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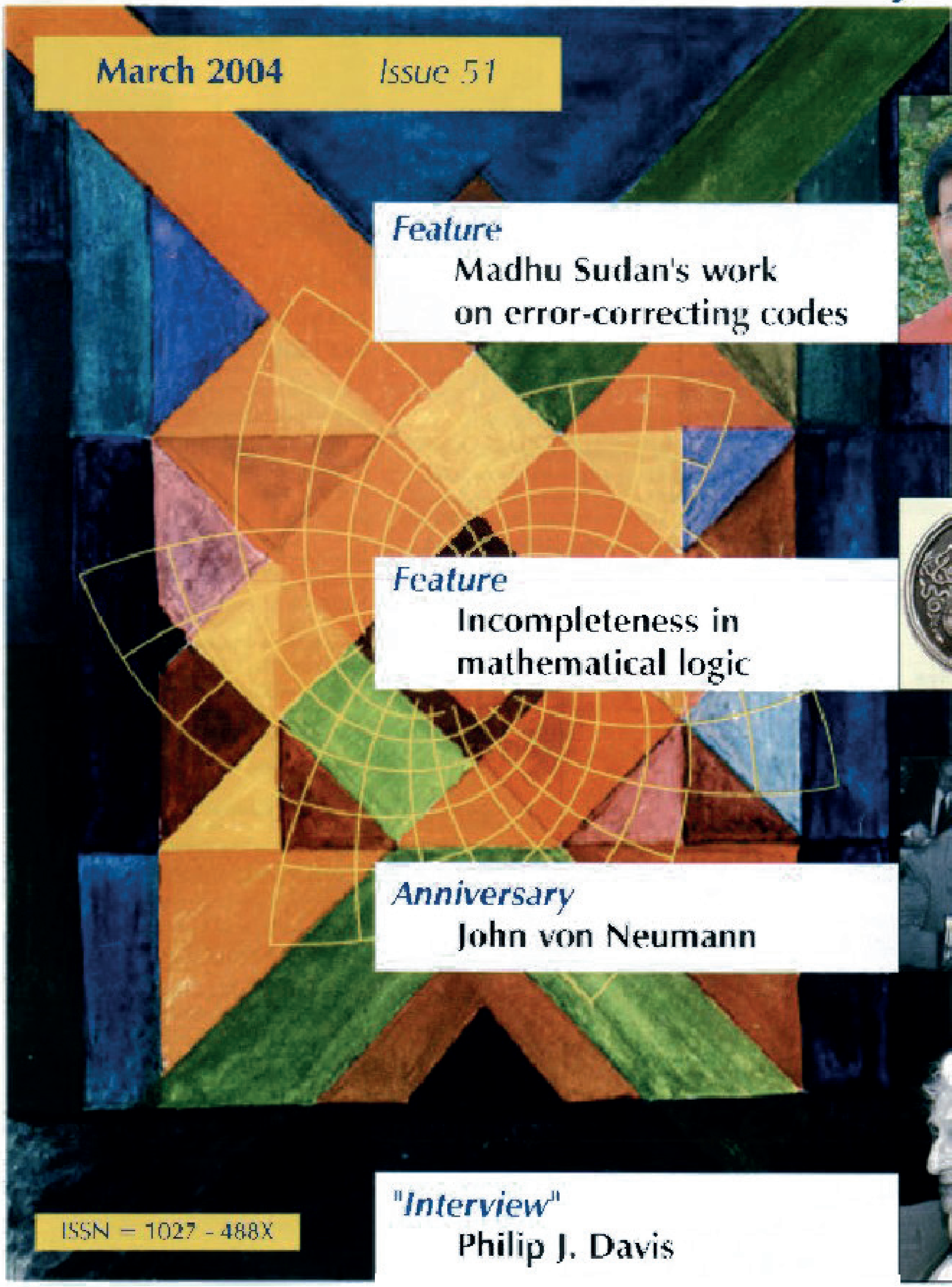

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NEWSLETTER

EUROPEAN MATHEMATICAL SOCIETY

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NOTICE FOR MATHEMATICAL SOCIETIES

Labels for the next issue will be prepared during the second half of May 2004. Please send your updated lists before then to Ms Tuulikki Mäkeläinen, Department of Mathematics, P.O. Box 4, FIN-00014 University of Helsinki, Finland; e-mail: tuulikki.makelainen@helsinki.fi

INSTITUTIONAL SUBSCRIPTIONS FOR THE EMS NEWSLETTER

Institutes and libraries can order the EMS Newsletter by mail from the EMS Secretariat, Department of Mathematics, P. O. Box 4, FI-00014 University of Helsinki, Finland, or by e-mail: (tuulikki.makelainen@helsinki.fi). Please include the name and full address (with postal code), telephone and fax number (with country code) and e-mail address. The annual subscription fee (including mailing) is 80 euros; an invoice will be sent with a sample copy of the Newsletter.

EMS Committee

EMS Agenda

2004

15 May
Deadline for submission of material for the June issue of the EMS Newsletter
Contact: Martin Raussen, email: raussen@math.aau.dk

19-24 June
EURESCO Conference
Symmetries and Integrability of Difference Equations: EuroConference on Analytic Linear and Nonlinear Difference Equations and Special Functions
Helsinki (Finland)

25 June
EMS Executive Committee meeting at Uppsala (Sweden)
Contact: Helge Holden, email: holden@math.ntnu.no

26-27 June
EMS Council meeting at Uppsala (Sweden)
Contact: Helge Holden, email: holden@math.ntnu.no
or Tuulikki Mäkeläinen email: tuulikki.makelainen@helsinki.fi

27 June - 2 July
4th European Congress of Mathematics, Stockholm
Web site: http://www.math.kth.se/4ecm

4-24 July
EMS Summer School at Cortona (Italy)
Evolution equations and applications
Contact: Graziano Gentili, email: gentili@unifi.it

15-23 July
EMS Summer School at Bedlewo (Poland)
Analysis on metric measure spaces
Contact: Olli Martio, email: Olli.Martio@helsinki.fi

30 August – 3 September
EMS Summer School at Universidad de Cantabria, Santander (Spain)
Empirical processes: Theory and Statistical Applications
Web site: http://www.eio.uva.es/ems/

3-5 September
EMS-Czech Union of Mathematicians and Physicists (Mathematics Research Section)
Mathematical Weekend, Prague (Czech Republic)
Web site: http://mvs.jcmf.cz/emsweekend/

5-6 September
EMS Executive Committee Meeting in Prague

2005

12-16 December
EMS-SIAM-UMALCA joint meeting in applied mathematical
Venue: the CMM (Centre for Mathematical Modelling), Santiago de Chile

2007

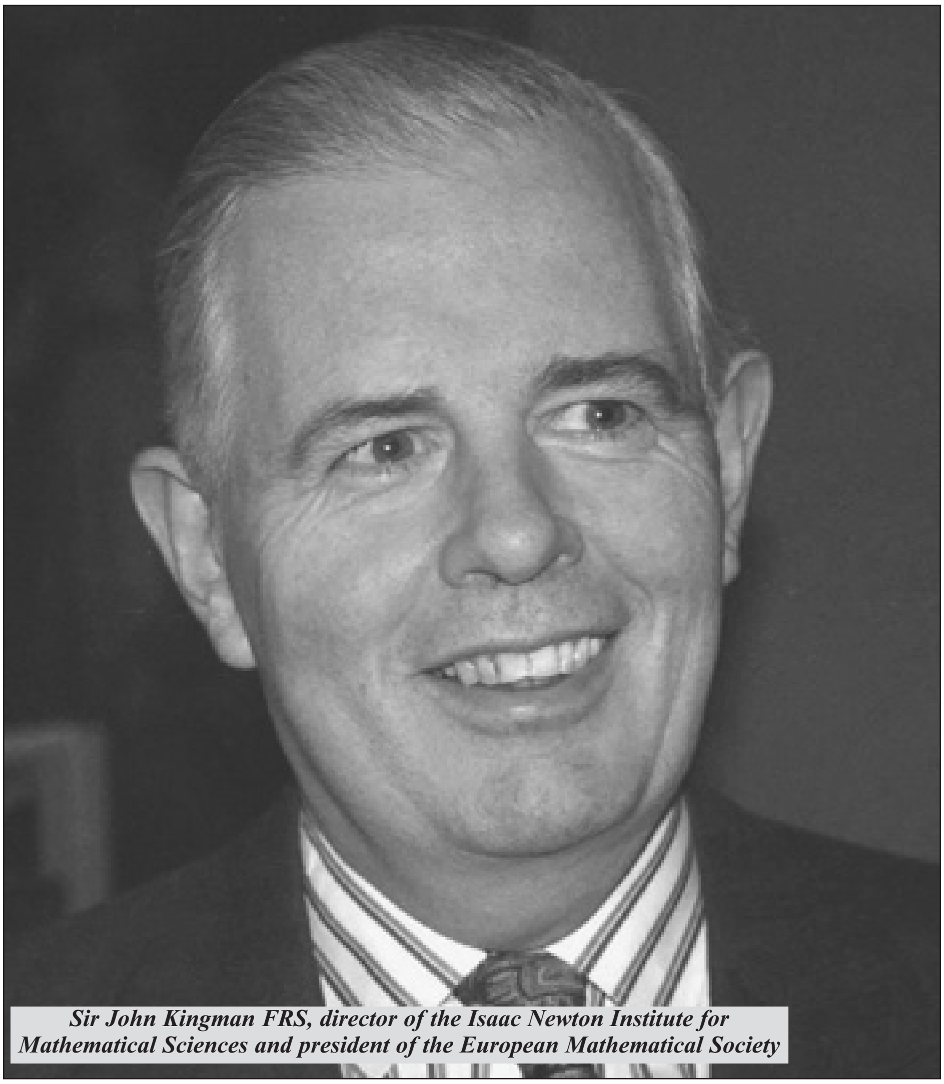
16-20 July
ICIAM 2007, Zurich (Switzerland)

Editorial

by

John Kingman

Letter from the President to the Individual Members of the European Mathematical Society



Sir John Kingman FRS, director of the Isaac Newton Institute for Mathematical Sciences and president of the European Mathematical Society

Dear Colleague,

I am taking the opportunity of this issue of the Newsletter, before the Society's Council meets in Uppsala in June, to send a message to all those who support the Society through individual membership. Although the corporate membership of the national mathematical societies is the backbone of the EMS, it adds immeasurably to our strength that individual mathematicians take the trouble to join and take part in our activities.

Some of you will be at the Uppsala Council, as representatives of the whole body of individual members, or as delegates of societies, and I look forward to hearing your views on the policies and priorities that your Society should pursue. Many more, I hope, will be at the European Congress in Stockholm from 28 June to 2 July. Ari Laptev and his team have produced a really exciting programme, covering a great range of mathematics and its applications, which will repay the attendance of any mathematician.

The richness of the subject, and the variety and importance of the fields to which it is now applied, is a major theme of the message we are trying to get across within Europe. There is a tendency among scientists to recognise the importance of traditional mathematics, but to feel that this is enough and that the search for new mathematics is an eccentric activity of little practical importance. They could not be more wrong. The challenges of the modern world, of its science and technology, require new mathematical insights and techniques, and an active mathematical research base is essential underpinning for a prosperous and civilised Europe.

As I write, the pace of debate about a new European Research Council is accelerating, and the EMS is taking an active part. We must ensure that any new structure for research support in Europe takes full account of the needs of the mathematical community, and of the contribution it can make to research in almost every field. This is not true only within the European Union, but across all our countries, so that funding bodies like UNESCO and the European Science Foundation are targets too.

I believe that the message is now increasingly being heard, but we need to take every opportunity to press the case, nationally and internationally.

Mathematicians need not only support for their research, for conferences and summer schools and the means to travel, but also ways of publishing their results, whether by traditional printing or electronically. One of the achievements of the Society has been to set up its own publishing house, dedicated to meeting the needs of the mathematical community rather than excessive profits for shareholders. We are now publishing journals including our own *JEMS*, and the first of what will be a significant series of books is now out. It is a challenge to compete with the long experience and sheer scale of the major commercial publishers, without large capital resources, but it is a challenge we shall meet in the interests of healthy mathematical publication.

The responsibility for all this and much more falls on a rather small group of people, to whom I should like to pay tribute. The Officers and Editors and the members of the Executive Committee have to fit in their Society functions with busy lives as working mathematicians. We could not function without the help of Tuulikki Mäkeläinen, who runs our Helsinki office with cheerful devotion and efficiency. Anyone who has experienced, for instance, contract negotiations with the EU or UNESCO, will understand the sort of pressures that come, often with the tightest of timescales. We should be able to spread the load more widely, and this is an issue I shall raise at the Uppsala Council.

I end as I began, with thanks to you all for your support of your Society. Together we can make a real difference to mathematics in Europe. I wish you all a successful year, with many hard problems solved and many exciting theorems proved.

Yours very sincerely,

John Kingman

Zentralblatt

as a portal to the mathematical literature: from 1868 to the present

Silke Göbel (FIZ Karlsruhe)
and Bernd Wegner (TU Berlin)

In spring 2003, the database ZMATH of Zentralblatt für Mathematik extended its services for information pertaining to mathematical literature from 1931 to the present to include also the information from the Jahrbuch back to 1868. With this unique user service, it is now possible to find references for the complete literature from more than 135 years, with the aid of a search menu. A fundamental requirement of the mathematical information services has thus been completed. As mathematical results have no expiration date, it is important for researchers in Mathematics and its Applications to have complete information on the literature and access to the corresponding text, even when the publications are older. In this respect, the extended offer of Zentralblatt is an important milestone in the development of the WDML (World Digital Library in Mathematics), which is backed by the IMU.

Traditionally, the data base ZMATH contained the information from Zentralblatt für Mathematik, which was founded in 1931. Even before the extension to the contents of the Jahrbuch, it was the leading reviewer service of mathematics. The contents of the Jahrbuch were captured in a data base over a period of several years as a part of the ERAM project, sponsored in part by the DFG (Deutsche Forschungsgemeinschaft, the German Research Council). The document registration was completed in the year 2003.

The Jahrbuch

The "Jahrbuch über die Fortschritte der Mathematik" was the first comprehensive review journal for the field of Mathematics. It contains information on nearly all publications in Mathematics and its Applications for the time period from 1868 to 1942. It was founded by the mathematicians Carl Orthmann and Felix Mueller, appeared yearly (with a few exceptions), and contained anywhere from 880 (in 1868) to 7000 (in 1930) references to mathematics literature. The reviews were written in part by such famous mathematicians as Felix Klein, Sophus Lie, Richard Courant, or Emmy Noether. In WW II the appearance of the Jahrbuch was delayed and after the war it was never taken up again. The concept of the Jahrbuch was marked by its service as a source of complete documentation information. It appeared only after all of the work of the year in question had been completed. This proved later to be a drawback, as the information was no longer up-to-date when it went to print.

Development of the Jahrbuch data base

The entire information contained within the printed version of the Jahrbuch was newly recorded and extended in a data base by the TU Berlin and SUB (Staats- und Universitätsbibliothek) Göttingen, as main partners in the ERAM (Electronic Research Archive for Mathematics) project. A structure was selected that allows the information to be searched for within the framework of the usual Zentralblatt search engine in order for the project to make use of the infrastructure of the editorial staff of Zentralblatt. To take into account the demands of a modern literature data base, classifications were added corresponding to the MSC2000 scheme, as were keywords and English translations of titles. These tasks were carried out by more than 70 experts from all over the world on a voluntary basis. They also added comments and hints to the importance of a given work.

The cooperation between Berlin and Göttingen covers further standardizations and extensions. A journal data base is under development that identifies sources whose names or abbreviations have been altered. Such identification is under construction for the entire area from 1868 to the present. The editorial work on the Jahrbuch data is not yet complete and will always remain open when one considers the commentaries. The cooperation of further experts is always welcome.

Links to the complete texts

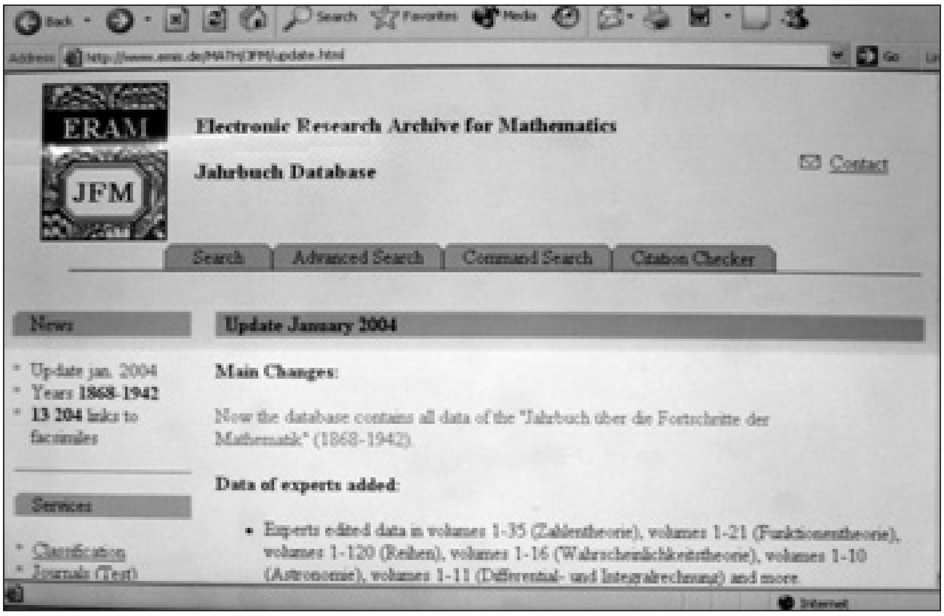
In addition to the recording of the Jahrbuch data, the ERAM project includes a second component, that of the digitization of mathematical publications, especially from the time period of the Jahrbuch. The capacity of the project is approximately 1.2 million pages. The texts are freely accessible. This capacity will be reached in the course of the current year. A series of important journals are already in the digital archive developed by Göttingen, including "Mathematische Annalen", "Mathematische Zeitschrift", "Inventiones Mathematicae", and "Commentarii Mathematici Helvetici", as well as the classical publications such as "Abhandlungen der Gesellschaft der Wissenschaften in Göttingen" and "Kleins Enzyklopädie der Mathematik". If the publication period of a journal exceeds that of the Jahrbuch, then the digitization covers the years until the journal appeared in electronic form over the internet.

When available, links from the Jahrbuch data to other electronic sources pertaining to a given publication, are provided by the data base ZMATH. The installation of these links from the Jahrbuch data base is also a part of the project and concerns mainly the digital archive developed for ERAM. There are presently more than 3300 such links. Links also exist to other digital archives such as the "Digital Library in Cornell", the "University of Michigan Historical Math Collection", the scanned articles in Project GALLICA, the "Monatshefte für Mathematik und Physik" at the Universität Graz and works from the "Berlin-Brandenburgischen Akademie der Wissenschaften". There are plans for further links to the archives of JSTOR and NUMDAM (France), "Bibliotheka Wirtualna Matematyki" (Poland), and the Einstein Archive (Israel). Analogous activities will also be carried out for ZMATH to encompass the time from 1931 to the present.

Connection between ZMATH and Jahrbuch

Historical aspects have led to the fact that the Jahrbuch data base, which contains additional information such as commentaries provided by experts, has more fields than the data base ZMATH is currently handling. This additional information is therefore not available through the ZMATH web service. To retrieve this type of information, there is a link from ZMATH to a separate site for the installation of the Jahrbuch data base.

Both data bases can be found on the internet at the FIZ Karlsruhe and the European Mathematical Society's EMIS server:
ZMATH: *http//www.emis.de/ZMATH*
ERAM: *http///www.emis.de/projects/JFM*
The latter address also provides access to the full texts offered by ERAM.



What the EMS did in 2003*

David Salinger, EMS Publicity Officer

There were two good-humoured and productive executive committee meetings: 8-9 February in Nice and 14-15 September in Lisbon.

The committees of the EMS are active and are listed on EMIS. Much of the Society's work is conducted via these committees, whose activities will be reported to Council in June.

The Society is represented on the steering committee of the Banach Centre (Poland), on the European Science Foundation's Euresco committee, the President attends the ESF standing committee for Physical and Engineering Sciences (PESC) and the Past President, Rolf Jeltsch, represented the Society at the International Congress on Industrial and Applied Mathematics (ICIAM) in Sydney.

The EMS has been actively engaged with the 6th Framework programme of the European Union and has vigorously emphasized the rôle of mathematics in discussions preparing the way for a possible European Research Council.

EMS obtained the support of UNESCO-Roste for the conference (on applied mathematics and the applications of mathematics) held in conjunction with the French mathematical societies SMF and SMAI, in Nice in February. UNESCO-Roste also supported the two Summer Schools: 'Dynamical Systems' in Porto and 'Stochastic Methods in Finance' in Bressanone, the latter run by the Centro Internazionale Matematico Estivo (CIME).

The first joint 'mathematical weekend' with a member society was held in Lisbon in September. This was an idea emanating from the Portuguese Mathematical Society, so the honour of organising the first one fell to them.

There was a joint meeting with SIAM and IPAM, in Los Angeles in May on 'Applied Inverse Problems'.

Preparations for the fourth European Congress of Mathematics are well in hand: all that remains is for you to attend, in Stockholm, 27th June to 2nd July 2004 (*www.math.kth.se/4ecm/*).

The first publications, both books and a journal, of the EMS Publishing House appeared in 2003 (and are now available from *www.ems-ph.org*).

The fifth volume of the Society's journal, JEMS, was published. Haim Brezis was appointed Editor-in-Chief in succession to Jürgen Jost.

Martin Raussen succeeded Robin Wilson as Newsletter editor.

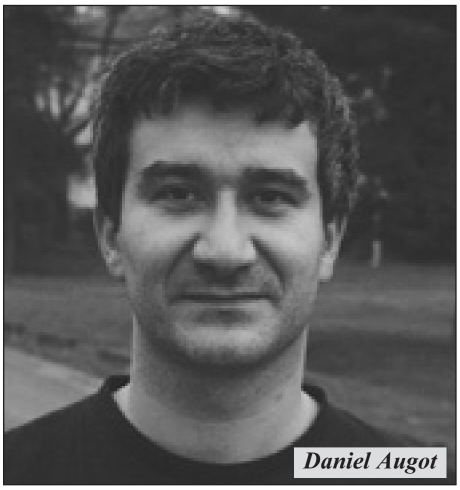
EMS continued as a participant in various electronic publishing activities: LIMES, EULER, the Co-ordinating Committee for Zentralblatt für Matematik (of course, Zentralblatt continues in a hard copy version as well).

EMS has 51 full, 1 associate, 2 institutional, and 20 academic institutional corporate members. Individual members number about 2300. In 2003, the Society collected 82,324 euros in membership fees, which was ahead of budget.

*Based on Tuulikki Mäkeläinen's annual 'Activity Report' to the auditors.

Madhu Sudan's contributions to the theory of error-correcting codes

Daniel Augot (INRIA-Rocquencourt, France)


Daniel Augot

Abstract

This paper presents aspects of Madhu Sudan's work on error-correcting codes, which, among other of his achievements[1], was the reason for awarding him the Nevanlinna Prize in August 2002[2], the equivalent of the Fields Medal within the domain of mathematical aspects of computer science. Madhu Sudan's major breakthrough is a decoding algorithm for Reed-Solomon and algebraic geometry codes; it decodes in the presence of far more errors than the usual decoding capability of these codes allow for. In fact, the algorithm returns a list of potential candidates instead of a unique solution. Such a decoder is called a list-decoder.

The algorithm is first of all a fundamental theoretical advance, whose practical and theoretical implications lie still in the future. In this survey, we introduce the practical background for coding theory, and we explain two classes of codes, the Reed-Solomon and the algebraic-geometric codes. In the end, we present Madhu Sudan's decoding algorithm, as a generalization of the Berlekamp-Welch decoding algorithm.

This paper is based on the presentation that Madhu Sudan himself gave in [6]. It appeared in French in *gazette des mathématiciens* **98**, oct. 2003, pp. 5 – 13. We thank the editor of *gazette* for the permission to reproduce it in this Newsletter. The author wishes to thank the copy editors of the Newsletter for assistance with the translation into English.

Introduction and Definitions

The theory of error-correcting codes is the discipline of applied mathematics whose subject is the reliable transmission of information through a noisy transmission channel, using combinatorial and algorithmic objects called *error-correcting codes*. To introduce the subject, we will first explain the basic notions of coding theory.

Let us follow a message on its way from the sender to the receiver, and let us observe the interesting notions which appear. There are several entities involved in the process: the sender, the receiver and the transmission channel. The aim of the sender is to transmit a *message* m to the receiver, where m belongs to a finite space $\mathcal{M}$, *the message space*. A noisy transmission channel is able to convey arbitrarily long sequences of symbols from an alphabet Σ, which is "small" (one of the most interesting cases being $\Sigma = \{0,1\}$). The space of messages to be encoded is $\mathcal{M} = \Sigma^k$, the set of sequences of length k of symbols from Σ.

Sender and receiver agree on the length n on the transmitted sequences, called the *length of the code*, and thus the messages that can be exchanged belong to Σ^n, which is called *the ambient space*. Moreover, sender and receiver have to agree on the encoding E, which is an injective function $E : \mathcal{M} \to \Sigma^n$, which is used to code the messages before transmission. The image $C = \{E(m), m \in \mathcal{M}\}$ is called the *code*. The fraction k/n, usually denoted by R, is the *transmission rate* of the code; in coding theory, it is considered as *the first fundamental parameter* of a code.

The transmission channel introduces noise on the transmitted messages. This noise can be seen as a function from the ambient space into itself. From now on, we suppose that the alphabet Σ is a field (Σ is most often a finite field of small size q), thus inducing a vector space structure on Σ^n. It is then convenient to consider *linear codes*, that is, codes which are the image of Σ^k under a *linear map* from Σ^k to Σ^n, which will always be supposed non-singular. We shall now specify a linear code C by its *generator matrix* (its rows form a basis of C), which describes a set of size q^k in a compact manner. The transmission channel produces a *noise vector* $e \in \Sigma^n$, such that the received word is $y = E(m) + e$ with m denoting the sender's message. Now the receiver has to apply a decoding function $D : \Sigma^n \to \mathcal{M}$. This decoding function D, which corresponds to an efficient algorithm, is such that $D(y) = m$ with probability close to one. Informally, the encoding introduces a redundancy by augmenting the length of the messages. This redundancy can then be used to decode the transmitted message, even in the presence of noise. From the point of view of reliable communication, the fundamental question of coding theory is

Given a probability distribution P for the transmission channel (that is, a probability distribution for additive errors), what are the best encoding and decoding functions ? In mathematical terms, what is the smallest error probability:

$$\min_{E,D} \left\{ \mathbf{E}_{m \in \mathcal{M}} \left(\Pr_{\eta \in P} [D(E(m) + \eta) \neq m] \right) \right\}$$

where $\mathbf{E}$ *denotes the mathematical expectation.*

[1] Madhu Sudan has important contributions to the field of computational complexity, and he is one of the author of the famous PCP theorem [8], related to the conjecture $P \neq NP$.

[2] http://www.maa.org/news/fields02.html/

Shannon studied the asymptotic properties of this quantity when the noise distribution on Σ^n is the cartesian product of a noise distribution on Σ itself. In this context, there exists a quantity $C_0 \in [0,1]$, depending on the transmission channel, such that, for any rate $R < C_0$ and any $\epsilon > 0$, there exists a couple coding/decoding, with a code of rate R, such that the error probability after decoding is at most ϵ – for n sufficiently large.

For the rest of this presentation, we shall only consider the case of a *q-ary symmetric channel*, defined as follows: each transmitted symbol $\sigma \in \Sigma$ is preserved with a probability $1-\delta$, or is transformed into another symbol $\sigma' \in \Sigma$ with probability $\delta/(q-1)$; moreover, these events have to occur independently from one transmitted symbol to another.

On the other hand, Hamming defined the notions of an *error-correcting code* and of an *error-detecting code*. Let us define the *Hamming weight* of a sequence $x \in \Sigma^n$ as the number of non-zero coordinates of x, and the *Hamming distance* between x and y as the weight of the difference $x-y$ (that is to say, the number of coordinates in which x and y differ). This defines in fact a metric on the message space. Furthermore, we define the *minimum distance* of a code C as the smallest distance between two distinct codewords in C. The transmission channel generally generates errors η of small weight, for instance errors whose weight is bounded by e. We shall say that a code C corrects e errors if the balls of radius e centered at the codewords do not intersect. Indeed, if the weight of the error is less than or equal to e, and if C is e-correcting, then there is a *unique* codeword closest to the received word. An error correcting capability equal to e implies that the minimum distance between two distinct codewords has to be bigger than or equal to $2e+1$. The minimum distance of a code is the *second fundamental parameter* of a code. We shall use the notation $[n,k,d]$-code, for a code of length n, dimension k and minimum distance d. From Hamming's point of view, the fundamental question is

Given an alphabet Σ of size q, and two positive integers n and k, $k < n$, which is the larger minimum distance d of a code $C \subseteq \Sigma^n$ with transmission rate k/n ?

We have to admit that the Hamming problem is not yet solved when the size of the alphabet is small compared to n (one of the most important cases is of course $q = 2$). There exists a satisfactory answer to the question when q is bigger than or equal to n (see the Reed-Solomon codes in the next section).

