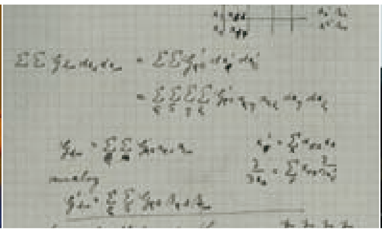
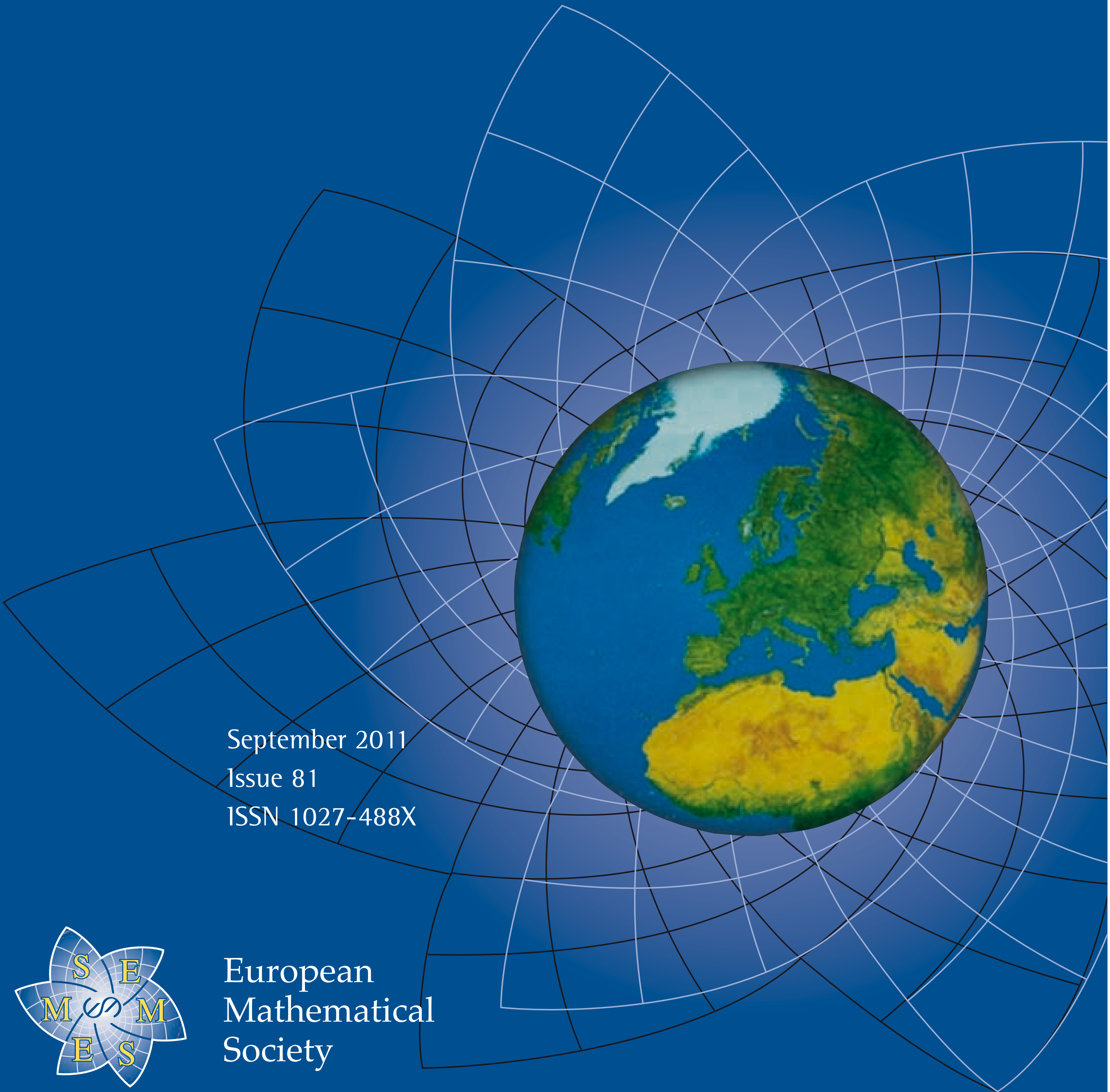
NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY





