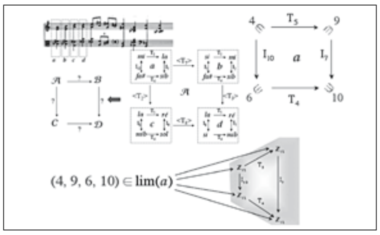
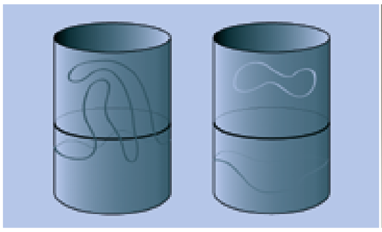
NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY



Feature
Music and mathematics
p. 15



Feature
Symplectic topology
p. 30

Interview
Panos Pardalos
p. 35

Centers
Independent Univ. Moscow
p. 38

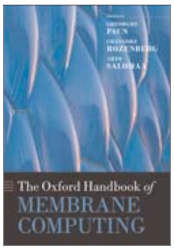
March 2010
Issue 75
ISSN 1027-488X

European
Mathematical
Society

Mathematics Books from Oxford

Don't forget that all EMS members can benefit from a **20% discount** on a large range of our Mathematics books. For more information please visit:

www.oup.com/uk/sale/science/ems



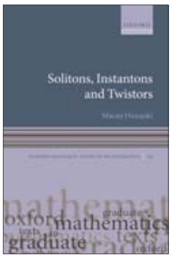
The Oxford Handbook of Membrane Computing

Edited by Gheorghe Paun, Grzegorz Rozenberg and Arto Salomaa

Provides both a comprehensive survey of available knowledge and established research topics, and a guide to recent developments in the field, covering the subject from theory to applications.

2009 | 696 pp | Hardback |978-0-19-955667-0

EMS member price: ~~£85.00~~ | £68.00



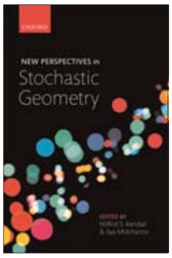
Solitons, Instantons, and Twistors

Maciej Dunajski

Provides a self contained and accessible introduction to elementary twistor theory; a technique for solving differential equations in applied mathematics and theoretical physics.

2009 | 376 pp | Paperback | 978-0-19-857063-9
Hardback | 978-0-19-857062-2

EMS member price: ~~£34.95/£65.00~~ | £27.96/£52.00



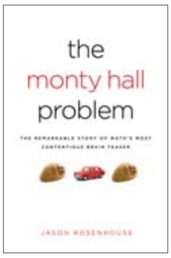
New Perspectives in Stochastic Geometry

Edited by Wilfrid S. Kendall and Ilya Molchanov

The collection lays the foundations for future research, providing a sense of the fresh perspectives, new ideas, and interdisciplinary connections now arising from Stochastic Geometry.

2009 | 608 pp | Hardback | 978-0-19-923257-4

EMS member price: ~~£75.00~~ | £60.00



The Monty Hall Problem

The Remarkable Story of Math's Most Contentious Brain Teaser

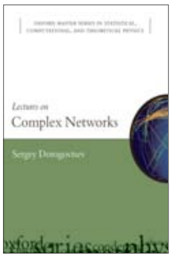
Jason Rosenhouse

The first book to discuss the many facets of an especially famous mathematical brainteaser, it introduces important mathematical ideas in a way that is approachable and engaging.

2009 | 208 pp | Hardback | 978-0-19-536789-8

EMS member price: ~~£15.99~~ | £12.79

NEW !



Lectures on Complex Networks

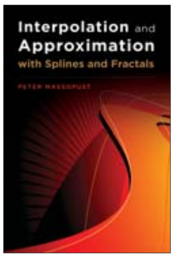
Sergey Dorogovtsev

This text is a concise modern introduction to the science of complex networks, and is based on lectures for university students and non specialists.

Feb 2010 | 144 pp | Paperback | 978-0-19-954893-4
Hardback | 978-0-19-954892-7

EMS member price: ~~£19.99/£39.95~~ | £15.99/£31.96

NEW !



Interpolation and Approximation with Splines and Fractals

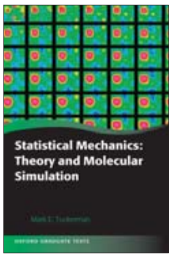
Peter Massopust

This textbook is intended to supplement the classical theory of uni and multivariate splines and their approximation and interpolation properties with those of fractals, fractal functions, and fractal surfaces.

Feb 2010 | 346 pp | Hardback | 978-0-19-533654-2

EMS member price: ~~£50.00~~ | £40.00

NEW !



Statistical Mechanics: Theory and Molecular Simulation

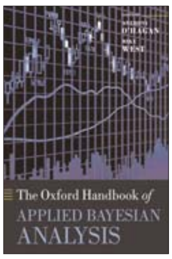
Mark Tuckerman

Useful both to students as a textbook and to practitioners as a reference tool, it treats both basic principles in classical and quantum statistical mechanics as well as modern computational methods.

Feb 2010 | 712 pp | Hardback | 978-0-19-852526-4

EMS member price: ~~£47.99~~ | £38.39

FORTHCOMING!



The Oxford Handbook of Applied Bayesian Analysis

Edited by Anthony O' Hagan and Mike West

Explores contemporary Bayesian analysis across a variety of application areas.

Mar 2010 | 928 pp | Hardback | 978-0-19-954890-3

EMS member price: ~~£85.00~~ | £68.00

DISCOVER NEW BOOKS BY EMAIL

Subscribe to Oxford e.news and receive a monthly bulletin from OUP with information on new books in mathematics. Just visit:

www.oup.com/uk/emailnews

ORDER DIRECT FROM OUP

Online: **www.oup.com/uk/sale/science/ems**

Tel: +44 (0)1536 741727

Don't forget to claim your EMS discount!

European Mathematical Society

Newsletter No. 75, March 2010

The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

ISSN 1027-488X

Published by the
EMS Publishing House
ETH-Zentrum FLI C4
CH-8092 Zürich, Switzerland.
homepage: www.ems-ph.org

EMS Executive Committee

President

Prof. Ari Laptev
(2007–10)
Department of Mathematics
South Kensington Campus
Imperial College London
SW7 2AZ London, UK
e-mail: a.laptev@imperial.ac.uk
and
Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm, Sweden
e-mail: laptev@math.kth.se

Vice-Presidents

Prof. Pavel Exner
(2005–10)
Department of Theoretical Physics, NPI
Academy of Sciences
25068 Rez – Prague
Czech Republic
e-mail: exner@ujf.cas.cz

Prof. Helge Holden
(2007–10)
Department of Mathematical Sciences
Norwegian University of Science and Technology
Alfred Getz vei 1
NO-7491 Trondheim, Norway
e-mail: holden@math.ntnu.no

Secretary

Dr. Stephen Huggett
(2007–10)
School of Mathematics and Statistics
University of Plymouth
Plymouth PL4 8AA, UK
e-mail: s.huggett@plymouth.ac.uk

Treasurer

Prof. Jouko Väänänen
(2007–10)
Department of Mathematics and Statistics
Gustaf Hällströmin katu 2b
FIN-00014 University of Helsinki
Finland
e-mail: jouko.vaananen@helsinki.fi
and
Institute for Logic, Language and Computation
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands
e-mail: vaananen@science.uva.nl

Ordinary Members

Prof. Zvi Artstein
Department of Mathematics
The Weizmann Institute of Science
Rehovot, Israel
e-mail: zvi.artstein@weizmann.ac.il

Prof. Franco Brezzi
Istituto di Matematica Applicata e Tecnologie Informatiche del C.N.R.
via Ferrata 3
27100, Pavia, Italy
e-mail: brezzi@imati.cnr.it

Prof. Mireille Martin-Deschamps
(2007–10)
Département de mathématiques
Bâtiment Fermat
45, avenue des Etats-Unis
F-78030 Versailles Cedex, France
e-mail: mmd@math.uvsq.fr

Prof. Igor Krichever
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027, USA
and
Landau Institute of Theoretical Physics
Russian Academy of Sciences
Moscow
e-mail: krichev@math.columbia.edu

Dr. Martin Raussen
Department of Mathematical Sciences, Aalborg University
Fredrik Bajers Vej 7G
DK-9220 Aalborg Øst, Denmark
e-mail: raussen@math.aau.dk

EMS Secretariat

Ms. Riitta Ulmanen
Department of Mathematics and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
FI-00014 University of Helsinki
Finland
Tel: (+358)-9-191 51507
Fax: (+358)-9-191 51400
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Prof. Vasile Berinde
Department of Mathematics and Computer Science
North University of Baia Mare
430122 Baia Mare, Romania
e-mail: vberinde@ubm.ro

EMS Agenda

2010

20–21 March
EMS EC Meeting, Edinburgh

17–18 April
Meeting of Presidents of European national mathematical societies, Bucharest, Romania

26–27 April
EMS-ESF Forward Look “Mathematics and Industry” - Consensus Conference (in the framework of the Spanish Presidency of EU), Madrid, Spain

1 May
Deadline for submission of material for the June issue of the EMS Newsletter
Vicente Muñoz: vicente.munoz@mat.ucm.es

16–22 May
ESF-EMS/ERCOM Conference on “Algebraic Methods in Dynamical Systems”, Będlewo, Poland

21–25 June
SIMAI (Societa Italiana di Matematica Applicata e Industriale) – SEMA (Sociedad Española de Matemática Aplicada) Joint Congress in Applied and Industrial Mathematics, Cagliari, Italy

28 June–3 July
ESF-EMS/ERCOM Conference on “Teichmüller Theory and its Interactions in Mathematics and Physics”, Bellaterra, Barcelona, Spain

2–7 July
Euroscience Open Forum under the slogan Passion for Science (ESOF2010), Torino, Italy, http://www.esof2010.org

11–13 July
EMS Council Meeting; Conference “Mathematics in Industry”; 20th Anniversary of European Mathematical Society, Sofia, Bulgaria

26–31 July
16th Congress of ECMI (European Consortium of Mathematics in Industry), Wuppertal, Germany

1 August
Deadline for submission of material for the September issue of the EMS Newsletter
Vicente Muñoz: vicente.munoz@mat.ucm.es

19–27 August
International Congress of Mathematicians, ICM2010, Hyderabad (India)
http://www.icm2010.org.in

12–17 September
ESF-EMS/ERCOM Conference on “Highly Oscillatory Problems: From Theory to Applications”
Cambridge, United Kingdom

2011

18–22 July
ICIAM 2011 Congress, Vancouver, Canada

Editorial

We are now heading toward our next EMS council meeting, which will take place on 10–11 July 2010 at the Metropolitan Hotel in Sofia: http://www.metropolitanhotelsofia.com/en/. The council meeting starts at 1pm on 10 July and ends at noon on 11 July. Our colleagues in Bulgaria are working hard on the organization of the official part of the meeting and also on the scientific conference "Mathematics in Industry" that will begin on the afternoon of 11 July and end on the evening of 12 July.

In 2010, we shall celebrate 20 years since the European Mathematical Society was established. The EMS Executive Committee is planning a round table devoted to this event and is inviting all former EMS Presidents to participate, as well as M. Atiyah.

The EMS council meeting in 2010 will be an important occasion where the EMS Executive Committee will give a report on what has been achieved during the last years since our previous council meeting in Utrecht.

Let me mention here some initiatives of the EMS committees.

About two years ago we had the idea of starting annual meetings of the presidents of the national mathematical societies. The French Mathematical Society kindly took responsibility for hosting our first meeting in Luminy, 06–07 April 2008. It was an extremely fruitful event where the presidents of the national mathematical societies were able to exchange ideas and learn more about how each other conducted their societies. It was decided that we should continue such meetings and indeed such a meeting was held on 9–10 May 2009 at the Banach Center in Warsaw. The next meeting will be on 17–18 April 2010 in Bucharest, Romania, where we will also participate in the ceremony "100 years of the Rumanian Mathematical Society".

The EMS EC believes that such meetings establish better contacts with national societies and give the EMS a better understanding of the needs of European mathematicians. Reports on these meeting are published on the EMS webpage.

After some persuasion, Brussels has agreed to acknowledge the notion of Infrastructure for Mathematics. At the end of July, Brussels published a €10M/4 years call on "Infrastructures for mathematics and its interfaces with science, technology and society at large". On behalf of the ERCOM and the EMS Applied Math. Committee and its project Math & Industry, a small committee was set up whose members are Jean-Pierre Bourguignon, Mario Primicerio and Ari Laptev. Later, this committee was increased in size to include Gert-Martin Greuel (Oberwolfach), who is the chairman of the EMS ERCOM, Maria J. Esteban (President of the SMAI) and Thibaut Lery (European Science Foundation). Our task was to prepare a proposal and also to initiate a broad discussion on how the European mathematical community views a large infrastructure in mathematics. Ultimately we had to make difficult choices in our effort to find a balance by taking into consideration complicated geographical, scientific and pure/applied aspects between the different institutions.

€10M for four years is not a large amount to support all European mathematics. Therefore, we have to think very carefully about how to spend the money in order to make European mathematics more visible and appreciated. We hope that if our project is approved, this would be a first step in the development of a virtual, intellectual infrastructure for mathematics in Europe. There is an indication that if this project is successful, the European Union would be willing to provide more substantial funding of European mathematics in the future. The first results of the approval process are expected at the end of February.

The EMS Committee for Women and Mathematics was recently involved in organizing the 14th General Meeting of European Women in Mathematics (EWM), 25–28 August 2009, at the Department of Mathematics and Informatics, University of Novi Sad, Serbia. Among the speakers was Professor Ingrid Daubechies, Princeton University, USA, who is also a 2009 EMS lecturer. The conference was extremely successful. One of the organizers of this meeting Dusanka Perisic has now been approved as the new Chair of the EMS Committee for Women and Mathematics. The next EWM general meeting in 2011 will be hosted by the CRM in Barcelona.

Another important initiative came from the European Science Foundation who, together with the CNRS, have suggested that the EMS should lead a Forward Look project on "Mathematics & Industry". This project has now been approved by the ESF and the Chair of the EMS Applied Math. Committee Professor Mario Primicerio, who has assembled a committee that is working hard on this project.

On 26–27 April 2010, the Consensus Conference of the EMS-ESF Forward Look on Mathematics and Industry will be organized in Madrid. This event is part of the official agenda of the Spanish Presidency of the EU. All information about this event is on the conference webpage: http://www.ceremade.dauphine.fr/FLMI/Consensus_Conference_Madrid/.

There was also a suggestion to have a Forward Look on the European Digital Mathematical Library. Finally, however, it has been decided to integrate efforts into an upcoming EC proposal on Infrastructure for Mathematics. After many years of unsuccessful EU applications, I was very happy to hear from the Chair of the EMS Committee for Electronic Publishing Ulf Rehmann that support for the EU-project concerning the EuDML has been accepted (€1.6M/2 years).

The EC EMS has decided to appoint a new Ethics Committee. Professor Arne Jensen has agreed to chair this committee. The committee's duties are still to be discussed but a need for such a committee has been approved.

The Committee for Developing Countries led by Tsou Sheung Tsun is, as usual, extremely active. It receives a large number of donations and it is involved in many different activities helping developing countries to be part of our community. The project of free access to ZMATH for universities from developing countries is running successfully.

The EMS Committee for Raising Public Awareness led by Ehrhard Behrends had a successful meeting in Brussels. The committee was able to secure some substantial funds for its future activities from a private foundation.

I would like to remind every individual EMS member of the possibility of having an individual password in order to gain free access to Zentralblatt. Please use this opportunity. Zentralblatt will naturally be included as part of the project on Infrastructure for Mathematics and we hope that its continuous development will serve our community in the best possible way. In order to receive a login + password please contact Olaf Teschke: teschke@zblmath.fiz-karlsruhe.de.

Our new webpage http://www.euro-math-soc.eu was developed by the University of Bremen and we are very grateful to D. Kozlov, who was able to provide funding for this service. This project is now completed and the files have been moved to Helsinki where they will have the necessary assurance of stability. Nowadays, webpages provide an important means of communication, data storage, information and services.

As a whole, I find the development of the European Mathematical Society very satisfactory and I am sure that its future prosperity is secured.

Finally, I would like to ask the society members to think of possible future EC candidates to be elected at the council meeting in Sofia. Please send them to me at a.laptev@imperial.ac.uk and to the EMS secretary Stephen Huggett at stephen_huggett@mac.com.

Ari Laptev
EMS President

Rectification

As Professor Jean Mawhin kindly pointed to us, Jean Leray was not correctly identified in the picture published in page 32 of the last issue of the EMS Newsletter (December 2009). As we realized later, in fact he is not in the picture: the second person from the left is Jean Bernard and the man next to Pierre Cartier is André Blanc-Lapierre. It is highly possible that the person sitting in the middle is Edgar Lederer, but we have not been able to identify it with certainty. The three of them (Bernard, Blanc-Lapierre and Lederer) were members of the Académie de Sciences. We sincerely apologize for this misunderstanding.

New editor in the Editorial Board

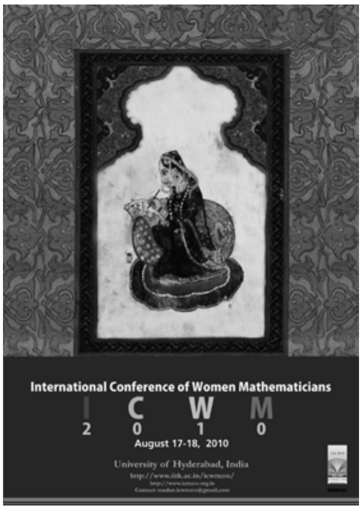
Eva Miranda obtained her PhD in Mathematics in 2003 at the Department of Algebra and Geometry at the University of Barcelona. She held an assistant position at the University of Barcelona and then a Marie Curie postdoctoral EIF position at the University of Toulouse and made several research stays at the Institut des Mathématiques de Jussieu in Paris, the University of Toronto, the Mathematisches Forschungsinstitut Oberwolfach (MFO) and MIT in Boston. Later, she came back to Spain with a Juan de la Cierva Junior Research Position at the Universitat Autònoma de Barcelona. She is currently a lecturer in mathematics at the Universitat Politècnica de Catalunya.

Her interests focus on several problems in differential geometry and mathematical physics, including symplectic and Poisson geometry and questions related to Hamiltonian dynamics. She has obtained several results on normal forms for singularities of integrable systems on symplectic, contact and Poisson manifolds and she has worked out some rigidity results for group actions and additional geometrical structures on these manifolds. Recently, she has also been working on geometric quantization.

Besides her research and teaching activities, she has been an active member of the international mathematical community, organizing ten workshops and a research programme at CRM-Barcelona.

ICWM 2010

International Conference of Women Mathematicians, Hyderabad August 17–18, 2010
http://www.icm2010.org.in/icwm2010.php



ICWM 2010 will take place at the University of Hyderabad over the two days immediately before the International Congress in 2010. The meeting is aimed principally at women mathematicians attending the ICM (though men are also very welcome to attend), and in particular at young women mathematicians and women from Asia and from developing countries. The talks will be colloquium style lectures aimed at a general mathematical audience, and it is hoped that participants will be provided with an opportunity to meet other women mathematicians about to take part in the ICM and to find out about some of the areas of research to be covered at the ICM.

There will be nine lectures of 45 minutes each from the following speakers:

Julie Deserti (Paris, France)
Frances Kirwan (Oxford, UK)
Maryam Mirzakhani (Stanford, USA)
Neela Nataraj (IIT Bombay, India)
Raman Parimala (Atlanta, USA)
Mythily Ramaswamy (TIFR Bangalore, India)
Maria Saprykina (KTH Stockholm, Sweden)
Nathalie Wahl (Copenhagen, Denmark)
Di Yana (CAS Beijing, China)

In addition to the lectures there will be a discussion forum and a conference dinner on the evening of 17 August.

Registration will begin on 1 January 2010. More information (on the venue, programme, accommodation etc) will be available on the website at http://www.icm2010.org.in/icwm2010.php by then.

The scientific programme has been planned by the EWM/EMS Scientific Committee, co-opting two mathematicians from India:

Ulrike Tillmann (Oxford, UK), chair
Viviane Baladi (ENS, Paris, France)
Eva Bayer (Lausanne, Switzerland)
Christine Bernardi (Paris VI, France)
Christine Bessenrodt (Hannover, Germany)
Antonella Grassi (U Penn, USA)
Ursula Hamenstaedt (Bonn, Germany)
Dusa McDuff (Stony Brook, USA)
Ragni Piene (Oslo, Norway)
Mythily Ramaswamy (TIFR Bangalore, India)
Sujatha Ramadorai (TIFR Mumbai, India)
Vera Sos (Renyi Institute, Budapest, Hungary)
Nina Uraltseva (St Petersburg, Russia)
Michele Vergne (Ecole Polytechnique, Paris, France).

EMS Executive Committee meeting in Istanbul, 17–18 October 2009

Vasile Berinde, EMS Publicity Officer

Venue and attendance

Perhaps in order to balance the times when the EMS Executive Committee (EC) meetings were mainly held in Western and Central European countries, 2009 was a year of South-Eastern European EC meetings. After its first meeting in Athens, Greece, in March, the EMS Executive Committee reconvened at the Marmara Pera Hotel in Istanbul, 17–18 October, at the invitation of the Turkish Mathematical Society. Present were: Ari Laptev (*President* and *Chair*), Pavel Exner and Helge Holden (*Vice-Presidents*), Stephen Huggett (*Secretary*), Jouko Väänänen (*Treasurer*), Zvi Artstein, Franco Brezzi, Igor Krichever, Mireille Martin-Deschamps and Martin Raussen (*EC members*) and, by invitation, Vasile Berinde (*Publicity Officer*), Vicente Muñoz (*Editor-in-Chief of the EMS Newsletter*), Mario Primicerio (*Chair of the EMS Committee for Applied Mathematics*) and Riitta Ulmanen (from the *Helsinki EMS Secretariat*).

Officers' reports, standing committees and newsletter

After the ratification of electronic votes since the last meeting, the president reported on the work on the infrastructure project MathEI (**Math**ematics **E**uropean **I**nfrastructure), which aims to build a virtual world-class pan-European research infrastructure in mathematics involving academia, researchers and industry. The project is mainly designed for the intellectual infrastructure in mathematics and would include: interdisciplinary conferences; research laboratories; research schools; research projects; support for postdoctoral researchers; support for research in small groups; websites for mathematical positions; and websites for doctoral theses. The regulations of the EMS/Springer History Prize as proposed by the working group were then presented by Martin Raussen. The EC approved them and decided that there would be just one prize awarded eve-

From left to right: Zvi Artstein, Vasile Berinde, Franco Brezzi, Mario Primicerio, Pavel Exner, Stephen Huggett, Mireille Martin-Deschamps, Ari Laptev, Martin Raussen, Riitta Ulmanen, Vicente Muñoz, Jouko Väänänen, Igor Krichever, Helge Holden.

ry four years at the European Congresses of Mathematics. Other reports came from the treasurer (the budget is healthy) and the publicity officer (a special event in 2010 marking the 20th anniversary of the EMS will be associated with the council meeting in Sofia).

The reports for each committee were presented and accepted by the EC. The terms of office of each Chair were also reviewed and agreed as follows. *Applied Mathematics*: Mario Primicerio, re-elected for (2010–2011); *Developing Countries*: Tsou Sheung Tsun (2007–2010); *Eastern Europe*: Jan Kratochvil (2005–2010); *Education*: Konrad Krainer (2009–2011); *Electronic Publishing*: Ulf Rehmann (2009–2012); *ERCOM*: Gert-Martin Greuel was approved as the new Chair for (2010–2013); *Raising Public Awareness*: Ehrhard Behrends (2009–2012); *Women and Mathematics*: Dusanka Perisic was elected for (2010–2013); *Meetings Committee*: Fred van Oystayen (2009–2011); and *Ethics Committee*: Arne Jensen (2010–2013). (Note that the dates are inclusive.)

It was agreed that the newly set up Ethics Committee would start with the following members: Jean-Paul Allouche, Tatyana Shaposhnikova, Graziano Gentili and Radu Nicolae Gologan. Franco Brezzi argued that the problem of evaluation is now so important that it might justify setting up an EMS Committee on Evaluation. The EC welcomed the suggestion and invited Franco to draft a preliminary document on this question that will be presented to the next EC meeting.

At the recommendation of Editor-in-Chief Vicente Muñoz, Eva Miranda was approved as a new editor of the EMS Newsletter. The EC also agreed that another new editor could be appointed if necessary.

Closing matters

The president closed the meeting by expressing the gratitude of all present to Betül Tanbay and Ali Ülger for the excellent arrangements that the Turkish Mathematical Society had made for the EMS EC meeting in Istanbul. The next EC meeting will be in Edinburgh, 20–21 March 2010.

The talks of the 2009 EMS Lecturer Ingrid Daubechies, delivered at the European Women in Mathematics meeting (Novi Sad, Serbia, August 25–28, 2009), are available online at the new website of the European Mathematical Society.

http://www.euro-math-soc.eu/node/467

www.mathematics-in-europe.eu

Ehrhard Behrends, chair of the rpa committee

For many years the rpa committee of the EMS has been concerned with "**r**aising the **p**ublic **a**wareness" of mathematics.

In the past, various activities have been realized, such as round-table discussions and competitions to find the best published article that raises the public awareness of mathematics.

At the beginning of 2009, some new members and a new chair were elected by the executive committee (EC) of the EMS. (The names of the present members of this committee can be found on the EMS homepage www.euro-math-soc.eu by following the links committees/committee list/Raising Public Awareness.)

It was the desire of the EC that a new rpa activity should be established: a popular mathematical web page under the auspices of the EMS. In this article we present a brief summary of the preparations to date.

Why a popular web page of the EMS?

In nearly all countries public understanding and appreciation of mathematics are far from satisfactory.

As a consequence, politicians and other shapers of public opinion generally have only a very vague idea why our field is of vital importance, and furthermore, many students hesitate to study sciences in which mathematics is essential.

Various activities to remedy this have been initiated at the national level, and it would clearly be useful to bundle these ideas in order to make rpa material more widely available and to provide a forum for the exchange of ideas.

As a side effect such a web page may help to increase the visibility of the EMS.

Preparations 1: The Brussels meeting

The first meeting at which ideas concerning the structure of such a popular webpage were discussed took place in May 2009 in Brussels, Belgium (local organizer: Thomas Bruss).

The participants of the Brussels meeting

Here are the most important decisions:

1. The name of the web page will be www.mathematics-in-europe.eu. Also www.math-in-europe.eu and www.maths-in-europe.eu will lead to this page.
2. It will be crucial to find a sponsor in order to be able to realize this project.
3. All of the contents will be provided in English, and contributions written in another European language will also be included (together with an English translation).
4. Many ideas concerning the contents, the design, and the structure were collected. (Some details can be found below.)

Our sponsor

For many years there has existed a sponsoring agreement between the Munich RE (see www.munichre.com) and the German Mathematical Society (DMV): this support was provided to create and host the popular German-language webpage www.mathematik.de.

The MR is the biggest reinsurance company in Germany.

This collaboration has been most satisfactory, and therefore it was natural to approach Munich RE again, this time with a proposal that they sponsor www.mathematics-in-europe.eu .

Negotiations came to a successful conclusion, and in December 2009 a contract between Munich RE and the EMS was signed. The Munich RE will provide annually a rather generous sum of money, which will be used to create and to run the popular EMS web page. As a partner of the EMS, the DMV is responsible for the administrative aspects of this contract.

The Kraków meeting

Soon after the contract had been signed by representatives of Munich RE, EMS, and DMV, a meeting of a subcommittee of the rpa committee was organized. It took place in Kraków, Poland, at the beginning of January 2010 (local organizer: Krzysztof Ciesielski). There, more detailed suggestions concerning content and structure, as well as a time schedule for the work of the following months, were discussed:

The structure and the contents of www.mathematics-in-europe.eu

In addition to static elements, the **starting page** will contain material that will be randomly generated every day, such as a quotation of the day, a historical reminder (e.g.: Niels Henrik Abel was born on this day in 1802), the first lines of a popular article on mathematics (together with a link to the full version), … The main menu will contain only a few items in order to keep the page uncluttered; it will be complemented by pop-up submenus.

As far as the **technical realization** is concerned, it is recommended that the page be hosted with a content management system (CMS) based on PHP and a database (e.g., JOOMLA!).

The many ideas proposed in Brussels were grouped as follows to structure of the **contents** of the web site:

1 News: What relevant news from the mathematical community might be interesting for a general audience (Abel Prize, Fields Medals, breakthroughs, …)?
2. Information: What is mathematics? Examples of contemporary research, philosophy of mathematics, the mathematical landscape, the mathematical calendar (e.g., an article “Gauss published the Gauss algorithm 200 years ago”), the history of mathematics, interesting links, …
3. Popularization activities: books, videos, and podcasts associated with mathematics, exhibitions, lectures, special events, survey of national rpa activities, posters, …
4. Competitions: Competitions for school children and students in high school and college, a mathematical advent calendar, …
5. Mathematical help (this is a working title only): first aid for school children (what should one know about the quadratic equation, etc.), mathematics in different languages (a dictionary), interesting suggestions for school teachers to be used in the classroom, …
6. Mathematics as a profession: interviews with mathematicians who work in the “real world,” a description of what they actually do (actuaries, mathematicians in finance, engineers, etc.)
7. Miscellaneous: FAQ, mathematical misconceptions, mathematical stamps, mathematical jokes, …

The next steps

The rpa committee will have its next meeting in Istanbul in May. Until then, the raw version of www.mathematics-in-europe.eu should be prepared to the point that the web page can go online not much later. Thus, during the coming months, it will be necessary to solicit bids for writing the requisite software and for the graphic design, and as much material as possible will have to be prepared by the rpa committee members .

It must be stressed that this article should be considered as nothing more than a preliminary information about this popular web project. More details will follow later, and all EMS members in all countries are kindly invited to collaborate.

Ehrhard Behrends [behrends@math.fu-berlin.de] is Professor for Mathematics at the Free University of Berlin. For more than 10 years he was the secretary of the German Mathematical Society. At present he is the chair of the committee “raising the public awareness of mathematics” of the EMS.

2009 Ramanujan Prize for Young Mathematicians from Developing Countries

The Ramanujan Prize Selection Committee has announced that the 2009 **Srinivasa Ramanujan Prize** will be awarded to Ernesto Lupercio, a researcher at CINVESTAV, Instituto Politécnico Nacional, Mexico.

Lupercio is being honoured for “his outstanding contributions to algebraic topology, geometry and mathematical physics. He is an expert in the theory of orbifolds (spaces with singularities arising from finite symmetric groups). He has fundamental results on K-theory, gerbes, and Chas-Sullivan type string topology operations. The prize is also in acknowledgement of the enormous contribution that Professor Lupercio has made to mathematics in Mexico, through his energy, enthusiasm and collaborations with young researchers.”

The Prize has been supported by the Niels Henrik Abel Memorial Fund, with the participation of the International Mathematical Union. The Prize is awarded annually to a researcher from a developing country less than 45 years old, who has conducted outstanding research in a developing country. The Prize carries a $15,000 cash award and travel and subsistence allowance to visit ICTP for a meeting where the Prize winner will be required to deliver a lecture.

See http://prizes.ictp.it/Ramanujan/ramanujan-prize-winner-2009.

The Białowieża Meetings on Geometric Methods in Physics: Thirty Years of Success and Inspiration

S. Twareque Ali and Theodore Voronov; Photos: Tomasz Goliński

Introduction

The Białowieża meetings, held under the general theme, "**Workshop on Geometric Methods in Physics**", have been organized every year since 1982. The meetings are organized through the **University of Białystok** and currently by the Department of Mathematical Physics of the university. From their inception, the main organizer of the workshops has been Professor **Anatol Odzijewicz** of Białystok. The meetings themselves are held in the primeval **Białowieża Forest**. As a rule, the workshops last a week (from a Sunday to a Saturday), coming either at the end of June or the beginning of July. The beginning or the end of each meeting coincides with the night of the regional folk festival of "Kupala", an event that has lent the colour of legend to the meetings. The first meeting, in 1982, was attended by about 15 physicists and mathematicians from several universities and institutes in Poland. Over the years, the annual meetings have grown, both in the number of participants and in the number of countries represented, and have become significant international events. Currently nearly a hundred participants from over 20 countries of Europe, North and South America, Asia, Australia and Africa participate each year. Created initially as a meeting place for researchers from the East and the West (in the times of the cold war), the Białowieża meetings are now a meeting ground for mathematicians and physicists from all over the globe. The participants have included some of the most established mathematicians and physicists of the world, as well as younger researchers and graduate students. In fact the participation of (post)graduate students is among the main objectives of the workshops. Proceedings of the meetings have been published for nearly all the workshops, starting from 1992. The Białowieża series of meetings is presently considered by professionals in the field, in terms of scientific quality and pedagogical content, to be one of the most serious of its kind. Held on the eastern border of the EU (the town of Białowieża is but a few kilometres from the Polish-Byelorussian borderline), these meetings have a distinct 'European flavour' and this is another aspect that makes them so attractive. While inside the lecture hall the official language is English, outside of it the language could easily move from German to French to Italian or even to Farsi or Bengali and of course, one also hears lots of Polish and Russian, as well as Czech and Slovak. Indeed, one can hear an entire Babel of languages that unites today's international scientific set.

There is a permanent **website** of the conference: http://wgmp.uwb.edu.pl/. It contains information about the current meeting, as well as links to pages with information on previous meetings (dating back to 1994).