Constructions of codes

We shall begin with the "family" of random codes (in fact a *non-construction*). Let $V_q(n, d-1)$ denote the volume (number of words of length n over an alphabet of size q) of the Hamming sphere of radius $d-1$. The Varshamov-Gilbert bound states that there exists a $[n,k,d]$ code if the parameters n, k, d satisfy:

$$q^k V_q(n, d-1) \leq q^n.$$

Now let $H_q(x)$ denote the q-ary entropy function: $H_q(\delta) = -\delta \log_q(\frac{\delta}{q-1}) - (1-\delta)\log_q(1-\delta)$. Taking logarithms at the base q and using the approximation $V_q(n, \delta n) \approx q^{H_q(\delta)n}$, we get that there exists codes on the following bound

$$R \geq 1 - H_q(\delta), \text{ with } R = \frac{k}{n} \text{ and } \delta = \frac{d}{n}.$$

This result is "in particular true" for random codes: When the length n grows, a random code lies on the Varshamov-Gilbert bound with a probability (exponentially) close to one. The question that immediately arises is whether codes exist *above* the Varshamov-Gilbert bound.

Apart from random codes, coding theory has tried – since the foundational work of Shannon and Hamming – to produce explicit and constructive families of good codes, whose parameters (dimension and minimum distance) can be explicitly computed. This led to a whole zoo of code families, some of which have found application in practice.

Let us cite another bound, the Singleton bound. This result states that the parameters of any $[n,k,d]$ linear code have to satisfy $k + d \leq n+1$. To prove it, let us consider a *parity check matrix* of such a code: it is an $(n-k) \times n$ matrix whose rows correspond to $n-k$ linear forms vanishing on C: any codeword c satisfies $Hc = 0$. The rank of this matrix H is at most $n-k$. Since the code does not contain any non-zero word of weight less than d, there exists no linear relations among any set of $d-1$ columns of H, i.e.: $d-1 \leq n-k$.

From a more constructive point of view, we will limit the presentation to Reed-Solomon codes, which are optimal with respect to the Singleton bound, and to geometric Goppa codes (also called algebraic geometry codes).

Reed-Solomon codes. An $[n,k]-$Reed-Solomon code of dimension k and length n is defined by n distinct elements $\alpha_1, \ldots, \alpha_n \in \mathbf{F}_q$, where $\mathbf{F}_q$ denotes the finite field of q elements. We shall call these elements *points*. Let $\mathbf{F}_q[x]$ denote the algebra of univariate polynomials in the indeterminate x, and let ev denote the map

$$\begin{array}{rcl} \mathrm{ev} : \mathbf{F}_q[x] & \rightarrow & \mathbf{F}_q^n, \\ \mathrm{ev}(f(x)) & = & (f(\alpha_1), \ldots, f(\alpha_n)). \end{array}$$

Then the Reed-Solomon code RS_k, of dimension k and *support* $\alpha_1, \ldots, \alpha_n$, is given by

$$RS_k = \{\mathrm{ev}(f(x));\ \deg(f(x)) < k\}.$$

It is easy to see that its dimension is k, and that the minimum distance is $d = n-k+1$. Indeed, any polynomial of degree less than k has at most $k-1$ zeros, and the vector $\mathrm{ev}(f(x))$ has at least $n-k+1$ non-zero coordinates. Thus for Reed-Solomon codes, we get the equation $k + d = n+1$, which corresponds precisely to the Singleton bound.

Dividing the code parameters by the length n of the code, we observe that a Reed-Solomon code with rate R has (approximately) a minimum distance of $1-R$. Hence these parameters are satisfactory, but it is a major drawback that it is not possible to let the length of code words grow to infinity while keeping the size of the alphabet finite.

Geometric Goppa codes. The geometric Goppa codes were introduced by Goppa in [1]. They are natural generalizations of the Reed-Solomon codes. We present a simplified version, known as "one point" algebraic geometry codes.

Let C denote a smooth irreducible algebraic curve, defined over $\mathbf{F}_q$. Let $P_1, \ldots, P_n$ be n distinct rational points on C and let P_∞ be another rational point distinct from $P_1, \ldots, P_n$. Let $L(lP_\infty)$ denote the function space in $\mathbf{F}_q(C)$ associated to the divisor lP_∞. Again let ev denote the evaluation map:

$$\begin{array}{rcl} \mathrm{ev} : L(lP_\infty) & \rightarrow & \mathbf{F}_q^n, \\ \mathrm{ev}(f) & = & (f(P_1), \ldots, f(P_n)). \end{array}$$

In analogy to the Reed-Solomon codes, we defined the geometric Goppa code $\Gamma(P_1,\ldots,P_n,lP_\infty)$ as follows:

$$\Gamma(P_1,\ldots,P_n,lP_\infty) = \{\mathrm{ev}(f);\ f \in L(lP_\infty)\}\,.$$

If the curve C has genus g and if l has been chosen such that $l \geq 2g-2$, then the Riemann-Roch theorem implies that the dimension $\Gamma(P_1,\ldots,P_n,lP_\infty)$ is $l-g+1$. In the same vein, it is easy to prove that the minimum distance d satisfies $d \geq n-k$. For these codes, we obtain $k+d \geq n-g+1$, and we see that the "default" with respect to Reed-Solomon codes and the Singleton bound is given by g, the genus of the curve. On the other hand, it is possible to construct codes of growing length, *over a fixed alphabet*, by constructing curves with more and more points.

Using a construction of a family of curves whose genus does not grow too fast, Tsfasmann, Vladut and Zink [7] showed the existence of codes with relative distance δ and transmission rate R greater than or equal to $1-\delta-\frac{1}{\sqrt{q-1}}$. When the size of the alphabet is a square larger than 49, these codes provide better parameters than random codes, that is to say, their parameters are *above* the Varshamov-Gilbert bound. This result came as a big surprise for coding theorists, who thought that the Varshamov-Gilbert bound was an optimal bound. Nevertheless, in the very important binary case ($q = 2$), it is still not known if the Varshamov-Gilbert bound is optimal or if codes exist with parameters above that bound.

Decoding algorithms

The problem. Decoding is a difficult task, and naive algorithms yield a very bad complexity[3]. For example, an *exhaustive search*, which enumerates all codewords in order to find the one closest to the received word, has an exponential complexity in terms of the length of the code (with the condition that the dimension of the code grows linearly with the length of the code).

It is not obvious how to give a precise statement for the decoding problem; in fact, there are various different formulations in the literature. The community of researchers considers – without having a satisfactory proof – that decoding of random codes is a difficult task. This means that, although random codes have good parameters (the Varshamov-Gilbert bound), it seems impossible to decode them in an efficient way.

It remains to define the problem for fixed families of codes (Reed-Solomon, Geometric Goppa codes). Following Madhu Sudan's presentation, we will stick to the following definitions:

NCP (Nearest Codeword Problem) The problem is to find the nearest codeword to the received word, with respect to the Hamming distance.

LD (List Decoding) An integer e is given. The problem is to find *all* (possibly none) codewords at distance e from the received word.

BDD (Bounded Distance Decoding) An integer e is given. The problem is to find *one* (possibly none) word among the words at distance e from the received word.

UD (Unambiguous Decoding) Here $e = (d-1)/2$ is given, where d is the minimum distance of the code. The problem is to find the (unique, if it exists) codeword at distance e from the received word.[4]

Classically, the most studied problem has been the last one (unambiguous decoding).

Unambiguous decoding This problem has been solved with efficient algorithms for both families introduced in this article. Note that each of these algorithms is non-trivial, and that the most spectacular progress has been to achieve the decoding of geometric Goppa codes (see the survey [3]). We shall not present these algorithms here, but let us note that they generally require quadratic complexity in terms of the length (or almost-quadratic[5]), and that every such algorithm is constructed ad hoc for each family of codes: a generic algorithm with good complexity does not exist.

As an example and serving as background for Sudan's decoding algorithm, we present the Berlekamp-Welch decoding algorithm which decodes Reed-Solomon codes. The problem can be stated as follows: given n points $\alpha_1,\ldots,\alpha_n$ in $\mathbf{F}_q$, n values $y_1,\ldots,y_n$ in $\mathbf{F}_q$, and an integer $e < \frac{n-k}{2}$, find all univariate polynomials $f(x)$ in $\mathbf{F}_q[x]$ of degree less than k, such that $f(x_i) = y_i$ for at least $n-e$ values of i. Note that for such a value of e, decoding is unambiguous, and hence there is at most one solution $f(x)$.

Let $f(x)$ be the polynomial solution to the problem, and consider the monic polynomial $E(x)$, of degree e (the error weight), such that $E(\alpha_i) = 0$ if there is an error at position i. Then, we can be certain that, for every i, either $y_i = f(\alpha_i)$ or $E(\alpha_i) = 0$. An equivalent algebraic statement is: For every i, $E(\alpha_i)y_i = E(\alpha_i)f(\alpha_i)$.

The Berlekamp-Welch decoding algorithm follows these two steps: First find two polynomials $E(x)$ ad $N(x)$ of degrees e and $e+k-1$, respectively, such that $E(\alpha_i)y_i = N(\alpha_i)$ for every i. This is evidently a problem of linear equations with the coefficients of $E(x)$ and $N(x)$ as the unknowns. Hence, it can be solved, for instance, by Gaussian elimination. In particular, there exists a solution when the number of unknowns is less than the number of equations, i.e., when $e+k+e < n$ or $e < \frac{n-k}{2}$, – the correction capability of the Reed-Solomon codes! The second step of the algorithm is to return the polynomial $N(x)/E(x)$ which can be checked to be a unique polynomium, regardless of the specific solution $(E(x), N(x))$ of the linear algebra problem.

List Decoding (LD) The list decoding problem occurs when one wants to decode e errors with $e > \lfloor\frac{d-1}{2}\rfloor$, where d is the minimum distance of the code. Indeed, in that case, there is no longer always a unique codeword at distance e, and it becomes necessary to return a *list* of candidates.

The principal contribution of Madhu Sudan [5] concerns the decoding of Reed-Solomon codes. He produced

[3]The *time complexity* of an algorithm is the number of elementary operations necessary for the algorithm to perform. For decoding algorithms, we shall distinguish between the *performance* of an algorithm, which is a measure of its capability to correct errors, and its *complexity*, which measures its execution time.

[4]For the sake of completeness, let us mention the existence of a large class of codes and corresponding decoding algorithm, which perform well in practice, the families of turbo-codes and of LDPC codes, together with iterative decoding algorithms. For these algorithms, the notion of minimum distance is of marginal importance.

[5]$O(n^{7/3})$

an algorithm able to decode much more than the error correction capability $\lfloor \frac{d-1}{2} \rfloor$ of the code. The problem is again simply formulated: given n points $\alpha_1, \ldots, \alpha_n$ in $\mathbf{F}_q$, n values $y_1, \ldots, y_n$ in $\mathbf{F}_q$, and an integer e, find all polynomials $f(x) \in \mathbf{F}_q[x]$ of degree less than k such that $f(\alpha_i) = y_i$ for at least $n - e$ values of i. But, in this case, the integer e may be greater than $\frac{n-k}{2}$, and thus there may be several solutions. The algorithm devised by Madhu Sudan is a generalization of the Berlekamp-Welch algorithm, based on the following.

Recall that the Berlekamp-Welch decoding algorithm looks for a polynomial $Q(x,y) = N(x) - yE(x)$ of degree 1 in Y such that $Q(x_i, y_i) = 0$ for every i, under the restrictions $\deg N(x) \leq k + e$ and $\deg E(x) \leq e$. The idea of Madhu Sudan is to look for a *bivariate* polynomial $Q(x,y) = \sum_i Q_i(x) y^i$ *of degree greater than or equal to* 1 in y such that $\deg Q_i(x) < n - e - i(k-1)$, and such that $Q(x_i, y_i) = 0$ for every i. Then, every univariate polynomial $f(x)$, that solves the decoding problem, satisfies $Q(x, f(x)) = 0$. Indeed, the polynomial $Q(x, f(x))$ has a degree strictly less than $n - e$, by construction of $Q(x,y)$. Moreover, since $f(x_i) = y_i$ for at least $n - e$ values of i and $Q(x_i, y_i) = 0$ for every i, we see that the polynomial $Q(x, f(x))$ has more roots than its degree. Hence it is identically zero.

As a consequence, Sudan's decoding algorithm consists of two steps:

1. Find a bivariate polynomial $Q(x,y)$ such that the above conditions hold.

2. Find all factors of $Q(x,y)$ of the form $y - f(x)$.

The first step is again clearly a linear algebra problem: The unknowns are the coefficients of $Q(x,y) = \sum_i Q_i(x) y^i$ such that $Q(x_i, y_i) = 0$ for every i, under the previous constraints on the degree of the $Q_i(x)$. To make sure that there always exists a non-zero solution $Q(x,y)$, we will ask that the following sufficient condition is satisfied: If N_Q is the number of monomials appearing in $Q(x,y)$, then $N_Q > n$. We note that the degree of the polynomial $Q(x,y)$ is at most $\lfloor \frac{n-e}{k-1} \rfloor$: this is an upper bound of the number of solutions of the equation $Q(x, f(x)) = 0$; thus it is also an upper bound for the number of solutions to our problem. A straightforward computation of the number of monomials of $Q(x,y)$ and the condition $N_Q > n$ leads to the following relation:

$$e < n - \sqrt{2kn} \tag{1}$$

This bound should be compared with the bound for Unambiguous Decoding, i.e.: $e < \frac{n-k}{2}$. Dividing by n, we get $\epsilon < 1 - \sqrt{2R}$, with $\epsilon = e/n$ and $R = k/n$. The performances of both algorithms are compared in Figure 1.

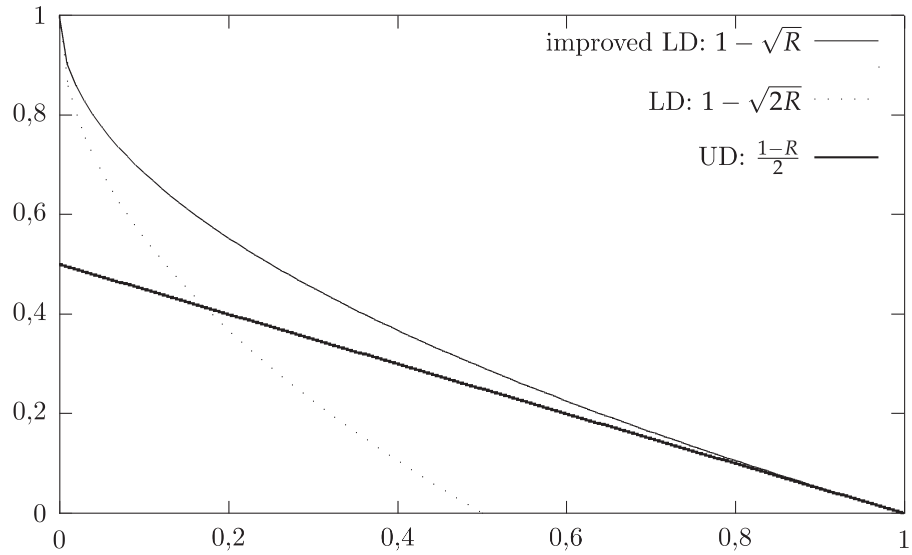


Figure 1: Performance of the Berlekamp-Welch, Sudan and Guruswami-Sudan algorithms. x-coordinate: the transmission rate R; y-coordinate: the error correcting capability.

The first (negative) remark is that the correction capability of Sudan's algorithm is *not always* greater than the capability of the Berlekamp-Welch decoding algorithm, in particular for codes with big rate R. The second (positive) remark is that the error correcting capability approaches 1, when the transmission rate is close to zero.

The second step in Sudan's decoding algorithm consists in finding the polynomials $f(x)$ which are solutions to the equation $Q(x, f(x)) = 0$, i.e., to find all factors of $Q(x,y)$ of the form $y - f(x)$. We recall that there exists efficient algorithms of polynomial complexity to factor multivariate polynomials over a finite field. Madhu Sudan thus concludes that his algorithm has a polynomial complexity. Of course, in practice, this is not sufficient, and research is ongoing to find better algorithms to perform the two steps of Sudan's algorithm in order to obtain a completely effective algorithm which could then be implemented in hardware.

Improved list decoding Venkatesan Guruswami and Madhu Sudan quickly proposed an algorithm extending the error capability up to the bound $1 - \sqrt{R}$, which is always better than the classical bound $\frac{1-R}{2}$ (see figure 1). Without going into the details of the article [2], we make the following remark. If $f_1(x)$ and $f_2(x)$ are two solutions to the decoding problem, then there may be indices i such that $y_i = f_1(x_i) = f_2(x_i)$. Since $y - f_1(x)$ and $y - f_2(x)$ both divide $Q(x, y)$, it follows that $Q(x, y)$ will have a multiplicity of order at least 2 at the point (x_i, y_i). This insight is point of departure for the improved algorithm: as for the simple algorithm, it searches for a polynomial $Q(x, y)$ that satisfies the condition $Q(x_i, y_i) = 0$ with a given (auxiliary) multiplicity r. If the degree of $Q(x, y)$ is well chosen (given by an auxiliary parameter l), then, for any solution to the decoding problem, we will get $Q(x, f(x)) = 0$. Optimizing the auxiliary parameters r and l leads to the bound $1 - \sqrt{R}$.

To conclude this section, we cite [2, 4] to remark that both algorithms (Sudan, Guruswami-Sudan) can be generalized rather easily to algebraic geometry codes. It comes as a surprise that these generalizations lead to algorithms which are conceptually simpler than the previously known algorithms designed for solving the classical decoding problem.

Conclusion

The discovery of an algorithm with such an error correcting capability has renewed the interest in applications of both the Reed-Solomon codes and the algebraic geometry codes, since their *performance* is greatly improved. Of course, there remains the problem of choosing a solution among the list of solutions given by the algorithm. But in practice, the returned list is of size one with probability close to one. Moreover, work has to be done to improve the implementation of the algorithm in order to obtain a speed that is suitable for applications.

The algorithm has already found application in cryptology (to cryptanalyse a symmetric encryption algorithm), and also in the domain of Intellectual Property Right Management. In both cases, the high level of correction, for codes with rate close to 0, is used. There is no doubt that, from a theoretical point of view, this algorithm will find many applications in various branches of applied mathematics, and it will become a classical algorithm in theoretical computer science.

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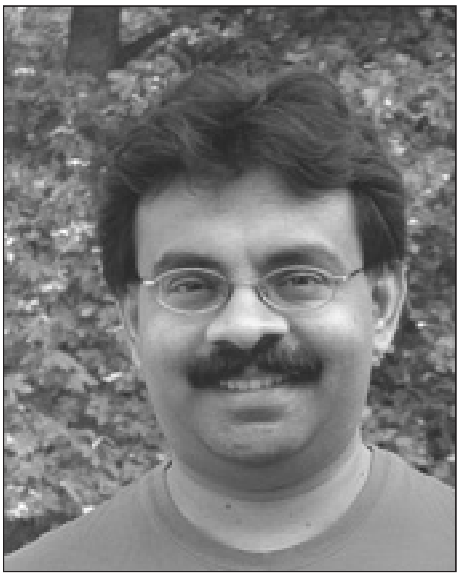


Madhu Sudan

Biography: MADHU SUDAN

Madhu Sudan recieved his Bachelor's degree from the Indian Institute of Technology at New Delhi in 1987 and his Ph.D. from the University of California at Berkeley in 1992. Madhu Sudan is a Professor of Computer Science and Engineering at the Massachusetts Institute of Technology. He was a Research Staff Member at IBM's Thomas J. Watson Research Center in Yorktown Heights, NY from 1992 to 1997 and has been at his current position at MIT since 1997. Madhu Sudan is spending the academic year 2003-2004 as a Fellow at the Radcliffe Institute for Advanced Study.

Madhu Sudan's research interests include computational complexity theory, algorithms and coding theory. He is best known for his works on probabilistic checking of proofs, and on the design of list-decoding algorithms for error-correcting codes. He has served on numerous program committees for conferences in theoretical computer science, and was the program committee chair of the IEEE Conference on Computational Complexity '01, and the IEEE Symposium on Foundations of Computer Science '03. He is a member of the editorial boards of the Journal of the ACM, SIAM Journal on Computing, and Information and Computation, and a member of the scientific board of the Electronic Colloquium on Computational Complexity (ECCC).

Madhu Sudan is a recipient of ACM's Doctoral Dissertation Award (1993), the Alfred P. Sloan Foundation Fellowship (1997), the NSF Career Award (1998), the IEEE Information Theory Paper Award (2000), the Gödel Prize (2001), the Nevanlinna Prize (2002), a Distinguished Alum Award from the CS division of the Univ. of California at Berkeley (2003), and the Radcliffe Fellowship (2003-2004).

Herculean or Sisyphean tasks?

Bernard R. Hodgson (Universit Laval, Qu´ebec, Canada)

Bernard Hodgson

A renewed look at the incompleteness phenomenon in mathematical logic

Hercules or Sisyphus? The hero endowed with exceptional strength and capable of tremendous exploits? Or the ill-fated facing an endless task, condemned to eternally roll up the side of a mountain a rock that hurtles down the slope as soon as the summit is reached? Which of these two characters from Greek mythology spontaneously comes to mind when thinking of some mathematical problems for which it is not known *a priori* whether their solution represents a feasible, though colossal, task, or on the contrary a totally unachievable task?

Let us for instance consider the possibility that an exponent n greater than 2 be such that the equation $x^n + y^n = z^n$ has non-zero integer solutions. A famous proposition, due to Pierre de Fermat (1601–1665), precisely states that such an exponent does not exist. Until the British mathematician Andrew Wiles finally obtained, in 1994,[1] a satisfactory proof of this "last theorem of Fermat", many felt that this might well be the typical case of a Sisyphean situation. Or maybe one should say a "doubly-Sisyphean" situation, since not only did the opponents of Fermat's thesis not succeed in identifying concrete values of n corresponding to solutions, but also all those who over the years had thought that a proof of Fermat's assertion was within reach, had seen their hopes deceived. It is now known, thanks to Wiles, that this problem is rather one whose solution represents a Herculean task: extremely difficult, but nonetheless achievable.

Other situations apparently of a Sisyphean nature, but in fact Herculean, have been brought out in recent years. Quite amazingly links were established between some of these problems and the phenomenon of *incompleteness of mathematical formalism* discovered in the early 30s by the Austrian logician Kurt Gödel (1906–1978). This led to the observation that, contrary to the opinion of a great number of mathematicians or epistemologists, incompleteness does not revolve only around more or less esoteric statements of no concern in the daily work of the mathematician. Recent research has shown that incompleteness is always around and can show up unexpectedly in a host of mathematical contexts. Gone, thus, the cosy indifference of mathematicians towards "metamathematical" considerations! But what exactly does this major change of perspective consist of?

Gödel for all

Metamathematics studies mathematics as it is practised, in particular as regards the nature and the role of formalized reasoning. Taking its roots in the "foundational crisis" triggered off in the early twentieth century by the discovery of inconsistencies in some nooks of the mathematical landscape (*see* the box entitled *Russell Paradox*), the metamathematical enterprise reached its peak in the so-called *Hilbert's Programme*, named after the great German mathematician David Hilbert (1862–1943) who attempted to prove rigorously the non-contradiction of formalized mathematics.

Already with Leibniz (1646–1716) one finds the idea of transposing reasoning to a formal framework through the introduction of appropriate symbolism and deductive rules. And when Hilbert proposes his programme, considerable progress has been accomplished towards this objective, especially since the mid-XIX$^{\text{th}}$ century. But in 1931, *coup de théâtre*, Gödel shakes the scientific community with the publication of his famous *incompleteness theorems*, which reveal strict limitations inherent to mathematical formalism and give, by the same token, a fatal blow to Hilbert's Programme.

The *first* of these theorems establishes the existence, in any formal system satisfying very general conditions — such a system encompasses for instance the theory of *elementary arithmetic* of addition and multiplication, which is really minimal if one is to do any mathematics — of a *statement true but unprovable* according to this system.

As to the *second* incompleteness theorem of Gödel, it provides a specific example of such a statement: namely, the statement expressing the consistency of the formal system itself. And there is no point in trying to remedy such deficiencies by simply adopting these unprovable statements as new basic principles (*axioms*), as new true and unprovable statements will immediately spring up.

[1] The proof announced by Wiles in 1993 contained a flaw which was finally corrected the following year.

Russell Paradox (1902)

Let us designate by C the collection of all sets X satisfying the following property: X is not an element of X. Membership in C thus amounts to making true the defining property; in other words, X is an element of C if, and only if, X is not an element of X. In the particular case where one considers the statement "C is an element of C", one deduces that this assertion is true if, and only if, a second one is true, namely "C is not an element of C". This contradictory situation leads to the conclusion that the defining property of C is unacceptable.
Discovered by the British mathematician and philosopher Bertrand Russell (1872–1970), this paradox shows that "naïve" set theory is inconsistent, so that is it not possible to work with sets without introducing some regulative devices. The goal of the numerous formalizations of set theory developed during the XX$^{\text{th}}$ century is precisely to restrict the rules of formation of sets in such a way to avoid inconsistencies. Metamathematical research, more generally, aims at strengthening the foundations of mathematics and at ensuring the mathematician of a working environment where he does not need to fear encountering a contradiction at any moment.

Reactions to the results of Gödel were quite varied. While some were fascinated by this new awareness of the intrinsic limitations of formalism, the "working" mathematician — namely, the mathematician unconcerned by epistemological matters — considered the incompleteness phenomenon as extraneous to his work. Because either it was the expression of a most reasonable constraint (second theorem) — to be credible, a consistency proof for a formal system can hardly take place inside that system — or else the unprovable statement devised by Gödel in his first theorem, a cunning variation on the theme of the *Liar paradox* (*see* the box *Gödel's unprovable statement*), was perceived as being unsusceptible of being connected to problems "normally" studied in mathematics. This is precisely the aspect about which matters have drastically changed recently. And this is where we meet Hercules and Sisyphus.

Kurt Gödel

Gödel's unprovable statement

It has been known since Antiquity that sentences referring to themselves can lead to odd situations. Such is the case with the assertion "This statement is false", ascribed to the Greek philosopher Eubulides (IV$^{\text{th}}$ century B.C.). If the assertion is true, then it becomes false, and reciprocally if it is false, then it becomes true. Also based on the notion of *self-reference*, Gödel's statement can be seen as a variant of the Liar paradox. Succinctly it says: "This statement is not provable". The detailed working-out of Gödel's statement requires a number of rather substantial technical constructions, but it is not too difficult to see why it is both *true* and *unprovable*. If Gödel's statement were provable, it would assert a truth, formalism having precisely been devised so to prove only true sentences. But what does the statement say? That it is unprovable. Hence, if it is provable, it is unprovable, a contradictory situation. One thus concludes that the statement must be unprovable, and consequently true, as it affirms its unprovability. For many, however, Gödel's statement is merely a kind of linguistic game with no link to actual mathematical practice.

When Hercules meets the hydra

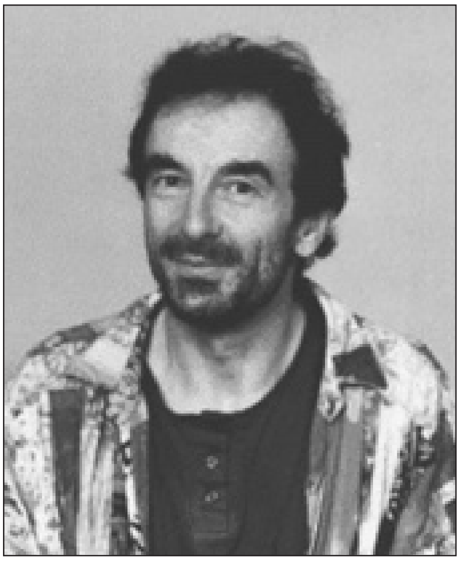
ANTICO (1460 – 1528)
Hercules and the Lernaean Hydra (bronze)

The second of the twelve labours imposed on Hercules was a fight against the Lernæan Hydra, a monster with nine heads, or even more according to some authors, which would grow again as soon as chopped off — in some versions the pruned head was even replaced by two. Legend tells how Hercules finally overcame the Hydra, with the help of his nephew Iolaos who would immediately burn the wound left by a chopped head so to prevent regrowth.

Logicians Laurie Kirby and Jeff Paris have devised a few years ago an even more diabolical hydra to oppose Hercules, as the number of sprouting heads grows as the battle unfolds (*see* the box *The rules of the battle*). In spite of their innocuous appearance the rules defining the battle have a truly dramatic effect as, after only a few steps, Hercules is facing an incredibly dense hydra. Can he nonetheless bring the hydra down? Bluntly said, will he behave as a mythical Hercules or will he be carried, as the pathetic Sisyphus, into an unending chore?

Reuben L. Goodstein (1912 – 1985)

Laurie Kirby



Jeff Paris

The answer is that *Hercules always wins the battle*, whatever the hydra he is confronting or the strategy he uses in cutting off heads (his reputation is thus not overrated!). This most stunning result of Kirby and Paris, based on a remarkable number theorem due to the logician Reuben L. Goodstein [3], fully contradicts the intuition gained from considering a few concrete battles, which appear simply endless — try it and you will see. During his fight against an hydra, Hercules will need to cut an incredibly large, but *finite*, number of heads. Even if the arrangement of heads becomes wider, it actually does not get taller as it forms a tree staying close to the ground, like a bonsai, and with branches connected closer and closer to the root. Patiently, Hercules will end up reducing the hydra to a set of heads directly linked to its root, whence he only needs to sever each head one by one, without any new one being grown.