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European Mathematical Society

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The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

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EMS Agenda

2011

5–9 September
15th General Meeting of European Women in Mathematics, CRM, Barcelona
www.crm.cat/ewm/

5–9 September
Fourth European Summer School in Financial Mathematics
Zurich, Switzerland
www.math.ethz.ch/finance/summerschool/

2–4 October
Meeting of the EMS Education Committee, Barcelona, Spain
Günter Törner: guenter.toerner@uni-due.de

7–9 October
EMS-RSME Mathematical Weekend, Bilbao, Spain
www.ehu.es/emsweekend/

20 October
Meeting of the EMS Applied Math Committee, Frankfurt, Germany
Mario Primicerio: primicer@math.unifi.it

31 October–4 November
The hyperbolic and Riemannian geometry of surfaces and other manifolds. A conference in honour of Peter Buser
Ascona, Switzerland
homeweb.unifr.ch/parlierh/pub/Buser/

19–20 November
Meeting of the RPA Committee, Bilbao, Spain
Ehrhard Behrends: behrends@math.fu-berlin.de

25–27 November
Executive Committee Meeting, Firenze, Italy
Stephen Huggett: s.huggett@plymouth.ac.uk

2012

17–19 February
Executive Committee Meeting, Slovenia
Stephen Huggett: s.huggett@plymouth.ac.uk

23–24 March
Meeting of ERCOM. Budapest, Hungary.

31 March–1 April
Meeting of presidents of EMS member mathematical societies, Prague, Czech Republic
Stephen Huggett: s.huggett@plymouth.ac.uk

14 April
Meeting of the EMS Committee for Developing Countries, Limoges, France
Tsou Sheung Tsun: tsou@maths.ox.ac.uk

30 June–1 July
Council Meeting of European Mathematical Society, Kraków, Poland
www.euro-math-soc.eu

2–7 July
6th European Mathematical Congress, Kraków, Poland
www.euro-math-soc.eu

Editorial

Stefan Jackowski (President of the
Polish Mathematical Society)

A dozen reasons to participate in the 6th European Congress of Mathematics in Kraków, 2–7 July 2012

Mathematicians travel a lot as there are many conferences and workshops that take place all around the world. What is special about the European Congresses of Mathematics? Five of them have already taken place, in Paris (1992), Budapest (1996), Barcelona (2000), Stockholm (2004) and Amsterdam (2008). Why should you consider attending the next meeting?

1. To be a mathematician means to be more than an analyst, a probabilist or a topologist. It also means having a broad picture of the entire field. The European Congress of Mathematics provides an excellent opportunity to obtain information about the current state of mathematics from experts. Such an overview of contemporary mathematics is particularly important for young researchers looking for new research horizons. (There will be grants to support the participation of young mathematicians at the 6ECM.[1])
2. The plenary and invited speakers have been carefully selected by the Programme Committee, which was appointed by the European Mathematical Society (EMS). Among the plenary and invited speakers are stars of 20th century mathematics, as well as new leaders taking our field into the 21st century. The lectures will cover a broad spectrum of mathematical disciplines.
3. A special lecture commemorating the late Professor Andrzej Pelczar, former Vice-President of the EMS, former President of the Polish Mathematical Society (PTM) and former Rector of the Jagiellonian University (and the initiator for holding the 6ECM in Kraków) will be delivered during the congress. The speaker will be nominated by a committee appointed by PTM.
4. EMS prizes for young mathematicians (up to 35 years old), which will be awarded at the opening of the 6ECM, honour mathematicians who are likely to play a crucial role in the development of our field in the coming decades. Among past EMS prize winners are several Fields Medallists, including three of the last four Fields Medal winners. Try to guess who will receive prizes in 2012 and come to Kraków to see whether you are right. You can listen to the winners' lectures and personally congratulate them. You may also submit your own candidates for EMS prizes!