Format of lectures

The lectures at each workshop are generally organized around a number of specific topics. The choice of topics depends mainly on the choice of each year's key speakers. About a dozen or more (it can be up to 20 but 15 or so is more usual) plenary talks are given each year by prominent, internationally recognized mathematicians and

One of the general photos of the participants.

physicists, covering the topics that form the focal point for that year. These lectures are usually held during the mornings. The lengths of the lectures are tailored to allow for a pedagogically oriented coverage of the topic. Apart from the plenary talks, a large number of contributed lectures of various lengths are also scheduled, with a significant number of contributions from graduate students.

In the lecture hall.

The participants are all housed in nearby hotels and guest houses. Meals are usually taken together and a good number of social events are organized, mainly during the evenings. These allow for a week-long shared scientific, academic and social experience.

At the campfire.

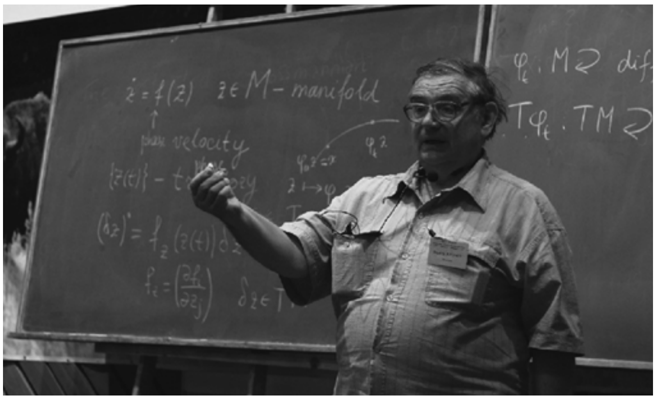
A large number of scientific collaborations, joint publications and exchanges have come out of the meetings. The accessibility of the lecturers to the other participants, in particular the students, is one of the most valuable aspects of this shared experience.

Topics in focus

Within the general theme of geometric methods in physics, the meetings focus on a number of specific topics each year. Among some of the major topics discussed in the workshops are quantization techniques, coherent states methods, Poisson and symplectic geometry, infinite-dimensional systems, harmonic analysis, non-commutative geometry, integrable systems, field theory and theoretical quantum optics. The organizers depend on an international advisory committee to suggest topics and speakers and to help contacting them each year. Consequently, the workshops deal with areas of mathematics and theoretical physics that are at the frontiers of current research, with the lectures being given by experts in the field.

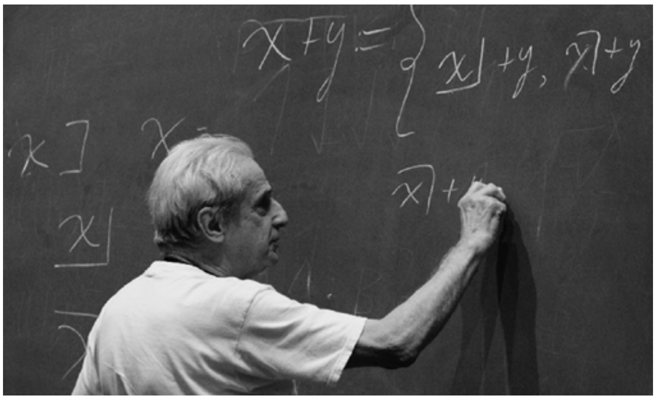
Profile of lecturers

The plenary speakers at the workshops have included Nobel laureates and other high profile mathematicians and physicists, including **D. Anosov, V. Buchstaber, F. Calogero, A. Cattaneo, D. Elworthy, G. Emch, B. Fedosov, R. J. Glauber, M. Kruskal, K. C. H. Mackenzie, G. W. Mackey, V. Mathai, M. Shubin, A. Veselov, Cecile DeWitt-Morette, Bryce DeWitt** and **S.L. Woronowicz**. (M. A. Shubin also played a substantial role in the organizing committee of the meetings over a period of years.)



A lecture by Dmitri Anosov.

Although a majority of speakers have been European (with a large proportion coming from Poland and Russia), there has been participation from a significant number of US, Canadian, Japanese and Latin American scientists. Additionally, participants have also come from Africa (Benin and Rwanda) and Asia (India, Bangladesh, Iran and South Korea).



A lecture by Martin Kruskal.

Organization of the meetings

Since the early times of the Białowieża meetings, they have gradually developed into a sort of 'informal **international research institute**' based on the participants of the meetings. As a rule, those who have attended the meeting once wish to come again and do indeed come when there is an opportunity. In the meantime, there are contacts and exchanges between the meetings, with an anticipation to meet 'next time in Białowieża', often culminating in some joint work conceived in the hospitable and inspiring

Białowieża atmosphere. Participants and organizers of the meetings, including those who serve on the scientific advisory board, act as editors for the proceedings, referees for the texts, etc., all making for an ever-growing network of mathematicians and physicists who see the Białowieża meetings as an important part of their research life and are dedicated to helping them continue.

Each year, typically on the third or fourth day of the conference, in the evening after the lectures have finished, a meeting of the informal advisory board is convened. The board does not have a fixed membership; some people involved with the conference for many years have a seat there but the rule is that each year's plenary speakers are asked to attend and they make up the majority of the advisory board for the coming year. The main task of this annual advisory board meeting is to discuss the list of invited speakers for the next conference. Each participant of the advisory board can suggest candidates, who are then discussed together. Typically the initial list consists of too many candidates and some "preference order" is assigned to them. After that, the job of each board member consists of contacting the candidates proposed by them (commonly approved by the advisory board and with a certain order of preference) and to recruit them as the next conference plenary speakers at Białowieża. When an informal acceptance is obtained, the formal side is taken up by the local organizers at the University of Białystok, who send out the official invitations. Two objectives are achieved by such an arrangement: (1) under the assumption that the conference plenary speakers in year N are truly brilliant, they (in their capacity as members of the advisory board) suggest no less brilliant *different* mathematicians or physicists as potential plenary speakers for the year N+1, so the high level is maintained together with the continuity of the principal topics; and (2) at the same time, since the advisers cannot obviously invite themselves, there is always a flow of new people invited to speak, who thus join the 'Białowieża community', while many of the invited plenary speakers of year N or one of the previous years come again to the meeting in year N+1 (at their own expense).

Proceedings of the meetings

The proceedings of the meetings have been published, more or less continuously, since 1992. The proceedings volumes have appeared through major publication houses, such as World Scientific, the AMS, Kluwer and Polish Scientific Publishers, and also in journals, such as the *Journal of Nonlinear Mathematical Physics* and *Reports on Mathematical Physics*. A special volume consisting of invited articles was brought out through World Scientific to commemorate the 20th anniversary meeting in 2002. Since 2007, the proceedings volumes have been published by the AIP (American Institute of Physics). It is worth mentioning that the editors of the proceedings have taken special care to ensure that all the papers appearing in the Białowieża proceedings volumes are properly peer-refereed. This is important for maintaining the high standard of this publication and also because it makes such papers "count" towards research profiles (something that younger researchers have to take into account).

Meeting location and history

Białowieża (pronounced "bye-lah-VYE-zhah", the name meaning "White Tower") is a village on the border of Poland and Byelorussia, some 60 km from the regional capital Białystok and about 260 km from Warsaw. This region of Poland, called **Podlasie**, has a distinct character due to its sizeable Russian minority (Eastern Orthodox in religion). One can see typical Orthodox churches with their cupola and eight-ended crosses. In the village of Białowieża there is one such church dedicated to St Nicholas and built by the last Russian Emperor Nicholas II.

Białowieża: St Nicholas Orthodox Church. Photo from Wikimedia Commons

For centuries, the village used to be the centre of a royal hunting reserve for Lithuanian grand princes, Polish kings and Russian Emperors in succession. In a beautiful park, actually on the pathway from the conference accommodation to the conference hall where the lectures take place, the participants can see an obelisk in memory of the particularly outstanding hunting of the Polish King Augustus III in 1752. The hotel containing the conference hall stands on the site of the former Tsar's palace built in 1894. It was damaged by a German shell during the last war and was completely demolished in the 1960s. There is an exposition available where visitors to Białowieża can see pictures of the palace; the surrounding park (the Palace Park) and the surrounding buildings that remain recall the beauty of the palace.

Białowieża is located inside a gigantic **primeval forest**, the **Białowieża Forest**, now within the borders of two countries, Byelorussia and Poland (possessing about two thirds and one third of the forest respectively). On the Polish part the ancient forest is partly protected by the Białowieża National Park. The social program of the Białowieża conferences has always included an excursion to the forest and the open air zoo where visitors can see various local animals including the famous Białowieża "**żubr**", i.e. *wisent* or *European bison* (a cousin of the American bison). The żubr remains the symbol of Białowieża. Wisents used to be regarded as extinct but were reintroduced back into nature in the course of the 20^{th} century. The Białowieża Forest is one of the very few places in Europe where one can see wisents in their natural environment.

Currently the village of Białowieża is one of the well-known Polish country resorts. It is still very quiet there.

During the conference, the participants live in several of the numerous Białowieża hotels and guest houses, all located very close to where the lectures are held. The participants are immersed in a homely rural atmosphere with excellent food and outstandingly friendly service. The hotels are quite simple but comfortable. The insides of the hotels, which are decorated with hunting trophies, vividly recall past times. Part of the social program of the conference is related to the private open air **Museum of Wooden Architecture and Domestic Life of the Russian People in Podlasie Region** (known as the "Białowieża Skansen"), which was founded by Professor Odzijewicz about the same time as the Białowieża conference. The museum has developed over time and currently includes several large wooden peasant houses suitable for living (and actually used as free accommodation for students coming from Białystok) and two windmills, all brought to the museum from their original locations in various Podlasie villages and re-assembled. The conference campfire takes place on the territory of the museum.

The conference hall, situated in the middle of the Palace Park, is very spacious and equipped with all modern presentation facilities, as well as (for the time of the conference) with the traditional blackboards that are preferred by some participants". The blackboards are brought from the University of Białystok.

Funding of the meeting

The University of Białystok has provided generous funding over the years for these meetings. Additionally, at different times, partial funding has been made available by the Embassy of France in Warsaw, the Stiftung für Deutsch-Polnische Zusammenarbeit and the Polish Ministry of Education.

It should be emphasized that over the years, a lot of work that has made these meetings possible has been done on a voluntary basis, including the skilful work of the editors of the proceedings and the work of maintaining the meeting **homepage** on the Internet. A huge share of the organizational and technical help, especially during the meetings, has been done by the young mathematicians and physicists (PhD students and postdocs) from the University of Białystok. Booking rooms almost a year in advance with the local guest-house owners has also helped to keep the costs lower. Some of the participants are housed in the museum ("Skansen") for free. That is how the museum sponsors the meeting. Occasionally there are larger and smaller donations from local businessmen who view these annual meetings as an important part of Białowieża life and take pride that their little town once a year becomes a prominent international scientific centre. The local organizers work very hard to keep the costs of the meetings low (never at the expense of the quality).

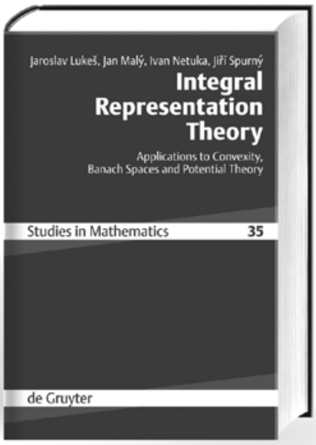
All that said, with prices in Eastern Europe steadily going up and approaching those of elsewhere, maintaining the conference without financial help from any "external" funding bodies becomes increasingly difficult. The organizers have already had to increase the registration fee for the participants to the present €450 (the invited speakers are exempt). The fee includes full board and lodging for the duration of the conference (7½ days including arrival and departure), as well as transportation from Warsaw and back, so it still looks low by average European standards. The organizers typically offer some support to students. However, any further increase of the registration fee may negatively affect that, as well as the participation of mathematicians and physicists from Eastern Europe. Lack of external funds also puts a severe restriction on the number of plenary speakers that the organizers can afford to invite. Sometimes most attractive suggestions for a speaker have to be put aside because of the impossibility of funding more in a given year. Obviously, getting regular funding from European sources can substantially change the situation. There is a unique opportunity to help this remarkable European scientific initiative that has been run by volunteers and abundantly proved itself over almost thirty years, and to allow it to continue with the same success, as a focus of inspiration and an informal research centre for a great number of mathematicians and physicists for many more years to come.

Syed Twareque Ali [stali@mathstat.concordia.ca] is a professor at the Department of Mathematics and Statistics at Concordia University, Montreal, Quebec, Canada. He received his PhD in mathematical physics (1973) from the University of Rochester. His areas of research include square integrable group representations, quantization theory, coherent states and wavelets. He is an author of three books, editor or co-editor of 15 volumes and author of over 100 research articles.

Theodore Voronov [theodore.voronov@manchester.ac.uk] is a Reader in pure mathematics at the School of Mathematics of the University of Manchester, UK. He was born in Moscow and received his undergraduate and graduate degrees from the Department of Mathematics of Moscow State University (MA in 1984 and PhD in 1989). He has worked at Moscow State University (1991–1996), the University of California at Berkeley (1997) and the University of California at San Diego (1998) and has held a position in Manchester since 1999. His areas of research include geometry, topology and mathematical physics, in particular supermanifolds, bracket geometry, quantization, index theorem, super linear algebra and integral geometry.

Tomasz Goliński [tomaszg@alpha.uwb.edu.pl] received his MSc from the University of Białystok. He is a researcher at the Department of Mathematical Physics of the University of Białystok, Poland. Besides mathematical physics, his interests include artistic photography.

DE GRUYTER



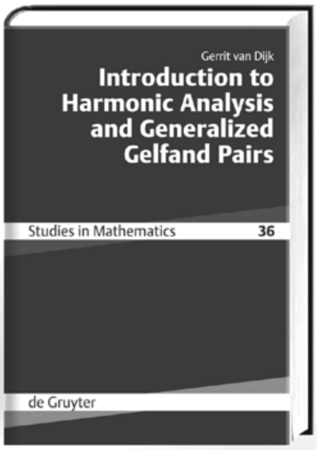
INTEGRAL REPRESENTATION THEORY

Applications to Convexity, Banach Spaces and Potential Theory

By Jaroslav Lukes, Jan Malý, Ivan Netuka, Jiri Spurný.

2010. xvi, 715 pages.
Hardcover. RRP € 99.95 [D] / * US$ 129.95
ISBN 978-3-11-020320-2
eBook. RRP € 111.00 [D] / *US$ 129.95
ISBN 978-3-11-020321-9
(de Gruyter Studies in Mathematics 35)

From the contents
- Geometry of convex sets
- Choquet theory of function spaces
- Affine functions on compact convex sets
- Perfect classes of functions and representation of affine functions
- Simplicial function spaces
- Choquet's theory of function cones
- Topologies on boundaries
- Several results on function spaces and compact convex sets
- Continuous and measurable selectors
- Construction of function spaces
- Function spaces in potential theory and Dirichlet problem
- Applications



Gerrit van Dijk

INTRODUCTION TO HARMONIC ANALYSIS AND GENERALIZED GELFAND PAIRS

2009. ix. 223 pages.
Hardcover. RRP € 69.95 [D] / *US$ 108.00
ISBN 978-3-11-022019-3
eBook. RRP € 78.00 [D] / *US$ 108.00
ISBN 978-3-11-022020-9
(de Gruyter Studies in Mathematics 36)

From the contents
- Fourier series
- Fourier integrals
- Locally compact groups
- Haar measures
- Harmonic analysis on locally compact abelian groups
- Theory and examples of Gelfand pairs
- Theory and examples of generalized Gelfand pairsx

Rene Schilling, Renming Song, Zoran Vondracek

BERNSTEIN FUNCTIONS

Theory and Applications

2010. xii, 313 pages.
Hardcover. RRP € [D] 79.95 / *US$ 124.00
ISBN 978-3-11-021530-4
eBook. ISBN 978-3-11-021531-1
RRP € 89.00 / *US$ 124.00
(de Gruyter Studies in Mathematics 37)

From the contents
- A self-contained and unified approach to the topic
- With applications to various fields of mathematics, such as probability theory, potential theory, operator theory, integral equations, functional calculi and complex analysis
- With an extensive list of complete Bernstein functions.

*for orders placed in North America
Prices are subject to change.
Prices do not include postage and handling.
eBooks are currently only available for libraries/institutes.

DE G

www.degruyter.com

How to theorize music today in the light of mathematics? A musician's point of view

François Nicolas [1]

This article is a translation of the article *Gazette des mathématiciens* [2], n°119, January 2009 [3]. Republished with permission.

"Despite all the experience that I could have acquired in Music,
as I had practiced it for quite a long time,
it's only with the help of Mathematics that I have been able to untangle my ideas,
and that light made me aware of the comparative darkness in which I was before."
Rameau (1722)

If the relationships between mathematics and music aren't limited to their theoretical dimensions – I had the opportunity on the last annual day of SMF [4] "Mathematics and Music" (21 June 2008) [5] to suggest that music and mathematics would gain a lot by considering the *making of* music from mathematics/mathematics from music – then it is obvious, since the meeting of Euler-Rameau in 1752 [6], that theorizing music in the light of mathematics is still the most productive approach. It is clear that 250 years after this meeting, the ways of implementing such a theorization have significantly changed.

The new theoretical configuration that has been reached (at the beginning of the 21st century) has been established for ten years [7] under the name of "*mamuphi*" (for ***ma****thematics-****mu****sic-****phi****losophy*) [8], including a seminar (Ens-Ircam-CNRS), a school (of mathematics for musicians) and various meetings and publications [9].

The *mamuphi* nebula gives us a better analysis of the various ways of theorizing music today in the light of mathematics and allows us to choose the best mathematical tools for that. In *mamuphi*, these tools converge: they are primarily those of algebraic geometry as redrawn by Grothendieck [10], and more specifically those of his topos theory [11]. Mathematicians, musicologists and musicians converge in *mamuphi* to prioritise this toposic approach [12]; however, they diverge on how to implement it.

Precise details

Theorizing music can be done in a number of ways: there are *acoustical, psychological, economical, sociological, ethnological* and *psychoanalytical* but also *philosophical, epistemological*, etc. theorizations of music as well as *mathematical, musicological and musical* theorizations. Only these last three methods are active in *mamuphi*.

If *mamuphi* registers philosophy in its workspace, it is not primarily for possible philwhichosophical theorizations of music – such as that which Adorno has produced. It is rather the following conviction: one can theorize music in the light of mathematics only in the shade of philosophy (more precisely, in the shade of a given philosophy, suitable for the chosen orientation). This shade of philosophy is due to the fact that what "theorizing" means does not come from itself: theorizing does not have a univocal meaning but depends not only on what is to be theorized but almost as much on *which* theorization is used (i.e. of its "subject" as much as its "object"). It is at the precise point where these various designs from theory should be articulated – these "theoricities" – that philosophy will play its part.

In this article, the philosophical aspect of *mamuphi* work will be left aside.

Mamuphi confronts three different ways of theorizing music (in the light of mathematics and in the shade of philosophy): a *mathematician*'s manner, a *musicologist's* manner and a *musician's* manner.

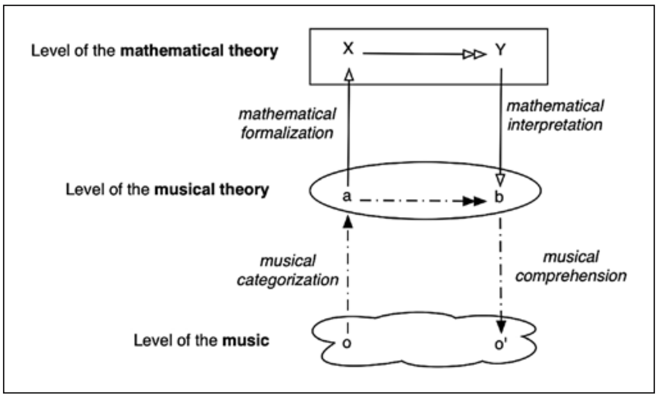
I. *Mathematician*'s manner of theorizing music

The first approach takes up, under contemporary mathematical conditions, the flag of the great Euler. Today, the work of Guerino Mazzola [13] extends this mathematical tradition. Quite naturally, the musician will find in Mazzola's work characteristics that were already present in Euler's theory of music [14].

Theory of a theory

A mathematical theory of music always originates from a pre-existent theory of music and this theory (which is used as a precondition by the mathematician) is of musical nature (as at the time of Euler) or *musicological* nature (like today [15] for Mazzola). Indeed, a mathematician cannot build his theory directly from musical scores (even if he can read these scores well, the mathematician will hardly plan to propose a new idea of them) but will instead work from pre-existent analyses of these scores and therefore from preformed musical theories, which will be used as a basis for the mathematician's work.

One can draw the scaffolding of the theories thus [16]:



For example, Mazzola's theory undertakes, in the course of its vast project, to formalize:

- The theory of counterpoint by Johann Joseph Fux (18th century).
- The theory of tonal harmony by Hugo Riemann (19th century).
- The analysis of the *Hammerklavier* sonata (Beethoven) by two musicologists Ratz & Uhde (20th century).
- The analysis of *Structures I.a* (Boulez) by Giorgi Ligeti.

To theorize mathematically is to formalize and thus to deform

To mathematically theorize an existing musical theory is to formalize it according to one's own mathematical requirements. This formalization, being neither a translation nor a simple transposition [17] thus implies a deformation; it requires a re-handling of the original theory so that the categories common to both theoretical faces will eventually have shifted significances.

One can understand this in the Eulerian design of consonance/dissonance relationships [18]. One finds this point in Mazzola's work, for example in his formalization of "cadenzas" and "modulations": between the mathematical concepts (of *cadenza* and *modulation*) and the homonymous musical concepts, the relations will be of intersection rather than of recovery.

Mazzola retains only two properties of musical modulation:

The existence of harmonic sequences able to affirm a particular tonality (those which will articulate a tonal cadenza, e.g. II-V-I).

The existence of harmonic sequences that are common to two close tonalities (it serves here to use enharmonic chords, carrying tonal ambiguity: for example II-I-IV in C major could be reinterpreted as VI→V→I in F major).

But while proceeding thus,

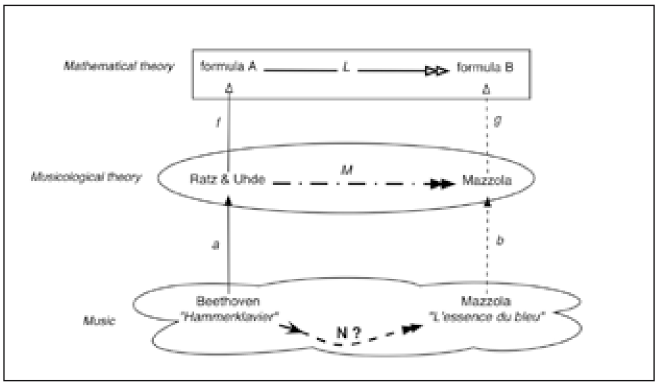
- Mathematical formalization remains indifferent to the order of the harmonic sequences: for example, the sequence VI→II→V→I (a perfect cadenza) and the sequence I→II→V→VI (a broken cadenza) are mathematically equivalent in the same unordered unit {I, II, V, VI}; this interpretation will astonish the musician…
- In the same way, mathematical formalization will consider that its "cadenza" {II-V} is equivalent to its "cadenza" {VII} since this last chord (B-D-F in C major), which can only appear in this tonality, is alone (among the other chords) in affirming C major. Again here, the musician will not recognize his music, his tonalities and his modulations; if, for the musician, the sequence IIthe sequence V is the gesture of a musical cadenza then the simple statement of VII is not the same since, quite to the contrary, this chord constitutes the prototype of the polymorphic pivot-chord [19], which is common to many tonalities.

In short, the musician does not recognize the cadenzas and modulations in the homonymous concepts of Mazzola, just as they would not recognize their own harmonic functions in the Eulerian formalization of musical pleasure.

This torsion concerns a structural law; it does not come from a mathematician's laziness or incompetence. Cohesion of musical experimentation and coherence of mathematical formalization – musical logic and mathematical logic – are clearly *two* separate things. They can approach and enter into resonance but they cannot amalgamate nor even overlap. [20]

Let us give another example in the way in which mathematical theorization tends to deform the musical neighbourhoods that it undertakes to formalize.

To test the mathematical formalization of a musicological theory (by Ratz and Uhde) of the sonata *Hammerklavier* (Beethoven), G. Mazzola wonders whether it is possible to find a musical equivalent with a mathematical formula such as B, formula deductible (within the framework of its mathematical theory) from formula A (which formalizes the sonata as theorized by the musicologists).



To carry that out, Mazzola composes a piece for piano (*L'essence du bleu*) whose musical analysis (arrow *b*) is carried out according to the same musicological principles as that for the sonata of Beethoven (arrow a) and then mathematically formalized (arrow *g*) according to the same logic as that which was used for the analysis of the Beethoven sonata (arrow *f*) and therefore leads to a related formalization B (arrow *L*) with starting formalization A.

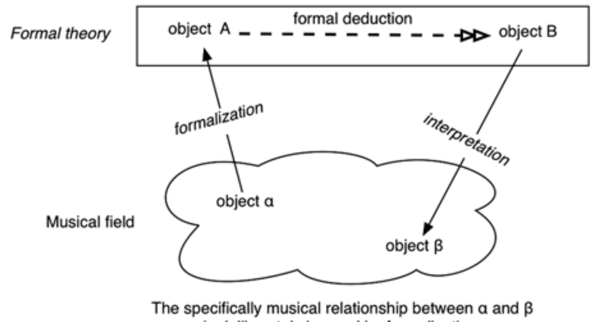
It is understood that this device can ensure that there exists, in the theory of Ratz & Uhde, an arrow M such that the top rectangle commutates (i.e. such that g°M = L°f), *since the new* piece of music (*L'essence du bleu*) was made up precisely so that its analysis is related (by *M*) with the analysis (by Ratz & Uhde) of Beethoven's sonata.

But the musician will address an additional question to the mathematician: does there also exist a kind of arrow N – an arrow which is specifically musical (and either musicological or mathematical) – such that the bottom rectangle (and thus also the complete rectangle) commutates i.e. such that $b°N = M°a$ (and $g°b°N = L°f°a$)? In other words, would this theoretical construction induce a bringing together of *Hammerklavier* and *L'essence du bleu* in a specifically musical nature?

For the musician – who is the only one able to come to a conclusion properly about the musical existence of such a relation [21] – such an "arrow" N does not exist in this precise case. Examining the two partitions (which we will not do here) indeed proves that there is hardly any musical relation between the sonata of Beethoven and the work composed ad hoc by Mazzola, which is not surprising. The fact that musicological analysis can bring closer (arrow M) the analytical structures of two musical works is not enough to musically connect these two works (just as two buildings could not be architecturally related as significant spaces just because their plans connect the same row of rooms or because one could count the same number of columns on their frontages).

This raises the question of how much such a mathematical theorization deforms the musical world; by founding formal neighbourhoods that do not have a musical counterpart, it brings closer musical objects that remain for the musician extremely distant, just as it puts distant and separates what for the musician constitutes a neighbourhood (see the families of harmonies built by Euler on the basis of his scale of softness – they separate harmonies that are musically close and bring closer musically distant harmonies).

This deformation of musical topology by mathematical formalization is not due to negligence of the mathematician. It is an effect of structure, which is due to the following point that legitimizes the logico-mathematical construction of a "model theory": mathematics takes the field (which it will undertake to formalize) as a discrete space of objects (they are their own neighbourhood). Mathematical formalization will thus be a formalization of the objects (in this case musical) but by no means the musical relations between the objects, relations which are *voluntarily* ignored [22]. The purpose of this formalization will be to build a new (theoretical) type of space where the new (mathematical) objects will be connected by deductive relationships between musical field and mathematical theory, by formalization and interpretation of the (musical and mathematical) objects but by no means their respective relationships.



The specifically musical relationship between α and β is deliberately ignored by formalization.

Technically, the theorization in question will thus not be *functorial*: formalization and interpretation will not be "functors" between the two categories. [23]

Thus, if the specific interest of any formalization precisely holds with the contrasting relationship between a starting field that is formally taken as *discrete* (without immanent relationships) and a theoretical field where the objects will be connected by formal deductions, it goes without saying that the musical relationships (that the musician knows well but that the theory is unaware of) will appear to the musician as deformed and not reflected by the theoretical construction in question. That the musician regards such a mathematical theorization with reserve will be inevitable.

A theory coordinating a sheaf of formalizations

To mathematically theorize music makes use of a great diversity of formalizations, which the mathematician will have to coordinate if he wants to build a theory of music and not to accumulate a cluster of local operations.

Mazzola carries out this coordination within the framework of Grothendieckian topos theory [24]. Euler, of course, did not have such a pre-existent framework and his theorization of music was useful to him – *inter alia…* – in unifying the mathematics of his time (then in a vast movement of diversification).

In both cases, a mathematical theory of music is not satisfied to collect disparate formalizations and to deal with their mathematical unification. It is understood that this requirement is specifically a mathematical desire and is not musical at all.

A theory serving mathematics rather than music

Such mathematical theories, which aim at mathematics much more than music and which are the subjective business of mathematicians, cannot be of real use for the *working* musician. Musicians – craftsmen of their art – will not be interested in these mathematical theories. Quite simply, they will not read them – it is not only that they would be perplexed by the technical detail; it is more that they do not need such a theory, in practice or in a possible desire for theorizing music.

A theory producing new knowledge on music

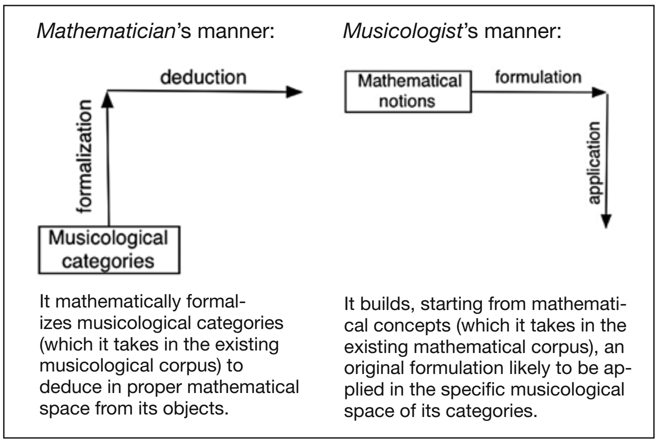
This by no means implies that such a mathematical theory of music remains uninteresting for the musician, at least not for the thoughtful musician [25]. For example, the formalization of Mazzola leads to this remarkable result: the theory of counterpoint by Fux and the theory of harmony by Riemann prove narrowly related by this theory according to the geometry of intervals. These two theories (of Fux and Riemann) remain separated, however, by chronology (respectively of the 18th and 19th centuries) and by the practice of musicians (in music, counterpoint and harmony give place to disjoined lessons, without theoretical unification [26]).

Thus, this mathematical theory reveals structural properties that have been up to that point unperceived by the musician and the musicologist. This theory makes it possible to extend the knowledge of music even if it does not make it possible to invent anything with regard to musical practices.

It is for that reason that this mathematical type of theory will interest musicologists rather than musicians if these groups can be classified as follows: musicologists trouble themselves with knowledge in externality, with music conceived like an object that is already there, whilst musicians consider knowledge in interiority, with music that they make.

II. *Musicologist*'s manner of theorizing music

The musicologist's manner of theorizing music operates in contrary to the mathematician's manner; it will develop an extant mathematical theory to apply to a musicological question. One can present the contrast of these two dynamics in the following way:

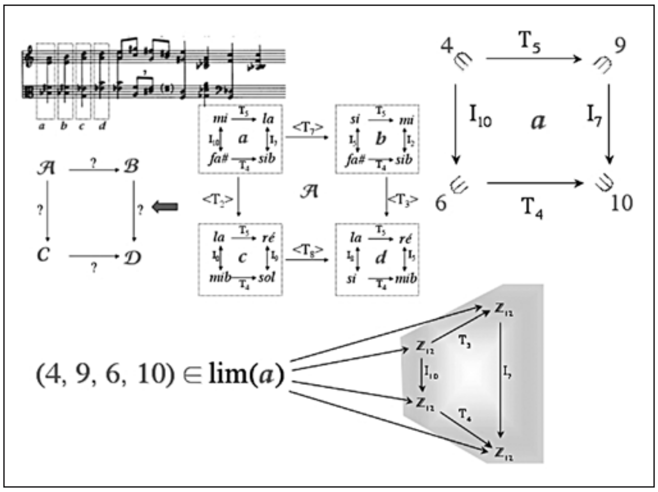


Altogether, the musicologist's manner of theorizing music with mathematics consists of building a "mathematical model" for a musicologically given problem. If a mathematician's formalization can be conceived like a "mathematization" of the music, the musicological approach will consist of a mathematical "modelization" of musicology [27]. Thus this last approach prioritises, in mathematics, its capacity of calculation rather than the power of its concepts.

This kind of musicological theory is committed to what is called "a computational musicology". In *mamuphi*, the most successful proponent of this orientation is Moreno Andreatta. [28]

The work of musicology is generally carried out in pure algebra (and primarily group theory) but an important part is now based on a modelization in term of topos. For example, this relates to what *music theory*, since David Lewin, has called the "transformational" approach of pitch networks.

This is initially a question of segmenting a score into pitch groups – or "chords" – connected to each other (*a transformational network*) by musical operations of *transposition* and *inversion* in turn to produce a total recovery of the score concerned. The resultant of this constitution of abstract space is the transformations in time (*a transformational progression*) of the constitutive groups of the network. This way of insisting less on the particular nature of the gathered pitchs than on the structure of the transformations to which these groups give rise creates a musicological problem that lends itself quite naturally to modelization of a categorical type, again prioritising the relations between objects.



