The existence proof of such systems is a *tour de force*: indeed, Newhouse [8] showed that arbitrarily close to the parameters identified by Benedicks and Carleson there are parameters for which future orbits from U split into a infinite family of different attractors. Non uniform hyperbolicity translates into the existence of *critical points*, i.e. points $z \in \Lambda_{a,b}$ admitting a vector $v \neq 0$ such that $\|Df^n_{a,b}(z).v\|$ decreases exponentially when n goes to $+\infty$ as well as to $-\infty$. *Homoclinic tangencies* are simple examples of critical points: in that case, z is a tangency point between two curves W^s (called stable) and W^u (called unstable) having the property of being contracted towards the same periodic orbit O both by positive respectively negative iterations. The

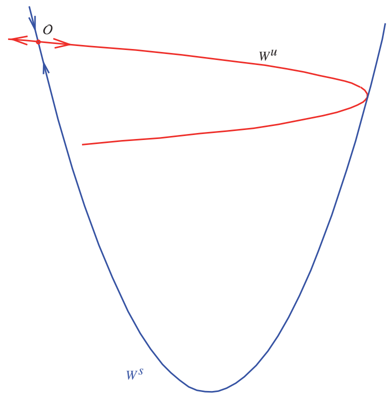


Figure 5. An homoclinic tangency between the stable, W^s, and unstable, W^u, curves of a fixed point O

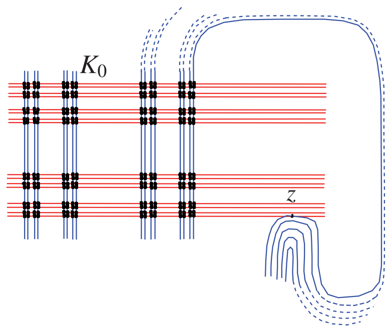


Figure 6. Tangency between the stable and unstable laminations of a horseshoe

presence of critical points makes the analysis of the dynamics especially difficult.

Non-uniformly hyperbolic horseshoes

Jean-Christophe quickly followed the development of non uniformly hyperbolic dynamics: in 1990 he gave a Bourbaki seminar [18] where he presented the work of Jakobson, Rees and Benedicks-Carleson. His 1994 talk [17] at the International Congress of Mathematicians in Zürich underlines the importance of this new topic. It is also the subject of a course which he gave at the ETHZ in 1996, of his inaugural lesson in 1997, and then of six of his courses at the Collège de France between 1998 and 2016.

His goal is clearly stated [17]:

> What we would like to do in the next few years is to obtain a conceptual theory of "weakly hyperbolic basic sets" (including of course the striking examples considered above). For a smooth diffeomorphism f of a manifold M, such a "weakly hyperbolic basic set" should again be a *compact, invariant, transitive, locally maximal* subset K of M satisfying moreover some kind (?) of *weak hyperbolicity* condition.

Several aspects of the non-uniformly hyperbolic sets described by Benedicks and Carleson in [1] and of their methods deserve to be mentioned. First of all, future iterations are favoured over past iterations: in this framework only the first condition in $\mathcal{H}$ is hard to obtain. Finally, in the proof we need to assume that the dynamics is *extremely dissipative* (the parameter b – the jacobian – is assumed to be very close to 0) so that the attractor is very thin, almost a curve.

Jean-Christophe considers a different approach to the study of non-uniform hyperbolicity, in the spirit of Jakobson's initial proof. As early as 1997-98 his course announces a "general theory" of non-uniformly hyperbolic systems for surface diffeomorphisms. His goal is to develop a technique that is symmetric with respect to time, which is not restricted to attractors, and which also helps us understand sets of large Hausdorff dimension.

Together with Jacob Palis, they propose to apply this programme to the study of homoclinic bifurcations of horseshoes: points whose future or past orbits are attracted by such a hyperbolic set define two laminations (a stable and an unstable one). A change in the dynamics allows the deformation of these laminations in order to create new intersections and thus enrich the initial dynamics. When the horseshoe is sufficiently thick some intersections between the leaves of the two laminations produce tangencies that persist for neighbouring systems: this was highlighted by Newhouse [8] at the end of the 1970's, and then made precise by Palis, Takens, Moreira and Yoccoz [7, 9].

Palis-Yoccoz theorem

In this theorem the authors consider a surface diffeomorphism f_0 with a horseshoe K_0 and also a quadratic tangency point z between a stable leave and an unstable leave. The tangency is assumed to be isolated: the union Λ_0 of K_0 with the orbit z has a neighbourhood U from which every orbit originating from $U \setminus \Lambda_0$ escapes. For a generic family (f_t) containing f_0 the goal is to describe the set of points Λ_t whose orbit remains in U in the future as well as in the past. One can also assume that the stable and unstable laminations still intersect in the neighbourhood of z for parameters $t > 0$.

We associate to K_0 the dimensions d^s and d^u transversal to the stable and unstable laminations. The Hausdorff dimension of K_0, equal to $d^s + d^u$, plays an essential role:

- When $d^s + d^u < 1$, the set of parameters $t > 0$ for which the dynamics is hyperbolic [9] has total density at 0.
- When $d^s + d^u > 1$, the set of parameters $t > 0$ for which the dynamics is not hyperbolic [7] has total density at 0.

Trying to understand how this "loss of hyperbolicity" comes about, Jean-Christophe and J. Palis focused on the case where $d^s + d^u$ is slightly bigger than 1. They showed that non-uniform hyperbolicity prevails in an explicit neighbourhood $D_{PY} \subset \{(d^s, d^u), d^s + d^u > 1\}$ of the locus $d^s + d^u = 1$.

Theorem (Palis-Yoccoz [10]). *Fix* $(d^s, d^u) \in D_{PY}$. *Then for a non-zero measure parameter set* $t > 0$ *the set* Λ_t *is a non-uniformly hyperbolic horseshoe.*

The notion of *"non-uniformly hyperbolic horseshoe"* is technical, however it satisfies two essential properties:

- The set of orbits attracted by Λ_t (in the past or in the future) has zero Lebesgue measure. In particular Λ_t is "saddle"-like and does not contain either an attractor or a repeller.
- *Most* (in a sense that will not be made precise here!) orbits of Λ_t equidistribute by future iterations to an invariant probability measure μ_t^+. Furthermore, μ_t^+-almost all points satisfy $\mathcal{H}$. (A similar property is satisfied for past iterations, with respect to another measure μ_t^-.)

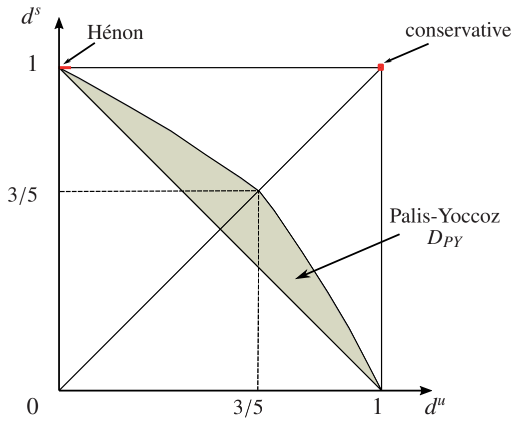


Figure 7. Stable and unstable dimensions of a horseshoe

The proof [10] took several years and fills a whole volume of the *Publications Mathématiques de l'IHES*. The impact left by Jean-Christophe on this topic is considerable and exceeds by far that monumental article.

Strong regularity

The analysis of non-uniformly hyperbolic dynamics proposed by Benedicks and Carleson depends on a careful control of the growth of the iterated derivatives at the critical points. However, Jean-Christophe understood that by introducing an *ad hoc* version of Yoccoz puzzles (see [5], by Xavier Buff) he would be able to isolate a structure allowing one to express the choice of parameters in combinatorial terms. It turns out this structure is sufficiently rigid to lead to the proof of the positivity of the measure of the set of parameters in question. This approach follows the spirit of Jakobson's proof in that the analysis of the dynamics is concentrated on a sequence of uniformly hyperbolic sets whose uniformity degenerates as they approach the critical value.

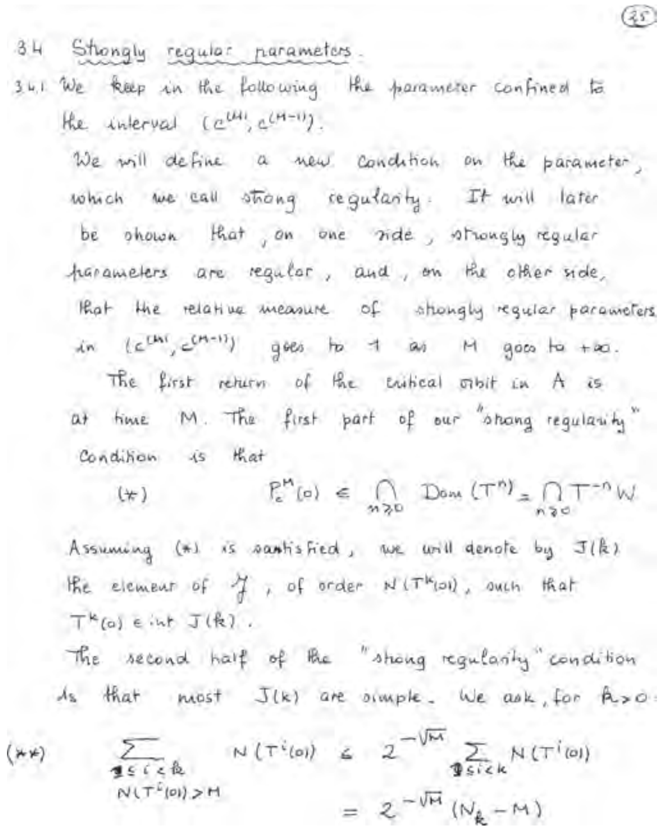
Strong regularity in one dimension

In order to understand these ideas in more detail, it is useful to revisit the one dimensional case.

When the dynamics is uniformly hyperbolic, the hyperbolic set K can be covered by a finite family I_i of disjoint intervals, each one being diffeomorphically mapped to a predetermined interval by an iterate $f_a^{n_i}$.

To study the dynamics of the map $f_a(x) = x^2 + a$ when $a \in [-2, 0]$, Jean-Christophe [13] considers the interval $A = [\alpha_a, -\alpha_a]$ where α_a is the negative fixed point of f_a. Again we can try to fill in the interval A with a (this time countable) family I_i of intervals. For the *simplest* intervals (away from the critical point 0), all the iterates $f^k(I_i)$ are disjoint from A when $1 \le k < n_i$. Near the critical point 0, there are smaller intervals that can return several times to the interval A before covering it.

This construction has an advantage: it allows for a successive approximation of the dynamics by the set K of orbits that only visit a finite family of intervals I_i; since K avoids a neighbourhood of the critical point, a result by Mañé shows that it is uniformly hyperbolic. Enriching the family I_i degenerates the hyperbolicity.



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3.4 Strongly regular parameters.

3.4.1 We keep in the following the parameter confined to the interval $(c^{(M)}, c^{(M-1)})$.

We will define a new condition on the parameter, which we call strong regularity. It will later be shown that, on one side, strongly regular parameters are regular, and, on the other side, that the relative measure of strongly regular parameters in $(c^{(M)}, c^{(M-1)})$ goes to 1 as M goes to $+\infty$.

The first return of the critical orbit in A is at time M. The first part of our "strong regularity" condition is that

$$(*) \qquad P_c^M(0) \in \bigcap_{n \geq 0} \mathrm{Dom}(T^n) = \bigcap_{n\geq 0} T^{-n} W$$

Assuming $(*)$ is satisfied, we will denote by $J(k)$ the element of $\mathcal{Y}$, of order $N(T^k(0))$, such that $T^k(0) \in \mathrm{int}\, J(k)$.

The second half of the "strong regularity" condition is that most $J(k)$ are simple. We ask, for $k>0$:

$$(**) \qquad \sum_{\substack{1 \le i \le k \\ N(T^i(0)) > M}} N(T^i(0)) \le 2^{-\sqrt{M}} \sum_{1 \le i \le k} N(T^i(0)) = 2^{-\sqrt{M}} (N_k - M)$$

Figure 8. Yoccoz's course notes on Jakobson's theorem

The decomposition of A in intervals I_i allows one to study the dynamics in a combinatorial way, and to code the orbits through the sequence of intervals it visits. To capture the dynamics which are not uniformly hyperbolic one must allow the critical orbit to return ever closer to the critical point. Jean-Christophe defined *strongly regular* maps: these are maps f_a, for which most returns of the critical orbit to the interval A belong to a simple interval I_i. The hypothesis that the sum of all *"deep"* return times of the critical orbit only represent a small fraction of all the iterates, guarantees that the family I_i indeed fills almost all the interval A, implying that the dynamics is non-uniformly hyperbolic.

Parameter selection

It remains to show that in the parameter set the strongly regular maps have positive Lebesgue measure. Following Adrien Douady's adage *sow in the dynamics space to harvest in the parameters space*, Jean-Christophe shows that the dependency of the puzzle structure induces a similar structure in the parameter set, and that the latter structure is slowly varying when one varies the parameter; a *transversality* phenomenon that allows one to take advantage, in the parameter space, of the estimates obtained in the dynamics space.

For the parameter $a = -2$ the second iteration of the critical point is fixed and all the intervals are simple. As a parameter $a > -2$ gets closer to -2, the part of A that is not filled by simple intervals is contained in an ever smaller neighbourhood of the critical point. The dynamics in the complement set is uniformly hyperbolic and the estimates on the expansion rate obtained for this set are good. The strong regularity con-

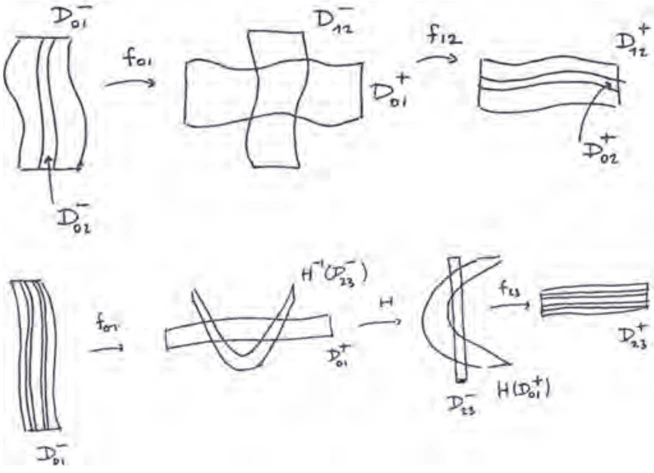

Figure 9. Maps of affine and fold types (Sketch by J.-C. Yoccoz)

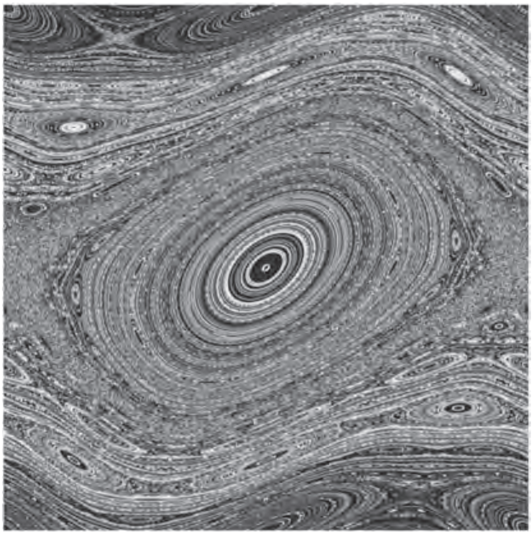

Figure 10. Some orbits of the standard map on the torus $\mathbb{T}^2$

dition allows for enough deep returns to exploit those good estimates in the parameter space and prove, by a large deviations argument, that the set of strongly regular parameters has a Lebesgue density point at $a = -2$.

Strong regularity on surfaces

In higher dimensions, iterating a uniformly hyperbolic surface diffeomorphism leads to composing *"affine- like"* maps that expand and contract along two transverse directions (see figure 9): the composition of such maps is still affine. However, when critical points are present, one must also consider *"fold-like"* maps, i.e., maps of quadratic type. We must then check that the compositions that show up in the system preserve this new class of maps. In general this is not the case, and one must select the parameters for which the folding effects do not accumulate by iteration. The shift to dimension two also brings other difficulties. Indeed, the sole critical point $c = 0$ is replaced by an infinite critical set: the parameter selection must then take each of its points into account. Furthermore, this set is not known *a priori* like in dimension 1: the definition of critical point introduced previously requires the analysis of the behaviour of arbitrarily long sequences $\|Df^n(z).v\|$, and hence a precise enough knowledge of the dynamics.

Since the critical set contains the tangency points between the stable and unstable laminations (as in figure 6), we expect its dimension to be bounded below by $d^s + d^u - 1$, which is the typical dimension for the intersection of Cantor sets. Therefore the critical set is all the more difficult to control, as the dimension of the horseshoe is large. The case of the dynamics of the Hénon attractor studied by Benedicks and Carleson would correspond to a dimension $d^s = 1$ and a dimension d^u extremely close to 0. By comparison, the hypothesis made on (d^s, d^u) by Palis-Yoccoz is completely explicit and allows us to get close to $d^s + d^u = 6/5$.

The proof of [10] did not fully meet Jean-Christophe's requirements: the definition of strong regularity that is stated there depends not only on the map, but also on the parametrisation of the family. The concept was taken up and reworked a few years later by Pierre Berger [4] in the context of attractors. This led Jean-Christophe to schedule his 2016 course at the *Collège de France* on a proof of Benedicks and Carleson's theorem which uses strong regularity and which relies heavily on [4]. Despite his illness, he attemped one of his most ambitious courses. Weakened, however, he only could present the first two lectures – clear and deep.

Conservative dynamics

Jean-Christophe wanted to "build a theory of (weakly) hyperbolic systems that would allow, in the long run, to deal with other still poorly understood examples, in particular in the conservative." [15] He had in mind systems that preserve a volume form. The model family of maps is the standard family on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:

$$(x, y) \mapsto (2x + a\sin(2\pi x) - y, x).$$

The goal being to answer this well-known dynamicists' problem [12]:

Conjecture (Sinai)**.** *The standard family satisfies property $(\mathcal{H})$ for a set of parameters a and of points $z \in \mathbb{T}^2$ of non-zero Lebesgue measure.*

The hypothesis of volume preservation leads to new phenomena. Indeed, the Kolmogorov-Arnold-Moser theorem shows the existence of families of closed invariant curves, whose union has non-zero volume, and such that on these curves property $(\mathcal{H})$ cannot be satisfied. Therefore it should be checked that this "elliptic" part of the dynamics can coexist with a non-uniformly hyperbolic part, or that it can even disappear when the parameter a is large enough.

Billiards

In a short interview [14], Jean-Christophe explains that Sinai's conjecture is his favourite open problem in dynamical systems. He decides to talk about it in a more "concrete" example.

If one plays billiards inside a convex domain U, one identifies each trajectory by taking note, at each reflection, of the point of impact on the edge of U and of the angle of incidence. One thus obtains a map of the ring $A := \partial U \times (-\pi/2, \pi/2)$ which associates to each reflection the next reflection. It is a diffeomorphism that preserves a volume form! In the very

Figure 11. Jean-Christophe Yoccoz and two of his students at Cetraro, in 1998

special case where the billiard table U is an ellipse, each orbit passing near the edge of the ring A belongs to an invariant curve. In the general case, one expects to observe the coexistence of invariant curves and non-uniform hyperbolicity:

Problem. Show that for the map associated with a convex billiard which is not an ellipse, the set of points verifying property $(\mathcal{H})$ has non-zero volume.

Horseshoes with dimension arbitrarily close to 2 appear within conservative dynamics. Yet, when $d^s + d^u$ increases, the recurrence of the critical set tends to be stronger. To show the non-uniform hyperbolicity of these systems, it seems necessary to consider frequent compositions of fold type maps leading to higher order maps.

The difficulty is considerable. Jean-Christophe said he was not hoping for a solution before the end of the 21st century!

Credits

The picture of figure 10 has been obtained with an interactive numerical experiment available at experiences.math.cnrs.fr and done by J.-R. Chazottes and M. Monticelli. The authors thank Stefano Marmi for the picture 11.

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This is an English translation by Rafael Sasportes of the French original article entitled "Un problème pour le XXI(I)ème siècle", published in April 2018 in the special issue of *La Gazette des Mathématiciens* dedicated to Jean-Christophe Yoccoz. The *EMS Newsletter* thanks the author, *La Gazette des Mathématiciens*, and the Société Mathématique de France for authorisation to republish this article.

Euler, Stirling, and Wallis: A Case Study in the Notion of Equivalence Between Theorems

Fausto di Biase (Università "G. D'Annunzio" di Chieti-Pescara, Pescara, Italy)

The notion that two theorems may be equivalent to each other is sometimes met with hesitation. In this article we tell a story that shows that there is something interesting and useful in this notion. We look at the following three results: Stirling's formula, Wallis' product formula, and the evaluation of the probability integral. The task of giving simple *proofs of these results is the object of unabated attention. In order to enhance our understanding of these results, we show in a precise way that these results are indeed* equivalent *to each other.*

1 Introduction

A controversial notion

The notion that two theorems may be equivalent is sometimes met with hesitation, since *a posteriori* i.e., *after* they have been proved, there is apparently not much more to say.

Example 1. Consider the following two statements:

- "The Fourier series of an L^2 function converges almost everywhere";
- "The maximal operator associated to the Fourier series is of weak type (2,2)".

Before the truth value of any of the two statements had been determined, in the mid-fifties it was understood that they are equivalent to each other – thanks to the work of A. Calderón and A. Zygmund, based on previous work of G. H. Hardy, and J. E. Littlewood.[1] This equivalence – whose roots lie in a work by Kolmogorov [8] – turned out to be very useful, for two reasons. Firstly, it pointed to a specific research agenda. Secondly, it was a hint that perhaps hidden in the background there was some more general result – which indeed was discovered a few years later by E. M. Stein in a seminal, important work [16]. This example shows how the equivalence between two *statements* can be useful if it can be established *before* the two statements have been proven.

Example 2. Consider the following two statements:

- Every continuous map from the closed disc into itself has at least a fixed point;
- The Hex game cannot end in a draw.

These two statements are equivalent [4]. As a matter of fact, each of these two statements is a theorem. In this example, the equivalence is considered of interest – even if it was establishe *after* the two results had been proved – since it points to a surprising connection between two apparently remote topics.

We would like to tell a story that will perhaps dispel the diffidence toward this notion.

1 [19, Theorem 1.22, v.2]

A general viewpoint

The general viewpoint that we would like to suggest in this paper is that a proof that two statements are equivalent will enhance our understanding of the subject, provided their equivalence assumes the form of a *theorem* which represents an *a priori, direct link* between the two statements, where

- the term *a priori* means that the connection is not contingent on the truth value of the two statements – a notion vaguely inspired by modal logic [9];
- the theorem is a *direct link* in the sense that it shows *at once*, in a visible and immediate way, and *without any further substancial work*, that the two statements must *necessarily* have the same truth value.

We should emphasize the fact that an a priori, direct link between two statements is *a theorem* – a positive result which adds something to our knowledge, not a purely conjectural result. In particular, even if the two equivalent statements at hand should one day turn out to be false, the a priori, direct link which we have determined will have increased our knowledge.

Now, we would like to give an example of an *a priori, direct link* between two well-known results. Firstly, recall that if F and G are sequences of positive real numbers, then the symbol $F_n \sim G_n$ means that $\lim_{n\to+\infty} \frac{F_n}{G_n} = 1$.

Example 3. Consider the following two statements:

$$n! \sim e^{-n} n^n \sqrt{2\pi n} \qquad \text{(S)}$$

$$\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{n\pi} \qquad \text{(W)}$$

The following result is an a priori, direct link between (S) and (W).

Theorem 4 (Moritz [13])**.**

$$n! \sim \frac{(2n)!!}{(2n-1)!!} e^{-n} n^n \sqrt{2} \qquad (\text{W}\leftrightarrow\text{S})$$

The tag (↔) chosen for this formula alludes to the fact that it shows *at once*, in a visible and immediate way, and *without any further substancial work*, that the two statements must *necessarily* have the same truth value. Moreover, the statement in Theorem 4 is not contingent on the truth value of the two statements (S) and (W). Hence, Theorem 4 is an a priori, direct link between (S) and (W). Theorem 4 is enlightening because it shows that (S) and (W) are different sides of the same object, since it points to a *necessary* connection between their truth values: It contains the heart of the connection between the two statements.