1 Grants are sponsored by the Foundation for Polish Science and the European Mathematical Society.

The medieval Market Square in Kraków with the Cloth Hall. ©iStockphoto.com/martin-dm

5. The Felix Klein Prize in the application of mathematics, established by the EMS and the Institute for Industrial Mathematics in Kaiserslautern, will be awarded. In addition, there is a new prize, the Otto Neugebauer Prize in the History of Mathematics, established by the EMS and Springer-Verlag. Again, everybody is welcome to submit candidates for these prizes!
6. At the 6ECM all participants have the opportunity to create part of the scientific program of the Congress by organising mini-symposia. There is a special two hours long session that consists of coordinated presentations on a selected topic proposed by the organiser. Get together with your mathematical friends and submit a proposal for a mini-symposium!
7. Everyone will have the opportunity to present a research poster, which will receive the attention and

Plenary Speakers of the 6ECM

Adrian Constantin (*Universität Wien, Austria*)
Camillo De Lellis (*Universität Zürich, Switzerland*)
Herbert Edelsbrunner (*Institute of Science and Technology, Vienna, Austria*)
Mikhail Gromov (*Institut des Hautes Etudes Scientifiques, France*)
Christopher Hacon (*University of Utah, USA*)
David Kazhdan (*The Hebrew University of Jerusalem, Israel*)
Tomasz Łuczak (*Adam Mickiewicz University in Poznan, Poland*)
Sylvia Serfaty (*Université Pierre et Marie Curie – Paris 6, France*)
Saharon Shelah (*The Hebrew University of Jerusalem, Israel*)
Michel Talagrand (*Université Pierre et Marie Curie – Paris 6, France*)

comments of experts. There will be a poster competition, the results of which will be announced at the closing ceremony. Submit your poster!

8. Besides the strictly scientific program, we expect several round table discussions that will cover a wide range of topics of general interest, such as the financing of mathematical research in Europe, the strategy of the EMS, mathematical education, and mathematics and emerging economies. Everyone is invited to submit a proposal for a round table discussion.
9. Combine your participation at the 6ECM with attendance at more specialised satellite meetings, which will be organised close in time to the 6ECM. We have already announced several satellite meetings devoted to various aspects of pure and applied mathematics, and we expect to have more soon. Traditionally, a joint meeting of the European Women in Mathematics and Women in Mathematics EMS-Committee is held the day before. We invite you to participate in the satellite meetings and to propose new satellite meetings.
10. The European Congresses help to form a community of mathematicians from European institutions. We have many topics to discuss. There are a growing number of exchange students taking mathematics courses at various institutions. Research projects are financed by diverse international sources and various ways of evaluating research results are used. University system reforms are spreading throughout Europe. These are some examples of the important issues that can be discussed during the 6ECM both formally and informally.
11. Enjoy Kraków, a city with a great past and a very interesting present. Kraków, the capital of Poland from the 14th to the 16th century, played an important role in European history as a political, religious, scientific and cultural centre. It was home to Pope John Paul II, who contributed to the recent great political change in Europe. Kraków was also home to a large Jewish community, one of the very few places in Poland where buildings in the old Jewish quarter remained intact after the Holocaust. In recent years it has been a place for the revival of Jewish cultural and religious life. Many museums in Kraków contain priceless objects of art; the best known is *Lady with an Ermine*, a painting by Leonardo da Vinci. The newest museum is the Museum of Contemporary Art in Kraków (MOCAK), which was designed by an Italian architect. It is built in a post-industrial zone close to Schindler's famous factory, which has also been converted into a museum. Every Summer Kraków hosts many exciting artistic events.
12. Last but not least, the European Congresses of Mathematics aim to be a showcase to present the unity of mathematics. Large meetings are important to foster public awareness of mathematics. If we want mathematics to be perceived as an important, well-defined field, not just an addendum or a tool in science and technology, we need to show the strength of the mathematical community to the public and to politicians who make decisions about financing research and higher education. So come to Kraków and take part in this important event!

And before coming, we invite you to visit the website www.6ecm.pl for more information about the Congress, the prizes to be awarded during the 6ECM, satellite meetings, tourist attractions and practical information. At the website you can also pre-register for the 6ECM and submit proposals for the various activities mentioned above. You are additionally invited to join us on Facebook.