Modelization of a musicological analysis by D. Lewin of Schoenberg's op. 11 n°2

More precisely, the musicologist, anxious to enumerate and classify these musical structures ("Klumpenhouwer networks"), will model them in a toposic way (see the *limit* in the last diagram). The result, once implemented by means of a computer [29], will be able to illustrate the good strategies of analysis concerning networks working in a score. Thus musicological modelization by topos will lead directly to a computer-assisted musicological analysis.

The feedback effect of musicology on mathematics should also be mentioned. Certain questions, addressed by this formalization to mathematics, will inspire new mathematical problems. This is what Moreno Andreatta likes to call a "*mathemusical*" problem: a musicological problem addressed by mathematics such that its formalization gives rise to new theorems, which open new musicological applications. [30]

III. Musician's manner of theorizing music

There remains a very different third approach of theorizing music in the light of mathematics: that of the musician – i.e. the *working musician* (there is not really any other type!).

The musician is distinguished from the two preceding orientations because theorization will not aim at producing a "theory". The theorization will instead relate to what Louis Althusser has called "theoretical practice", i.e. an intervention whose aim is not the constitution of a theoretical system that is stable and transmissible but the release of an idea for the music. [31] For this reason, one is able to say that the musician's theorization is an *ideation*. [32]

Methodologically, the recourse to mathematics to theorize music will thus be carried out under the heading of what will be called, following Gaston Bachelard, an *experimentation* of thought. It will be, at the same time, formalizing and interpreting musical categories and mathematical concepts to put a discursive thought-proof together with mathematical coherence. The following example is borrowed from the author's own work.

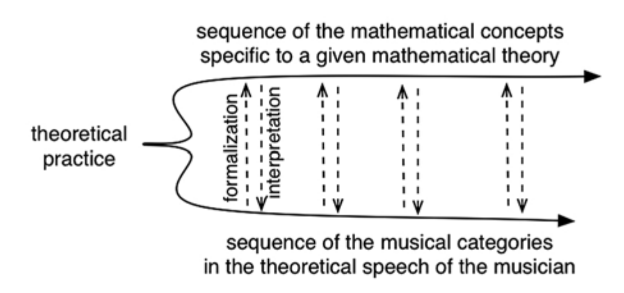
To theorize a Music-*world like topos…*

Let us suppose that a thoughtful musician feels the need to theorize how music can form a particular world; a worthy motivation is an opposition to the new practice of using the label "the musics" instead of the ancestral musician's expression "music". [33]

The author would like to support the thought that there exists a world of music (and not only one area, which one can roughly delimit as a general universe) and only one, and that this world, though internally diversified (like any world!), remains connected (all that occurs in some place in this world relates potentially to any other place). In short, the musician would like to be able to say of music what Alain Connes says of mathematics: "*there is only one mathematical world*" [34] and "*this mathematical world is connected*" [35].

But how can such an idea of one and only one musical world be founded? The musician, then, will be able to turn to mathematics while saying that [36] "the Grothendieckian concept of topos provides a strong contemporary mathematical idea of what is a world; thus let us put our idea of a musical world-proof against this mathematical idea of topos".

The musician will then start a theoretical practice that will simultaneously explore the double sequence of mathematical concepts and musical categories according to the following progression:



In our example (of how to theorize, in the light of the Grothendieckian mathematics of topos, music like a world), this experimentation [37] will lead the musician to the following tasks:

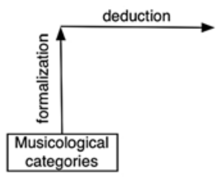
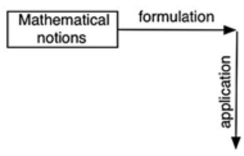
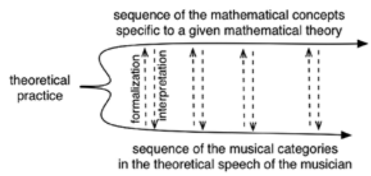
1. Formalize a piece of music like a sheaf of executions of its score.
2. Formalize the library of the scores of music like a *site* of its *quodlibets*.
3. Formalize the world of the music like a *category* of the works extracted from this library.
4. Formalize the world of the music like a *topos* of all these works-sheaves.
5. Draw, in the course of the work, all useful and relevant conclusions concerning the musical objects and their relations.

It is accepted that the author's experimentation of mathematical concepts will hardly interest mathematicians, since the effects of such a theorization will remain intrinsically musical. This experimentation will no more interest the musicologists who will not recognize the procedures regulating their "objective" production of knowledge. [38]

There is a difficulty here, specific to confrontations that animate the *mamuphi* meetings. It does not follow that the theorizations of the mathematician, the musicologist and the musician can be mutually interesting. Thus the specific challenge of the *mamuphi* project is to put in the resonance of these distinct theoricities, objectively as well as subjectively (it is here that the shade of philosophy is required).

On the whole…

Let us summarize the three main trends (see table at the

	Mathematization or mathematician's *formalization*	**Modelization** or musicologist's *application*	**Experimentation** or musician's *theoretical practice*
			
Aims of this theorization:	to make mathematics while widening the power of mathematics and consolidating the components	to produce, in objectifying externality, new knowledge on music	to deepen, in subjectifying interiority, musical knowledge
Result of this theorization:	(mathematical) theory of music	(musicological) theory of music	(musician's) idea of the music
The music is:	an indirect origin (via musicology)	an indirect target (via musicology)	a sensible space of thought
The mathematics is:	a target	an origin	a conceptual space of thought
The mathematics concerned takes the form of:	*theories*	*formulas & equations*	*concepts*
The music-mathematics ratio prioritisation:	formalizations	interpretations	resonances, therefore *mathems*

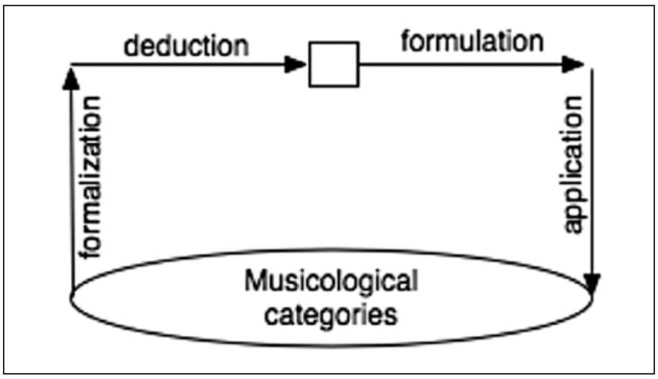
bottom of this page).

Relations between these three theorizations

Even if one understands where the personal preferences of the author of this article (musician) lie, it is clear that each orientation discussed here has its own coherence and there does not exist any overlap [39], which would indicate an hierarchical basis to the three theorizations. However, the preceding table shows that the three orientations can be broken down to 2+1 in three ways.

***Complementarities* – mathematicians/musicologists**

Firstly, the positions of the mathematician and the musicologist arise in our table like duals – they are complementary. This complementarity inspires a new approach (this time *mixed*) of theorizing music, an approach that



connects *mathematization* and *application* [40]:

***Complicities* – mathematicians/musicians**

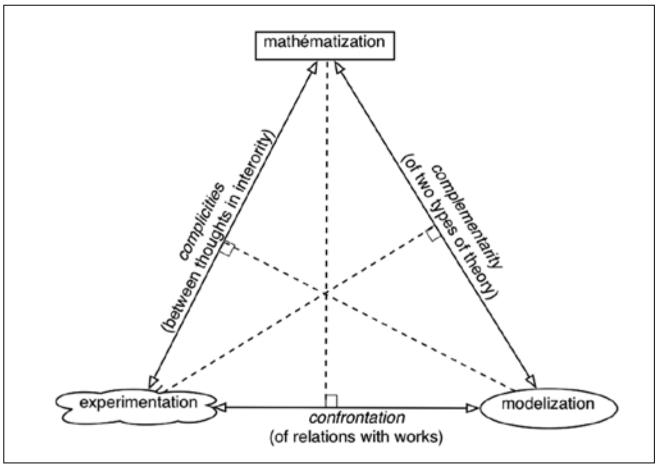
Secondly, one notices that the orientations of the mathematician and the musician give more intimate complicities between thoughts in interiority than the musicological practice of technical modelization, which prioritizes the computing power of mathematics and exteriorizes the "objective" dimension of music.

***Confrontations* – musicians/musicologists**

Thirdly, the theoricities of the musicologist and the musician meet around scores since they give the same direct attention to them. This will maintain between them what will be called here, with a nice euphemism, a healthy competition.

General geometry

The musician's theory is "orthogonal" to the complementarity of the mathematical and musicological theories, as musicological modelization is orthogonal to the complicities between thoughts in interiority (mathematician and musician) and as mathematization is orthogonal to the musician/musicologist confrontations relating to scores, and the *mamuphi* geometry that proceeds from these relations could thus be drawn (see figure next column).



A counterpoint…

On the whole, and according to a musical metaphor, the relations between the three theoricities give to the polyphonic development of *mamuphi* the pace of a counterpoint.

As musicians know well, it is the dissonances – not the consonances – which make music, and these dissonances, at least since Schoenberg, do not need to be solved to remain musical.

Thus, this musician will be able to wait for the best of these *mamuphi* dissonances and orthogonalities. Here, it was necessary for him to make them clearly heard and so to restore them to *mezzo-forte* (*mf*) rather than *pianissimo (pp)*…

Notes

[1] Compositor (*École normale supérieure/Ircam*) http://www.entretemps.asso.fr/Nicolas.
[2] http://smf.emath.fr/Publications/Gazette.
[3] http://smf.emath.fr/Publications/Gazette/2009/119/smf_gazette_119_35-49.pdf.
[4] *Société mathématique de France*: http://smf.emath.fr.
[5] http://smf.emath.fr/VieSociete/JourneeAnnuelle/2008, http://smf.emath.fr/VieSociete/JourneeAnnuelle/2008/Resumes.html.
[6] *Gazette*, July 2008, n°117: http://smf.emath.fr/Publications/Gazette/2008/117, http://smf.emath.fr/Publications/Gazette/ 2008/117/smf_gazette_117_35-47.pdf.
[7] The opportunity to start *mamuphi* was provided by an initiative (at the end of 1999) of the EMS (*European Mathematical Society*), which, within the framework of its *Diderot forum* (http://emis.math.ecnu.edu.cn/etc/diderot4.html), had chosen "logic" as the issue to be debated in workshops with Ircam.
[8] http://www.entretemps.asso.fr/maths.
[9] Consult the two inaugural books of *mamuphi* for more details: *Mathematics and Music (A Diderot Mathematical Forum)*; ed. G. Assayag, H. G. Feichtinger, J. F. Rodrigues; Springer-Verlag, 2002 – http://www.maa.org/reviews/mathmusic.html. *Penser la musique avec les mathématiques?*; ed. G. Assayag, G. Mazzola, F. Nicolas; Delatour, 2006 – http://www.ircam.fr/598.html?&tx_ircamboutique_pi1[showUid]=172&cHash=bb50400732.
[10] http://www.entretemps.asso.fr/Grothendieck, http://www.grothendieckcircle.org.
[11] I.e. the fifth of the twelve "great ideas", which were released in "*Récoltes et Semailles*" (2.8) http://www.math.jussieu.fr/~leila/grothendieckcircle/RetS.pdf.
[12] My own reference books on the matter are *Topoi. The Categorial Analysis of Logic* of R. Goldblatt (North-Holland, 1984) and *Sheaves in Geometry and Logic. A First Introduction to Topos Theory* of Saunders Mac Lane & Ieke Moerdijk (Springer-Verlag, 1992).
[13] See the two reference books: *The Topos of Music*, Birkhäuser, Basel, 2002. *La vérité du beau dans la musique*, Delatour, Paris, 2007.
[14] Here, one will not systematically present this vast mathematical theory. It is here only a question of reading this theory as a musician, i.e. remaining more attached to divining its matter and to distilling its mathematician subjectivity than to exploring the proper mathematical depth.
[15] Musicology was invented only during the 19th century, under the

double influence of German historicism and French positivism…

[16] Let us specify that this "diagram" (and those that follow) only gives an indication of a guiding idea. Thus it has only illustrative value: the points and arrows that appear here have only metaphorical relationships with the objects, morphisms and functors of category theory.

[17] The philosopher Charles Alunni, co-organiser of the *mamuphi* seminar, has proposed regarding it as a *tra(ns)duction*.

[18] *Gazette*, July 2008, n°117 (op. cit.).

[19] Technically, this chord VII concerns a diminished seventh. This chord illustrates tonal uncertainty by avoiding any cadential logic. Thus the less (musically) "cadential" chord corresponds, in mathematical formalization, with the more (mathematically) "cadential" chord…

[20] As indicated, it is at this place that the shade of philosophy is necessary. Music and mathematics are radically *two*, without any possibility - other than in a (neo)-positivist or scientific way - of uniting them. That is due to the irreducible singularity of the musical work of art. As always, it will be an axiomatic choice:
– One supports that "there are works of art" (Hegel) and that these works are in art the true subjects; in this case mathematics could only be unaware of this specificity to analyse work-subjects only by their ontic dimension (that of simple "pieces of music").
– Or one supports that there is no meaning to distinguishing between pieces of music and musical works, that there is no place in music for a figure of "subject" for something like a "musical subject"; in this case mathematics will be able "to seek" to formalize music completely in the same way that it can legitimately seek to completely formalize the movement of planets, the reproduction of ants and the food preferences of human animals. But does such a project (to reinstall music under the supervision of mathematics) constitute for mathematics a *real* ambition? Without coming to the same conclusion about the question of works, Euler knew, in all cases, to avoid such covetousness and to respect the autonomy of the world of the music, without losing (quite to the contrary!) the power of thought suitable for mathematics.

[21] We point out Euler: "*In music, as in all the fine arts in general, it is necessary to be aligned on the opinion of those which have at the same time an excellent taste and much of judgment. Consequently it is necessary to hold account only of opinion of people which, having received nature a delicate ear, perceives with accuracy all that this body transmits to them, and is able to judge some in a healthy way*." It is thus a question "of consulting the metaphysicians [in this case musicians] that this search relates to." (*Tentamen*…, chap. II)

[22] From this point of view, the particular case where mathematics formalizes a pre-existent "empirical" theory (in this case a musicological one) – a field that is not "discrete" this time, since it is equipped with internal relations (of proximity, distance, sequence, etc.) and thus with neighbourhoods not reduced to only one point – constitutes only one alternative, since formalization and interpretation will continue to relate only to objects and not to morphisms. The theorization thus considered will not produce more functors between musicological and mathematical theories: the musicological theory being used as a starting field remains too empirical to be truly formalizable in a mathematical category.

[23] A fortiori, one cannot have adjunction between a musical field and a mathematical theory. An important aspect of the *mamuphi* internal debates relates precisely to this point…

[24] Let us note its systematic reinterpretation of categorial morphisms like addresses (similar to theoretical informatics), x-y being rewritten as x→y...

[25] The musician tends to become pensive "when the music stops" (Th. Reik), the musician finding himself temporarily vacant from the musical world. It is this moment when he is naturally led to reflect on what arrived to him, to charge his musical experiment and to encourage him to continue his to and from (in and out of the Music-world).
– Like mathematicians, musicians are regularly subjected to nihilist temptation: the temptation of "What good is it?", "in vain" (Nietzsche). That the abandonment of their cause often takes the form not of a desertion but of an academization does not take anything from the fact that it is indeed a subjective resignation.

[26] The unification is carried out only practically, for example by a school choral exercise and a fugue…

[27] Let us recall that the definition of "model" gets busy here, in this case contrary to the meaning that this word has in a (logico-mathematical) "model theory". In "model theory", the word "model" indicates the original to copy; in "mathematical modelization", the word "model" indicates the reduced model, the model to be interpreted. For a discussion of the philosophical meaning of this (neo-positivist) inversion, one can read the book of A. Badiou: *The Concept of Model*, translated by Zachery Luke Fraser & Tzuchien Tho (Melbourne: repress, 2007).

[28] Let us indicate that this computational musicology finds a natural extension in the seminar, related to *mamuphi*, that is held in Ircam under the name *MaMuX*: http://www.ircam.fr/equipes/repmus/mamux. One will find many contributions to this new type of musicology in the *Journal of Mathematics and Music*: http://www.tandf.co.uk/journals/titles/17459737.asp.

[29] To see *From a Categorical Point of View: K-Nets ace Limit Denotators* (G. Mazzola and Mr. Andreatta, *Prospects of New Music*, 44-2, 2006) and, more generally, works of the *Musical representations* team in Ircam, go to http://recherche.ircam.fr/equipes/repmus.

[30] It would thus, in my opinion, be a question of a "mathemusicological" problem…

[31] This *musician's* idea of music is distinct, of course, from the musical idea: that which, in the course of the work, takes the shape of a musical object, for example a theme.

[32] I call this musician's ideation "musical intellectuality". We thus do not mislead on the theoretical work of Rameau, the pioneer of musical intellectuality. Its evolution stresses that it was a question for him of intervening theoretically for the benefit of a certain (harmonic) idea of music, badly established in his time; his "theory" was thus a (theoretical) manner of pleading his cause of an "harmonic" musician rather than a melodic one by giving to this "theory" strong bases, rooted in the rationality (in particular Cartesian) of his time.

[33] It is about a musician's concern for the unity of music. This concern is equivalent to the Eulerian concern for preserving the unity of mathematics over the beneficial diversity of its practices.

[34] *A View of Mathematics*: http://www.alainconnes.org/docs/maths.ps.

[35] *Les déchiffreurs*, p. 14, Belin, 2008.

[36] The musician will be aware of this operation from the book of Alain Badiou (*Logics of Worlds: Being and Event, Volume 2*, translated by A. Toscano; New York: Continuum, 2008) since this book supports that the philosophical concept of world must be established today under the condition of the mathematical category of topos.

[37] For more details, one can refer to a first presentation of this *work in progress*: http://www.entretemps.asso.fr/Nicolas/2008/Faisceaux.htm.

[38] So that a musicologist can be interested in an "idea musician", it is necessary initially for him to clarify it in a "musicological object"…

[39] Philosophy does not concern more from a Sirius point of view…

[40] This approach operates very directly in *mamuphi*: the works of Mazzola aim to informatically implement the theory and stress the computational repercussions of the mathematical theory, and the musicological work of Andreatta has roots in the Mazzolian theory of the music.

François Nicolas [fnicolas@ens.fr], composer and PhD in philosophy, is a former student of the École Polytechnique. He studied organ, piano and musical writing. His musical experience led him first to the jazz scene before he reached contemporary music. He has worked with IRCAM, perfecting the Modalys software program (synthesis via physical models) and the Timée (multi-source loudspeakers). Jobert publishes his works. F. Nicolas associates composition with theoretical work. Co-founder in 1986 of the Entretemps revue, he holds at Ircam and Ens-Ulm the Samedis d'Entretemps meetings (meetings on books about music), mamuphi *seminar (mathematics, music and philosophy) and the Entretemps seminar (on contemporary music). From 2003, he works also at the École Normale Supérieure (Ens-Ulm) in Paris as associated professor and associated researcher.*

Applied Platonism

Zvi Artstein

In some colloquium lectures I have given I have posted a slide reading "Nature is a good approximation of Mathematics" and, on other occasions, "Nature is a very good model of Mathematics" (Doron Zeilberger kindly incorporated it in his list of quotes – see shttp://www.math.rutgers.edu/~zeilberg/quotes.html). The audience generally liked it, taking it as a reasonable joke. That made me explain that no joke was intended, to which many responded: "Oh, you are a Platonist". I denied it but didn't always manage to explain my point. While following the recent series of articles about Platonism in the EMS Newsletter (Davies, June 2007; Hersh, Mazur, June 2008; Mumford, Davis, December 2008; and Gardner, Davies, June 2009), the matter has become clear to me: I am an applied Platonist.

A striking appearance of applied Platonism will be displayed in the next paragraph, while here we start with some straightforward examples. Quoting Gardner (EMS Newsletter, June 2009): "The primality of 17, in an obvious way, is out there in the behaviour of pebbles in much the same way that the elliptical orbit of Mars is out there..." Now, the orbit of Mars is not an ellipse; in fact, no star or planet follows an exact elliptical orbit. Rather, the actual orbit of Mars is an *approximation* of a mathematical ellipse. The elliptical orbit is a mathematical entity. I propose calling it a mathematical reality (no need to recruit Plato). Planet Mars exhibits a good approximation of the mathematical ellipse. An even better approximation would be exhibited by a body that was the sole planet in a solar system – so the mathematics predicts. A perfect match with an elliptical orbit would be in a universe consisting of two bodies, in the coordinate frame of one of them. We shall never be able to check if this perfect approximation is indeed realizable. With regard to the primality of 17, a perfect approximation is achieved when the primality of 17 is checked by trying to place 17 pebbles in a non-trivial rectangular grid. Still, there is an advantage in viewing that as a perfect approximation rather than as *the* mathematical reality. For instance, alluding to the Hersh-Gardner dispute, I would propose that climbing 5 stories and reaching floor 13 from floor 8 is a perfect approximation of the mathematical reality of $8 + 5 = 13$, whilst reaching floor 14 from floor 8 in a hotel that skips floor 13 would be a lousy approximation of the said mathematical reality. I once stayed in a hotel that marked floor 13 as 12.5; this constitutes a better approximation to the mathematical reality of addition of whole numbers. The mathematical reality that has been discovered or invented, depending on whether Brian Davies' suggestion to let Platonism die is adopted or not (see Davies, EMS Newsletter, June 2007), in the two examples could be described as abstraction, or idealization, of the physical world. Likewise, the Mandelbrot sets form an abstraction of self similar-like phenomena in nature. In particular, the abstract mathematics mentioned here could, in principle, become a physical reality. In that respect, applied Platonism subsumes the traditional approach of idealization and abstraction of physical phenomena.

But applied Platonism goes beyond that. Mathematical reality may defy physical reality and even contradict basic physical laws, yet nature may exhibit a good approximation of it. A prime example of such phenomena was alluded to in the interview with John Ball (EMS Newsletter, March 2009). The specific mathematics, developed by John Ball and Dick James[1] examines the variational problem reflecting the energy minimum of an elastic body. The mathematical minimizer contradicts the possible physical reality; indeed, the mathematics requires that the local arrangement of the body exhibits, simultaneously, distinct phases distributed according to prescribed probabilities. This is not physically possible, yet the patterns exhibited in nature form a very good approximation, interpreted in the so-called convergence in the sense of Young measures. Although the mathematical solution is not physically possible, the approximation is so accurate that it carries a predictive power. And the Ball-James solution is not the only appearance of nature approximating mathematical reality that contradicts the basic laws of physics.

I wish to emphasize the approximation issue. The approximation process is not necessarily symmetric. To explain what modifications in a given mathematical model will produce a better description of a prescribed physical reality is not the same as explaining what elements of a given physical reality make it deviate from a prescribed mathematical model. Both processes are relevant when employing mathematics in the understanding of nature. The relevance of the analysis, qualitative or quantitative, of the deviations of nature from a mathematical reality, and the predictive power of it, is what an applied Platonist believes in. In particular, an applied Platonist believes that nature can be interpreted as an approximation of mathematical statements, even when the mathematical statement may not be feasible in the physics.

Is an applied Platonist a Platonist? To this end, one should have a clear description of what Platonism is. Such a description is not available (and has not emerged, in my opinion, from the recent debate in the EMS Newsletter). However, regardless of whether Platonism is rooted in the belief that mathematical truth is culture-free or not, once a person understands that a mathematical statement stands alone, and is relevant to physical reality via an approximation, that person is an applied Platonist. It

1 See *Archive Rational Mechanics and Analysis* 100 (1987), 13–52 and *Phil. Trans. R. Soc. Lond A* 338 (1992), 389–450.

is interesting to note that prominent physicists warn us from adopting the mathematical statements as being the physics, rather than forming a tool to depict facets of, or providing a computational tool to, physics (see David Mermin, "What's bad about this habit" in Physics Today, May 2009). Thus, physicists may consider the mathematics as a tool only, (pure) mathematicians may consider mathematics as the real thing and applied Platonists see the connection between the two realities.

I have refrained from referring to the issue of mathematics versus applied mathematics, yet a comparison is called for. In my opinion there are no applied mathematicians. There are mathematicians who care about the mathematical aspects of the science they are doing (hence including most of the so-called applied mathematicians) and there are those that apply mathematics but care only of the application. The deviation of physical reality (nature) from the mathematical reality (Plato or not) is primarily a mathematical issue. Those I know that actively follow applied Platonism are mathematicians.

Does the notion of applied Platonism shed light on mathematical Platonism? It may. Consider, for instance, the question of whether there exist chapters in mathematics that cannot take part in the exploration of nature via the applied Platonism route. Would measurable cardinals form such a chapter? No, or so I believe; any mathematical pattern that emerges in the human brain is, potentially, a component in the applied Platonism paradigm. But this is probably the source of a new debate.

Zvi Artstein [zvi.artstein@weizmann.ac.il] received his PhD from the Hebrew University of Jerusalem under the supervision of Robert J. Aumann. Following a post-doctoral position at Brown University he joined the Weizmann Institute of Science where, on top of his research, he held several administrative positions. He is currently a member of the Execute Committee of the EMS.

Nominalism versus Realism

David Corfield

This brief essay will certainly not be an interpretative exercise to discern Plato's philosophical views on the nature of mathematics. That would require the kind of subtle exegesis that can be found in Colin McLarty's "'Mathematical Platonism' Versus Gathering the Dead" (McLarty, 2005), which makes an excellent case for saying that the position closest to contemporary Platonism is voiced not by Socrates but by Glaucon, his interlocutor for much of *The Republic*. Instead, the essay will discuss what should be made of two different uses in contemporary philosophy of mathematics of the distinction: nominalism *versus* realism. First, a description is needed of what will be called here the *external* and *internal* forms of this distinction.

As the term suggests, participants in the *external* nominalism/realism debate look on at mathematics from the outside. They see a great uniformity amongst different pieces of mathematics, whether it be adding 2 and 2, calculating the Fourier transform of a function or proving Fermat's Last Theorem. The philosophically salient activity of mathematicians appears to the externalist to be that of establishing the truth of certain propositions. The question then is what makes these propositions true. Such statements appear to refer to entities and to state properties that hold for them. But what then are these entities? Those who take them to be existing abstract objects are termed 'realist' or 'Platonist'; those who would think we've been misled by the outward grammar of the propositions and that we need make no reference to any form of entity are termed 'nominalist'.

Now when it comes to the *internal* nominalism/realism distinction, it is not so much that we need to take sides in a debate. What interests those who make the distinction above all is rather the thought that some pieces of mathematical theory are worthier than others. Let us explore this thought in the hands of Imre Lakatos. Forgetting all you may have heard about Lakatos as one of the first social constructivists about mathematics, consider the following quotation:

> *"As far as naive classification is concerned, nominalists are close to the truth when claiming that the only thing that polyhedra have in common is their name. But after a few centuries of proofs and refutations, as the theory of polyhedra develops and theoretical classification replaces naive classification, the balance changes in favour of the realist." (Lakatos, 1976: 92n)*

This Footnote appears in Lakatos' famous dialogue 'Proofs and Refutations' and asserts his claim that if we properly subject our mathematical reasoning to a thorough toughening-up process ('dialectic' to give it its fancy name) then we can arrive at more adequate conceptions. Poincaré's late nineteenth century definition is better justified than those of his predecessors earlier in the century.

So it is not that an individual is an internal realist or nominalist, just that there is a distinction to be made within the practice of mathematics between different assertions, definitions and ideas. Indeed, the distinction can be applied to one's own work. In the letter he wrote to his sister, the philosopher Simone Weil, André Weil (1940) describes how, when devising the axioms of a uniform space, it resembled to him the activity of a sculptor working with snow – the material did not resist. What unites uniform spaces is the mere fact that they all satisfy some axioms, rather than that they share a common essence. By contrast, in a further fragment of letter, attached in his *Collected Works* to the previous letter, Weil likens his work on the analogy between function fields and number fields to a sculptor working on hard stone, releasing the form from its prison. Now we are reaching for essential properties behind varied surface appearances.

So how are we to characterise the grounds for such an internal distinction? In the case of the sciences, we might imagine that we are right to make internal-to-practice distinctions between the reality of oxygen and non-reality of phlogiston, or perhaps between the reality of the inferiority complex construct and non-reality of the Oedipal complex, and we are right precisely because of external reality. We believe our concepts to have grasped something in the world. We like to ground our sense of the internal reality of aspects of a practice on external reality. But what can mathematicians count on to play this role? Physical interpretation may be thought to warrant the reality of some mathematics – e.g. natural numbers and beads or group representations and particles – but not all. We might then extend this warrant to include realisation in a game governed by symbolic rules. But in doing this we open the floodgates to all formal manipulation. Even in the case of recognised games, we might say that were I to prove something about chess, it is made true by the set of possible legal games. However, according to the internal distinction I'm alluding to, chess is not a part of mathematics, or at any rate far from what is most real. If the axioms of uniform spaces were devised so easily, then the arbitrariness of the rules of chess must strike us as all the more contingent. Where we expect a real concept to prove its mettle by leading us on to surprising discoveries elsewhere, Vaughan Jones' towers of subfactors and knot invariants being a good case, we don't expect a result concerning chess to be relevant to anything else.

Staying on the theme of games, the mathematician Alexandre Borovik once told me he thinks of mathematics as a *Massively-Multiplayer Online Role-Playing Game*. If so, it would show up very clearly the difference between internal and external viewpoints. Inside the game people are asking each other whether they were right about something they encountered in it – "When you entered the dungeon did you see that dragon in the fireplace or did I imagine it?" But someone observing them from the outside wants to shout: "You're not dealing with anything real. You've just got a silly virtual reality helmet on." External nominalists say the same thing, if more politely, to mathematical practitioners. But in an important way the analogy breaks down. Even if the players interact with the game to change its functioning in unforeseen ways, there were the original programmers who set the bounds for what is possible by the choices they made. When they release the next version of the game they will have made changes to allow new things to happen. In the case of mathematics, it's the players themselves who make these choices. There's no further layer outside.

What can we do then instead to pin down internal reality? Let us take as a starting point something that has often been noted about mathematics – its conservatism. Mathematics has been going on for an awfully long time. We recognise mathematical thinking in a culture that, over four millennia ago, could ask for the length of a field given that 11 times its area added to 7 times its length is 6 and 15/60, following a recipe which gives the answer 30/60 or 1/2. But note that this is not just a case of a result being recognisable by us. We also take it to be the kind of thing a young person should learn today. That mathematical topic is still important. Anyone with any hope of becoming a mathematician had better understand the formula for the quadratic equation. It may be one speck in a research mathematician's mind but it is still a recognisably good piece of mathematics, one capable of multiple elaborations, which may lead into deep waters, e.g. the formulae for the roots of cubic and quartic equations but not for the quintic, Galois theory and so on. On the other hand, there are things that are perfectly true about entities that are properly mathematical but which don't have this status. The fact that the number represented by 37 in our normal denary system is prime and remains prime when the digits are reversed is true but I would not say it is a part of mathematics. Just as I should say that a contingent physical fact that a neutrino from the sun passed through my body within a fraction of a second of a photon from Sirius being absorbed by my eye according to my frame of reference is not the concern of physics.

Mathematics is the historical course of mathematical activity. Any good, sufficiently complete history of mathematics will tell the story of the solution of the quadratic. No good history would mention the fact about 37 and 73. Similarly any good, even rather brief, history of mathematics will tell the story of the complex numbers. This shows that another philosopher, José Benardete, although alive to the internal sense of realism, is hopelessly wrong when he says:

> *"Stated in realist terms, the extended number system [of the complex numbers – DC] is presumed in effect to stake out a 'natural kind' of reality. Far from 'carving reality at the joints', however, the system can be shown to feature a flagrantly gerrymandered fragment of heterogeneous reality that is hardly suited to enshrinement at the centre of a serious science like physics, not to mention a rigorous one like pure mathematics. Couched in these ultra-realist terms, the puzzle might be thought to be one that someone with more pragmatic leanings – the system works, doesn't it? – need not fret over; and in fact such a one might*

even look forward to exploiting it to the discomfort of the realist. Fair enough. I should be happy to have my discussion of this Rube Goldberg contraption (as the extended number system pretty much turns out to be) serve as a contribution to the quarrel between anti-realist and realist that is being waged on a broad front today." (Benardete 1989: 106)

If you pass the complex numbers off as pairs of real numbers, each of which is a Dedekind cut of rationals, each of which is an equivalence class of pairs of integers, each of which is an equivalence class of pairs of natural numbers, then gerrymandering is easy to argue for. But to do so you must ignore the whole story of mathematics.

What is it to assert that a piece of contemporary mathematics, say the attempt by Jacob Lurie to devise a homotopic geometry, is good mathematics? It is to say that "Time will tell" and if it does choose to tell, it will do so as a chapter in the story of mathematics. We can construe what Weil was saying above as the claim that the axioms for uniform spaces may not have that honour; they will appear at best as a minor character, or may be superseded by a better notion devised at a later date. His work on function and number fields, on the other hand, he predicts to have an historical permanence.

But whether actual history retains something isn't quite enough because we also work with a notion that a practice may make mistakes. It may dismiss things later seen as important, it may linger on things later seen as trivial and so on. But if this took place against a backdrop of rapidly and radically changing views as to the best organisation of mathematical thinking, there would be little sense that one's decisions could be right or wrong. What we have then is real history located somewhere between two extremes.