The result in (S) is due to James Stirling (1692–1770), who obtained it while working on problems of probability theory by completing a previous result by Abraham De Moivre (1667–1754). It is very important in probability theory and it has deep connections to many other topics.

The result in (W) was proved by John Wallis (1616–1703) in Proposition 191 of his book *The Arithmetic of Infinitesimals*, published in 1656 [17]. Wallis was interested in the application of the new ideas of infinitesimal calculus to the computation of areas, in particular, the area of a circle, and was led to this result by a clever use of the idea of interpolation – an idea that was later used by Euler with great virtuosity. Wallis' book had a great influence on Newton, although it had been met with sharp criticism by Fermat, Huygens and Hobbes.

The probability integral

This name refers to the following result (due to Euler [2], [3]).

$$\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi} \tag{E}$$

It was rediscovered by Laplace [10], [11] and played a fundamental role in the work of Gauss [5]. Indeed, it plays an important role in the method that Gauss used to predict the position of Ceres on the basis of very few observations [6].

A tale of three stories

The task of giving *simple* proofs of (S), (W), and (E) is the object of unabated attention. There are currently many *simple* proofs of these results, of which [7], [12], [13], [14], [18] are just a small sample.

Below we illustrate the general viewpoint of Sec. 1 by looking for a priori, direct links between these results.[2] In particular, our main intent is *not* to obtain yet another simple proofs of these results.

We have already seen that Theorem 4 looks at the first two results – (S) and (W) – from the general viewpoint of Sec. 1, and establishes an a priori, direct link between them. In this paper we try to do for the pair of statements (E) and (W) what Moritz did for (S) and (W).

2 A priori, direct links between (E) and (W)

Lemma 5. *If l_n is any sequence of positive numbers such that* $\lim_{n\to+\infty} l_n = +\infty$ *then*

$$\int_{\mathbb{R}} e^{-x^2}\,dx = \lim_{n\to+\infty}\int_{-l_n}^{l_n} (1+x^2/n)^{-n}\,dx \tag{1}$$

Proof. Observe that $e^{-x^2} \le (1+x^2/n)^{-n} \le (1+x^2)^{-1}$. Indeed, $e^z \le (1-z)^{-1}$ for $z \le 0$, then let $z = -x^2/n$. The sequence $(1+x^2/n)^{-n}$ converges uniformly to e^{-x^2} in each bounded interval, as can be seen by the Taylor series of ln and e^x. Now, given $\epsilon > 0$, choose $c > 0$ such that

$$I := \int_{\mathbb{R}\setminus[-c,c]} e^{-x^2}\,dx < \epsilon$$

and

$$II := \int_{\mathbb{R}\setminus[-c,c]} (1+x^2)^{-1}\,dx < \epsilon$$

as we may, due to the absolute integrability of the functions. If n is large enough, then $l_n > c$ and

$$III := \int_{[-c,c]} \left|\left(1+x^2/n\right)^{-n} - e^{-x^2}\right| dx < \epsilon$$

due to uniform convergence, thus

$$\left|\int_{[-l_n,l_n]} \left[\left(1+x^2/n\right)^{-n} - e^{-x^2}\right] dx\right| \le I + II + III < 3\epsilon \qquad \square$$

Corollary 6. *There is an a priori, direct link between* (E) *and the following statement*

$$\lim_{n\to+\infty} \sqrt{n}\int_0^{\pi} (\sin\theta)^{2n}\,d\theta = \sqrt{\pi} \tag{E.1}$$

Proof. It suffices to show that

$$\int_{\mathbb{R}} e^{-x^2}\,dx = \lim_{n\to+\infty}\sqrt{n}\int_0^{\pi} (\sin\theta)^{2n}\,d\theta \tag{E↔E.1}$$

Let $l_n = \sqrt{n}$ in Lemma 5 and let $\frac{x}{\sqrt{n}} = z$ and $z = \cot\theta$ in this order. Then

$$\int_{-\sqrt{n}}^{\sqrt{n}} (1+x^2/n)^{-n}\,dx = \sqrt{n}\int_{-1}^{1} (1+z^2)^{-n}\,dz$$
$$= \sqrt{n}\int_{\pi/4}^{3\pi/4} (\sin\theta)^{2n-2}\,d\theta$$

Now observe that

$$\sqrt{n}\int_{\pi/4}^{3\pi/4} (\sin\theta)^{2n-2}\,d\theta = \sqrt{n}\int_0^{\pi} (\sin\theta)^{2n-2}\,d\theta + o(1)$$

since the difference is bounded by $\pi\sqrt{n}2^{-n}$. Finally, integration by parts yields the familiar reduction formula

$$n\cdot S_n = (n-1)\cdot S_{n-2} \quad (n \ge 2) \tag{2}$$

for the sequence $S = \{S_n\}_n$ defined by

$$S_n = \int_0^{\pi} (\sin\theta)^n\,d\theta \tag{3}$$

which implies that

$$S_{2n} \sim S_{2n-2} \tag{4}$$

We have thus proved that $\int_{\mathbb{R}} e^{-x^2}\,dx = \lim_{n\to+\infty}\sqrt{n}S_{2n}$, hence the conclusion. $\square$

Corollary 7. *The statement in* (E) *is equivalent to*

$$S_{2n} \sim \sqrt{\frac{\pi}{n}} \tag{E.2}$$

Observe that the second order homogeneous difference equation (2), together with the initial values S_0 and S_1, uniquely determines the sequence S. A more general point of view yields a better view. Let $\mathbb{S} = (0,+\infty)^{\mathbb{N}}$ be the set of all sequences of strictly positive numbers. If $F = \{F_n\}_n \in \mathbb{S}$ and $p, q \in (0,+\infty)$, then we define $F^*(p,q) = \{F_n^*(p,q)\}_n \in \mathbb{S}$ as

2 We will try to use in our proofs as little machinery as we can and, moreover, we will try to offer proofs conforming to the standards of *clarity* described in [1]. An example *in the negative* is given by proofs by contradiction, which are opaque, as observed in [15, p.3], since they shed little insight into the link between hypothesis and conclusion.

the unique solution of the second order homogeneous difference equation

$$F(n)F_n^*(p,q) = F(n-1)\cdot F_{n-2}^*(p,q) \quad (n \ge 2) \tag{5}$$

with initial conditions $F_0^*(p,q) = p$, $F_1^*(p,q) = q$. The assignment $F \mapsto F^*(p,q)$ is a map $\mathbb{S} \to \mathbb{S}$: This map is *surjective* but *not* injective. We call $F^*(p,q)$ the sequence *associated to* F, with initial data (p,q). If a statement does not depend on the special choice of the initial data, we denote $F^*(p,q)$ by F^*.

Example 8. Let $I = \{I_n\}_n \in \mathbb{S}$ be the *identity sequence*, defined by $I_n = n$. Since $S_0 = \pi$ and $S_1 = 2$, then (2) says that the sequence S defined in (3) is the sequence $I^*(\pi,2)$ associated to I with initial data π and 2.

Remark 9. Observe that the sequence I, which enters in the reduction formulas (2), acts as a "parameter". We think that a better understanding of our topic could be achieved by allowing a change in the parameters envolved, in order to discern the mutual relations between the various relevant properties. For these reasons, we have introduced the space $\mathbb{S}$ and the map $F \mapsto F^*$, since these constructs allow us to change the "parameters" of our phenomena, so to speak, and express the link between the truth values of various statements. Lemma 12 will show that the equivalence between (E) and (W) is a special case of a general equivalence, which has been expressed by introducing the space $\mathbb{S}$ and the map $F \mapsto F^*$, since anything which holds for every $F \in \mathbb{S}$ will also hold for I.

If $F \in \mathbb{S}$, we define $F^\sharp \in \mathbb{S}$ as the unique solution of the second order homogeneous difference equation

$$F_n^\sharp = F_{n-2}^\sharp \cdot F_n \cdot \quad (n \ge 2)$$

with initial data $F_0^\sharp = 1$ and $F_1^\sharp = F_1$. Hence

$$F_0^\sharp = 1,\ \ F_1^\sharp = F_1,\ \ F_2^\sharp = F_2,\ \ F_3^\sharp = F_3\cdot F_1,\ \ F_4^\sharp = F_4\cdot F_2,$$

Example 10. If $I \in \mathbb{S}$ is the identity sequence, defined in Example 8, then $I_n^\sharp \equiv n!!$

Lemma 11. *If $F \in \mathbb{S}$ and $p,q \in (0,+\infty)$ then*

$$F_{2n}^*(p,q) = p\frac{F_{2n-1}^\sharp}{F_{2n}^\sharp} \quad (n \ge 1) \tag{6}$$

Proof. Induction. If $n = 1$ then $F_{2n}^*(p,q) = F_2^*(p,q) = F_0^*(p,q)\frac{F_1}{F_2} = p\frac{F_1}{F_2} = p\frac{F_1^\sharp}{F_2^\sharp}$. Assume that (6) holds for n. Then

$$\begin{aligned} F_{2(n+1)}^*(p,q) &= F_{2n+2}^*(p,q) \\ &= F_{2n}^*(p,q)\cdot\frac{F_{2n+1}}{F_{2n+2}} \\ &= p\frac{F_{2n-1}^\sharp}{F_{2n}^\sharp}\cdot\frac{F_{2n+1}}{F_{2n+2}} = p\frac{F_{2n+1}^\sharp}{F_{2n+2}^\sharp} \end{aligned}$$ □

Lemma 12. *If $F \in \mathbb{S}$ and $p,q \in (0,+\infty)$ then the following conditions are equivalent:*

$$\frac{F_{2n}^\sharp}{F_{2n-1}^\sharp} \sim \sqrt{p\frac{F_{2n}}{q}} \tag{W'}$$

$$F_{2n}^*(p,q) \sim \sqrt{p\frac{q}{F_{2n}}} \tag{E'}$$

Proof. It suffices to apply (6) □

Corollary 13. (E) *and* (W) *are equivalent.*

Proof. We specialize Lemma 12 to $F = I$, $p = \pi$, and $q = 2$. Then (W′) says that $\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{\pi n}$, and this is (W). On the other hand, since $I^*(\pi,2) = S$, as seen in (3) and Example 8, then (E′) says that $S_{2n} \sim \sqrt{\frac{\pi}{n}}$, and this is (E.2), which is equivalent to (E). □

3 A proof of (E) and (W)

Since we have established that (E) and (W) are equivalent, in order to prove them it suffices to prove one of them. In Lemma 12 we have seen that the equivalence between (E) and (W) is a special case of a more general equivalence, which has been expressed in terms of the space $\mathbb{S}$ and the map

$$F \mapsto F^*$$

Now, we would like to understand the meaning of the properties (W′) and (E′) which appear in Lemma 12. Observe that both (W′) and (E′) contain the initial data (p,q) in their statement. Thus, we would like to understand those properties in terms of the *asymptotic properties* of the sequence F^*, i.e., in terms which do *not* depend on the initial data but only on the behavior of F_n^* for large values of n. We will now show that this may be achieved by using the following notions.

We say that $G \in \mathbb{S}$ has *step 1* if $G_n \sim G_{n+1}$. For example, powers of n and polynomials in n have step 1. If G has step 1 then $G_n \sim G_{n+2}$, but this condition is strictly weaker: We say that a sequence with the latter property has *step 2*. The next result shows that monotonicity acts as a reinforcing condition.

Lemma 14. *If $G \in \mathbb{S}$ has step 2 and it is and monotone then it has step 1.*

Proof. If G is decreasing then

$$\begin{aligned} \frac{G_n}{G_{n-2}} &< \frac{G_n}{G_{n-2}}\cdot\frac{G_{n-2}}{G_{n+1}} \\ &= \frac{G_n}{G_{n+1}} = \frac{G_n}{G_{n+2}}\cdot\frac{G_{n+2}}{G_{n+1}} < \frac{G_n}{G_{n+2}} \end{aligned}$$

hence the result. If G is increasing the inequalities are reversed. □

If $G \in \mathbb{S}$ has step 1 then, in particular, $G_{2n+1} \sim G_{2n+2}$, but the latter condition is strictly weaker than the former, and we say that a sequence which satisfies the latter condition has *partial step 1*.

Lemma 15. *If $G \in \mathbb{S}$ has step 2 then it has step 1 if and only if it has partial step 1.*

Proof. From $G_{2n+1} \sim G_{2n+3}$ and $G_{2n+1} \sim G_{2n+2}$ we deduce that $G_{2n+2} \sim G_{2n+3}$, which means that $G_{2n} \sim G_{2n+1}$ and since we know that $G_{2n+1} \sim G_{2n+2}$, the conclusion follows. □

Monotonicity acts as a reinforcing condition also with respect to the notion of having partial step 1, but actually a weaker notion of monotonicity will do. Observe that if $G \in \mathbb{S}$ is monotone then the subsequences $\{G_{2n}\}_n$ and $\{G_{2n+1}\}_n$ are either both increasing or both decreasing, but this condition is strictly weaker than monotonicity, and we call *separately monotone* a sequence which satisfies the latter condition.

Lemma 16. *If $G \in \mathbb{S}$ is separately-monotone then it has step 1 if and only if it has partial step 1.*

Proof. Since the property of having partial step 1 is weaker than the property of having of step 1, it suffices to show that if it has of partial step 1 and separately monotone then it has step 1. The proof of this fact is similar to that of Lemma 14 and will be omitted. □

Now we show that F^* shares some of the properties of F, in a weaker form.

Lemma 17. *If F has step 1 then F^* has step 2. If F is monotone then F^* is separately-monotone.*

Proof. It suffices to observe that (5) may be written as

$$\frac{F_n^*}{F_{n-2}^*} = \frac{F_{n-1}}{F_n} \quad (n \geq 2)$$

□

Lemma 18. *If $F \in \mathbb{S}$, $p, q \in (0, +\infty)$, and $F_1 = 1$, then the following two conditions are equivalent:*

1. *one of the equivalent conditions of Lemma 12 holds for F, p, and q;*
2. *F^* has partial step 1.*

Proof. The following identity can be shown by induction.

$$F_{2n+1}^*(p,q) = qF_1 \frac{F_{2n}^\sharp}{F_{2n+1}^\sharp} \quad (n \geq 1) \tag{7}$$

The proof is similar to that of (6) and is omitted. Then (6) and (7) imply that

$$\frac{F_{2n+1}^*(p,q)}{F_{2n+2}^*(p,q)} = \frac{q}{p} F_1 \frac{F_{2n}^\sharp F_{2n+2}^\sharp}{\left(F_{2n+1}^\sharp\right)^2} = \frac{q}{p} F_1 \left[\frac{F_{2n+2}^\sharp}{F_{2n+1}^\sharp}\right]^2 \frac{1}{F_{2n+2}} \tag{8}$$

If we assume that $F_1 = 1$, then (8) shows that (W') is equivalent to $F_{2n+1}^*(p,q) \sim F_{2n+2}^*(p,q)$. □

Lemma 18 shows that the properties (W') and (E'), appearing in Lemma 12, may be expressed in terms of asymptotic properties of the sequence F^* (and say that F^* has partial step 1). Thus, we now have to show that I^* has partial step 1. Observe that, by Lemma 16, since I^* has step 2, then it has partial step 1 if and only if it has step 1. On the other hand, Lemma 14 shows that, since I^* has step 2, it is sufficient to show that I^* is monotone, in order to show that it has step 1. The fact that I^* is monotone follows from the equality $I^*(\pi, 2) = S$ and from the fact that S is monotone, and this fact in turn follows from its "explicit" representation given in (3).

4 Conclusion

The story about Euler, Stirling, and Wallis is not yet over, we think. The reader has no doubt noticed that the equivalence between (E) and (W) has been obtained through a sequence of intermediate statements, for each of which there is an a priori, direct link with the following one. Is it possible to achieve the equivalence in just one step? Is there an a priori, direct link (or a sequence of these) between (E) and (S) which does not go through (W)? Is it possible to refine the argument involving the map $F \mapsto F^*$?

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Grothendieck: The Myth of a Break

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Ask the man in the street about Alexander Grothendieck, and he will most likely answer you: "Alexander Grothendieck was a mathematical genius who, at the age of forty, became mad, abandoned mathematics and developed a mystic delirium." If that man happens to be a mathematician, he will probably add: "Before he left the stage he drew attention to himself with some leftist extravagance and the writing of a long, delusional pamphlet in which he 'settled his accounts' with the community: *Récoltes et semailles*."

This is the general idea that uninformed people tend to have. Let us examine what various professionals have to say about him. For example, the science historian Leo Corry, in *Writing the Ultimate Mathematical Textbook: Nicolas Bourbaki's Elements of Mathematics:*[1]

> *As it happened, however, Grothendieck left the group in around 1958–59 while some of the members, above all Serre, Schwartz and Dieudonné, continued to be close friends and collaborators. Later on, in 1970, he completely retired from public scientific life, as he discovered that I.H.É.S. was partly funded by the military.*

More recently, the French scientific magazine *Pour la Science*, read by a wide audience, published an issue devoted to the mathematician's life on the occasion of his passing.[2] In the first lines of the editorial of that issue, the managing editor of the magazine, Maurice Mashaal, says:

> *In the fifties and seventies, in France, an exceptionally brilliant mathematician impressed all of his colleagues.*

followed by the mention of his works in a few lines, and then, in the middle of his presentation:

> *After his sudden withdrawal from the scientific scene in 1970 right up until the last period of his life, which was in total solitude, he wrote tens of thousands of unpublished pages.*

Further, Grothendieck's biography is written by the mathematician Winfried Scharlau, who summarises his article in three points (section entitled "The essentials"):

- Son of a Russian anarchist and a German immigrant, and for a long time stateless, Alexander Grothendieck grows up in various shelters in Germany and then France.
- During the postwar period, he establishes himself as one of the greatest mathematicians of the century. He develops a new vision in algebraic geometry, his main discipline.
- In 1970, he suddenly turns his back on research. He is interested in radical ecology, then leaves permanently to live a reclusive life, limiting all contact with the outside world to a bare minimum.

If I am to believe the spirit of the time when I am writing this article, the "common opinion"– that Grothendieck effectively turned his back on the mathematical community in 1970 – was formed a few years after his disappearance.

However, with a little curiosity, it is easy to see that this vision is *completely false*! Let us search a little.

P. Cartier was a fellow mathematician and friend, both during and after the glorious years. This is what he said in 2000 in *Un pays dont on ne connaîtrait que le nom*. (*Grothendieck et les motifs*)[3]

> *(…) In the meantime, Grothendieck had dropped everything in 1970, after twelve years of unchallenged scientific rule over the I.H.É.S. Until his official retirement in 1988 at the age of sixty, he would only work sporadically, leaving behind a not insignificant "posthumous" work, of which three major writings stand out. The first,* A la poursuite des champs[4]*, written in 1983, is a six hundred-page reflection on multi-dimensional categories. Here, combinatorics, geometry and homological algebra are mixed together in a grandiose project. After fifteen years of effort, only three definitions have been created that are probably equivalent (or almost so) to multidimensional categories (in a broad sense*[5]*. Its stake is not just pure mathematics, since a good theory of assemblies has many potential applications (theoretical computing, statistical physics, etc.).*
> *The second,* Esquisse d'un programme*, is a text written in 1984 in support of a CNRS job application. Grothendieck sketched the construction of a tower (or a game of Lego), describing the deformations of algebraic curves. And finally,* La longue marche à travers la théorie de Galois*, written in 1981, gives partial indications on the constructions claimed in the Sketch.*

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1 Handbook of the History of Mathematics, Oxford University Press 2009.

2 *Pour la Science*, September 2016, no 467.

3 Preprint IHES/M/00/75, November 2000 and (written slightly differently) INFERENCE, International revue of science, http://inference-review.com/article/un-pays-dont-on-ne-connaitrait-que-le-nom. My quote is from the second document.

4 Footnote of P. Cartier: The mathematician Ronald Brown explains the complex story of this document in the paper "The origins of Alexander Grothendieck's *Pursuing Stacks*" (…).

5 In a footnote P. Cartier explains what the issues are.

Perhaps one will object that even though Grothendieck continued to produce mathematics, as Cartier testifies, he had completely broken with the community. However, by inquiring a little more, one can quickly learn that from 1972 to 1984 Grothendieck was professor at Montpellier, further from 84 to 88 a researcher at CNRS. The 1984 text *Sketch of a programme* has been published, it is cited by contemporary works and a whole colloquium[6] was devoted to it. Moreover, during this same period he had several PhD students at Montpellier, of which at least two, to my knowledge (maybe more), became university professors.

Récoltes et semailles is a very long text (about 1000 pages) by Grothendieck, which was mimeographed by the university of Montpellier (1985–86); it is a partly autobiographical memoir describing Grothendieck's life as a mathematician. A persistant rumour says that this writing is only a settlement of accounts with his former students, which would explain why it never found a publisher.[7] But, in the same *Preprint IHES/M/00/75* containing the paper of P. Cartier quoted above, we find a different analysis in *Découvrir et transmettre* by the historian of mathematics Alain Herreman. You will read in the conclusion of *Découvrir et transmettre*:

> *Whether it is the influence of his elders, the reception of his work and its transmission by his students, or the process of discovery,* Récoltes et semailles *is on all levels a confrontation with the collective dimension of mathematics and at the same time an attempt to elaborate it conceptually.(…)*

At this stage, now warned that this text is very different from just a "gunfight at the O.K. corral", you might want to examine it more closely and you will find many passages where Grothendieck evokes his professional activity from 1970 onwards:

> *As you know, I left the mathematical "great world"[8] in 1970 as the result of an issue of military funds in my home institution (the I.H.É.S.). After a few years of anti-militaristic and ecological activism in a "cultural revolution" style, of which you have probably had some form of echo here and there, I practically disappeared from the traffic, lost in a provincial university God knows where. Rumor has it that I spent my time tending sheep and drilling wells. The truth is that, apart from many other occupations, I was bravely going, like everyone else, to do my lecture courses in the faculty (that was my unoriginal bread and butter). It even happened to me, here and there, for a few days, even a few weeks or a few months, that I did maths again at "brin de zinc"[9] – I have boxes full of my scribblings, which I must be the only one who can decipher.*
> *R&S, p. 75*[10]

or:

> *(…) for more than ten years[11], my friend[12] remained for me (in a self-evident way) my main discussion partner in mathematics; or more precisely, between 1970 and 1981, he was the only discussion partner (except during one episode) to whom I wanted to speak during the periods of my sporadic mathematical activity, when I was in need of an interlocutor.*
> *R&S, p. 307*

If we recall the work output that Grothendieck was capable of, it goes without saying that his sporadic activity is certainly equivalent to the activity of one or even more ordinary mathematicians!

To conclude this point, here is the testimony of Yves Ladegaillerie[13] who was his student at Montpellier:

> *As a professor in the Faculty of Science, he did the same work as others; with care, availability and dedication. He typed on his old machine the mimeographied texts which were generously distributed to everyone (…). As soon as he arrived in Montpellier in 1973, I taught with Grothendieck, and he quickly offered me to do some work in research with him (…). In the 80s, we had a small working seminar with him, Malgoire and Contou-Carrère, at Montpellier.*

These testimonies are enough to show that the assertion of Grothendieck's break with the mathematical world is inaccurate. If there is a break, it is not with the totality of the mathematical community but only with a part of it, perhaps the one he calls "great world" in the quotation above.

The story of Grothendieck's break with the mathematical community is therefore clearly a myth. However, this myth is currently being revised, as shown by at least

6 CIRM, Luminy, conference on *Les dessins d'enfants*, April 19–24, 1993. See the book *The Grothendieck theory of dessins d'enfants*, Cambridge University Press (1994).

7 See note 10.

8 In French it is "grand monde"; it is something like "high society" with an ironic connotation. I come back later to what Grothendieck means with "grand monde".

9 Grothendieck's style is very literary, sometimes poetic, sometimes slangy, sometimes difficult to understand. Here the term "brin de zinc" could be a different writing of the slang "brindezingue" (the prononciation is the same) which means "completely drunk" or a deformation of "a tout berzingue" which means "very fast". R&S contains eleven ocurrences of "brin de zinc" in different settings. I give up any attempt to translate it.

10 *Récoltes et semailles* is not yet published but available on the web. My references (denoted by R&S) are from the version online at: https://www.quarante-deux.org/archives/klein/prefaces/Romans1965–1969=Recoltesetsemailles:pdf. The numbering of the pages corresponds to that of the pdf document, not of the original document. To my knowledge R&S is not translated.The translations of the quotes are mine.