Sisyphean on its surface, the battle of Hercules against the hydra is thus in fact a genuinely Herculean task: absolutely tremendous in its scope, but still feasible.

Some explosive sequences

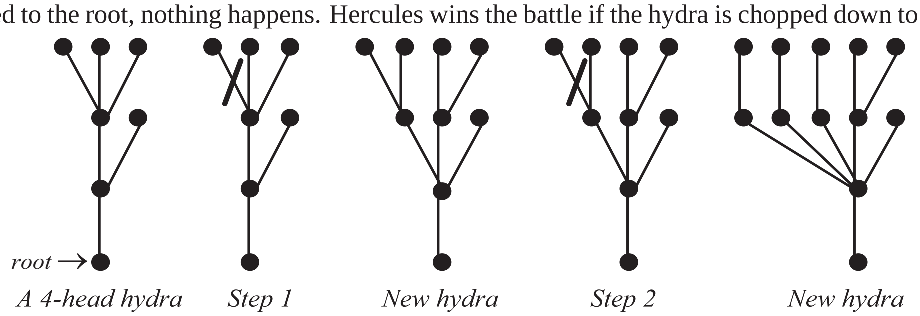
Leaving Hercules to his hydra, let us consider another context of Herculean, but at first glance Sisyphean, tasks. One could think here of Hercules trading the sword for a pencil in order to compute the terms of a numerical sequence.

The number $3 \cdot 2^{402\,653\,211} - 3$ is a gigantic number[2] which made its appearance in the mathematical literature a few years ago. Here is its history.

In 1944, the British logician R. L. Goodstein introduced a process for generating sequences of natural numbers that, against all expectations, inevitably end up in 0. The number $3 \cdot 2^{402\,653\,211} - 3$ is precisely the number of steps needed until the *Goodstein sequence* obtained starting with 4 as the initial value finally reaches 0.

The rules of the battle

A hydra is a tree, that is a configuration of points, each being connected through a unique path to a specific point called the root. A head is a point other than the root located at the end of a path. On the n^{th} step of the battle, Hercules chops off one of the heads (indicated below with an oblique mark), after which the hydra regenerates itself by growing, from the point located two levels below towards the root, n copies of the section containing the pruned head; if the head is directly connected to the root, nothing happens. Hercules wins the battle if the hydra is chopped down to its root.



[2] This number is of the order of $10^{121\,210\,695}$ and comprises more than 121 000 000 digits in its usual decimal representation. If one were to write those digits at the rate of 100 digits per line and 50 lines per page, this would yield a book of more than 24 000 pages!

Weak Goodstein sequences

Let b be a positive integer ≥ 2 (the *base*). Any natural number can then be written (uniquely) in the form

$$m = k_1 \cdot b^{a_1} + k_2 \cdot b^{a_2} + \cdots + k_n \cdot b^{a_n},$$

where the exponents $a_1, a_2, \ldots, a_n$ are strictly decreasing positive integers ($a_1 > a_2 > \cdots > a_n$) and the coefficients $k_1, k_2, \ldots, k_n$ are base-b " digits " — each coefficient is thus a natural number smaller than b. For instance, with $b = 2$ and $m = 266$, one gets

$$\begin{aligned} 266 &= 1 \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\ &= 2^8 + 2^3 + 2^1. \end{aligned}$$

The *weak Goodstein sequence* beginning with 266 is defined as follows. Its first term m_0 is precisely the preceding representation of 266. In order to obtain m_1, the second term of the sequence, one first increases the base by 1, going from 2 to 3, and then subtracts 1 from the result:

$$m_1 = 3^8 + 3^3 + 3^1 - 1 = 3^8 + 3^3 + 2 = 6\,590.$$

One proceeds analogously for computing the other terms of the sequence. After each subtraction of 1, the representation is rewritten, if necessary, as a sum of multiples of powers of the current base — see the computation of m_4 below. Here are the first terms of the weak Goodstein sequence beginning with 266:

$$\begin{aligned} m_0 &= 2^8 + 2^3 + 2^1 = 266 \\ m_1 &= 3^8 + 3^3 + 2 = 6\,590 \\ m_2 &= 4^8 + 4^3 + 1 = 65\,601 \\ m_3 &= 5^8 + 5^3 = 390\,750 \\ m_4 &= 6^8 + 6^3 - 1 \\ &= 6^8 + 5 \cdot 6^2 + 5 \cdot 6^1 + 5 = 1\,679\,831 \\ m_5 &= 7^8 + 5 \cdot 7^2 + 5 \cdot 7^1 + 4 = 5\,765\,085 \end{aligned}$$

A restricted version of Goodstein's process is obtained from the representation of numbers in a specific base. Given a natural number $b \geq 2$, any integer m can be written as a sum of multiples of powers of b. In the case where $b = 10$, this simply yields the usual notation for natural numbers: $266 = 2 \cdot 10^2 + 6 \cdot 10^1 + 6 \cdot 10^0$.

Let us now modify this base-b representation of m by systematically replacing b with $b + 1$, and then subtracting 1 from the result thus obtained. This leads to a new number which can be subjected to a similar process, replacing this time $b+1$ with $b+2$ and then subtracting 1 again (*see* box *Weak Goodstein sequences*).

The successive numbers thus produced appear to grow larger and larger. But this is a misleading observation, as the phenomenon is temporary: any such sequence, if pursued long enough, will unavoidably get to 0. This is without any doubt a most astonishing result, and it reflects the fact that in spite of the increase in the base, the subtraction of 1 gradually " eats away " one after the other, all the terms appearing in the successive expressions.[3]

The process studied by Goodstein is in fact more general and involves much more spectacular numbers. It rests on the notion of the *complete representation* in base b of a natural number: as in the preceding case, one writes this number as a sum of multiples of powers of b, but then does the same with the exponents occurring in this representation, as well as with the exponents of these exponents, etc., until the whole representation stabilises. For example, the complete base-2 representation of 266 ($= 2^8 + 2^3 + 2^1$) is $2^{2^{2+1}} + 2^{2+1} + 2^1$.

Goodstein's process consists in replacing once again by $b + 1$ all instances of b in the complete base-b representation of an integer, and then subtracting 1. The growth which can be observed in the first terms of the resulting sequences (*see* the box *Goodstein sequences*) is simply phenomenal when compared to weak sequences, the terms rapidly becoming of a truly breathtaking size!

Still here, despite the fantastic explosion that can be observed, a " Hercules " computing the successive terms of such a sequence cannot avoid eventually obtaining 0, a fact which, it must recognised, is totally counterintuitive. Goodstein [3] has indeed proved that any Goodstein sequence reaches 0 — but this will in general take a very long time! For instance, beginning with $m = 4$, the rank of the term m_r where the sequence finally gets to 0 is precisely the number $r = 3 \cdot 2^{402\,653\,211} - 3$ mentioned above.[4] And the effect is still more fantastic if one takes a larger integer as the source of the sequence .

This is yet another instance of a task apparently of a Sisyphean nature, but which is in fact Herculean: computing the successive terms of a Goodstein sequence seems to be a never-ending job, and it will indeed be extremely long, but it is inevitably finite as one always gets to 0... as long as one is patient enough!

[3] Note that even if some computations make the terms sometimes more numerous — see the passage from m_3 to m_4 in the box *Weak Goodstein sequences* —, their exponents get smaller. This situation is a reminder of the heads of the hydras, which grow in number but inexorably get closer to the ground.

[4] This result is not too difficult to establish and is the topic of a guided exercise in [4].

Goodstein sequences

Given the natural number m, the *Goodstein sequence* beginning with m is the sequence of integers $m_0, m_1, m_2, \ldots$ defined as follows:

- m_0 is the complete base-2 representation of m;
- m_1 is obtained from m_0 by replacing all 2's by 3, and then subtracting 1;
- m_2 is obtained from m_1 by replacing all 3's by 4, and then subtracting 1;

etc., the process terminating if a term takes the value 0.
The first terms of the Goodstein sequence beginning with $266 = 2^{2^{2+1}} + 2^{2+1} + 2^1$ are

$$
\begin{aligned}
m_0 &= 2^{2^{2+1}} + 2^{2+1} + 2^1 \; = \; 266 \\
m_1 &= 3^{3^{3+1}} + 3^{3+1} + 3^1 - 1 \\
&= 3^{3^{3+1}} + 3^{3+1} + 2 \\
&= 443\,426\,488\,243\,037\,769\,948\,249\,630\,619\,149\,892\,886 \;\approx\; 10^{38}, \\
m_2 &= 4^{4^{4+1}} + 4^{4+1} + 1 \;\approx\; 10^{616}, \\
m_3 &= 5^{5^{5+1}} + 5^{5+1} \;\approx\; 10^{10\,921}, \\
m_4 &= 6^{6^{6+1}} + 6^{6+1} - 1 \\
&= 6^{6^{6+1}} + 5\cdot 6^6 + 5\cdot 6^5 + 5\cdot 6^4 + 5\cdot 6^3 + 5\cdot 6^2 + 5\cdot 6^1 + 5 \\
&\approx 10^{217\,832}, \\
m_5 &= 7^{7^{7+1}} + 5\cdot 7^7 + 5\cdot 7^5 + 5\cdot 7^4 + 5\cdot 7^3 + 5\cdot 7^2 + 5\cdot 7^1 + 4 \\
&\approx 10^{4\,871\,822}.
\end{aligned}
$$

It is not easy to provide a simple explanation for the strange behaviour of Goodstein sequences. It should be stressed however that while the exponential part of Goodstein's process seemingly gives rise to a real numerical outburst, this is nonetheless a limited phenomenon, as the boundless growth is only apparent. As in the weak case, subtraction of the unit will eventually gobble up the gigantic numbers resulting from the successive changes in the base — but this may happen only at the end of a very long journey, because of the staggering values met along the road. Let us observe for instance the impact of this “ -1 ” on the Goodstein sequence beginning with 3, an exceptionally short one:

$$
\begin{aligned}
m_0 &= 2^1 + 1\cdot 2^0 \\
m_1 &= 3^1 + 1\cdot 3^0 - 1 \;=\; 3 \;=\; 1\cdot 3^1 \\
m_2 &= 1\cdot 4^1 - 1 \;=\; 3 \;=\; 3\cdot 4^0 \\
m_3 &= 3\cdot 5^0 - 1 \;=\; 2 \;=\; 2\cdot 5^0 \\
m_4 &= 2\cdot 6^0 - 1 \;=\; 1 \;=\; 1\cdot 6^0 \\
m_5 &= 1\cdot 7^0 - 1 \;=\; 0.
\end{aligned}
$$

As the term $m_2 = 3\cdot 4^0 = 3$ is, so to say, independent of the base 4, the next terms are not affected by the following changes in the base, so that the sequence merely decreases to 0. The very same phenomenon will take place for any Goodstein sequence, but with terms of a much higher rank: it is for instance remarkable how much longer the Goodstein sequence beginning with 4 is, when compared to that whose origin is 3.

The behaviour of Goodstein sequences is reminiscent of the battles of Hercules with the hydra. In both cases one faces a situation which appears to explode uncontrollably. The fact is, however, that while both contexts are extraordinarily complex, they relate to processes which are finite as they do terminate after a certain time — extremely long! Apparently Sisyphean, the corresponding tasks are thus genuinely Herculean.

But there is more to it: the Herculean-Sisyphean dichotomy, whether in the context of the hydra or of Goodstein, takes on a new flavour when approached from another perspective bringing the incompleteness phenomenon to the fore.

And what about incompleteness?

The number of heads chopped off by Hercules during a battle is absolutely enormous: finite, but exceeding without any doubt the wildest imagination. For instance, even if Hercules would eliminate heads at the furious pace of one head severed every second, a fight against a modest hydra with only a few heads at the outset could last so long that the number of seconds since the Big Bang would in comparison appear trivial! And it is precisely in this aspect of the situation that incompleteness lies.

Taking, as suggested above, elementary arithmetic as a framework, one has the following double phenomenon. It is possible, on the one hand, to simulate in that context Hercules' battles via “ number crunching ”: this is a technical, but not too difficult fact (one associates to each hydra a suitable numeric code). It is however *impossible* to prove in this setting that Hercules always wins: a rigorous proof of this invincibility requires richer contexts, such as the theory of *transfinite ordinal numbers*. It can in fact be proved that there is a limit to the growth rate of functions which can be dealt with in elementary arithmetic, and the function giving the length of the battles precisely exceeds this limit: it grows much too quickly! In a similar way, the function expressing the length of Goodstein sequences, that is the number of terms needed to reach 0, always takes finite values, but its growth goes well beyond the framework of elementary arithmetic.[5]

While our “ working ” mathematician tends to judge as artificial the statement devised by Gödel, he perceives the Hercules *vs* hydra fights or the computations of Goodstein sequences as *bona fide* mathematical problems — belonging to the branch of mathematics called *combinatorics*. This mathematician knows how to prove with appropriate tools that the battles always end in Hercules' victory and that Goodstein sequences all converge to 0, but he cannot do that in elementary arithmetic, which is nevertheless the natural formal setting for combinatorics. He is thus facing a proposition which is *true* but *unprovable* in the framework provided by the theory to which it belongs. While the battles against hydras or the computations *à la* Goodstein are in fact Herculean, they will appear Sisyphean for anyone taking elementary arithmetic as the “ system of reference ”. This state of affairs is thus tinged with a kind of relativity: the task appears endless for an observer located “ inside ” elementary arithmetic, but it is in fact finite — though colossal —, which can be noticed by looking at things from “ outside ” arithmetic, in a more powerful formal setting.

Fermat's last theorem belongs as well, by content, to elementary arithmetic; but the proof given by Wiles rests on high-level mathematical tools, quite beyond that theoretical framework. Is there hope that Fermat's assertion, while Herculean, could be proved one day to be regarded as Sisyphean when considered from the viewpoint of elementary arithmetic? Such a result would be most revealing, as it would thus provide a kind of measure of the intrinsic difficulty of this theorem. While this possibility is not excluded, research in mathematical logic is however not quite yet at that stage.

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To go further

Hofstader [5] presents a popular exposition of Gödel's results and also the development of an analogy with the music of Johann Sebastian Bach and the drawings of the Dutch artist Maurits C. Escher (1898–1972). Our mythological analogies take their origin in the technical paper [7], in which Kirby and Paris presented their results, as well as in Gardner's popular account [2]. The distinction between weak and general Goodstein sequences can be found in Cichon [1]. Paper [6] explores the possibility of using properties of Goodstein sequences to define formal frameworks where one can prove the finiteness of some algorithmic processes studied in theoretical computer science.

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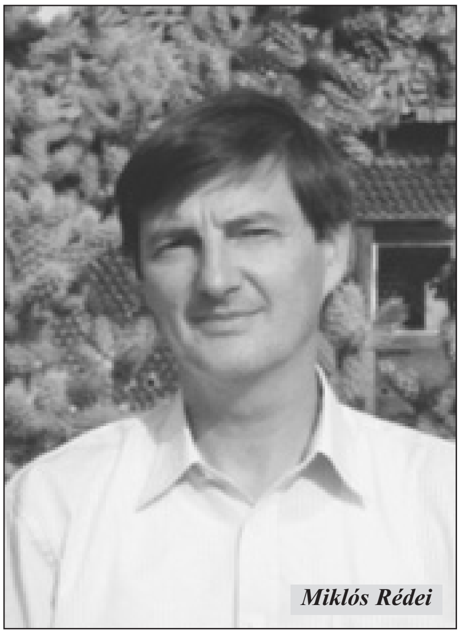
[5] On the contrary, elementary arithmetic allows to prove that weak Goodstein sequences are always finite. The underlying process is thus somewhat complex, but in practice much less than in the case of general Goodstein sequences.

John von Neumann 1903-1957

Miklós Rédei (Budapest)



Miklós Rédei

If the influence of a scientist is interpreted broadly enough to include impact on fields beyond science proper, then John von Neumann was probably the most influential mathematician who ever lived. Not only did he contribute to almost all branches of modern mathematics and created new fields, but he also changed history after the Second World War by his work in computer design and by being a sought-after technical advisor to the post-war military-political establishment of the U.S.A. To celebrate John von Neumann's 100th birthday, the international "Von Neumann Centennial Conference" took place in Budapest, Hungary in October 15-20, 2003. Part of this event was the "Linear operators and foundations of quantum mechanics" conference, where von Neumann's legacy in operator theory was reviewed and discussed by leading experts in this field. During the conference the American Mathematical Society and the János Bolyai Mathematical Society unveiled a commemorative plaque on the house in Budapest where von Neumann was born and raised. To remember von Neumann the present note sketches von Neumann's life and career and recalls briefly some of his views on the nature of mathematics.

Childhood and Education

John von Neumann (known in his native tongue as Neumann János) was born in Hungary in 1903 to a well-to-do family. His father, Max von Neumann, a successful banker, established the family's wealth during the calm and economically prosperous years of the Austro-Hungarian Monarchy, which followed the so-called "self rule" in 1867 that secured Hungary's semi-independence within the Monarchy. Accordingly, von Neumann had a first rate education. This education started by home schooling and included language instruction in the form of the presence of German speaking maids in the household in downtown Budapest, where von Neumann was raised. German, being von Neumann's primary second language, remained superior to his English until about the mid thirties. When the time came, von Neumann enrolled in the famous, expensive private Protestant high school in Budapest. László Rátz, von Neumann's high school mathematics teacher, recognized his talent in mathematics there. Rátz asked for, and received, permission from von Neumann's father to arrange tutoring in mathematics for von Neumann by faculty members of the Technical University in Budapest, insisting at the same time that von Neumann attend regular mathematics classes, which von Neumann did. As a result of this private tutoring, von Neumann had already been prepared in high school to become a professional mathematician; yet, after graduating from high school the von Neumanns made the decision to enrol John von Neumann in the chemical engineering program of the Eidgenössiche Technische Hochschule (ETH) in Zürich, Switzerland. Chemical engineering was a very popular field at the time. In addition, a chemical engineer had a far better chance of landing a job than did a mathematician, a consideration that weighed heavily in the eyes of von Neumann's father, a practically minded person. However, simultaneously with his education at ETH, von Neumann also studied mathematics in Berlin and in Budapest, and he finished his formal university studies by receiving his PhD in mathematics (axiomatic set theory) in Budapest.

Career

In 1926 von Neumann went to Göttingen on a Rockefeller fellowship to work as Hilbert's assistant. Göttingen was not only one of the centres of mathematics but it was also a Mecca of theoretical physics; thus in Göttingen von Neumann could familiarize himself with the latest developments concerning quantum mechanics. Hilbert himself gave lectures on the mathematical foundations of quantum mechanics in the academic year 1926-1927. Von Neumann attended these lectures, and he worked out the notes taken during those lectures. This led to a joint publication [4] and eventually to von Neumann's three ground breaking papers [9, 10, 11] on the mathematical foundations of quantum mechanics that served as the basis of his book [12].

After a short stay in Berlin and Hamburg as non-tenured faculty ("Privatdozent"), von Neumann was invited by Princeton University in 1929 to lecture. In January 1930 von Neumann was offered a permanent professorship in Princeton University, which he declined; he was then entrusted to substitute for the Jones professorship in mathematical physics for five years. Finally, in January 1933, he accepted the invitation to become one of the first six permanent professors of the Institute of Advanced Study (IAS), established in Princeton in 1933.

Contrary to the still widespread belief that von Neumann had left Europe for fear of political prosecution, von Neumann was keen on emphasizing that he had taken up residency in the USA before the political situation in Europe became unbearable and that he therefore never considered himself a refugee scientist (von Neumann's letter to M.R. Davie, May 3, 1946 [17]). In harmony with his being a non-refugee scientist at IAS von Neumann was regularly visiting Europe - and Hungary in particular - during the thirties; it was only in the face of imminent threat of war that he decided not to visit Europe any more.

Von Neumann retained his academic position in Princeton until the end of his life, but he spent a lot of time travelling and lecturing at different universities and research institutes. During the war von Neumann became increasingly involved in military-related research, among other things he participated in the Manhattan Project. After the war von Neumann's military and governmental consulting activity expanded tremendously both in volume and significance. He was serving on a number of very influential committees that shaped post-war U.S. military policy. At the peak of his power he lists the following appointments as the most significant advisory positions: consultant to the U.S. Atomic Energy Commission; consultant to the Armed Forces Special Weapons Project; member of the Scientific Advisory Board of the U.S. Air Force; member of the Scientific Advisory Committee of a classified Air Force project connected with the Office of the Special Assistant (Research and Development) of the Secretary of the Air Force; member of the Technical Advisory Board on Atomic Energy connected with the Office of the Assistant Secretary of Defence (Research and Development) and member of the Scientific Advisory Committee of the U.S. Army Ordnance Corps, Ballistic Research Laboratories, Aberdeen Proving Ground (letter to colonel H.H. Rankin, August 26, 1954, [17]). His "exceptionally outstanding service" to the United States and its Navy during the war was acknowledged by awarding him the *Medal for Merit* (October, 1946) and the *Distinguished Service Award* (July, 1946).

After the war von Neumann received offers from different universities, especially from MIT and UCLA. MIT's offer included the promise to fund an electronic computer project at MIT, making available MIT's extensive engineering know-how. Computer design was in the focus of von Neumann's academic interest after the war, and he was urging IAS' leadership to make IAS the home of an electronic computer development project. His efforts and

John von Neumann

persistency paid off. IAS, with support from Radio Corporation of America (RCA) and Princeton University, decided to carry out a computer project with von Neumann as director. Having secured IAS' support for the computer project von Neumann declined MIT's offer - but he did this with great regret, anticipating what later became increasingly obvious, namely that IAS was not the ideal place to carry out the engineering-heavy computer project. It is fair to say that IAS was not the most suitable institution for von Neumann after the war. IAS' ivory-tower-like intellectual climate was not ready to accommodate von Neumann's application oriented interests. In addition, von Neumann always had a very extensive and diverse consulting activity not only in government but also in the private sector (he was consultant to the Standard Oil Company and IBM for instance), which does not seem to have been welcomed by IAS. Von Neumann must have realized this while he was on official leave from IAS as Atomic Energy Commissioner from 1954, because he did not intend to return to IAS after this job. In 1956, he was again offered a position at MIT and at the same time, he was negotiating a position at UCLA. In March 1956 von Neumann finally decided to accept the offer to be appointed as professor at large at UCLA, however he was never able to take the position because he died of incurable cancer in Washington D.C. on February 8, 1957. He is buried in the Princeton cemetery.

Given the unparalleled diversity and depth of von Neumann's contribution to both pure and applied mathematics, it is impossible to list even his major achievements in a short biographic sketch. Rather than trying to do the impossible, I will recall von Neumann's general views about mathematics and its relation to the physical sciences below.

Von Neumann's views on the nature mathematics

Von Neumann's interpretation of Gödel's incompleteness theorems

Von Neumann started his career as a mathematician with work on axiomatic set theory, and he also worked on Hilbert's program aimed at proving consistency of mathematics by finitistic means. The turning point in the history of Hilbert's program was the famous Second Conference for Epistemology of the Exact Sciences that took place between 5 and 7 of September 1930 in Königsberg. It was in the discussion session on September 7 that Gödel announced the first version of what became known as Gödel's first incompleteness theorem: every sufficiently rich and consistent axiom system contains meaningful statements that are undecidable within that system. Von Neumann immediately grasped the significance of Gödel's result for the axiomatic foundation of mathematics. He pressed Gödel for further details and, as his letter to Gödel (November 20, 1930, [2], [17])

Hungarian stamp

shows, shortly after the Königsberg conference, he apparently obtained what is known as Gödel's second incompleteness theorem. The second incompleteness theorem says that the consistency of a sufficiently rich axiomatic theory cannot be proved within the system itself: the statement expressing consistency of the system in the system is an undecidable proposition in the system. Von Neumann informed Gödel of this result, only learning that Gödel himself had already established this consequence of the first incompleteness result. Von Neumann then acknowledged Gödel's priority and did not publish anything on the topic. (For a history of the Königsberg conference, see [1], [3] and the references therein.)

From the second incompleteness theorem von Neumann had drawn a very strong conclusion for the Hilbert program: "... there is no rigorous justification for classical mathematics" (letter from von Neumann to Gödel, November 29, 1930). On this point von Neumann strongly disagreed with Gödel. In his letter to Carnap (June 7, 1931, [3] and [17]) von Neumann writes:

> Thus I am today of the opinion that
> 1. Gödel has shown the unrealizability of Hilbert's program.
> 2. There is no more reason to reject intuitionism (if one disregards the aesthetic issue, which in practice will also for me be the decisive factor).
>
> Therefore, I consider the state of the foundational discussion in Königsberg to be outdated, for Gödel's fundamental discoveries have brought the question to a completely different level. (I know that Gödel is much more careful in the evaluation of his results, but in my opinion on *this* point he does not see the connections correctly).

Von Neumann on mathematical rigor

The unrealizability of Hilbert's program was a decisive development shaping von Neumann's views on the nature of mathematics: in his view this development showed that there is no immovable notion of rigor in mathematics, and that one cannot justify classical mathematics by mathematical means:

> Whatever philosophical or epistemological preferences anyone may have in this respect, the mathematical fraternities' actual experiences with its subject give little support to the assumption of the existence of an a priori concept of mathematical rigor.
>
> I have told the story of this controversy [debate about the foundations of mathematics] in such detail, because I think that it constitutes the best caution against taking the immovable rigor of mathematics too much for granted. This happened in our lifetime, and I know myself how humiliatingly easily my own views regarding the absolute mathematical truth changed during this episode, and how they changed three times in succession! [14][p. 6]
>
> It is not necessarily true that the mathematical method is something absolute, which was revealed from on high, or which somehow, after we got hold of it, was evidently right and has stayed evidently right ever since. To be more precise, maybe it was evidently right after it was revealed, but it certainly didn't stay evidently right ever since. There have been very serious fluctuations in the professional opinion of mathematicians on what mathematical rigor is. To mention one minor thing: In my own experience, which extends over only some thirty years, it has fluctuated so considerably, that my personal and sincere conviction as to what mathematical rigor is, has changed at least twice. And this is in a short time of the life of one individual! [13][p. 480]
>
> The variability of the concept of rigor shows that something else besides mathematical abstraction must enter into the makeup of mathematics. [14][p. 4]

The "something else" is empirical science, physics in particular:

> ... some of the best inspirations of modern mathematics (I believe, the best ones) clearly originated in the natural sciences. [14][p. 2]

Von Neumann mentions geometry and analysis as examples of mathematical disciplines that clearly have empirical origins, but he firmly believed that all mathematical disciplines have an empirical origin however remote, and that mathematics' becoming detached from its empirical roots carries with it a risk:

> As a mathematical discipline travels far from its empirical source … it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art* [14] [p. 9].

The field is then in danger of developing along the line of least resistance and will "separate into a multitude of insignificant branches" [14] [p. 9].

> Whenever this stage is reached, the only remedy seems … to be a rejuvenating return to the source: the reinjection of more or less directly empirical ideas. [14] [p. 9].

But the relation between mathematics and science is a two-way one: that science fertilizes mathematics is just one aspect of their rich mutual dependence. The other side of their relationship is that mathematics also permeates science:

> The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or, more generally, to any

science which interprets experience on a higher than purely descriptive level. [14] [p. 1].

In modern empirical sciences it has become a major criterion of success whether they have become accessible to the mathematical method or to the near-mathematical methods of physics. Indeed, throughout the natural sciences an unbroken chain of pseudomorphoses, all of them pressing toward mathematics, and almost identified with the idea of scientific progress, has become more and more evident. [14] [p. 2].

Von Neumann on the axiomatic method in physics

In harmony with his relaxed attitude about mathematical rigor in mathematics, von Neumann also took a moderate position about mathematical precision in physics. Specifically, he saw that the axiomatic method couldn't be practiced in physics the way it can in mathematics, von Neumann embraced what is dubbed in [7], "opportunistic soft axiomatization" (see also [6] and [8]). Its explicit formulation can be found already in the 1926 joint paper by Hilbert, Nordheim and von Neumann on the foundations of quantum mechanics [4]. This paper contains a relatively lengthy passage on the axiomatic method in physics. The main idea is that a physical theory consists of three, sharply distinguishable parts: (i) physical axioms, (ii) analytic machinery (also called "formalism") and (iii) physical interpretation.

The physical axioms are supposed to be semi-formal requirements (postulates) formulated for certain physical quantities and relations among them. The basis of these postulates is our experience and observations. The analytic machinery is a mathematical structure containing quantities that have the same relation among themselves as the relation between the physical quantities. *Ideally*, the physical axioms should be strong and rich enough to *determine* the analytic machinery *completely*. The physical interpretation connects then the elements of the analytic machinery and the physical axioms. But the ideal situation never occurs (hence the terminology "opportunistic soft axiomatization"):

In physics the axiomatic procedure alluded to above is not followed closely, however; here and as a rule the way to set up a new theory is the following.