1. The history of a practice that demonstrates the ability to understand the path that led to the current situation. Profound conceptual transformations take place but only when justified by an explanation of what was partial about earlier views. They lead to unexpected discoveries in what appear to be unrelated fields. The practice uses historical research not to justify its present position but to challenge its current conceptions. Practitioners are ready to understand partiality in their own viewpoints by exposing their ideas to other practitioners. They make an effort to understand other viewpoints. There is a dynamic exchange with practices that use its results.
2. The field is divided into isolated communities looking to protect their own theories from outside scrutiny. Any conservatism is to be attributed to sociological and intellectual inertia. When changes take place it is due to arbitrary fashion. Conceptual changes never lead to light being thrown in unexpected places.

There is much more to be said here but the point is that, despite some appearances, the actual history of mathematics more closely resembles (1) than (2). To take one part of it, in the past I may have been more open to the idea that large parts of earlier mathematical thinking have been lost to us but I am less inclined to believe that now. Useful for me on this score was putting the following claim to the test:

"The foregoing analysis of the early geometric works of Klein and Lie is far from exhaustive. Nevertheless, it should suffice to make clear that during this relatively brief period they developed a wealth of interesting ideas, techniques, and results that are all but forgotten today." (Rowe, 1989, 264)

In a weblog discussion (Corfield 2009), it became abundantly clear that Rowe had overstated the case and that, as far as algebraic geometry is concerned at least, we should agree with Matthew Emerton's assertion:

"Overall, my sense is that algebraic geometers are aware of the richness of the past of their subject (not all individually of course, but collectively, as a group of researchers), and have made continual efforts over the decades, as their subject developed, to go back to past literature and comb it for ideas that have been temporarily forgotten or misunderstood, if only for the hope of finding a technique that will help them solve open problems of current interest." (Corfield, 2009)

To conclude, mathematical reality in the internal sense may be thought to be that which allows mathematics to proceed as a tradition of intellectual enquiry, that is, allows it to approximate the first of the two descriptions of practice given above.

References

Benardete J. (1989) *Metaphysics: The Logical Approach*, Oxford, New York: Oxford, University Press.

Corfield D. (2009) Geometric Help Wanted, weblog discussion at http://golem.ph.utexas.edu/category/2009/03/quadric.html.

Lakatos I. (1976) *Proofs and Refutations: The Logic of Mathematical Discovery*, Worrall J. and Zahar E. (eds.), Cambridge University Press.

McLarty C. (2005) *'Mathematical Platonism' Versus Gathering the Dead: What Socrates teaches Glaucon*, Philosophia Mathematica 13(2): 115–134.

Rowe D. (1989) The Early Geometrical Works of Sophus Lie and Felix Klein, in *The History of Modern Mathematics I: Ideas and Their Reception*, Rowe D. and McCleary J. (eds.), Harcourt Brace: 209–273.

Weil A. (1940), Letter to Simone Weil, in *Oeuvres Scientifiques*, Volume 1 (New York: Springer, 1979), pp. 244–255.

David Corfield [D.Corfield@kent.ac.uk] is a senior lecturer in philosophy at the University of Kent, author of Towards a Philosophy of Real Mathematics (Cambridge University Press, 2003) and host of the blog The n-Category Café (http://golem.ph.utexas.edu/category/).

Mathematical Rejections

Philip J. Davis

To the Memory of Richard Hamming

This article is a reaction to a reader of my recent review of a biography of Ernst Zermelo in SIAM NEWS.[1] This reader picked up on the fact that Zermelo did not believe Gödel's proof and wrote to me to say that Zermelo was not an isolated case in the history of mathematics.

Indeed he was not. Over millennia, many mathematical concepts, constructions and theorems have been questioned or denied. What follows here is a brief sketch of some of the doubts that have been expressed and the manner in which individual users and creators of mathematics have come to terms with these doubts in one way or another. Among the concepts that have been doubted are (not in chronological order) irrationals, negatives, zero, one, imaginaries, infinitesimals and infinity. These objections were most often of a semantic and/or ontological nature; that is, the meaning and existence of the objects given names or symbols and the manipulation of symbolic sentences were questioned. The well-established adjectives currently in use, such as 'negative', 'irrational', 'surd' and 'imaginary', are residues of ancient doubts.

Though the boundary line between pure and applied mathematics is exceedingly fuzzy – some commentators have even insisted that there is no such thing as pure mathematics – I shall not deal with the difficulties and conflicts associated with the mathematical modelling of real world phenomena. Moreover, I shall limit myself to mathematics of the 'Western tradition'.

The doubts, dilemmas and angst surrounding the concepts just mentioned have been treated *in extenso* by a variety of authors[2] but allow me to recall briefly and simplistically a few of the objections that have been raised:

Zero – common sense complains; how can nothing be something?

One – a number must express numerosity or multiplicity.

Fractions – how can 1/2 = 2/4 when half a pie is not the same as two pieces of pie cut into quarters?

Negative numbers – how can less than nothing be something? Scepticism about negatives lasted far into the 19th century. António José Teixeira, mathematics professor in Coimbra, wrote in 1890 that he did not like the proportion 1:–1:: –1:1 and asserted that "the negative quantities do not possess any arithmetical existence".

Irrational numbers – √2 exists as the length of the diagonal of the unit square. It was held to be a line and had no existence as a number.

Imaginary (complex) numbers – this has troubled mathematicians for a long time. In a famous quotation, Leibnitz wrote: "Imaginary numbers are a fine and wonderful refuge of the divine spirit almost an amphibian between being and non-being." Even Gauss bit his nails; in his first proof of the fundamental theorem of algebra, he is thought to have deliberately removed complex numbers both from the formulation and from the proof.

Infinity – how can there be an infinity, particularly a "completed infinity", when the concept is germinated, set forth or defined by a finite number of symbols?

Infinitesimals – according to the philosopher George Berkeley, infinitesimals are "the ghosts of departed quantities".

Moving to mathematical functions (i.e. curves and graphs), we come across another objection…

Dirac Function – how can a function that is zero on $(-\infty, +\infty)$ except at one point have a positive area 'underneath it'?

Operator calculus and umbral calculus were both pursued usefully by parlous procedures considered illegitimate by purists with a strict sense that rigorous mathematics should be pursued and of what it should consist of.

These ideas, particularly that of infinity, have been fertile fields for philosophers, theologians, neo-Platonists, mystics and even cranks but the surprise is that even mathematicians of international reputation have come forward with doubts. Over the years – sometimes it has taken centuries – these problematic concepts have been totally absorbed into mainstream mathematics by having been embedded within axiomatic, deductive formalizations of a traditional type. Prior to such "regularizations" – and this is of great importance – these concepts proved useful to science, technology and even to mathematics itself, as well as a wide variety of humanistic concerns.

Thus, it emerges that both within and without mathematics, utility confers ontological reality and justificatory legitimacy. An increase in utility is accompanied by additional legitimacy and an abatement of scepticism.

Now the concepts of utility, and the idea of more utility and less utility, are admittedly vague and the relation of utility to the acceptance and legitimization of theoretical material is variable. Thus, in astronomy, though the Ptolemaic system of the solar system can produce accurate descriptions of planetary motion, Ptolemy gave way to the explanatory strength of Newtonian dynamics, which, in turn, gave way to Einstein. But acceptance and application depends on what you want to do. NASA, in planning its space trajectories, can ignore Einstein.

It is also the case that utility as a pre-condition for legitimacy stands in low regard in certain portions of the mathematical community. Wasn't it Euclid who is quoted as saying: "Give the student a coin for he demands to profit from what he learns." Luckily, there are many criteria besides utility for justification, legitimacy and acceptance, e.g. the process of mathematical proof or physi-

1 SIAM NEWS, Vol 41, No. 1, Jan/Feb 2008.
2 See the references to Brian Rotman, Gert Schubring, Underwood Dudley, Hal Hellman, Imre Lakatos and Philip J. Davis.

cal verification. In addition, non-deductive methods have been employed and all of these should be considered modes of theorematic evidence.

*****

When historically we reach the ideas of Georg Cantor, set theory and certain aspects of mathematical logic, all ushered in at the end of the 19th century, the picture changes substantially. Though of utility within mathematics, e.g. the ideas of Cantor easily imply the existence of transcendental numbers (and this implication now appears as an example in many elementary texts on set theory), I do not think that e.g. transfinite cardinals nor even Zermelo's famous and notorious Axiom of Choice have had substantial "real world" applications. There is, however, feedback into applications in terms of certain popular and approved kinds of notations and expositional rhetoric.

The scepticisms of Poincaré, Zermelo, Brouwer, Hermann Weyl, Wittgenstein, Errett Bishop, etc. derive from this period and from this corpus of mathematics. While each of these authors have most certainly had individual metaphysical qualms, the lack of substantial backup from the world outside of mathematics must certainly have contributed unconsciously to their angst.

There are, of course, coteries of musicians, artists and poets whose works show little attraction for the larger public. If certain areas or developments of mathematics do not achieve significance in the larger world then what we have is an in-group dialogue that maintains its own criteria of meaning, validity and importance; we have material that is pursued as *art pour l'art* and is capable of stimulating great controversy. What supports such efforts with the public purse is partly that art is regarded as absolutely necessary for the cultural life of a nation and partly the hope that ultimately such material will have thrust upon it great significance for the material demands of the multitudes by as yet unborn, brilliant imaginations. In the case of Cantorism and its many sequels, it has created and is surrounded by an aura of religious mysticism pursued by those who attribute limited horizons of the human imagination to Cartesian rationalism and wish to throw off its shackles.

Sometimes the door of rejection is not slammed shut but kept ajar just a bit. Referring back to Poincaré's talk of 1908, David Mumford and Philip Davis wrote:[3]

> *'One senses in this section considerable ambivalence of Poincaré towards Cantor's ideas. While acknowledging that "(His) services to science we all know," he ends the paragraph by saying "(with this theory) we can promise ourselves the joy of the physician called in to follow a beautiful pathological case!" It seems that uppermost in his mind in this short paragraph are the paradoxes that arise in this field, the apparent contradictions which "would have overwhelmed Zeno… with joy." As we know, from our vantage point, it was Gödel's ingenious use of exactly these paradoxes that led to the deepest result in the foundations of mathematics, to Gödel's magnificent incompleteness theorem, whose philosophical significance continues to reverberate… But, on the other hand, there is also a widespread feeling among working mathematicians that measurable cardinals and the like, that is to say, present day set theory, are indeed some kind of "pathological case"… So Poincaré perhaps caught the future mainstream reaction to this area as well as pinpointing its arguably most significant idea.'Now to provide a deeper feeling for the bubbling ferment of the Age of Cantor, let me provide a few short, incisive and well-documented quotations.*

[3] *Henri's Crystal Ball*, Notices of the AMS, April, 2008.

Kronecker didn't like Cantor's stuff. In Eric Temple Bell's colourful words, Kronecker believed that Cantor had created a world of mathematical insanity (I use the word 'stuff' not as a pejorative but as a device allowing me to avoid having to pinpoint the exact material). Kronecker recognized the brilliance, power and strength of Lindemann's proof of the transcendendency of the number ϖ –but did not even accept the concept of irrationality. Indeed, for numerical calculations and applications, the concept of a polynomial ideal which he accepted would have been sufficient.

Emile Picard didn't like Cantor's stuff. He said: "Some believers in set theory are scholastics who would have loved to discuss the existence of God with Saint Anselm…"

Poincaré, despite the ambiguity expressed in his 1908 speech, didn't really like Cantor's stuff. He said: "Later generations will regard set theory as a disease from which one has recovered." Nor did he like Russell's stuff. He said that mathematical logic creates mathematical monsters.

Brouwer didn't like Hilbert's stuff. He denied the logical principle of the excluded middle. Hilbert struck back by writing: "Taking the Principle of the Excluded Middle away from the mathematician is the same as prohibiting the boxer from using his fists."

Borel and Baire didn't like Zermelo's stuff. He told poet Paul Valéry that he'd given up on set theory because it fatigued him and he foresaw serious illness for himself if he persisted in it. Baire believed that the real number system could not be well ordered (which was implied by Zermelo's Axiom of Choice).

Hilbert wrote famously of his "dream" that all meaningful problems in mathematics can be settled one way or another:

> *"There is the problem. Seek the solution. You can find it by pure reason. In mathematics there is no* igorabimus.*"*

Gödel, in an amazing *coup de theatre*, reduced Hilbert's dream to a logical impossibility. But any number of people, including Zermelo and Wittgenstein, didn't like Gödel's stuff. Zermelo wrote: "…I publicly assert that Gödel's much admired 'proof' is nonsense," probably reflecting a conflict between Zermelo's Platonism and Gödel's Formalism.

Wittgenstein called Gödel's results *Kunststücken* (just tricks).

The antagonisms cited look like a good old-fashioned Western movie barroom dustup. One instance has even led to a lawsuit. And now that the dust has settled, what can be found when looking within the wider mathematical playing field? Before Bolyai and Lobachevski, there was only one geometry: that of Euclid, and now, following Riemann, there is an infinity of geometries equally valid from the deductive point of view. Thus the "truth" of any particular geometry and a fortiori of any mathematical theory is a meaningless concept.

We find that there are a number of conflicting interpretations of randomness and probability with heated arguments across dividing aisles. There is now a salad bar of different logics or set theories. We see a continued search for new axioms as part of an endless search for an absolute foundation for mathematics. To some commentators an absolute foundation for mathematics cannot be found and, in fact, mathematics has no need for such a thing. All of this feeds into the philosophy of the subject. Prior to the end of the 19th century, the philosophy of mathematics was simple enough: it was that of Platonism. We now easily have five distinguishable varieties of mathematical philosophy together with variations that exhibit the Freudian narcissism of slight differences.

*****

Thus, as has been seen over the millennia, many mathematical concepts, constructions, manipulations and theorems have been questioned or denied by brilliant thinkers. How do I navigate in such ambiguous waters? I come now to a description of what has been my personal mathematical practice. I take courage from a well-known quote of Richard Hamming, who was a brilliant techno-realist:

> *"I know that the great Hilbert said: 'We will not be driven out of the paradise Cantor has created for us,' and I reply: 'I see no reason for walking in!'"*

I generalize and reformulate Hamming's assertion in the following way as a piece of personal pragmatism.

"What works for you may or may not work for me – and vice versa".

What do I accept? What do I reject? What works for me? Stressing the positives and limiting consideration to the mathematics of a deductive or computational nature that I have personally produced, I would say that a mathematical idea, construction, object, definition, phraseology, theorem or manipulation works for me if I have used it in a technical paper or if I have advocated it in a book or a lecture. It certainly doesn't work for me if on a rare occasion I have polemicized against it. Moreover, there is a large grey area of personal indeterminacy here and I confess that I have not been consistent in my judgments (what person is absolutely consistent?).

If I were asked which specific mathematical ideas, objects, etc. work for me, I would be unable to compile a detailed and complete list. I would be unable to say that such and such constitutes a list of my personal axioms and that what works for me are all the consequences of these axioms. Could anyone? The only way I could answer the question is to look at and analyze what is in my papers, programs, talks, suggestions, etc.

In a famous quotation Cantor asserted that the essence of mathematics lies in its freedom.[4] One has the freedom to dream, to create structures of thought, to reduce them to symbols so as to communicate them to others and then, like Ezekiel prophesying over dry bones, say that one has breathed interpretive life into them. A second freedom, a reaction to this and occurring over and over again, is that we all have the freedom to reject anything we can't swallow.

[4] "Das Wesen der Mathematik liegt gerade in ihrer Freiheit."

Acknowledgement

Thanks to Bernhelm Booss-Bavnbek for a number of significant observations.

Bibliography

Alan Baker, *Non-Deductive Methods in Mathematics*, Stanford Encyclopedia of Philosophy.

Philip J. Davis, *Mathematics and Common Sense*, AK Peters, 2006.

Underwood Dudley, *Mathematical Cranks*, MAA, 1992.

Heinz-Dieter Ebbinghaus, *Ernst Zermelo*, Springer, 2007.

Paul Ernest, *The Philosophy of Mathematical Education*, Falmer, 1991.

Loren Graham and Jean-Michel Kantor, *Religious Heresy and Mathematical Creativity in Russia*, Mathematical Intelligencer, 2008.

E. Grosholz and H. Breger, eds., *The Growth of Mathematical Knowledge*, Kluwer Academic, 2000.

Hal Hellman, *Great Feuds in Mathematics*, John Wiley, 2006.

Imre Lakatos, *Proofs and Refutations*, Cambridge Univ. Press, 1976.

Paolo Mancosu, *Mathematical Explanation: Why it Matters*, The Philosophy of Mathematical Practice, Oxford University Press, 2008.

Helena M. Pycior, *Symbols, Impossible Numbers and Geometric Entanglements*, Cambridge University press, 1997.

Brian Rotman, *Signifying Nothing: The Semiotics of Zero*, St. Martin's Press, 1987.

Gert Schubring, *Conflicts Between Generalization, Rigor, and Intuition: Number Concepts Underlying the Development of Analysis in 17–19th Century France and Germany*, Springer 2005.

B.Sriraman & S. Goodchild (eds) *Relatively and Philosophically Earnest: Festschrift in honor of Paul Ernest's 65th Birthday*. Information Age Publishing, Charlotte, NC.

Alasdair Urquhart, *Mathematics and Physics: Strategies of Assimilation, The Philosophy of Mathematical Practice*, ed. Paolo Mancosu, Oxford, 2008.

Philip J. Davis [Philip_Davis@brown.edu] received both of his undergraduate and doctoral degrees from Harvard in the field of pure mathematics. He was Chief, Numerical Analysis, National Bureau of Standards, Washington, D.C., for five years. He joined the faculty of Applied Mathematics at Brown University, Providence, Rhode Island, in 1963 and is now Professor Emeritus.

The symplectic topology of cotangent bundles

Tim Perutz

Introduction

Symplectic manifolds entered mathematics through the geometric formulation of Hamilton's approach to dynamics. In Hamiltonian mechanics, the phase-space of a physical system can be viewed as the *cotangent bundle* T^*L of a manifold L. The evolution of points in phase-space is controlled by a differential equation involving a function $H: L \to \mathbb{R}$. The procedure for deducing this differential equation from H can be understood in terms of a geometric structure on T^*L – its symplectic structure.

Symplectic geometry evolved from its Hamiltonian origins into a branch of topology with a distinctive character, targeting global problems concerning compact manifolds [2]. Some symplectic topologists have returned to the subject's roots, revisiting cotangent bundles with a view to topological rather than dynamical questions. The object of their interest, and the subject of this article, is the interaction between the topology of L as a smooth manifold and the symplectic topology of T^*L – how can smooth topology inform symplectic topology? Can symplectic topology inform smooth topology?

Cotangent bundles

We begin with (symplectic) cotangent bundles. Every finite-dimensional real vector space V has a *dual* vector space $V^* = \mathsf{Hom}(V,\mathbb{R})$ of linear maps $V \to \mathbb{R}$. It has the same dimension as V. In differential geometry, vector spaces come in parametric families. If M is a smooth n-dimensional manifold, one has a family of n-dimensional vector spaces $\{T_qM\}_{q\in M}$, in which T_qM is the vector space of tangent vectors at q. Collectively, these vector spaces form the tangent bundle TM, which is itself a manifold, of dimension $2n$. The cotangent bundle of M is the union of all the *duals* to the tangent spaces, $T_q^*M := (T_qM)^*$. It is another $2n$-dimensional manifold. The cotangent bundle to the n-dimensional sphere $S^n = \{q \in \mathbb{R}^{n+1} : |q| = 1\}$ is

$$T^*S^n = \{(q, p) \in \mathbb{R}^{n+1} \times \mathsf{Hom}(\mathbb{R}^{n+1},\mathbb{R}) : |q| = 1,\ p(q) = 0\}.$$

A *1-form* λ on a manifold M is a choice of cotangent vector $\lambda(q) \in T_q^*M$ for each point q, varying smoothly with q. Since the cotangent bundle is also a manifold, one can talk about its 1-forms. This has a tautological feel which is vindicated by the existence of a *natural* 1-form λ_M on T^*M, sometimes called '$p\,dq$' form. Consider the projection map $\pi: T^*M \to M$, $(q, p) \mapsto q$. Its derivative at (q, p) is a linear map $D\pi: T_{(q,p)}(T^*M) \to T_qM$. To evaluate λ_M on a tangent vector $v \in T_{(q,p)}(T^*M)$, evaluate $p \in T_q^*M$ on $(D\pi)(v) \in T_qM$. Alternatively, introducing coordinates $(q_1,\dots,q_n)$ on a patch of M, there are corresponding coordinates $(p_1,\dots,p_n)$ for the cotangent directions, and one has, in differential geometers' standard notation, $\lambda_M = \sum_i p_i\,dq_i$.

A *2-form* ω is a bilinear product ω_q on each tangent space T_qM, skew-symmetric in the two inputs: $\omega_q(u, v) = -\omega_q(v, u)$. One can differentiate a 1-form λ to get a 2-form $d\lambda$. In particular, we can construct the 2-form $\omega_M = -d\lambda_M$ on T^*M: in coordinates, $\omega_M = \sum_i -d(p_i\,dq_i) = \sum_i dq_i \wedge dp_i$. This 2-form is the prototypical example of a *symplectic form*: a 2-form that is non-degenerate as a bilinear form, and also *closed* (locally of the form $d\alpha$ for a 1-form α). In Hamiltonian dynamics, one works on a phase-space which is T^*M for some manifold M, and ω_M governs the dynamics.

The symplectic manifold (T^*M, ω_M) – the smooth manifold with its symplectic 2-form – will be the protagonist of this article.

Lagrangians in cotangent bundles

Lagrangian submanifolds are at the heart of symplectic topology. A Lagrangian submanifold of a $2n$-dimensional symplectic manifold (X, ω) is an n-dimensional embedded submanifold L so that $\omega(u, v) = 0$ whenever u and v are tangent to L. Lagrangian submanifolds of T^*M are plentiful. For instance, there are the cotangent fibres T_q^*M, for any $q \in M$. There is the zero-section $M \subset T^*M$ (the zero cotangent vector at each point $q \in M$).

Diffeomorphisms of T^*M that preserve its symplectic structure are abundant, and these diffeomorphisms take Lagrangians to Lagrangians, thereby generating new examples.

The simplest interesting compact manifold is the circle, $M = S^1$. The cotangent bundle T^*S^1 is a cylinder $S^1\times\mathbb{R}$, and a compact Lagrangian in T^*S^1 is just a collection of disjointly embedded loops in the cylinder.

We now impose more stringent conditions on our submanifolds of T^*M. We shall look at compact, *exact* Lagrangians L: those such that the $p\,dq$ form λ_M is exact on L, i.e., the line-integral of λ_M around a loop in L is always zero. In T^*S^1, Stokes' theorem implies that exactness is the condition that, for each component of the Lagrangian, the signed area between that loop and the zero-section is zero (see Figure 1). If Λ is a compact, exact Lagrangian submanifold of T^*S^1 then

- Λ is connected. (This is because any two exact loops intersect one another at least twice.)
- Λ has degree ±1, i.e., its algebraic intersection number with each cotangent line $T_q^*S^1$ is ±1. (As an embedded circle in the cylinder, its degree must be 0 or ±1. But if the degree were zero, Λ would bound a disc of positive area, and this is forbidden by exactness.)
- Λ can be deformed, through exact Lagrangians, to the zero-section $S^1 \subset T^*S^1$. (Invoke some standard arguments in surface topology.)

- Λ has Maslov class zero. This is a topological property, but specifically a symplectic one. It means that, in any loop $\gamma\colon S^1 \to \Lambda$, the signed count of points $t \in S^1$ so that $T_{\gamma(t)}\Lambda$ shares a tangent with the cotangent line $T^*_{\gamma(t)}M$ is zero. (This is true of the zero-section, and remains true under deformations through Lagrangians.)

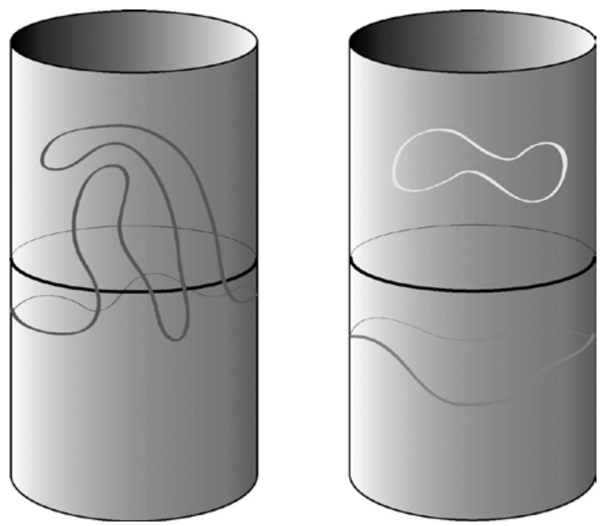


Figure 1. Two pictures of (part of) T^*S^1, each showing the zero-section as a black loop. Of the remaining loops, the one on the left is exact (the area between it and the zero- section is taken to be zero) but the two on the right are not.

Vladimir Arnol'd [1] began to probe the question of whether exact Lagrangians Λ in cotangent bundles behave as simply as they do in T^*S^1. He posed part (a) of the following question.

Question 1 ('Weak Arnol'd conjectures')**.** *Let Λ be a compact, exact, orientable Lagrangian in T^*L, where L is also compact and orientable. (a) Does Λ intersect each cotangent fibre once, when intersections are counted algebraically, with signs? (b) Does Λ necessarily have Maslov class zero?*

A positive answer to the following bolder question from [8] would imply positive answers to (a) and (b).

Question 2 ('Strong Arnol'd conjecture')**.** *Can Λ be deformed through exact Lagrangians to the zero-section?*

The principal tool for making headway on these questions has been the theory of *Floer cohomology*. To describe this, we need to know that an *exact symplectomorphism* is a self-diffeomorphism ϕ of M such that $\phi^*\lambda_M - \lambda_M$ is the derivative of a function; this implies that $\phi^*\omega_M = \omega_M$. As part of a more general theory, Floer [5] assigned a vector space $\mathsf{HF}(L_0, L_1)$ over $\mathbb{F}_2$, the field of 2 elements, to any pair of compact, exact Lagrangian submanifolds in T^*M, such that

(i) $\mathsf{HF}(L_0, L_1) \cong \mathsf{HF}(\phi(L_0), L_1)$ when ϕ is an exact symplectomorphism;

(ii) $\dim \mathsf{HF}(L_0, L_1) \leq \#(L_0 \cap L_1)$ providing that L_0 and L_1 intersect transversely; and

(iii) $\mathsf{HF}(L_0, L_0)$ is isomorphic to the singular cohomology $H^*(L_0; \mathbb{F}_2)$.

The existence of such an assignment immediately implies Gromov's result that in $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ there are *no* compact exact Lagrangians. If L were such a Lagrangian then, whenever ϕ is an exact symplectomorphism one would have, by (i) and (iii), $\mathsf{HF}(\phi(L), L) \cong \mathsf{HF}(L, L) \cong H^*(L; \mathbb{F}_2) \neq 0$, hence by (ii) $\phi(L) \cap L \neq \emptyset$. But we can take ϕ to be a very large translation in $\mathbb{R}^n$, so that $\phi(L) \cap L = \emptyset$.

Elaborating this argument, Lalonde–Sikorav [8] showed that there are no compact, exact, embedded Lagrangians in $T^*(M \setminus \{q\})$. A compact exact Lagrangian embedded in T^*M must therefore project surjectively to the zero-section. This is not as strong as wrapping once round the zero-section, but it is a step in that direction. They also proved cases of (b).

The following theorem represents the state of the art on (a), and goes beyond it in some cases.

Theorem 1 (Fukaya–Seidel–Smith [6] ; Nadler [10])**.** *Let L be a compact, simply connected manifold. Let $\Lambda \subset T^*L$ be a compact, orientable, exact Lagrangian, with Maslov class zero. Assume that both L and Λ are spin manifolds. Then the projection $\Lambda \to L$ is a map of degree ± 1 inducing an isomorphism on singular homology.*

(Degree ± 1 is just another way of saying the algebraic intersection property (a). The 'spin' condition stems from a Floer-theoretic technicality, and is unlikely to be essential. Spheres and orientable surfaces are spin, but the complex projective plane is not.) Nadler's and Fukaya–Seidel–Smith's proofs use Floer cohomology in a more sophisticated way than the proof of Gromov's theorem about $\mathbb{R}^{2n}$. The data from many Floer vector spaces, and a good deal of further data, are assembled into an intricate algebraic structure called the *Fukaya A_∞-category*. The objects in this category are compact, exact Lagrangians. The vector space $\mathsf{HF}(L_0, L_1)$ is interpreted as a space of 'maps' from L_0 to L_1, whilst pseudo-holomorphic polygons – solutions to certain geometric PDE related to the pattern of intersection points between the Lagrangians – lead to notions of 'composition of maps'. Fukaya categories were invented in the early 1990s, but putting the construction on a sound footing is a complicated business that has taken a long time, and it is only recently that they have become an effective tool.

The theorem of Fukaya–Seidel–Smith and Nadler is proved by finding a basis, of sorts, for the Fukaya category.

Suppose I want to decide whether two vectors u and v in an n-dimensional vector space, equipped with an inner product, are equal. I need only check whether $e_i \cdot u = e_i \cdot v$ for $i = 1, \ldots, n$, where $(e_1, \ldots e_n)$ is any basis. If the basis is actually orthonormal, then one has the formula $u = \sum_i (e_i \cdot u)e_i$. The coefficient $e_i \cdot u$ measures how much u 'overlaps' with e_i. Analogously, if I want to identify a representation of a finite group G on a finite-dimensional complex vector space U, I need only compare U with the irreducible representations $V_1, \ldots, V_N$. One has the formula $U = \bigoplus_i \mathsf{Hom}_G(V_i, U) \otimes V_i$.

In T^*L, one can find a finite collection of non-compact Lagrangians $(L_1, \ldots, L_N)$ such that, if one want to identify a compact Lagrangian Λ as an object of the Fukaya category, one need only see how much it 'overlaps' with $L_1, \ldots, L_N$, where the overlap is measured by $\mathsf{HF}(L_i, \Lambda)$. One cannot hope for an 'orthonormal' basis here, and the situation is more subtle than the representation-theoretic one, but the analogy is still helpful. Fukaya-Seidel-Smith and Nadler find 'bases' with different geometric origins, and use them to show that Λ and the zero-section L are isomorphic objects in the Fukaya category. It then follows that $\mathsf{HF}(\Lambda,\Lambda) \cong \mathsf{HF}(L,L)$, which implies $H^*(\Lambda) \cong H^*(L)$ by property (iii) of HF, and that $\mathsf{HF}(\Lambda, T^*_q M) \cong \mathsf{HF}(L, T^*_q M) \cong \mathbb{F}_2$, from which one can deduce that Λ wraps exactly once around the the zero-section L.

Cotangent bundles and smooth topology

Symplectic topologists dream of proving new theorems in manifold topology via the symplectic topology of cotangent bundles. A natural target for such theorems – but equally the most formidable target – is the structure of 4-dimensional smooth manifolds, since we understand less about them than about manifolds of any other dimension.

If L_0 and L_1 are smooth manifolds then a diffeomorphism $f\colon L_0 \to L_1$ – that is, a smooth map with a smooth inverse – gives rise to a smooth map $f^*\colon T^*L_1 \to T^*L_0$. This map sends a cotangent vector λ at $f(x)$ to the cotangent vector $f^*\lambda$ at x which evaluates on a tangent vector v at x as $(f^*\lambda)(v) = \lambda((D_x f)(v))$, where $D_x f\colon T_x L_0 \to T_{f(x)} L_1$ is the derivative of f at x. The map f^* is itself a diffeomorphism, for $(f^*)^{-1} = (f^{-1})^*$. Thus, using the symbol $\cong$ to signify the existence of a diffeomorphism, we have that $L_0 \cong L_1$ implies $T^*L_0 \cong T^*L_1$.

Does the converse hold? – Are manifolds L_0 and L_1 with diffeomorphic cotangent bundles themselves diffeomorphic? They must certainly resemble one another topologically: there must be a *homotopy equivalence* $f\colon L_0 \to L_1$, i.e., a continuous map which has an inverse-up-to-homotopy g. So $f \circ g$ can be continuously deformed to the identity map on L_1, and $g \circ f$ to the identity map on L_0.

The subject of *surgery theory* [7] is relevant here, because it measures the difference between homotopy equivalence and diffeomorphism. It only works in dimension 5 and higher. Suppose we have two compact, simply connected d-manifolds, M_1 and M_2, and a homotopy equivalence $h\colon M_1 \to M_2$. Surgery theory is a two-step process. The first step is to determine whether h 'respects tangential structure'. The exact meaning of this is subtle, but part of it is that h should be a *tangential homotopy equivalence*. That is, there exist linear isomorphisms $T_x M_1 \to T_{h(x)} M_2$ varying continuously with $x \in M_1$. If h were a diffeomorphism, its derivative Dh would provide such isomorphisms. When $M_1 = S^d$, the first step amounts to determining whether M_2 bounds a manifold with trivial tangent bundle. The second step is to determine whether an h that respects tangential structure is homotopic to a diffeomorphism. This step involves 'surgically' modifying manifolds. A typical result of surgery theory is that there are exactly seven non-standard homotopy 9-spheres; of those, only one bounds a 10-manifold with trivial tangent bundle.

The answer to our question about diffeomorphism of cotangent bundles is 'no': if $T^*L_0 \cong T^*L_1$ then it need not be the case that $L_0 \cong L_1$. Homotopy n-spheres *always* have diffeomorphic cotangent bundles. However, we can ask a more refined question. If L_0 is diffeomorphic to L_1 then T^*L_0 is symplectomorphic to T^*L_1. Does the converse hold?