11 After his departure from I.H.É.S. My note.

12 Pierre Deligne. My note.

13 *Grothendieck after 1970*, online at http://www.grothendieck circle.org

two recent texts.[14] On the one hand, Jean-Paul Allouche in his review of the book *Alexandre Grothendieck: A mathematical portrait*[15] in the *Newsletter of the European Mathematical Society* of March 2015, questions the so-called madness of Grothendieck:

> *What strikes me in the view that mathematicians have of Grothendieck's work and life is the huge gap between the interest and fascination for his mathematical work and – at least for a large majority of mathematicians – the fact that they are rejecting the rest of Grothendieck's thoughts (should we accept military grants, the unbearable explicit hierarchy that mathematicians build among themselves, the urgency of working in ecology and the like instead of dealing with mathematics, etc.). It is so easy to declare that he was "a bit mad" or "deeply depressed" or even "suffering of psychosis" rather than to think that he could well have been very much in advance also in these subjects.*

There is also a similar reflection in D. Nordon's "Bloc-note" in the journal *Pour la Science*[16]

> *When Grothendieck began to criticize the scientific institution in the 1960s and 1970s, some discredited his objections by saying that they where fuelled solely by his frustration. He had not been able to demonstrate Weil's conjectures and consoled himself by vilifying the institution. The problem with such psychological explanations is that they are always plausible, and never interesting.*
> *Frustration is the most shared thing. We have all been – and more than once – the fox facing the inaccessible grapes.*[17] *So there is not a whistleblower that you can't accuse of being frustrated, but it doesn't say anything about the value of what he is saying. If only arguments from authors virgins of any frustration had a chance to be judicious, no wise argument would ever have been considered since the world began! To disqualify an argument by interpreting it as the disguised expression of bitterness by its author is an avatar of an old process, as unfair as it is effective: to discredit an individual is easier than to refute his arguments. Especially if they are relevant.*

A myth is an imaginary construction developed to ensure a certain social cohesion around the defense of particular interests. My hypothesis is that the myth of the break was coined by this mathematical "great world" that Grothendieck left in 1970, which poses several questions:

- What is the "great world" actually?
- If there is indeed a mathematical "great world", is it actually responsible for the myth?
- If a group had indeed created the myth, what were its motives? What noble or less noble interests did it try to defend?

I am not claiming to know the answer to these questions, which go far beyond my competence and, I believe, that of any individual. Only a collective work involving competent mathematicians, historians and sociologists could enlighten us in a relevant way. My only ambition is to convince the reader that an open-minded reading of R&S could open up interesting research paths on the function and evolution of the French mathematical community between the immediate post-war period and the present day.

A short account of more or less-known facts

The hurried reader who knows the history of Grothendieck from before 1970 can skip this paragraph.

Montpellier, the arrival in Paris, the thesis in Nancy

Grothendieck studied mathematics at the University of Montpellier, where he did not find enough to satisfy his attraction to mathematics.

> *He passed his bachelor's degree at the faculty of sciences in Montpellier without his professors taking notice of him (nor did he notice his professors). He then went to Paris to become a mathematician.*[18]

Schwartz's slightly scornful opinion in referring to his colleagues at Montpellier does not seem completely justified. Indeed, Grothendieck did have a certain affection for "Monsieur Soula", who was trying to convince him that since Lebesgue there was nothing more to do in mathematics:

> *Mr. Soula, my "differential calculus" teacher, was a kind and willing man towards me. But I do not think he convinced me. There must have already been within me the preconceived idea that mathematics is an unlimited thing in length and depth.*
> *R&S, p. 34*

One can also think that Mr. Soula was aware of the Montpellierian shortcommings because he helped Grothendieck to "go up" to Paris and to meet Henri Cartan by providing him with a letter of recommendation.[19] More-

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[14] See also on the website of the Collège de France, as part of the symposium for the start of the 2016 academic year, *Migrations, refugies, exil*, a reading by Alain Connes of several pages of R & S, as well as *La clef des songes* that will make the listener want to read more.

[15] Leila Schneps (ed.), International Press of Boston.

[16] March 2016.

[17] This is a reference to Jean de Lafontaine's fable *The fox and the grapes* where the fox covets grapes that are inaccessible.

[18] Laurent Schwartz, *Un mathématicien aux prises avec le siècle*, Editions Odile Jacob, Paris (1997) p. 292.

[19] Here is a testimony of J-P. Kahane who knew Mr. Soula: One day, with Soula and Turrière, the conversation turned to Grothendieck, whom they had as a student. They took out license exam copies for me, and all of a sudden my respect for them took a huge leap forward. These copies were illegible. An examiner might have refused to read them. But these old gentlemen, Soula at first I think, had felt what was behind Grothendieck, and he passed his examinations. Jean-Pierre Kahane, *Grothendieck et Montpellier*. Images des Mathematiques, CNRS, 2014..

over Grothendieck, if he says that Montpellier could not satisfy his thirst for mathematics, also says that he does not have a totally negative memory of it:

> *Yet now, thinking back to those three years, I realise that they were not wasted. Without even knowing it, I learned in solitude what the essence of the mathematician's job is – what no teacher can really teach. (…)*
> *To put it another way: I learned in those crucial years to be alone. By this I mean conversing with my own mind about the things I want to know rather than relying on the ideas and consensus, whether expressed or implied, that come to me from a larger or smaller group that I feel myself to be a member of, or that for any other reason would be invested by me with some authority.*
> *R&S, pp. 34–35*

In Paris, Grothendieck discovers the world of mathematics "in the making", follows a course by Leray, attends the Cartan seminar:

> *In the year that followed, I hosted a Cartan course at "L'École"*[20] *(on the differential formalism on manifolds; also that of the "Séminaire Cartan", to which I clung firmly, amazed to witness the discussions between him and Serre, with big shots of "spectral sequences" (brr!) and drawings (called "diagrams") full of arrows covering the whole picture. (…) I had been to see Mr. Leray at the Collège de France to ask him (if I remember correctly) what his course would entail. I do not remember any explanations he gave me, nor did I understand anything about it (…)*
> *R&S, p.140*

Conscious of the value of Grothendieck, A. Weil and H. Cartan sent him to…

> *(…) Nancy, which at that moment was a bit like Bourbaki's headquarters; with Delsarte, Dieudonné, Schwartz, Godement (and a little later also Serre) teaching at the University.*
> *R&S, p. 145*

He wrote his thesis, which he defended in 1953, under the direction of L. Schwartz who tells us:

> *It was the most beautiful of "my" theses. (…) The collaboration with this talented young man was a fascinating and enriching experience.*[21]

From 1953 to 1956 he took multiple trips to universities abroad.

At I.H.É.S. 1958–1970

I.H.É.S. was created in 1948 by the industrialist Léon Motchane based on the model of the Institute for Advanced Studies at Princeton. Jean Dieudonné and Alexandre Grothendieck were the first full professors of the institute. From 1960 to 1969, a seminar was organised by Dieudonné and Grothendieck, the "Séminaire de Géométrie Algébrique", which led to a series of Lecture Notes in Mathematics: the famous S.G.A. This seminar was attended by elite, young researchers, almost exclusively from E.N.S. They would write down the content of Grothendieck's weekly presentations with the help of Dieudonné, and Grothendieck would supervise. It is quite an exceptional situation that a seminar leader does not write his own material by himself, or at least with one or two close associates. Instead, Grothendieck set out the main points of his theories and half a dozen smart students wrote the S.G.A.s. more or less anonymously. This situation is obviously close to that of Bourbaki, but with a one major difference: the founders of Bourbaki – Weil, Cartan, Delsarte, De Possel and Dieudonné … are well-known mathematicians and peers with equal rights and duties (even if some are a little more equal than others), and this collegiality "without a leader" seems to continue with the younger generations (Cartier, Serre,…); while for the S.G.A., it is young people preparing a thesis who write down mathematics that is only partially theirs. At the same time, between 1960 and 1967, Grothendieck wrote, in collaboration with Dieudonné, the *Elements of Algebraic Geometry* (E.G.A.), which appears in the form of eight volumes of I.H.É.S. publications, solely under his name.

In 1966 he refused to go to Moscow to receive the Fields Medal in protest against USSR policy in Central Europe. Soon after, he began getting involved in supporting Vietnam against the United States.

The opening to the world

In the wake of the events of 1968, Grothendieck engaged in political activism. His two main commitments were the denunciation of scientific work for military purposes and the participation in the creation of the movement "Survivre et vivre", the first movement of political ecology in France.

In 1970, discovering that the I.H.É.S. had received some small military funding, he asked that this subsidy cease immediately. Not receiving the support he had hoped from his colleagues, he resigned. In 1971 his course at the College of France, where he had been appointed for two years, was not renewed because of his intention to devote sessions on the theme *Science and Technology in the current evolutionary crisis: are we going to continue scientific research?* in addition to his course on "La théorie de Dieudonné des groupes de Barsotti–Tate". Also on this occasion, he did not receive the expected support from his scientific community.

What is the "great world"?

In my first quote from R&S, Grothendieck talks about his departure from the "great world" of mathematics. In

[20] In the French mathematical community everybody understands that "École" is "École Normale Supérieure de la Rue d'Ulm".

[21] Laurent Schwartz, *Un mathématicien aux prises avec le siècle*, Editions Odile Jacob, Paris (1997) p. 294.

R&S I counted the expression "great world" twenty-nine times, and almost as many times for "beautiful world." He gives a definition in a footnote:

> *There was also, in the background, the thought of a certain spirit in the world of mathematicians, and more particularly in what might be called (without sarcastic or mocking intonation) the "great world" of mathematics: the one which "sets the tone" for deciding what is "important" or "lawful" and what is not, and which also controls publications and, to a large extent, careers.*
> *R&S, p. 388*

but I do not know what to think about it in so far as he specifies that here he uses the expression the "great world" "without a sarcastic or mocking intonation", whereas, for the most part, his intention is, at least, ironic. That is why I am going to specify what I personally mean by "great world", but without claiming that this is precisely what Grothendieck intended.

Promoting the progress of mathematical knowledge is a complex collaborative undertaking led by a community of people. Like any human community of any size, the mathematical community has operating rules that are written in regulatory texts and traditions. Like any community, it has managers: laboratory directors, members of the C.N.U.[22], members of the Academy, leaders of scientific societies, etc. They are responsible for making the institution work at its best by exercising the power they are invested in. They are leaders but they are not necessarily part of what I call the "great world".

I believe that the "great world" is to be sought elsewhere. In addition to the transmitting of "well-established knowledge", the mission of the mathematical community is, as for any scientific community, to distinguish what precisely has the vocation to become this "well-established knowledge" among the mass of accumulated results. This essential mission cannot be assumed by the political power and the state, as Lysenkoism has shown in the USSR, or as anti-Darwinism has shown in the USA. In Western-style democracies, it is entrusted to what is known as "peer judgment". "Peer judgment" is exercised on two crucial occasions.

The first one is, as is well known, the process of reviewing the articles submitted to mathematical journals and the second occasion is when people are recruited to academic positions where questions of balance between disciplines and teaching requirements can compete with scientific excellence. In theory, the voice of each peer has the same value, but it is natural that the voice of leading mathematicians carries more weight than that of smaller players.

It happens that some mathematicians or groups of mathematicians have such a high opinion of their excellence that they think that their vision of mathematics must prevail at all costs. It is much more than a mere corporatist attitude, where you defend your discipline and try to "fit in" your friends. It is a much more disturbing attitude that is close to fanatism. They feel personally invested with the mission to defend what they consider to be mathematics ("La mathématique", as Bourbaki says), to propagate the "true faith" and behave like real crusaders as Arnaud Denjoy says of Bourbaki.[23]

> *I fear your absolutism, your certainty of holding the true faith in mathematics, your mechanical gesture of drawing the sword to exterminate the infidel to the Bourbaki Qur'an. […] We are many to judge you as despotic, capricious, sectarian.*

It is these mathematicians, often of great renown, but not always, and certainly with an oversized ego, that I call the "great world" or the "beautiful world". So my "great world" is not what Alain Herreman calls (in Grothendieck's words) the "microcosm", made up of about twenty colleagues or students who have had professional relations with the mathematician, even if some could be part of it. Nor do I seek, as I have already said, to demonstrate here that this is indeed what Grothendieck means by "great / beautiful world" but simply to use this concept a little vaguely to read some passages of R&S.

Before 1970: Grothendieck in the "great world"…

The partly fanatical behaviour that I attribute to individuals in my concept of "great world" does not enable me to decide lightly if Grothendieck was a member of it or not; and I will not do that. I do not know if he was willing to cover up bad cases in the name of mathematics' defense, all I know is that, before 1970, his whole life was devoted to mathematics, to his own mathematics, which is a respectable choice. But it is indisputable that his privileged position at I.H.É.S. made him very close to the "great world" and adopt in a more or less conscious way certain of its attitudes. In R&S he severely criticises this behaviour, which he now considers (after 1983) as harmful. What is interesting here is that this criticism, which includes a large part of self-criticism, cannot be swept away on the pretext that it is only a vulgar "settlement of accounts". Thus he explains at length some of his attitudes as his "contempt", that retrospectively he severely judges:

> *I did not say to myself back then that if the pupil[24] was indeed unproductive, it was a reason to advise him to do something else and to stop working with him, but not to treat him with contempt. I had identified myself with being "strong at maths" such as this prestigious elder, at the expense of "nobodies" that it would be lawful to despise.*
> *R&S, p. 146*

[22] The national organism that rules careers in France.

[23] Arnaud Denjoy to Henri Cartan (May 22, 1954). Archives de l'Académie des sciences, fonds Montel, carton 1. Quoted in: Anne-Sandrine Paumier, David Aubin, *Polycephalic Euclid?: Collective Practices in Bourbaki's History of Mathematics,* 2013. ¡hal-00871784v3¿.

[24] Grothendieck relates the attitude of one of his prestigious elders towards one of his students.

This question of respect for the person and the importance of respect in the process of mathematical creation holds a huge place in R&S but this is not the point I want to address.[25] Beyond the necessarily subjective appreciation of the moral attitudes of the "great world", there are more objectively appreciable practices of which here is an example:

> *Towards 1960 or 1961 I offered Verdier the development of new foundations of homological algebra as a possible thesis work (…) His work on foundations continued satisfactorily, materialising in 1963 with a "State 0" on the derived and triangulated categories, mimeographed by the I.H.É.S. (…)*
> *If its defense[26] did not take place in 1963, but in 1967, it is because it was unthinkable that this 50-page text, the embryo of a foundation work still to come, could constitute a doctoral[27] dissertation – and of course the question did not even arise. For the same reason, at the thesis defense on 14 June 1967 (in front of a jury consisting of C. Chevalley, R. Godement and myself, who presided), it was out of the question to present this work as a thesis. The text submitted to the jury of 17 pages (+ bibliography) was presented as an introduction to a large-scale work in progress. (…)*
> *R&S, p. 352*

We are very far from the tradition which at the time required, for the "thèse de doctorat d'état", the defense of a "second thesis" which consisted of the oral presentation of a work far from the specialty of the candidate! Grothendieck recognises this:

> *If the title of doctor of science was awarded to J.-L. Verdier on the basis of this 17-page text, sketching some ideas that he himself admitted had not all come from him, then it was clearly a goodwill contract between the jury and himself: he committed himself to completing and making available to the public this work of which he presented a brilliant introduction.*
> *R&S, p. 352*

That this testimony by Grothendieck is made in the context of his "settling of accounts" with his former student does not detract from the objectivity of facts that are nevertheless important because they concern the successor of H. Cartan as Director of study at E.N.S. That Verdier was a high-quality mathematician at that time is probably not doubtful to the specialists of the discipline – I do not know – but was that reason enough to go over the rules and traditions and just accept that he gets away with minimal drafting work? Grothendieck, the author of R&S, therefore the Grothendieck after the break, actually considers that on this occasion he has shown lightness:

> *I accept the entire responsibility as J.-L. Verdier's thesis supervisor and president of the jury, for my folly in having awarded him (together with C. Chevalley and R. Godement, both trusting the guarantee I gave) the title of doctor for a work which had not yet been entirely completed (…) To this responsibility I should also add not having ensured during the two years that followed (before my departure from the mathematical scene) that Verdier did indeed fulfil the contract he had made.*
> *R&S, p. 353*

Grothendieck is certainly at fault but less, it seems to me, than Chevalley and Godement, and more generally than the microcosm of the S.G.A., who let it go. Since he lived only for the mathematics that he identified with the activity of the S.G.A., he was excusable of this contempt of tradition; a tradition that he did not know very well because of his atypical career.

…and then on the other side: after 1970

A large part of R&S is devoted to a "case" in which Grothendieck accuses several members of what he calls the "great world" to have, first, ignored the work of a mathematician who was not part of the brotherhood, and then, in a second instance, to have looted it. There are certainly some interesting things to understand in this testimony of Grothendieck, but that presupposes a very good understanding of the mathematics involved and the truth of the facts asserted, which is not at all my case. This is the part where Grothendieck engages most in personal attacks, whose foundation can only be appreciated by sharp specialists. I will not talk about it. On the other hand, the following testimonies, which concern his work as an "ordinary mathematician" in Montpellier, speak for themselves.

Despite the fact that he has decided to leave the "great world", Grothendieck still intends to continue to do mathematics. During his stay in Montpellier he is interested in a few young mathematicians and experiences the ordinary mathematician's life. Before going into detail, let's say that these are instances where he can see that giving his approval to the work of young mathematicians no longer has any effect on the work's reception by the "great world", since he is no longer a member of it.

As the works in question are related to his own body of work (naturally, since it concerns his testimony on their value), he attributes their rejection to the "burial" enterprise of which he feels the victim. But there is another reading, less centered on Grothendieck and more simply factual, that we can try here.

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[25] In *Découvrir et transmettre*, cited above, A. Herreman offers a fascinating analysis of Grothendieck's entire reflection on the collective process of mathematical creation, and thus of this particular point.
[26] of Verdier's thesis
[27] Grothendieck says "thèse de doctorat d'état": in the sixties, in France, there were two thesis: a small one (two years of research) called "thèse de troisième cycle") which is more or less equivalent to our present "mastere thesis" and a "big thesis", the "thèse de doctorat d'état" (approximately 5 years of research) which is roughly equivalent to our present "habilitation thesis".

The difficulty in publishing

In the 1970s, Olivier Leroy was a young man from Montpellier who impressed Grothendieck with his skills:

> *He was a young man of maybe twenty who must have had just a smattering of diagrams, a bit of topology and topos, and had handled a lot of infinite discrete groups, I think… It was three times nothing, to be honest, and yet with that he managed to fill in all the blanks anyway and to "feel" without any effort what I, an old veteran, told him at full speed for two or three hours, based on a fifteen-year familiarity with the subject. I had never seen anything like it, or at most with Deligne, and perhaps with Cartier, who was also quite extraordinary in that respect at such a young age.*
> *R&S, p 406*

O. Leroy wrote some of his work in the form of a note to the C.R.A.S.[28] The note was rejected. Grothendieck has an opinion on this rejection:

> *One of my friends and companions of yesteryear gently explained to me recently that, in the course of time, alas, and with the immeasurable increase of mathematical output that we are aware of, "we" are absolutely obliged, whether we like it or not, to make a strict selection of the papers that are written and submitted for publication, and to publish only a small part. He said it with a sincere, desolate look, as if he himself were also victim of this inevitable fatality – the same look that he had when saying that he was himself also a member, yes it's unfortunate but that's how it is!, of the "six or seven people in France" who decide which articles are going to be published, and which ones are not (…)*
> *R&S, p. 187*

In this passage, Grothendieck offers us a fairly accurate portrait of a member of the "great world". He returns to this character:

> *One or two months later I learned that this colleague had refused a few years ago to recommend the publication of a certain note to the "comptes rendus", whose author as well as theme (one that I had proposed to him roughly seven or eight years before) were dear to me. (…)*
> *I think he did an excellent job (presented as a "thèse de 3e cycle). I had never been this young researcher's boss, brilliantly gifted as he is (I do not know if he will continue to apply his gifts to mathematics, given his welcome…), and he completed his work without any form of contact with me. But it is also true that the origin of the theme that was developed could not be doubted; he was in big trouble the poor guy, and possibly without suspecting anything! This colleague said that in a civilized manner, there is at least that and I would not have expected less from him, "sincerely sorry but you understand…". Two years of work by a highly motivated researcher just starting out versus the three-page CR cost – how much public money would it have cost?*
> *R&S, pp. 188–189*

and understands that for some time now this work has struggled to be recognised. He then gives us a juicy but totally imagined description of the censor's behaviour:

> *This same draft of notes to the "Comptes Rendus" had the honour of being submitted to another one of the "six or seven people in France…", who sent it back to the author's "boss" because these mathematics "did not entertain him" (textual!). (The boss, revolted but cautious, himself in a rather precarious position, preferred to say nothing rather than saying anything displeasing…) Having had the opportunity to discuss it with this colleague and ex-student, I learned that he had taken the trouble to read the note carefully and consider it (it must have brought back many memories), and that he had found that some of the statements could have been presented in a more helpful way for the user. He did not deign, however, to waste his precious time submitting his comments to the person in question: fifteen minutes of the renowned man's time versus two years of work by an unknown young researcher! Maths amused him enough to seize this opportunity to reconnect with the situation studied in the note (which could not fail to generate in him, as well as in myself, a rich tapestry of various geometric associations), to assimilate the given description, then, with very little effort given his experience and his means, to detect some clumsiness or deficiencies. He did not waste his time: his knowledge of a certain mathematical situation clarified and enriched thanks to two years of conscientious work by a researcher taking his first steps; work that the Master would certainly have been able to do (in broad outline and without demonstrations) within a few days. This being gained, we remember who we are – the cause is judged, two years of work by Mr. Nobody are good for the rubbish bin…*
> *R&S, p. 188*

In the following case, Grothendieck is more personally concerned, since it refers to one of his students:

> *Yves Ladegaillerie started working with me in 1974. (…) and he grasped it more or less, until the day when it ended up "tilt" with him, I do not know when and why. (…) From the moment Yves had grasped it, he did his thesis in a year, a year and a half, results, writing, everything, and moreover dressed to the nines. It was a brilliant thesis, less thick than most of those which had been written with me, but as substantial as any of those eleven theses. The defense was done in May 1976. The thesis is still not published today. (…) The central result will finally appear, nine or ten years later and reduced to the bare bones, in a short article of Topology (hush – I have an accomplice in the editorial board of this estimable journal).*
> *R&S, pp. 399–400*

[28] Comptes Rendus de l'Académie des Sciences.

The ordinary mathematician of my generation who has not rubbed shoulders with the "great world" will not be surprised by this testimony, which he will probably have read with jubilation. It's somewhat the "sprinkler sprinkled". Grothendieck, whose opinion was to be solicited as the divine word when he was the guru of the "great world", does not even manage to make a poor little "note" now that he is no longer recognised!

Admittedly, it was not always easy to publish, especially if one was not part of the batch of a great feodalism having some power, but I would not want to give the impression that in these years it was not possible to publish a note to the Academy outside the servile allegiance. Allow me a little personal testimony. At the beginning of the 1980s the little mathematics school I belonged to was disputed and had trouble getting anything published. I was in contact with R. Thom, whose hostility to the ideas of the school in question I knew, but whose listening qualities I appreciated. On my advice, a young colleague asked him to present a note to the "Comptes Rendus", which he did. Surprised, I took the opportunity to tell him that I was delighted to see that he was convinced by the point of view of my school of thought: "Not at all…", he answers me.

– "But did you not send the note! Then surely you agree? I do not understand."

– "It's very simple, though. Mr. X seems to me to be an entirely competent mathematician. He did not convince me, it's a fact, but others than me could be convinced. By what right would I deprive the community of appreciating those ideas that may be relevant?"

This is not the supposed style of Grothendieck's "companion of yesteryear".

Professionnal instances

Contou-Carrère is a mathematician with a non-traditional background. As a holder of a foreign doctorate he is the only candidate for a position as professor at the university of Perpignan[29] and supported by local mathematicians. The "Comité Consultatif des Universités" (C.C.U.) rejects the proposition.