One typically conjectures the analytic machinery before one has set up a complete system of axioms, and then one gets to setting up the basic physical relations only through the interpretation of the formalism. It is difficult to understand such a theory if these two things, the formalism and its physical interpretation, are not kept sharply apart. This separation should be performed here as clearly as possible although, corresponding to the current status of the theory, we do not want yet to establish a complete axiomatics. What however is uniquely determined, is the analytic machinery, which - as a mathematical entity - cannot be altered. What can be modified - and is likely to be modified in the future - is the physical interpretation, which contains a certain freedom and arbitrariness. [4] [p. 106], translation form [7].

A closer look at how von Neumann actually treated and presented quantum mechanics reveals that he did indeed follow the methodology of opportunistic soft axiomatization in his work on quantum theory [7].

One may wonder what made von Neumann so successful not only in pure mathematics but in a wide variety of other disciplines as well. While there is no easy and simple answer to this question, it seems plausible that his exceptional talent was combined with a broad education that avoided narrow-minded concentration on mathematics and this made him appreciative and receptive of the problems of the real world.

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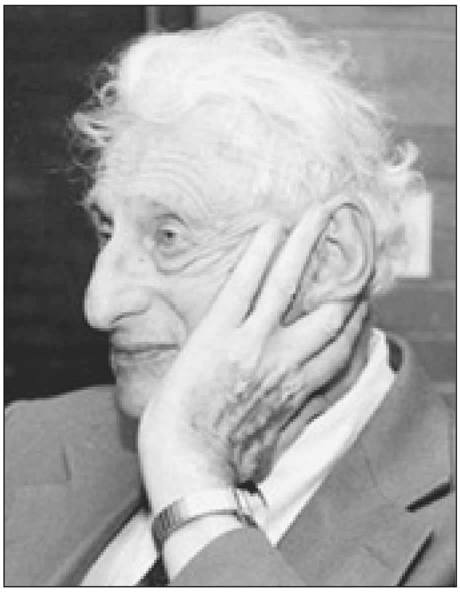
A Letter to Christina of Denmark

Philip J. Davis (Brown University, Providence, R.I., USA)



Philip Davis

The great Swiss mathematician Leonhard Euler wrote a highly admired three-volume popularization of science entitled *Letters to a Princess of Germany* (1768-72). Sometime ago, my friend Christina (pseud.) in Denmark sent me a letter containing a number of questions about mathematics, a subject about which she had heard much, but knew little. I believe that Christina's questions are among those frequently asked by the general literate public. I have formulated some answers to Christina's questions and titled them as I have, in admiration both of Christina and Euler. My answers are, alas, incomplete, but I hope they will elicit more questions.

Q1. *What is mathematics?*
A1. Mathematics is the science and the art of quantity, space, and pattern. Its materials are organized into logically deductive and very often computational structures. Its ideas are abstracted, generalized, and applied to concerns other than mathematics itself.

When mathematics mixes with "outside concerns", the mixture is called applied mathematics, and for reasons that are by no means clear and may even remain a perpetual mystery, mathematics has been found to be of utility and an indispensable aid to the physical sciences. It has often been called the "Handmaiden of the Sciences." Some authorities have identified theoretical physics as that which is mathematical in character.

Applied mathematics exhibits descriptions, predictions and prescriptions. Description replaces a real world phenomenon with a mathematical surrogate: the lampshade casts a parabolic shadow on the wall. Prediction makes a statement about future events: there will be a total eclipse of the sun on July 11, 2010. Its duration will be 5 minutes and 20 seconds. Prescription (or formatting) organizes our lives and actions along certain lines: traffic lights control the flow of automobiles in a periodic fashion.

Mathematics has intimate relations with philosophy, with the arts, with language, and with semiotics (theory of signs and their use). Since the development of mathematics is often inspired and guided by aesthetic considerations, mathematics is an "amphibian": it is both a science and one of the humanities.

Q2. *Why is mathematics difficult and why do I spontaneously react negatively when I hear the word?*
A2. There are many reasons why the average person finds mathematics difficult. Poor, uninformed teaching; boring, doctrinaire presentations; over-concentration on the deductive aspect of the subject; presentations that fail to connect mathematics with the day-to-day concerns of average people. These are some of them.

Mathematical thinking and manipulations are cerebral activities that simply don't engage everyone. And then, let's face it, the material ***can*** be difficult. Common sense often seems to be irrelevant. While mathematical skills and understanding can be learned and developed, I believe there is such a thing as innate talent for mathematics. Just as not everyone has the talent to be creative in art, or to write a great book, or has the body to be a ballet dancer or to break athletic records, not everyone can scale the heights of mathematical understanding. The very fact that professional mathematicians make long lists of unsolved problems attests to the fact that ultimately, all professionals reach their own limits of mathematical accomplishment.

Q3. *Why should I learn mathematics? History widened my horizons and deepened my "roots." When I learned German, it opened up cultural treasures to me. Karl Marx explained (if not changed) the society I live in. What does mathematics have to offer?*
A3. Mathematics is one of the greatest human intellectual accomplishments. Mathematical education is partly to enable us to function in a complex world with some intelligence, and partly to train our minds to be receptive to intellectual ideas and concepts. Even as natural languages such as German or Chinese embody cultural treasures, mathematics does also. But one has to learn the "language" of mathematics to appreciate these treasures and to learn a bit of the history of mathematics and appreciate its growth within the history of general ideas.

Looking about us, if we are perceptive, we can see not only the natural world of rocks, trees, and animals, but also the world of human artifacts and of human ideas. The ideas of Karl Marx (and of other thinkers) explain the world along certain lines, each necessarily limited, but these ideas have often changed the world beyond recognition.

Mathematics also, is capable of explaining the world in remarkable but limited ways. It is capable of formatting our human lives in useful ways, and it is capable, again in a limited way, of allowing us to look into the future and to make prudential judgments. It, too, has often changed the world beyond recognition. Every educated person should achieve some appreciation of the historical role that mathematics plays in civilization in order that the subject be given both intelligent support as well as intelligent resistance to its products.

Q4. *How has mathematics changed in the last 100 years? What have been the dominant trends?*
A4. First of all, the amount of new mathematical theories created since 1900 is enormous. During that time, mathematics has become more abstract and deeper than it was in the 19th Century. By this I do not mean that simple things such as arithmetic have to be viewed by the general public in a more abstract or deeper way than before, but that the conceptualization of old mathematics and the creation and applications

of new mathematics by the professionals have had that abstract character.

Mathematical logic is now firmly on the scene, as is set theory. A new and more abstract algebra has grown mightily and consolidated itself. Infinitary mathematics (i.e., the calculus and its elaborations) has grown by leaps and bounds, but as opposed to the 19th Century, has increasingly had to share the stage with finitary mathematics.

Geometry, which began in antiquity as visual and numerical (lengths, areas, volumes), moved into an abstract axiomatic/deductive mode and now embraces such studies as algebraic, combinatorial, and probabilistic geometry whose expositions often contain no visual figures whatever.

A major trend since the late 1930's is to view mathematics as the study of deductive structures. Mathematics has shared the structuralist point of view with numerous non-mathematical disciplines ranging from linguistics to anthropology to literary criticism.

There is no doubt that the electronic digital computer which emerged in the middle 1940's, and is in many ways a mathematical instrument, has changed our day-to-day life noticeably. The computer has revitalized or widened the scope and increased the power of a number of traditional branches of mathematics. It has also created new branches of the subject, and all in all, has had a revolutionary effect throughout science and technology.

Q5. *Tell me about the "chipification" of mathematics.*

A5. A good fraction of the mathematics that affects our day-to-day life is now performed automatically. It is built into computer chips that are in our wristwatches, automobiles, cash registers, automatic teller machines (ATM's), coffee machines, medical equipment, civil engineering equipment, word processors, electronic games, telephone and media equipment, military equipment, ID plastic... the list has become endless. The word "chips" and the idea of inserting them to do clever things has become commonplace.

The mathematics in chips is hidden from public view. The average person, though greatly affected by "chipification", need pay no conscious attention to it as mathematics. This would be the job of a corps of experts who design, implement, monitor, repair, or improve such devices. The "chipification of the world" is going forward at a dizzying rate. One of the disturbing side effects of this trend may very well be that in the near future Sally and Johnny will not have to add or read or study in the traditional way. Will this put us back several hundred years, when literacy and numeracy were relatively rare achievements? Not at all. Modes of communication and interpretation of social arrangements have changed in the past and will change in the future. It does not mean that communication will disappear.

Q6. *Where are the centers where new mathematics is developed? Who is "hip?" A fan of good rock music knows where to go to hear it. Where would a math fan go these days?*

A6. There are mathematical research centers in all the developed countries of the world. They are located in universities, governmental agencies, research hospitals, scientific, economic or social "think tanks", and industries. All of these centers tend to be organized into groups and tend to concentrate their research on very limited and specific problems. There are also many mathematical researchers who work independently and productively outside of groups.

Depending on what sort of "mathematical music" you would like to hear, you would visit one of these groups or talk to one of these individuals. Although these groups exhibit a great deal of esprit de corps and self esteem, and can be influential beyond their own walls, there is, in my judgment, no single group that predominates. This diversity is a very good thing.

And don't forget: there are now websites, available in the home, that cater to every level of mathematical interest and engagement.

Q7. *How is mathematics research organized? Who is doing it, who is paying for it and why? Lonely, harmless riders, or highly efficient, highly organized, secret, threatening groups. Should one be scared?*

A7. The forward movement of mathematics is driven by two principal forces: forces on the inside of the subject and forces on the outside.

Forces on the inside perceive certain questions or aspects that are incomplete or unanswered and call for answers. Or new mathematical ideas arising from free mental play come into prominence. "Lonely riders" very often make substantial contributions.

Forces on the outside call for the application of available mathematics to the outside world of people or things. Such forces may result in the development of genuinely new mathematics.

The work is paid for by institutions or organizations in anticipation of profit, or of scientific, social, national or cultural gain. The economically independent mathematical researcher exists, but is quite rare.

NASA, the space agency, for example, is a source of considerable mathematical support. In the years following 1939, and particularly in the U.S.A., a great deal of mathematical research has been paid for by the military, and individual mathematicians have profited by this support independently of their private views of military or foreign policy. Since war appears to be endemic to the human condition, military support will undoubtedly go on at some level.

Support from the medical/health sector will gain in importance. The productivity of the United States gets a considerable boost from the "entertainment industry" and movies are using computer graphical software which has a considerable mathematical underlay.

Some of the work of the various groups is restricted or confidential. This may be for reasons of national security or for reasons of industrial confidentiality in a free and competitive market.

In an ideal democratic society, all work must ultimately come into the open so that it may be judged both as regards its internal operation and its effects on society. Free and complete availability of information is one of the hallmarks of ideal science. While this has not always been the case in mathematical practice, judged over the past several centuries, the record is pretty good.

Doing mathematics, developing new mathematics, is simply one type of human activity among thousands and thousands of activities. As long as constant scrutiny, judgment and dissent flourish, fear can be reduced.

Q8. *Other sciences have had their "breakthroughs" in the past 10-20 years. What breakthroughs does mathematics claim?*

A8. We might begin by discussing just what might be meant by the phrase "breakthrough in mathematics." If we equate breakthroughs with prizes, and this equivalence has a certain merit, then the work of e.g., the Fields Medal winners should be cited. This would cover pure mathematics where this prize, given since 1936, is spoken of as the mathematical equivalent of the Nobel Prize. Similar prizes of substantial value exist in various pure mathematical specialties, in applied mathematics, statistics, computer science, etc.

The work of Fields medal winners over

the past decades, people such as K.F. Roth, R. Thom, M. Atiyah, P.J. Cohen, A Grothendieck, S. Smale, A. Baker, K. Hironaka, S.P. Novikoff, and J. Thompson, cannot, with some very few exceptions, be easily explained in lay terms. In view of their arcane nature, the public rarely hears about such things; and should a particular accomplishment reach the front page of the newspapers, its meaning is often mutilated by packaging it in the silver paper of sensation. Sensation is always easier for the public to grasp than substance.

Judging from the selection of Fields Laureates, the criteria for the honor seem to be: (1) the solution of old and difficult mathematical problems, (2) the unification of several mathematical fields through the discovery of cross connections and of new conceptualizations, (3) new internal developments.

A few years ago the world of research mathematicians was electrified by the public announcement that the Clay Mathematics Institute, a private organization, was offering one million dollars each for the solution of seven famous mathematical problems. In view of the public's general indifference, if not dislike, of mathematics, perhaps this is not an unreasonable price to pay for a bit of publicity.

We live in an Age of Sensation. As a result, only the "sensational" aspects of mathematics get much space in the newspapers: e.g., Fermat's "Last Problem", "the largest prime number now known is ...." or the solution of other so-called "big" unsolved problems, big international students' contests, etc.

On the other hand, mathematical applications that are directly and currently useful, e.g. the programming and chipification of medical diagnostic equipment, are rarely considered as "breakthroughs". They do not make the front pages. Historic research or philosophic discussions of the influence of mathematics on society have yet to make an impact on society and to be honored in prestigious manners.

Yet, from the point of view of affecting society, the major mathematical breakthrough since the end of World War II is the digital computer in all its ramifications. This breakthrough involved a very strong mix of mathematics and electronic technology and is not the brainchild of a single person. Hundreds, if not hundreds of thousands of people have contributed and still contribute to computerization.

Consider the recent flight to Mars that put a "laboratory vehicle" on that planet. Whether or not that event excites you, you must admit that it is an almost unbelievable technological accomplishment. Now, from start to finish, the Mars shot would have been impossible without a tremendous underlay of mathematics built into chips and software. It would defy the most knowledgeable historian of mathematics to discover and describe all the mathematics that was involved. The public is hardly aware of this; it is not written up in the newspapers. And yet, a historian would be very reluctant to call the Mars shot a mathematical breakthrough because it resulted from the steady accumulation over centuries of mathematical knowledge and techniques.

Q9. *Medical doctors fight cancer, AIDS and SARS. What is now the greatest challenge to modern mathematics?*

A9. One might distinguish between "internal" problems and "external" problems. The former are problems that are suggested by the operation of the mathematical disciplines within themselves. The latter are problems that come from outside applications, e.g., what is the airplane shape of minimum drag when the airplane surface is subject to certain geometrical conditions? What are the aerodynamic loads on the plane structure during maneuvers? There are tremendous challenges that confront biomathematicians.

In 1900, the great mathematician David Hilbert proposed a number of very significant unsolved problems internal to mathematics. Hilbert's reputation and influence was so great that these problems have been worked on steadily, and most of them have been resolved. Setting up, as it did, a hierarchy of values as to what was important (every mathematician creates his own list of unsolved problems!), their solutions have had a considerable influence on the subsequent progress of mathematics. The solvers, in turn, have gained a great reputation for themselves in the mathematical community.

Within any specific field of mathematics, its practitioners will gladly tell you what they think the major unsolved problem (or challenge) is. Thus, if a topologist is queried, the answer probably is to prove the general Poincaré Conjecture. If an analyst is queried, the answer might be to prove the Riemann Hypothesis.

As regards "external problems", a fluid dynamicist might say: devise satisfactory numerical methods for processes such as occur in turbulence or in weather that develop over long periods of time. A programming theorist might say: devise a satisfactory theory and economical practice for parallel computation. Physicists create new mathematics and then call their attention to mathematicians as challenges for treatment according to mathematicians' criteria.

If your question is answered in terms of specific problems, it is clear that there are many challenges and there is no agreement on how to prioritize them. Researchers can be drawn to specific problems by the desire of fame, money, or simply that their past work suggests fruitful approaches to the many problems that remain unsolved.

Your question can also be answered at a higher level of generality (vagueness!). In 1992, for the Committee on Science, Space, and Technology of the United States House of Representatives, the American Mathematical Society identified five interrelated long term goals for the mathematical sciences:

- "to provide fundamental conceptual and computational tools for science and technology
- to improve mathematics education
- to discover and develop new mathematics
- to facilitate technology transfer and modeling
- to promote effective use of computers." [*Notices of the AMS,* Feb., 1992]

While believing that these goals are still valid, I should like to go up one more rung on the ladder of generality and answer you that the greatest challenge to modern mathematics is to keep demonstrating to society that it merits society's continued support.

Mathematics can and has flourished as a harmless amusement for a few happy aficionados both at the amateur and professional levels. But to have a long and significant run, mathematics must demonstrate an intent that engages the public. If the intent is simply to work out more and more private themes and variations of increasing complexity and of increasing unintelligibility to the general public, then its support will be withdrawn.

The long history of mathematics exhibits a variety of mathematical intents. Some of these have been: to discover the Key to the Universe, to discover God's will (thought to be formulated through mathematics), to act as a "handmaiden" to science, to act as a "handmaiden" to commerce and trade, to provide for the defense of the realm, to provide social formats of convenience and comfort, to develop a super brain — an intelligence amplifier of macro proportions.

The public demands a return for its support, but the place and the form of an acceptable return cannot be specified in advance. Perhaps a mathematical model of brain operation will be devised and will lead to insight and ultimately to alleviation of mental disease. It is not too fatuous to think that many of the common problems that beset humankind such as AIDS, cancer, hunger, hostility, envy, might ultimately be aided by mathematical methods and computation. While efforts in these directions are praiseworthy, some, of course, may come to naught.

By way of summary, the greatest challenge to the mathematicians of the world is to keep its subject relevant and to keep up the pressure that its applications promote human values.

Q10. *What can you say about militarization, centralization, regionalization, politicization of mathematics?*
A10. This is a very large order to answer: the first topic by itself would require books, and there are such books in print. Since around 1940, one of the major financial supporters of mathematical research in the USA has been the military or the "military industrial complex." A similar assertion can be made of all the advanced nations. There are essential mathematical underlays to new and sophisticated weaponry both offensive and defensive that are vital. Related economic, demographic, strategic studies and predictions often involve mathematics at the highest levels of complexity.

Since the end of the Cold War, when, hopefully, military options will be reassessed and diminished, one looks forward to developments and applications of mathematics supported increasingly by such fields as medicine, biology, environment, transportation, finance, etc.

Centralization and regionalization: in the pre-computer days it used to be said that a mathematician didn't require much by way of equipment; just a few reams of paper, a blackboard and some penetrating ideas. Today, although much research is still done in that mode, we find increasingly that mathematicians require the aid of supercomputer installations. However, this is still far less costly than, say, the billions of dollars of laboratory equipment required by high-energy physicists or astrophysicists.

Centralization of mathematics occurs as the result of a number of factors: (1) the willingness and ability of society to support mathematical activity, (2) the desire and the utility of having mathematicians work in groups, centers, etc. The notion of the critical inter-communicating mass of creative individuals is at work here. New systems of rapid intercommunication and transport of graphical and printed material may in the future affect the clumping of centers for research and development.

Politicization: While the content of mathematics is abstract, mathematics is created by people and is often applied by people to people's concerns. It is to be expected then, that the creation and the application of mathematics should be subject to support, pressures, monitoring, and suppression by governmental, political or even religious institutions. The interaction between mathematics and human institutions has a long and documented history.

Q11. *Give me ten points that worry a concerned mathematician.*
A11. A concerned mathematician will worry about the abuse, misuse, or misinterpretation of mathematics or of its applications. But remember that men and women who have been satirized or pilloried as technocrats, computocrats or mathocrats may still be all concerned people.

Insofar as we are living in a thoroughly mathematized civilization, the number of concerns is necessarily vast and it is natural that most concern should be focused on "life and death" issues. If, for example, a mathematical criterion were developed via computerized encephalography for determining whether a person is "brain dead", then this would engender a great deal of concern. The existence of high precision "surgical" warfare (whose mathematical underlay is substantial) and the claims made for it are worrisome.

Q12. *I read a statement attributed to the famous physicist Max Born that the destructive potential of mathematics is an immanent trend. If that is so, why then should I, an average person, know a whole lot more about mathematics?*
A12. You should know more about it for precisely that reason.

All creative acts have destructive potential. Though we try to reduce the risk, an electrical outlet in the average home is not totally risk-free. The destructive or revolutionary potential of graphical arts, of literature, or of the television is well documented. Even as mathematics solves many problems, it creates new problems, both internal to itself and external to it.

To live is to be at risk, and no number of insurance policies can reduce the psychological risk to zero. Moreover, to live at the very edge of risk is thought by some to be "truly alive."

The more the average person knows, the better off that person will be to make judgments. Some of those judgments will be how to balance risk with prudence. In a world in which scientific, technological and social changes occur rapidly, a democratic society cannot long endure in the presence of ignorance.

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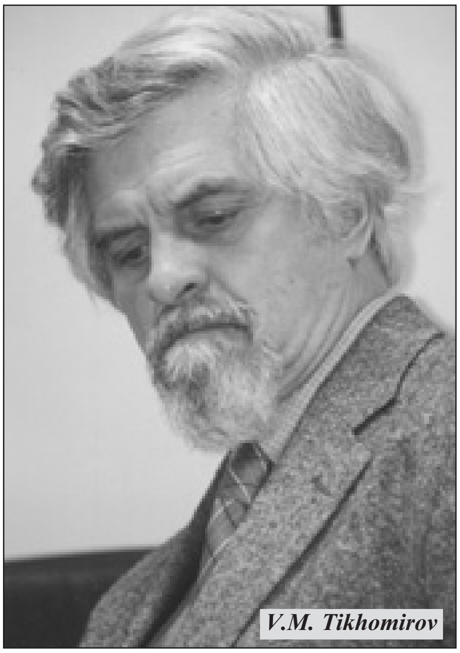
Kronborg, Elsinore, Denmark

The Moscow Mathematical Society

Part II

S.S. Demidov, V.M. Tikhomirov, T.A. Tokareva (Moscow)

S.S. Demidov



V.M. Tikhomirov

T.A. Tokareva

This is the second and final part of the authors' article on the history of the Moscow Mathematical Society. Part I including Sections 1-3 on the early years of the Society and short CVs of the authors appeared in the preceding issue 50 of the Newsletter, pp. 17-19.

4. The Soviet Government and the mathematical community

During the years of the Revolution and the Civil War the Bolshevik authorities did not meddle in the organization and the development of scientific studies and they put aside the question of the "organization" of science. However, from the mid – 1920s this "organization" became one of their main goals.

In 1925 the Academy of Science celebrated its 200th anniversary. On 27 June 1925 the Soviet government decided to "recognize the Russian Academy of Sciences as the highest scientific institution of the U.S.S.R.". According to this decision the Academy was transformed into a national institution: the Academy of Sciences of the U.S.S.R. The Academy was subordinated to the Soviet of the People's Commissars (Soviet of Ministers) of the U.S.S.R., giving the Soviet authorities the ability to interfere directly in scientific life. I. V. Stalin planned that the monumental edifice of Soviet science would be topped by the reorganized Academy of Sciences. For this it was necessary to revise the statute of the Academy. This project was elaborated and confirmed by the Politburo on 26 May 1927. In June 1927 the new statute was approved.

According to the project of "Sovietisation of the Science" all scientific and educational institutions had to be reconstructed. The authorities focused their attention on the oldest institution: the Moscow University.

In the 1920s D. F. Egorov was the leader of the Moscow mathematical community and president of the Moscow Mathematical Society. He was a corresponding member (from 1924) and honourary member (from 1929) of the Academy of Sciences of U.S.S.R. This very religious man disapproved of Marxist ideology and of the Soviet regime. Consequently he very quickly became the object of embittered criticisms from the followers of the new ideology. In 1930 he was arrested according to the affaire of the so-called veritable Orthodox Church and in 1931 he died in exile in Kazan [5,6].

D.F. Egorov

The Moscow Mathematical Society took great risks and the Moscow mathematicians did everything in their power for its salvation. Therefore the Society was reorganized. E. Kolman[1], a prominent functionary of the Party and an unimportant mathematician, was "elected" president of the Society for a short period. In 1932 P. S. Aleksandrov (1896-1982) was elected president of the Society and remained on this post until 1964. He kept the Society's traditions, which were reinforced by D. F. Egorov.

The "Egorov Affair" marked a new stage in the life of the national mathematical community [6] - from now on open dissent could not be tolerated among its members and especially not among its leaders.

5. The birth of the Soviet Mathematical School

According to Stalin's plan the Academy of Sciences of the U.S.S.R., "the headquarters of Soviet science", must be "at hand". That is why the Academy was transferred

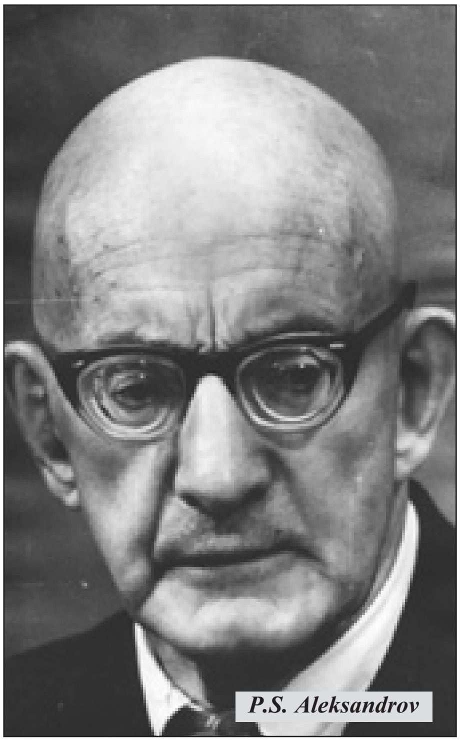
Session of the Moscow Mathematical Society, May 1975

to Moscow in 1934. At the same time several leading institutes of the Academy including the Steklov Mathematical Institute were also transferred to Moscow. Reforms can have unexpected results which are not foreseen by their executors. Thus, the transfer of the Steklov Mathematical Institute to Moscow had an extremely fruitful influence on the development of mathematics in the country. It became a centre, which was unique according to the wide range of activities and to the concentration of mathematical talents. Egorov's disciple Nikolai Nikolaevich Luzin played the principal role in the organization of this scientific centre. His pupils and the pupils of his pupils[2] became the founders of many famous schools in the theories of functions, probability, topology, differential equations, number theory, functional analysis, mathematical logic, complex analysis, algebra and applied mathematics. The amalgam was realized, the synthesis of the traditions of the old schools - the Petersburgian school of mathematical physics (S. Sobolev), of Chebyshev's directions of development of probability theory and approximation theory (S. Bernstein), of the algebraic Kiev School of D. A. Grave (B. N. Delone, O. Yu. Shmidt) - with the newly formed Moscow schools was a success. It is necessary to add that the Moscow school of differential geometry – the principle direction of the pre-revolutionary Moscow mathematics, ascending to K. M. Peterson and D. F. Egorov - was enriched by the ideas of tensor geometry by the school of V. F. Kagan.

As a result of all these changes a powerful research potential was formed around the Steklov Mathematical Institute, the Department of Mathematics and Mechanics of Lomonosov Moscow State University and the Moscow Mathematical Society. This potential became the origin of one of the leading mathematical schools in the second half of 20th century, the "Soviet Mathematical School".

6. The Moscow Mathematical Society after the end of the Second World War

The Moscow Mathematical Society held an exceptional position in Soviet mathematical life. The Society was located in the capital and it was natural that it became the centre of principal events. The most important Soviet mathematical results were presented here and prominent foreign mathematicians visiting Moscow gave their lectures here. After Stalin's death such visits became a regular occurrence. The Society was actively involved



P.S. Aleksandrov

A.N. Kolmogorov

I.M. Gelfand

in editing the Matematicheskii Sbornik, the Uspekhi Matematicheskikh Nauk and its Transactions (Trudy). The Society was also involved in the preparation of such fundamental works as "Mathematics in the U.S.S.R. for 15 (30, 40) years" and "Mathematics in U.S.S.R. 1958-1967" (Vol. II). The Society also organized conferences and seminars, and awarded special prizes. Among the prize winners we find some of the country's best mathematicians.

The Party and the state authorities of course controlled the activities of the Moscow Mathematical Society, as they controlled every public activity in the Soviet State.

However, the control of the Society, a public organization, was not so severe as the control of the state institutions. A state functionary is always more vulnerable than an elected member on a board of a public organization. Therefore, at the end of Stalin's era and at the beginning of the "Khrushchov thaw", the Moscow Mathematical Society started to take liberties such as inviting speakers who opposed the authorities, for example the dissident A. S. Esenin-Vol`pin. In 1970, in the comparatively mild Brezhnev time, the Society took the liberty to elect as its president I. R. Shafarevich, who was an active dissident.

The centenary of the Society was celebrated in 1964 at Moscow University. P.S. Aleksandrov, president of the Society for 32 years, gave the opening address. The following presidents were: A. N. Kolmogorov (1964-1966), I. M. Gelfand (1966-1970), I.R. Shafarevich (1970-1973), A. N. Kolmogorov (1973-1985), S. P. Novikov (1985-1996) and V. I. Arnold (from 1996).

Nowadays the Society continues to support multi-faceted activities. Almost every Tuesday the scientific milieu gathers to hear lectures at which new results are presented. The Society supports summary lectures, as well as discussions on existing problems in the mathematical community. As in the past, the Society remains very concerned about national problems connected with mathematics.