Question 3. *Does the symplectic structure of the cotangent bundle capture the smooth structure of the original manifold?*

This would be another consequence of the strong Arnol'd conjecture. For if $T^*L_0 \cong T^*L_1$ symplectically, then L_0 embeds as an exact Lagrangian in L_1. If the strong Arnol'd conjecture is true, then this implies that $L_0 \cong L_1$ smoothly!

A very encouraging development is a preprint of Abouzaid [3], which shows that if a $(4k+1)$-dimensional manifold Σ has $T^*\Sigma \cong T^*S^{4k+1}$ symplectically then Σ is a homotopy-S^{4k+1} which bounds a manifold with trivial tangent bundle. In conjunction with surgery theory, Abouzaid's theorem implies that there is at most one non-standard homotopy-sphere Σ which shares a symplectic cotangent bundle with T^*S^{4k+1}.

Lagrangian immersions

So far, everything we have said about Lagrangians in cotangent bundles points to positive answers to the Arnol'd conjectures. Even so, it is instructive to think about how one might seek a counterexample. Take a tangential homotopy equivalence $h\colon \Lambda \to L$. Can we find an exact Lagrangian embedding $h'\colon \Lambda \to T^*L$, homotopic to the composite of h with the zero-section $L \hookrightarrow T^*L$? We can do something very close to this. There is an 'h-principle' [4, 16.3.1] – a method for exploiting the flexibility of certain situations in differential geometry – to homotope h to an *exact Lagrangian immersion* h'. This means that if U is any small open set in Λ then $h'(U)$ is a Lagrangian submanifold of T^*L_0. However, the image $h'(\Lambda)$ may cross over itself at a finite number of *double-points*. The h-principle typically produces extremely wiggly immersions with many double-points.

Question 4. *Let $\Lambda \to L$ be a homotopy equivalence that respects tangential structures. What can one say about the least number $n(\Lambda, L)$ of double points of an exact Lagrangian immersion $\Lambda \to T^*L$?*

The strong Arnol'd conjecture predicts that if $n(\Lambda, L)$ is zero, we should have $\Lambda \cong L$. In dimensions 5 and higher, does $n(\Lambda, L)$ have something to do with the surgery obstruction to homotoping h to a diffeomorphism? In dimension 4, surgery theory breaks down, but Smale's h-cobordism theory [12] tells us that if we are given a homotopy-equivalence $h\colon L_0 \to L_1$ then there is a unique smooth 5-dimensional manifold M with boundary $L_0 \amalg L_1$ which realises h in the sense that the inclusions $i_0\colon L_0 \to M$ and $i_1\colon L_1 \to M$ both have homotopy-inverses k_0, k_1, and the composite $k_1 \circ i_0$ is homotopic to h. It is tempting to think that $n(\Lambda, L)$ should be related to the complexity of M, as measured for instance by the least number of gradient flow-lines between the critical points of a Morse function $f\colon M \to [0,1]$ such that $f^{-1}(i) = L_i$ for $i = 0, 1$. This would link the unsolved problem of classifying 4-manifolds with the symplectic topology of cotangent bundles.

Unrecognisability of symplectic manifolds

This final section tours through mathematical logic, algebra and topology before looping back to symplectic topology.

Undecidable problems worm their way into geometry through Turing's notion of algorithmic solvability. An algorithm is an idealised computer program which, whatever numerical input one feeds, performs a sequence of steps and then terminates. Programs can themselves be numbered (this is how they are recorded on your hard disc), and Turing's archetypal decision problem T is: will the program numbered m eventually terminate when fed the input n? His theorem is that no algorithm will decide.

Turing's problem can be embedded into other decision problems P, in the sense that an algorithm to solve P would lead to the impossible algorithm to solve T. Those problems too must then be algorithmically unsolvable. A famous example is the word problem W for groups: one can form the free group $\mathbb{F}_p$, consisting of words w in letters $g_1, \ldots, g_p$ and their inverses, such as $w = g_4^{28} g_{89}^{-3} g_4^{-1}$. No algorithm will decide whether a chosen collection of words $w_1, \ldots, w_q$ in a chosen $\mathbb{F}_p$ normally generates the whole of $\mathbb{F}_p$.

Having embedded Turing's logical problem into an algebraic one about groups, one can next embed the algebraic problem into topological ones. Given an n-manifold N, is there an algorithm that will decide whether it is diffeomorphic to the n-sphere S^n? If $n \geq 4$, the answer is no, because no algorithm will decide whether the fundamental group $\pi_1(M)$ is trivial. This is because one can embed inside this problem the unsolvable word problem W. This example needs some clarification: the idea of an algorithm in connection with non-discrete objects such as manifolds is perplexing. The point is that there are finite ways to encode the manifold, or at any rate, enough information about the manifold as to make the problem unsolvable. The most familiar procedure would be to give a triangulation of the underlying space. Triangulations can be specified in a purely combinatorial way.

The fundamental group obstructs algorithmic recognition of manifolds, but it is essentially the only obstruction [9]: any simply connected, compact smooth manifold of dimension > 4 is, in theory if not in practice, algorithmically recognisable among manifolds of the same class. What about recognising symplectic manifolds, up to symplectomorphism (that is, diffeomorphisms respecting the symplectic forms)?

Theorem 2 (Seidel [11]). *There is a simply connected, exact symplectic 12-manifold M which is not algorithmically recognisable among manifolds of that class.*

To be precise, we should say that there is a finite method of encoding a class of exact symplectic 12-manifolds, and recognition is not possible among manifolds of this class. The manifold M is not compact, but the non-compactness is of a mild sort; it is the symplectic structure of M (not the smooth topology) that is not algorithmically recognisable.

Seidel's argument goes as follows. Take a finite presentation P for a group G by generators and relations: $G = \langle g_1, \ldots, g_p \mid r_1, \ldots, r_q \rangle$. There is an algorithm that builds from P a 6-manifold S_P whose homology is that of the 6-sphere, but whose fundamental group $\pi_1(S_P)$ is canonically isomorphic to G (technically, for this one wants $H_1(G) = H_2(G) = 0$). No algorithm will determine whether S_P is diffeomorphic to S^6, because no algorithm decides whether $\pi_1(S_P) = G$ is trivial. Next one looks at the symplectic cotangent bundle T^*S_P. That still has fundamental group G, but there is a process of 'handle attachment' which produces another exact symplectic 12-manifold M_P which is simply connected. If T is the trivial presentation $\langle g_1, \ldots, g_p \mid g_1, \ldots, g_p \rangle$, then $M = M_T$ is algorithmically unrecognisable. Indeed, there is a gadget SH^0, the *symplectic cohomology* – an $\mathbb{F}_2$-vector space – which detects triviality of G. This works as follows: $SH^0(T^*S_P)$ has as basis the set of conjugacy classes of G, so its dimension is > 1 when G is non-trivial. The handle addition process kills π_1 but leaves SH^0 invariant, whence $SH^0(M_P) \cong SH^0(S_P)$.

Algorithmic recognisability of closed symplectic manifolds is open, and so is that of the simplest exact symplectic manifolds, such as $\mathbb{R}^6$. Seidel's theorem shows plainly that comparing manifolds symplectically and not just smoothly adds an additional layer of subtlety. We have no systematic understanding of this new layer, but we may suspect that in the case of cotangent bundles T^*L, its complexity should be correlated to subtleties in the smooth topology of L.

Acknowledgment

The author thanks Ivan Smith, whose ideas on this subject have shaped his own.

Bibliography

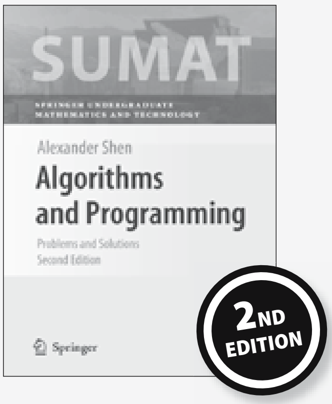
[1] V. Arnol'd, *First steps of symplectic topology.* VIIIth international congress on mathematical physics (Marseille, 1986), 1–16, World Sci. Publishing, Singapore, 1987.
[2] V. Arnol'd, *Symplectic geometry and topology*, J. Math. Phys. 41 (2000), no. 6, 3307–3343.
[3] M. Abouzaid, *Framed bordism and Lagrangian embeddings of exotic spheres*, 2008 preprint, arXiv:0812.4781.
[4] Y. Eliashberg, N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, 48. American Mathematical Society, Providence, RI, 2002.
[5] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. 28 (1988), no. 3, 513–547.
[6] K. Fukaya, P. Seidel, I. Smith, *Exact Lagrangian submanifolds in simply-connected cotangent bundles*, Invent. Math. 172 (2008), no. 1, 1–27.
[7] W. Lück, *A basic introduction to surgery theory*, ICTP Lecture Notes Series 9, Vol. 1, Abdus Salam International Centre for Theoretical Physics, Trieste. Available at http://www.math.uni-muenster.de/u/lueck/homepages/wolfgang_lueck/
[8] F. Lalonde, J.-C. Sikorav, *Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents*, Comment. Math. Helv. 66 (1991), no. 1, 18–33.
[9] A. Nabutovsky, S. Weinberger, *Algorithmic aspects of homeomorphism problems*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), 245–250, Contemp. Math., 231, Amer. Math. Soc., Providence, RI, 1999.
[10] D. Nadler, *Microlocal branes are constructible sheaves*, 2006 preprint, arXiv:math/0612399.
[11] P. Seidel, *A biased view of symplectic cohomology*, Current developments in mathematics, 2006, 211–253, Int. Press, Somerville, MA, 2008.
[12] S. Smale, *On the structure of manifolds*, Amer. J. Math. 84 1962, 387–399.

Tim Perutz [perutz@math.utexas.edu] is an assistant professor at The University of Texas at Austin. He obtained his Ph.D. from Imperial College London in 2005, and went on to postdoctoral positions at Cambridge and Columbia. His research interests centre on the use of Floer-theoretic techniques, both in symplectic topology and in the topology of manifolds of dimension 4 and lower.

Springer
the language of science

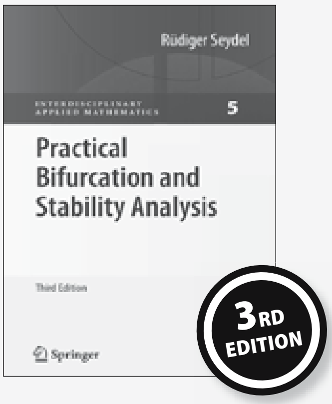
springer.com

Highlights in Springer's eBook Collection



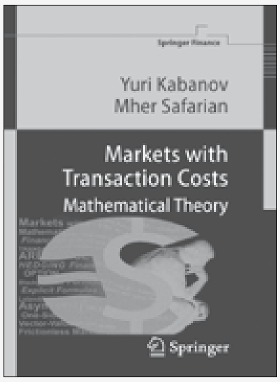
Reviews of the 1st Edition
► *The book is addressed both to ambitious students and instructors looking for interesting problems [and] fulfills this task perfectly, especially if the reader has a good mathematical background.*
► **Zentralblatt MATH**

2nd ed. 2010. XII, 272 p. 68 illus. (Springer Undergraduate Texts in Mathematics and Technology) Hardcover
ISBN 978-1-4419-1747-8
► **€ 49,95 | £44.99**



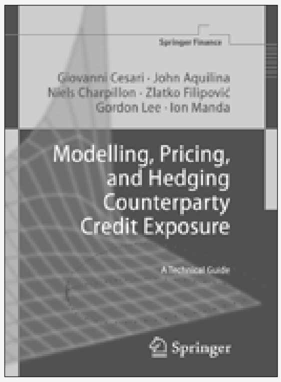
This book contains computational methods for numerically computing steady state and Hopf bifurcations. It is probably the first textbook to describe these types of numerical bifurcation techniques. The book requires only a basic knowledge of calculus.

3rd ed. 2010. XVIII, 483 p. 200 illus. (Interdisciplinary Applied Mathematics, Volume 5) Hardcover
ISBN 978-1-4419-1739-3
► **€ 69,95 | £59.99**



This book presents a unified treatment of various problems arising in the theory of financial markets with friction. It gives a succinct account of arbitrage theory for financial markets with and without transaction costs based on a synthesis of ideas.

2009. XIV, 294 p. 1 illus. (Springer Finance) Hardcover
ISBN 978-3-540-68120-5
► **€ 69,95 | £59.99**



This volume offers practical solutions to the problem of computing credit exposure for large books of derivatives. It presents a software architecture that allows the computation of credit exposure in a portfolio-aggregated and scenario-consistent way.

2009. XX, 254 p. (Springer Finance) Hardcover
ISBN 978-3-642-04453-3
► **€ 69,95 | £59.99**

For access check with your librarian

The Mathematics of Medical Imaging

A Beginner's Guide

T. G. Feeman

This text explores medical imaging, one of the most significant areas of recent mathematical applications, in a concise manner accessible to undergraduate students. The author emphasizes the mathematical aspects of medical imaging, including not only the theoretical background, but also the role of approximation methods and the computer implementation of the inversion algorithms.

2010. X, 141 p. 20 illus. (Springer Undergraduate Texts in Mathematics and Technology) Hardcover
ISBN 978-0-387-92711-4 ► **€ 39,95 | £35.99**

Option Prices as Probabilities

A New Look at Generalized Black-Scholes Formulae

C. Profeta, B. Roynette, M. Yor

The Black-Scholes formula plays a central role in Mathematical Finance; it gives the right price at which buyer and seller can agree with, in the geometric Brownian framework, when strike K and maturity T are given. This yields an explicit well-known formula, obtained by Black and Scholes in 1973. The present volume gives another representation of this formula in terms of Brownian last passages times.

2010. X, 250 p. (Springer Finance) Softcover
ISBN 978-3-642-10394-0 ► **€ 59,95 | £53.99**

Easy Ways to Order for the Americas ► **Write:** Springer Order Department, PO Box 2485, Secaucus, NJ 07096-2485, USA ► **Call: (toll free)** 1-800-SPRINGER ► **Fax:** 1-201-348-4505 ► **Email:** orders-ny@springer.com or **for outside the Americas** ► **Write:** Springer Customer Service Center GmbH, Haberstrasse 7, 69126 Heidelberg, Germany ► **Call:** +49 (0) 6221-345-4301 ► **Fax :** +49 (0) 6221-345-4229 ► **Email:** orders-hd-individuals@springer.com
► Prices are subject to change without notice. All prices are net prices.

014554x

Interview with Panos Pardalos

by Themistocles M. Rassias (Athens, Greece)

Panos Pardalos

Dr Panos Pardalos is a Distinguished Professor at the University of Florida. He is also the Director of the Center for Applied Optimization.

Dr Pardalos obtained a PhD degree from the University of Minnesota in Computer and Information Sciences. He has held visiting appointments at Princeton University, the DIMACS Center, the Institute of Mathematics and Applications, the FIELDS Institute, AT&T Labs Research, Trier University, the Linkoping Institute of Technology and Universities in Greece.

He has received numerous awards including University of Florida Research Foundation Professor, International Educator Award, Doctoral Dissertation Advisor/Mentoring Award, Foreign Member of the Royal Academy of Doctors (Spain), Foreign Member of the Lithuanian Academy of Sciences, Foreign Member of the Ukrainian Academy of Sciences, Foreign Member of the Petrovskaya Academy of Sciences and Arts (Russia) and Honorary Member of the Mongolian Academy of Sciences.

Dr Pardalos has received the degrees of Honorary Doctor from Lobachevski University (Russia) and the V. M. Glushkov Institute of Cybernetics (Ukraine). He is a fellow of AAAS, a fellow of INFORMS and in 2001 he was awarded the Greek National Award and Gold Medal for Operations Research.

Dr Pardalos is a world leading expert in Global and Combinatorial Optimization. He is the Editor-in-Chief of the *Journal of Global Optimization*, the *Journal of Optimization Letters* and the *Journal of Computational Management Science*. In addition, he is the managing editor of a number of book series and a member of the editorial board of several international journals. He is the author of several books and edited volumes. He has written numerous articles and has developed several well-known software packages. His research is supported by the National Science Foundation, the National Institute of Health and other government organizations.

How do you see mathematics research changing in the 21st century?

Interdisciplinary work is reshaping mathematics research. A great mathematician like A. Turing was also an influential computer scientist. Von Neumann was also a great physicist. Most recent examples include mathematicians like Steve Smale, who did fundamental work in economics, dynamical systems, machine learning and cooperative systems.

In which field do you see the most influence of mathematics in the 21st century?

Mathematics has entered new dimensions with profound impact. These include biomedical sciences, drug design and social networks. You can see this trend in the announcements of several funding agencies. For example, government agencies have announced funding on several high scope problems with the first one to be research in "the mathematics of the brain".

Please explain some of your recent work in biomedical science.

In the last few years, I have been working with a group of students, engineers and neuroscientists on brain dynamics. A problem we have studied extensively is the dynamics of the epileptic brain. Some fundamental questions we investigated include the prediction and control of epileptic seizures based on electro-encephalogram (EEG) data analysis. This work involves chaos theory, mathematics of networks, statistics and mathematical programming. For our work on epilepsy we received the "William Pierskalla Award" for research excellence in health care management science from the Institute for Operations Research and the Management Sciences (INFORMS). In addition, several patents have been issued with our new techniques in understanding brain dynamics.

We live in the information revolution. From the Internet to iPhones and personal computers, information flow has already made a great impact and change in our lives. What in your opinion is the contribution of mathematics in this revolution?

There is no doubt that we have moved from the industrial age to the information age. Understanding the dynamics of information systems can be accomplished with mathematical tools and data analysis. New journals and new network models have been established with many specific new scopes. The structure of the Web (the Internet) and its dynamics is the source of challenging mathematical questions. Random graph theory has been complimented with models of large-scale power-law distribution networks.

Can you give me an example where your work in this area has had an impact?

The proliferation of massive data sets brings with it a series of special computational challenges. The "data avalanche" arises in a wide range of scientific and commercial applications. With advances in computer and information technologies, many of these challenges are beginning to be addressed. A variety of massive data sets (e.g. the web graph and the call graph) can be modelled as very large multi-digraphs with a special set of edge attributes that represent special characteristics of the application at hand.

Understanding the structure of the underlying digraph is essential for storage organization and information retrieval. Our group was the first to analyze the call graph and to prove that it is a self organized complex network (the degrees of the vertices follow the power law distribution). We extended this work for financial and social networks. Our research goal is to have a unifying theory and develop external memory algorithms for all these types of dynamic networks.

In my recent joint work with DingZhu Du and Ron Graham, we introduced a new method which can analyze a large class of greedy approximations with non-submodular potential functions, including some longstanding heuristics for Steiner trees, connected dominating sets and power assignment in wireless networks. There exist many greedy approximations for various combinatorial optimization problems, such as set covering, Steiner tree and subset-interconnection designs. There are also many methods to analyze these in the literature. However, all of the previously known methods are suitable only for those greedy approximations with submodular potential functions. Our work will have a lasting impact in the theory of approximation algorithms for many network problems.

You mentioned before that understanding the dynamics of information systems can get serious help from mathematics. Can you give me some examples?

We try to understand the potential influence information has on the system and how that information flows through a system and is modified in time and space. Concepts that increase our knowledge of the relational aspects of information as opposed to the entropic content of information are an important area of research. Dynamics of Information plays an increasingly critical role in our society. Networks affect our lives every day. The influence of information on social, biological, genetic and military systems must be better understood to achieve large advances in their capability and understanding of these systems. Applications are widespread and include design of highly functioning businesses and computer networks, modelling the distributed sensory and control physiology of animals, quantum entanglement, genome modelling, multi-robotic systems and industrial and manufacturing safety. Classical Information Theory is built upon the notion of entropy, which states that for a message to contain information it must dispel uncertainty associated with the knowledge of some object or process. Hence, large uncertainty means more information; small uncertainty means less information. For a networked system, classical information theory describes information that is both joint and time varying. However, for networked systems, information theory can be of limited value. Entropy does not attend to the value or influence of information: in a network some information, though potentially large in its entropy, could have little value or influence on the rest of the network, while another, less entropic, piece of information may have a great deal of influence on the rest of the system. How information flows and is modified through a system is not dependent upon entropy but more likely on how potentially useful the information is. How the value of information is linked to the connectedness of the network (and vice versa) is critical to analyzing and designing high performing distributed systems, yet it is not well studied.

You have published several books on the mathematics of optimization and, in particular, global optimization. What is a global optimization problem and why are such problems considered hard?

Most existing methods in optimization focus on computing feasible points that satisfy optimality conditions. Under certain convexity assumptions these points are locally optimal. Finding globally optimal solutions is the key objective of global optimization. The distinction of global (optimal) versus local, with its various connotations, has found a home in almost all branches of the mathematical sciences. In many applications, checking the convexity of an objective function is a very difficult problem. From the complexity point of view, many problems in global optimization are very hard. Computational complexity can be used to analyze the intrinsic difficulty of many aspects of optimization problems and to decide which of these problems are likely to be tractable. In addition, the pursuit for developing efficient algorithms also leads to elegant general approaches for solving optimization problems and reveals surprising connections among problems and their solutions.

Global optimization has been expanding in all directions at an astonishing rate over the last few decades. At the same time one of the most striking trends in optimization is the constantly increasing interdisciplinary nature of the field.

I am working on all aspects of global optimization with several PhD students.

Which of your books has had the greatest influence?

My textbook, *Introduction to Global Optimization* (there is also a Chinese translation by Tsingua University Press), has been used as a graduate textbook in many universities around the world. This is one of the first textbooks in the field of global optimization in English. In addition, I co-edited two handbooks of global optimization. Furthermore, for several years I have been involved with the *Encyclopedia of Optimization*.

What was your motivation for working on a multivolume Encyclopedia of Optimization?

At the onset, I had no plans for editing an Encyclopedia. Such a mega-project evolved that way after an invitation from the publisher. Developing and working in such a project involves many challenging issues, such as designing a framework of the desired product, involving the best people to advise, identifying outstanding authors and referees, and dealing with the production team and the publisher. Since an encyclopedia is never in a final form, dealing with such a project is a lifelong, demanding activity. On the other hand, it is very satisfying to see that the *Encyclopedia of Optimization* is used by a wide audience of researchers.

As an editor-in-chief of the main journal in the field of global optimization, what do you see to be the new directions?

Global optimization has expanded to include several areas, such as generalized convexity, variational problems and problems with equilibrium constraints.

What are your thoughts on basic mathematics education at universities today?

In general, the situation is disappointing. You meet MBAs who cannot do basic algebra and graduates of engineering schools who do not know how to solve differential equations. There is a great need for good mathematical knowledge in engineering, medicine and social sciences. This is driven by the demand and funding of interdisciplinary research.

From our previous discussions you mention that you like philosophy and poetry. What are your interests in philosophy and poetry?

I have always been fascinated by the pre-Socratic philosophers. They touched all deep questions humans try to answer. All these philosophers were polymaths. The great ancient mathematicians like Pythagoras and Euclid not only studied mathematics but also the connections of mathematics with music, aesthetics and architecture.

Many times in my life I have written poetry. Mathematics expresses the rigorous part of your character. In poetry we express things that we cannot formulate precisely; poetry, like music, is needed for understanding and communicating different parts of ourselves. I very much enjoy reading Elytis, Seferis and Kavafis.

Is there any magic behind the word mathematics?

The word *μαθηματικά* (pronounced: mathematica, meaning: mathematics) originates from the verb *μανθάνειν* (to learn, to feel, to watch, to understand, to realize).

From the same verb originates the word *μάθησις* in the Attic dialect, which has the form *η μάθα* in the Doric, Aeolic and Macedonian dialects and the form *ο μάθος* in the Ionic dialect. It is the *πράξις του μανθάνειν* that is the process, the action of learning, the learning, the knowledge, the education but also the teaching.

Hence, *το μάθημα* is what somebody is learning, is taught, the knowledge and the science. Therefore, *μαθητής* is the person who learns something, the person being taught.

In plural, *τα μαθήματα* meant for the ancient Greeks the mathematical sciences, the mathematics, since it was necessary to *μανθάνειν* in order to excel in that. Thus, *ο μαθηματικός* (the mathematician) came to mean the person elaborating on the *μαθηματικά*, the mathematical sciences, and consequently is what the mathematician works on.

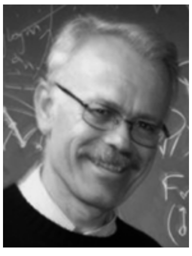
Surprisingly, the word *η μούσα* and therefore the words *το μουσείον* (the museum), *η μουσική* (the music) and *ο μουσικός* (the musician) all seem to have common radix with the verb *μανθάνειν*.

For example, Hesichius saved in his lexicon the word *η μεθήρη* with the meaning "care" and it is believed that the word *η μούσα* (the one who cares about and takes care of the music) originates from this rare word. It seems that the radix *μεν–θ* or *μαν–θ*, depending on whether the dialect is northwestern or southeastern, were used to construct words having the meaning "I turn my mind to something, I care about something, I take care of something, I attempt to achieve something".

Actually, if one accepts the assumption of a common Indo-European language base, this same radix seems to appear in other languages with a similar meaning. For instance, if I recall correctly, in Sanskrit, the word "medha" means "wisdom".

Last but not least, what are your plans for the future?

I will continue my work on research, teaching and advising my graduate students. A couple of years ago I was the Doctoral Mentoring Award Winner at the University of Florida. The best reward for a teacher is to see his students succeed. It is important to be honest, friendly and available to students, to create the opportunity for students to develop short- and long-range educational goals, to understand themselves, to explore the world of research, to foster critical thinking and decision-making skills and to engage in academic planning. In these processes, the advisor serves as an expert in his field and as a provider of general and specific program information. An advisor should also create a positive research atmosphere, reward achievements and maintain an enthusiasm for learning. After all, this is the real meaning of mathematics and a mathematician.



Themistocles M. Rassias [rassias@math.ntua.gr] is a Professor of Mathematics at the National Technical University of Athens, Greece. He received his PhD at the University of California at Berkeley under the supervision of the Fields Medalist Steve Smale. An example of Dr Rassias' contribution in the field of Mathematical Analysis is "Hyers-Ulam-Rassias stability" and "Cauchy-Rassias stability", and in Geometry the "Aleksandrov-Rassias problem". He is a well known author of several articles and books, mainly in the areas of Mathematical Analysis, Global Analysis, Geometry and Topology. He is a member of the Editorial Board of several international mathematical journals including the EMS Newsletter.

The Independent University of Moscow

Yu. S. Ilyashenko and A. B. Sossinsky

Walking along Bolshoi Vlasevsky, a small by-street in the historic centre of Moscow, you will probably notice a new four-storey house, partially hidden by the trees of a tiny square that separates it from the street, a house that looks rather like the office building of some successful financial centre. But its appearance is deceptive; it actually houses the Independent University of Moscow (IUM).

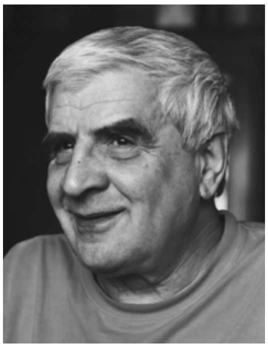
Many Western mathematicians have heard of this university and know that it plays an important role in Russian mathematics and that it has a rather high international reputation. There is lot of talk about the IUM and a great deal has been written about it but very few people can explain what this institution actually is and how it functions. And no wonder; the IUM is extremely unusual, it resembles no other place where mathematics is taught and it has no prototype or clone.

In this article, we shall try to give a brief and balanced account of what the IUM is really like, covering its achievements and its problems.

How it all began

In 1991, perestroika was well underway. For most Russian mathematicians, and many other people, it was a first glimpse of freedom, a time of hopes and hesitation. Should one look for a position abroad? Should one give up mathematics and go into business? Should one try to survive in the new environment, in which salaries of university professors were less than one fifth of the subsistence wage? Or should one passively observe the Russian mathematical school deteriorating as the result of the brain drain?

There was one person, however, who did not hesitate – Nikolay Konstantinov, the influential teacher and organizer of various unofficial mathematics-oriented structures (at one time branded "anti-Soviet" by the communist bureaucracy), e.g. mathematics circles, specialized schools and classes, olympiads and other competitions, and summer institutes[1]. Konstantinov's priority was to revive or create a university capable of teaching mathematics at the level of *mekhmat* (the Mechanics and Mathematics Department of Moscow State University) during the "Golden Years of Soviet Mathematics", described in the book of that title published by the AMS in 2006.



Nikolay Konstantinov – Founder of the IUM

In July 1991, Konstantinov invited the leading Russian mathematicians to Moscow to a classroom in the famous School #57 for a discussion about what should be done about university mathematics education. Practically all of them came: V. I. Arnold, A. A. Beilinson,† R. L. Dobrushin, B. A. Dubrovin, L. D. Faddeev, A. G. Khovansky, A. A. Kirillov, S. P. Novikov, A. N. Rudakov, M. A. Shubin, Ya. G. Sinai and V. M. Tikhomirov.

A. B. Sossinsky was also there, witnessing a heated and fascinating discussion that, unexpectedly, ended in a consensus. It was agreed that *mekhmat* was beyond repair. Indeed, in the 1980s, almost all the best students (most of whom chose either Arnold, Gelfand, Kirillov, Manin, Novikov or Sinai as their research advisor) were not given positions at MSU by the communist administration after obtaining their PhDs. That, together with the brain drain, the disappearance from the research scene of the great mathematicians of the previous generation and the inept and reactionary administration, led to a degradation of the scientific and educational level of the department. On the other hand, a large team of the best alumni of *mekhmat* (who were not involved in the academic world) were full of energy and ready to teach. So, it was further agreed that the only solution was to create a new elite university, a kind of École Normale Supérieure rue d'Ulm in a Russian style, which was not dependent upon the official educational authorities but was a university whose mathematical department would provide a high level research-oriented curriculum for the very best students.

Thus the Independent University of Moscow was born; in September 1991, classes opened in a school near MSU, on the sheer enthusiasm of its founders, without any source of support.

We shall return to our account of the 18 year history of the IUM below.

What the IUM is today

The IUM is not really a university; it is a small elite school training future research mathematicians. Varying in time, there are from 40 to 50 undergraduates[2] and from 10 to 15 graduate students. There are no tuition fees; in fact, the IUM pays all its official students a small stipend. Dur-

[1] To give an idea of the atmosphere of such gatherings, let us quote one of Konstantinov's followers (A. Leman), who taught a class in one of the mathematical schools in Moscow. Answering a question about the mathematical level that his students achieve, he said: "We don't teach people to be mathematicians – we teach them to be free".

[2] Actually, the number of freshman students in the middle of the autumn semester reaches 100 but they are free listeners and only 20–30 of them survive the first examination session and become official students.

ing each academic year, 40–60 different mathematicians[3] give courses at the IUM. Classes take place in the evenings as most students are simultaneously matriculated at other institutions (mostly at *mekhmat* MSU). On average, only 4–5 students finish the 5-year course of study[4] each year at the IUM.

Despite its small size, the IUM is one of the most active mathematical centres of Russia. Here are its main regular activities.

The Globus seminar

This is a general seminar covering all of mathematics and is meant to be accessible not just to experts in the topic under discussion. The talks are like colloquium talks in the US, except that they last two academic hours rather than one. As in the case of the Bourbaki seminar, the proceedings of the Globus seminar are regularly published.

The Math in Moscow program

This is a program mainly aimed at North American students, who come to Moscow to study mathematics "in the Russian way" (but taught in English) for a semester; they can choose several courses from the large range offered and they are credited for completing them at their home institutions.

Contests supporting young mathematicians

The IUM conducts several nationwide contests: two Möbius contests (for undergraduates and graduate students), the Deligne and Dynasty contests for young PhD students and the Dobrushin contest, which is especially aimed at the IUM: it sponsors five stipends for undergraduates and a full professorship for a year at the IUM.

The Moscow Mathematical Journal

This is a relatively new international mathematics journal, published in English and distributed by the American Mathematical Society, which now has the highest (by far) citation index among Russian mathematical journals.

Publication of textbooks and monographs

Many of the courses taught at the IUM then appear as textbooks or monographs published and distributed by MCCME[5] publishers. Fifteen of them have been translated into English. Besides textbooks and monographs, the IUM teachers (and our colleagues from the MCCME) have authored some 50 popular science brochures in mathematics and its applications.

The IUM has also been active in creating other mathematical structures, with which it remains in close contact.

The Mathematics Department of the State University – Higher School of Economics (HSE)

Created only last year, this is a department of fundamental mathematics in the framework of an institution of higher learning that, unlike the IUM, bestows a state approved diploma. Most of the teaching staff come from the IUM. This new faculty has already positioned itself as a viable competitor of *mekhmat* and of the Department of Computational Mathematics of MSU.

The Moscow Center of Continuous Mathematical Education

The MCCME was created (just as the IUM) with the initiative of N. Konstantinov. It is a very active independent organization involved in mathematics education in middle and high schools, olympiads and other contests, specialized mathematics schools and classes, teacher training and other related activities. It is now in charge of the building in which the IUM is housed and actually administers the logistics of the IUM (accounts department, office supplies, computer network, cleaning and repairs, etc.).

The Poncelet Laboratory

This is a French-Russian mathematics lab jointly run by the CNRS and the IUM. Besides six French researchers working in Moscow for a year, it hosts a large number of short-term visitors and organizes up to nine small international conferences every year. The present director is M. A. Tsfasman, the first director being A. B. Sossinsky.

There is a lot more to say about both the MCCME and the Poncelet Lab; more information may be found at http://www.mccme.ru/ and http://www.poncelet.ru. Other activities are described in more detail below.