On behalf of the European Mathematical Society and the local organisers of this congress, the Polish Mathematical Society and the Jagiellonian University in Kraków, I cordially invite you to participate in the 6ECM.

See you in Krakow!
Да сустрэчы ў Кракаве!
Ще се видим в Краков!
Vidimo se u Krakowu!
Ens veiem a Cracòvia!
Vidimo se u Krakowu!
Uvidíme se v Krakově!
På gensyn i Krakow!
Tot ziens in Krakau!
Wir sehen uns in Krakau!
Näeme Krakow!
Nähdään Krakovassa!
Rendez-vous à Cracovie!
იხილეთ თქვენ კრაკოვში!
Θα σας δούμε στην Κρακοβία!
Nézze meg Krakkóban!
Sjáumst í Krakow!
Féach tú i Krakow!
להתראות בקרקוב!
Ci vediamo a Cracovia!
Tiekamies Krakovā!
Pasimatysime Krokuvoje!
Се гледаме во Краков!
До встречи в Кракове!
Se deg i Krakow!
Vê-lo em Cracóvia!
Ne vedem în Cracovia!
Uvidíme sa v Krakove!
Se vidimo v Krakow!
Nos vemos en Cracovia!
Vi ses i Krakow!
Krakow görüşmek üzere!
До зустрічі в Кракові!
Do zobaczenia w Krakowie!

The Meeting of the Presidents of the National Member Societies of the European Mathematical Society, 7–8 May 2011 in Bilbao

Martin Mathieu (Vice-President of the Irish Mathematical Society)

The Royal Spanish Mathematical Society (RSME) commemorates the centennial of its founding in 2011. In this connection, the society invited the presidents of the national member societies of the EMS to a meeting on the weekend of 7–8 May 2011 in Bilbao. This was the 4th meeting of a very informative and successful series of meetings (Luminy, 2008; Warsaw, 2009 and Bucharest, 2010, although the latter had a much restricted participation due to travel difficulties at the time).

There were 41 participants present at the meeting representing 34 mathematical societies and organisations. A long and varied agenda was presented and discussed in an intense but informal work atmosphere at the New Auditorium of the University of the Basque Country. The discussions were very effectively led by the EMS president Professor Marta Sanz-Solé of the University of Barcelona, supported by the secretary Dr Stephen Huggett (Plymouth) and other support staff.

The first activity of the meeting turned out to be a guided tour through the nearby Guggenheim Bilbao Museum, which was once described as the "… greatest building of our time" by Philip Johnson. Apart from the opportunity to see some fantastic artwork – such as Richard Serra's "Matter of Time" or Jenny Holzer's LED installation – this provided time to meet the other participants in a welcoming and informal environment. Personally, I valued this a lot as this was my first participation in a meeting of this kind and I hardly knew anybody before.

After lunch, the formal business started with a brief welcome and an introduction of each of the national representatives. Two presentations followed, which were given by the host organisation (Antonio Campillo) and on the work of ICIAM (Rolf Jeltsch), in particular on the IMU-ICIAM working group on bibliographic metrics, stirring up quite some discussion (not unexpectedly). It was suggested to formulate a best practices document that could guide the evaluation of individual researchers, which has now become so customary in many European countries.

In her subsequent president's report, Marta Sanz-Solé highlighted new collaborative agreements with the Mathematical Union of Latin American and Caribbean Countries and with the International Association of Mathematical Physics. She also described several new EMS committees, among them a new ethics committee, the remit of which will be to formulate a code of practice on mathematical publications. Her report ended by drawing attention to the EMS position paper on the next EU framework programme (FP7).

Various EMS business issues followed, on membership (where it was suggested that each member society nominates a corresponding member), the society's website, the meetings committee, the EMS publishing house and the EMS Newsletter.

The President of the Polish Mathematical Society Professor Stefan Jackowski described in detail the current

Even the light rain didn't diminish the enjoyable tour at the Guggenheim Museum.

RSME President Antonio Campillo at his welcoming speech.