One of the problems is the mathematical education in primary and secondary schools. The Society has focussed on this issue from its foundation. In the first volumes of the Mathematicheskii Sbornik a special section was dedicated to professors of secondary level education. In 1934, the Society established a special division for high schools. The division's principal goals were to extend the boundaries of mathematical instruction, to exchange didactic experience and to establish permanent connections among the professors of secondary and higher education. An important initiative of the Society was to establish the mathematical Olympiads for secondary schools. The first mathematical Olympiad was organized in the autumn of 1935.

During its history the Society has often discussed the problems concerning instruction programs and textbooks for schools. One of the most rigorous and crowded conferences of the Society was the conference of 27 November 2001, where mathematicians examined the reforms for mathematical instruction proposed by the Russian Ministry of Education.

7. Conclusion

The Moscow Mathematical Society was established in Russia during the reform era of Alexander II in a period of economical and cultural expansion. During its long history the Society has been sufficiently conservative on questions concerning its organization as well as the expressions of its activities. At the same time the Society has reacted very severely to the epoch's aspirations.

The Society started as a group of mathematicians who mutually supported their scientific occupations. The Society was then transformed into an informal institution, which coordinated and organized the activities of the national mathematical community. With varying success the Society played this role up to the 1930s. After the reorganization of the Soviet mathematical life, the Society took the leading role in public mathematical organization - a role that, to a certain degree, was independent from the Party and state authorities. We refer here to the mathematical Society of the capital, which in the super–centralized soviet society held a special position. The Moscow Mathematical Society manoeuvred between administrative institutions that did not always act in accordance with each other. Institutions such as the Mathematical department of the Academy of Sciences of U.S.S.R. and the Steklov Mathematical Institute, on the one hand, and the Ministry of higher and special secondary education, leadership of Lomonosov Moscow State University and its faculty of mathematics and mechanics, on the other hand. The Society conquered the problems that occurred on its way, preserving a high scientific reputation and authority.

These dignities probably helped the Moscow Mathematical Society to surmount the difficulties of the perestroika epoch, in which the statute of the public scientific society was also promoted. It is not necessary for the Society to have important financial support for its activities and the Society does not depend on the emigration of leading scientists, as does for example the Moscow State University. Even after their "removal" the members preserve their relations with the Society, they continue to publish their articles in its editions and to present their papers on its conferences during their visits to the homeland. In its long history with periods of prosperity and of disruption, the Society has pursued its activities in spite of all.

The fruitful continuity of mathematical studies in Russia is a traditional target of the Moscow Mathematical Society. On its centenary P. Aleksandrov wrote [7, p.9]: "… the Moscow Mathematical Society always cultivated … a many-sided development of mathematics, the Society didn't cultivate a pre-established framework and systems of values. For a long period the Moscow Mathematical Society was the place where mathematical inventions, researches and creative emotions of generations of Muscovite mathematicians appeared and existed. The Moscow Mathematical Society was not only the locality where mathematical discoveries were registered and where popular lectures on mathematics were given. The Moscow Mathematical Society was also the school for mathematical aesthetics of the very rigorous mathematical taste and the school of scientific ethics as well".

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[1] On this election we do not possess the related documents. All the Proceedings of the Society of that period were destroyed. In the 60s, L. A. Lyusternik gave this information at the seminar on the history of mathematics at Lomonosov Moscow State University.

[2] V. V. Golubev, V. V. Stepanov, I. I. Privalov, D. E. Men`shov, M. Ya. Suslin, A. Ya. Khinchin, P. S. Aleksandrov, P. S. Uryson, L. A. Lyusternik, M. A. Lavrent`ev, N. K. Bari, P. S. Novikov, I. G. Petrovskii, A. N. Kolmogorov, L. V. Keldysh, S. M. Nikol`skii, L. G. Shnirel`man, A. O. Gel`fond, A. N. Tikhonov, A. G. Kurosh, L. S. Pontryagin, A. I. Mal`tcev, M. V. Keldysh, I. M. Gelfand.

Problem Corner
Contests from Bulgaria Part III

Paul Jainta

With good reason I have adjourned the current series on maths contests held in Bulgaria. In December 2002 the legendary flagship among mathematical journals focusing particularly on teenagers with a special talent for mathematics or physics, the Hungarian KÖMAL, has procreated an English-speaking baby. The new offspring was christened 'Mathematical and Physical Journal for Secondary Schools', and turned out from the start to be a great pleasure to all those admirers of the parent paper who are not good with the Hungarian language. The excellent reputation of KÖMAL is worthy of this interlude at any rate.

Now, we will continue describing the occurrences in Bulgarian mathematics circles. Many helpers are doing a good job there in identifying and furthering mathematically able young aces. So, a consideration of the species-rich mathematical biotopes in Bulgaria will undoubtedly be a rewarding undertaking. I want to express my warm thanks to **Prof. Sava Grozdev**, Institute of Mechanics, Bulgarian Academy of Sciences, Sofia, for thinning out for us the unclear maze of stimulating young compatriots. In Bulgaria the multi layered spectrum of further education of young people who are greedy for maths food, contains such a lot of material it would be sufficient for half a dozen additional Corners.

The Patron Saints of Contests (Sava Grozdev)

Year after year about 30 different mathematics competitions are organized in Bulgaria and to boot some international events like the "European Kangaroo" competition, the Australian Mathematical Contest, the Mediterranean Contest or the Tournament of Towns, which are not counted in that figure. A couple of these competitions are named after outstanding Bulgarian mathematicians, Ivan Salabashev (1853 - 1924) for instance, whom I have mentioned in a former part.

Nikola Obreshkov (1896 - 1963) is another remarkable representative of the mathematicians' guild of Bulgaria. He is well-known for his scientific results in exploring the distribution of zeroes of entire and meromorphic functions, or for working on the summation of non-convergent series and Diophantine approximations as well as on other fields. While still an Assistant Professor, Nikola Obreshkov formulated and proved a generalisation of the Descartes Theorem for positive zeroes of algebraic equations, which has kept its original appearance for about 300 years. Also he solved an open question related to the Borel constant. The competition in honour of Nikola Obreshkov, is distinguished by the fact that apart from problem solving it includes special answering to some entertaining puzzles. During the Awarding Ceremony, which takes place in one of the numerous Sofian schools (named after Obreshkov, too), students have to answer questions circling the History of Mathematics and are asked to solve a handful of problem riddles. These, and similar initiatives, are of extreme importance for popularizing Mathematics in Bulgaria, for they are suited to dissipate the public prejudice that Mathematics is always a dry science. Usually, the Awarding Ceremony ends in an artistic program with lively music provided by a band from the host school, as background to the final. Occasionally the band's name is borrowed from the world of Mathematics, like "Algorithm" for example.

There are some other outstanding Bulgarian mathematicians who act as Patron Saints for competitions: Ljubomir Chakalov (1886 - 1963), Cyril Popov (1880 - 1966) and Virgil Krumov (1930 - 1990). The first two have been scientists and university professors, as Nikola Obreshkov was, while the third worked at a school as an ordinary mathematics teacher. The origin of the "Virgil Krumov" contest is the city of Silistra, on the river Danube, which is the birthplace of its patron. The competition is addressed to students from grades 5 – 12, and about 500 students from all parts of the country are participating annually. The papers for grades 5 – 8 each include 9 problems, while in higher classes only 3 problems are posed. The winners will receive cash and objective prizes. One of the peculiarities of this contest is that the coaches of the winners are awarded too. The ulterior motive is obvious, for stimulating teachers is the basis for an effective widespread impact in recruiting more mathematics students. On the one hand this is a natural acknowledgment of the teacher's efforts and on the other hand an incentive for future works. The best students representing Bulgaria at international Olympiads are at the apex of the pyramid. The base of this pyramid is formed by the mass of students whose instruction and flaming up are in the hands of teachers from all parts of the country. So, Bulgarian teachers are putting up scaffolding for a stable basis, year by year.

Furthermore, some competitions are named after historical Bulgarian personalities. For instance, Petar Beron (1800 - 1871), was a leader of the Bulgarian national rebirth and the author of the first Bulgarian primer (1824). A mathematics contest which is associated with this name is carried out anually for students up to 7th grade. Chernorizets Hrabar was another notable and refined person. Living in the IX Century, he belonged to the first Bulgarian 'men of letters'. The "Chernorizets Hrabar Tournament" was dedicated to his achievements in writing. Some other established mathematics competitions are connected with important incidents and anniversaries of the Christian calendar. Especially worth mentioning are those competitions on the occasion of Christmas, Easter, Saint George (6th May) and Saint Nikola (6th December). A lot of competitions only play a role on a regional scale. Bulgaria, which has a territory of about 110 000 km^2, is divided into 28 administrative districts. Each region has its own administrative centre and a so-called Education Inspectorate that acts as the current right-hand man of the national Ministry of Education and Science. By the assistance of the corresponding Inspectorates, mathematics competitions are organized in the cities of Sofia, Shumen, Vidin, Vratsa, Montana, Kardjali and so on. Actually, the competition held in Shumen is named after 'Preslav the Great' in memory of one of the former capitals of the old Bulgarian Kingdom. Some of these competitions are with international participation. For example, the mathematics community in Vidin maintains active contacts with colleagues from Romania, while the organisers from Kardjali are always inviting contestants from Turkey to brood over problems together with the home competitors. The relations are based on an equivalent exchange principle and annually, Bulgarian students and teachers start off for return visits respectively. Similar initiatives are held with the neighbour country Macedonia too.

Two other competitions should be noted as well. They are organized under a more diverse principle. In each administrative centre there is at least one school, known to be the Foreign Language School. In such a school one of the following foreign languages are being studied, profoundly: English, French, German, Spanish, Italian or Russian. Most of the subjects are being taught and deepened in the chosen language. For students from these schools an extra National Mathematics Contest is carried out each year in one round. The number of participants (8th - 12th grade) varies continually between 300 and 400, and all contestants have to undergo a three-hour paper consisting of three problems. Finally, they are ranked individually and as a team.

Well, you see, Bulgaria provides an effective forge for future mathematicians. In this market-place one is haggling over problems. Now you are invited to assess the quality of this merchandise yourself, on the basis of six new finished products. All questions are selected from the National Olympiad. The authors and the corresponding year of first appearance are given in brackets.

158 (S. Grozdev, 1986) Given a n-gon (not necessarily convex) and a point S in its interior, a light source is located at S such that no side of the polygon can be illuminated by the source entirely. What polygon with the smallest number of edges ($n \geq 3$) satisfies this property?

159 (O. Muchkarov, 1996) Find all prime numbers p, q, such that the number $\dfrac{\left(5^p - 2^p\right)\left(5^q - 2^q\right)}{pq}$ is an integer.

160 (A. Ivanov, 1997) If a, b, c are positive real numbers with abc=1, prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}$$, with equality only if a = b = c = 1.

161 (E. Kolev, 1998) $A_1, A_2, \ldots, A_n$ are n equidistant collinear points in the plane. They are coloured in two colours. What is the smallest number of points ($n \geq 3$) such that there exist three points A_i, A_j, A_{2j-i} ($1 \leq i < 2j-i \leq n$) which are of the same colour?

162 (N. Nikolov, 1998) Prove that there exists no function $f: \mathbf{R}^+ \mapsto \mathbf{R}^+$, which satisfies the inequality $(f(x))^2 \geq f(x+y)(f(x)+y)$ for any $x,y \in \mathbf{R}^+$.

163 (S. Grozdev and S. Doichev, 2002) Given a triangle with side lengths a,b and the lengths of the corresponding angular bisectors l_a, l_b, find the smallest rational number k such that $\dfrac{l_a + l_b}{a+b} < k$.

Solutions to some earlier problems

Now we will get to work on the problems set in issue 47, March 2003. Unfortunately, the inflow of solutions has been limited this time.

146 (S. Grozdev, 1987: Winter Competition) Given 4 arcs on a circle, such that each pair of arcs has exactly one point in common, prove that at least one arc contains the midpoint of another arc.

Solution by Erich N. Gulliver, Schwäbisch Hall, Germany. Also solved by Niels Bejlegaard, Copenhagen, Denmark, and J. N. Lillington, Dorchester, UK.

Let us prove a stronger conclusion from a weaker assumption, namely: Given 4 arcs on a circle, such that each pair has at least one point in common, then one of the arcs is contained in the union of the other three.

To prove this, assume that the fourth arc is not contained in the union of the first three, so some point of the fourth arc is not contained in that union. We cut the circle at that point and 'straighten' it. Then the first three arcs can be mapped onto three intervals, say I, J, K, on a straight line. One of these intervals, say I, contains the leftmost part of their union, and one, say I or J, contains the rightmost part of their union. But $I \cap J \neq \{\ \}$, so $I \cup J$ is also an interval and $K \subset (I \cup J)$.

147 (S. Grozdev, 1988: Atanas Radev Tournament Special Prize) Given 100 positive integers $x_1, x_2, \ldots, x_{100}$, such that $1/\sqrt{x_1} + 1/\sqrt{x_2} + \ldots + 1/\sqrt{x_{100}} = 20$, prove that at least two of the integers must be equal.

Solution by J.N. Lillington, Dorchester, UK. Also solved by Niels Bejlegaard, E. N. Gulliver, J. N. Lillington, Gerald A. Heuer, Concordia College, Moorhead, MN, USA, E. Macias and A. Sotelo, Santiago de Compostela, Spain, and Dr Z. Reut, London.

Prove first of all that $\dfrac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}$, $n \geq 1$. Now $4n^2-4n < 4n^2-4n+1$ or $4n(n-1) < (2n-1)^2$ and

$2\sqrt{n(n-1)} < 2n-1$ i.e. $1 < 2n - 2\sqrt{n(n-1)}$ which leads to $\dfrac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}$. (1)

Suppose now that $x_1, x_2, \dots, x_{100}$ are all distinct. Then

$$\frac{1}{\sqrt{x_1}}+\frac{1}{\sqrt{x_2}}+\dots+\frac{1}{\sqrt{x_{100}}} \le \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{100}} < 1+2\sqrt{2}-2\sqrt{1}+2\sqrt{3}-2\sqrt{2}+\dots+2\sqrt{100}-2\sqrt{99} = 1\text{-}2+20 = 19$$

(applying (1) for $n \ge 2$). Hence if $\frac{1}{\sqrt{x_1}}+\frac{1}{\sqrt{x_2}}+\dots+\frac{1}{\sqrt{x_{100}}} = 20$, then at least two of $x_1, x_2, \dots, x_{100}$ are equal for they cannot all be distinct.

(*Ed*- Bejlegaard, Gulliver, Heuer, Macias and Sotelo, have chosen an alternative approach by use of the integral method. I will give Heuer's solution: Let $s = \sum_{k=1}^{100} \frac{1}{\sqrt{x_k}}$. If all the x_k are different, then $s \le \sum_{k=1}^{100} \frac{1}{\sqrt{k}}$. It suffices, then, to show that $\sum_{k=1}^{100} \frac{1}{\sqrt{k}} < 20$. While one could do this by brute force numerical approximation, a more elegant way is to use the fact that for $k-1 \le x \le k$ we have $\frac{1}{\sqrt{k}} \le \frac{1}{\sqrt{x}}$. It follows then that $\sum_{k=1}^{100} \frac{1}{\sqrt{k}} \le 1 + \int_1^{100} \frac{dx}{\sqrt{x}} = 19$).

148 (S. Savchev, 1990: Winter Competition) Let S be an infinite set of points in the plane with the following property: if A, B, C are different points of S, then the distance from A to the line BC is an integer. Prove that all the points of S are collinear.

Solution by E. N. Gulliver. Also solved by Niels Bejlegaard and J. N.Lillington.

First we observe that for any two distinct points A and B there exists only a finite(*) number of lines through B with integer distance from A. This is because the distances in question are bounded above by the distance between A and B, and for each distance there are at most two such lines. Now assume that three points X,Y,Z of S are not collinear. Applying the first observation to each pair of points out of X,Y,Z, we find that one can draw only a finite number of lines through each of the points X,Y,Z, that contain points of S other than X,Y,Z. The number of triplets of such lines (through X, Y, Z respectively) is finite, too. Thus, each triplet has at most one point in common because X,Y,Z are not collinear. This contradicts the assumption that S is finite.
(*Ed*- (*) *One can see this fact as follows: Let B a point of S , $B \ne A$ and $|AB| = d$. Take another point C in S which is different from A and B. Denote by l the line AC and let r be the distance from B to l. Obviously, $r \le d$, and since r is an integer, then r can take only the values 0,1, … , [d], where [d] denotes the integer part of d. There are exactly two lines through A with a distance r from B. Both lines are tangents to the circle with centre B and radius r. Consequently, there exists at most 2[d]+1 possibilities for line l with a distance 0 from B (including line AB)*).

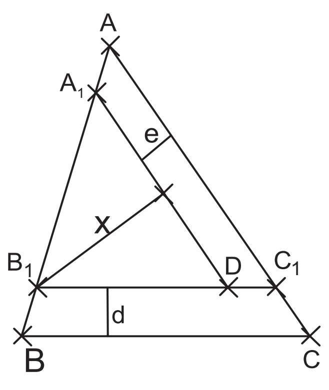
149 (S. Grozdev, 1989: Spring Tournament) A triangle ABC cannot be covered by any two smaller triangles that are similar to it. Find the angles of the triangle.

Solution by Niels Bejlegaard, Copenhagen. Also solved by J. N. Lillington.

In the figure, triangle ABC has side lengths $a<c<b$; so for the corresponding altitudes we get $h_a>h_c>h_b$ (*).
Triangle AB_1C_1 is homothetic to triangle ABC, with A the centre of homothecy and $k<1$ the ratio of corresponding sides. Draw line segments A_1D parallel to AC with $A_1D = BC = a$, and B_1C_1 parallel to BC. Both segments intersect at point D.
Denote d as the distance between the parallel sides B_1C_1 and BC, and e as the distance between (parallel) sides A_1D and AC. Now $d = h_a - kh_a = (1\text{-}k)h_a$.
The ratio of similitude of the similar triangles A_1B_1D and ABC gives

$$\frac{x}{A_1D} = \frac{h_b}{AC} \text{ or } x = \frac{a \cdot h_b}{b}$$ (x … Altitude from B_1 to side A_1D).

So, $e = k \cdot h_b - h_b \cdot \frac{a}{b} = k \cdot h_b - k \cdot h_b \cdot \frac{a}{k \cdot b} = k \cdot h_b (1 - \frac{h_b}{k \cdot h_a})$.



For quadrilateral A_1AC_1D to cover quadrilateral BB_1C_1C we must have $d < e$, which implies $(1-k)h_a < k \cdot h_b \left(1 - \frac{h_b}{k \cdot h_a}\right)$ and after some algebra, the inequality simplifies to $1 > k > \frac{h^2+1}{h^2+h}$ with $h = \frac{h_a}{h_b}$. It is fulfilled when (*) holds, which means that k exists if and only if $h_a=h_c=h_b$ i.e. the triangle ABC is equilateral.

150 (A. Ivanov, 2000: Atanas Radev Tournament Special Prize) Given 4 points in the plane, such that the distance between any two of them is an integer, prove that at least one of the distances is divisible by 3.

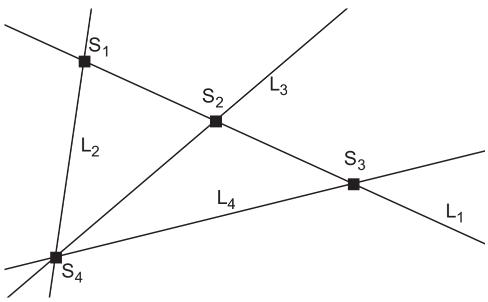
Solution by Erich N. Gulliver. Also solved by Niels Bejlegaard and J. N. Lillington.

Given any four points P,Q,R,S in a three dimensional Euclidean space with distances a=PQ, b=RS, c=PR, d=QS, e=PS, f=QR, these vectors span an n-simplex with vertices P,Q,R,S, whose volume V is given by the expression $144V^2=(a^2+b^2)(c^2d^2+e^2f^2)+(c^2+d^2)(a^2b^2+e^2f^2)+(e^2+f^2)(a^2b^2+c^2d^2)-a^2c^2f^2-a^2d^2e^2-b^2c^2e^2-b^2d^2f^2-a^4b^2-b^4a^2-c^4d^2-d^4c^2-e^4f^2-f^4e^2$. If P,Q,R,S are points in a plane, the value of V equals zero.

Now assume that a,b,c,d,e,f are integers. If none of them were divisible by 3 then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv f^2 \equiv 1$ (mod 3), and the above polynomial would take only the value 2 (mod 3) rather than zero.

(Ed- In an informal sense, an n-simplex can be thought of as an n dimensional generalisation of a triangle. Thus a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. Niels Bejlegaard has excavated another rare erratic boulder to avoid much tedious computing. In the 'Acta Arithmetica' LXII, 1992, he found the following cumbersome relation $a^2x^4+b^2y^4+c^2z^4+a^2b^2c^2 = (a^2+b^2-c^2)(x^2y^2+c^2z^2)+(b^2+c^2-a^2)(y^2z^2+a^2x^2)+(c^2+a^2-b^2)(x^2z^2+b^2y^2)$, *with side lengths a,z,x,c and y,b the lengths of the diagonals of a plane quadrilateral, established by T.G. Berry in an article titled 'Points at rational distances from the vertices of a triangle'. All his further considerations are similar to those of E. Gulliver).*

151 (S. Grozdev, 1994: Atanas Radev Tournament Special Prize) In a city all tramway routes have the following properties: there are at least 2 stops on each line; at least one line connects each pair of stops; each pair of lines has exactly one stop in common. Prove that there is the same number of stops on each tram route.



I have to concede here some difficulties with the wording of the problem. Obviously, the translation of the text from Bulgarian language into English speech has left a gap somewhere, which may have caused a bit of confusion to readers. J.N. Lillington comments that this question is ambiguous. The result is false if the question means 'Given any two stops there is a requirement that some line joins them'. For, consider four stops as shown in the figure. L_1 has three stops S_1, S_2, S_3. All other lines have two stops.
However if the question means 'Given two stops there is a line joining them with no stops in between' the result is true. For in this case no line can have three stops, for if this is the case and a line has stops S_1, S_2, S_3, then there must exist another line with stops S_1 and S_3. Thus each line has exactly two stops only.
Niels Bejlegaard has also attempted to access the problem, but had to give up. 'This problem has given me increased amounts of grey hair', he sighed.

I hope that the new set of questions above will not be the cause for any other reader to go grey. That concludes this issue of the Corner. Please keep sending me Olympiad Contests and your nice solutions.

THE FOURTH EUROPEAN CONGRESS OF MATHEMATICS

Stockholm, June 27 – July 2, 2004

Ari Laptev, Chairman, 4ECM Organization Committee

The Fourth European Congress of Mathematics (4ECM) will take place in Stockholm, Sweden, June 27 to July 2, 2004. It will be the major international mathematical event of the year 2004. The theme of the Congress is "Mathematics in Science and Technology". By now the programme of the Congress is more or less settled and can be found at:

http://www.math.kth.se/4ecm/program/scientific.programme.html

There will be seven Plenary Lectures, thirty three Invited Lectures, twelve European Network Lectures and Science Lectures. Two of the planned six Science Lectures have not yet been finalized. It has been rather surprising for us to discover how difficult it is to find speakers among physicists. In particular, it is a great disappointment that Gerard't Hooft, who first agreed to give a talk at the 4ECM, has withdrawn.

There will be ten EMS Prizes of 5.000 Euro each to young mathematicians who have made a particular contribution to the progress of Mathematics. The closing date for nominations for the EMS Prizes was the 1st of February and the nominations will be considered by the members of the Prize Committee chaired by N.N.Uraltseva (St.Petersburg). The list of prize winners will be announced at the opening ceremony of the 4ECM on the 28th of June.

I remember that about a year or so ago there was a certain scepticism regarding the openness of the procedure of nominations of the candidates for the EMS prizes. It was agreed then that nominations may be made by anyone, including members of the Prize Committee or by the candidates themselves. Some people thought that such a procedure would lead to nominations of too many non-serious candidates and thus create difficulties for the work of the Prize Committee.

At the end of October 2003 there were only about ten nominations and there was some speculation about whether the prizes had been publicised enough. However, by February 1 there were **54** young mathematicians who had been nominated for the prizes. Almost all of them are serious mathematicians who have made substantial contributions in their areas of mathematics and most of them are definitely worthy of a prize. I wish the members of the Prize Committee all the best in their task which though hard I am sure will prove to be very rewarding.

On reflection, Professor N.Uraltseva and I think that the open nominating procedure has been extremely successful and I would strongly recommend following such a procedure as a standard rule for future EMS Congresses.

We are about to consider the first round for contributed papers (posters) for which the first deadline was the 15th of February. It will not be too late to apply for a poster session even after the 15th of February, but it should not be done later than the 20th of April. So far there are 232 applications for poster sessions. It seems very likely that we will ultimately match the corresponding number of applications accepted at the 3ECM in Barcelona (about 300 posters).

In view of the fact that many of the participants of the 4ECM will need a Swedish visa, we have created a special link "Letter of Invitation" on our home-page, *http://www.math.kth.se/4ecm*. Here one can find a simple form which has to be completed in order to obtain an invitation. It is important to emphasize that the Letter of Invitation is in no way an acceptance of financial responsibility by the organizers for the participant. We look at new applications every day and Letters of Invitations are sent to the applicants within a couple of days after their applications have been received.

The deadline for applications for 4ECM Grants is the 29th of February, see *http://www.math.kth.se/4ecm/grants.html*. The applicants will be informed on whether they have received a 4ECM grant or not in the middle of March. Grant holders should confirm their registration before April 1, 2004. The registration fees for those having obtained type A grants will be paid automatically by the organizers. Type B and C grants will be paid to their holders on arrival at the Congress.

About Stockholm, Royal Institute of Technology and Stockholm University

Once again I should like to remind you that Stockholm is one of the most beautiful cities in the world. It is the Royal Capital of Sweden and was founded by Birger Jarl in 1252 around a fortress. Built on 14 islands you are never far from the water as shown on the 4ECM poster, which we are happy to hear has been widely appreciated. Although we have sent it to more than 900 Mathematics Departments in Europe, Northern America, Australia and Japan, I continue to receive many letters from people who would like to have an extra copy of such a poster for their Department or for themselves.

In the next issue of the EMS Newsletter we are planning to write an article about Swedish

Ari Laptev

Mathematics and Swedish Universities. Here I would just like to remind you that the 4ECM is organized by the Royal Institute of Technology (Kunliga Tekniska Högskolan (KTH)) in collaboration with Stockholm University. Both schools are relatively young compared with other Universities in Europe and in Sweden. The oldest University in the Nordic countries is Uppsala University founded in 1477. It might be interesting to note that the second Swedish University was founded in Tartu/Dorpat (Estonia) in 1632. The University of Turku/Åbo (Finland) was founded slightly later in 1640 and the University of Lund in 1668.

The origins of the Royal Institute of Technology in Stockholm go back to 1827, when the "Teknologiska Instituttet" began to offer education in technological subjects with a strong professional touch. Stockholm University (SU) is an even younger Institution. It was founded as a private University (Högskoler) in 1878 and was brought under state control in 1904.

The Mathematics Departments of KTH and SU are very strong at the moment with a number of excellent mathematicians who are actively involved in different European projects and contribute greatly to the development of modern mathematics.

Welcome to the 4ECM in Stockholm

The City Hall at Stockholm. Photo: Christer Lundin

Fourth European
Congress of Mathematics
Mathematics in Science and Technology
4ecm
STOCKHOLM 2004
Stockholm, Sweden
June 27 – July 2, 2004
VETENSKAP OCH KONST
KTH
Organized by
Royal Institute of Technology in Stockholm
UNIVERSITET STOCKHOLMS
in collaboration with
Stockholm University
under the auspices of
European Mathematical Society
For further information: Http://www.math.kth.se/~4ecm
E-mail address: 4ecm@stocon.se

ICME-10, Copenhagen 2004

H.C. Hansen (Danish ICMI-representative)

The ICME congresses are held every fourth year under the auspices of the International Commission on Mathematical Instruction (ICMI). The first ICME was held in 1969 in Lyon, France.

ICME-10 in Copenhagen 2004 is expected to gather around 3000 mathematicians, including researchers in education, teachers, and other interested parties from over 100 different countries. The congress will maintain and develop the ICME traditions but will also introduce a number of new elements to the scientific programme. The scientific programme is planned by the International Programme Committee (IPC), which consists of 21 members from all over the world.

Tradition and innovation

The main part of the scientific programme contains the best elements from former ICMEs but the IPC has added a number of new ideas into the structure. The resulting programme at ICME-10 includes the following events:

- 8 Plenary activities. Besides lectures, this includes reports from some of the Survey Teams that will give a review of the state-of-the-art with respect to a certain theme.
- Four scores of regular lectures. These will cover a wide spectrum of topics, themes and issues.
- 29 Topic Study Groups, each organized by prominent experts in the specific field. 8 of these are organized according to educational levels, 13 according to content related issues and the rest to overarching perspectives and meta-issues.
- 24 Discussion Groups with genuine interactive discussion and no oral presentations.
- A thematic afternoon with five parallel mini-conferences: Teachers of mathematics, Mathematics education in society and culture, Mathematics and mathematics education, Technology in mathematics education and Perspectives on research in mathematics education from other disciplines.
- Workshops and Sharing Experiences Groups. These can range from new approaches to teaching and new mathematical topics, to obstacles to innovation and PhD students sharing approaches.
- Posters - and time slots for presentations of these. The posters will be grouped and particular Round Table sessions will be scheduled for such groups.
- National presentations on the state of and trends in mathematics education in the following countries: all the Nordic countries, Mexico, Korea, Rumania and Russia.
- Presentation of papers. However not all the papers accepted by the organizers can be accommodated in the oral programme, so the idea of Presentation by Distribution, which was invented for ICME-9, will also be adopted for ICME-10.