> *The fact remains that Contou-Carrère's candidacy was ruled inadmissible by the "Comité Consultatif des Universités" and the file was returned. The thing that baffled me was that, in the absence of an official explanation, neither the President of the CCU (the national board that made the decision) nor any of the members had the minimum of respect to write personally, either to Contou-Carrère himself, or at least to the director of the Perpignan Institute of Mathematics, to give a few words of explanation as to the meaning of this vote, which could only be perceived as a stinging disavowal of the choice of Perpignan's colleagues, as well as a disavowal of their sole candidate as being capable of honourably filling the post he had been offered.*
> *R&S, p. 400*

The contempt displayed by the C.C.U. is indicative of the spirit of the "great world". Of course the C.C.U. may have been right regarding the case's substance, but it is totally unacceptable to not explain the reasons for the rejection. Finally, let me show a last incident that humiliated Grothendieck, being reduced to the rank of standard mathematician.

In 1983 there was a "Commission des Thèses de Mathématiques des Universités Parisiennes", without an institutional existence, whose members were not known; the mathematical community agreed to have to obtain its approval for a defense to take place in a Parisian university. One of the peculiarities of this commission is that it did not motivate its decisions.[30]

In 1983, Grothendieck wanted Contou-Carrère's thesis to be defended.

> *It is all the more remarkable [Grothendieck refers to Verdier's own defense] that J.-L. Verdier refused my proposal to be part of the Contou-Carrère Thesis Jury in December 1983, with J. Giraud and myself having the role of research director, estimating that the thesis (entirely written and yet carefully read by J. Giraud) and the jury would not offer the guarantee of sufficient seriousness without referring to the control of the "Commission des Thèses de Mathématiques des Universités Parisiennes" (Sic).*
> *R&S, p. 353*

In R&S, Grothendieck mainly attributes his new situation after 1970 to a degeneration of mores. He attributes to his elders, Weil, Cartan, Dieudonné and Leray…, all sorts of moral virtues that are absent in his contemporaries and his pupils, who would indulge in the worst turpitudes. In fact, what has mainly changed for Grothendieck between the sixties and the seventies is not so much the mathematical world as his position in the world. But an evolution of this world is not excluded. Anyway, once again his testimony deserves attention.

Who benefits from the myth?

In consideration of these testimonies we are obliged to take note of the existence of some serious shortcom-

[29] A small town in the south of France.

[30] Here is the response received by the supporters of a candidate who had been denied the defense of his thesis on a subject (unrelated to the work of Grothendieck, let us specify): "I answer you on behalf of the "Commission des Thèses" to the letter addressed to the President of the University of Paris VII concerning the functioning of the "Commission des Thèses". This letter has of course been sent to us. (…) In this letter, you raise the case of the thesis of Mr X for which the "Commission des Thèses" asked the opinion of a second reviewer following a first report which did not allow the Commission to take a final decision. You are asking us to send you this latest report. The Commission was unanimous in finding your request inadmissible. It is a fundamental principle of the functioning of the Commission that the confidential nature of the reports requested, which guarantees the freedom of judgment of the reviewer. (…) In *Et pourtant ils ne remplissent pas* N*!*, C. Lobry, Aléas, 1989).

ings in the implementation of the "judgment of peers" in the French mathematical community. The testimony of Grothendieck is not a revelation and at the time of Grothendieck's diffusion of the 250 copies of R&S mimeographed by the University of Montpellier there were other testimonies of serious dysfunctions of this kind, but the personality of their authors, often modest mathematicians, meant that they did not get much attention. The "great world" had good grounds to challenge them by highlighting the alleged mediocrity of their authors. But with Grothendieck this type of argument was no longer adequate.

So the "great world" invented the myth of the break. Grothendieck's uncompromising personality lent itself remarkably to the operation: his break with the "great world" was confused with a withdrawal from the whole community, his uncompromising analysis of the mathematicians' collective work with a "settling of accounts", the expression of a particular spirituality with a "mystical delirium".

The "great world" who knows well that the community, as a whole, does not approve of Grothendieck's militant provocations:

- it is not well-regarded to challenge the mathematician L.S. Pontryagin at the Congress of Nice on the potential military use of his works,
- one does not transform a chair at the Collège de France into a leftist tribune in the style of sixty-eighters,

thinks, rightly, that not too many questions will be asked. Time will do the rest.

Now let us hold ourselves back from virtuously booing the "great world" that we have just described and which no longer exists in this form. It has indeed become fashionable to distance oneself from it, even to despise it, for example through a critique, *a posteriori* of the "mistakes of Bourbaki" who failed to recognise the importance of mathematical physics, the probability theory and failed to predict the importance of computing. For the current mathematical community it would be the same unfortunate mistake as refusing to take into account the testimony of one of its most distinguished members. We must remember the exceptionally rapid growth of the community between 1950 and 1980. If my recollections are correct, there were about fifteen mathematics teachers at the faculty of sciences of Grenoble in 1960 when I was studying there. When I left it in 1970, there were about one hundred and fifty. Before the war, only a few former students of the École Normale Supérieure, often after a passage at secondary schools, would go to research.

The situation did not change much immediately after the war and then, suddenly, during the sixties, things accelerated. It was normal to worry about the consequences of the "massification" of the profession of mathematician. The arrogance of the "great world" individuals was certainly not the right answer, but it is in this context that it must be analysed.

We must not forget that this "great world" with a high social position is also the one who created the I.H.É.S., this small institute with an annual budget of a few million which housed no less than 11 Fields medals in half a century of existence. It is definitely not nothing! Especially since some disciplines that Bourbaki had "missed" are now justly represented. And if one thinks that my reference to the I.H.É.S. is too elitist to look at, then consider the CIRM, this perfectly democratic instrument at the service of the wider community, which we owe to a great extent to the energy of the most elitist among the elitists: Jean Dieudonné.

The role of the "great world" is to be appreciated in the context of the era which is one of a transition between the feodal world and the capitalist world. In the post-war period and until the 1970s, the French mathematical world functioned as in the feodal world where legitimacy stems from lineage. Twenty years later it had clearly turned to the capitalist world where legitimacy stems from the money one is able to make. This transition took place with the confrontation of the old world, that of pure mathematics, with the new world of applied mathematics and computer science. The decline of the Bourbachic empire saw the expansion of a new form of domination that proposed its new ways of organising research. There were no more large pyramidal organisations supporting the big names of the moment, but small competing teams animated by a single dynamic scientific personality. The INRIA[31], among others, and Jacques-Louis Lions, emblematic figure in the rise of mathematics related to computer sciences, successfully proposed this new model of development in mathematics for their entry into techno-science. New dynamics where put into place, control by the "great world" has given way to that of the "evaluations" and the arrogance of brilliant mathematicians to that of no less brilliant "managers". Who knows if the resistance of the "great world", in spite of its reprehensible excesses, has not partly protected us (temporarily?) from a limitless commercialism which will, for sure, lead to the end of science as we know it since Galileo? There are still too few studies on the evolution of power structures in the French mathematical community at the turn of the 1970s. A good knowledge of the mechanisms that led to the present situation is obviously essential to better apprehend the future and a critical but honest reading from R&S can help. I wrote this paper simply in the hope of arousing some vocation.

The myth of Grothendieck's break with science is useless now that the "great world" which he was supposed to protect no longer exists.

Acknowledgements

I would like to thank A. Herreman whose critic of a first version of this text allowed me to remove some ambi-

[31] Institut National de la Recherche en Informatique et Automatique, a French institute devoted to computer sciences.

guities, J-P. Allouche, J.-L. Gautero, J.-M. Lévy-Leblond, D. Nordon, B. Philippe, A. Rapaport, J.M. Salanskis, T. Sari, and G. Wallet for fruitful exchanges.

A French abridged version of this text was published by the review Alliage *(http://revel.unice.fr/alliage/) No 79 (2018) as* Grothendieck, le mythe d'une rupture*, p. 46–58. I thank Alliage for authorising me to try its translation (with the help of R. Nielsen).*

Claude Lobry is professor (retired) at the University Nice-Sophia Antipolis. From 1995 to 2000 he was the director of CIMPA and later (2003–3008), with Bernard Philippe («directeur de recherche» at INRIA) co-head of the French program SARIMA devoted to development of mathematics and computer sciences in Africa. His current scientific interests are mathematical system theory, theoretical ecology, non-standard analysis and epistemology.

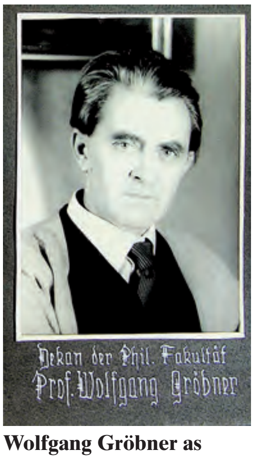
Historical Traces of Austrian Mathematicians in the First Half of the 20th Century

Robert Frühstückl (University of Vienna, Austria)

The aim of this paper is to provide a selective overview of available historical records and archival material in relation to Austrian mathematicians. I am limiting the scope of this exposition both geographically and historically, focusing on the cities of Vienna and Innsbruck in roughly the first half of the 20th century. Naturally, this does not imply that these are the only places in Austria where historians or mathematicians could find material of interest, but is simply due to having to choose concrete examples, as an exhaustive coverage would be impossible in this context.

Not long ago, as a start to the archive series in the EMS Newsletter, David Rowe reported on the extensive collection of German mathematicians' papers at the Göttingen State and University Library (SUB), where a Central Archive for German Mathematics Bequests was established in a joint project with the German Mathematical Society (DMV).[1] Unfortunately, a similar arrangement does not exist in Austria. To my knowledge, there are only a small number of collections that are open to researchers, some of which will be presented in the following.

I will begin with Wolfgang Gröbner's (1899–1980) papers, stored at the Archives of the University of Innsbruck. This collection is quite extensive and thus provides a unique opportunity for studying various aspects of Gröbner's work as a mathematician, such as his personal relations within the scientific community, the development of his ideas and – in some cases – the political struggles that his work involved as well.



Wolfgang Gröbner as dean of the Faculty of Philosophy of the University of Innsbruck, 1950. (Innsbruck University Archive)

Born in what is today South Tyrol in 1899, Gröbner went to Vienna to study mathematics, where he frequently attended courses taught by Wilhelm Wirtinger (1865–1945) and Philipp Furtwängler (1869–1940).

Both Wirtinger and Furtwängler hold a significant position in the history of Austrian mathematics and both substantially shaped the Mathematical Seminar at the University of Vienna well into the 1930s.[2] Wirtinger is famous for his work in complex analysis, whereas Furtwängler is remembered mostly for his work in algebraic number theory. Gröbner received his doctorate from the University of Vienna in 1932 under the supervision of Furtwängler.

After his studies in Vienna and a study visit to Göttingen, Gröbner's career took him to Rome, and only briefly back to the University of Vienna. In Rome, Gröbner was a member of the Institute for Applied Mathematics under its director Mauro Picone.[3] Probably at this time, Gröbner began several long-lasting acquaintances with Italian mathematicians, which can be reconstructed on the basis of his correspondence. He was also handpicked to lead a research group on industrial mathematics in 1942, which was part of a bigger aviation research facility funded by the NS regime during the Second World War, before receiving a position at the University of Innsbruck in 1947, where he was active until 1970.[4]

Gröbner's papers cover a wealth of material, including not only correspondence from about the 1930s onwards until well into the 1970s, but also various manuscripts of his mathematical work and teaching material, as well as items from Gröbner's personal library. They include not only mathematical literature but also works of philosophy, for example William James' *Pragmatism* and several volumes by Robert Reininger. Considering this apparent interest in philosophy, it is not surprising that the collection also contains manuscripts of Gröbner's own philosophical writings and lectures, of which he had published a significant number since the 1960s. There are also several documents relating to his infamous non-mathematical seminar course on *Grenzprobleme* (boundary problems) at the University of Innsbruck, which was intended as a critical reflection on the epistemology of knowledge claims in theology and which raised considerable controversy among members of the theological faculty in Innsbruck.[5]

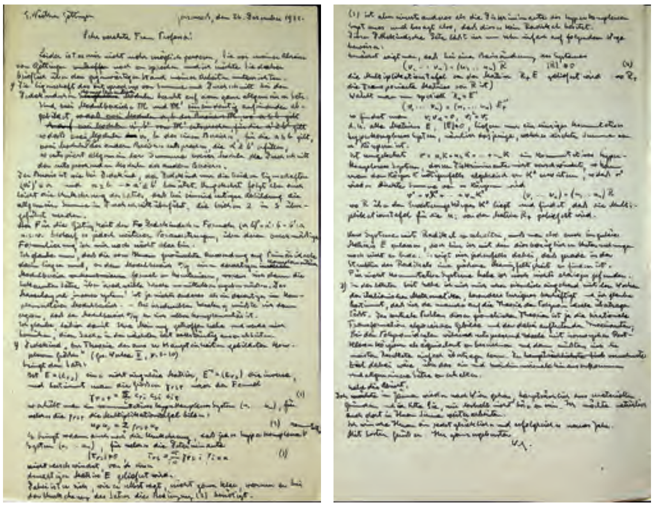
Overall, the collection consists of 16 boxes, among them six boxes exclusively containing correspondence with a wide range of mathematicians especially from Germany and Italy. The papers are grouped according to their years of origin and (partially) alphabetically

1 Rowe 2016.
2 Sigmund 2015.

3 Epple et al. 2005.
4 Goller und Oberkofler 1993. On Gröbner's work during WW2 see esp. Epple et al. 2005.
5 A description of Gröbner's quarrels with the theological faculty at the University of Innsbruck is given (in German) by Goller und Oberkofler 1993.

ordered. In addition to the wide range of correspondents, it is also the sheer quantity of papers available that makes the collection special. I did not undertake to count the entire material, but the first box of the six containing Gröbner's correspondence alone contains several hundred sheets of paper.

It is perhaps interesting to note that one of the oldest items of this correspondence is a letter the young Gröbner sent to Emmy Noether in 1932, right after he had returned from a study visit to the University of Göttingen, in which he summarised and reported on his latest work.



Gröbner's letter to Emmy Noether from December 26, 1932. (Innsbruck University Archive)

At that time, Gröbner was interested in algebraic geometry and just about to develop his own views on ideal theory. Unfortunately, the collection does not contain Noether's reply, if there was one.

The oldest items in the collection also include handwritten excerpts from works by Noether, but also Richard Dedekind and Bartel van der Waerden, which give us a glimpse into the scope of Gröbner's studies in the early 1930s.

His letter to Noether points to an interesting general aspect of the history of Austrian mathematics from the late 19th to the first decades of the 20th century: Gröbner was not the only student from Vienna who was awarded the opportunity to visit Göttingen. From about the second half of the 19th century on, it was established practice to provide excellent doctoral students with grants that enabled them to go to Göttingen, Berlin, Milan or Paris for some time to continue their studies at one of the most important centres for mathematical research in Europe.[6] As a matter of fact, Wilhelm Wirtinger, for example, visited Berlin and Göttingen after having received his doctorate from the University of Vienna in 1887.[7] Unfortunately this tradition of cooperation between Vienna and Göttingen seems to have declined during the 1930s for reasons which are not entirely clear, but might have been due to the increasing political tensions between Austria and National Socialist Germany at that time.

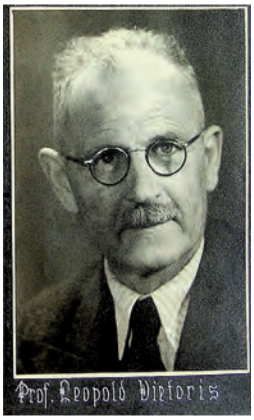
As already indicated, Gröbner corresponded widely with several important and prominent figures in the history of mathematics. Among these are Wilhelm Blaschke, Bartel van der Waerden, Gerrit Bol, Helmut Hasse, Gerhard Gentzen, Oskar Perron and Ludwig Prandtl, to name just a few.

Because Gröbner was fluent in Italian, he was destined to take on the role of an intermediary between German and Italian mathematicians, translating not only between languages but apparently also between research cultures.[8] Thus, the collection also contains letters written in Italian that Gröbner exchanged with Mauro Picone and Francesco Severi.

This particular position as well as his stay at Picone's Institute for Applied Mathematics in Rome was very likely one of the reasons that made Gröbner a prime candidate for the organisation of an applied mathematics institute, the so-called *Arbeitsgruppe für Industriemathematik* (Working Group for Industrial Mathematics) at the aeronautical research facility in Braunschweig in 1942 (see above).

In addition to the correspondence, the collection certainly deserves attention for the numerous manuscripts and offprints relating to one of Gröbner's greater research projects in Innsbruck after the war, which is the development of the method of Lie-series for the numerical solution to differential equations. Of particular interest in this regard could also be that Gröbner received funding for this research from NASA and other US government institutions as well as private companies, such as General Electric. This switching between work in pure and applied mathematics is certainly one of the many interesting features of Gröbner's career.

Just in passing I want to mention the papers of another Austrian mathematician who held a position at the University of Innsbruck: Leopold Vietoris (1891–2002).



Leopold Vietoris as dean of the Faculty of Philosophy of the University of Innsbruck, 1947. (Innsbruck University Archive)

Vietoris received his doctorate from the University of Vienna, where Gustav von Escherich and Wilhelm Wirtinger were his supervisors. In 1925, Vietoris was awarded a grant by the Rockefeller foundation[9] and went to Amsterdam to study with L.E.J. Brouwer. After his return and some intermediate career steps, Vietoris finally settled at the University of Innsbruck in 1930. Today Vietoris is known for, among other things, his contri-

[6] Binder 2003.

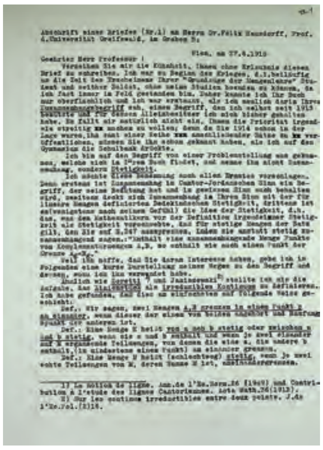
[7] Cf. Einhorn 1985, pp. 7–8.

[8] As pointed out to the author by Peter Goller.

[9] The Rockefeller Foundation funded several European mathematicians, esp. during the 1920s. For a historical analysis of its activities within the field of mathematics see Siegmund-Schultze 2001.

butions to the field of topology as well as his work on probability theory, which he only took up in the 1980s.[10] Together with Wolfgang Gröbner, who would join him after the war, they would both shape the Institute of Mathematics at the University of Innsbruck in the second half of the 20th century.

Leopold Vietoris' papers contain his correspondence from 1945 to 2001, which comes alphabetically ordered in several boxes, as well as scientific manuscripts, several offprints and teaching material on differential geometry, logic and set theory, topology and analytic geometry, among others. One of the highlights of the collection is surely a letter to Felix Hausdorff; the collection also contains Vietoris' personal copy of Hausdorff's *Grundzüge der Mengenlehre*, first published in 1914.[11]



First page of a letter from Vietoris to Felix Hausdorff, June 27, 1918. (Innsbruck University Archive)

The material can be consulted at the University Archives, located in the University's main building near the city centre of Innsbruck. The staff of the Archive, especially its director Peter Goller, are very supportive and strive to facilitate a pleasant working experience. It should also be mentioned that Peter Goller, because of his rich and comprehensive knowledge of the history of the University of Innsbruck, can provide a good deal of additional helpful and illuminating information, missing links and advice for research on the Gröbner and Vietoris collections.

The history of mathematics in Austria is naturally closely connected to the history of the University of Vienna, which celebrated its 650th anniversary in 2015 and is thus one of the oldest academic institutions in the German-speaking countries. The University of Vienna was a stage in the career paths of many Austrian mathematicians and the Mathematical Seminar functioned as the central point of the discipline, not only because the *Österreichische Mathematische Gesellschaft* (Austrian Mathematical Society) was founded there but also because the *Monatshefte für Mathematik und Physik* were edited and published in Vienna. Unfortunately though, collections as extensive as the Gröbner and Vietoris bequests are not available at the University Archives. Scholars can, however, turn to several other historical sources on Austrian mathematicians. The archives of the University of Vienna and the Technical University of Vienna contain staff records of mathematicians who either held a position at the faculty or obtained their *venia legendi* (awarded after successfully completing the *habilitation*) there. Although the amount of material contained in these records does vary, all in all they contain a good deal of information on such distinguished figures in the history of Austrian mathematics as Wilhelm Wirtinger, Philipp Furtwängler, Hans Hahn, Kurt Gödel, Karl Menger, Eduard Helly, Alfred Tauber and Johann Radon, to name just a few.

The history of mathematics at the University of Vienna is of particular interest not only to practitioners of the field but to anyone interested in the history of science, culture and society in the turbulent decades following the fin-de-siècle.

Strikingly, for some time during the first half of the 20th century, mathematicians and philosophers came into closer contact with one another in Vienna. Although this is by no means an exclusive characteristic of the Viennese tradition – after all, the developments in mathematical logic and set theory as well as the well-known foundational crisis in mathematics after the First World War brought with them an increase in philosophical interest in mathematics almost everywhere in Europe – Vienna around the 1920s was the place where the Vienna Circle, a discussion group of philosophers, physicists and mathematicians, was formed. The outcome of the work of this group, the philosophical position of Logical Empiricism, became one of the most influential schools of philosophy after its founders emigrated to the United States following the political turmoil of the 1930s in Europe. One of the founding members of the Vienna Circle was the mathematician Hans Hahn (1879–1934), who held one of the three chairs of mathematics at the University of Vienna at that time, his colleagues being the aforementioned Wilhelm Wirtinger and Philipp Furtwängler.

Hans Hahn, date unkown. (Vienna University Archive, 106.I.43.)

Hahn played a quintessential role in the history of science and mathematics as he made important contributions to modern analysis and is counted among the creators of functional analysis.[12] But Hahn was not only a distinguished mathematician and promoter of a scientific world conception through the founding of the Vienna Circle, but also teacher and sponsor of two exceptionally talented students in the 1920s: Karl Menger (1902–1985) and Kurt Gödel (1906–1978).

The Archives of the University of Vienna contain Gödel's staff records, which allow some insight into the rather complex process of his emigration to the United States after the Nazis' seizure of power in Austria.[13]

[10] Reitberger 2002.

[11] A reproduction of Vietoris' letter as well as Hausdorff's reply is available online at the website of the Archives of the University of Innsbruck: https://www.uibk.ac.at/universitaetsarchiv/copy_of_dokumente-aus-dem-universitaetsarchiv/.

[12] Sigmund 2015.

[13] Vide Sigmund et al. 2006.

However, in the following I want to focus on Karl Menger, since in addition to his staff records, the University Archives also hold a few of his personal papers. This collection was originally given to the Technical University of Vienna by Menger's daughter Rosemarie Menger-Gilmore, but was transferred to the University of Vienna Archives in 2012, where it is now stored. As already indicated, these records do not contain a very large amount of material but there are two items which should be of general interest.

Menger is an important figure in the history of Austrian mathematics, as well as in the broader cultural history of Vienna. He was the son of the economist Carl Menger, who is famous for his groundbreaking work in economics, especially on the development of the theory of marginal utility. He also worked on the methodology of economics as a branch of the theoretical social sciences, where he developed an account that would emphasise its dissociation from the so-called historical school of economics prevailing at that time. The works of Carl Menger are also widely regarded as essential in the formation of the Austrian school of economics.

His son Karl studied mathematics, physics and philosophy at the University of Vienna and received his doctorate in mathematics in 1924 under the supervision of Hans Hahn, who seems to have had considerable impact on the young Menger. In his *Reminiscences of the Vienna Circle and the Mathematical Colloquium*, Menger recalls his first impressions of attending a seminar with Hahn:

> *In March 1921, just after Hahn's arrival in Vienna, I read a notice that he would conduct a two-hour advanced seminar every Wednesday during spring semester, entitled Neueres über den Kurvenbegriff (News about the concept of curves). […] Hahn went right to the heart of the problem. Everyone, he began, has an intuitive idea of curves […]. But anyone who would try to make that idea precise, Hahn said, would encounter great difficulties.*[14]

Hahn later generalised this observation of the difficulties involved in specifying the intuitive notion of curve to an extensive criticism of the concept of intuition in the epistemology of mathematics. His paper titled *Die Krise der Anschauung* (The Crisis of Intuition) was first published in a series entitled *Krise und Neuaufbau in den exakten Wissenschaften* (Crisis and Rebuilding of the Exact Sciences) co-edited by Hahn, Menger, the physicist Hans Thirring and chemist Hermann Mark.[15] Remarks Hahn had made more than ten years prior to this extensive discussion seem to have had a stunning effect on Menger:

> *I was completely enthralled; and when after that short introduction, Hahn set out to develop the principal tools used in those earlier attempts – the basic concepts of Cantor's point set theory, all totally new to me – I followed with the utmost attention. […] I left the seminar room in a daze. Like everyone else, I used the word 'curve' and had an intuitive idea of curves – mental pictures associated with the term. Should I not be able spell out [sic!] articulately how I used the word and to describe clearly what I saw?* [16]

This far-reaching experience set the stage for Menger's work on the theory of dimension and curves, which he began immediately after having encountered the problem in Hahn's seminar. A typescript of Menger's work on the definition of the concepts of curve and dimension from 1921 is part of his personal papers held at the Archives of the University of Vienna. It is the very same document that Menger himself deposited at the Austrian Academy of Sciences in 1923. The collection also contains a notification concerning the reopening of the sealed documents eight years later in 1929.