Some of the participants during one of the breaks.

state of organisation of the 6th European Congress of Mathematics in Kraków in July 2012 and gave us a tour of their website. He pointed out that pre-registration was already open. Marta Sanz-Solé described the prizes to be awarded, invited national mathematical societies to consider providing support for young mathematicians to attend the congress and outlined the procedure by which the congress programme had been put together.

A longer discussion on funding of mathematical research in Europe developed. Marta Sanz-Solé described the work of the ERC as a real success story and drew attention to a plan for grants for small research groups. She also introduced a new project of the Executive Committee to prepare a map of research funding for mathematics in Europe for which input from the national societies will be needed.

Mats Gyllenberg (ESMTB and PESC/ESF), having first given an account on the history of the European Science Foundation (ESF), revealed a rather sad picture for the future. The ESF is likely to become defunct in the very near future, removing yet another funding opportunity for mathematics.

A variety of special projects initiated by national societies was presented on the second day. This included the 7th Congress of Romanian Mathematicians, the situation of mathematics in Georgia, the South-East European Doctoral Year of Mathematical Sciences 2011, the future of the Erwin Schrödinger Institute in Vienna and an outline proposal for a submission on inquiry based teaching methods to FP7. Several presidents immediately expressed their support for this idea.

This was followed by an open discussion and an expression of severe concern on the proposed closure of the geometry section in the VU Amsterdam. A draft letter expressing strong opposition to this proposal was circulated and signatures were collected to be sent along with this letter.

In my opinion, the aim of this meeting in Bilbao – to establish further links between national mathematical societies, to foster networking and to make us aware that mathematics always needs more lobbying – were fully achieved.

The next Meeting of Presidents is planned to be held in Prague in Spring 2012 on the 150th anniversary of the Union of Czech Mathematicians and Physicists.

More detailed information as well as copies of the slides of the presentations can be found on the EMS website at http://www.euro-math-soc.eu/node/1045.

The above pictures were taken by Ms Miren Zubeldia Plazaola.

The First EMS-ERCE Label Attributed to ASSMS Lahore*

Michel Waldschmidt (Vice-President EMS-CDC, Paris) and Alla Ditta Raza Choudary (Director, ASSMS, Lahore)

The first ERCE (Emerging Regional Centre of Excellence of the European Mathematical Society) label has been attributed to the Abdus Salam School of Mathematical Sciences (ASSMS) in Lahore (Pakistan) on the recommendation of the Committee for Developing Countries (CDC).

The Abdus Salam School of Mathematical Sciences, GC University, in Lahore was created in 2003 by the Government of the Punjab to act as a centre of excellence for the mathematical sciences. ASSMS aims to be recognised regionally, nationally and internationally as a distinctive provider of high quality teaching, learning and research. It is dedicated to producing a substantial output of doctoral students with research experience at international level and to supporting mathematics throughout Pakistan. It takes pride in being a quality institution of higher education in Pakistan. It aims to provide liberal education to the youth of the subcontinent irrespective of caste, colour or creed.

ASSMS offers a fulltime MPhil/PhD programme in mathematical sciences maintaining outstanding international standards. The school has distinguished, non-Pakistani faculty members who are known worldwide for their professional excellence. At present the Higher Education Commission (HEC) of Pakistan is the main sponsor of the school.

ASSMS has established the strongest and the largest exclusive time (more than fulltime) MPhil/PhD mathematics programme in the country with about 100 PhD students. Among these, there are 30 female students enthusiastically pursuing their PhD degrees.

Before the existence of ASSMS, Pakistani mathematicians used to go abroad for post-doctoral fellowships. Now, here at ASSMS, things are happening the other way round. Many young mathematicians with MPhil/PhDs from prestigious institutions around the world are choosing ASSMS to do their post-doctoral fellowships. This success of the post-doctoral fellowship programme shows the high quality of the faculty at ASSMS.

The school regularly organises international conferences on the theme "21st Century Mathematics". ASSMS also runs a large number of research seminars, workshops, Schools (especially CIMPA Research Schools), lecture series, colloquia, professional enhancement seminars and intensive courses for researchers and the university faculty. The conferences have been pioneering in their nature and at this level within Pakistan in the field of mathematics.