Some more history

The first classes at the IUM were held in September 1991 in the Lyceum of Information Technologies (a ten minute walk away from Moscow University) on weekday afternoons. The lecturers were E. B. Vinberg (algebra), A. A. Kirillov (calculus) and A. B. Sossinsky (geometry). The IUM had no official status and no financial support at all. In fact, in order to cover the lyceum's increased electricity bill and extra hours for the cleaning women, the teaching staff actually used money out of their own pockets – a unique situation, in which professors were not paid for teaching but instead had to pay to be able to do it!

During the first years of the IUM's existence, the technical part of the administration consisted of friends and followers of Konstantinov: M. Vyalyi (organization of studies), V. Imaikin and V. Prasolov (lecture notes and publications) and S. Komarov (finances). The first Rector (i.e. President) of the IUM was the outstanding mathematical physicist and leading research fellow of the Steklov Institute, the late M. K. Polivanov, also known for his collaboration with Solzhenitsyn. He was one of the authors of the collection of papers (illegal in Soviet times) *Iz Pod Glyb* (From Under the Boulders). Solzhenitsyn writes about him in his autobiography.

---

[3] Most of them hold permanent positions at other institutions and the IUM pays them symbolic honoraria for their courses on a per hour basis.

[4] The MS-level diploma delivered by the IUM does not have "accreditation", i.e. the official seal of approval of the Ministry of Education. Nevertheless, students with the MS-level diploma have been accepted on PhD-track programs at e.g. Harvard and the Steklov Mathematical Institute.

[5] The Moscow Center of Continuous Mathematical Education (see below).

At first the Independent University had two subdivisions: the College of Mathematics and the College of Mathematical Physics. In the latter, the first Dean was the late O. V. Zavialov; he was followed by A. N. Kirillov. Among the most active professors, let us note A. Pogrebkov and V. Pavlov. In the early years, mathematics courses were taught by S. P. Novikov and D. V. Anosov, the classes taking place in the Steklov Institute. However, the College of Mathematical Physics only survived for about ten years. Since 2005, the Steklov Institute has created its own educational centre (headed by V. Pavlov and D. Treschev), which is, in a sense, a continuation of the College of Mathematical Physics but not formally part of the IUM.

The first Dean of the Mathematics College was A. N. Rudakov; after his departure to Norway, he was replaced by Yu. S. Ilyashenko in 1994. In 2000, Ilyashenko was elected President of the IUM, which by then consisted de facto of the mathematics college only[6]. From 1992 to 1996, the mathematics college classes were hosted by the famous specialized School #2, whose principal was then P. V. Khmelinsky.

In 1995, a minor miracle occurred, a miracle without which the IUM would probably not have been able to survive: the Prefect of the Central District of Moscow A. I. Muzykansky, one of the leading political figures of Moscow in the early days of perestroika, convinced the Moscow Major Yu. M. Luzkov to provide the IUM with a building of its own – the one mentioned at the beginning of this article. More precisely, the building was allocated to the newly created Moscow Center of Continuous Mathematical Education[7].

The newly reconstructed MCCME-IUM building

The building was reconstructed and furnished at the city's expense and opened in an official ceremony in the presence of Muzykansky (representing the Mayor), the President of the Russian Academy of Sciences Yu. S. Osipov, the Rector of MSU V. A. Sadovnichi, Academician V. I. Arnold and other personalities.

In the new building, the IUM continued and extended the teaching process to include a graduate school (created thanks to the efforts of Victor Ginzburg and Alexander Beilinson) and progressively widened the spectrum of its other activities (listed above).

Looking back at the history of the IUM, one should not forget the crucial help extended to us by several foreign colleagues. Among them are Michiel Hazewinkel of the Netherlands and H. Samuelson of the US (both of whom contributed math books to our library), Pierre Cartier (who hosted our students in Paris during the ENS-IUM exchanges), Jean-Michel Kantor (who organized the support of the Société Mathématique de France to the IUM), William Faris, who organized for the IUM "Bill Faris emergency Foundation" to give support in case of disaster, William H. Jaco, John Ewing, Sergei Gelfand and Galina Kovaleva (who organized different kinds of support by the American Mathematical Society), Dan Stroock (who headed a corporation for the support of Russian mathematicians in the early 90s), and Jaco Palis, who supported the IUM in his capacity of the President of the IMU, Mr and Ms Clay, founders of the Clay Mathematics Institute (CMI), Arthur Jaffe and James Carlson, former and present Directors of the CMI, who provided long lasting support to the IUM, Felix Browder and Christiane Rousseau, who initiated the stipends of the AMS and the CMS for the American and Canadian participants of the MIM program, and last but not least, Robert MacPherson and Pierre Deligne, whose practical and moral support during all these years was invaluable and whom we regard, together with the mathematicians who decided to create the IUM, as founding fathers of the Independent University.

Our faculty, teaching and research

In the first years of the IUM's existence, classes were conducted by D. V. Anosov, V. I. Arnold, A. A. Kirillov, S. P. Novikov, A. N. Rudakov, V. M. Tikhomirov, E. B. Vinberg and other outstanding lecturers of that generation. At the present time, our lecturers include Victor Vassiliev (Vice-President of the Moscow Mathematical Society, plenary lecturer at the ICM in 1994 and Member of the Russian Academy of Sciences), Boris Feigin (plenary lecturer at the ICM in 1990), Maxim Kazaryan (whose research work in mathematics was declared to be the best in the Russian Academy of Sciences in 2005), Sergei Natanzon (recipient of the Dobrushin fellowship and a leading expert in complex analysis and Teichmüller

Opening ceremony of the MCCME–IUM building: S. Gusein-Zade, N. Konstantinov, A. Khovanski, Yu. Ilyashenko, A. Sossinsky, P.P. Baskevich, V.Arnold.

[6] The original IUM project was to create a real university with several colleges. At different times of its history, attempts were made to create a college of biochemistry and a college of philology and linguistics but these attempts proved unsuccessful.

[7] To some extent, the MCCME was created specifically for this purpose because, according to the existing rules, the city authorities could only support primary and secondary education, higher education being under the auspices of the federal authorities.

Opening ceremony of the MCCME–IUM building: V. Arnold speaking, A. Gonchar, Yu. Ilyashenko, A. Muzhykantski (seated left to right)

spaces), Alexander Belavin (one of the world leaders in quantum field theory), Alexander Kuznetsov (winner of the Möbius and Deligne contests and awarded the European prize of Best Young Mathematician in 2008 and one of the four Prizes of Best Young Scientist in 2009 by President Medvedev), Sabir Gusein-Zade, Sergei Lando, Stefan Nemirovski, Askold Khovanski and many others.

The IUM has no permanent positions in the sense used in regular universities. The faculty members of the IUM listed on our webpage are those who have given at least a one semester course over the whole history of the IUM. This list includes more than a hundred names. Among them are those who have given courses occasionally, those who have given a series of courses from time to time and then quit for a while and those who have given various courses regularly, forming, as it were, the backbone of the Independent University.

The IUM encourages Russian mathematicians now working in the West to come to the IUM and to give intensive "crash courses". In two or three weeks, the equivalent of a semester course may be taught in this way. Such courses have been given by A. Katok, I. Krichiver, A. Kuksin and M. Shubin among others. This tradition helps to maintain the participation of the Russian mathematical diaspora in the mathematical life of their homeland.

From time to time we invite our young alumni, or even our students, to give basic or higher level courses. This is a bold practice but has proved to be successful. Several of our young alumni have become part of the backbone of the IUM in the 2000s.

The courses are addressed to very strong listeners and are really intensive. Yet the goal is that the material should be grasped on the spot. The lectures are accompanied by a list of problems that are discussed with the students at the exercise classes by the instructors and their assistants. The problems are not mere exercises but rather allow the participants to rediscover parts of the material included in the courses.

There are several research seminars at the IUM, which are sometimes organized as research groups with a flexible meeting schedule. In one of the classrooms, you may often meet Boris Feigin talking to a group of his students about various topics of modern mathematics. Grigory Olshanski has taught his graduate students at the IUM from the beginning of the 1990s and continues today. One of his first students at the IUM was Andrey Okunkov. A seminar headed by Victor Vassiliev took place at the IUM for about ten years. The seminar by M. Kazarian and S. Lando on combinatorics and topology has been taking place at the IUM for many years. M. Tsfasman has trained several young number theorists at the IUM via small informal seminars. The seminar on Riemann surfaces, Lie algebras and mathematical physics headed by S. Natanzon, O. Sheinman and O. Shwartzman has been running for over a decade. These and many other groups and seminars have been the training ground for numerous young researchers. They have become winners of the Pierre Deligne and Dynasty contests, faculty members of the IUM, young researchers at the Steklov Institute and have found teaching and research positions in various other institutions, including the HSE.

Our alumni

In 1996, the first 8 students of the IUM were each awarded its MS-level diploma. In all, 58 students have graduated from the Independent University, an average of only 4–5 graduates per year. Almost all of them have become research mathematicians. Most of them also graduated from *mekhmat* MSU. However, several IUM students did not study in parallel at any other institutions and, although IUM diplomas are not accredited by the Russian authorities, all of them were accepted onto PhD-track programs at prestigious institutions: N. Markaryan to the Steklov Institute in 1995, V. Vologodsky to Harvard in 1996, V. Kirichenko to the University of Toronto in 2002 and R. Travkin to MIT in 2007.

Let us add a few words about the exceptional case of Roman Travkin. He has suffered since early childhood with cerebral palsy. Confined to a wheelchair, his manual coordination is insufficient to use an ordinary computer keyboard, his slurred speech is difficult to understand and he is only able to function with the constant assistance of his father Mikhail Travkin, who devotes all his time, in fact his whole life, to Roman. Roman took first place in the Russian national olympiad but no Russian university would accept him – for medical reasons – except for the IUM, where he got his MS diploma (completing the five-year course in four years) and which organized financial support for him and his father during his studies in Moscow, in particular from the Dynasty Foundation headed by D. B. Zimin and from the Clay Institute. At present, Roman Travkin is a second year graduate student at MIT, working with R. Bezrukavnikov. Recently, he passed his qualifying exam with a well above average score and has two research papers in publication.

The Globus Seminar

The Globus Seminar, headed by M. A. Tsfasman and Yu. S. Ilyashenko, has been running since 1997. At first, it was intended as a seminar for students, with the aim of broadening their mathematical culture. The first lecturer was V. I. Arnold. Progressively, it became one of Moscow's leading mathematics seminars. At different times, talks have been given by S. P. Novikov (Fields medallist, 1970), Ya. G. Sinai, Yu. I. Manin, P. Deligne (Fields medallist, 1978), Steve Smale (Fields medallist, 1966), M. Kontsevich (Fields medallist, 1998), L. Lafforgue (Fields medallist, 2002), A. Okounkov (Fields medallist, 2006), J.-P. Serre (Fields medallist, 1962) and many other outstanding mathematicians.

The talks are registered, then written out by V. V. Prasolov and, after the authors' corrections, published in the form of seminar proceedings. Six such volumes have appeared to date. The first two have been translated into English by Cambridge University Press.

The Math in Moscow (MiM) program

The program started functioning in 2001. Since then, over 150 North American and European undergraduates have participated in it, including students from Harvard, Princeton, MIT, Berkeley, Cornell, Yale, McGill, Toronto and Montreal. The American students are supported by 10 NSF grants every year (awarded by the American Mathematical Society) and the Canadian students by 3 NSERC (National Science Education Research Center) grants. Though the majority of the students of the MiM program come from the US and Canada, the program encourages the participation of European students too. At present, MiM is a joint program of the IUM, the Higher School of Economics and the MCCME. The IUM carries on the mathematical part of the program; from September 2008, it has shared this job with the new Mathematics Department of the HSE. Moreover, the HSE provides the dormitory rooms and visa support whilst the MCCME resolves other logistical problems.

Here are some reactions of Math in Moscow students.

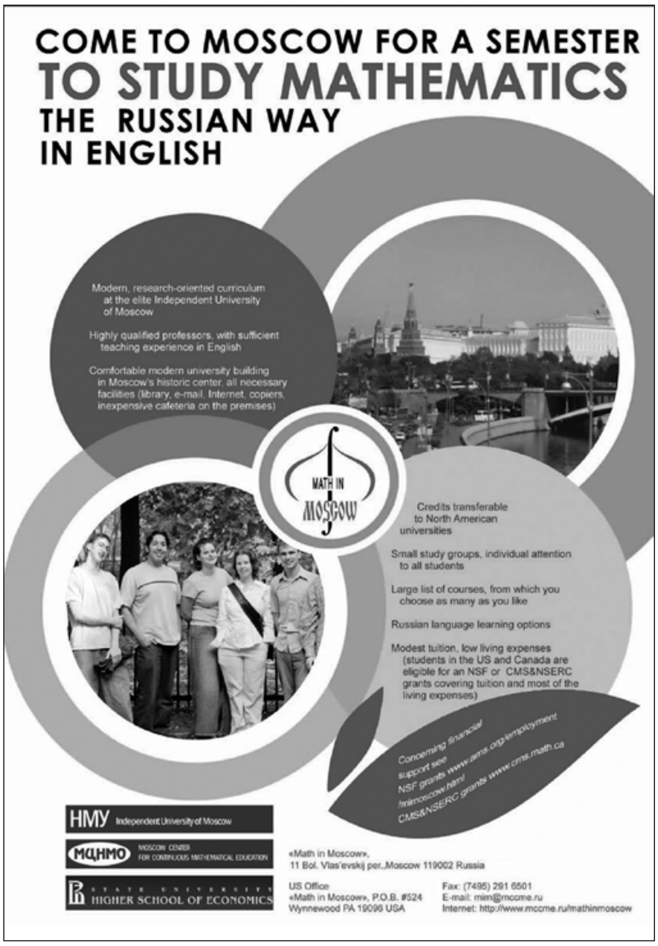
I think I learned as much or more than I would have had I stayed in the US. Certainly I wouldn't have experienced as much emotional growth. Being in another country really changed me.
Ian Le, Harvard University, Autumn 2002

The department here is the friendliest and closest I've ever found. It's been an absolute joy to be here, and I can't wait to come back.
Tom Church, Cornell University, Autumn 2005

Thank you very much for everything: great mathematics, wonderful experience and lots of fun. Moscow looked more beautiful for me than I expected it to be and your university is much better. I've heard that the IUM is a very interesting place but my stay here was even more pleasant than I thought.
Andrei Negut, Princeton University, Spring 2007

During his stay in Moscow, Negut began to do some serious research with Yu. S. Ilyashenko; this research is still in progress.

It should be understood that the program is not a bed of roses; we have had problems with some (actually very few) of the students and, until this year, finding adequate living quarters for them was quite a headache for the administrators of the program, headed by Irina Paramonova (Shchepochkina). Fortunately, that problem seems to have been resolved by means of the dormitories of the State University-Higher School of Economy. More about the IUM-HSE cooperation will be mentioned below.



A poster of the MiM program.

For more details about the MiM program, see the website www.mccme.ru/mathinmoscow.

The Moscow Mathematical Journal (MMJ)

The Moscow Mathematical Journal was first published in 2001. Its founders are Yu. S. Ilyashenko and M. A. Tsfasman, and together with S. M. Gusein-Zade, they are the Editors-in-Chief. The Editorial Board of the MMJ includes four Fields Medal laureates (L. Lafforgue, G. Margulis, S. Novikov and S. Smale) and several other famous mathematicians. It is published in English (we have been unsuccessful in trying to obtain financial support for a Russian version) and distributed by the American Mathematical Society. To our surprise, the journal has been rather successful: according to data from the AMS, it has the highest citation index among the Russian mathematical journals (0.6 as compared to 0.34 for *Uspekhi* and 0.33 for *Funct. Anal. Appl.*).

Among the authors published in the MMJ, let us note V. I. Arnold, Yu. I. Manin, S. P. Novikov, Ya. G. Sinai, R. A. Minlos, A. G. Khovansky, B. L. Feigin, W. Brieskorn, P. Cartier, J.-P. Serre, L. Lafforgue and M. Kontsevich.

Support for young mathematicians

The Möbius Contest was organized in 1997 by two successful businessmen A. Kokin and V. Balikoev, who were former students of the applied mathematics department of the Moscow Institute of Electronics and Mathematics. At first it was aimed at supporting one undergraduate or graduate student of the IUM. Vadim Kaloshin, now Michael Brin Chair Professor at the University of Maryland, raised additional funds in the US for the contest and thus increased the number of awards. Pierre Deligne contributed some funds from his Balzan Prize as well. At present the Möbius contest awards five biannual stipends aimed at supporting graduates and undergraduates from all over Russia.

The Pierre Deligne Contest. In 2004, Pierre Deligne was awarded the Balzan Prize, worth one million Swiss franks, half of which had to be used according to the rules of the prize to support some mathematical project. Deligne decided to support, as he wrote in a letter to Yu. S. Ilyashenko, "Russian mathematics struggling for survival". Together with Ilyashenko, he created the "Pierre Deligne Contest" for under-35 mathematicians with a PhD degree and convinced the Balzan Foundation to contribute the funds to Russia. The contest has awarded 16 three-year fellowships in the period 2005–08. The funds are now exhausted but Deligne is resolute about continuing to fund the contest from his own sources.

The Dynasty Contest. In May 2006, Arnold and Ilyashenko convinced D. B. Zimin, a prominent businessman and the organizer and head of the charity foundation "Dynasty", to organize an annual "Dynasty Foundation Contest" for three winners. The jury of both contests is the same and the mathematical interests of its members cover almost all the landscape of modern mathematics. The contests support young mathematicians who are well-known to the community but also reveal new brilliant names.

The Dobrushin Stipend. One of the admirers of the talent of the late R. Dobrushin established a stipend for five IUM students. It is awarded every six months for a half year period. In parallel, a one-year Dobrushin fellowship for one IUM professor is awarded.

The mathematics department of the HSE

The State University – Higher School of Economics is a new institution and one of the most popular universities of economics and humanities in Russia (see http://www.hse.ru/lingua/en/about.html). In 2007, the president of the HSE Ya. I. Kuzminov put forward the idea of a broad cooperation between the HSE and the IUM. As a response, the IUM suggested creating a Department of Fundamental Mathematics as part of the HSE.

The new department includes a part of the faculty of the IUM and uses the teaching style and programs of the IUM, slightly modified if needed. The goal is to make the new department as strong as the Mechanics and Mathematics Department of the MSU. The Dean is S. K. Lando. Three chairs, Algebra, Geometry and Discrete Mathematics, are headed by A. N. Rudakov, V. A. Vassiliev and S. K. Lando, respectively.

The new department started teaching about 40 freshman students on 01 September 2008. We strongly hope that the department will become an important new mathematical centre in Russia, having taken much of its expertise and spirit from the IUM.

The legal status of the IUM

From the very beginning, the administration of the Independent University strived to transform the informal educational institution to a *bona fide* university with all the usual legal attributes of an ordinary Russian institution of higher learning, namely:

(i) Registration and licence (which would make the teaching process legal).
(ii) Accreditation (after which the IUM MS-level diploma would be officially recognized).
(iii) Freeing the students from the draft.
(iv) Holding classes in the daytime (rather than in the evening).
(v) Running a graduate school.
(vi) Having a scientific council authorized to deliver PhDs.

Battling the educational bureaucracy to obtain these legal rights, the present IUM administration, as well as the previous ones, have proved to be remarkably ineffective. Only items (i) and (v) have been achieved. Moreover, in 2006, we lost item (i); the right to teach mathematics legally was taken away from us by the Ministry of Education.

At present the IUM has changed its status, becoming, from a legal point of view, a subdivision of the MCCME. As such, its collaborators are allowed to teach but they can only deliver a diploma for "additional education" (which is officially recognized by the state but is not legally equivalent to an ordinary MS degree). But, overall, the program taught at the IUM is still equivalent to a university MS program.

Our budget and our sponsors

The IUM is not supported by the Ministry of Education nor by the city of Moscow; it is privately endowed and functions with a very small budget. We are grateful to our Russian sponsors: A. Vavilov, President of the Human Capital foundation, A. Volozh, President of the Yandex company, D. Zimin, President of the Dmitry Zimin Dynasty foundation, for their past and present support of the IUM. The main sources of income are:

- Support from the Clay Mathematics Institute, continued from 1998.
- The Math in Moscow program.
- Overheads of various research grants.
- Support from the Yandex company (now suspended).
- Support from the Human Capital Foundation (now stopped).

In total, the budget of the IUM is appoximately 30 times smaller than that of a mathematics department of a similar size in an American university.

An indirect, yet very important, support is provided to the IUM by the Steklov Mathematical Institute (director: V. V. Kozlov) and the Institute for Information Transmission Problems (director: A. P. Kuleshov).

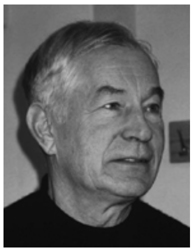
Perspectives

Originally, the IUM was meant to be a small, privately endowed, elite university consisting of several colleges, granting MS and PhD degrees officially recognized by the Russian authorities but independent of the Ministry of Education. This ambitious project was never realized. Moreover, its only remaining college, the mathematical one, was not only refused accreditation by the ministerial bureaucracy but had its teaching licence cancelled.

And yet the Independent University still exists and remains one of the leading mathematical centres of Russia. Its parallel *alter ego*, the Mathematics Department of the HSE, brings together the advantages of a college in a state-approved, accredited university with a high-class teaching staff coming from the IUM and carrying on the development of a modern, research-oriented curriculum.

We are optimistic about the near future. We do not agree with those who fear that the creation of the HSE Mathematics Faculty, competing with the IUM, will lead to the latter's degradation. We believe that the IUM is stable and that it has its own ecological niche. It is difficult to imagine the Moscow mathematical scene today without the Independent University, the building it shares with the MCCME, its Globus seminar, its Math in Moscow students, its nationwide support of young mathematicians via the Möbius, Deligne, Dynasty and Dobrushin contests, the Moscow Mathematical Journal, the Poncelet Lab and, above all, its tiny group of talented and highly motivated undergraduates and graduate students, its underpaid, first-class teachers and researchers who work with an enthusiasm and flair that defies, to our mind, any logical explanation.

Yulij S. Ilyashenko [yulij@math.cornell.edu] is President of the Independent University of Moscow. He is a professor at the Department of Mathematics of the Cornell University (half-time position for the autumn semesters), a professor at the Mechanical Mathematical Department of Moscow State University, a leading scientist of the Steklov Mathematical Institute of the Russian Academy of Sciences (half position) and Vice-President of the Moscow Mathematical Society.



Alexey Sossinsky [asossinsky@yandex.ru] was born in Paris in 1937 and has a French high school education, a BS degree from NYU and an MS and a PhD from Moscow State University. Basically a research mathematician (known for his work in topology, in particular knot theory), he has always had a strong interest in mathematics education and in the popularization of mathematics. He began his mathematical career as an associate professor at MSU, was forced to leave MSU for political reasons in 1974 and worked for 13 years on the popular science magazine "Kvant". At present, he is a professor at, and the Vice-President of, the Independent University of Moscow. Sossinsky is the author of over 50 research articles, several mathematical monographs and popular science books, including a book on knot theory that has been translated into six languages.

ICMI column – http://www.mathunion.org/icmi

Mariolina Bartolini Bussi (Member of the Executive Committee of the International Commission on Mathematical Instruction)

Mathematics Education and Technology – Rethinking the Terrain

The 17th ICMI Study
Series: New ICMI Study Series, Vol. 13
Hoyles, Celia; Lagrange, Jean-Baptiste (Eds.)
2010, XIV, 494 p. 50 illus., Hardcover

Mathematics Education and Technology – Rethinking the Terrain revisits the important 1985 ICMI Study on the influence of computers and informatics on mathematics and its teaching. The focus of this book, resulting from the 17th Study led by the ICMI, is the use of digital technologies in mathematics teaching and learning in countries across the world. Specifically, it focuses on cultural diversity and how this diversity impinges on the use of digital technologies in mathematics teaching and learning. Within this focus, themes such as: mathematics and mathematical practices; learning and assessing mathematics with and through digital technologies; teachers and teaching; design of learning environments and curricula; implementation of curricula and classroom practice; access, equity and socio-cultural issues; and connectivity and virtual networks for learning, serve to organize the study and bring it coherence.

Providing a state-of-the-art view of the domain with regards to research, innovating practices and technological development, *Mathematics Education and Technology – Rethinking the Terrain* is of interest to researchers and all those interested in the role that digital technology plays in mathematics education.

The list of contents includes:

Section 1.
2 Design of learning environments and curricula.
3 Designing software for mathematical engagement through modelling.
4 Designing digital technologies and learning activities for different geometries.
5 Implementing digital technologies at a national scale.

Section 2.
6 Learning and assessing mathematics with and through digital technologies.
7 Integrating technology into mathematics education: theoretical perspectives.
8 Mathematical knowledge and practices resulting from access to digital technologies.
9 The influence, and shaping, of digital technologies on the learning – and learning trajectories – of mathematical concepts.
10 Micro-level automatic assessment supported by digital technologies.
11 Technology, communication, and collaboration.

Section 3.
12 Teachers and technology.
13 Working with teachers: context and culture.
14 Teachers and teaching: theoretical perspectives and issues concerning classroom implementation.
15 Teacher education courses in mathematics and technology: analyzing views and options.

Section 4.
16 Implementation of curricula: issues of access and equity.
17 Some regional developments in access and implementation of digital technologies and ICT.
18 Technology for mathematics education: equity, access, and agency.
19 Factors influencing implementation of technology-rich mathematics curriculum and practices.

Section 5.
20 Future directions.
21 Design for transformative practices.
22 Connectivity and virtual networks for learning.
23 The future of teaching and learning mathematics with digital technologies.

The list of content is really impressive and answers most questions raised by teachers and teacher educators all over the world. It bears witness to the ICMI's actions in the field of information and communication technologies.

Section 1 focuses on the issues and challenges involved in designing mathematics learning environments that integrate digital technologies, while recognizing that the tools made available in such environments can and do shape mathematical activity in ways that to some extent are predicable and in some not. In addition to considering the specific opportunities and constraints of different digital technologies for structuring mathematical learning experiences (including various software packages, hardware configurations and the Internet), the implications of design decisions on tools, curriculum, teaching and learning are considered.

Section 2 focuses on developing understanding of how technologies might enhance or constrain the learning and teaching of mathematics, and the implications for assessment practices. The focus includes consideration of how digital technologies might be employed to develop a learner's knowledge and on how interactions with digital tools mediate learning trajectories. Additionally, the theme addresses the challenges involved in balancing the use of mental, paper-and-pencil and digital tools in both assessment and teaching activities.

Section 3 discusses how the integration of any new artefact into a teaching situation could be expected to alter its existing equilibrium and require teachers to undergo a complex process of adaptation, with modifications in the case of digital technologies likely to be particularly pronounced. Various frameworks, drawing from both theory and practice, are currently employed to analyze the role of the teacher in orchestrating technology-integrated mathematics learning. Complementarities and contrasts between these frameworks are considered. The way that the frameworks are operationalized in the face of ever-evolving resources and the implications of these complex issues for a teacher's professional development are discussed.

Section 4 starts from the observation that access to, and use of, digital technologies differs between countries, and within countries, according to socio-economic, gender and cultural factors. The focus is on the way to understand how cultural practices in technology-integrated mathematics enhance, or erode, equity and agency in mathematics education.

Section 5 contains three chapters where the authors look at the overall landscape concerning the potential and impact of digital technologies on mathematics teaching and learning, and consider future prospects and challenges.

It really is a pity that it was not possible to include the text of the opening plenary of the Study Conference held in Hanoi (December 2006) given by Seymour Papert, a scholar of vision, experience and stature in the field of mathematics, mathematics education and technology. Unfortunately, Papert had a terrible accident the following day and is still undergoing extensive rehabilitation. A short abstract of his speech is given by the editors, who claim, however, that the spirit of his talk was the central idea of both the conference and the volume.

Hopefully, this volume will be a reference for the many teachers and teacher educators all over the world. Indubitably, digital technologies are one of the major focuses of instruction (and not only mathematical instruction) everywhere, due to the increasing availability of cheap computers. It is worthwhile mentioning that at the symposium on *The first century of the International Commission on Mathematical Instruction (1908–2008)*, celebrated in Rome in 2008 (Menghini et al., 2008), Working Group 4 was dedicated to *Resources and technology throughout the history of ICMI*. In that case, however, the historical perspective was emphasized, putting digital technologies in the trend of resources available for teachers according to Klein's original intentions. This historical reconstruction, summarized in the quoted proceedings, is now published in an extended way in the first issue of *ZDM – The International Journal on Mathematics Education* (Borba & Bartolini Bussi, 2010); the issue collects together revised versions of the papers presented in Rome in Working Group 4.

To end this column about the ICMI's actions to promote the effective use of digital technologies, it is worthwhile quoting the 2009 UNESCO award won by Alexei Semenov, a member of the ICMI Executive Committee until December 2009 and Rector of the Moscow Institute of Open Education. The purpose of the UNESCO King Hamad Bin Isa Al-Khalifa Prize for the Use of Information and Communication Technologies in Education is to reward projects and activities of individuals, institutions, other entities or non-governmental organizations for excellent models, best practice and creative use of information and communication technologies to enhance learning, teaching and overall educational performance. The citation reads as follows: *Under the leadership of its Rector, Alexei Semenov, the Moscow Institute of Open Education has provided in-service training to about 30,000 teachers annually for the past 16 years. Professor Semenov has developed exemplary programmes to enable teachers to include ICTs in their work, as well as textbooks and teacher guides used widely in the Russian Federation and other countries*.

References

Borba M. & Bartolini Bussi M. G. (eds.) (2010) Historical aspects of the use of technology and devices in ICMEs and ICMI. *ZDM – The International Journal on Mathematics Education*, vol. 42, n. 1.

Hoyles C. & Lagrange J. B. (2010) *Mathematics Education and Technology – Rethinking the Terrain*. The 17th ICMI Study, Springer.

Menghini M., Furinghetti F., Giacardi L. & Arzarello F. (eds.) (2008) *The First Century of the International Commission on Mathematical Instruction (1908–2008). Reflecting and Shaping the World of Mathematics Education*. Rome: Istituto della Enciclopedia Italiana fondata da Giovanni Treccani.

Young researchers in mathematics education will meet at the fifth European Summer School

Paolo Boero, Scientific Advisor for YESS-5

In the previous issues of this newsletter, information has been given about the history and the aims of the European Society for Research in Mathematics Education (ERME), about its last conference CERME 6 (Lyon, 2009) and about the two main ERME initiatives for young researchers in mathematics education (the YERME day and the YERME Summer School), jointly promoted by ERME and YERME (the ERME community of Young European Researchers in Mathematics Education).

This article will give some information on the preparation of the next Summer School (YESS-5, taking place near Palermo, Sicily, 18–25 August 2010; for further information, see http://math.unipa.it/~grim/YESS-5/Home_YESS-5.html) and the situation of doctoral studies in mathematics education in Europe (with a glance outside Europe) as the analysis of "application forms" for YESS-5 is ending.

111 people submitted their application forms: 7 Masters students, 102 PhD students and two postdocs. 82 are from the universities of 17 European countries, 18 are from the universities of four Mediterranean non-European countries, 10 are from the universities of Canada, USA and Mexico, and one is from an Australian university.

These data are interesting because they show the increasing interest of PhD students from Europe (and other countries as well) for the European Summer School: the number of applications has constantly increased from 62 for YESS-1 in Klagenfurt (Austria) to 95 for YESS-4 in Trabzon (Turkey) to 111 for YESS-5 this year, with an increasing number of applications from non-European countries (28 this year compared to 6 for YESS-1).

A difficult problem (to be solved over February 2010) will concern the choice of participants: in the previous school it was not difficult to choose participants according to easy-to-apply criteria (like the exclusion of those Masters students who had not yet matured a precise research orientation). This year, the choice of 60 participants out of 111 applicants (102 are PhD students in mathematics education) will be much more difficult!

The application forms contain interesting data about the situation of PhD studies in mathematics education in Europe (and also outside Europe). Indeed, applicants must provide the Program Committee (which is responsible for admission to the Summer School) with essential information concerning their supervisor, the subject of their thesis, their scientific interests and preparation.

Most PhD students of our sample are from the same country as their university; however, in 16 cases they come from another country. The country that hosts a higher number of foreign PhD students is France.

Concerning the preparation of the PhD students, while in some cases PhD studies seem to be mostly devoted to the preparation of a thesis (in general, this happens when general preparation is ensured by a specific Masters curriculum in mathematics education or in mathematics and sciences education), in other cases PhD studies include courses not specifically aimed at the preparation of a thesis. Also, the scope of the thesis shows important differences between different countries: the subject may concern a rather broad subject - like, for instance, the use of ICT in the teaching and learning of calculus – or a narrower topic – like the evaluation of the learning of the concept of a derivative. According to our knowledge of PhD studies in other disciplines, all these differences seem to be related to differences concerning the same issues in other fields.

The most interesting data concern the subject of the PhD thesis. From available application forms (and also learning from the experience of the previous summer schools) we can see how different conceptions about mathematics education between different countries (and, in some cases, between different universities in the same country) result in important differences concerning:

- The content. It may concern very different subjects, for example: evaluation and assessment in mathematics instruction; the teaching and learning of a specific (more or less narrow) subject; modelling of teaching and learning situations; curricular innovations; and comparative studies.
- The relevance of the personal commitment in the elaboration of the theoretical framework. (In some cases it consists of mere references to existing epistemological elaborations, learning theories, etc. In other cases a strong personal elaboration seems to be required.)
- The nature of the theoretical framework (important differences concern the involved disciplines, from history and epistemology of mathematics to psychology, sociology of education, etc., and their importance in the framework).
- The importance and depth of the analysis of the mathematical content dealt with in the thesis.
- The methodology. Different methodological choices concern the design and analysis of experimental activities, the nature and analysis of collected data, etc.