For details and registration information, please visit *www.ICME-10.dk*.

FIRST ANNOUNCEMENT

The Czech Mathematical Society and the European Mathematical Society organize the 2nd Joint EMS Mathematical Weekend September 3 – 5, 2004 Prague, Czech Republic

The minisymposium follows the scenario of a series of smaller more frequent, meetings in specific areas of mathematics suggested by the EMS: the EMS Mathematical Weekends. They offer a chance for European and other mathematicians working in related areas to get together and interact on a more permanent basis. This is the second meeting in a newly established series, following the first EMS Mathematical Weekend organized by the Mathematical Society in Lisbon in 2003.

The meeting will consist of 4 plenary lectures followed by a programme in 4 special sessions:

- Complexity of Computations and Proofs
  Chair: Jan Krajíček (Czech Academy of Sciences)

- Discrete Mathematics and Combinatorics
  Chair: Jaroslav Nešetřil (Charles University)

- Mathematics of Fluid Mechanics
  Chair: Eduard Feireisl (Czech Academy of Sciences)

- Mathematical Statistical Physics
  Chair: Roman Kotecký (Charles University)

Local information

The mathematical weekend will take place in the newly reconstructed building of the School of Computer Science of the Faculty of Mathematics and Physics at Charles University, located in the historical centre of Prague, Mala Strana.

Accommodation can be reserved in nearby hotels and guest rooms in student residences upon request.

For further information please see *http://mvs.jcmf.cz/emsweekend/* or mail to: *mvs-jcmf@kam.mff.cuni.cz*

Forthcoming conferences

compiled by Vasile Berinde (Baia Mare, Romania)

*Please e-mail announcements of European conferences, workshops and mathematical meetings of interest to EMS members, to one of the following addresses: **vberinde@ubm.ro** or **vasile_berinde@yahoo.com**. Announcements should be written in a style similar to those here, and sent as Microsoft Word files or as text files (but not as TeX input files). Space permitting, each announcement will appear in detail in the next issue of the Newsletter to go to press, and thereafter will be briefly noted in each new issue until the meeting takes place, with a reference to the issue in which the detailed announcement appeared*

April 2004

29 March-1: 4th IMA Conference on Modelling Permeable Rocks, Southampton, UK
Theme: An international meeting bringing together geologists and mathematicians, statisticians and engineers working on the modelling and analysis of the spatial patterns observed in permeable rocks
Scope: The emphasis will be on new developments in the mathematical modelling of geological patterns for the purposes of improving fluid flow prediction
Topics: Faults and fractures, influence of geomechanics; using geophysics to identify heterogeneity; process modelling/pattern formation; quantitative techniques for data collection; geostatistical techniques; uncertainty in modelling heterogeneity; integration of data in models
Main speakers: B. Bodvarsson (L. Berkeley Nat. Lab.), B. Noetînger (IFP), D. Zhang (Los Alamos National Lab.), A. Journel (Stanford Univ.)
Format: Keynote lectures and contributed presentations
Sessions: Plenary lectures
Organizers: The Institute of Mathematics and its Applications
Organising committee: A.H. Muggeridge (Imperial College, London), P. Abrahamsen (Norwegian Computing Centre, P. Burgess (Shell), J.P. Chiles (ENSMP), J. Gomez-Hernandez (UPValencia), R. Jolly (BP), P.R. King (Imperial College), S. Knight (Pelican), C. Mijnssen (PDO, Oman), G. Pickup (Heriot-Watt), C. Ravenne (Institut Francais du Petrole), J. Walsh (Dublin), M. van Bracht (TNO), C. White (Louisiana State)
Location: University of Southampton, UK
Information: e-mail: conferences@ima.org.uk
Web site: http://www.ima.org.uk

5-7: 5th International Conference on Modelling in Industrial Maintenance and Reliability - Impacting on Practice, University of Salford, UK
Information: e-mail: conferences@ima.org.uk
Web site: www.ima.org.uk
[For details, see EMS Newsletter 50]

19-24: CHT-04 International Symposium on Advances in
Computational Heat Transfer; on cruise ship MS Midnatsol between Kirkenes and Bergen, Norway
Information:
e-mail: cht04@cfd.mech.unsw.edu.au
URL: http://cht04.mech.unsw.edu.au
[For details, see EMS Newsletter 49]

May 2004

30-June 3: Fifth European Conference on Elliptic and Parabolic Problems: A Special Tribute to the Work of Haim Brezis.
Gaeta, Italy
Topics: The panel of speakers includes, in particular, numerous former students of H. Brezis. Gaeta, where the conference will take place, is in the gulf of Serapo, between Rome and Naples. The hotel is located directly by the sea, just a ten minute walk from the old city of Gaeta. Besides Elliptic and Parabolic issues, the topics of the conference include Geometry, Free Boundary Problems, Fluid Mechanics, Evolution Problems in general, Calculus of Variations, Homogenization, Control, Modelling and Numerical Analysis.
Invited speakers:
see URL: http://www.math.unizh.ch/gaeta2004
Organising Committee: C. Bandle (Basel), H. Berestycki (EHESS), B. Brighi, A. Brillard (Mulhouse), M. Chipot (Zurich), J.-M. Coron (Paris Sud), C. Sbordone (Naples), I. Shafrir (Haifa), V. Valente (IAC, Roma), G. Vergara Caffarelli (Roma)
Information: gaeta@math.unizh.ch

30-June 5: Commutative rings and their modules, Incontro INdAM, Cortona, Italy
Information: e-mail: cortona4@mat.uniroma3.it; web site: http://www.mat.uniroma3.it/seminari/conferenze/cortona2004.htm
[For details, see EMS Newsletter 50]

30 - June 6: SPT2004 - Symmetry and Perturbation Theory
Cala Gonone (Sardinia, Italy) (follows SPT96, SPT98, SPT2001 and SPT2002 conferences)
Information: Web site http://www.sptspt.it
[For details, see EMS Newsletter 49]

June 2004

2-4: Mathematical problems in Engineering and Aerospace Sciences, The West University of Timisoara, Romania (ICNPAA2004)
Information: e-mail: SeenithI@aol.com;
Web site: www.icnpaa.com
[For details, see EMS Newsletter 49]

3-10: Sixth International Conference on Geometry, Integrability and
Quantization, Sts. Constantine and Elena Resort (near Varna), Bulgaria
Aim: This sixth edition of the conference aims, like the previous ones, to bring together experts in Classical and Modern Differential Geometry, Complex Analysis, Mathematical Physics, and related fields in order to assess recent developments in these areas and to stimulate future research
Organizers: Ivailo M. Mladenov (Sofia) and Allen Hirshfeld (Dortmund)
Location: Sts. Constantine and Elena Resort (near Varna), Bulgaria
Information: e-mail: mladenov@obzor.bio21.bas.bg; hirsh@physik.uni-dortmund.de
Web site: http://www.bio21.bas.bg/conference/

7-9: International Conference "Trends in Geometry - In Memory of
Beniamino Segre", Roma (Italy)
Aim: The conference is organized to mark the hundredth anniversary of the birth of Beniamino Segre (1903-1977). Keeping in mind the wide range of mathematical areas to which Beniamino Segre contributed, the aim of the conference is to present the state of the art in Geometry, in its algebraic, differential, arithmetical and combinatorial aspects. The intention of the organizers is to honour Beniamino Segre by inviting some outstanding mathematicians of today. The talks will be addressed to a general mathematical audience.
Speakers: M. Berger, P. J. Cameron, J.H. Conway, W. Fulton, P. Griffiths, M. Gromov, J.W.P. Hirschfeld, D. Jungnickel, G. Korchmaros, Y.I. Manin, E. Sernesi, J.A. Thas, G. Thorbergsson, G. Tomassini, J. Zaks
Scientific Committee: E. Arbarello, P.V. Ceccherini, C. de Concini, D.
Ghinelli, S. Marchiafava, P. Maroscia, C. Procesi, E. Vesentini.
Sponsors: Accademia Nazionale dei Lincei; Università di Roma `La Sapienza´; GNSAGA dell'INdAM; Dipartimento di Matematica `G. Castelnuovò dell'Università di Roma `La Sapienza´; COFIN: Geometria delle varietà differenziabili, Gruppi, Grafi e Geometrie, Spazi di moduli e teorie di Lie (all Univ. di Roma `La Sapienza´)
Information: e-mail: segre2004@mat.uniroma1.it
Web site: www.mat.uniroma1.it/segre2004

7-11: Conference on Poisson Geometry, Luxembourg City, Grand-Duchy of Luxembourg
Information: http://www.cu.lu/Poisson2004
[For details, see EMS Newsletter 49]

10 - 17: 1º CIME Course - "Representation Theory and Complex Analysis", Venezia, Italy
Speakers: M.G. Cowling (Univ. of New South Wales): *Applications of representation theory to harmonic analysis of Lie groups*; E.V. Frenkel (Univ. of California at Berkeley): *Recent developments in the geometric Langlands Program;* M. Kashiwara (Research Institute for Mathematical Sciences): *Equivariant derived category and representation of semisimple Lie groups;* D.A. Vogan Jr. (Massachusetts Institute of Technology): *Unitary representations and complex analysis;* N.R. Wallach (University of California at San Diego): *Representation theory of Lie groups and applications*
Course scientific directors: E. Casadio Tarabusi; A. D'Agnolo; M.A. Picardello.
Sponsors: The European Commission, Division XII, TMR Programme "Summer Schools"; Consiglio Nazionale delle Ricerche; Ministero dell' Istruzione, dell'Universita' e della Ricerca; Ministero degli Affari Esteri - Direzione Generale per la Promozione e la Cooperazione - Ufficio V.
CIME Scientific Committee: V. Capasso, C. Cercignani, P. Colli, R. Conti, M. Manetti, E. Mascolo (C.I.M.E. Secretary), A. Maugeri, V. Vespri, C. Viola, G. Zampieri, P. Zecca (C.I.M.E. Director).
Information: e-mail: cime@math.unifi.it
Web site: *http: //www.math.unifi.it/CIME*
Address: Fondazione C.I.M.E. c/o Dipart.di Matematica "U. Dini"
Viale Morgagni, 67/A - 50134 FIRENZE (ITALY)
Tel. +39-55-434975 / +39-55-4237111 —

FAX +39-55-434975 / +39-55-4222695

14 – 16: SEFI Mathematics Working Group Seminar, Vienna, Austria
Themes: What do we mean by understanding mathematics? How should we assess engineering mathematics? New ideas in teaching mathematics to engineers
Scientific organizing committee: Marie Demlova (Praha), L. Mustoe (Loughborough) (chair and co-chair); Brita Olsson-Lehtonen (Espoo), C.-H. Fant (Chalmers), Daniela Velichova (Bratislava), D. Lawson (Coventry)
Local organizers: H.K. Kaiser (hans.kaiser @tuwien.ac.at), Martina Lederhilger-Widl (mlederhilger@zv. tuwien.ac.at)
Location: The conference will take place at the main building of Vienna University of Technology, A- 1040 Vienna, Karlsplatz 13.
Format: 15 minute papers (plus discussion); Poster presentations; Working group discussions; Invited plenary lectures
Abstracts: are still very welcome, please send them as soon as possible to Marie Demlova or Leslie Mustoe (L.R.Mustoe@lboro.ac.uk)
Deadline: (acceptance of papers) end of March 2004
Registration: Please register until the end of April 2004; Conference fee: 250 Euro (a limited reduction is possible upon application)
Information:
e-mail: mlederhilger@zv.tuwien.ac.at;
hans.kaiser@tuwien.ac.at;
demlova@math.feld.cvut.cz

16-23: **5th International Conference on Functional Analysis and Approximation Theory (FAAT 2004), Acquafredda di Maratea, Potenza, Italy**
Information: http://www.dm.uniba.it/faat2004
http://www.dm.unile.it/faat2004
[For details, see EMS Newsletter 50]

18-23: Mathematical Foundations of Learning Theory, Universitat Pompeu Fabra, Barcelona
Information:
http://www.crm.es/MathematicalFoundations
e-mail: MathematicalFoundations@crm.es
[For details, see EMS Newsletter 50]

19-24: Symmetries and integrability of Difference Equations. Euro Conference on Analytic Difference Equations, Special Functions & Quantum Models on the Lattice, Helsinki, Finland
Aim: This meeting is the second in a series of two devoted to discrete systems and their integrability and symmetries (the first took place in Giens, France, 2002). This second conference will emphasise linear and nonlinear special functions, associated quantum problems and geometry
Topics: analytic difference equations and spectral theory; difference bispectral problems; difference Galois theory; Q-hypergeometric and elliptic modular functions; representation theory and orthogonal polynomials; algebraic curves and addition formulae of Abelian functions; discrete and quantum geometry; quantum models on the lattice; quantum mappings
Programme Committee: Chair: Jarmo Hietarinta (Turku Univ., FI); Vice-Chair: Frank Willem Nijhoff (Leeds Univ., UK)
Speakers: (will include) K. Aomoto (Nagoya, JP); R. Askey (Wisconsin-Madison, US); A. Bobenko (TU Berlin, DE); A. Doliwa (Warsaw, PL); S. Elaydi (Trinity, San Antonio, US); V. Enolskii (Herot-Watt, Edinburg, UK); L. Fadeev (Steklov, St. Petersburg, RU); A. Grunbaum (Berkeley, US); L. Haine (Louvaine, BE); M. Ismail (U. South Florida, US); T. Koornwinder (Amsterdam, NL); I. Krichever (Columbia, US); I. Laine (Joensuu, FI); F. Marcellan (U. Carlos III, Madrid, ES); M. Noumi (Kobe, JP); O. Ragnisco (Roma Tre, IT); J.-P. Ramis (U. Paul Sabatier, FR); V. Roubtsov (Angers, FR); S. Ruijsenaars (CWI, NL); P. Santini (Roma La Sapienza, IT); M. van der Put (Groningen, NL); J. Felipe van Diejen (talca, CL); P. Vanhaecke (Poitiers, FR); A. Zhedanov (Donetsk, UA)
Deadline: (for applications) 19 March 2004
Information: e-mail: euresco@esf.org
Web site: http://www.esf.org/euresco/04/pc04185

20-27: 42nd International Symposium on Functional Equations, Opava, Czech Republic
Topics: Functional equations and inequalities, mean values, functional
equations on algebraic structures, Hyers-Ulam stability, regularity properties of solutions, conditional functional equations, iteration theory, functional-differential equations, applications of the above, in particular to the natural, social, and behavioural sciences
Organizers: J. Smítal, M. Stefánková, Math. Institute, Silesian
Univ., CZ-746 01 Opava, Czech Republic (ISFE42@math.slu.cz)
Scientific Committee: J. Aczel (Honorary Chair; Waterloo, ON, Canada), Z. Daroczy (Debrecen, Hungary), R. Ger (Katowice, Poland), J. Raetz (Chair; Bern, Switzerland), L. Reich (Graz, Austria), and A. Sklar (Chicago, IL, U.S.A.).
Information: Participation at these annual symposia is by invitation only.
Those wishing to be invited should send details of their interest and, preferably, publications (paper copies) and/or manuscripts, with their postal and e-mail addresses to Professor Jürg Rätz, Mathematisches Inst., Univ. Bern, Sidlerstr. 5, CH-3012 Bern, Switzerland (math@math-stat.unibe.ch) before April 6, 2004. A short history of the symposia can be found on the website http://riesz.math.klte.hu/~isfe

21-24: Functional Analysis Workshop, Joensuu, Finland
Theme: Functional analysis
Topics: These include Banach spaces and operator theory, Frechet spaces, spaces of analytic functions
Main speakers: These include K. Bierstedt (Paderborn), J. Bonet (Valencia), A. Borichev (Bordeaux), G. Godefroy (Paris), Chen Huaihui (Nanjing), S. Kislyakov (St. Petersburg), R. Meise (Dusseldorf), A. Nicolau (Barcelona), E. Odell (Austin), D. Preiss (London), E. Saksman (Jyvaskyla), J. H. Shapiro (East Lansing), D. Vogt (Wuppertal)
Format: Plenary lectures, invited sectional lectures and contributed talks
Sessions: Plenary lectures and parallel sessions
Scientific committee: J. Taskinen (Joensuu, chair), R. Aulaskari (Joensuu), M. Lindstršm (Abo), H-O. Tylli (Helsinki)
Local organising committee: J. Taskinen, M. Kotilainen, V. Ramula, R. Sinkkonen (all Joensuu)
Location: Department of Mathematics, University of Joensuu, Joensuu, Finland
Deadlines: 30 April for abstracts, 20 May for accommodation
Information: http://www.joensuu.fi/mathematics/workshop2004

21-25: Conference "Nonlinear Analysis", In honour of Haim Brezis, on the occasion of his 60th birthday, Le Carré des Sciences, Paris, France
Speakers: A. Aftalion, L. Ambrosio, G.I. Barenblatt, F. Bethuel, J. Bourgain, X. Cabre, L. Caffarelli, A. Chang, Y. Choquet-Bruhat, P.G. Ciarlet, P. Constantin, L.C. Evans, F. Hamel, S. Klainerman, J.-F. Le Gall, Y.Y. Li, E.H. Lieb, F.-H. Lin, P.-L. Lions, H. Matano, Y. Meyer, M. Mimura, S. Müller, N. Nadirashvili, F. Otto, P.H. Rabinowitz, S. Serfaty, G. Sivashinsky, E. Stein
Organizing Committee: H. Berestycki, M. Bertsch, F. Browder, M. Chipot, M. Comte, J.-M. Coron, I. Diaz, Y. Maday, I. Shafrir, D. Smets, L. Véron
Scientific Committee: A. Ambrosetti, A. Bahri, H. Berestycki, J.-P. Bourguignon, F. Browder, J.-M. Coron, G. Da Prato, M. Giaquinta, D. Kinderlehrer, L. Nirenberg, B. Peletier, J. Serrin, R. Temam
Registration: free but required (via web site)
Information:
web site: http://www.ann.jussieu.fr/HB2004/
e-mail: hb2004@ann.jussieu.fr

21 – 29: 2º CIME Course - Nonlinear and Optimal Control Theory, Cetraro, Cosenza, Italy
Speakers: A. Agrachev (SISSA, Trieste): *Differential Geometry of Optimal Control Problems and Hamiltonian Systems;* A.S. Morse (Yale Univ.): *Lectures on Control Using Logic and Switching;* E. Sontag (Rutgers Univ.): *Input to State Stability and Related Notions;* H. Sussmann (Rutgers Univ.): *Flows, Generalized Derivatives, Variations and Necessary Conditions for an Optimum for Curve Minimization Problems;* V. Utkin (Ohio State Univ.): *Sliding Modes Control: Mathematical Tools, Design and Applications*
Course scientific directors: E. Casadio Tarabusi; A. D'Agnolo; M.A. Picardello
CIME Scientific Committee: V. Capasso, C. Cercignani, P. Colli, R. Conti, M. Manetti, Elvira Mascolo (C.I.M.E. Secretary), A. Maugeri, V. Vespri, C. Viola, G. Zampieri, P. Zecca (C.I.M.E. Director)
Sponsors: The European Commission, Division XII, TMR Programme "Summer Schools"; Consiglio Nazionale delle Ricerche; Ministero dell' Istruzione, dell'Universita' e della Ricerca; Ministero degli Affari Esteri - Direzione Generale per la Promozione e la Cooperazione - Ufficio V.
Scientific Direction: Gianna Stefani; P. Nistri
Information: e-mail: cime@math.unifi.it
Web site: *http: //www.math.unifi.it/CIME*
Address: Fondazione C.I.M.E. c/o Dipart.di Matematica "U. Dini" Viale Morgagni, 67/A - 50134 FIRENZE (ITALY)
Tel. +39-55-434975 / +39-55-4237111;
FAX +39-55-434975 / +39-55-4222695

25-27: New Developments in Electronic Publishing of Mathematics -
a workshop integrating mathematicians, libraries, editors and publishers
Organisers: K. Stange (Royal Library, Stockholm); H. Becker (SUB Göttingen); B. Wegner (TU Berlin). Organized as an official satellite conference to the ECM 2004 at KTH Stockholm; will be in combination with the 5th EMANI workshop and the 3rd WDML workshop
Topics: Electronic publishing in mathematics, enhanced authoring and mark-up tools for mathematics, electronic mathematics knowledge management, mathematics in the semantic web, electronic mathematical libraries, web offers in mathematics, search engines for mathematical information, mathematics databases, digitisation of mathematical publications, long-term preservation of digital mathematical documents, access structures to mathematical offers, mathematics portals
Scientific and **Program Committee**: J. Airong (Tsinghua University Library, Beijing); T. Bouche (CMD, Grenoble); V. Coti Zelati (UMI, University of Naples); N. Hungerbühler (University of Fribourg, CH); A. de Kemp (Springer-Verlag, Heidelberg); Y. Laurent (CMD, Grenoble) ; E. Macias (USC, Santiago de Compostela) ; O.

Martio (EMS, Helsinki University); Ai-ling Ong (CWI, Amsterdam); J. Poland (Cornell University Library, Ithaca); Alf van der Poorten (CEIC/ceNTRe, Sydney); R. Schwänzl (IWI, Osnabrück); A. Zemskov (GPNTB, Moscow)
Preliminary registration: and proposals for papers or talks should be sent to Bernd Wegner: wegner@math.tu-berlin.de or editor@zblmath.fiz-karlsrhe.de. They also can be submitted through any member of the Program Committee. Further details will come with the 2nd Announcement
Deadline: for submission will be May 15, 2004; Final registration: TBA
Proceedings: are planned, details TBA

26-July 1: 7th International Conference of The Mathematics Education into the 21st Century Project, Ciechocinek, Poland.
Information: please e-mail:
arogerson @vsg.edu.au for all information
Web site (of previous conferences only): http://math.unipa.it/~grim/21project.htm
[For details, see EMS Newsletter 50]

July 2004

1-15: Moonshine - the First Quarter Century and Beyond.
A Workshop on the Moonshine Conjectures and Vertex Algebras,
International Centre for Mathematical Sciences, Edinburgh UK
Aim: Moonshine and related topics have been active research areas since the late 1970s. The aim of this Workshop is to review the impact of this research area on mathematics and theoretical physics and to highlight possible new directions.
Topics & format: The first part of the meeting will be expository, including such areas as Borcherds's proof of the Conway-Norton conjecture, Construction of the Monster, Vertex (operator) algebras, Modular Moonshine, BKM algebras and automorphic forms, FLM's construction and proof of the McKay-Thompson conjecture. The second part of the meeting will consist of invited talks on current research.
Speakers: Expository Talks will be given by the following:
*J. Bruinier (Cologne); C. Dong (Santa Cruz); *T. Gannon (Alberta); R. Griess (Michigan); Y.-Z. Huang (Rutgers); E. Jurisich (Charleston); H. Li (Rutgers); A. Meurman (Lund); M. Miyamoto (Tsukuba); A. Ryba (CUNY); (* awaiting confirmation)
Organisers: International Centre for Mathematical Sciences
Scientific Organising Committee: A. Baker (Glasgow), A. Ivanov (Imperial College), J. Lepowsky (Rutgers), J. McKay (Concordia), V. Nikulin (Liverpool), M. Tuite (Galway)
Sponsors: The meeting is supported by the Engineering and Physical Sciences Research Council
Information: e-mail: icms@maths.ed.ac.uk
Web site: http://www.ma.hw.ac.uk/icms/meetings/2004/moonshine

3-10: Conference on Symplectic Topology ECM Satellite Conference
Stare Jablonki, Poland
Aim: To survey recent developments and new directions of research in symplectic topology. We plan also some survey talks comprehensible to students and non-specialists.
As a complement to the conference, an Introductory Mini-course (about 12 hours of lectures) for students will be organized before the conference. It will take place in Warsaw, July 01-03, 2004
Main speakers: M. Abreu, D. Auroux, P. Biran, Ana Cannas da Silva, M. Entov, K. Fukaya, Y. Karshon, D. Kotschick, F. Lalonde, E. Lerman, D. McDuff, K. Ono, D. Salamon, M. Schwarz, I. Smith
Format: Invited talks and contributed presentations. Sessions
July 3 (arrival day) - July 10 (departure day). The lectures will start on July 4 in the morning and will end in the afternoon on July 9.
Organizers: Banach Centre, Committee on Mathematics of the Polish Academy of Sciences, University of Szczecin, University of Warmia & Mazury, University of Wroclaw
Organising committee: B. Hajduk (Wroclaw), A. Tralle (Olsztyn), J. Kedra (Munich/Szczecin). Scientific advice: L. Polterovich (Tel Aviv).
Location: Stare Jablonki (Old Little Apple Trees) is a typical village in the Warmia region. It is located in a beautiful area near the lake Szelag Maly. The conference will take place in Hotel Anders.
The Introductory Mini-course will take place at the Mathematical Institute of the Polish Academy of Sci. in Warsaw, Sniadeckich Str. 8.
Grants: Several grants for young mathematicians will probably be available.
Deadlines: (for registration) May 15th, 2004
Information: e-mail: symp@univ.szczecin.pl
Web site: http://symp.univ.szczecin.pl

26-31: 6th World Congress of the Bernoulli Society and the 67th Annual Meeting of the Institute of Mathematical Statistics, Barcelona (Spain)
Information: e-mail: wc2004@imub.ub.es
Web site: http://www.imub.ub.es/events/wc2004/
[For details, see EMS Newsletter 49]

August 2004

18-21: The Thirteenth International Workshop on Matrices and Statistics, in Celebration of Ingram Olkin's 80th Birthday, Bedlewo, near Poznan, Poland
Information: e-mail: matrix04@main.amu.edu.pl
Web site: http://matrix04.amu.edu.pl/
[For details, see EMS Newsletter 50]

23-September 2: International Conference-School on Geometry and Analysis dedicated to the 75th anniversary of Academician Yu. G. Reshetnyak, Novosibirsk, Russia
Aim: The main goal of the meeting is to provide students and young mathematicians with a possibility to attend lectures of internationally recognized researches in geometry, quasiconformal analysis, nonlinear potential theory, and variational problems and to present results of their own research
Topics: Riemannian geometry in the large. Quasiconformal analysis. Nonlinear potential theory and Sobolev spaces. Variational problems and related equations.
Format: Mini-courses in geometry and analysis intended for students and young mathematicians (up to five 90 min. lectures), plenary lectures (90 min.), and short communications (20 min.)
Programme committee: M. Agranovskii (Israel), J. Ball (Great Britain), A. A. Borisenko (Ukraine), V. N. Berestovskii (Russia), Yu. D. Burago (Russia), V. I. Burenkov (Great Britain), V. N. Dubinin (Russia), F. Gehring (USA), V. Gol'dshtein (Israel), J. Heinonen (USA), T. Iwaniec (USA), S. S. Kutateladze (Russia), V. I. Kuz'minov (Russia), O. Martio (Finland), V. Maz'ya (Sweden), V. M. Miklyukov (Russia), I. Nikolaev (Russia & USA), M. Shubin (USA), V. D. Stepanov (Russia), I. A. Taimanov (Russia), V. A. Toponogov (Russia), A. M. Vershik (Russia), V. V. Vershinin (Russia)
Organizers: Sobolev Institute of Mathematics and Novosibirsk State University
Location: Novosibirsk, Russia
Grants: Some financial support for young scientists is expected, see the conference website for updated information
Deadlines: 1st May for registration, 15th July for submission of an abstract
Information: contact Sergei Vodopyanov (Chairman of the Organizing Committee),
e-mail: angeom@math.nsc.ru
Web site: http://www.math.nsc.ru/conference/ag2004/indengl.htm

24-27: International Conference on Nonlinear Operators, Differential Equations and Applications (ICNODEA-2004), Cluj-Napoca, Romania
Information: e-mail: nodeacj@math.ubbcluj.ro
Web site: http://www.math.ubbcluj.ro/~mserban/confan.htm
[For details, see EMS Newsletter 49]

September 2004

8-11: Dixiemes journees montoises d'informatique theorique, a Liege (Tenth Mons theoretical computer science days, in Liege)
Information: e-mail: M.Rigo@ulg.ac.be;
Web site: www.jm2004.ulg.ac.be
[For details, see EMS Newsletter 50]

13 – 18: 3º CIME Course - Stochastic Geometry, Martina Franca, Taranto, Italy
Speakers: A. Baddeley (Univ. of Western Australia, Perth – Australia):
Spatial Point Processes and their Applications; I. Barany (Univ. College, London, England, Hungarian Acad. of Sciences Budapest): Random Points, Convex Bodies, and Approximation; W. Weil (Univ. of Karlsruhe, Germany): Recent Trends in Integral Geometry; R. Schneider (Univ. of Freiburg, Germany): Random Sets (in particular Boolean Models)
Course scientific director: Prof. W. Weil
CIME Scientific Committee: V. Capasso, C. Cercignani, P. Colli, R. Conti, M. Manetti, Elvira Mascolo (C.I.M.E. Secretary), A. Maugeri,
V. Vespri, C. Viola, G. Zampieri, P. Zecca (C.I.M.E. Director)
Sponsors: The European Commission, Division XII, TMR Programme "Summer Schools"; Consiglio Nazionale delle Ricerche; Ministero dell' Istruzione, dell'Universita' e della Ricerca; Ministero degli Affari Esteri - Direzione Generale per la Promozione e la Cooperazione - Ufficio V.
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Tel. +39-55-434975 / +39-55-4237111
FAX +39-55-434975 / +39-55-4222695