Knowing Hans Hahn was also instrumental in getting Menger to attend the weekly meetings of the Vienna Circle, where especially the philosopher-physicist Moritz Schlick and the philosopher-logician Rudolf Carnap expressed interest in discussing Menger's work on curves and dimension. His frequent participation in these meetings would soon lead to him distancing himself from this group, although he never broke entirely with the Circle. Instead he remained at its periphery and started his own discussion group, the *Mathematisches Kolloquium* (Mathematical Colloquium), whose results were published in a separate periodical called the *Ergebnisse eines Mathematischen Kolloqiums* (Results of a Mathematical Colloquium), which was co-edited by Kurt Gödel, who has already been mentioned, and Georg Nöbeling, who studied with Menger and received his doctorate under his supervision in 1931.[17]

Mathematicians interested in logic and model theory may find it interesting that it was also Menger who invited Alfred Tarski to Vienna to participate in the *Mathematische Kolloquium* in the late 1920s. But the scope of mathematical fields discussed in the *Kolloquium* was much broader. Regular participants included Abraham Wald, who made significant contributions to mathematical statistics and econometrics, and Olga Taussky-Todd who, after receiving her doctorate from Philipp Furtwängler, gained an excellent reputation for her work in algebraic number theory. The example of Wald and Taussky-Todd also shows that Menger's role in organising the *Kolloquium* did not stop at providing a platform for mathematical discussion but also involved active financial support. Together with Hahn, Menger organised a series of scientific lectures for the general public in

[14] Menger 1994, pp. 38–39.

[15] Mark et al. 1933. For a reprint of Hahn's contribution see Hahn 1980.

[16] Menger 1994, pp. 39–41.

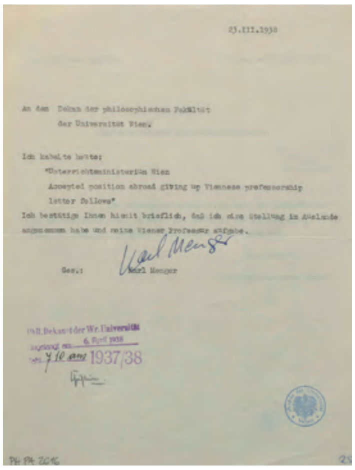
[17] On the history of Menger's relation to the Vienna Circle and the formation of the Mathematical Colloquium see Stadler 2015a, pp. 201ff (for an English version see Stadler 2015b). The papers of the *Ergebnisse eines Mathematischen Kolloquiums* were edited and published by Egbert Dierker and Karl Sigmund in a comprehensive volume, see Dierker und Sigmund 1998.

Vienna, which received much attention at that time. The money collected from the entrance fees to these talks was used to support scholars like Taussky-Todd and Wald.[18]

However, before all that took place, Menger left Vienna in 1925 to go to Amsterdam and continue his work with L.E.J. Brouwer, having been awarded funding from the Rockefeller Foundation. Just about a year later, Menger earned his habilitation with a work on the theory of curves. Although the Amsterdam period was apparently stimulating and productive for Menger, he got into a serious conflict with Brouwer just a short time later, after Brouwer had raised grave accusations against Menger over the issue of priority in the development of the theory of dimension. The dispute that followed lasted several years and in 1930, Menger saw himself forced to directly reply to Brouwer's accusations in volume 37 of the *Monatshefte für Mathematik und Physik*,[19] the principal journal of Austrian mathematics founded in 1890 by Gustav von Escherich and Emil Weyr. Menger's personal papers at the University Archives also contain traces of this dispute, which take the form of a manuscript detailing different versions of a reply to Brouwer from about the late 1920s.

After the quarrels with Brouwer, Menger left Amsteram for good and returned to Vienna to take over the chair for geometry at the University that had previously been held by Kurt Reidemeister. In his *Reminiscences of the Vienna Circle and the Mathematical Colloquium*, Menger recalls: "The Vienna of 1927 had again become a highly interesting and intellectually lively city."[20]

These words stand witness for a vibrant and creative intellectual atmosphere in Vienna, even in the late 1920s – an atmosphere that, already frail at that time, would soon collapse as the city entered in 1930 into the new decade which would see two regime changes in Austria. Disaffected by the increasingly repressive climate and the murder of Moritz Schlick in 1936, Menger decided to leave Austria for the United States and started teaching at Notre Dame in 1937. When the Nazis seized power in Austria in 1938, Menger formally resigned from his position at the University of Vienna. The ministerial decree effectuating his dismissal followed three months later.


23.III.1938

An den Dekan der philosophischen Fakultät
der Universität Wien.

Ich kabelte heute:
"Unterrichtsministerium Wien
Accepted position abroad giving up Viennese professorship
letter follows"
Ich bestätige Ihnen hiemit brieflich, daß ich eine Stellung im Auslande angenommen habe und meine Wiener Professur aufgebe.

Gez.: Karl Menger

Phil. Dekanat der Wr. Universität
eingelangt am: 6. April 1938
1937/38

PH PA 2616

Menger's letter of resignation to the dean of the Faculty of Philosophy of the University of Vienna, March 23, 1938. (Vienna University Archive, PH PA 2616)

Menger was not the only mathematician who lost his position because of the racist political agenda of the new regime. In addition to Menger, eminent mathematicians such as Kurt Gödel, Eduard Helly and Alfred Tauber were also affected.[21]

The fate of Alfred Tauber (1866-1942) is particularly tragic as he was not able to emigrate after 1938 and was subsequently deported to the Theresienstadt concentration camp, where he died in 1942.[22]

Tauber had made significant contributions to analysis: the concept of Tauberian theorems in particular originates in his work. Nevertheless, he had been unable to find a suitable academic position in Vienna, and his main teaching activity was the introductory courses in insurance mathematics. His principal income came from a position as a mathematician in an Austrian insurance company, a fate he shared with Eduard Helly.[23]

Alfred Tauber, date unknown. (Vienna University Archive, 106.I.91)

Karl Menger did not return to Europe after 1945, although this option was considered by the University of Vienna after the war. However, it seems that several objections were raised against the suggestions to reinstall Menger in his position. One of them was voiced by natural scientists – especially in the field of mineralogy – on the grounds that the lectures in geometry should be taught as "close to intuition" as possible.[24] This was just 25 years after Menger's efforts to specify the intuitive notion of curve were initiated in Hahn's seminar.

Additional material on Menger's professional biography can be found in his staff records at the University Archives.

Finally, I want to point out that in addition to archival material there is also a significant body of published literature which is an excellent source for the history of Austrian mathematics. Thus I would like to briefly turn the reader's attention to the aforementioned *Monatshefte für Mathematik und Physik*, which were published until 1944 and functioned as the main communication platform for the discipline in Austria. The papers in this journal provide a unique perspective on the development of mathematics in Austria as well as its intellectual community. The *Monatshefte* continued to appear after the war and are still published today under the title *Monatshefte für Mathematik*. All volumes of the *Monatshefte* are fully digitized and can thus be easily explored through the respective databases (to which university libraries will usually provide access). In addition to original papers, the *Monatshefte* also contained extensive literature reviews, which provide a glimpse into the exchange and reception

[19] According to Einhorn, based on an oral report from the Austrian mathematician Edmund Hlawka in Einhorn 1985, p. 184. Vide also the section on Olga Taussky-Todd in Sigmund 2001.
[19] Menger 1930.
[20] Menger 1994, p. 9.

[21] Reiter 2001.
[22] Binder 1984, Sigmund 2015.
[23] Sigmund 2004.
[24] Cf. Einhorn 1985, pp. 185–186.

of ideas within the broader intellectual network of which Austria was just one part. The inventory of the University Archives can be searched through an online Archive Information System.[25] However, this only provides an overview of the items since the online system is not complete. Additional search tools are available on site in the archive's reading room. Also, the staff at the University Archives are very helpful in handling research requests and provide excellent and friendly support to visitors.

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Appendix

Selection of personal papers at the Archives of the University of Innsbruck (UAI)
- Gröbner, Wolfgang
- Vietoris, Leopold

Selection of staff records at the Archive of the University of Vienna (UAW)
- Furtwängler, Philipp
- Gödel, Kurt
- Gröbner, Wolfgang
- Hahn, Hans
- Helly, Eduard
- Menger, Karl
- Radon, Johann
- Tauber, Alfred
- Wirtinger, Wilhelm

Additional Material at the Archive of the University of Vienna
- Menger, Karl, Nachlassfragment

Robert Frühstückl recently obtained his doctorate in the history of science from the University of Vienna with his work on mathematical institutions in Vienna between 1930 and 1945. His studies were based on the history and philosophy of science esp. the Vienna Circle and the history of science in National Socialism. In 2016 he was visiting pre-doctoral research fellow at the Max-Planck-Institute for the History of Science in Berlin.

[19] https://scopeq.cc.univie.ac.at/Query/suchinfo.aspx

The Importance of Ethics in Mathematics

Maurice Chiodo (King's College, Cambridge, UK) and Toby Clifton (Fitzwilliam College, Cambridge, UK)

Mathematics is useful because we can find things to do with it. With this utility ethical issues arise relating to how mathematics impacts the world. Now more than ever, we mathematicians need to be aware of these, as our mathematics, and our students, are changing society. In the first of a two-part series on Ethics in Mathematics, we address why, as mathematicians, we need to consider the ethics of what we do.

Mathematics and the world

We study one of the most abstract areas of human knowledge: mathematics, the pursuit of absolute truth. It has unquestionable authority. But, in some sense, absolute truths have absolutely no meaning. The statement "2+3=5" is an absolute truth, but what does it mean? Its meaning and utility are added later when people who understand the statement reconcile it with the physical world. It is the mathematically trained who interpret and apply mathematics to the real world and thereby assign it meaning; through this it becomes useful.

Indeed, it is clear that mathematics is one of the most useful and refined tools ever developed. When something is useful, however, it can often also be harmful; this can be either through deliberate misuse or ignorance. The humble knife provides an illustration of the principle; in order to use such a tool responsibly, one must be made aware – often by those who first introduce one to it – of the potential dangers. If a primitive tool like a knife can be so useful and harmful at the same time, then what is mathematics capable of? Mathematics has many more applications, and by the same reasoning must also have a greater potential to do ill. As mathematicians we are seldom warned of this. Other disciplines such as law, medicine and engineering have, for a long time, addressed the potential for harm within their field. We, as the practitioners and wielders of mathematics, need to be similarly aware, and adjust our actions accordingly, otherwise we can, and sometimes do, cause harm with our work. But how could mathematics possibly be harmful, and what exactly might this harm be?

In this article our emphasis is on the experiences of pure mathematicians, although our arguments apply equally to applied mathematicians, statisticians and computer scientists. Many of us (although certainly not all) are motivated to study mathematics by its beauty and intrinsic interest rather than its applications in science and industry. It's as though we are studying a form of abstract art; far from real-world impact or considerations, and only fully appreciated by a small number. Despite all this, government and industry pay for our work; one suspects they don't just do this for the sake of our intellectual stimulation. If our work is completely abstract and detached – an art form, so to speak – then shouldn't we seek funding from those who fund abstract art? So what might be the value that science councils and industry see in what we do? It is not only the mathematical results that we produce, but also the mathematicians we train. Our mathematics makes a difference, and our students go out and do real things with their training. If it is the case that our work is being funded because it has perceived impact, then surely we should query and understand why we are being paid to do it.

There is already much discussion of ethics in the mathematical community. However, these discussions usually focus exclusively on issues *within* the community. These are important and many of us are already familiar with them: from improving diversity and inclusivity, to widening participation in mathematics, to addressing instances of plagiarism and publishing irregularities. These are pressing concerns. Every discipline engages with such *intrinsic* ethical issues. These, however, are not our focus in this article. Mathematics is one of the few disciplines that fails to address *extrinsic* ethical issues; those concerning how the community impacts wider society. This particularly includes ethical implications relating to the applications of mathematics and work of mathematicians. It is these extrinsic ethical issues that we are trying to raise awareness of. Our concern is not so much that mathematicians are deliberately malign, but instead that they fail to recognise these extrinsic ethical issues. Indeed, most of the mathematicians we have come across would baulk at the thought of acting unethically; the problem, instead, is that many do not recognise that mathematical work *can* have such an effect.

Some case studies

Having recognised that mathematics is useful because it can be applied, and that with these applications come extrinsic ethical issues, we now consider two concrete examples: the global financial crisis (GFC) of 2007–8 and targeted advertising.

The GFC was one of the defining events that shaped the modern global economy. Its repercussions have been felt around the world, with many suffering a decline in living standards. The causes of the GFC are complex; however, there is consensus that mathematical work played a vital role. An important factor is thought to have been the misuse of Collateralised Debt Obligations (CDOs). These saw mathematicians pool large collections of interest-bearing assets (mostly mortgages), then 'cut the pool into pieces' to form a collection of interest-bearing products. Mathematically these products had less overall risk and thus higher value than the original assets. They were traded wildly. The mathematics behind their construction is highly non-trivial, requiring stochastic calculus, differential equations, etc. Research mathematicians, beginning with the work of Black and Scholes, and later Li, derived a model and pricing formula for CDOs. Though it took a deep understanding of mathematics to derive these models, only a more superficial understanding (at the undergraduate level) was required to apply and to trade them. As a result, their users may not have fully appreciated their limitations or inner workings. Mathematics – which by itself is sure and certain – seemed to explain their value, and so most were happy. Unfortunately, some of the assumptions did not hold. For example, the model assumed there wasn't tail dependence in the default risk of underlying assets, but there was; for instance when two mortgaged houses were on the same street. In the end, the risk was not properly accounted for, and when house prices declined it led to the write-down of $700 billion of CDO value from 2007 to 2008. The rest is history.

Our next example is targeted advertising. Adverts have always been placed so as to catch the eye of their desired audience. However, now that people possess portable internet-connected devices and social media accounts, it has become possible to target adverts at the individual level. Nowadays, these can be tuned to fit very specific demographics, and as such it's now possible to specify who *doesn't* see an advert. This allows advertising campaigns that are selective, contain adverts that contradict each other, and that are impossible to externally scrutinise. In short, adverts can now be used to manipulate individual people. This becomes particularly dangerous when applied to political advertising. Using large data sets obtained through social media it is possible to profile the political persuasions and preferences of an individual. Machine learning has become the main tool of the trade here, and it is the mathematically trained doing it [3]. These adverts can even deceive by appearing non-partisan. For instance, one can send an advert saying "Voting is important; make sure you vote" only to those who might be inclined to vote for your party.

Whatever the strategy, these types of adverts are increasingly prevalent, and it is thought that such tactics influenced the 2016 US election and UK referendum on EU membership. It is we mathematicians who make all this possible. Cambridge Analytica, one of the organisations alleged to have been involved in such advertising, had a small team of no more than 100 data scientists [4], some of whom were trained mathematicians. Regardless of one's political persuasion, it is clear that this sort of work is deceptive and dangerous, and that mathematicians are enabling it. Ultimately, it is mathematicians who make up part of the teams specifying how such targeting works and carrying it out.

The impact of mathematicians

As a result of the pace and scale at which modern technology operates, through use of internet connectivity and readily-available fast computation, the consequences of the actions of mathematicians are more quickly realised and far-reaching than ever before. A mathematician in a big tech company can modify an algorithm, and then have it deployed almost immediately over a user base of possibly billions of people. Even on a smaller scale, we have seen that a small number of mathematicians, despite limited resources, can have a vast impact on the world; targeted advertising exemplifies this.

If you model a physical system, such as gravity, then your model is falsifiable. If the model does not accurately reflect the physical system, then on application it clearly fails – your rocket doesn't launch properly. You know when a model was good because the rocket makes it to the moon and back. Modelling a financial system is more difficult, as the system is affected by the application of the model. A pricing algorithm, if widely used to buy or sell a product, influences the market for the product in question. How does a model model its own impact?

So now what happens if you are modelling the future behaviour of people by predicting something like: 'How likely is a particular individual charged with a crime to reoffend with a serious offence, a non-serious offence, or not reoffend, in the next 24 months?' Furthermore, what if that is being used to determine what prosecution and sentencing mechanisms are applied to that person?[1] If you predict that a person will reoffend seriously in 24 months, and they don't (after being released or acquitted), then you might observe that. But what if they are found guilty and sentenced to 25 months with the choice of judicial process based on your prediction? How do you test whether your prediction was correct? Now we have a serious ethical issue: we are using mathematical reasoning to make decisions about people that impact their lives, and in many of these cases we can never know whether the decisions made were desirable or appropriate. Is it right to use mathematics in such a way without careful reflection?

We now face an ethical dilemma. Do we limit ourselves to falsifiable claims, or do we allow ourselves to make claims, make decisions and initiate actions that are unfalsifiable? We are of course entitled to do the latter; however, we should then bear in mind that we have lost mathematical certainty. Furthermore, if we do this, we should broaden our perspective and training so that we can incorporate as many aspects of society as possible.

Concerns for the future

So what is on the horizon for mathematicians? Is it sufficient to simply look at the above list of cases and avoid those specific actions or industries entirely? Unfortunately not; new mathematics produces new ethical issues every day. Such a future example may lie in alternative credit scoring. This is starting to be done by new companies who lack access to standard datasets that established credit-scoring agencies have (such as financial records, bill payment history, etc.). They instead use different datasets such as social media profiles, in some cases requesting full access to social media accounts by asking

[1] The Harm Assessment Risk Tool (HART) developed by the Durham Constabulary, which uses random forest machine learning, is used to make such predictions, and then determine if an accused criminal is to be offered the opportunity of going through the *Checkpoint* program (tinyurl.com/y4vxrd77) which is an alternative to criminal prosecution aimed at reducing re-offending.

for login credentials [2]. While this sounds undesirable to the point that most people will not be interested, it must be remembered that some people will be sufficiently desperate for credit so that there will always be some takers. These companies scrape an applicant's social media looking for actions they perceive to reflect creditworthiness. These could include places the person visits, the hours they sleep, the 'quality' of the friends they have and so on. This approach is unfalsifiable, lacks proper regulation and has the potential to harm society since the extension of credit is a mechanism of social mobility. If such a process, one that is enabled by mathematically trained people, starts having negative impact, who is accountable? Ultimately, we must live in the world that we and our students create, and we must ponder whether there is a sense in which we are partially responsible.

Do these ethical issues arise in academia?

But what about mathematicians working in academia: are any of these ethical issues relevant to them? Consider a pure mathematician, a number theorist, say. Suppose they develop an algorithm for fast factorisation. Should they publish it? If so, when, where and how? If not, what should they do? Should they have thought about it beforehand? We have asked many mathematicians this exact question, and a typical response is: "I would publish it on arXiv immediately. It's my right to publish whatever mathematical work I do." (Not all mathematicians give such a response, but many do.) When pressed on the consequences of publishing such an algorithm in that way – for instance the breaking of RSA encryption in a chaotic manner and the ensuing collapse of internet commerce and the global economy that would follow – one explained: "Well, it's their fault for using RSA. It's not my problem." Of course, responsible disclosure is a complicated topic, and one that is heavily debated by security researchers. But with an example like this, ethics has crept into the world of the pure mathematics researcher in academia.

If an area as abstract as number theory is not 'safe' from ethical considerations, is there any mathematical work that is? Can a pure mathematician hide from ethical issues in academia? What about a statistician, or an applied mathematician? Or do ethical questions arise for all mathematicians regardless of where we do our mathematical work?

Why management can't guide us

Some mathematicians (academic or industrial) may think that, since they are not directly involved in the application of their work, they need not consider its extrinsic ethical implications. After all, we just do the maths, and so it's 'not our problem'. This oft-held belief is generally associated with the perception that there are people and structures above us (managers, supervisors, advisory boards, etc.) who will intervene to prevent us from doing anything that we ought not to. *We* work on the abstract problems, *they* worry about why. But can we rely on management to do this effectively? Will they vet our work to ensure that its use is aligned with the values of society? At each stage of separation from mathematical work some understanding of it is lost. It is difficult for a manager to understand *all* of the mathematical work we do and its limitations when applied and used. It is the nature of management that managers will only have partial knowledge of the work being done. There would be no point in a manager reproducing the work of all of the people under them, and mathematics is such that if you don't 'do it for yourself' then there is a chance you may not fully understand it. Given this fact, there is always an onus on the individual mathematician to consider the ethical implications of what is being done. Of course, it must also be considered that managers might have other values, perhaps more aligned with the objectives of the organisation than of wider society. We should understand and anticipate this.

Some managers may go so far as to try to manipulate us. For instance, if we voice objection at what we have been asked to do, they may try to quash it with the classic argument: "If you don't do it, then someone else will". At a first glance, this seems convincing; however, it fails on two counts when referring to mathematical work. First, there are not that many mathematicians in the world. We possess a unique set of skills and abilities, and it requires years of training to produce a good mathematician, even when starting with someone who has the right interests and reflexes. Given the scarcity of mathematicians, this argument fails in practice. Moreover, as mathematicians we understand its contrapositive; the original statement is equivalent to: "If no one else does it, then you will". This is, of course, absurd. The argument has as the implicit underlying assumption that the task being requested will definitely be completed. If no one else builds me a nuclear bomb, then will you? What we should really be considering here is the argument: "If you don't do it, then someone else *might*". True, someone else might, but they may not be easy to find, or even exist at all. Now the power of meaningful objection has returned to the mathematician. Whether you choose to take the pragmatic perspective that there are not many mathematicians or the logical perspective of the contrapositive, your objection means something. Some mathematicians take this idea even further, and take a conscious decision to take a seat at the table of power, effecting positive change from the managerial level. This happens in various areas: in academia, in industry and even in politics. This is discussed in more detail in [1] as 'the third level of ethical engagement'.

Why the law can't guide us

The problem extends beyond management. We may think that the law provides a clear description of what is and is not acceptable to society, and thereby presumably what is and is not ethical. However, this misses the point for several reasons. Firstly, the law is not an axiomatised system; it is interpreted by courts rather than by machines. This is a type of system with whose details mathematicians are generally not familiar. Furthermore, there is the problem that the law will always lag behind technological development; we cannot expect lawmakers to have done our mathematics before we do it ourselves. Additionally, the processes by which laws are made are (deliberately) slow, requiring public consultation, votes and implementation periods. Consider the case of the General Data Protection Regulation (GDPR). It started to be written in 2011, only came in to effect in 2018, and is thought by many to be already out of date. Finally, it can be the case that lawmakers lack a full understanding of the fine details of the subject at hand. For example, a member of the UK Science and Technology Select Committee, Stephen Metcalfe, declared at a public outreach event that "one solution to algorithmic bias is the use of algorithms to check algorithms, and the use of algorithms to check training data". Ultimately, the law is not there to serve as moral advice; there are plenty of immoral things one can do that do not break any laws. As such, it is not well suited as a source of ethical advice.

Thus, if we can't rely on management and we can't rely on lawmakers and regulators, then who can we rely on? The answer is as obvious as it is difficult to admit: ourselves. The only way mathematicians can try to prevent their work from being used to do harm is if they think about it themselves. No one else can, so we must.

A growing awareness of Ethics in Mathematics

Awareness that mathematicians need to consider extrinsic ethical issues is building in the community. In 2018 the head of mathematics at Oxford, Professor Michael Giles, commented at a panel discussion event: "Cambridge Analytica is interesting from one point of view in that, if you'd asked me 20 years ago whether mathematicians at the PhD level needed to be exposed to ideas of ethics, I would have said 'Clearly, that is irrelevant to mathematicians'. Now I really think that this is something we have to think about. In the same way that engineers have courses looking at 'What it means to be a professional engineer', and 'Ethics, and your responsibilities as an engineer', I think that is something that we have to think about as mathematicians now." Moreover, arxiv.org is currently revising the description of their mathematics tag *History and Overview* to include "Ethics in Mathematics" as a sub-category.