From our knowledge of different countries, these differences seem to depend both on different research orientations and theories, and on different institutional constraints. Concerning this issue, several PhD students in our sample come from departments of mathematics, some others come from departments of mathematics and science education, others come from faculties of sciences of education. Moreover, in some cases a systematic collaboration exists between the academic institutions and the school system and in other cases no interaction exists.

Given this situation, the task of the organizers of the European Summer School, and especially of the "experts" who will act as coordinators of the Thematic Working Groups of the school, will not be easy at all (as it was not easy in the previous schools!). Our commitment concerns the general aims of the school, intended to promote cooperation and collaboration among young researchers and the development of a European space of research in mathematics education.

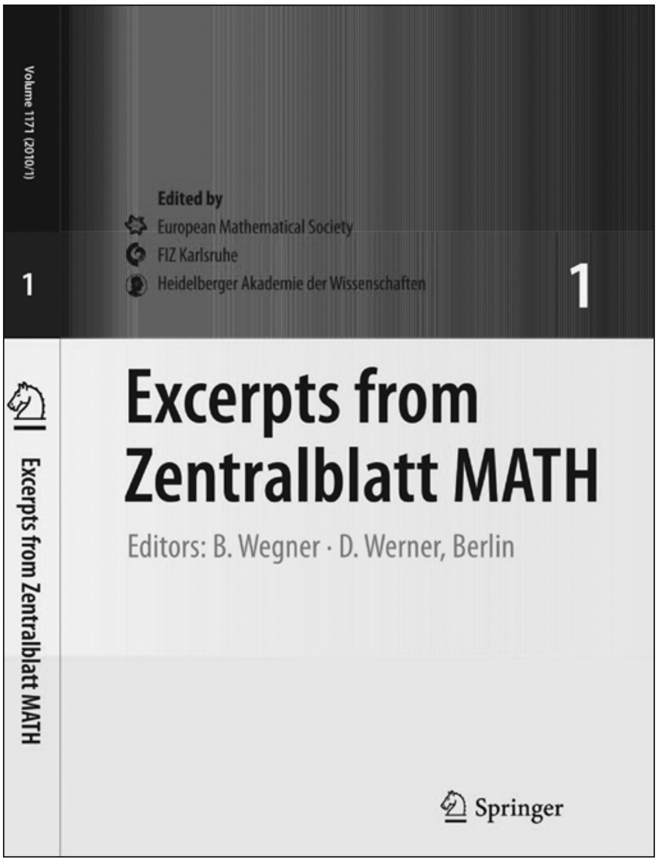
In particular, a strong effort is needed to exploit the present diversities as opportunities to improve the quality of European research in mathematics education through the comparison of how some crucial problems concerning the teaching and learning of mathematics (e.g. teacher preparation, the use of technology in mathematics education, and the teaching and learning of proof) are dealt with in different countries and research teams, in the perspective of the mutual enrichment of research tools and methodologies.

Zentralblatt Column

Dirk Werner

In January 2010, the first issue of Zentralblatt's new print service *Excerpts from Zentralblatt MATH* was published.

Excerpts appears monthly with 150 pages and, unlike its predecessor, will present only a cross section of the reviews in the ZBMATH database. Of course, full coverage of the reviews will still be available from the database.



Until 2009, the printed version of Zentralblatt included approximately 70% of the entries in the database in the core fields of mathematics and approximately 30% of those in the applied areas like statistics, computer science, mathematical physics and economics. This resulted in twenty-four 600-page volumes per year that were physically difficult to handle by readers and librarians alike.

Experience has shown that fewer and fewer readers have been using the print version rather than the database to support their work. Indeed, the search facilities that hardcopy volumes provide are very limited and for leisurely browsing the printed copies are not exactly handy. In response to this, the old print service has been discontinued and replaced with the new *Excerpts*, which will offer a selection of reviews representing all areas of mathematics and its applications. Basically, all book reviews from our database will be presented in addition to reviews of journal articles with more than a narrow interest for the mathematical community. Reviews of the nth-edition of a monograph with little or no changes with respect to previous editions may well be skipped. As for journal articles, the choice does not follow a mathematical algorithm. Should the Riemann hypothesis ever be proved, a review of the ensuing article will obviously appear in *Excerpts*; but, even below this millennial level, there is a lot of interesting mathematics that is described in informative reviews, some of which will be found in *Excerpts*. We are confident that this selection, made by the editors and the scientific staff at Zentralblatt, will appeal to a wide audience of mathematicians.

Excerpts will be printed in a two-column format in a size not unlike the EMS Newsletter, allowing leisurely reading, ideally in an armchair environment.

Number Theory. Algebra. Algebraic Geometry. 19

Number theory. Algebra. Algebraic Geometry

1175.11001
Mollin, Richard A.
Fundamental number theory with applications. 2nd ed.
Discrete Mathematics and its Applications. Boca Raton, FL: CRC Press (ISBN 978-1-4200-6659-3/hbk). x, 369 p. $ 94.95; £ 44.99 (2008).

The second edition of this very interesting book includes a revision of its contents and a pledge for the publication of a second volume with advanced material for a second course in number theory. The new structure of this book consists of seven chapters: Arithmetic of the integers, modular arithmetic, primitive roots, quadratic residues, simple continued fractions and Diophantine approximation, additivity - sums of powers and Diophantine equations.

Each chapter includes many interesting applications and the book concludes with six significant appendices, solutions, bibliography and a list of symbols.

For the review of the first edition (1998) see Zbl 0943.11001. *Panayiotis Vlamos (Athens)*

1175.20001
Dresselhaus, Mildred S.; Dresselhaus, Gene; Jorio, Ado
Group theory. Application to the physics of condensed matter.
Berlin: Springer (ISBN 978-3-540-32897-1/hbk). xv, 582 p. EUR 64.95/net; SFR 113.50; $ 89.95; £ 50.00 (2008).

This is an excellent text whose contents would be better reflected by a title such as "Application of Group Theory to the Physics of Condensed Matter". It is not meant to be a general well rounded introduction to group representation theory that is illustrated by applications, but rather is focused on condensed matter physics from the beginning. The book originates from lectures by Charles Kittel and J. H. van Vleck in the 1950s and much of the material was presented in courses by the authors over the last three decades. The material is meant for Electrical Engineering and Physics students at the graduate level who have a firm grounding in basic quantum mechanics and solid state physics. No specific knowledge of group theory is assumed, but applied linear algebra is prerequisite. In the mathematical introduction the basic structure and representation theory for finite groups is derived, (with real proofs!) up to the orthogonality relations and character tables.

As an indication of the narrow focus of the book from the viewpoint of general applied group representation theory, I did not find the terms Lie group or Lie algebra in the index. Almost all of the groups considered are finite or discrete, and rigorous results from representation theory are limited primarily to finite groups. The only major exception to this is the full rotation group in three space where the principal results are imported from quantum mechanics books without detailed proof. For a book of this length there are not many references and very few historical comments. There are no proofs of such basic results as the crystallographic restriction or the Wigner-Eckart Theorem, and no rigorous derivations of the point groups or the crystallographic or magnetic groups. However, there is an immense amount of detailed material about these groups, their representations and their applications to condensed matter physics, including character tables.

Whereas most books on applied group representation theory would give only a few illustrative examples of applications to the analysis of energy levels for quantum mechanical systems, this book treats the problem for systems with a lattice structure in great detail. The emphasis is on classification of energy levels, using a perturbation theory approach based on symmetry breaking and branching laws for the restriction of irreducible representations of a group to direct sums of irreducible representations of a subgroup. The symmetry breaking is introduced in stages.

The authors start with a single atom in a rotational symmetric background and then subject it to a crystal environment. Mathematically this involves restriction of irreducible representations of the rotation group to a point subgroup. Coupling of states, direct products of representations and selection election rules are considered.

At the next level of complication atoms are combined into molecules, and molecular energy levels are studied. One starts with rotational symmetry for a molecule and then breaks the energy

2 Looking back

Looking back

1175.01034
Hausdorff, Felix
Main features of set theory.
(Grundzüge der Mengenlehre.
Mit 53 Figuren im Text.) (German)
Leipzig: Veit & Comp. 473 p. (1914).

F: Hausdorff's 1914 treatise on set theory [Grundzüge der Mengenlehre. Leipzig: Veit & Comp. (1914; JFM 45.0123.01)] is one of the great books of mathematics. A list of its eminent qualities might begin with readability, clearness, conciseness, liveliness, ingenuity, wittiness, diversity, comprehensiveness, and nonchalance. The command of language is that of a brilliant writer. Through a classical approach shines an idiosyncratic modernity. Marvellous presentations of textbook material are complemented with countless new ideas, which turned out to be seminal not only for set theory, but also for topology and measure theory.

The German term "Grundzüge" can be translated as "main features" or "outlines". It denotes a broad treatment of a subject which might stop at a certain level of complexity, but which gives a full picture of what is considered to be characteristic. Hausdorff speaks of "Hauptsachen der Mengenlehre" (main issues of set theory) in his foreword and addresses a wide audience consisting of all "who possess some abstraction of thinking". "Grundzüge" is definitely not to be read as "Grundlagen" (foundations), and thus the title already points at Hausdorff's understanding of set theory, which is explained in the first chapter of the book:

"Die Mengenlehre ist das Fundament der gesamten Mathematik. Über das Fundament dieses Fundamentes ist eine vollkommene Einigung noch nicht erzielt worden. Den Versuch, den Prozeß der uferlosen Mengenbildung durch geeignete Forderungen einzuschränken, hat E. Zermelo unternommen. Da indessen diese äußerst scharfsinnigen Untersuchungen noch nicht als abgeschlossen gelten können und da eine Einführung des Anfängers in die Mengenlehre auf diesem Wege mit großen Schwierigkeiten verbunden sein dürfte, so wollen wir hier den naiven Mengenbegriff zulassen, dabei aber tatsächlich die Beschränkungen innehalten, die den Weg zu jenem Paradoxon abschneiden."

(Set theory is the foundation of all mathematics. A complete agreement about the foundation of this foundation has not yet been reached. The attempt to delimitate the process of the boundless formation of sets by adequate postulates has been undertaken by E. Zermelo. But since this keen-witted analysis cannot be presumed to be completed, and since an introduction of the beginner into set theory along these lines should be linked with major difficulties, we want to allow the naive notion of a set here, but in doing so we in fact keep to the limitations which cut off the way to that paradox.)

So the teacher is aware of the paradoxes of naive set theory, but nevertheless teaches naive set theory. Hausdorff's book marks the beginning of what has been done ever since: Beginners are not confronted with a – by now well-understood – axiomatic system, but they are taught naive set theory with a hint at Russell's paradox. In 1914, when Zermelo's first axiomatic system of 1908, including his axiom of choice, was still discussed controversially, completed and made precise, Hausdorff's attitude is of crucial importance: While the foundations of the new foundation of mathematics had to be clarified and disseminated, an outstanding mathematician was there writing a dauntless 476 page book about set theory, substantially advancing the subject and its impact for all of mathematics. The effect was stabilizing. No one reading the book is left with the impression that set theory is something vague or inconsistent. The book is about the fascinating mathematics of infinity. After reading it, one might be eager to see how this rich theory can be given a proper foundation. Then Zermelo's system and its extensions by Abraham Fraenkel and others naturally supply the theory presented in Hausdorff's book with axioms. Thus Hausdorff, not interested in axiomatics himself, helped to promote axiomatic set theory.

Hausdorff's achievement appears even greater when we look at the treatises on set theory written before 1914. Cantor presented his theory in two lengthy journal articles in 1895 and 1897, and these remained the main sources of

In presenting the material, a coarser grid than the MSC codes is being used; the material has been grouped into nine sections: General mathematics, history, foundations; Number theory, algebra, algebraic geometry; Real and complex analysis, functional analysis and operator theory; Differential, difference and integral equations; Discrete mathematics; Topology and geometry; Probability theory, statistics, applications to economics and biology; Numerical analysis, modelling, computer science, algorithms; and Mathematical physics.

Since the bibliographical information also gives the Zentralblatt accession number (e.g. 1175.20001), finer navigation will still be possible.

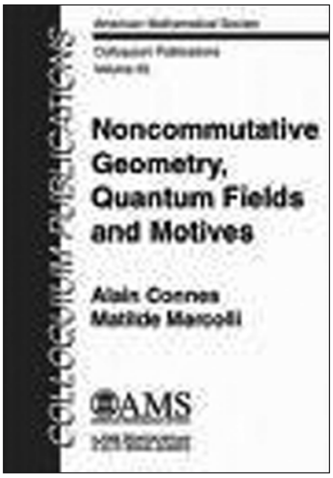
Readers specialising in one field of mathematics are therefore invited to browse the reviews from adjacent fields as well, which are typically found in the same section, and certainly those from not so adjacent fields in other sections. For example, aficionados of functional analysis might like to take a peek at operator theory, abstract harmonic analysis and partial differential equations.

Another new feature of *Excerpts* is the "Looking Back" section opening each issue in which contemporary reviews of classical mathematical works are presented. These reviews will be expressly commissioned from renowned experts for this section. The first issue carries a review of Hausdorff's "Grundzüge der Mengenlehre"; later issues in spring 2010 offer reviews of papers by Abel and Liouville. In addition, seminal papers and modern classics from the second half of the 20th century will be considered, like Mumford's "Abelian Varieties".

The editors hope that you will enjoy the new reader-friendly format and browse the *Excerpts*, taking a break every now and then to study a review in detail. Needless to say, your comments are most welcome.

Dirk Werner [werner@math.fu-berlin.de] teaches mathematics at Freie Universität Berlin. His field of specialisation is functional analysis. Since 2009, he has been the Deputy Editor-in-Chief at Zentralblatt in charge of the new print edition.

Book Reviews



Alain Connes and
Matilde Marcolli

Noncommutative Geometry, Quantum Fields and Motives

American Mathematical Society Colloquium Publications Volume 55
American Mathematical Society, Providence, RI, 2007
785 pages
ISBN-10: 0821842102

Reviewed by Frédéric Paugam

This book explains, in a beautiful patchwork style, various relations between quantum field theory, number theory and noncommutative geometry. It is composed of four main chapters:

1. The first chapter covers the conceptual explanation of the BPHZ renormalization procedure in terms of the Connes-Kreimer algebraic group scheme of Feynman graphs, its relation with differential Galois theory and the Riemann-Hilbert problem, and the motivic reason why one gets periods of algebraic differential forms when one renormalizes Feynman diagrams. Finally, it explains the noncommutative geometric approach to the classical standard model by the spectral action principle.
2. The second chapter explains the noncommutative geometric approach to the spectral interpretation of the zeroes of Riemann's zeta functions.
3. The third chapter studies in a noncommutative geometry setting the dynamics of Hecke operators and the relation of the corresponding quantum statistical mechanics systems with explicit class field theory.
4. The fourth chapter finishes by giving an interpretation of Weil's explicit formula as a trace formula and a dictionary between the chosen viewpoint and Weil's proof of the Riemann hypothesis for function fields.

The following is a description of those four chapters in more detail.

The first chapter, entitled "quantum fields, noncommutative spaces, and motives", starts in Sections 1 to 6 with an overview of perturbative quantum field theory and Feynmann diagrams. These are simply notations for some integrals of rational functions that enter into the computation of the average values of observables in particle physics. These notations give combinatorial tools to check if the given integrals are divergent, and to replace them in a coherent way by some associated convergent integrals. This is the hard combinatorics of the renormalization (so-called BPHZ) procedure. The Connes-Kreimer Hopf algebraic description of this method is then described in great detail, introducing the group of diffeographisms, which is the pro-algebraic group of characters of the Hopf algebra of (dressed) Feynman graphs. The authors make extensive use of the dimensional regularization scheme, which gives a way, through change of variable to polar coordinates and the use of the meromorphic continuation of the gamma function, to regularize some divergences of the integrals in play.

These methods are completely standard in particle physics but the Connes-Kreimer description gives a very concise and neat way to explain them. Moreover, it allows clarification of the relation of the renormalization procedure to differential and motivic Galois theory. Indeed, the group of diffeographisms is shown in Sections 7 to 8 to be related to the differential Galois group of the family of flat equisingular connexions in a universal way. This also allows the authors to associate a Galois group to every theory (by use of particular flat equisingular connexions), which gives interesting information on its renormalizability properties. The Riemann-Hilbert correspondence then gives a relation of these differential systems to (families of) periods of differential forms on algebraic varieties, which are now widely known as a special incarnation of Grothendieck's notion of motives. This last fact, which explains the relation of the renormalization procedure to number theory, is covered in Section 8.

The first parts of Chapter 1 would have already been enough to make an exceptional opus on the mathematics of quantum field theory but the authors continue their road through this subject with a short survey of particle physics. They then explain the relation of the (classical Euclidean) standard model of elementary particles coupled with gravity to noncommutative geometry. The coupling with gravity is explained by defining a noncommutative space whose "diffeomorphisms" give the full gauge symmetries of gravity coupled to matter, i.e. the semi-direct product of the diffeomorphism group of spacetime with the gauge symmetry group. This non-commutative space is encoded in the tensor product of a finite dimensional matrix algebra (finite noncommutative space) with the algebra of smooth functions on spacetime. The full apparatus of noncommutative geometry in the Connes approach (spectral triples, cyclic cohomology) is used to study in detail the classical standard model and the Higgs mechanism (which allows interaction bosons to be given a mass). The authors finish in Section 19 with a noncommutative geometric description of the dimensional regularization procedure by studying "complex dimensional subspaces" that have a natural meaning in noncommutative geometry through some spectral zeta function from the dynamical point of view.

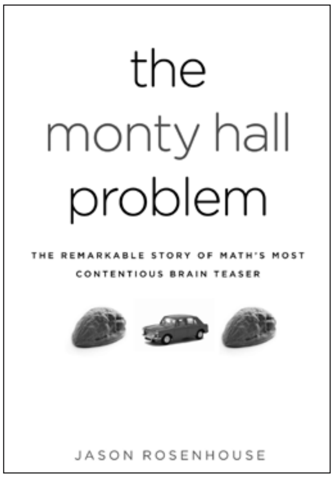
The second chapter, entitled "Riemann zeta function and noncommutative geometry", explains the spectral approach to the study of Riemann's zeroes. The basic idea, which is also present in the automorphic setting, is that the right definition of the zeta function is not as a simple formula but as the greatest common divisor of a family of mellin transforms of test functions of adelic space. This fact is at the heart of Connes' spectral interpretation of the zeroes of zeta and also explains why automorphic L-functions are easier to study than arithmetic zeta functions. As the spectral interpretation is of dynamical nature, noncommutative geometry also gives nice tools and intuitions to study it, through the adele class space A/Θ^*. This also allows the description of Weil's explicit formula through a trace formula, which is a good starting point for the study of deeper problems on Riemann zeta. This chapter gives quite a complete introduction to these advanced topics, which is a neat way to approach analytic number theory.

The third chapter, entitled "quantum statistical mechanics and Galois symmetries", gives a survey of the author's work on the noncommutative geometric description of the dynamics of Hecke operators on modular curves, and its relation with explicit class field theory in the complex multiplication case. These dynamical properties of Hecke operators have been the recent source of various works outside noncommutative geometry and thus give an interesting relation between number theory, dynamical systems and noncommutative geometry. The presentation is given in the setting of adelic Shimura varieties, so that it can also be a good introduction to the geometric theory of automorphic forms.

The fourth chapter, entitled "endomotives, thermodynamics, and the Weil explicit formula", starts from the relations between the second and the third chapter and finishes with a cohomological trace formula and a noncommutative geometric description of the mathematical objects present in Weil's proof of the Riemann hypothesis for function fields. The notion of endomotives is introduced to give a conceptual (motivic) explanation of the relation between quantum statistical mechanics and explicit class field theory. The relation between quantum statistical mechanics of lattices and the spectral interpretation is then given, so that the "Frobenius" flow in chararacteristic zero (which gives a spectral interpretation of the poles of the Gamma factor) is seen as a noncommutative thermodynamic (temperature) flow. This gives nice analogies between number theory and physics. The explicit formula is then given a cyclic cohomological explanation by the use of the cyclic module associated to the embedding of the idèle class group in the adèle class space. The chapter ends with a dictionary between Weil's proof of the Riemann hypothesis for function fields and noncommutative geometry. In particular, the points of the curve with value in an algebraically closed finite field are shown to sit in the adèle class space.

To conclude, even if this book has a very wide scope, from number theory and arithmetic geometry to physics to noncommutative geometry, and despite the fact that its chapters can almost be read separately, one becomes completely convinced that the interactions give precious tools to better understand the three main topics. This book can be used either as an introduction to physics for mathematicians or as an introduction to the developing subject of noncommutative arithmetic geometry. It can also, to quote the authors, be read in the perspective of a "walk in the noncommutative garden".

Frederic Paugam [frederic.paugam@math.jussieu.fr] is maître de conférence at the University of Paris 6. His thesis was completed in Rennes in 2002 and was on the arithmetic of Abelian varieties. He has had postdoctoral positions at the University of Regensburg, MPIM-Bonn and IHES before he attained a permanent job in Paris. He now works on arithmetic geometry and mathematical physics, with main interests in global analytic geometry and functional equations of L-functions on the one side and formalization of standard physics on the other side.



Jason Rosenhouse

The Monty Hall Problem

208 pp., Oxford University Press, New York, 2009
ISBN-10: 0195367898

Reviewed by Paulo Ventura Araújo

The subtitle of this book reads '*The remarkable story of math's most contentious brain teaser*'. It is a bit unsettling to learn that something you have never heard of is the most famous or contentious of its kind. Nowadays it is easy to fill the gaps with a bit of *googling* and you are an instant specialist before anyone notices anything amiss. However, the reviewer candidly acknowledges his ignorance: before picking up this book, he didn't know who Monty Hall was and what problem he contended with. After finishing the book, the reviewer, albeit learning only the most sketchy facts of Monty Hall's life (he was an American television presenter), has acquired a staggering amount of information about the problem that bears his name. Which is how it should be!

Although it was never featured in the television show that Monty Hall hosted from 1963 to 1977, the problem seems to be aptly named. The game goes as follows. There is one contestant and Monty Hall, the host. The contestant is faced with three identical closed doors, one door hiding a car and each of the other two hiding a goat. The contestant chooses a door. Monty Hall opens another door instead and discloses a goat. Then he offers the contestant the opportunity to switch his choice to the other unopened door. It is understood that he will be offered the prize behind the door of his final choice. Should he stick to his first choice or switch? Does switching increase his chances of winning the car?

There are a number of unstated assumptions that need clarifying: first, that the word *identical* implies that each door has probability 1/3 of hiding the car; second, that the contestant has no clue as to which door actually hides the car; third, that Monty Hall knows where the car is (otherwise he could not be sure of opening a goat-concealing door); and fourth, that Monty chooses randomly which door to open whenever he has the option to do so.

Very little knowledge of probability theory seems to be necessary to work out this problem. Suppose the contestant is stubborn and, as such, will inevitably stick to his first choice of door. There is probability 1/3 that this door conceals the car. Therefore, if he follows the strategy of sticking, his probability of winning is 1/3. If, on the other hand, he is determined to switch then he wins exactly when the door of his first choice conceals a goat, an event which occurs with probability 2/3. It looks as if the strategy of switching doubles his chance of winning.

The foregoing reasoning appears to be faultless. Only a few lines and we are done with the problem – so how can anyone write a full-length book about it? Well, let us try another approach. To begin with, all three doors are equal: each one stands an equal chance of hiding the car and therefore the total probability 1 is equally divided amongst them. The contestant makes his choice, but as long as all doors remain shut the probabilities involved are not affected. Then Monty shows a goat behind one of the doors. This door has now probability zero, and so the surplus probability 1/3 must be evenly transferred to the remaining closed doors. Thus, each unopened door has now probability 1/2 of hiding the car. Since the contestant gains no advantage by switching, why should he bother? Switching *and* losing would make him feel like a fool.

It should be clear where this argument goes astray, but the point is that many people believed it (or believed other equally fallacious arguments). As the book shows, the overwhelming majority of people, even mathematically educated people, give the wrong answer (*do not switch*) when first acquainted with the problem. Even so great a mathematician as Paul Erdős was befuddled by it.

Although the Monty Hall problem has been around since 1975 – and in a different formulation, as *the three prisoners problem*, since 1959 when Martin Garden introduced it in his *Scientific American* column – only in 1990 did it attract widespread notice in the US. Marilyn vos Savant, writing in response to a reader in her Questions & Answers section in *Parade* magazine, argued correctly that switching has a 2/3 chance of winning, against a 1/3 chance for sticking. What followed is, as Jason Rosenhouse mildly puts it, '*one of the strangest chapters in the history of mathematics*'. Many angry readers wrote to correct vos Savant – not a few of them holding PhDs in mathematics and urging her to admit her error, lest her carelessness should seriously damage the already ailing mathematical literacy in America.

It was not long before a number of readers reported experimental evidence supporting vos Savant's opinion. Classes of all levels in schools across the country simulated the game, and their findings were in agreement with the solution proposed by vos Savant. In July 1991, the story was front page news in the *New York Times*. After all, it is not every day that so many mathematicians are proved wrong by a lay person boasting no university degree in mathematics.

This is undoubtedly a gripping story with a twist: mathematicians, instead of being the clever guys, are for once proved to be no better than the average Joe, or even a little worse, if we consider their insufferable arrogance.

Having gained such enormous publicity, many people took up the Monty Hall problem. Mathematicians analysed countless variations of the problem in a collective effort to assuage the profession's wounded self-esteem. Cognitive scientists and psychologists wondered just what was wrong with the human brain, given its notorious incompetence in probabilistic reasoning. Philosophers saw in the Monty Hall problem an opportunity to ponder the very notion of probability. Even physicists and economists dabbled with this seemingly humble problem.

Divided into eight chapters, Rosenhouse's book deals thoroughly with all these multidisciplinary aspects of the problem, with the mathematical portion occupying four chapters of varying difficulty. Around the Monty Hall problem and its variations, the author weaves a fine, highly motivated introduction to elementary probability theory, building up from the basic concepts to conditional probabilities and to Bayes' theorem. The different versions of the problem involve changes in the way Monty chooses his door, changes in the number of doors, changes in the number or value of the prizes, a change from one to two contestants and even (a bit disrespectfully) a change from one to two Montys.

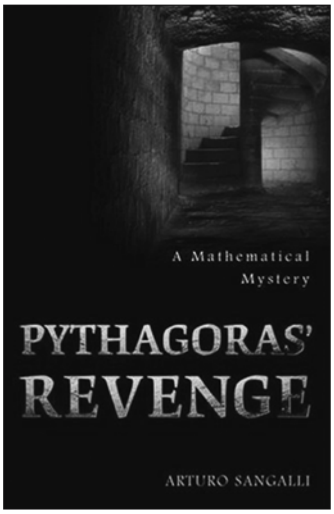
These variations are all great fun, but they are not the stuff of bedtime reading. As with all mathematical books, it is better to work out the solutions for yourself rather than reading straight through the book. For the sake of illustration, here is one of the easiest variations. Suppose there are again three doors, one door hiding a car and the other two hiding goats. This time, however, both Monty and the contestant ignore which door conceals the car. Otherwise, things proceed much as before. The contestant chooses a door and Monty randomly opens a different door, again revealing a goat (he was lucky, but we need not concern ourselves with how things would go on had he disclosed a car). Monty offers the contestant the chance to switch doors. Does he increase his probability of winning the car by accepting the offer?

Surprisingly, the answer is no. Here is an explanation. Number the doors from 1 to 3 in such a way that the car is behind door 1. This numbering is of course unknown to both the contestant and Monty, so that in choosing their doors they may pick up any ordered pair such as (1, 2) – meaning the contestant chooses door 1 and Monty chooses door 2 – or (2, 1) – meaning the other way around. There are six possible pairs, all with the same probability of being picked up. For exactly two such pairs, (1, 2) and (1, 3), the contestant wins by sticking and for exactly two other pairs, (2, 3) and (3, 2), he wins by switching. Therefore the strategies of sticking and switching have exactly the same chance of winning.

The non-mathematical portions of the book, and even the least demanding mathematical portions, are very good reading and are suitable for a large non-specialist audience. In the cognitive chapter, for instance, we are made aware of two basic types of faulty probabilistic reasoning: that which is innate in the human mind and that which arises from misapplication of imperfectly learned mathematical tools. The conclusion, perhaps, is that a smattering of badly digested learning is often worse than no learning at all. The philosophical chapter delves deeply into a disturbing question: could it be that the best strategy in the long run, as shown by computing the relevant probabilities, is not the most advisable to adopt if you are only in for a single round of the game? Thankfully the answer is no, but before reaching this conclusion the author does a masterful job of dissecting a philosopher's arguments to the contrary.

And, since the author is a mathematician (but also a very skilled writer), it is somehow a vindication of our profession, after the embarrassing events that surrounded the Monty Hall problem, that he came to write this fine book.

Paulo Ventura Araújo [paraujo@fc.up.pt] was born in 1966. He has been teaching mathematics at the University of Porto (Portugal) since 1991. He is the author of two textbooks on geometry and has also written or co-written three books on other things – mostly on trees (real trees, not mathematical objects).



Arturo Sangalli

Pythagoras' revenge

Princeton University Press
2009, 183 p.
ISBN: 978-0-691-04955-7

Reviewed by Adhemar Bultheel

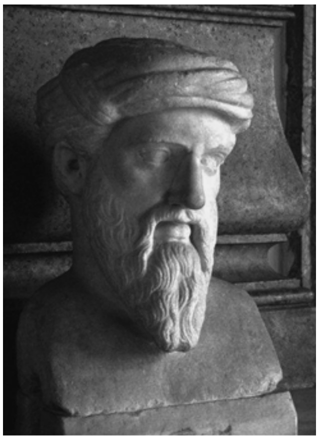
Pythagoras was born on Samos around 570 BC and has been most influential in mathematics, politics, religion and philosophy. Pythagorean philosophy is dominated by numbers and mathematics and it is generally accepted to have greatly influenced later philosophers such as Plato. There were two kinds of followers: the *akousmatikoi* or "listeners" and the *mathematikoi* or "learners". The mathematikoi were considered to be more advanced and better skilled in the fundamental theory. Pythagoreans were convinced that numbers rule nature (from the music of the planets to the scales of music). They also believed in the reincarnation of the soul in another animal life form, which is why they were vegetarian. The Pythagoras adepts had a lot of opposition and there was even a general uprising against them. In the second and first centuries BC, the original ideas were revived by the Neo-Pythagoreans.

Arturo Sangalli

Although Pythagorean philosophy may be somewhat less well-known to a general public, his influence on mathematics and mathematical teaching has been very important. The Pythagorean Theorem is one of the items in mathematics that seems to be accepted as belonging to the cultural backpack of anyone who pretends to be intelligent. Most of the time even politicians know the theorem, or at least it does not show bad taste to confess that one has heard of it. Ironically, the Pythagorean adepts believed in the ratio of integers and it is exactly his theorem that exposes, in the isosceles case, the square root of 2. Therefore Pythagoreans tried to cover this up and the Greek preferred geometry over numbers until Descartes restored numbers in the 16th century. Even though Pythagoras has had a big impact on Western civilization, there is no extant written document left from his hand. So all we know is second-hand information, which is the source of many mysteries and legends. For example, there is a legend saying that Pythagoras' enemies set his house on fire so that he had to flee; suddenly he halted, turned around and said that he'd rather die than run away, whereupon his pursuers cut his throat.

In the wake of the hype caused by Dan Brown's *Da Vinci Code*, Arturo Sangalli has found inspiration in all the previous characteristics of Pythagoras and the Pythagoreans to write his mathematical analogue. As Pythagoras is so popular or at least so well-known, the

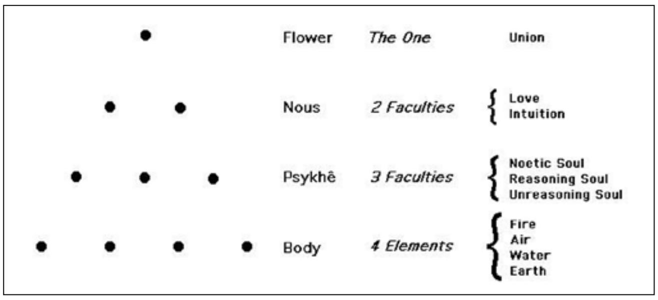
Pythagoras' statue on Samos



Pythagoras

mysteries about his person and the sect-like and memorable (Neo-)Pythagoreanism are indeed a good choice. Sangalli has a PhD in mathematics and as a freelance journalist has previously authored books on the interface between mathematics and computer science. Here, he has designed a plot where a Neo-Pythagorean sect "the Beacon" believes that Pythagoras predicted that he would be reincarnated around the middle of the 20th century. So they start looking for their "Dalai Lama" via the Internet.

They believe that Norton Thorp, a world renowned mathematician is "The One". On the other hand, there is an Oxford professor in classical history, Elmer Galway, who happens to discover a parchment book containing an Arabic translation of an old text that refers to a scroll allegedly written by Pythagoras, the Master himself. The other part of the book is discovered by the sect, so that both parties are in search of the other half of the information. Finally the papyrus scroll is found by Norton in some underground basilica in Rome[1] , thanks to a carving of the tetractys, the 10-dotted triangle, a Pythagorean symbol. The parallel with the *Da Vinci Code* is striking: the mystic locations, the sect of "bad guys", the intelligent "hero" Galway and the legacy of a secretly hidden message passed on by a great historical figure.