20-24: 12th French-German-Spanish Conference on Optimization,
Avignon, France
Information: e-mail: alberto.seeger@univ-avignon.fr;
Web site: http://www.fgs2004.univ-avignon.fr
[For details, see EMS Newsletter 50]

23-26: 4th International Conference on Applied Mathematics (ICAM-4), Baia Mare, Romania (previous editions in 1998, 2000 and 2002)
Information:
e-mail: marietag@ubm.ro; icam4@ubm.ro
Web site: http://www.ubm.ro/site-ro/facultati/departament/manifestari/icam4/index.html
[For details, see EMS Newsletter 49]

Recent books

edited by Ivan Netuka and Vladimír Souček (Prague)

Books submitted for review should be sent to the following address:
Ivan Netuka, MÚUK, Sokolovská 83, 186 75 Praha 8, Czech Republic

A. Barvinok: *A Course in Convexity,* *Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, 2002, 366 pp., US$59, ISBN 0-8218-2968-8*
As the author says in the introduction, it is impossible to cover the whole of convexity in one textbook. Nevertheless, this book shows that basic principles can be presented in an accessible way and demonstrated on various examples and applications. After introductory results on general convex sets in Euclidean spaces and general topological vector spaces (theorems of Cadatheorory, Helly and Radon, separation results and the Krein-Milman theorem), the author goes on to applications in discrete and combinatorial convexity. The role of duality in convexity is illustrated on examples. The further topics are the ellipsoidal approximation, combinatorial structure of polytopes and interaction of convex bodies with lattices. The applications range over different areas, including analysis, probability, algebra, combinatorics and number theory. The presentation is very clear, and suitably chosen applications illustrate abstract results and make the reading more attractive. (jra)

G. Belitskii, V. Tkachenko: *One-dimensional Functional Equations,* *Operator Theory Advances and Applications, vol. 144, Birkhäuser, Basel, 2003, 206 pp., €98, ISBN 3-7643-0084-1*
This specialized monograph is devoted to a detailed study of functional equations of the form $g(x, f(x), f(F_1(x)), \ldots, f(F_n(x)))=0$ on the real line R or on the unit circle T. Here f is an unknown function, F_k, $k = 1,\ldots,n$, are given mappings from R or T into itself and g is a given mapping into R. Various equations are included, studied in literature as special cases. Two main questions studied are local solvability at every point and the global solvability of the equation. The content can be characterized by names of the chapters: Implicit functions (a special case), Classification of one-dimensional mappings, Generalized Abel equations, Equations with several transformations of argument, and Linear equations. This carefully written condensed text is accompanied by numerous examples. It contains new results and concepts, which may be applied also to multidimensional functional equations. The book will be of interest to specialists in functional equations as well as to those working in related fields. It can be recommended for libraries. (jve)

J. Bertin, J. P. Demailly, L. Illusie, C. Peters: *Introduction to Hodge Theory,* *SMF/AMS Texts and Monographs, vol. 8, American Mathematical Society, Providence, 2002, 232 pp., US$65, ISBN 0-8218-2040-0*
This is an English translation of the text, which was originally published in French by the Société Mathématique de France as vol. 3 of the series Panoramas et Synthèses in 1996. It consists of three parts: Hodge Theory and Vanishing Theorems (by J. P. Demailly), Frobenius and Hodge Degeneration (by L. Illusie), and Variations of Hodge Structure, Calabi-Yau Manifolds and Mirror Symmetry (by J. Bertin and Ch. Peters). Its main aim is to make the reader familiar with the contemporary state and development of the Hodge theory. The authors were well aware of the fact that the text would require prerequisites from several branches of mathematics. Consequently, they prepared the exposition in such a way that it is as easy as possible for the reader. Their attempt was quite successful, nevertheless a potential reader must have a basic knowledge in several directions (e.g. analytic and algebraic geometry, sheaves, homological algebra). But it is worth the effort, because he (she) can penetrate into the contemporary state of affairs in the Hodge theory, and it is not easy to find other texts having the same qualities. Moreover, some recent publications on this topic are written in the language of physics, which need not always be completely understandable to all mathematicians. Numerous references contribute to a good orientation in the subject. We find also many open problems showing possible ways of further research to young scientists. (jiva)

M. Berger: *A Panoramic View of Riemannian Geometry,* *Springer, Berlin, 2003, 824 pp., 425 fig., €59.95, ISBN 3-540-65317-1*
The present book of the famous French differential geometer is not a monograph and also not an encyclopaedia. The author says in the Preface: "This book is not a handbook of Riemannian geometry, nor a systematic awarding of prizes. We give only the best recent results, not all of the intermediate ones. However, we mention when the desired type of results started to appear, this being of historical interest … We present open problems as soon as they can be stated. This encourages the reader to appreciate the difficulty and the current state of each problem." The author continues: "Since the text is unusual, it is natural to expect unusual features of presentation. First, references are especially important in a book about mathematical culture. But there should be not too many… The interested reader will be able to trace back to most of the standard sources… The immensity of the field poses a problem of classification: in our division into chapters, necessarily arbitrary, we did not follow any logical or historical order. We have tried to follow certain naturalness and simplicity…In a panorama, you can see the peaks, but you do not climb them. This is a way of saying that we will not prove the statements we quote. But, in a panorama, sometimes you can still see the path to a summit; analogously in many cases we will explain the main ideas or the main ingredients for the proof…". Except for in the first chapter, the author usually limits himself to compact manifolds without boundary. His "not too many references" includes 1310 items. Topics treated in the book include Euclidean geometry, a description of surfaces from Gauss to present days, metric geometry and curvature, spectrum of the Laplacian, geodesic dynamics, relation between curvature and topology, holonomy groups and Kähler manifolds. The last chapter collects formal definitions of notions, which are otherwise treated in a rather informal way. I finish this review quoting the author again: "The present book will bring pleasure and be of help to professional Riemannian geometers as well as those who want to enter into the realm of Riemannian geometry, which is an amazingly beautiful, active and natural field of research today." (ok)

A. Bonnet, G. David: *Cracktip is a Global Mumford-Shah Minimizer,* *Astérisque 274, Société Mathématique de France, Paris, 2001, 259 pp., FRF 300, ISBN 2-85629-108-2*
One of the most topical models studied in the modern calculus of variations is the Mumford-Shah functional. It is motivated by the question of best approximation by piecewise smooth functions. Similar problems are studied in mathematical theory of image processing. A competitor for the Mumford-Shah functional is a function u of two variables, which is smooth except for a singular set, where it can jump. The Mumford-Shah functional is a sum of two terms: the first term is the ordinary Dirichlet integral over the regular part of the domain of u, and the second term is the length (precisely, one-dimensional Hausdorff measure) of the singular part of the domain. Any minimizer of the Mumford-Shah functional must be a harmonic function in the regular part. The cracktip is a canonical example of a harmonic function in the complement of the half-line $y=0$, $x<0$. In polar coordinates, $u(r \cos t, r \sin t) = \text{const}\ r^{1/2} \sin (t/2)$. In his paper from 1991, E. De Giorgi raised the conjecture that the cracktip could be a global minimizer of the Mumford-Shah functional. Although the conjecture was supported by experiments, the question of a rigorous proof became a famous problem. Now, Alexis Bonnet and Guy David present the solution of the problem. The fact that they need the extent of a monograph to describe the proof certifies that the problem was really considerably deep. The method of the proof exploits a careful analysis of the harmonic conjugate to the competitor and its level set. Blow up techniques and monotonicity of the energy functional are also used. In the existence part, weak compactness properties in SBV are bypassed and all is done in the framework of strong minimizers and competitors with closed singular sets. The

presentation is well organized so that the reader can recognize where to learn the strategy of the proof and where to look for particular technical details. The aim of the monograph is to give evidence that the problem is solved. Certainly, the book is a valuable source of inspiration for researchers who try to attack problems of a similar nature. (jama)

F. T. Bruss, L. Le Cam, Eds.: *Game Theory, Optimal Stopping, Probability and Statistics,* *Lecture Notes Monograph Series, vol. 35, Institute of Mathematical Statistics, Beachwood, 2003, 324 pp., US$66, ISBN 0-940600-48-X*
His colleagues, scientific collaborators, students and friends wrote this special volume in honour of Thomas S. Ferguson on the occasion of his 70th birthday. Thomas Ferguson is a mathematical statistician who remarkably contributed to several parts of mathematical statistics. The book contains interesting papers mostly connected with areas where Ferguson was working, but there are also papers not so closely connected with his areas of interest. The volume is divided into 4 parts. The first two parts consist mostly of papers related Ferguson's interests. Part I contains papers on Amazons, on the square root game, on games against prophets, and on randomized distributions. Papers on secretary problems, optimal control, selecting monotone sub-sequences, and on lattices with applications to martingale theory are included into Part II. Part III includes papers discussing boundary crossing probabilities, efficient coupling, and almost sure number of pairwise sums of random integer subsets. Part IV contains papers from different parts of statistics, e.g. Cox proportional hazard models, density estimates in metric space, nonparametric methods, and sequential analysis. All papers were refereed. (mah)

G. Cohen: *A Course in Modern Analysis and its Applications,* *Australian Mathematical Society Lecture Series 17, Cambridge University Press, Cambridge, 2003, 333 pp., £24.95, ISBN 0-521-81996-2, ISBN 0-521-52627-2*
This textbook is designed for a one-semester course at a senior undergraduate level, not only for students who are majoring in some area of mathematics, but also for future teachers of high school as well as to those who need to learn some mathematical analysis for use in other areas. The exposition does not use anything from complex functions theory, measure and Lebesgue integration, or modern algebra. The introductory part is devoted to basic notions such as sets, mappings, countability, point sets, sequences and series, uniform convergence and some linear algebra. Then metric spaces are studied (convergence, completeness, compactness) and several applications of the fixed point theorem are given. Basic notions of topological spaces and normed vector spaces are included with applications to some approximation theory, integral equations and numerical analysis. The last two chapters deal with inner product spaces and Hilbert spaces. The material covered in the book is standard; each subject is accompanied by exercises (with selected solutions at the end of the book). (in)

J. H. Conway, D. A. Smith: *On Quaternions and Octonions,* *A.J. Peters, Natick, 2003, 159 pp., US$29, ISBN 1-56881-134-9*
The topic treated in the book is the geometry and arithmetic of quaternion and octonion algebras. As is written in the introduction: "Our attempt to understand the `cosmic significance' of octoninos has led to this book!" Nearly two thirds of the book are devoted to octonions, their geometry, arithmetic, and "triality symmetry". The introductory chapter with historical notes on quaternions and octonions is followed by a discussion of complex numbers. In particular, their relation to the geometry of the Euclidean plane serves as a model for corresponding relations of quaternions and octonions to higher-dimensional Euclidean spaces. The next part contains a short discussion of Gaussian, Eisenstein and Kleinian integers. Octonions are constructed as elements of composition algebra of dimension eight. Also Moufang loops are formed by composition algebras, which demonstrate the triality symmetry when applied to an eight-dimensional orthogonal group. Arithmetic of integral octonions is studied in detail. Authors clarify the arithmetic and geometry of divisors of an arbitrary octonion integer using Rehm's new form of the Euclidean algorithm. This leads to a new theory of octonion factorisation that appears here for the first time. The book is intended for mathematicians and other scientists interested in low-dimensional spaces and their symmetries. It can serve as a text for graduate courses in geometry, group theory, algebra, and number theory. (mer)

E. B. Davies: *Science in the Looking Glass. What Do Scientists Really Know?* *Oxford University Press, Oxford, 2003, 295 pp., £25, ISBN 0-19-852543-5*
The book is written by a professor of mathematics at King's College, London. He published almost 300 articles and four books and his scientific interests cover topics from a broad area ranging from quantum theory to pure mathematics and philosophy of science. The author discusses the development of science in historical context, in order to explain why many long established "facts" have turned out not to be so certain. From the author's preface: "My conclusion is surprising, particularly coming from a mathematician. In spite of the fact that highly mathematical theories often provide very accurate prediction, we should not, on that account, think that such theories are true or that nature is governed by mathematics. In fact, scientific theories most likely to be around in a thousand years time are those, which are the least mathematical - for example evolution, plate tectonics, and the existence of atoms." The ideas presented in the book can be useful, interesting and fascinating for a wide audience. (lb)

M. Emmer, M. Manaresi, Eds.: *Mathematics, Art, Technology and Cinema,* *Springer, Berlin, 2003, 242 pp., €79.95, ISBN 3-540-00601-X*
The presented book deals with ideas relating mathematics, art, technology and images, films and cinema. The main aim of the book is to describe relations between mathematical ideas and culture, to discuss differences between two worlds - exact sciences and humanism - and to show that these differences are the matter of past. The book is divided into two parts. The first part offers a collection of papers, which were presented in the conference *Mathematics and Culture* (Bologna, October, 2000). The discussion contains many interesting topics (classical descriptive geometry and CAD, geometry of sight, mathematics and modern arts, technologies and computer animators, etc.) describing connections between mathematics and culture as well as important applications of mathematics. The second part contains articles presented in the conference *Mathematics and Cinema* (Bologna, December, 2000). Here the reader can find a discussion of films devoted to certain mathematical ideas, their inventors, discoveries, mathematical proofs, etc. (including Fermat's last theorem, the fantastic world of Escher, etc.), about the life and work of some excellent mathematicians (Möbius and others) and there are also some interesting interviews with screenwriters (for example, V. Natali). The book is full of interesting illustrations and photographs. It is written not only for mathematicians but also and, in particular, for many others who are not interested in mathematics and who sometimes feel a fear of mathematics. It can reveal to them the nature and beauty of mathematics. (mbeè)

A. Etheridge: *A Course in Financial Calculus,* *Cambridge University Press, Cambridge, 2002, 196 pp., £21.95, ISBN 0-521-89077-2*
The title of the book explains very properly its connection to Financial Calculus by M. Baxter and A. Rennie. Following the publication of the latter book in 1996, the author of the present review and evidently a number of further university teachers of stochastic analysis were influenced by its readability to prepare a course on stochastic financial models. To this purpose it was necessary to supply the framework of the book with some theory of stochastic analysis and to provide a mathematical explanation of the notions used. This is what the book by Etheridge does in a commendable way. It complements the book by Baxter and Rennie, adapting it into a university course. The book under review begins by explaining, on single period models, notions like arbitrage, complete market, and risk neutral probability measure. The Sharpe binomial model is used to introduce the evolution of a stock price and the call or put options. The basic theory of martingales in discrete time is then explained. A chapter on the Wiener process (Brownian motion) presents a comprehensive collection of its properties. The theory of stochastic integrals and differentials follows, including the Girsanov theorem and the Brownian martingale representation theorem. A reference

to the Feynman–Kac representation in treating the connections between stochastic and partial differential equations is somewhat misleading with regard to the genesis of the material presented. The applications of the Black–Scholes approach are covered at first to the same extent as by Baxter and Rennie. More sophisticated options, models with several stocks and stock prices having jumps, are dealt with in the last two chapters. Interest rate models are not discussed. The twenty exercises in average per chapter are a positive feature of the book. They mostly extend the material presented. (pm)

S. Fajardo, H.J. Keisler: *Model Theory of Stochastic Processes,* *Lecture Notes in Logic 14, A.K. Peters, Natick, 2002, 135 pp., US$32, ISBN 1-56881-172-1*
The book is a general study of stochastic processes on adapted or L-adapted spaces, where L is the time line (index set), which is usually the real interval [0,1], and its subset of dyadic rationals or a finite set; an adapted space is, by definition, a [0,1]-adapted space whose filtration is right continuous. Important notions, such as saturated and rich L-adapted spaces, are defined and studied. Several examples illustrate possible applications of the described results in stochastic analysis. The reader may find theorems that guarantee an existence of certain adapted processes. A few basic ideas, including saturation of adapted spaces, come from model theory. Moreover, nonstandard analysis is widely and substantially used. The exposition is relatively self-contained but a background in classical and nonstandard probability theory is assumed. The book is nice, interesting and inspiring. It demonstrates how non-classical approaches can provide excellent tools for deep analysis of complicated mathematical structures. (jmlè)

K. Fritzsche, H. Grauert: *From Holomorphic Functions to Complex Manifolds,* *Graduate Texts in Mathematics, vol. 213, Springer, New York, 2002, 392 pp., 27 fig., €64.95, ISBN 0-387-95395-7*
The book is a nice introduction to the theory of complex manifolds. The authors' intention is to introduce the reader in a simple way to the most important branches and methods in the theory of several complex variables. The discussion starts with the theory of complex manifolds and holomorphic vector bundles. These concepts are illustrated on many examples, including coverings of C^n, quotient manifolds of C^n (tori, Hopf manifolds), projective spaces and Grassmannians. There is a very interesting and detailed exposition of the Stein theory. The book contains a discussion of many important topics, including differential forms, the Dolbeault and the de Rham cohomology, the Serre duality, Hodge and Kodaira decomposition of the Dolbeault cohomology and their relations to differential geometry, real methods in complex analysis, applications of Kähler theory and harmonic forms to the study of pseudoconvex subdomains of complex manifolds, the Kobayashi and the Bergmann metrics and boundary behaviour of biharmonic maps. The book is written in a very readable way; it is nice introduction into the topic. (jbu)

B. A. Fusaro, P. C. Kenschaft, Eds.: *Environmental Mathematics in the Classroom,* *The Mathematical Association of America, Washington, 2003, 255 pp., £29.95, ISBN 0-883-85714-6*
The book belongs to the 'Series Classroom Resource Materials', providing supplementary materials for students - laboratory excercises, projects, historical information, textbooks with unusual approaches, etc. The volume contains 14 contributions devoted to various problems in environmental mathematics, such as how to understand data concerning environmental challenges, how to model emission of various media (for instance sulphur dioxine), evolution of ground water pollution, modelling of groundwater flow, modelling of ground-level ozone, mathematics of populations, infections and invasions, models of population growth, models of measuring the benefit of protecting the environment and the mathematics of oil spills. Authors also present some very interesting examples, which are not in the main stream of environmental research, such as a prediction of the frequency of albinos in buffalo population, influence of population bottlenecks upon genetic variations or lead poisoning in humans. In most of the contributions, a corresponding mathematical model is formed and its results are confronted with the present and past reality, including a formulation of some future trends. Among models used are linear and exponential models, models formed by tables, matrix models and models formed by logical trees. High school mathematics is sufficient for most of the problems. Elementary functions and their graphs are used, especially linear and exponential graphs. Powers and roots, algebraic expressions, sequences and series are also needed. This very interesting and valuable book can be a useful source and good inspiration for all mathematics teachers, showing possible applications of mathematics in practical questions. It can be used as a base for a special mathematical course as well as supplementary material in an ordinary mathematics course. It could also serve as a base for project integrated teaching of mathematics and natural sciences and also as an interesting book for self-reading. (ood)

G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott: *Continuous Lattices and Domains,* *Encyclopedia of Mathematics and its Applications 93, Cambridge University Press, Cambridge, 2003, 591 pp., £75, ISBN 0-521-80338-1*
The main subjects of this volume are directed complete partially ordered sets (structures generalizing complete lattices) and, in particular, domains, i.e. continuous directed complete partially ordered sets. Points of view from various areas of mathematics in which the study of continuous lattices has been motivated (computation theory, general topology, category theory, algebra and analysis) are employed. The book extends and revises the authors' "A Compendium of Continuous Lattices" (1980), while retaining its structure; results about complete lattices have been generalized to directed complete partially ordered sets wherever possible. Chapters 0 and I expose the theory of ordered sets, especially lattices and domains. Chapters II and III deal with the topological structure of directed complete partially ordered sets. Results on sober spaces (most notably the Hofmann-Mislove theorem) and on the connection between domains and Cartesian closed categories, which succeeded publication of the Compendium, are presented. Categorical behaviour of various classes of lattice-like structures and functions between them is studied in Chapter IV. The self-duality of domains is proved here and a new concept of power domains is introduced. Chapter V is devoted to the spectral theory of continuous lattices. The material on domain environments, which can be applied to characterize Polish spaces, has been added here. In the last two chapters continuous lattices are treated as (compact) topological semilattices, i.e. as objects of topological algebra. The text is supplemented by a comprehensive bibliography and many exercises. Every section ends with notes on the history of the research. Being written in an expository and almost self-contained way, the book can serve as a handbook for those working in the area as well as a student textbook. (emu)

A. Greven, G. Keller, G. Warnecke, Eds.: *Entropy,* *Princeton Series in Applied Mathematics, Princeton University Press, Princeton, Oxford, 2003, 358 pp., £46.95, ISBN 0-691-11338-6*
The mysterious, subtle, and still not completely understood notion of entropy that ultimately governs all complex physical, chemical, and biological systems, is approached in this book in a three-fold way, namely physical, stochastical, and information-theoretical, reviewing both historical development and recent trends and results. In part one, I. Müller and H. -O. Georgii expose fundamental macroscopical and statistical-physics concepts respectively. Part 2 collects expositions on the macroscopical approach in the context of continuum mechanics (by I. Müller and by K. Hutter and Y. Wang), on fine mathematical issues in hyperbolic equations (by C. M. Dafermos) and on irreversibility and its relation with entropy (by J. Uffink and E. H. Lieb and J. Yngvason). Part 3 then addresses the second mentioned approach, i.e. stochastic processes. Contributions by S. R. S Varadhan, F. den Hollander, E. Olivieri, and C. Maes address relations of entropy with large deviations, random motion in a random medium, metastability, or entropy production in driven spatially extended systems, and ends with a short philosophical treatment by J. L. Lebowitz and C. Maes. Finally, Part 4 consists of information-theoretical contributions by F. Benatti, J. Rissanen, L. S. Young and M. Keane, dealing with entropy in the context of dynamics and information, complexity and information in data, dynamical systems, and ergodic theory. The unique feature, resulting from a symposium held at MPI

in Dresden in 2000, is a composition of different viewpoints and methodologies linked by the common object: entropy. The editors have succeeded in creating a marvellous book, which will be of a great interest to a wide audience. This will include physicists working in continuum or statistical mechanics and thermodynamics, mathematicians working on physical applications and particularly in continuum thermodynamics, in stochastic processes, and probability theory in general, and information theorists. (trou)

J. R. Harper: *Secondary Cohomology Operations,* *Graduate Studies in Mathematics, vol. 49, American Mathematical Society, Providence, 2002, 268 pp., US$64, ISBN 0-8218-3198-4*
In this book, secondary cohomology operations are approached through the idea of secondary compositions. The first two chapters include a geometric discussion of primary operations with a summary discussion of features of the Steenrod operations, the Steenrod algebra and the cohomology of Eilenberg-McLane spaces. Chapter three presents the basic geometric theory including a version of the Peterson-Stein formula and the Milnor filtration. Chapter four develops a basic theory of secondary operations and indicates how to apply the general theory to operations of order higher than the secondary operations. Chapter five contains several examples, where the Milnor filtration comes into the play. In particular, Adams and Liulevicius-Shimada-Yamanoshita operations on the cohomology ring of complex projective space are treated in details. Chapter six goes through results on the Hopf invariant, and the last (seventh) chapter develops detailed information about the cohomology structure of fiber spaces using Steenrod operations. Many examples in this chapter are relevant in obstruction theory. (pso)

F. Hélein: *Harmonic Maps, Conservation Laws and Moving Frames,* *second edition, Cambridge Tracts in Mathematics 150, Cambridge University Press, Cambridge, 2002, 264 pp., £47.50, ISBN 0-521-81160-0*
This is the second edition of the book, originally published in French in 1996 and in the English translation in 1997. The book offers an introduction to the theory of harmonic maps between Riemannian manifolds with the emphasis on some recent topics: regularity of weakly harmonic maps, weak compactness of weakly harmonic maps, a role of symmetries and Noether conservation laws, the compensation compactness phenomena of the Jacobian determinant and its Hardy space regularity, including also the method of moving frames. The first chapter contains various notions of weak solutions and the covariant version of Noether's theorem and monotonicity formulae. Then regularity results for weakly harmonic maps are described. The second chapter concerns harmonic maps with values in a symmetric manifold (for example a two dimensional sphere). These questions are related to completely integrable systems. The third chapter is devoted to a description of the compensation phenomena for the Jacobian determinant and to the use of Hardy and Lorentz spaces. These properties are then used to prove the partial regularity of weakly harmonic stationary maps. The fourth chapter deals with harmonic maps with values into manifolds without symmetry, using the method of Coulomb moving frames. The fifth chapter contains an excursion to the study of conformal parameterizations of surfaces. This topic uses the compensation results and Hardy spaces results for the Jacobian determinant. This is an excellent book. It contains the main part of harmonic maps theory, regularity of harmonic maps and then all this is related to the new direction of Noether's conservation laws, the compensation phenomena and harmonic analysis. Relations between harmonic maps and symmetry and conservation laws are the main new topic. The book is very well written and it contains truly beautiful geometrical analysis. It also contains a quick, direct introduction to the current research. (jsou)

C. Hertling: *Frobenius Manifolds and Moduli Spaces for Singularities,* *Cambridge Tracts in Mathematics 151, Cambridge University Press, Cambridge, 2002, 270 pp., £45, ISBN 0-521-81296-8*
Frobenius manifolds are complex metric manifolds with an associative and commutative multiplication on the holomorphic tangent bundle, which satisfies some axioms involving also the unit and Euler vector fields. Dubrovin introduced Frobenius manifolds in 1991 in his work on topological field theory. Later, they turned out to play an important role in quantum cohomology, singularity theory and mirror symmetry. The present book is a concise and practically self-contained introduction to Frobenius manifolds and their applications in singularity theory. It is addressed to researchers and students from graduate level onwards and assumes some preliminary knowledge of algebraic geometry and singularity theory. The first part of the book is devoted to the general theory of manifolds with a multiplication on the tangent bundle, in particular to Frobenius manifolds as special cases of these manifolds. The second part presents a construction of Frobenius manifolds in the singularity theory, developing ideas pioneered by K. Saito. (mm)

H. Heyer, N. Jacob, I. Netuka, Eds.: *Heinz Bauer: Selecta,* *Walter de Gruyter, Berlin, 2003, 597 pp., €128, ISBN 3-11-017350-6*
The Selecta of Heinz Bauer (1928 - 2002) consists of 27 of his most important papers, which are collected within the three main topics of his work, namely within "Measure and integration", "Convexity" and "Potential theory". They represent an overview of the interactions between the fields Heinz Bauer worked in. A number of his papers are anticipated by erudite and comprehensive essays on Bauer's contribution to these branches written by S. D. Chatterji, D. A. Edwards and I. Netuka. The Selecta are completed by curriculum vitae of Heinz Bauer, by the list of his PhD. students and by bibliography consisting of textbooks and monographs (10 items), lecture notes (6 items), research papers (39 items), surveys, conference and seminar contributions (27 items), preliminary communications (6 items) and other publications (28 items). The book was to be handed over to Heinz Bauer on the occasion of his 75th birthday. Unfortunately, he died on the eve of the release of his Selecta and did not live to see this creditable achievement. The present volume is designed to offer to the appreciative reader Bauer's main ideas and can be recommended to everybody interested in these fields. The mathematical community will surely find in it a wealth of material that is pleasurable to read. (jl)

S. J. Hogan et al., Eds.: *Nonlinear Dynamics and Chaos: Where Do We Go From Here?,* *Institute of Physics Publishing, Bristol, 2002, 358 pp., £40, ISBN 0-7503-0862-1*
Low dimensional nonlinear systems are now fairly well understood. On the contrary, real systems, like biological or chemical ones, are more complex and it is not clear how to investigate them. The main goal of the conference "Nonlinear Dynamics and Chaos", which was held in Bristol in June 2001, was to collect the opinions of fourteen leading experts on the future development of this field. These experts were also asked to present examples of new phenomena that have not yet benefited from the "classical" nonlinear dynamics. The proceedings collect thirteen enlarged invited lectures. There are several common topics among them: pattern formation due to instability (driven by breaking symmetry or by diffusion) and reverse processes (i.e. stabilization via instability); limits of reduction method, when both slow and fast dynamics are presented; new bifurcation schemes in multiple time scales and hereditary systems and, of course, the nature of chaos. Examples are primarily from physics and biology but also from telecommunication. All lectures have a survey character; hence the book will be useful not only for researchers in dynamical systems but also for graduate students with interests in evolution systems and applied mathematics. (jmil)

H. N. Jahnke, Ed.: *A History of Analysis, History of Mathematics,* *vol. 24, American Mathematical Society, Providence, 2003, 422 pp., US$89, ISBN 0-8218-2623-9*
This book appeared first in German four years ago, published by Spectrum Verlag. It contains thirteen chapters devoted to various aspects of development of analysis from antiquity to the 20th century. Different authors, mostly well-known specialists in the history of mathematics, write chapters that are addressed to a broad mathematical community. The content of all chapters summarize facts from the history of analysis, which should be known to every educated mathematician and every analyst. It is rather difficult to complete such a task on about four hundred pages but the result is very good: the book is a well-written text accompanied by a corresponding bibliography to every chapter and two registers, which help a lot with orientation. The book is a good source of infor-

mation for students having mathematics in their programmes of study. The book will find its way easily onto the shelves of libraries, since it corresponds to present trends and demands. It was a timely decision to translate the book into English. (jive)