As part of their formal training, few mathematicians have ever been told about extrinsic ethics before. Previous generations of mathematicians have evaded this crucial point, and in the process have possibly let society down. It rests on the current and upcoming generations to pick up this idea before it's too late.

Mathematicians always take a generation or more to accept a new and fundamental idea about the nature of their subject; debates about the admissibility of zero as a number provide such an example. We're at a similar juncture again. Now some say: "Surely there's no use in considering ethical issues in mathematics", but by the time our students are professors and industry leaders, they may well be saying: "*Of course* we should be considering ethical issues in mathematics!" But why hasn't the mathematical community taken this on board already? Why wasn't this done 100 years ago, by the likes of Gödel and Russell? Two reasons come to mind. Firstly, the dangers were less proximate, since much of today's technology simply did not exist. Secondly, every mathematics undergraduate was already exposed to philosophy, as it formed part of every university education. Thus, exposure to Ethics in Mathematics, in its own right, was less urgently needed.

So if Ethics in Mathematics has become so important to mathematicians, then how might we teach it to them in a relevant and useful way without foisting an entire philosophy degree upon them? Disciplines such as law, medicine and engineering have long taught their undergraduate students about extrinsic ethics in their respective fields. In the following article of this *Newsletter* we'll further explore why such teaching has not yet occurred in mathematics, and outline how one might go about giving such directed teaching of Ethics in Mathematics (EiM).

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Teaching Ethics in Mathematics

Maurice Chiodo (King's College, Cambridge, UK) and Piers Bursill-Hall (The University of Cambridge, UK)

In the previous article in this issue of the *Newsletter* we addressed why mathematicians should consider the ethics of what they do. Here we outline, based on our experiences, three key elements for *teaching* Ethics in Mathematics (EiM): (1) a lecture series on ethical issues in mathematics; (2) exercises with an ethical component in problem sheets of other maths courses; and (3) a supportive environment so students perceive value in this teaching.

Why is this an issue now?

While some may argue that mathematicians will inevitably develop ethical skills when they begin to work in industry, we would like to suggest (see [2]) that the mathematical community actively encourages mathematical professionals to either regard mathematics as beyond ethics (Platonism), or that social and ethical consequences are just "not a mathematician's problem" (exceptionalism). Compare this to law, medicine, physical sciences; etc., which all teach profession-specific ethics. We suggest that, because mathematicians engage in sophisticated technical work which lies well above the level of public scrutiny, they should be actively trained to deepen the awareness of their social and ethical responsibilities (see [1]). But why hasn't it been done before?

Firstly, until the middle of the 20th century most people studying mathematics at post-secondary institutions in the West also received a robust training in philosophy, and thus were equipped with enough philosophical and ethical literacy to deal with professional, ethical questions. The second reason is more profound and lies in the newfound immediacy of the work of mathematicians. Until recently there was a genuine separation between people who did abstract mathematics (mathematicians), and those who applied such work in the physical world (physicists and engineers). There was a discernible professional and temporal gap between those who produced new theorems, and those applying them decades, even centuries, later. This reduced the appearance of ethical responsibility of mathematicians and gave everyone more time to consider the ethical issues. However, the digital revolution has reduced this gap.

The amount of pure mathematics used in finance, surveillance, big data and decision-making systems is vast and growing rapidly. Mathematics has not only become their foundation, but it is being 'brought to market' and has a potential social impact within months or even weeks. The power of new mathematics in ethically-laden industries means the professional and temporal gap between its creation and its application has reduced so much that the ethical consequences of mathematical work cannot be obscured or blamed on someone else. For the *first time ever*, mathematicians are *uniquely* responsible for the immediate social consequences of their work.

Constructing a course in EiM

Teaching Ethics in Mathematics (EiM) turns out to be non-trivial. Since no one has done it before, there is no body of precedent, resources, textbooks or lecture notes from others to build on.[1] Introducing it as an undergraduate course is necessarily a complex process: its ideas are new to your university, it is unlike theorem-based courses, and sometimes it is hard to argue that EiM should supplant any given traditional maths course, as teaching time and resources are already limited. What follows is only indicative, based on our experiences and conversations with colleagues around the UK and elsewhere in Europe. There is no unique or established way to teach EiM, and you will need to tailor the discussions here to your situation. Treat what follows as suggestions, not rules. We have introduced EiM in Cambridge as an informal, non-examinable course (of which there are already well-attended examples in our Faculty). Students were used to this format, but it might not be the right model for your institution; seminars, compulsory modules or project courses might work better.

It may be best – and easiest – to introduce EiM in a slow, evolutionary way, starting with 1–2 lunchtime seminars before developing it further. It helps your colleagues to get used to the idea, and if it proves to be popular, this may provide its own rationale for the course to be accepted into the curriculum. You will need to consider your audience carefully. Are your students studying mostly maths, or maths with physics or computer science? Different allied disciplines will suggest different case studies.

Teaching EiM is quite different from ethics courses in other disciplines. They do not follow the same exceptionalism and are already aware of the existence of ethical issues. Unfortunately, as mathematicians, we do not have this luxury. Indeed, you probably need to assume that most of your audience does not initially and intuitively accept the premise that there are ethical issues in mathematics. Thus, we strongly suggest not starting an EiM course with generic philosophical discussions on ethics because you can lose your audience as a result. We have found that most of our students are generally not receptive to the conceptual structures, language and approach of 'real' philosophy; what philosophers talk about is not always easily translated to an undergraduate mathematician. Hence, we strongly recommend resisting the temptation to ask a philosopher to teach this. Students need to see profession-specific ethical issues and discussions in a familiar language. Of course, engage with philosophers and ethicists to help design your course, and go to other disciplines (law, social sciences, engineering, etc.) to

[1] We have constructed a website ethics.maths.cam.ac.uk hosting resources for anyone who wants to construct an EiM course.

get advice and insight. However, we believe that such a course needs to be delivered by mathematicians even if they are not professionally trained ethicists, just like linear algebra lecturers need not be experts in algebra. You probably know more about it than your students, and you speak the same professional language.

A natural structure for such a course would be to split it into two sections: "*There exist* some ethical issues in mathematics", and then "*For all* mathematics that we do, there are ethical issues". It may seem pedestrian, but an array of case studies prove their existence. To get students to appreciate it you only need to present explicit and varied examples of work that mathematicians have done which have raised ethical issues. Your audience must reach the point where they accept that there are indeed ethical issues in *all* branches of mathematics. Giving just one example may lead them to think that it was a one-off. You can find a long list of case studies at ethics.maths.cam.ac.uk/cases; such mathematicians were probably not deliberately acting maliciously, but instead overlooked ethical consequences. It is important to emphasise this; teaching EiM should not be a platform for criticising others, or you risk putting your students on the defensive.

Now you are ready to move to the second stage: "*For all* mathematics that we do, there are ethical issues". Such generalisation is harder to accept. Students may think that "there are places X, Y, Z where mathematicians might do unethical things, so if I just avoid those, I'm safe." We have had this reaction from our students regularly. You need to dispel this and show them that there is nowhere to hide, not even in academia. Obviously *for all* statements cannot be proved by example but require more profound arguments. These can build on the lack of sufficient 'external' control mechanisms (weak regulation) and on the fact that mathematicians are trained and encouraged to strip away non-mathematical aspects of problems (which inevitably leads to issues). It can also include more social aspects such as there are people who will deliberately set out to exploit others and their labour, playing on their unwillingness to think about ethical consequences. Your students are about to enter an industrial economy which is set up and organised to work in ways that can obscure the ethical context and can enable moral disengagement.

This is your 'proof for ethics'. No matter how supposedly pure your (mathematical) work is, someone is inevitably paying you to do this work for their interests. When working mathematicians ask: Who is paying me? Why are they paying me? How will they use my work? How will they use me?[2]. …then an ethical self-examination has begun.

The course we give

Our course involves 20 contact hours, divided between lectures, interactive exercises and often lively and challenging discussions. We recommend you encourage interaction so students explore and digest these new ideas. Discussions are useful and necessary to develop an ethical understanding. We cover eight topics. The first half illustrate the existence of ethics in mathematics before we move on to argue for its universality, where we have found ourselves appealing to other disciplines (psychology, law, social science) to understand the inner workings of mathematical community and its interactions with the world. On many occasions these extra-mathematical observations proved to be the most interesting and persuasive for our audience. Descriptions, and a recording, of our lectures, are available at ethics.maths.cam.ac.uk/course/lectures.

In our lectures we cover the following.

(1) *Introduction to EiM*.
All mathematics is done in a social context. It sits at the heart of technological advancement and industrial progress. Understanding that it can be used for good, and ill, is the first step to ethical awareness.

(2) *Mathematics and modelling.*
Mathematical models are necessary to understand the world. We draw on examples from fields such as finance to teach the process of modelling and its limitations. The global financial crisis demonstrates that poorly understanding models can have devastating consequences.

(3) *Cryptography, surveillance and privacy.*
Mathematicians can enable the infringement of privacy by breaking strong encryption, collecting troves of personal data or through carelessness.

(4) *Fairness and impartiality in algorithms and AI.*
We talk about the ethics behind automated decision-making systems and related problems of fairness and impartiality by drawing on examples from predictive policing, prison sentencing, targeted advertising and mathematical fairness measures.

(5) *Regulation, accountability and the law.*
Industrial mathematics is very close to its social impact (e.g., credit scoring via machine learning) and hence mathematicians need to reconsider their responsibility, understand laws and regulations, and learn to self-regulate when lawmakers are behind the times.

(6) *Understanding the behaviour of the mathematical community.*
All fields, including mathematics, have a sense of community, conventions and values. Abstraction and the art of mathematical thinking may not necessarily lead to ethical solutions to industrial or social problems.

(7) *Psychology 101 – how to survive as a mathematician at work.*
Mathematicians encounter other issues, conflicts and dangers arising in the workplace. Their focused and dedicated nature means they may overlook instances of exploitation and manipulation of them and their work. Students must learn to identify these and to protect themselves.

(8) *A look into the future, what are the next steps?*
Being aware of the ethical issues is not the last step to take. We talk about ways to engage in moral behaviour by talking to colleagues, getting involved with

[2] As the now-resigned director of the MIT Media Lab, Joi Ito, recently found out; tinyurl.com/yyowldy9.

decision-making processes and by identifying and calling out unethical and harmful mathematics.

Be interactive!

We found it extremely fruitful to engage students in interactive demonstrations to show that, even though they are very logical in their thinking, and the problems they work on are well-defined with 'exact' solutions, they are still people with vices, shortcomings and weaknesses. When mathematicians do maths, they do not suddenly become perfect Platonic logical machines. It is essential to dispel the myth that "we're not people, we're mathematicians". For example, you can ask the audience to break up into groups, each producing an impartial plagiarism testing algorithm. Get them to present it to the class, and then proceed to pick apart all the value-judgements presented. If you are lucky, a few students will notice that there is no impartial plagiarism tester! The literature on the psychology of groups is full of valuable (and entertaining) tests and exercises to show how easily one can yield to unspoken social pressures.

Another activity is the 'oil pipe problem' [3, p. 124]. Start by drawing a an oil rig in the ocean and a refinery on a straight shoreline, giving the cost of piping under water and on land. Then ask your students to discuss and compute the optimal pipe path from the rig to the refinery. They may treat it as a first-year calculus problem at which point you should ask what other information might be relevant; are there coral reefs or protected habitats in the vicinity? It teaches students to include soft constraints alongside time and money. Our students quickly became engaged in lively discussions in these examples.

Teaching EiM: Politics or not?

Will you try to explain what the 'right' ethical conclusions are, expound on moral frameworks, or restrict yourself to only raising ethical awareness without offering answers or solutions? We regularly have students ask us, unsurprisingly, for the 'right answer' or the 'axioms and algorithms of ethics'. While we tried to avoid drawing ethical conclusions, this desire comes up regularly. We strongly suggest aiming to avoid ethical conclusions, and instead getting students to face the difficult job of coming to their own conclusions for their own reasons. By making it political, an anti/pro-capitalist rant, or a mission for social justice, you risk alienating students and colleagues. Many are simply not interested in a political agenda, but do care about not harming people.

Some mathematicians realise that maths has ethical consequences; others do not particularly care whether they cause harm. But most just lack well-developed ethical awareness. They may want to do maths, have fun in the process, and earn a living without causing harm to others; you can thus raise *their* ethical consciousness, as well as change how they view their work. You do not have to teach them political conclusions; this isn't part of mathematics, but part of the ordinary political discourse citizens have about their political world.

One reason mathematicians shy away from ethical discussions is that mathematics seeks timeless, absolute truths. The apparent perfection of mathematical truth can be its primary attraction. But ethics doesn't have the same binary clarity or timelessness. Different people may come to different conclusions or hold different moral values which are all reasonable, and mathematicians facing profession-specific ethical challenges have no universally-agreed ethical framework to use, because there isn't one. Unsurprisingly, suggesting that mathematicians need to be aware of ethical issues sometimes gets the response that ethics is imperfect and a matter of opinion, and moreover "*Whose ethics?*" which we would answer with "*Yours!*" We do not suggest that teaching EiM should give all the answers to ethical problems, but we do suggest that it is our duty to educate our students about it. The hard work of solving the questions remains and is an individual's social responsibility. The political debate that follows is part of what informed citizens frequently do.

Resources, exercise sheets and assessment

Setting assessment will depend on your course and department. If you do (we didn't), we would suggest setting essay(s) with an emphasis on analysis, reasoning, identification and exploration of ethical issues and (mathematical) sources. Judge contextualisation and line of reasoning, rather than the final conclusion. You can even ask students to present several solutions or options to a particular scenario (hypothetical, or drawing from real life).

There is something more important than assessment. Mathematics is not a spectator sport; every day, maths students go home after lectures and spend many hours on traditional exercise sheet questions. We all know that the value of doing this is to understand the mathematics at a deeper level by 'doing it for yourself'. In the same vein, students need to 'go home and practice' thinking about the ethical issues that can arise when 'doing' mathematics. What we propose is to give students mathematical exercises with *real* mathematical content, which also have an *ethical component*. One could assemble a collection of such questions into a set of dedicated sheets, and one might even make this the system of assessment of an EiM course. However, this still compartmentalises the learning process. ***So let us make a modest proposal:*** Beyond exercises for an EiM course, we believe it would be more effective if, when doing mathematical exercises in *other courses*, students encountered questions that require ethical considerations. This could help normalise ethical awareness in everyday mathematics. Its impact could be as large, or larger, than a standalone EiM course. While this is a different order of ethical engagement on the part of the department, it requires minimal effort on the part of your colleagues. If some of the exercise sheets in some of the courses contained a problem or two with an ethical flavour, this might serve to painlessly normalise the ethical engagement and awareness for many students. For first and second-year courses, we have prepared such a collection of questions, which can be found at ethics.maths.cam.ac.uk/course.

Students need to train their ethical reasoning just like they train mathematical reasoning via exercises. This proposition has the benefit that it requires no alteration to the core lecture content, beyond simply highlighting in lectures that some example sheet questions are designed to train not only technical and abstract understanding but also the interpretation of mathematics. However, whatever the mechanism your institution uses to give feedback on exercise sheets, you would need to instruct your teaching assistants about these questions. Don't expect them to instantly understand it; they are, after all, mathematicians who probably haven't had much training in ethical awareness. Providing written explanations helps. If example classes are predominantly led by graduate students, then as well as attending your EiM lectures, they can get involved with these EiM questions through teaching them.

Faculty support

Faculty support is critical to setting up an EiM course, but it can be hard to get. You do not necessarily need your colleagues' time or energy, you just need them to acquiesce to an experiment in EiM, even though it isn't about theorems or applications. In academia where resources are stretched so thinly that we struggle to teach all the mathematics we would like to, you will need to give good arguments to allocate resources to training in ethics. If we are trying to produce the best mathematicians possible and not just maximise the number of theorems taught, we have a duty to teach our students how to use this power and their mathematics responsibly. Otherwise, why are we teaching it to them at all?

Dismissive colleagues can damage the effectiveness of teaching EiM. Phrases such as "Why waste your time going to EiM lectures?", or even more subtle assertions ("Oh, don't worry about question 4; it's one of those ethics questions.") are damaging as they're quickly picked up by students, and it is essential to get departmental leadership on board to encourage colleagues to avoid (directly or indirectly) undermining the credibility of this teaching.

The objections can be orthogonal. One person might say "There is no EiM, so no need to teach it" and another might say "It is obvious that there is EiM, so no need to teach it". However, the most significant objection is an entirely reasonable argument: "We're a maths department, why are we teaching ethics? It's not precise; it's a matter of opinion". As we have repeated *ad nauseum*, other fields teach profession-specific ethics within their university training. Medical ethics is not medicine, but it makes doctors better doctors. Ethics *is* a matter of opinion, but that does not mean it cannot be addressed. Mathematicians deal with matters of opinion all the time. We discuss the beauty of mathematics, the elegance of proofs, letters of reference, partial marks on exams, and promotions. When refereeing papers, we fill our reports with value-judgements and opinions beyond mathematical accuracy. When every other profession faces ethical issues and trains professionals to deal with these issues, how can we exclude ourselves from it?

Concluding remarks

To be eligible for funding for a Centre for Doctoral Training (CDT) from EPSRC, applicants must demonstrate the provision of appropriate training in ethics for all doctoral students. An EiM course would give a convincing response to any such application, demonstrating that the applicants and department genuinely care about ethics and take it seriously. Referees will likely give more weight to an established EiM course than a simple statement of intent to teach ethics, or a reference to an external provider of such 'Responsible Research and Innovation' training (with no specific focus on mathematics).

We have had students from our EiM course tell us they had spoken to large tech companies who were extremely impressed that mathematicians were learning about ethics. It is a highly desirable skill, and as part of your teaching, you may consider providing students with a 'letter of participation'. This may not seem like much, but to employers, a mathematician with *any* ethical training can be a real asset in today's data-driven economy.

Recently a major UK broadsheet published an editorial arguing that mathematicians *need* to consider ethics [4]. And the 2019 Royal Institution Christmas lectures, to be delivered by Hannah Fry, will essentially focus on 'ethical issues in mathematics'. If the editors of a newspaper, and the general public, are aware of these issues and of the social responsibilities of mathematicians, surely the time has come to start teaching it to our students.

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A Presentation of the Austrian Mathematical Society

Barbara Kaltenbacher (Alpen-Adria-Universität Klagenfurt, Austria)

The Austrian Mathematical Society ÖMG was founded in 1903 by the university professors Ludwig Boltzmann, Gustav von Escherich and Emil Müller under the name *Mathematische Gesellschaft in Wien* (Mathematical Society in Vienna). The role of Boltzmann is historically not completely clear; at the first meeting on 14 January 1904, von Escherich, Müller and Wirtinger were elected as the managing committee of the Society, as reported in the annual reports of the German Mathematical Society DMV. More details about these early years can be found in the article *Vor 100 Jahren: Mathematik in Wien* (Hundred years ago: Mathematics in Vienna) by Christa Binder in the *Internationale Mathematische Nachrichten* (International Mathematical News) IMN 193 (2003), pp.1–20, http://www.oemg.ac.at/IMN/imn193.pdf.

After the Second World War, the Society reassumed its activity in May 1946. On 10 August 1946 it was formally reestablished by Rudolf Inzinger. An account of this can be found in the very first issue of the *Internationale Mathematische Nachrichten* IMN 1 (1947), http://www.oemg.ac.at/IMN/imn001.pdf.

The first president of this Society was Prof. Dr Rudolf Inzinger (TU Vienna, at that time Technische Hochschule Wien), the first and second deputy presidents were Prof. Dr. Johann Radon (University of Vienna) and Franz Prowaznik (State Superintendent of Schools in Vienna), respectively. Also Doz. Dr. Ludwig Hofmann (TU Vienna) and Doz. Dr. Edmund Hlawka (University of Vienna) were members of the board at that time.

From 19 to 22 May 1948, the first mathematical congress took place in Vienna; a detailed report can be found in the *Internationale Mathematische Nachrichten* IMN 4 (1948), http://www.oemg.ac.at/IMN/imn004.pdf. In order to emphasise its role as a national representative of mathematicians, the board decided to change the name to *Österreichische Mathematische Gesellschaft* (Austrian Mathematical Society) on 23 April 1948. This change was confirmed in the general assembly on 29 October 1948 and Rudolf Inzinger passed the presidency on to Johann Radon – well known for, e.g., the Radon–Nikodym Theorem or the Radon Transform. Today, an Institute of the Austrian Academy of Sciences is named after him, the Johann Radon Institute for Computational and Applied Mathematics https://www.ricam.oeaw.ac.at/

The journal *Internationale Mathematische Nachrichten of the Austrian Mathematical Society* was founded in 1947 by Rudolf Inzinger as *Nachrichten der Mathematischen Gesellschaft in Wien* (News of the Mathematical Society in Vienna) and edited for the first time in June 1947. In 1952 it was renamed *Internationale Mathematische Nachrichten* and served until 1971 as the official gazette of the International Mathematical Union. From 1953 to 1977 the IMN continued to be edited by Walter Wunderlich, who had been on board as editor of the Nachrichten since the journal was founded.

Since the reestablishment of the *Monatshefte für Mathematik* (Mathematical Monthly) by Johann Radon in 1947, the ÖMG has also contributed to its redactions; first of all up until 1955 and currently again since 2010.

As specified in its bylaws, the ÖMG makes an important contribution to the "promotion of mathematical sciences and support for mathematicians" through its conferences and awards.

Besides participation in the organisation of bi- and multilateral conferences such as the CSASC (Joint Meeting of the Czech, Slovenian, Austrian, Slovak and Catalan Mathematical Societies) and other joint conferences with neighbouring mathematical societies of similar size, the ÖMG also runs a biennial series of conferences, conferences (the most recent one in September 2019 in Dornbirn), where every second one is held jointly with the German Mathematical Society DMV as a large Mathematics Congress every four years. The last one took place in Salzburg in September 2017, the next one will be held in Passau in September 2021.

Awards are an important means of supporting young mathematicians in their careers and increasing the visibility of mathematics and mathematicians. The ÖMG provides yearly awards at several different career levels. The introductory one is a prize for high school pupils in their final year and the highest one is the research promotion award of the Austrian Mathematical Society. The latter is awarded along with the Inzinger Medal (see Figure) to excellent young mathematicians, who are usually at the stage of post docs, senior scientists or assistant/associated professors within two to ten years after their PhD, for their outstanding mathematical research work, the majority of which should have been carried out in Austria. A list of awardees of the Inzinger Medal can be found at http://www.oemg.ac.at/Preise.html#fp and contains a considerable number of distinguished colleagues working in academia within or outside of Austria.

The Inzinger Medal.

Group photo at the first Early Student Award meeting in Strobl, September 2018. (Photo by Wolfgang Wöss, TU Graz, Austria)

For the stages in between there are two further prizes: the Studies Prize for Master or PhD theses and the newly established Early Student Award, which also includes a meeting of some of the most talented first- and second-year mathematics students from all over Austria, with talks by experts from academia and industry and ample time for networking and discussion, in order to provide the students with insight into perspectives for mathematics graduates.

Last but not least, since 2016 the Austrian Mathematical Society has held its annual general assembly in the framework of the Tag der Mathematik (Mathematics Day), where the above mentioned prizes are also awarded and two - often recently appointed - colleagues from Austrian Universities present their research to the mathematical public in colloquium talks.

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About the Regions and Regional Conferences of Europe

On the occasion of the Third Caucasian Mathematics Conference (CMC)

Betül Tanbay (Boğaziçi University, Istanbul, Turkey)

Good old Europe, good Old World. Who are you? What are you? Living in Istanbul means for millions commuting back and forth between Europe and Asia every day. But instead of the long traffic queues on the bridges, it is more advisable to take a Bosphorus boat between these continents, especially if you want to think about geography or mathematics.

Mathematicians seek precise definitions, yet Europe remains quite undefinable. Political changes have too great an influence for a stable definition. The European Mathematics Society (EMS) was probably the projection to Mathematics of the EU project. It is comforting to see that the EMS is not destabilised as easily and that it has a wider and more stable understanding of Europe. At the 25th anniversary of the EMS, I remember its President Pavel Exner talking about 2000 years of mathematics in Europe, and evidently he was including Asia Minor, Mesopotamia and the Mediterranean Sea. Yet, national histories have different accounts. I often ask during my lectures what it is that makes a student from Izmir not feel like they are from the same country as Thales.

Two recent conferences made me feel like writing about this Europe in a larger context: the third Caucasian Mathematical Conference (CMC) and the Women Mathematicians of the Mediterranean Shores. The latter was in Bled, Slovenia in September and brought together mathematicians from all around the Mediterranean Sea. Another occasion to share common values, be it maths or great food.