Tetractys

Without giving away the story, I can try to explain the title. Thorp is traced by the sect with the help of Jule Davidson, a young mathematician who is solving difficult mathematical puzzles on the Internet. But Thorp turns out to be an Anti-Pythagoras (in the sense of an Anti-Christ). Johanna, Jule's twin sister, is a specialist in computer security, which is related to number theory and random numbers. She happens to attend one of Thorp's lectures where he is preaching that nature is essentially randomness, i.e. completely unpredictable, and therefore randomness is also at the heart of mathematics. "Solvable problems are like a small island in an ocean of undecidable propositions." This is, of course, the opposite of Pythagoras' views, which is, de facto, a complete catastrophe for the sect because what Pythagoras predicted, according to their beliefs, was in fact his own anti-self.

Impossible 15-puzzle

The book has interesting expositions about philosophy, history and of course mathematics. The latter are easily accessible for non-mathematicians too. There are some appendices going a bit deeper into some of the mathematics but, often, Sangalli lets one of the characters of the book explain it. So you can find something about the unsolvable 15-puzzle and combinatorics, and about random numbers and how they are generated, etc. Even lovers of mystery tales may like this story, although the "Indiana Jones" adventure-value is rather minimal and some portions of the text might be reminiscent of tedious mathematics lessons.

After all, this is a fiction novel and sometimes I found some parts of the text closely resembling lecture notes of a popular course on mathematics. Especially when one of the characters is "teaching", it feels a bit artificial. Also, the author could have saved on the number of characters. Some are introduced just to let them tell their part of the story without playing a role in the rest of the novel. The suspense is kept at a good level, though, and several unexpected twists in the story keep you reading on, even with the interfering "expositions" of a more philosophical or mathematical nature.

This review appeared first in the issue 76 of the Newsletter of the Belgian Mathematical Society, January 15, 2010. Reprinted with permission

Adhemar Bultheel [Adhemar.Bultheel@cs.kuleuven.be] is an emeritus professor at the Computer Science Department of the K.U. Leuven, Belgium. He received his PhD in mathematics in 1979. His professional interests are numerical analysis, rational approximation, orthogonal functions and structured linear algebra problems. He has been Vice-President (1999–2002) and President (2002–2005) of the Belgian Mathematical Society.

[1] This basilica near the *Porta Maggiore* on *Via Praenestina* in Rome does indeed exist and was used by the Neo-Pythagoreans in the first century AD; it was only discovered in 1915.

Solved and Unsolved Problems

Themistocles M. Rassias (Athens, Greece)

There are two basic approaches to Mathematics:
(i) Problem solving.
(ii) Theory development.
The British School of Mathematics and the French School of Mathematics in the early to mid 20th century represented (i) and (ii) respectively. But, as with most good categorizations, it involves a useful oversimplification. Of course, the two approaches interact in obvious ways.

In recent years, in all countries, the trend of Mathematics Schools has been to avoid this polarization between (i) and (ii) in teaching and research, instead stressing their complementary roles.

I. Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

59. Let a, b, c be positive real numbers such that $c \geq a \geq b$ and $a^2 \geq bc$. Prove that the following inequality holds.

$$\frac{\sqrt{a^2b+b^2c}}{a+c}+\frac{\sqrt{b^2c+c^2a}}{a+b}+\frac{\sqrt{c^2a+a^2b}}{b+c} \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{2}.$$

(Tuan Le, Fairmont H. S., California, USA)

60. Let a, b, c be positive real numbers. Prove that the following inequality holds.

$$\frac{16}{27}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^3+\sqrt[3]{\frac{abc}{(a+b)(b+c)(c+a)}} \geq \frac{5}{2}.$$

(Tuan Le, Fairmont H. S, California, USA)

61. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy the functional equation

$$f(x+y+xy)=f(x)+f(y)+f(x)f(y)$$

for all $x,y \in \mathbb{R}$.

(Prasanna K. Sahoo, University of Louisville, USA)

62. Let $f:\mathbb{R}\to\mathbb{R}$ be a function with Darboux property. If f^2 has limit in any point of $\mathbb{R}$ then f is continuous on $\mathbb{R}$.

(Dorin Andrica, "Babeş-Bolyai" University, Cluj-Napoca, Romania, and Mihai Piticari, "Dragoş-Vodă" National College, Câmpulung Moldovenesc, Romania)

63. Let $\alpha > 0$ be a positive real number and let $\beta \geq 2$ be an even integer. Let $x \in \mathbb{R}$ be a fixed real number and let $f:\mathbb{R}\to\mathbb{R}$ be a bounded function on $\mathbb{R}$ such that $\lim_{t\to x} f(t) = L$. Prove that

$$\lim_{n\to\infty} n^{\frac{\alpha}{\beta}} \int_{-\infty}^{\infty} \frac{f(t)}{1+n^{\alpha}(t-x)^{\beta}}dt = \frac{2\pi L}{\beta \sin\frac{\pi}{\beta}}.$$

(Ovidiu Furdui, Campia Turzii, Cluj, Romania)

64. A function $f(x,y)$ is defined on the unit square $[0,1]^2$ and is continuously-differentiable in each variable separately, i.e. $f(\cdot,y) \in C^1[0,1]$ for any y and $f(x,\cdot) \in C^1[0,1]$ for any x. Is it possible that $\int_0^1 f(x,y)\,dx > 0$ for any y, and $\int_0^1 f(x,y)\,dy < 0$ for any x?

Author's comment. There is an obvious but incorrect solution for this problem. Applying the Fubini-Tonelli theorem, we get

$$0 < \int_0^1 \left(\int_0^1 f(x,y)dx\right)dy = \int_0^1 \left(\int_0^1 f(x,y)dy\right)dx < 0,$$

which is a contradiction. So, the answer is "No". But actually the answer is "Yes". Moreover, there are even infinitely smooth examples of such functions f. The mistake in the reasoning above is that Fubini and Tonelli's theorem may not be applicable for our function. Fubini's theorem assumes that the modulus $|f(x,y)|$ is summable on the square and Tonelli's theorem assumes that both the repeated integrals exist and the function f is bounded at least from one side (either from below or from above). Our function f may not satisfy these assumptions, although it is smooth in each variable.

(Vladimir Protasov, Moscow State University, Russia)

II. Two New Open Problems

65*. **A conjecture on integer arithmetic.**
Let $\Delta((x_1,\ldots,x_n),(y_1,\ldots,y_n))$ denote the formula

$$(\forall i \in \{1,\ldots,n\}\ (x_i = 1 \Rightarrow y_i = 1)) \wedge$$
$$(\forall i,j,k \in \{1,\ldots,n\}\ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$
$$(\forall i,j,k \in \{1,\ldots,n\}\ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$$

Prove or disprove *the conjecture* [2, p. 3, Conjecture 2b]: if integers $x_1,\ldots,x_n$ satisfy $\max(|x_1|,\ldots,|x_n|) > 2^{2^{n-1}}$ then $\Delta((x_1,\ldots,x_n),(y_1,\ldots,y_n))$ for infinitely many $(y_1,\ldots,y_n) \in \mathbb{Z}^n$.
Author's comment. As the author has proved [2, p. 8, Corollary 2], the conjecture implies that if a Diophantine equation $D(x_1,\ldots,x_p)=0$ has only finitely many integer solutions then each such solution $(x_1,\ldots,x_p)$ satisfies

$$|x_1|,\ldots,|x_p| \leq 2^{2^{(2M+1)^{(d_1+1)\cdot\ldots\cdot(d_p+1)}-1}}.$$

Here, M stands for the maximum of the absolute values of the coefficients of $D(x_1,\ldots,x_p)$ and d_i denotes the degree of $D(x_1,\ldots,x_p)$ with respect to the variable x_i.
The Davis-Putnam-Robinson-Matiyasevich theorem states that every listable set $\mathcal{M} \subseteq \mathbb{Z}^n$ has a Diophantine representation, that is

$$(a_1,\ldots,a_n) \in \mathcal{M} \Longleftrightarrow \exists x_1 \in \mathbb{Z},\ldots,\exists x_m \in \mathbb{Z},$$
$$D(a_1,\ldots,a_n,x_1,\ldots,x_m) = 0$$

for some polynomial D with integer coefficients. Such a representation is said to be finite-fold if for any integers $a_1,\dots,a_n$ the equation $D(a_1,\dots,a_n,x_1,\dots,x_m)=0$ has at most finitely many integer solutions $(x_1,\dots,x_m)$. It is **an open problem** whether each listable set $\mathscr{M}\subseteq\mathbb{Z}^n$ has a finite-fold Diophantine representation. An affirmative answer to this problem would falsify the conjecture, see [1, p. 42].

[1] Yu. Matiyasevich, *Hilbert's tenth problem: what was done and what is to be done*. Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), 1–47, Contemp. Math. 270, Amer. Math. Soc., Providence, RI, 2000.

[2] A. Tyszka, *A hypothetical upper bound for the solutions (the number of solutions) of a Diophantine equation with a finite number of solutions*, http://arxiv.org/abs/0901.2093.

(Apoloniusz Tyszka, Hugo Kołłątaj University, Kraków, Poland)

66*. Find all mappings $T:\mathbb{R}^3\to\mathbb{R}^3$ such that

$$\|T\mathbf{u}\times T\mathbf{v}\|=\|\mathbf{u}\times\mathbf{v}\|$$

for all vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$. Note that $\times$ denotes vector product and $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^3$.

(Themistocles M. Rassias,
National Technical University of Athens, Greece)

III. SOLUTIONS

51. Find all continuous functions $f:\mathbb{R}\to[0,\infty)$ such that

$$f^2(x+y)-f^2(x-y)=4f(x)f(y) \tag{1}$$

for all real numbers x,y.

(Titu Andreescu, University of Texas at Dallas, USA)

Solution by the proposer. Setting $x=y=0$ yields $f(0)=0$. For $x=y$ we obtain $f^2(2x)=4f^2(x)$ and then $f(2x)=2f(x)$, since $f(x)\ge 0$.

We prove that

$$f(nx)=nf(x),n\ge 1.$$

Assume that $f(kx)=kf(x)$ for all $k=1,2,\dots,n$. We have

$$f^2\big((n+1)x\big)-f^2\big((n-1)x\big)=4f(nx)f(x).$$

Then

$$f^2\big((n+1)x\big)=[(n-1)^2+4n]f^2(x)$$

and hence

$$f\big((n+1)x\big)=(n+1)f(x),$$

as desired.

It follows that if p,q are positive integers then

$$qf\left(\frac{p}{q}\right)=f(p)=pf(1),$$

so

$$f\left(\frac{p}{q}\right)=\frac{p}{q}f(1)$$

and $f(r)=rf(1)$ for any positive rational r.

Setting $x=0$ in the initial condition gives

$$f^2(y)-f^2(-y)=0.$$

Then

$$f(y)=f(-y)$$

for all real y, hence

$$f(r)=|r|f(1)$$

for all rational numbers r.

We prove that $f(x)=|x|f(1)$ for all real numbers x. Let x be an arbitrary real number and let $(r_n)_{n\ge1}$ be a sequence of rational numbers with $\lim_{n\to\infty}r_n=x$. Because of the fact that

$$f(r_n)=|r_n|f(1)$$

and f is a continuous function, it follows that

$$\lim_{n\to\infty}f(r_n)=\lim_{n\to\infty}|r_n|f(1)=f\left(\lim_{n\to\infty}r_n\right),$$

hence

$$f(x)=f(1)|x|.$$

Note that $a=f(1)\ge 0$ and therefore the possible functions are $f(x)=a|x|$ for some $a\ge0$. Replace in the initial relation and get $4axy=4a|xy|$, i.e. $a(xy-|xy|)=0$ for all real numbers x and y. Take, for example, $x=1,y=-1$ to obtain $-2a=0$ and hence $a=0$. Therefore, the desired function is $f(x)=0,x\in\mathbb{R}$. □

Also solved by Con Amore Problem Group (Copenhagen, Denmark), Samuel Holmin (Royal Institute of Technology, Stockholm, Sweden) John N. Lillington (Wareham, UK), S. E. Louridas (Athens, Greece), Abbas Najati (University of Mohaghegh Ardabili, Iran), and Rafael Paya (Universidad de Granada, Spain).

Remark. Abbas Najati, (University of Mohaghegh Ardabili, Iran) in addition, solved the following problem. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f^2(x+y)-f^2(x-y)=4f(x)f(y) \tag{2}$$

for all real numbers x,y.

Solution. It is clear that $f\equiv0$ is a solution of (2). We assume that f is nonconstant. Letting $x=y=0$ in (2), we get $f(0)=0$. If we put $x=0$ in (1), we conclude that $f^2(y)=f^2(-y)$ for all $y\in\mathbb{R}$. Letting $y=x$ in (2), we get $f^2(2x)=4f^2(x)$ for all $x\in\mathbb{R}$. Therefore

$$f^2(x)=4f^2\left(\frac{x}{2}\right) \tag{3}$$

for all $x\in\mathbb{R}$. It follows from (2) that

$$f^2(x)-f^2(y)=4f\left(\frac{x+y}{2}\right)f\left(\frac{x-y}{2}\right)$$

for all real numbers x,y. By (3), we get

$$f^2(x)-f^2(y)=f(x+y)f(x-y) \tag{4}$$

for all $x\in\mathbb{R}$. If $f(x)=0$ for some $x\in\mathbb{R}$, by induction on n and using (4), we get $f(nx)=0$ for all integers n. Hence by induction on n and using (4), we get $f^2(nx)=n^2f^2(x)$ for all $x\in\mathbb{R}$. So $f^2(rx)=r^2f^2(x)$ for all rational numbers r. Since f is continuous, we have $f^2(x)=x^2f^2(1)$ for all $x\in\mathbb{R}$. □

Note. Samuel Holmin (Royal Institute of Technology, Stockholm, Sweden) provided a similar solution.

52. Let $(a_n)_{n\ge1}$ be a convergent sequence. Evaluate

$$\lim_{n\to\infty}\left(\frac{a_1}{n+1}+\frac{a_2}{n+2}+\cdots+\frac{a_n}{2n}\right).$$

(Dorin Andrica, "Babeş-Bolyai" University, Cluj-Napoca, Romania, and Mihai Piticari, "Dragoş-Vodă" National College, Câmpulung Moldovenesc, Romania)

Solution by the proposers. Let $a = \lim_{n\to\infty} a_n$ and we will show that the desired limit is $a\ln 2$. We consider the following cases.

Case 1. Let $a \geq 0$. Since

$$\lim_{x\to 0}(a-x)(\ln 2 - x) = \lim_{x\to 0}(a+x)(\ln 2 + x) = a\ln 2,$$

it follows that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that

$$a\ln 2 - \frac{\varepsilon}{2} < (a-x)(\ln 2 - x) < a\ln 2 + \frac{\varepsilon}{2} \tag{5}$$

for any $x \in (0,\delta)$. Consider $\varepsilon' < \min(\delta, \ln 2, a)$ and $\varepsilon' > 0$. There is a positive integer N such that for any $n > N$ we have $a - \varepsilon' < a_n < a + \varepsilon'$. For $n > N$ we have

$$\begin{aligned} &A_n(N) + (a-\varepsilon')\left(\frac{1}{n+N+1} + \cdots + \frac{1}{2n}\right) \\ &\quad < \frac{a_1}{n+1} + \cdots + \frac{a_N}{n+N} + \frac{a_{N+1}}{n+N+1} + \cdots + \frac{a_n}{2n} \\ &\quad < A_n(N) + (a+\varepsilon')\left(\frac{1}{n+N+1} + \cdots + \frac{1}{2n}\right), \end{aligned} \tag{6}$$

where

$$A_n(N) = \frac{a_1}{n+1} + \cdots + \frac{a_N}{n+N}.$$

Using the fact that

$$\frac{1}{n+N+1} + \cdots + \frac{1}{2n} \to \ln 2$$

for $n \to \infty$, it follows that there is a positive integer N_1 such that for any $n > N_1$ we have

$$\ln 2 - \varepsilon' < \frac{1}{n+N+1} + \cdots + \frac{1}{2n} < \ln 2 + \varepsilon'.$$

For $n > \max(N, N_1)$ we obtain

$$\begin{aligned} &A_n(N) + (a-\varepsilon')(\ln 2 - \varepsilon') \\ &\quad < \frac{a_1}{n+1} + \cdots + \frac{a_n}{2n} < A_n(N) + (a+\varepsilon')(\ln 2 + \varepsilon'). \end{aligned} \tag{7}$$

From $A_n(N) \to 0$ for $n \to \infty$, it follows that there exists a positive integer N_2 such that for any $n > N_2$,

$$-\frac{\varepsilon'}{2} < A_n(N) < \frac{\varepsilon'}{2}.$$

From (5) and (7) we get for any $n > \max(N, N_1, N_2)$,

$$a\ln 2 - \varepsilon' < \frac{a_1}{n+1} + \cdots + \frac{a_n}{2n} < a\ln 2 + \varepsilon',$$

i.e. the desired limit is $a\ln 2$.

Case 2. If $a < 0$, we work with the sequence

$$-\frac{a_1}{n+1} - \cdots - \frac{a_n}{2n}$$

and get the same result. □

Also solved by Con Amore Problem Group (Copenhagen, Denmark), Samuel Holmin (Royal Institute of Technology, Stockholm, Sweden), John N. Lillington (Wareham, UK), S. E. Louridas (Athens, Greece) and Abbas Najati (University of Mohaghegh Ardabili, Iran).

53. Find all nonconstant functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation

$$g(x)\,h(y) = f\left(\sqrt{x^2+y^2}\right) \tag{8}$$

for all $x, y \in \mathbb{R}$.

(Prasanna K. Sahoo, University of Louisville, USA)

Solution by the proposer. The general nonconstant solution of the functional equation (8) is

$$f(x) = ab\,e^{A(x^2)}, \quad g(x) = a\,e^{A(x^2)} \quad \text{and} \quad h(x) = b\,e^{A(x^2)}, \tag{9}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and a, b are arbitrary nonzero real constants. Note a function $A : \mathbb{R} \to \mathbb{R}$ is an additive function if and only if it satisfies $A(x+y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}$.

It is easy to verify that (9) satisfies (8). It is left to show that (9) is the only nonconstant solution of (8). Letting $y = 0$ in (8) we see that

$$f(|x|) = h(0)\,g(x) \tag{10}$$

for all $x \in \mathbb{R}$. Since f is nonconstant, $h(0) \neq 0$, otherwise f is identically 0. From (10) and (8) we obtain

$$h(0)\,g(\sqrt{x^2+y^2}) = g(x)\,h(y). \tag{11}$$

Notice that the left side of the above equation is symmetric in x and y. Hence we get

$$g(x)\,h(y) = g(y)\,h(x). \tag{12}$$

Also note that $g(0) \neq 0$ for the same reason as $h(0) \neq 0$. Putting $x = 0$ in (12) we get

$$g(y) = \frac{g(0)}{h(0)}\,h(y). \tag{13}$$

From (13) and (11) it follows that

$$h(0)\,h(\sqrt{x^2+y^2}) = h(x)\,h(y), \tag{14}$$

which in fact reduces to

$$F(\sqrt{x^2+y^2}) = F(x)\,F(y), \qquad x, y \in \mathbb{R}, \tag{15}$$

where

$$F(x) := \frac{h(x)}{h(0)}. \tag{16}$$

From (15) it is easy to see that F is an even function on $\mathbb{R}$. Furthermore, letting $y = x$ in (15) and using the fact that F is even, we get $F(x) \geq 0$ for all $x \in \mathbb{R}$. Also note that F is nowhere 0. If F is 0 at some x_0 then we get $F(z) = 0$ for all $z > |x_0|$ and eventually $F(x) = 0$ for all $x \in \mathbb{R}$. Thus $F(x) > 0$ for all $x \in \mathbb{R}$. We define

$$G(x) := \ln F(\sqrt{x}), \quad x \geq 0. \tag{17}$$

Hence (15) reduces to

$$G(x^2+y^2) = G(x^2) + G(y^2). \tag{18}$$

From [1] (Theorem 7, pp. 19-20) we see that

$$G(x^2) = A(x^2), \tag{19}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function. Using (17) we get

$$F(x) = e^{A(x^2)}. \tag{20}$$

From (20), (16), (13) and (10) we obtain the asserted form of f, h and g. □

[1] J. Aczél, A Short Course on Functional Equations, D. Reidel Publ. Co., Dordrecht-Boston-Lancaster-Tokyo, 1987.

Also solved by Abbas Najati (University of Mohaghegh Ardabili, Iran) and Samuel Holmin (Royal Institute of Technology, Stockholm, Sweden).

Remark. Abbas Najati (University of Mohaghegh Ardabili, Iran), in addition, considered the case where $f(g$ or $h)$ is continuous.

54. Let $f:[a,b]\to\mathbb{R}$ be a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Let $p\in(0,\infty)$ and assume for a given $x\in(a,b)$ that

$$M_p(x):=\sup_{u\in(a,b)}\left\{|x-u|^{1-p}\left|f'(u)\right|\right\}<\infty. \tag{21}$$

Then the following inequality holds.

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{p(p+1)(b-a)}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]M_p(x). \tag{22}$$

(Sever S. Dragomir, Victoria University, Melbourne, Australia)

Solution by the proposer. Let $x\in(a,b)$ and define the mapping $g_{1,x}:(a,x)\to\mathbb{R}$, $g_{1,x}(t)=(x-t)^p$.

Applying the Cauchy mean value theorem, for any $t\in(a,x)$ there exists a $\eta\in(t,x)$ such that

$$[f(t)-f(x)]\,g'_{1,x}(\eta)=\left[g_{1,x}(t)-g_{1,x}(x)\right]f'(\eta),$$

i.e.

$$(-p)(f(t)-f(x))(x-\eta)^{p-1}=(x-t)^p f'(\eta)$$

from which we obtain

$$|f(t)-f(x)| = \frac{(x-t)^p\left|f'(\eta)\right|}{p(x-\eta)^{p-1}}\le\frac{(x-t)^p}{p}M_p(x),\ \ t\in(a,x). \tag{23}$$

We define the mapping $g_{2,x}:(x,b)\to\mathbb{R}$, $g_{2,x}(t)=(t-x)^p$. Applying the Cauchy mean value theorem, we can find a $\xi\in(x,t)$ such that

$$[f(t)-f(x)]\,p(\xi-x)^{p-1}=(t-x)^p f'(\xi)$$

from which we get

$$|f(t)-f(x)| = \frac{(t-x)^p\left|f'(\xi)\right|}{p(\xi-x)^{p-1}}\le\frac{(t-x)^p}{p}M_p(x),\ \ t\in(x,b). \tag{24}$$

In conclusion, by (23) and (24) we may write

$$|f(t)-f(x)|\le\frac{1}{p}M_p(x)\,|t-x|^p\ \text{ for all }\ t\in(a,b). \tag{25}$$

Integrating (25) over t on $[a,b]$, we get

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \le\ &\frac{1}{b-a}\int_a^b|f(t)-f(x)|\,dt\le\frac{1}{p}M_p(x)\frac{1}{b-a}\int_a^b|t-x|^p\,dt\\ =\ &\frac{1}{p}M_p(x)\frac{1}{b-a}\left[\int_a^x(x-t)^p\,dt+\int_x^b(t-x)^p\,dt\right]\\ =\ &\frac{1}{p}M_p(x)\frac{(x-a)^{p+1}+(b-x)^{p+1}}{(p+1)(b-a)}\end{aligned}$$

and the inequality (22) is proved. □

Also solved by M. Bencze (Brasov, Romania) and John N. Lillington (Wareham, UK)..

55. Find all nonnegative solutions of the following system of equations:

$$\left(x_1+\cdots+x_k\right)\left(x_k+\cdots+x_{2009}\right) = 1,$$

where $k=1,\ldots,2009$.

Author's comment. This problem gives two wrong impressions at first glance. Firstly, it seems to be simple and routine, which is not the case at all (this is seen from the answer). Secondly, it seems that the problem is algebraic, which is not the case either. Its solution (the only solution I know) combines real analysis and plane geometry.

(Vladimir Protasov, Moscow State University, Russia)

Solution by the proposer. Answer. The solution is

$$x_k=\frac{\sin^2\left(\frac{\pi}{2011}\right)}{\sin\left(\frac{k\pi}{2011}\right)\sin\left(\frac{(k+1)\pi}{2011}\right)},\quad k=1,\ldots,2009.$$

Solution. Let us first show that all numbers x_i are positive. From the first equation of the system it follows that $x_1>0$. If k is the smallest index such that $x_k=0$ then combining the kth equation of the system with the $(k-1)$th, we get

$$\begin{aligned}&\left(x_1+\cdots+x_{k-1}\right)\left(x_{k+1}+\cdots+x_{2009}\right)\\ &\qquad=\left(x_1+\cdots+x_{k-1}\right)\left(x_{k-1}+\cdots+x_{2009}\right).\end{aligned}$$

Hence $x_{k-1}=0$, which contradicts the assumption. Now observe that $x_i=x_{2010-i}$ for all i. Indeed, using the notation $a=\sum_{i=1}^{2009}x_i$ we get $x_1=1/a=x_{2009}$. Then, combining the second equation of the system with the 2008th, we obtain $x_1+x_2=1/(a-x_1)=1/(a-x_{2009})=x_{2008}+x_{2009}$, which implies $x_2=x_{2008}$, and so on. The next step is to show that $a<2$. Since $x_1+\cdots+x_{1005}>\frac{a}{2}$, we see that

$$\begin{aligned}\left(\frac{a}{2}\right)^2&<\left(x_1+\cdots+x_{1005}\right)^2\\ &=\left(x_1+\cdots+x_{1005}\right)\left(x_{1005}+\cdots+x_{2009}\right)=1.\end{aligned}$$

Therefore $a<2$ and there exists an isosceles triangle with sides $1,1,a$. Denote its vertex by S and the vertices of its base by A_0 and A_{2009}. Let $A_1,\ldots,A_{2008}$ be points on the base such that $A_{i-1}A_i=x_i$, $i=1,\ldots,2009$. The first equation of the system now yields the similarity of the triangles $A_0SA_1\sim A_0A_{2009}S$, hence $\angle A_0SA_1=\angle A_0=\gamma$. From the symmetry it now follows that $\angle A_{2008}SA_{2009}=\angle A_{2009}=\gamma$. Now applying the second equation, etc., we get step by step: $\angle A_1SA_2=\cdots=\angle A_{2007}SA_{2008}=\gamma$. Thus, the segments SA_i split the angle A_0SA_{2009} into 2009 equal parts and each part equals the base angle γ. Since the sum of all these angles is π, it follows that $\gamma=\frac{\pi}{2011}$. Now applying the sine law, we get the answer. □

Also solved by Knut Dale (Telemark University College, Norway).

56. Consider the set of logarithmic derivatives of all algebraic polynomials that have no real roots:

$$S = \left\{ \frac{P'}{P} \,\middle|\, P \text{ is a polynomial without real roots} \right\}.$$

Clearly, this set is in the space $L_p(\mathbb{R})$ for any $p \in (1, +\infty)$. Is S dense in that space ? In other words, examine whether any function from $L_p(\mathbb{R})$ can be approximated by elements of S?

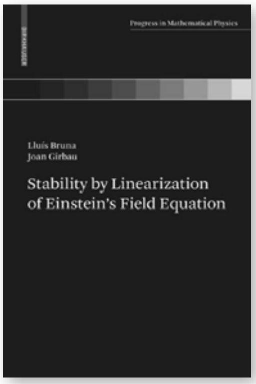
Author's comment. By the Weierstrass theorem, algebraic polynomials are dense in the space L_p on a finite segment. For the real line this is not true, since polynomials are not in $L_p(\mathbb{R})$. Nevertheless, some approximation bases for $L_p(\mathbb{R})$ can be made of polynomials, e.g. some classes of rational functions (Padé approximations, etc.). In this context, the class S is a natural pretender to constitute an approximation basis in $L_p(\mathbb{R})$. This class, for instance, is dense in L_p on a segment (this follows from the same Weierstrass theorem) in the space $C_0(\mathbb{R})$ of continuous functions vanishing at infinity, etc.

(Vladimir Protasov, Moscow State University, Russia)

Solution by the proposer. Answer. No.
Solution. For the sake of simplicity we consider only real-valued functions on $\mathbb{R}$. There are several approaches to solving this problem. Here we show one of them that involves the Hilbert transform on the real line: $F(f)[x] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{t-x}\,dt$, where the integral is in the sense of principal value. This is a continuous and continuously invertible operator in $L_p(\mathbb{R})$ for any $p \in (1, +\infty)$. The image $\mathscr{F}(f)$ of any function $f \in S$ is nonnegative almost everywhere on $\mathbb{R}$, hence the set $\mathscr{F}(S)$ is not dense in $L_p(\mathbb{R})$ and neither is S. To show that, we first note that any function $f \in S$ has the form $f(x) = \sum_{k=1}^{m} \frac{1}{x-z_k}$, where $z_1, \dots, z_m$ are complex roots of the corresponding polynomial P. Further, it is easily shown by direct calculation that $\mathscr{F}\left(\frac{1}{x-z_k}\right)$ equals $\frac{-i}{x-z_k}$ if $\operatorname{Im} z_k > 0$ and $\frac{i}{x-z_k}$ if $\operatorname{Im} z_k < 0$. In any case we see that the real part of the function $\mathscr{F}\left(\frac{1}{x-z_k}\right)$ equals $\frac{|\operatorname{Im} z_k|}{|x-z_k|^2} \geq 0$. Therefore, $\mathscr{F}(f) \geq 0$. □

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Number Theory*.

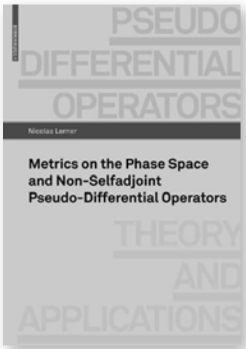


Stability by Linearization of Einstein‘s Field Equation

Bruna, Lluís
Girbau, Joan

2010, Approx. 225 p., Hardcover
ISBN: 978-3-0346-0303-4
EUR 59.95 / CHF 99.00
Progress in Mathematical Physics, Vol. 58

The concept of linearization stability arises when one compares the solutions to a linearized equation with solutions to the corresponding true equation. This requires a new definition of linearization stability adapted to Einstein‘s equation. However, this new definition cannot be applied directly to Einstein‘s equation because energy conditions tie together deformations of the metric and of the stress-energy tensor. Therefore, a background is necessary where the variables representing the geometry and the energy-matter are independent. This representation is given by a well-posed Cauchy problem for Einstein‘s field equation. This book establishes a precise mathematical framework in which linearization stability of Einstein‘s equation with matter makes sense. Using this framework, conditions for this type of stability in Robertson-Walker models of the universe are discussed.

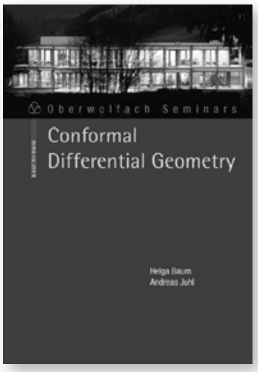


Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators

Lerner, Nicolas

2010, XII, 397 p., Softcover
ISBN: 978-3-7643-8509-5
EUR 69.95 / CHF 109.00
Pseudo-Differential Operators, Vol. 3

This book is devoted to the study of pseudo-differential operators, with special emphasis on non-selfadjoint operators, a priori estimates and localization in the phase space. We expose the most recent developments of the theory with its applications to local solvability and semi-classical estimates for nonselfadjoint operators. The first chapter is introductory and gives a presentation of classical classes of pseudo-differential operators. The second chapter is dealing with the general notion of metrics on the phase space. We expose some elements of the so-called Wick calculus and introduce general Sobolev spaces attached to a pseudo-differential calculus. The third and last chapter, is devoted to the topic of non-selfadjoint pseudo-differential operators. After some introductory examples, we enter into the discussion of estimates with loss of one derivative, starting with the proof of local solvability with loss of one derivative under condition (P). We show that an estimate with loss of one derivative is not a consequence of condition (Psi). Finally, we give a proof of an estimate with loss of 3/2 derivatives under condition (Psi). This book is accessible to graduate students in Analysis, and provides an up-todate overview of the subject, hopefully useful to researchers in PDE and Semi-classical Analysis.



Conformal Differential Geometry

Baum, Helga
Juhl, Andreas

2010, Approx. 165 p., Softcover
ISBN: 978-3-7643-9908-5
EUR 19.95 / CHF 34.90
Oberwolfach Seminars, Vol. 40

Conformal invariants are of central significance in differential geometry and physics. Well-known examples of conformally covariant operators are the Yamabe, the Paneitz, the Dirac and the twistor operator. These operators are intimely connected with the notion of Branson's Q-curvature. The aim of these lectures is to present the basic ideas and some of the recent developments around Q -curvature and conformal holonomy.
The part on Q -curvature starts with a discussion of its origins and its relevance in geometry and spectral theory. The following lectures describe the fundamental relation between Q -curvature and scattering theory on asymptotically hyperbolic manifolds. Building on this, they introduce the recent concept of Q -curvature polynomials and use these to reveal the recursive structure of Q -curvatures.
The part on conformal holonomy starts with an introduction to Cartan connections and its holonomy groups. Then we define holonomy groups of conformal manifolds, discuss its relation to Einstein metrics and recent classification results in Riemannian and Lorentzian signature. In particular, we explain the connection between conformal holonomy and conformal Killing forms and spinors, and describe Fefferman metrics in CR geometry as Lorentzian manifold with conformal holonomy SU(1,m).

Birkhauser Basel / Springer Basel AG
Viaduktstrasse 42
4051 Basel / Switzerland

Tel. +41 61 205 07 77
e-mail: sales@birkhauser.ch
www.birkhauser.ch