J. Jost: *Riemannian Geometry and Geometric Analysis,* *third edition, Universitext, Springer, Berlin, 2002, 532 pp., 24 fig., €44.95, ISBN 3-540-42627-2*
The aim of the book is a systematic and comprehensive introduction to Riemannian geometry and methods of geometric analysis. The third edition offers a new description of Morse theory and Floer homology. The first part presents the standard material from the theory of differential manifolds, Lie groups and algebras and from Riemannian geometry. Further discussion contains parts devoted to the de Rham cohomology groups and their main properties, the theory of connections and the corresponding curvatures, as well as to geodesic and Jacobi fields and the Rauch theorem. The second part presents more advanced topics, such as the theory of symmetric spaces, Morse theory, Floer homology, spin geometry and Dirac operators, the Seiberg-Witten equations and properties of harmonic maps between Riemannian manifolds, including their existence. It is a nice introduction to the theory covering a broad spectrum of topics, including an investigation of analytic questions arising from geometry as well as several applications of analytical techniques to solutions of geometric problems. Each part of the book ends with notes containing historical comments with references. (jbu)

V. Kac, P. Cheung: *Quantum Calculus,* *Universitext, Springer, New York, 2002, 112 pp., €34.95, ISBN 0-387-95341-8*
In this book, the two types of quantum calculus are developed - the q-calculus and the h-calculus. They correspond to difference equations (for $q\ ^{1}\ 1$, resp. $h\ ^{1}\ 0$) as substitutes of standard limit formula for derivative. During the course of quantum calculus along the lines of traditional calculus, many important results and notions of combinatorics, number theory and other fields of mathematics are introduced. Establishing q-binomial and q-Taylor formula, many results from the 18-th and 19-th centuries are treated in detail - for example Euler's identities for q-exponential functions, Gauss' q-binomial formula and Heine's formula for a q-hypergeometric function. As an application of these formulas, Euler's recurrent formula for the classical partition function, Gauss' formula for the number of sums of two squares and Jacobi's formula for the number of sums of four squares etc., are derived. The h-calculus is rather different, e.g. h-Taylor formula is Newton's interpolation formula and h-integration by parts is simply the Abel transform. The book is based on lectures and seminars given by the first author at MIT; it is addressed mainly to undergraduate students. (pso)

W. J. Kaczor, M. T. Nowak: *Problems in Mathematical Analysis III: Integration,* *Student Mathematical Library, vol. 21, American Mathematical Society, Providence, 2003, 356 pp., US$49, ISBN 0-8218-2050-8*
The book is divided into two parts: problems and their solutions. Even though the text covers standard parts of mathematical analysis, almost all sections start with an introductory summary of relevant definitions and a review of basic results. The first chapter consisting of 260 problems is devoted to the Riemann-Stieltjes integral with respect to monotone functions (or functions of bounded variation). Two sections deal with proper and improper integrals, in another section integral inequalities are studied, including also less known results such as Opial's and Steffensen's inequalities. A section on Jordan content closes chapter 1. The second chapter with 226 problems concentrates on the Lebesgue measure and integration on the real line. It covers the interchange of limit and integral, various modes of convergence of measurable functions, absolute continuity (including Banach-Zarecki's theorem), the relation of differentiation and integration, approximate continuity and basic problems on Fourier series. The book is primarily addressed to undergraduate students. It can be used for problem-solving seminars and exercises and it represents a valuable resource for teachers of mathematical analysis. (in)

B. Kawohl, O. Pironneau, L. Tartar, J.-P. Zolésio: *Optimal Shape Design,* *Lecture Notes in Mathematics 1740, Springer, Berlin, 2000, 388 pp., US$69.80, ISBN 3-540-67791-5*
The book consists of four parts, which cover topics of lectures given at the joint C.I.M./C.I.M.E. Summer School held in Tróia, Portugal, June 1998, and a well-written introduction by A. Cellina and A. Ornelas. The content can be generally described as "Optimal shape design" in a wide sense of the term. Individual chapters vary significantly by the criterion of the optimality, the nature of the studied problem and methods used for studying it. The first chapter, "Some nonconvex optimization problems" (written by B. Kawohl), starts with two fascinating problems - the opaque square problem and Newton's problem of minimal resistance - and describes the complicated and sometimes surprising path leading to today's state of their solution. The second chapter, "An introduction to the homogenization method in optimal design" (written by L. Tartar), is a very interesting and personal overview of the development of methods and ideas connected with homogenization, G and H convergence and the use of them in optimal shape design. The third chapter, "Shape analysis and weak flow" (written by J. P. Zolésio), is devoted to the domain variations in partial differential equations. The moving domain here is the image of the given domain by a non-autonomous vector field. The shape continuity is studied and, in the last two sections, applied to fluid-shell models. In the last chapter, "Optimal shape design by local boundary variations" (written by O. Pironneau), techniques for actual numerical implementation are explained both from the theoretical point of view and on well chosen model examples. The choice of material covering different aspects of optimal shape design, as well as the attractive way of exposition, makes the book accessible to a wide audience. (jsta)

S. G. Krantz, H. R. Parks: *The Implicit Function Theorem: History, Theory and Applications,* *Birkhäuser, Boston, 2002, 163 pp., €82.24, ISBN 0-8176-4285-4, ISBN 3-7643-4258-4*
This small book consists of six chapters and is entirely devoted to one of the most important results in analysis - the implicit function theorem and its variations. The book starts with historical comments on the evolution of ideas leading to the theorem. In particular, it contains a description of contributions by Newton (Newton's polygons), Leibniz (implicit differentiation), Lagrange (a solution of the Kepler equation by power series) and Cauchy (complex analysis version of the Lagrange method). The third chapter describes two basic proofs of the real-variable case (by induction and by iterations). Several applications are given in the fourth chapter (numerical homotopy methods, equivalent definitions of smooth surfaces and smoothness of the signed distance function). For the infinite-dimensional version, more sophisticated techniques are needed. The fifth chapter offers a sample of them (the Weierstrass preparation theorem in the complex case and ideas close to normal forms in the real case). The use of the Brouwer or the Schauder fixed point theorems implies that the differentiability conditions can be replaced by a certain weaker form and a compactness assumption. In the last chapter, a global inversion function theorem is proved and applied to the problem of characterization of Euclidean spaces, which are also normed division algebras. The last topic concerns the case when the derivative takes values on a larger space, hence the standard iterations can not be used. A version of the Nash-Moser procedure is described. The book will appeal to a large part of the mathematical community since everybody will find there some far going generalizations of classical results in a very readable form. (jmil)

A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi: *Classical and Quantum Computation,* *Graduate Studies in Mathematics, vol. 47, American Mathematical Society, Providence, 2002, 257 pp., US$59, ISBN 0-8218-2161-X*
The book is divided into two parts, describing separately classical and quantum computation. In the first part, the classical computation is described with the emphasis on complexity theory: Turing machines and complexity classes, Boolean circuits and their relation to Turing machines, non-deterministic Turing machines and the class NP, probabilistic algorithms and the class BPP and finally the hierarchy of complexity classes. The second part contains the main topic, the quantum computation in detail. Firstly, there is an introduction to the quantum computation: the tensor product algebra and the quan-

tum gates and circuits, the correspondence between classical and quantum computation, quantum circuits, quantum algorithms and the class BQP, quantum probability and density matrices, transformations of density matrices and definition of measuring operators and their properties. Secondly, there is a description of quantum algorithms: the quantum algorithms for Abelian groups and the hidden subgroup problem for Zk, the quantum analogues of the complexity classes and the class BQNP and comparison of classical and quantum codes and quantum error correction codes and the Shor code. The text contains a very clear, complete and detailed description of the topic together with many solved problems. Comparison between classical and quantum cases is also very valuable. The book can be recommended for a serious study of quantum computation (containing the necessary parts of classical computation theory). (jso)

I. Laba, C. Shubin, Eds.: *Lectures on Harmonic Analysis,* *University Lecture Series, vol. 29, American Mathematical Society, Providence, 2003, 137 pp., US$31, ISBN 0-8218-3449-5*

The book is based on graduate courses in Fourier analysis taught by Thomas H. Wolff. Izabella Laba completed the manuscript after Wolff's death. The content of the book is somewhat unconventional and represents a mixture of the very basic facts of harmonic analysis, that are relevant to the Kaki needle problem, with current research topics in this area. Early chapters cover standard background concerning the Fourier transform, convolution, the inversion theorem and the Hausdorff-Young and the Kchincin inequalities. Following chapters introduce the uncertainty principle and the stationary phase method. The choice of topics is focused to problems discussed in later chapters - the restriction conjecture and the Kakeya conjecture, distance sets and Fourier transforms of singular measures. The book ends with a comprehensive chapter on recent work connected with the Kakeya problem. The book is intended for a wider mathematical community. Both graduate students and experts will find in it a lot of material and will surely benefit from this excellent book. (jl)

M. V. Lawson: *Finite Automata,* *Chapman & Hall/CRC, Boca Raton, 2003, 307 pp., US$62,96, ISBN 1-58488-255-7*

`Finite automata' is a nice textbook intended for an undergraduate lecture. The book is divided into two parts. The first one is devoted to basic properties of finite automata and languages recognized by finite automata. The second part develops an algebraic theory of these languages. All presented results are illustrated by many simple examples. Nearly 200 exercises reinforce presented results and concepts. In the first part of the book, deterministic, non-deterministic and e-transition automata are studied. They are completely described and their equivalences are presented. The main topic is the language recognized by these automata. This part culminates by the Kleene theorem and by characterization of minimal automata for languages recognizable by finite automata. A method for solution of linear equations for languages is described. The second part of the book is motivated by transition monoids of automata. It is proved that the transition monoid of the minimal automaton recognizing a given language is isomorphic to the syntactic monoid of the language. This very fact switches the semigroup theory to the study of languages recognizable by finite automata. Basic results of semigroup theory are proved. This part culminates by the Schützenberger theorem characterizing star-free languages recognizable by finite automata and by the Eilenberg theorem connecting the varieties of recognizable languages and the pseudovarieties of finite monoids. The author purposefully deals only with a certain pedagogically justifiable selection of topics and facts; for example, he does not include two-way automata, examples characterizing the effectiveness of transformations between different types of finite automata, the effectiveness of generalized regular expressions, the complexity of algorithms, etc. These concepts are only briefly mentioned in remarks. The book is self-contained and easy to read. It can be recommended as a textbook for undergraduate lectures about finite automata. (vko).

I. P. Lebedev, I. I. Vorovich: *Functional Analysis in Mechanics,* *Springer Monographs in Mathematics, Springer, New York, 2003, 238 pp., €69.95, ISBN 0-387-95519-4*

The book is a revised and extended translation of the Russian edition. It consists of three chapters (Metric spaces, Elements of the theory of operators, and Elements of nonlinear functional analysis). These chapters cover the standard material taught in introductory courses on functional analysis, yet the book is different from usual textbooks on this part of mathematics. The main feature of the text is a systematic emphasis on applications in mechanics. For instance, various energy spaces are considered with motivation from bending of a bar, elastic membranes, bending of a plate and linear elasticity. Sobolev spaces and generalized solutions in mechanics are studied, in particular the existence of energy solutions is emphasized. The first chapter also includes the Ritz and Bubnov-Galerkin methods as well as the Bramble-Hilbert lemma. In the second chapter, a detailed exposition of the spectral theory of compact (and, in particular, self-adjoint) operators on Hilbert spaces is presented. Applications deal with 2D and 3D linear elasticity (membranes, plates and bodies). Courant's minimax principle is also included. Chapter 3 contains Fréchet and Gâteaux derivatives, critical points of functionals, degree theory and several applications, partly based on the results of the second author. They include, for instance, the von Karamán equations of a plate, buckling of a thin elastic shell and steady-state flow of a viscous liquid. The book is suitable for readers in continuum mechanics and can be recommended to graduate students and researchers in mathematics, physics and engineering. (in)

C. R. Leedham-Green, S. McKay: *The Structure of Groups of Prime Power Order,* *London Mathematical Society Monographs New Series 27, Oxford University Press, Oxford, 2002, 334 pp., £60, ISBN 0-19-853548-1*

This monograph is devoted to the study of structural properties of finite p-groups. In the first chapter, basic notions are introduced and fundamental theory of finite p-groups is developed. The second part deals with some group-theoretical constructions and in the next one, the p-groups of maximal class are investigated. Finite p-groups acting uniserially are studied in chapter 4 and in the chapter 5, Lie algebras are used for finding some bounds for a certain class of groups. The next part contains the proof of Conjecture A: For some function $f(p,r)$, every finite p-group of coclass r has a normal subgroup K of class at most 2 and index at most $f(p,r)$. If $p=2$, one can require K to be abelian. Chapter 7 is devoted to pro-p-groups and chapter 8 gives some constructions of finite p-groups. The homological methods are explained in part 9. The following two chapters are devoted to uniserial p-adic space groups and to the structure of some classes of finite p-groups. The last part works with some finiteness conditions on pro-p-groups and with the Grigorchuk and the Nottingham groups. (lbi)

O. Moeschlin, E. Grycko, C. Poppinga, F. Steinert: *Discrete Stochastics + CD-ROM,* *Springer, Berlin, 2003, 104 pp., €49.95, ISBN 3-540-14193-9*

This textbook presents a rigorous treatment of probability theory in the discrete case that culminates by the weak law of large numbers. It is suitable for everyone who wants to understand probability and avoid corresponding measure theoretic complications. It can also be useful for a beginner in probability or statistics who has not yet mastered measure and integral theory. The authors assume a reader with passable skills in calculus. The textbook offers examples accompanied frequently by illuminating pictures, some of the exercises are solved separately at the end of the book. The enclosed CD, containing an electronic version of the text, is ready to install the corresponding program automatically in case you have MS Windows installed on your computer. (pd)

J. A. Pelesko, D. H. Bernstein: *Modeling Mems and Nems,* *Chapman & Hall/CRC, Boca Raton, 2003, 357 pp., US$69.95, ISBN 1-58488-306-5*

Microelectromechanical and nanoelectromechanical systems (MEMS and NEMS respectively) represent a challenging and rapidly developing area in modern technology, where modeling plays a greater and greater role complementing traditional experimental research. This book exposes a wide range of topics in the area. The initial-boundary-value problems use basic tools that rely on continuum mechanics, exposed in chapter 2, with the focus on heat conduction, elasticity, linear thermoelasticity, fluid dynamics and electromagnetism. Scaling and the difference between usual macro-scale and micro/nano-devices are explained in the third chapter. The following chapters 4-9 expose a great variety of MEMS and NEMS and present relevant mathematical models, sorted according to specific common features. Namely, it concerns thermally driven systems, elastic structures, coupled thermo-elastic systems, electrostatic-elastic systems, magnetically actuated systems, and microfluidic devices. Although the exposition is primarily intended for the engineering community, because neither rigorous mathematical setting nor definite results (like existence of solutions of considered partial differential equations) are involved, applied mathematicians can also get a lot of inspiration. This is particularly true in the building of mathematical models in the area of MEMS and NEMS, if one forgets some imperfections like Hamilton's principle used to minimize the Lagrangean (p. 33, while simple counter-examples show that, in general, one cannot expect more than only stationary trajectories) or that the interval [a,b] is a Banach space (p. 349). The large number of exercises makes it suitable to some extent also as a textbook for graduate students. Besides, the authors created a web page related to this book, namely *www.modelingmemsandnems.com*. (trou)

L. Pook: *Flexagons Inside Out,* *Cambridge University Press, Cambridge, 2003, 170 pp., £19.95, ISBN 0-521-81970-9, ISBN 0-521-52574-8*
The book is devoted to flexagons, paper models that can be bent in different ways to change their shape. The classical flexagons studied in the book are the hinged polygons. They are made up from a paper net and have some freedom of motion. It is easy to let them move in surprising ways. The set of all possible positions is non trivial and can be used as a base for various puzzles as well as being precisely theoretically studied. The book starts with the description of flexagons. It also includes a collection of nets, with assembly instructions, for a wide range of paper models of flexagons. Then the author gives a short overview of the history of the study of flexagons since their discovery in 1939 until now and an overview of existing literature about the topic. The main advantage of the book is that it gives a mathematical analysis of flexagons by using only elementary mathematical concepts. Thus this book will be very useful for anybody who is not a professional mathematician but wants to get some insight into the mathematical structure of these exciting puzzles. (zš)

C. Pritchard, Ed.: *The Changing Shape of Geometry: Celebrating a Century of Geometry and Geometry Teaching,* *Cambridge University Press, Cambridge, 2003, 541 pp., £24.95, ISBN 0-521-53162-4, ISBN 0-521-82451-6*
The book consists of articles on geometry and its teaching that were previously published in journals of the Mathematical Association Teaching Committee. Distinguished mathematicians and educationalists, including H. S. M. Coxeter, M. F. Atiyah, G. H. Hardy, B. Russel, S. Singh, A. Gutierrez, J. Hersee and Pulitzer price Winner D. Hofstadter, write contributions. Chris Pritchard in his general introduction wrote: "As children we build sandcastles and snowmen, … later we hang wallpaper, negotiate narrow spaces in our cars ... A surprising number of us use specially-honed spatial skills to earn a living, ... But regardless of the extent to which our spatial talents are developed, from the cradle to the grave, we are all geometers". The book contains thirty "Desert Island Theorems" that bring about some surprising and beautiful results providing resources for mathematics teachers. The articles deal with the history of geometry, Euclidean and non-Euclidean geometry, topology, the golden ratio, recreational geometry, geometrical physics and teaching of geometry. The book will provide many hours of enjoyment for any enthusiast. (lbo)

H. A. Priestley: *Introduction to Complex Analysis, second* *edition, Oxford University Press, Oxford, 2003, 328 pp., £19.95, ISBN 0-19-852562-1, ISBN 0-19-852561-3*
The first version of the book appeared in 1985; this is the substantially revised second edition. The book is a well-structured and condensed introduction to the subject starting from scratch; a basic knowledge of complex numbers and a certain familiarity with real analysis is required. The first 9 pages contain an introductory review of basic notions and the following early pages describe the background needed from topology and geometry. The next 16 chapters deal with material usually included in most complex functions courses and the closing 3 chapters are devoted to Laplace and Fourier transforms, harmonic functions and some additional examples of conformal mappings. A recommendation for further study is included in the Appendix. The book contains numerous exercises and will be especially appreciated by students. (jive)

I. K. Rana: *An Introduction to Measure and Integration,* *second edition, Graduate Studies in Mathematics, vol. 45, American Mathematical Society, Providence, 2002, 424 pp., US$59, ISBN 0-8218-2974-2*
This book is a revised version of an earlier edition published in 1997. It starts with a detailed review of the Riemann integration theory (including the Lebesgue characterization of Riemann integrable functions). The author then shows the drawback of the theory, which leads to an introduction of an extension of the Riemann integral. After a short outline of the Riesz-Daniell approach, the Lebesgue approach is discussed in the rest of the book. After traditional chapters on abstract measurable sets and functions, there is a chapter on the Lebesgue integration (both on the real line and in abstract spaces). Further chapters are devoted to product spaces, different modes of convergence, theory of L_p- spaces, and the Radon-Nikodým theorem with applications to signed and complex measures. The text is accompanied by 200 exercises and seven appendices (real numbers, axiom of choice and continuum hypothesis, the Urysohn lemma, singular value decomposition of a matrix, functions of bounded variation and differentiable transformations). The book can be recommended to everybody interested in measure and integration theory. Graduate students will surely find in it a lot of material on this area and also find it a pleasure to read as well. (jl)

A. Ranicki: *Algebraic and Geometric Surgery,* *Oxford Mathematical Monographs, Clarendon Press, Oxford, 2002, 373 pp., £65, ISBN 0-19-850924-3*
Surgery theory, starting with the Kervaire-Milnor classification of exotic differentiable structures on spheres, is now the standard tool in the classification of manifolds (mainly of the dimension ≥ 5). The aim of this book is to develop machinery to prove the main result of the theory - the surgery exact sequence, computing the structure set of a differentiable manifold *M* in terms of topological *K*-theory of vector bundles over *M* and the algebraic *K*-theory of quadratic forms over the fundamental group ring. Along the way, the basics of Morse theory, embeddings and immersions, handle-bodies, Steenrod squares, Poincaré duality, vector bundles, cobordism, etc., are given. On the other hand, many results and applications are not covered, including Novikov's theorem on the topological invariance of the rational Pontryagin classes, surgery on piecewise linear and topological manifolds, the algebraic calculations of the *L*-groups for finite groups, and the geometric calculations of the *L*-groups for infinite groups etc. The book grew out of the author's joint lecture course with J. Milgram at Göttingen in 1987. (pso)

J. D. Sally, P. J. Sally, Jr.: *TriMathlon: A Workout Beyond the School Curriculum,* *A.K. Peters, Natick, 2003, 250 pp., US$30, ISBN 1-56881-184-5*
The book presents a collection of mathematical problems, which are intended to develop the mathematical skills and abilities of the reader. As the title indicates the whole book is conceived as a mathematical analogy of a physical training program. It is divided into three main parts. The first part (Arithmetic) deals with three games based on integers and their elementary properties. The second part (Numbers and Symmetry) presents games and problems in which integers are put in some geometrical configurations, such as magic squares. The last part (Geometry) deals with purely geometrical exercises, such as tessellation, packing circles in the plane, lattice geometry and study of figures by mean of their decomposition and subsequent composition. The presented problems and games are quite classical and belong to typical topics contained in collections of recreational mathematics. They are nevertheless presented in an interesting form. Each chapter starts with elementary exercises and their difficulty is gradually increased. The graphical presentation is beautiful. As for possible readers, the authors do not give any information about that and the concise indication "ages 10 and up" printed on the cover does not seem to be sufficient. The majority of

presented material corresponds to the level of the last grades of primary school. If it were intended for high school students some more advanced topics should be included. On the other hand the presentation and the layout of the book correspond to the age of high school students. The book can nevertheless be recommended as a source of inspiration for teachers, to high school students and for those younger children who will be closely followed by a tutor. (zš)

E. Sonnendrücker, Ed.: *Three Courses on Partial Differential Equations,* *IRMA Lectures in Mathematics and Theoretical Physics 4, Walter de Gruyter, Berlin, 2003, 162 pp., €36.95, ISBN 3-11-017958-X*
This volume contains enhanced versions of three advanced lectures given during a special week of introductory graduate courses traditionally organized at the University Louis Pasteur in Strasbourg. All these three courses address very important and interesting, but very different aspects motivated by the mathematical modelling of physical problems. M. Chipot writes the first contribution. At the beginning, the equilibrium of one or several heavy 2D-objects (disks, squares) rolling on a wire is studied. Further, the equilibrium of 3D-balls rolling on a membrane is considered. In all cases, the author discusses existence and uniqueness of the equilibrium. At the end of each chapter, the open problems are listed. The second course (written by J. Garnier) deals with problems arising from acoustics and geophysics where waves propagate in complicated media, the properties of which can only be described statistically. It turns out that if the different scales presented in the problem can be separated, there exists a deterministic result. O. Kavian is the author of the third course, which is dedicated to the inverse problem of identification of unknown parameters in various boundary value problems for partial differential equations from, e.g., the measured values on the boundary of the domain. The question of how much information is necessary to determine the unknown parameter, is answered in different settings. The text is addressed to students and researchers with a basic background in partial differential equations. (oj)

J. L. Taylor: *Several Complex Variables with Connections to Algebraic Geometry and Lie Groups,* *Graduate Studies in Mathematics, vol. 46, American Mathematical Society, Providence, 2002, 507 pp., US$74, ISBN 0-8218-3178-X*
The book is meant for graduate students interested in algebraic geometry or representation theory of semisimple Lie groups. The author has tried to make the text as self-contained as possible. The first part consists of selected topics from function theory of one complex variable, several possible definitions of holomorphic functions of several complex variables and their equivalence, and local behaviour of holomorphic functions, the Nullstellensatz and its proof. The second part is devoted to the study of local properties of regular and holomorphic functions and holomorphic varieties, basic homological algebra, sheaves and the sheaf cohomology, coherent analytic sheaves, and their direct and inverse images. Also Stein spaces are introduced and studied. Properties of spaces of meromorphic and holomorphic functions, functional calculus and sheaf calculus are discussed in the third part. The basic properties of projective varieties, the corresponding sheaves of regular and holomorphic functions and sections of some holomorphic vector bundles on projective varieties, and criteria for a variety to be projective or quasi-projective are developed in the fourth part together with a comparison of algebraic and analytic Serre's theorem. The last part is devoted to the study of complex semisimple Lie groups and their finite dimensional representations. It is a survey but it also contains a proof of the Peter-Weyl theorem. Special attention is paid to algebraic groups and their representations, semisimple groups and Borel subgroups as well as to the Borel-Weil-Bott theorem and its applications. Each chapter contains a number of exercises. The choice of material is very good and the book can be recommended for the study of the presented topics and their interactions. (jbu)

G.-W. Weber: *Generalized Semi-Infinite Optimization and Related Topics,* *Research and Exposition in Mathematics, vol. 29, Heldermann, Lemgo, 2003, 361 pp., €40, ISBN 3-88358-229-6*
This monograph, based on the author's habilitation, addresses optimization problems on the finite-dimensional space R^n having a single criterion to be minimized and an infinite number of equality and inequality constraints indexed (or parameterized) by a parameter from an infinite index set. The adjective "generalized" refers to the index set, which may depend on the state variable itself. In the first chapter, after presenting some motivating industrial applications, the author exposes 1st-order necessary and sufficient optimality conditions of F. John or Kuhn-Tucker type. Geometrical and topological aspects are investigated in the second chapter in context of stability of semi-infinite problems with respect to data perturbation. Here, Kuhn-Tucker conditions come again into play. Iteration procedures and their convergence are presented in chapter 3. Finally, chapter 4 applies semi-infinite optimization to certain infinite-dimensional situations, namely optimal control problems for ordinary differential equations and time-minimal control problems. The book is primarily written for experts in optimisation theory but PhD-level student in this area will find interesting material here as well. (trou)

C. D. Wensley, Ed.: *Surveys in Combinatorics 2003, London Mathematical Society Lecture Note Series 307*, *Cambridge University Press, Cambridge, 2003, 370 pp., £34.95, ISBN 0-521-54012-7*
This volume contains nine invited talks presented at the 19th British Combinatorial Conference held at the University of Wales, Bangor, in July 2003. Besides them, it contains a paper on W. T. Tutte (who died in May 2002) written by N. Biggs. The nine survey articles deal with graph theory, computer science, matroids and Coxeter groups, finite geometries and block designs, enumeration of geometric objects , Ramsey theory, and symmetric functions. L. D. Andersen and C. A. Rodger survey applications of the amalgamation method in decomposing edge sets of complete graphs in spanning subgraphs with prescribed properties. P. Hell surveys problems and results on algorithmic aspects of graph homomorphisms. S. R. Blackburn discusses combinatorial schemes for protecting digital content (codes with identifiable parent property, frameproof codes, secure frameproof codes, traceability schemes and related concepts). A. V. Borovik writes on ways how Coxeter groups arise in matroid theory and on Coxeter matroids, symplectic matroids, and buildings. D. M. Donovan, E. S. Mahmoodian, C. Ramsay and A. P. Street describe results on defining Sets (parts of an object determining it uniquely) in block designs, graphs and related structures. D. Ghinelli and D. Jungnickel survey the present state of knowledge on existence and structure of finite projective planes with large abelian collineation group. V. Kaibel and G. M. Ziegler discuss estimates for numbers of uni-modular triangulations of the m times n planar grid. I. Leader considers those linear homogeneous systems with integral coefficients, which always have monochromatic solutions (in positive integers). The article surveys some results in the finite case, which is well understood, and then discusses some features of the infinite case. Finally, the contribution of K. Nelsen and A. Ram is devoted to Kostka-Foulkes polynomials and Macdonald spherical functions. (mkl)

H. S. Wilf: *Algorithms and Complexity,* *second edition, A.K. Peters, Natick, 2002, 219 pp., US$39, ISBN 1-56881-178-0*
This is the second edition of a very successful introduction to the classical theory of algorithms. Examples in the exposition are mainly taken from graph theory, number theory and discrete mathematics. They are selected with great care and enable the reader to concentrate directly on the main features of demonstrated topics. Their plasticity makes it possible to follow the author's recommendation to write running programs and thus to realize ideas behind them. No special prerequisites are needed to follow the author's attractive exposition, apart from the feeling to understand the beauty of interconnections between various mathematical branches. Besides highlights of the first edition, such as basis of the theory of computation, the fast Fourier transform and public key cryptography, the new edition includes network flow algorithm and a report on the breakthrough result in primality testing proved by M. Agrawal, N. Kayal and N. Saxena. The exposition is complemented by a large amount of problems, to most of which the reader can find hints for solutions. The book can be warmly recommended to those willing to learn (or teach) basic ideas of computational mathematics. (špor)