CMC is already 5 years old. At the opening of the third one last August in Rostov-on-Don, Russian Federation, the CMC-III-Chair Armen Sergeev said: "A first is an occurrence, a second a coincidence, but a third is tradition." We all hope that it does become a tradition. The mathematical signs are so far positive.

The idea of a CMC was first pronounced at the EMS presidents meeting in Aarhus, Denmark in 2013, and the then EMS President Marta Sanz-Solé immediately asked the Georgian Mathematical Union President Roland Duduchava and EMS Vice-President Armen Sergeev to form a founding team with myself, at that time President of the Turkish Mathematical Society. It took us one meeting at the Istanbul Center for Mathematical Sciences overlooking the Bosphorus to set up the structure of CMC and its "raison d' être": Under the auspices of the European Mathematical Society, the mathematical societies of Armenia, Azerbaijan, Georgia, Iran, Russia and Turkey come together to create the Caucasian Mathematical Conference (CMC) project and to get to know each other, share their work, research and results, and jointly navigate new areas of research in mathematics.

There are three committees: the steering committee with representatives from the six national mathematical societies and from EMS; the scientific committee chosen from previous invited speakers, which chooses seven "invited speakers" and twelve "young speakers" from the propositions made by these societies; and the organizing committee.

The first Caucasian Mathematical Conference took place in 2014 in Tbilisi, Georgia with more than 150 registered mathematicians from these six countries and more. At the conference dinner, mathematicians from Azerbaijan, Armenia and Turkey joined each other to perform the famous Caucasian dances; living together was not such an impossible task! We could do mathematics, eat, drink and dance!

In September 2014, our first CMC was certainly not the centre of public attention. The world was busy with the NATO Summit that was held in Wales, UK the same week. Mathematics conferences and political summits are really different in essence. The goal of the summit was to provide "common security, prosperity and values", to come up with solutions to social and political nodes. In public, the rulers filled their discourse with

CMC-I, 5 September 2014, Tbilisi, Georgia.

CMC-II, 22 August 2017, Van, Turkey.

CMC-III, 26 August 2019, Rostov-on-Don, Russian Federation.

the words "freedom," "human rights," "democracy" and "peace". Meanwhile, CMC shareholders made neither political nor hypocritical promises, but decided to hold the second conference in 2016 in Turkey, and chose the Yüzüncü Yıl University in Van as the location: a meaningful barycenter for all the countries involved.

The events of Summer 2016 in Turkey obliged the organisers to postpone CMC-II until the following August. The meeting was held in 2017 with almost 200 participants from all countries. Can mathematical peace set an example for political peace?

Meanwhile, no NATO summit and no European Union summit has found solutions to nodes as promised, because the real decisions are taken behind closed doors. We have had to witness ugly negotiations around millions of people who have lost their homes. People having to leave their land… is this the destiny of the region?

Meanwhile, the CMC continues with the goal of becoming a tradition. At the Southern Federal University in Rostov this summer, mathematicians from 15 countries, including the "six", got together to listen to excellent talks, the EMS speaker being Ragni Piene, the first female invited speaker of the CMC series. I will skip the story of the President of the EMS becoming a Cossack on a boat ride on the quietly flowing Don, at least until our retirement. The steering committee had a meeting in a simple room at the Don Quixote Hotel where most participants were staying. All members were present. The first round of talks may have had some reminiscence of

Opening CMC-III, 26 August 2019, Rostov-on-Don, Russian Federation.

national reflexes, but it took less than a quarter of an hour to have the Azerbaijan representative Asaf Ghadiev proposing funding solutions for a summer school we would organise prior to CMC-IV to be held in Yerevan, Armenia in 2021. The steering committee also decided to have CMC-V in Azerbaijan, and the next in Iran. Hopefully, or "inşallah" as we commonly say in my neighbourhood.

What politicians discuss in meeting rooms is secret. What mathematicians discuss in meeting rooms is not secret but not easily communicable to the general public. Nevertheless, the fact that they set a "good example" by working together, overcoming the burden of borders and nationalities, generating positive energy in an atmosphere where the priority is the good of humanity and science deserves to become common knowledge.

While CMC is becoming a tradition, there are rumours that a Balkan Mathematics Conference will start, a BMC? From AMC to ZMC, I believe the European Mathematical Society has every reason to support such regional gatherings; they may not always achieve the prestige of an ECM, but they can be part of the sources that create the flow of excellent mathematicians.

In mathematics, the richer a theory, the harder it is to keep it both complete and consistent. Not very mathematical I know, but in this very case, I would compromise consistency. Instead of avoiding paradoxes, maybe we can just try to minimise them. The planet's most serious problems are becoming more and more independent of nationalities, so it might be a good idea to keep on working on getting together.

Betül Tanbay is a professor in Functional Analysis at the Boğaziçi University in Istanbul. She was founder and first co-director of the Istanbul Center for Mathematical Sciences. She was the first female president of the Turkish Mathematical Society, and she has also served and serves in many committees of the IMU or EMS. Tanbay received her undergraduate degree from ULP, Strasbourg in 1982, and graduate degrees from UC Berkeley in 1989

ICMI Column

Jean-Luc Dorier (Université de Genève, Switzerland)

The 2019 Felix Klein, Hans Freudenthal and Emma Castelnuovo ICMI Awards

ICMI is proud to announce the eighth recipients of the Klein and Freudenthal Awards and the second Emma Castelnuovo. We give some key elements below, the three full citations can be found at:
https://www.mathunion.org/icmi/awards/icmi-awards

Tommy Dreyfus, Professor Emeritus at Tel Aviv University, Israel receives the 2019 Felix Klein Award.

The *Felix Klein Award 2019* is awarded to **Professor Tommy Dreyfus** in recognition of his outstanding contributions to research as well as his leading role in shaping and consolidating the research community and in fostering communication between researchers.

For four decades, Tommy Dreyfus's research has been systematically deepening our understanding of mathematics learning. Trained as a mathematical physicist, Tommy has drawn in this work on his deep understanding of mathematics and his first-hand familiarity with ways in which mathematical ideas transpire and evolve. From the late 1970s and for the following two decades, his research focused on students' conceptualisation of mathematical objects such as function, and on the role of intuition, visualisation and aesthetics in mathematical thinking. These efforts resulted in the theory known as AiC – Abstraction in Context, which he developed with Baruch Schwarz and Rina Hershkowitz. Conceived in the late 1990s, the AiC framework has become increasingly influential. Since its inception, it has generated much empirical research all over the world. The theory has also been found to be useful to teachers, whom it provides with tools for monitoring student learning.

Another outstanding part of his work is his ongoing project of shaping and consolidating the international community of research in mathematics education; a goal that he has strived to attain in multiple ways. First and foremost, he has set standards and given directions for research in mathematics education through his extensive editorial work. His association with Educational Studies in Mathematics, which spans three decades including a three-year term as editor-in-chief, has been particularly influential. Professor Dreyfus has also served in, and shaped, numerous professional organisations, with PME (the international group for the Psychology of Mathematics Education) and ERME (the European Society for Research in Mathematics Education) among them. In addition, he has played key roles in numerous professional committees in Israel, Europe and America. His influence on research and on policy directly affecting mathematics teaching can be felt all over the world.

Moreover, Professor Dreyfus has contributed to changing the dominant narratives about theoretical diversity. With his help, the multiplicity of research discourses is now seen to be less a problem to solve than an opportunity to embrace.

To sum up, over the 40 years of his career, Professor Dreyfus has contributed to our collective endeavour of promoting mathematics education in great many ways: as a researcher, as an editor, as an organiser and policy adviser, and as a teacher and mentor. For all this and his many other contributions to our community, Tommy Dreyfus is an eminently worthy candidate for the Felix Klein Award.

Gert Schubring, a long-time member of the Institut für Didaktik der Mathematik at Bielefeld University, Germany, and an extended visiting professor at the Universidade Federal do Rio de Janeiro in Brazil receives the 2019 Hans Freudenthal Award.

The *Hans Freudenthal Award 2019* is awarded to **Professor Gert Schubring**, in recognition of his outstanding contribution to research on the history of mathematics education.

Gert Schubring's research of over four decades has opened new, important avenues of research into the phenomenon of mathematics education. Trained as a mathematician, Gert has been a member of the *Institut für Didaktik der Mathematik* (IDM) since 1973, when this interdisciplinary research institute for mathematics education was founded. In his doctoral dissertation, defended in 1977, Gert Schubring wrote on the genetic principle in approaching historical research in mathematics. Afterwards, he extended his interests, producing wide-ranging writings on the history of mathematics education within and across countries, and publishing on the history of mathematics.

Another, related but separate, strand of Gert's pioneering work was the study of textbooks, which he began during his investigations on the evolution of mathematics teaching in Latin America. This is yet another area of research that he helped to recognise as worthy of attention. In 2017 he also chaired the International Program Committee for the Second *International Conference on Mathematics Textbook Research and Development* held in Rio de Janeiro, Brazil.

Gert Schubring also laid out the formal structures that helped turn the study of the history of mathematics education into an academic field. He was the founding co-organiser of the *International Conference on the History of Mathematics Education* (ICHME), a forum that has already met six times since 2009.

For decades, Gert Schubring has been actively promoting the study of the history of the field of mathematics education, while simultaneously conducting significant historical studies of his own. No other researcher has had a greater impact on establishing the social history of mathematics education as a dynamic field of scholarly endeavour. His work has not only made us aware of the past of mathematics education; it has provided important insights into mathematics education as it stands today and has set directions for its future. It informs current teaching by showing ways in which historical mathematical texts can inspire pedagogy. It makes us aware of future possibilities and of the fact that they do not have to be merely determined by the past, but can be moulded by new understanding of past practices, values and ways of thinking. All these important contributions make Professor Gert Schubring an eminently deserving recipient of the Hans Freudenthal Medal for 2019.

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The 2020 Emma Castelnuovo Award for Outstanding Achievements in the Practice of Mathematics Education goes to the National Council of Teachers of Mathematics (NCTM), USA.

The *2020 Emma Castelnuovo Award for Outstanding Achievements in the Practice of Mathematics Education* goes to the **National Council of Teachers of Mathematics (NCTM)**, USA, in recognition of 100 years of development and implementation of exceptionally excellent and influential work in the practice of mathematics education.

Founded in 1920, NCTM is the world's largest mathematics education organisation, with 40,000 members and more than 230 affiliates. In their nomination of NCTM, the chair of the United States National Commission on Mathematics Instruction, John W. Staley, gave four reasons for nominating NCTM:

- NCTM has served the mathematics education community (nationally and internationally) for almost 100 years by providing leadership, publications and resources, professional development and networking opportunities.
- NCTM has served its membership by supporting and growing educators and involving them in many of the organisation's initiatives and projects, and providing various opportunities to develop members' leadership skills.
- NCTM continues to advocate high-quality mathematics teaching and learning for each and every student. This advocacy extends to the work that helps educators who choose to advocate with their elected officials and policymakers.
- NCTM continues to build and value collaborative relationships with educators throughout the world.

The Award Committee found much evidence in support of the reasons mentioned above, and further criteria also had to be met in order to achieve the Emma Castelnuovo Award. In the following, some exemplary activities of NCTM's past 30 years are highlighted. These activities fall in a wide range of domains – in particular, principles and standards as foundations for policy and practice, publications including research journals, professional development, legislative and policy leadership, and international collaboration.

ERME Column

Stefan Zehetmeier (University of Klagenfurt, Austria) and Jason Cooper (Weizmann Institute of Science, Israel)

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME) holds a bi-yearly conference (CERME), in which research is presented and discussed in Working Groups (TWG). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME Thematic Working Group 18 – Mathematics Teacher Education and Professional Development

Group leaders: Stefan Zehetmeier, João Pedro da Ponte

Thematic Working Group rationale

The study of mathematics teacher education and professional development has been a central focus of research over recent decades. In previous ERME conferences, various research activities regarding this topic have been presented and discussed. Within this Thematic Working Group (TWG) the focus lies on mathematics teacher education (both pre-service and in-service), professional development and growth. Moreover, models and programmes of professional development, as well as their respective contents, methods and impact have been described and analysed. Research increasingly focuses not only on the participating teachers, but also on the roles of teacher educators and academic researchers. These two roles are often fulfilled by one and the same person. The research community is attempting to develop theoretical and methodological frameworks to both describe and explain the complex topic of mathematics teacher education and professional development.

ERME conferences aim to promote communication, cooperation and collaboration in research on mathematics education, to learn about research and interests in different countries and to create opportunities for international cooperation between researchers in collaborative projects. In keeping with these aims, the TWG offers a communicative, collegial and critical forum for the discussion of the above-mentioned issues, which allows for diverse perspectives and theoretical approaches, and which contributes to the development of our knowledge and understanding as researchers, educators and practitioners.

TWG History

Research of mathematics teacher education and professional development has been an integral part of ERME conferences from the very beginning. In CERME 1 (1998 in Osnabrück, Germany), a working group named "From a Study of Teaching Practices to Issues in Teacher Education" was established. Due to this initial working group's success and participants' interest, this topic was also provided in the following ERME conferences. From CERME 2 until CERME 8, such a TWG was part of the conferences' respective scientific programmes. The number of both participating researchers and presented papers grew from conference to conference. For example, in CERME 4 (2005 in St. Feliu, Spain), 21 papers were presented in this TWG; in CERME 8 (2013 in Antalya, Turkey), 45 papers were presented. These increasing numbers indicated the respective topics' increasing relevance; yet this also raised the challenge of offering a communicative forum for an increasing number of TWG participants. At CERME 9 (2015 in Prague, Czech Republic), 75 papers were accepted for presentation in this TWG. This huge number led to the conference IPC's decision to split the TWG into three new sub-groups. One of them was called "mathematics teacher education and professional development", which was provided in CERMEs 9, 10 and 11. However, there was still an ongoing increase in the number of participants and papers. This led to the decision in CERME 11 (2019 in Utrecht, the Netherlands) to split this TWG yet again into two sub-groups: TWG18a (with a particular focus on teachers and teacher educators) and TWG18b (with a particular focus on professional development settings).

TWG topics

The following section provides several exemplary key questions and issues which emerged in TWG "mathematics teacher education and professional development" during the 2019 ERME conference. These topics relate in particular to pre-service teacher education, professional development programmes, in-service teacher classroom practice and collaboration.

Concerning pre-service teacher education, one central question is what pre-service teachers can learn from theory. Here, the role of mathematical concepts and theories is of particular relevance. The relationship between the mathematics that a teacher has to teach at school and the mathematics a teacher has to learn at university is far from being self-evident or clear. Research on this relationship offers fruitful opportunities for cooperation between research mathematicians and teacher educators. Another question is how to challenge previous experiences of teaching and learning mathematics that pre-service teachers bring with them when entering the teaching profession. For example, how can teachers be supported in being exploratory and inquiry-based in both their teaching and learning?

With regard to professional development programmes, the core questions are: How does one research the impact of professional development? Is there any chain of effects? How can impact and effect be conceptualised? How can impact be sustained over time? With regard to teacher educators, several central questions are moving into focus: How do educators' theoretical perspectives change over time? Which assumptions are taken for granted about how mathematics should be taught? What are possible implications for the professional development of mathematicians as lecturers in professional development programmes? Moreover, there are various reflections about whether and how the philosophy of mathematics and the philosophy of mathematics education coincide; e.g., How can dual roles as both teacher educator and researcher be managed? How can research done by mathematics specialists and teacher education specialists infuse and support each other?

Concerning in-service teacher classroom practice, the issue of incorporating inquiry into learning mathematics is central. Yet, many different notions of inquiry are being used. What are efficient strategies for implementing inquiry approaches in mathematics lessons? Furthermore, there is the need to explicitly connect inquiry with particular mathematical content. What mathematical knowledge do teachers need for inquiry approaches in mathematics lessons to be effective? What contributes to teachers' confidence in teaching mathematics (particularly with an inquiry-based approach)? What contributes to students' confidence in doing mathematics in the classroom? Also here, close cooperation between the scientific communities of mathematics and teacher education seems highly promising. Yet another issue is the combination of inquiry topics that are shared between different subjects (e.g. mathematics and science), such as problem solving or modelling.

Professional development of teachers takes place in various kinds of collaborative settings. One emerging line of research investigates what kinds of settings are conducive to combining collegial work with individual growth. A different line of research focuses on the collaboration of academics and teachers in professional development. The roles and (a)symmetries of these parties, when engaging in professional development and in research, are crucial. In particular, the roles of educational researchers (often acting as teacher educators) and of expert mathematicians are in the main focus. What motivates mathematicians and mathematics teacher educators to be involved in collaborative projects? What are the consequences for the respective contents and methods? Joint research activities by research mathematicians and teacher educators show great promise in bringing these issues forward.

TWG Future

In the upcoming ERME conference (2021 in Bolzano, Italy), even more TWGs concerning these topics will be established: besides a TWG on "mathematics teacher education and professional development" it is planned to also provide TWGs concerning "mathematics teaching and teacher practice(s)", "mathematics teacher knowledge, beliefs and identity" and "the professional practices, preparation and support of mathematics teacher educators". This wide range of topics and TWGs demonstrates the growing interest and relevance of these topics within the scientific community. In particular, it creates opportunities for researchers in the fields of mathematics and mathematics education to cooperate and learn from each other in fruitful ways. Both mathematicians and teacher educators are invited to actively participate in the next ERME conference in 2021, to present their research and to further develop our scientific communities' knowledge about the relationship between mathematics and teacher education.

Stefan Zehetmeier is an associate professor at the University of Klagenfurt (Austria). His research interests include mathematics teacher education, school development, evaluation and impact analysis of teacher professional development programmes. He holds a master degree in mathematics and physics education, a doctoral degree in education, and a habilitation in teacher education.

Jason Cooper's photo and CV can be found in previous Newsletter issues.

References to Research Literature in QA Forums – A Case Study of zbMATH Links from MathOverflow

Fabian Müller, Moritz Schubotz and Olaf Teschke (all FIZ Karlsruhe, Berlin, Germany)

Since August 2017, MathOverflow [1] users can seemlessly detect literature references in zbMATH [2] and insert them into discussions. Likewise, backlinks are automatically generated. This feature has been used in more than 500 distinct questions since then. In this column, we discuss the details of this tool and analyse its usage data during the last two years.

1 Motivation

Q&A services like MathOverflow and review services like zbMATH are both driven by active communities. Although they address quite different needs of a working mathematician, there are many occasions where it is highly useful to have both information facets available. As a typical example, we may refer to our earlier column [3] on mathematical pseudonyms in zbMATH author profiles, which greatly benefited from a respective MathOverflow thread [4] (and, vice versa, we could add a lot of further answers to it derived from our author database). Another helpful thread was a discussion on the documentation of academic misconduct in review services [5]. It generated valuable feedback which helped to identify and document several cases of plagiarism.

The most natural motivation for interlinking both sources obviously stems from the need of supporting questions and answers in the platform with references to the literature, ideally with minimal barriers for users. Backlinks generated in this course would add interesting information beyond the reviews. As such, the realisation of such an interface would be very much in the spirit of building the envisioned Global Digital Mathematics Library of interlinked services [6].

2 Linking MathOverflow and zbMATH

In order to do so, we started a collaboration in 2016 with MathOverflow to set up an intuitive and easy-to-use interface for citing literature sources by harvesting the power of the zbMATH database.

For this, we were able to make use of the existing technology we use for various kinds of matching tasks, like finding the zbMATH document which a certain reference from a document refers to. This is a non-trivial task that uses modern methods of machine learning in connection with domain-specific indexing technologies. The case for MathOverflow matching is actually an easier one: when people search for a mathematical paper, they will usually enter one or more author names plus a couple of relevant terms from the title of the paper. It thus suffices to search for these terms in the author and title fields and return the most promising results. In a mechanical task without human intervention this wouldn't be quite enough to ensure a high level of accuracy, but since there is the additional step of the user selecting the paper they want from a short list of top hits, it is entirely enough for this type of use.

On the zbMATH side, we realised this functionality by implementing a small wrapper API around the existing matching functionality. We have done this in the past for other services as well, including e.g. ProjectEuclid [7] or EuDML [8].

On the MathOverflow side, Scott Morrison implemented a small front-end that can be reached via the newly made "Insert Citation" button (see Figure 1) when composing a new question or answer. The search results are displayed including preview links to the articles at zbMATH and, if available, in other places like arXiv or the publisher's website, so if the user is not entirely sure it is the article they want they can easily have a look and check. On confirmation the citation is inserted, correctly formatted, into the MathOverflow question or answer, including bibliographic information, a zbMATH link and a DOI link if available.

Finally, having the information on MathOverflow is helpful for the community there, but it is similarly interesting for the zbMATH users to discover that a certain article has been discussed on MathOverflow. A regularly running script collects these using the StackExchange API[1] and displays them as backlinks together with the cited zbMATH articles. It thus contributes a small but

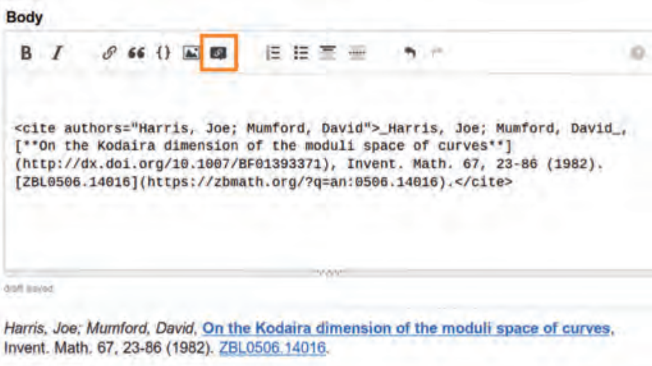

Fig. 1. Inserting a citation on MathOverflow.

[1] https://api.stackexchange.com/ (StackExchange is the network of Q&A sites for various fields of interest of which MathOverflow is a part).

helpful further step towards findability of mathematical ideas and interoperatibility of mathematical research tools. In the following, we investigate how this feature has been used in the past in Section 3 and discuss future perspectives in Section 4.

3 Statistical Analysis of Established Links

The tool described in Section 2 was quickly adopted by the community. With Fields medalist Terry Tao being an early adopter (see Figure 2), by now 223 registered user accounts at MathOverflow have used the feature.[2] As outlined in Figure 3, the average usage has increased over time (though the rate increases only slowly, also due to the fact that the initial usage after its introduction was already quite high). If one normalises the chart with regard to the overall number of posts, one sees a slight increase in the relative number of posts over the years (2017 (incubation year) 1.23%, 2018 (full year) 1.45%, 2019 (ongoing year) 1.34%). Note that the MathOverflow users are most active in April and their activity drops to about 85% of the April activity in June and July.

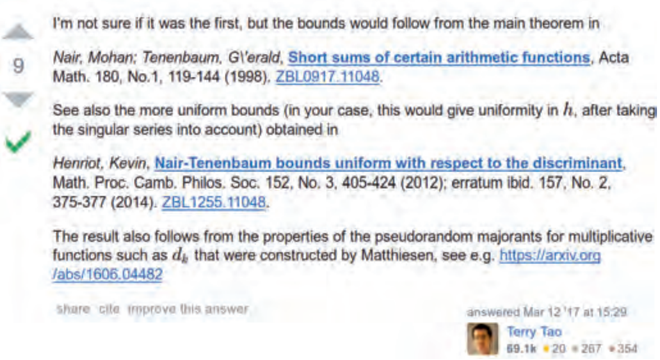

Fig. 2. Fields medalist Terry Tao uses the new feature [10].

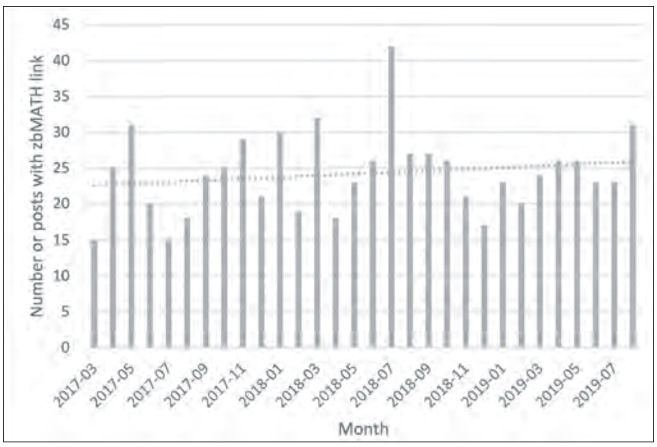

Fig. 3: Number of MathOverflow posts with a zbMATH link.

Links between MathOverflow and zbMATH

Overall 680 distinct zbMATH entries link to MathOverflow questions. The book *A primer on mapping class groups* [9] is the entry with most links (4) to discussions on MathOverflow.

In the reverse direction, 512 questions link from MathOverflow to zbMATH. Of those links 16% appear in the question and 83% in the answers. Here, one post [12] has 14 links to zbMATH.

Subjects according to MSC

The subjects of these posts (as reflected by the main area of the cited literature according to the Mathematical Subject Classification) reflect the most active communities in MathOverflow: algebraic and differential geometry; logic and category theory; manifolds and topology; and number and group theory account for more than half of these contributions.

Overall, we think that further investigation of the linking behaviour is of interest to the mathematical information retrieval community. Therefore, we provide the dataset together with a short usage guide on GitHub (see https://purl.org/zb/9).

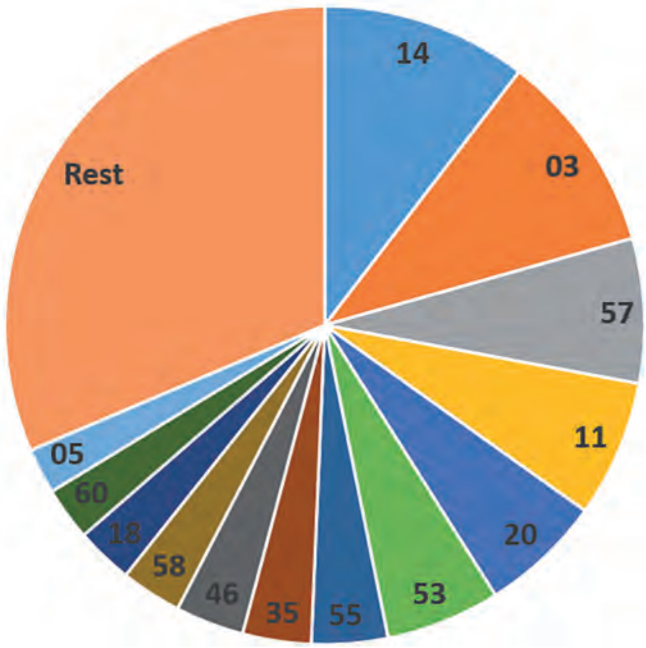

Fig. 4: Main MSC areas of zbMATH entries cited in MathOverflow.

4 Conclusion and future work

While linking academic literature from a question answering site such as MathOverflow was realised and adopted by the community, only 1% of the threads use that new feature. Questions in particular often do not reference academic literature. Here, advances in mathematical information retrieval technology might become helpful and suggest relevant academic literature related to the questions. While Corneli and Schubotz [13] envisioned that interlinking maths questions and Wikipedia articles might automatically resolve some questions, and a similar argument holds for MathOverflow questions and relevant academic literature. However, until now

[2] One may add that for almost as many users, their MathOverflow profile has been added as link to their zbMATH author profile, which is possible through the zbMATH community interface available at https://zbmath.org/author-profile/edit/ of from the author profile pages, cf. [11].

the machine understanding of mathematical questions is not yet developed enough to reasonably suggest relevant literature if the author or title of the publication are unknown to the author of the questions. Intelligent mathematical information retrieval technology and clever combinations of text and maths search engines might be able to suggest relevant literature in the future and thus help future question authors to resolve some questions by themselves.

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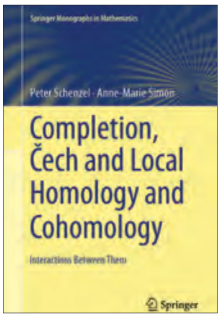
Pictures and CVs of the authors can be found in previous Newsletter issues.

Book Reviews



Peter Schenzel
Anne-Marie Simon
Completion, Čech and Local Homology and Cohomology
Springer, 2019
xvii, 346 p.
ISBN 978-3-319-96516-1

Reviewer: Siamak Yassemi

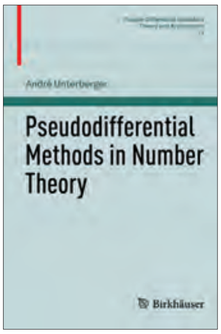
The Newsletter thanks zbMATH and Siamak Yassemi for permission to republish this review, which originally appeared as Zbl 1402.13001.

The aim of this book is a thorough study of the adic-completion, its left derived functors and their relations to the local cohomology functors. In the past there are several monographs about local cohomology, the derived functors of torsion. In a certain sense, completion is the dual notion of torsion. The book provides a first systematic study of completion (mainly in the non-Noetherian case) and its derived functors not only for modules but also for complexes. The authors focus is the case of arbitrary commutative rings with a view towards finitely generated ideals resp. ideals generated by weakly pro-regular sequences. A main technical tool is the Čech complex of a system of elements and its free resolution. In several results, the authors construct explicit morphisms that are quasi-isomorphisms between complexes describing isomorphisms in the derived category.

The book is divided into three parts. The first one is devoted to modules, the second part is devoted to the study of complexes and the part three is mainly concerned with duality.

In Chapter 1 the authors deal with preliminaries about complexes and also recall direct and inverse limits. Chapter 2 starts with basic results on adic completion. the authors study the left drived functors of the completion and then they study the class $\mathcal{C}_{\mathfrak{a}}$ of modules with the property that the left derived functors of the $\mathfrak{a}$ adic completion vanish in positive degree, while in degree aero it coincides with the $\mathfrak{a}$-completion functor. In Chapter 3, the authors give a generalization completeness criterion due to Jensen: "Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then, a finitely generated R-module M is complete in its $\mathfrak{m}$-adic topology if and only if $\operatorname{Ext}^i_R(F,M)=0$ for all $i>0$ and any flat R-module F". In Chapter 4, they summarize the work of Avramov and Foxby on unbounded complexes. In addition, it is presented a study of minimal injective resolutions of unbounded complexes, including the case of not necessarily Notherian rings. Chapter 5 is devoted to Koszul complexes and the notions of Ext-depth and Tor-codepth of a complex with respect to an ideal $\mathfrak{a}$ of a commutative ring R. In Chapter 6 the authors investigate Čech (co)homology and some related classes and give the Ext-depth and Tor-codepth sensitivity of the Čech complex. In Chapters 7–9 the authors investigate the left derived functors (called local homology functors) of the $\mathfrak{a}$-adic completion functors row $\varprojlim(R/\mathfrak{a}^n \otimes_R,-)$. Dual to Grothendieck's theory of local cohomology, these derived functors were first studied by E. Matlis. Then, motivated by topology, came the work of J.P.C. Greenlees and J.P. May (in a more general context). The authors consider these derived functors in several different directions. In Chapter 10, the authors study some versions of the Grothendieck Local Duality Theorem and dualizing complex. They prove some variations of the Grothendieck Local Duality Theorem. As a conclusion, this book is carefully and clearly written and is directed to readers interested in recent progress in Homological Methods in Commutative Algebra. The bibliography and index are excellent and so it will surely become a basic reference book for the subject. In addition, there are examples that illustrate the necessity of the assumptions of several of the statements. Parts of the book as well as the whole book might be used as a base for seminars with graduate students.

Siamak Yassemi works in commutative algebra and its connections to other areas of mathematics such as homological algebra and combinatorics. He received his Ph.D. under the supervision of H.B. Foxby from the University of Copenhagen, Denmark in 1994 and is now a full professor at the University of Tehran, Iran. He is a member of The Word Academy of Science (TWAS).



André Unterberger
Pseudodifferential Methods in Number Theory
Birkhäuser, 2018
178 p.
ISBN 978-3-319-92706-0

Reviewer: Anatoly N. Kochubei

The Newsletter thanks zbMATH and Anatoly N. Kochubei for permission to republish this review, which originally appeared as Zbl 1411.11004.

The book is devoted to applications of pseudodifferential calculus to analytic number theory, aimed to new approaches to the Riemann hypothesis (RH). After a brief description of the book in Chapter 1 (Introduction), in Chapter 2 the author explains the pseudodifferential Weyl calculus and the main properties of the Eisenstein distributions.

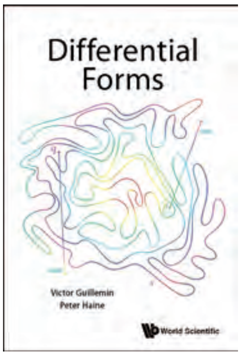
In Chapter 3, in the author's words, he introduces "the automorphic distribution $\mathfrak{T}_N^1$ in the plane and a collection of related one- and two-dimensional measures, as well as their limits – once a bad term has been deleted – as $N \to \infty$". "The distribution $\mathfrak{T}_\infty^1$ decomposes in terms of the Eisenstein distributions $\mathfrak{E}_{-\rho}$, where ρ runs through the set of non-trivial zeros of zeta". "The end of the chapter is devoted to various versions of the main criterion, which characterizes the Riemann hypothesis in terms of a collection of estimates regarding the operators, the symbols of which are rescaled versions of $\mathfrak{T}_\infty^1$ or $\mathfrak{T}_N^1$. The structure of these operators will be elucidated in Chapter 4".

It is shown in Chapter 5 that "it is precisely the Weyl calculus that provides the canonical map from modular distributions to modular forms of the non-holomorphic type". Here the author considers modular spaces of the metaplectic representation and number-theoretic constructions related to harmonic oscillator. In Chapter 6, the author gives "a discussion of pseudodifferential operators with symbols interesting from the point of view of arithmetic". This includes a deeper study of modular distributions.

In Chapter 7, the Fuchs calculus on the half-line is used (instead of the Weyl calculus). This gives, in particular, a new interpretation of the Weil positivity criterion for the RH. Chapter 8 looks for new approaches to the RH based on the p-adic and adelic methods. Note that there exists a well-developed theory of p-adic pseudodifferential equations connected with many branches of mathematics. See, for example, the recent book by A.Yu. Khrennikov et al. [Ultrametric pseudodifferential equations and applications. Cambridge: Cambridge University Press (2018; Zbl 1397.35350)] containing further references.

The book by A. Unterberger will be interesting and useful both for number theorists looking for new techniques, and for specialists in pseudodifferential operators interested in new application areas.

Anatoly N. Kochubei (born in 1949) is a Department Head, Institute of Mathematics, National Academy of Sciences of Ukraine. His research interests include fractional calculus, non-Archimedean analysis, partial differential equations, operator theory, mathematical physics.



Victor Guillemin and Peter Haine

Differential Forms

World Scientific Publishing, 2019
272 pp.
ISBN 978-981-3272-77-4 (hardcover)
ISBN 978-981-121-377-9 (softcover)
ISBN 978-981-3272-79-8 (ebook)

Reviewer: Andrew Bucki (Langston University, USA)

The Newsletter thanks zbMATH and Andrew Bucki for the permission to republish this review, originally appeared as Zbl 1418.58001.

There are many excellent graduate textbooks on the theory of differential forms. However, in the undergraduate program this subject is briefly touched only in multivariable calculus. The first encounter with differential forms is the change of variables formula,

$$\int_U f^*\varphi |\det J_f| dx = \int_V \varphi\, dy, \qquad (*)$$

where U and V are bounded open subsets of $\mathbb{R}^n$, $\varphi : V \to \mathbb{R}$ is a bounded continuous function, $f : U \to V$ is a bijective differentiable map, $f^*\varphi = \varphi \circ f : U \to \mathbb{R}$, and $\det J_f(x)$ is the determinant of the Jacobian matrix

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right).$$

The presence of "dx" and "dy" in $(*)$ can be justified by the fact that in single-variable calculus, with

$$U = (a, b), \quad V = (c, d),$$
$$f : (a, b) \to (c, d), \quad \text{and} \quad y = f(x)$$

a C^1 function with positive first derivative, the equation

$$\frac{dy}{dx} = \frac{df}{dx}$$

can be rewritten in the form $d(f^*y) = f^*dy$ and $(*)$ can be written as

$$\int_U f^*(\varphi dy) = \int_V \varphi dy. \qquad (**)$$

In this book, the authors' first goal is to legitimize this interpretation of equation $(*)$ in n dimensions and in fact, more generally, to show that an analogue of this formula is true when U and V are n-dimensional manifolds. Another related goal is to prove an important topological generalization of the change of variables formula $(*)$. If the assumption that f is a bijection is replaced by the requirement that f is proper, i.e., that preimages of compact subsets of V are compact subsets of U, then the formula $(*)$ can be replaced by

$$\int_U f^*(\varphi dy) = \deg(f) \int_V \varphi dy, \qquad (***)$$

where $\deg(f)$ is a topological invariant of f that counts, up to the sign, the number of preimage points of a generically chosen point of V. This degree formula connects the theory of differential forms with topology. Another goal of this book is to explore some of the other examples. For instance, if $U \subset \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$, then we define vector spaces

$$\Omega^0(U) = \{f\},$$
$$\Omega^1(U) = \{f_1\, dx_1 + f_2\, dx_2\},$$
$$\Omega^2(U) = \{f dx_1 \wedge dx_2\},$$

with $f, f_1, f_2 \in C^\infty(U)$, and $\Omega^k(U) = \{0\}$ for $k > 2$. The operator $d : \Omega^i(U) \to \Omega^{i+1}(U)$ is defined as

$$df = \frac{\partial f}{dx_1} dx_1 + \frac{\partial f}{dx_2} dx_2 \quad \text{for } i = 0,$$

$$d(f_1 dx_1 + f_2 dx_2) = \left(\frac{\partial f_2}{dx_1} - \frac{\partial f_1}{dx_2}\right) dx_1 \wedge dx_2 \quad \text{for } i = 1$$

and

$$d = 0 \quad \text{for } i > 1.$$

It is easy to see that $d^2 : \Omega^i(U) \to \Omega^{i+2}(U)$ is zero. Hence $\mathrm{im}(d\Omega^{i-1}(U) \to \Omega^i(U)) \subset \ker(d\Omega^i(U) \to \Omega^{i+1}(U))$ and this allows to define the de Rham cohomology groups of U as the quotient vector space

$$H^i = \frac{\ker(d\Omega^i(U) \to \Omega^{i+1}(U))}{\mathrm{im}(d\Omega^{i-1}(U) \to \Omega^i(U))}.$$

It turns out that these cohomology groups are topological invariants of U and are, in fact, isomorphic to the cohomology groups of U defined in the algebraic topology. Minor generalization of definitions of the vector spaces $\Omega^i(U)$allows to define these groups for open subsets U of $\mathbb{R}^n$ and, with a bit more effort, for arbitrary C^∞ manifolds. Their existence enables to describe interesting connections between problems in multivariable calculus and differential geometry on the one hand and problems in topology on the other. The book consists of five chapters and three appendices. A rigorous exposition of the theory of differential forms requires a lot of algebraic preliminaries, and that is what the reader will find in Chapter 1. The authors begin with: the definition of vector space, the notions of basis, of dimension, of linear mapping, of bilinear form, and of dual space and quotient space. Then, they present the main topics of this chapter, the concept of k-tensor and the concept of alternating k-tensor. Those k-tensors come up in fact in two contexts: as alternating k-tensors, and as exterior forms, i.e., in the first context as a subspace of the space of k-tensors and in the second as a quotient space of the space of k-tensors. Both descriptions of k-tensors will be used in later applications. Finally, the second half of Chapter 1 is mostly concerned with exploring the relationships between these two descriptions and making use of these relationships to define a number of basic operations on exterior forms such as the wedge product operation, the interior product operation, and the pullback operation. The authors use these results to define the notion of an orientation for an n-dimensional vector space, a notion that will allow to get rid of the absolute value sign in the term $|\det J_f|$ of the formula $(*)$.

The concept of k-tensor is the key ingredient in the exposition of the theory of differential forms in Chapter 2. The elements of definitions of $\Omega^k(U)$ and the operator d are typical examples of differential forms. However, the authors begin this chapter with the following more precise definition: If U is an open subset of $\mathbb{R}^n$, then a k-form ω on U is a function $\omega : U \to \Lambda^k(T_p^*U)$, where T_pU is the tangent space to U at p, T_p^*U is its vector space dual, and $\Lambda^k(T_p^*)$ is the k th-order exterior power of T_p^*U. Differential 1-forms are perhaps best understood as the dual objects to vector fields, so the authors discuss this observation and recall for future use some standard facts about vector fields and their integral curves. Then they turn to the topic of k-forms and discuss a lot of explicit examples to initiate the discussion on three fundamental operations on differential forms, namely the wedge product operation, the exterior differential operation, and the pullback operation. Also, they add to this list the interior product operation of vector fields on differential forms, a generalization of the duality pairing of vector fields with 1-forms. It is shown how these operations are related to the div, curl, and grad operations in calculus. Finally, the authors show some interesting applications of the material above to physics, to electrodynamics and Maxwell's equation, as well as to classical mechanisms and the Hamilton-Jacobi equations.

Chapter 3 is devoted to the integration of forms. The change of variables formula $(*)$ in integral calculus is a special case of a more general result, the degree formula. In this chapter the authors give a proof of this result. If U is a connected open set in $\mathbb{R}^n$ and $\omega = f dx_1 \wedge \cdots \wedge dx_n$ is a compactly supported n-form on U, then the authors prove that the integral $\int_U \omega = \int_U f dx_1 \ldots dx_n$ of ω over U is zero if and only if $\omega = dv$, where v is a compactly supported $(n-1)$-form on U. An easy corollary of this result is the following weak version of the degree theorem. If U and V are connected open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a proper mapping, then there exists a constant $\delta(f)$ with the property that $\int_U f^*\omega = \delta(f) \int_V \omega$ for every compactly supported n-form ω on V. Thus to prove the degree theorem it suffices to show that this $\delta(f)$ is the $\deg(f)$ in formula $(***)$. The degree formula has a lot of interesting applications and at the end of Chapter 3 the authors present two of them. The first is the Brouwer fixed point theorem, and the second is the fundamental theorem of algebra.

The differential forms that have been considered in Chapters 2 and 3 have been forms defined on open subsets of $\mathbb{R}^n$. In Chapter 4, the authors make the transition to forms defined on manifolds. They begin this chapter with a brief introduction to the theory of manifolds including the definition of n-dimensional manifolds as n-dimensional submanifolds of $\mathbb{R}^N$, i.e., as n-dimensional generalizations of curves and surfaces in $\mathbb{R}^3$. First, they define the crucial notion of the tangent space T_pX to a manifold X at a point p and show how to define differential forms on X by defining a k-form to be, as in Chapter 2, a function ω which assigns to each $p \in X$ an element ω_p of $\Lambda^k(T_p^*X)$. Next, they define what it means for a manifold to be oriented and show that if X is an oriented n-dimensional manifold and ω is a compactly supported n-form, then integral of ω is well-defined. Finally, the authors prove a manifold version of the change of variables formula $(*)$, manifold versions of two standard results in integral calculus: the divergence theorem and Stokes theorem, and the manifold version of the degree theorem. A number of applications of this theorem is presented.

Chapter 5 deals with cohomologies via forms. If X is an n-dimensional manifold, $\Omega^k(X)$ is the space of differential forms on X of degree k, $d : \Omega^k(X) \to \Omega^{k+1}(X)$ is exterior differentiation, and $H^k(X)$ is the k th de Rham cohomology group of X, i.e., the quotient of the vector space $Z^k(X) = \ker(d : \Omega^k(X) \to \Omega^{k+1}(X))$ by the vector space $(B^k(X) = \operatorname{im}(d : \Omega^{k-1}(X) \to \Omega^k(X))$, then the authors make a systematic study of these groups and show that some of the results about differential forms that they proved in earlier chapters can be interpreted as cohomological properties of these groups. One problem these cohomology groups is that since the spaces $\Omega^k(X)$ are infinite dimensional in general, there's no guarantee that the same is the case for cohomology groups $H^k(X)$. The authors show that if X has finite topology, then its cohomology groups are finite dimensional and, secondly, they show that many manifolds have finite topology. The remainder of the Chapter 5 is devoted to the Poincaré duality, Lefschetz, and Künneth theorems, cohomology of good covers, and the Čech cohomology groups.

The book is very well written and could be readable and usable for some undergraduates.

Andrew Bucki [ajbucki@langston.edu] is a Mathematics Professor at the Langston University. He obtained his PhD at the Maria Curie-Skłodowska University in Lublin. His research focuses mainly in differential geometry. He has written 45 papers, 2 books and about 2400 reviews.

Personal Column

Please send information on mathematical awards and deaths to newsletter@ems-ph.org.

Awards

Luigi Ambrosio has been awarded the prestigious **Balzan Prize** 2019.

The **Hans Schneider Prize in Linear Algebra** is awarded every 3 years by the International Linear Algebra Society. We congratulate the 2019 winners **Lek-Heng Lim** and **Volker Mehrmann.**

Xavier Tolsa is the recipient of the 2019 **King Jaume I Award** in basic research by the Valencian Government and the Valencian Foundation for Advanced Studies.

Xavier Ros-Otón is the recipient of the 2019 **Princess of Girona Foundation Scientific Research Award** for being "one of the most brilliant young mathematicians with the greatest global impact".

Yoav Benjamini, the Nathan and Lily Silver Professor of Applied Statistics at Tel Aviv University was awarded the 2019 **Pearson Prize**.

Irit Dinur, Professor at the Department of Applied Math and Computer Science of the Weizman Institute has been awarded the 2019 **Godel Prize** for her proof of the PCP Theorem.

Joaquim Serra Montolí (ETH Zürich) has been awarded the **José Luis Rubio de Francia Prize** 2018 for young researchers of Real Sociedad Matemática Española (RSME).

Marisa Fernández Rodríguez (University of the Basque Country, Spain), **Jesús María Sanz Serna** (Universidad Carlos III de Madrid, Spain), and **Sebastià Xambó** (Universitat Politècnica de Catalunya, Spain) have been awarded the 2019 **Medals of Real Sociedad Matemática Española**.

Daniel Álvarez Gavela (Princeton University, USA), **María Ángeles García-Ferrero** (Max Planck Institute for Mathematics, Leipzig, Germany), **Xabier García Martínez** (Universida de Vigo, Spain), **Umberto Martínez Peñas** (University of Toronto, Canadá), **Carlos Mudarra Díaz-Malaguilla** (Aalto University, Finland) and **Marithania Silvero Casanova** (University of the Basque Country, Spain) have received the **Vicent Caselles Research Awards** 2019 of the Real Sociedad Matemática Española and Fundación BBVA.

ETH Zürich awarded the **Heinz Hopf Prize** 2019 to **Ehud Hrushovski** (University of Oxford) for his outstanding contributions to model theory and their application to algebra and geometry.

The President of Armenian Mathematical Union, **Yuri Movsisyan**, has been awarded the **Medal of Anania Shirakatsi** 2019, Armenia's highest state award for the development of science and for outstanding research in algebra and applications.

The **Prize for Young Contribution "Ramiro Melendreras"** was awarded at the Spanish conference of the Statistics and Operation Research Society in September 2019 to **Carmen Minuesa** for research on inference in population-size-dependent branching processes.

The **Jaroslav and Barbara Zemánek Prize** in functional analysis with emphasis on operator theory for 2019 is awarded to **Martijn Caspers** (TU Delft, Netherlands) for his work on operator algebras and operator theory and proof (jointly with Denis Potapov, Fedor Sukochev and Dmitriy Zanin) of the Nazarov–Peller conjecture.

The first **David Goss Prize** in number theory was awarded in 2019 to **Alexander Smith** of Harvard University at the First JNT Biennial Conference:

Deaths

We regret to announce the deaths of:

Erio Castagnoli (9 January, 2019, Mantova, Italy)
Victor Arzumanyan (8 April 2019, Yerevan, Armenia)
Yurii Makarovych Berezansky (7 June 2019, Kiev, Ukraine)
Igor Zaslavsky (12 October 2019, Yerevan, Armenia)

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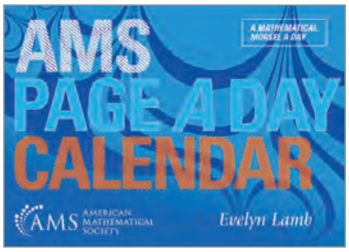

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Evelyn Lamb

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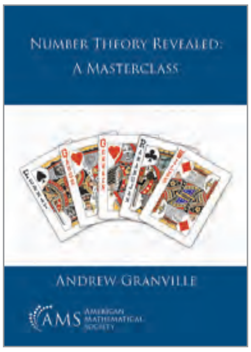

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