The PhD students are engaged in quality research and their papers are regularly published in prestigious, world-class journals. Research areas include different branches in analysis and several fields in algebra, combinatorics, graph theory, algebraic geometry, algebraic topology, differential geometry, mathematical modelling, fluid mechanics, control theory, stochastic processes, financial mathematics, dynamical systems, quantum theory and computational mathematics.

Oral and written communication skills are extremely valuable assets to do well in mathematics. ASSMS has introduced a professional communication skills training programme to increase the confidence level of its students and also to improve their ability to speak and write effectively.

Now ASSMS is taking some effective and practical steps to promote and popularise mathematics in schools in Pakistan. In this direction, concrete steps have been taken that are providing much needed encouragement and incentives to Pakistani students in schools and colleges. The faculty at ASSMS regularly organise training camps for students from schools and colleges. The participants of the camps are prepared and selected for the national team of Pakistan to compete at the International Mathematical Olympiad (IMO). In 2005, the National Team of Pakistan took part for the first time in the IMO and Pakistan won its first medal at the IMO in 2007 and its second in 2009. Furthermore, ASSMS has organised several mathematics contests at the national level. These contests have sparked considerable enthusiasm among the schools and colleges of Pakistan.

With this ERCE label, it is expected that this school will play an even more important role in the region by attracting young mathematicians from neighbouring countries eager to pursue their studies and reach the PhD level in a renowned centre, without needing to emigrate to a Western country.

The programme of ERCE is described on the website of the EMS-CDC and a call for applications has been issued by this committee and was published in the March 2011 issue of the EMS Newsletter. This is the very first step of a long, ongoing process, initiated by EMS-CDC, which aims to promote high-level mathematics in developing countries.

Reference websites:
ASSMS: http://www.sms.edu.pk/
EMS-CDC: http://ems-cdc.org

* In the section Research Centres of this issue (pages 42–44) there is a description of the Abdus Salam School of Mathematical Sciences ASSMS in Lahore.

Cooperation agreement between the EMS and UMALCA

Carlos A. Di Prisco

A cooperation agreement has recently been signed between the European Mathematical Society (EMS) and the Mathematical Union of Latin America and the Caribbean (Unión Matemática de América Latina y el Caribe, UMALCA) in order to promote the interaction of mathematicians from Europe and Latin America, including advanced graduate students and post-docs. The agreement contemplates some specific actions, e.g., establishing joint projects and activities, such as schools and workshops, and in particular giving support to UMALCA's Programme of Mathematical Schools (EMALCA), which has been active for over ten years. The agreement also seeks to promote the exchange of printed and electronic publications.

UMALCA was created in a meeting held at the Institute of Pure and Applied Mathematics of Rio de Janeiro (IMPA) in July 1995, with the participation of the Presidents of the Mathematical Societies of Argentina, Brazil, Chile, Colombia, Cuba, Mexico, Uruguay and Venezuela, and a representative from Peru. It was proposed as an instrument to improve and facilitate the connections between the different research groups existing in the Latin American region and to stimulate the scientific exchange of researchers and students.

UMALCA started its activities with a programme of small travel grants with the purpose of enhancing scientific cooperation between mathematicians from the region. There are three calls every year for applications to this programme. UMALCA's Scientific Committee evaluates the proposals and selects the recipients of the travel grants. On average some 20 travel grants have been assigned each year since 1996.

The Mathematical School of Latin America and the Caribbean, or Escuela Matemática de América Latina y el Caribe (EMALCA), is nowadays one of the most prominent of the programmes developed by UMALCA. It consists of a series of mathematical schools offering short introductory courses in order to stimulate interest in mathematical research among students, especially from the mathematically less developed areas of the subcontinent. The EMALCA programme was based on the experience developed in Mexico and Venezuela, countries where there is a long tradition of mathematical schools that have been important for the promotion of mathematical activity. The programme started in 2001, with a school in Cuernavaca, Mexico, followed by another in Mérida, Venezuela, the following year. Since then, EMALCAs have been organised in these two countries in alternating years. In addition, since 2005, EMALCAs have been organised in different countries, especially those with a more incipient mathematical community. EMALCAs have been held in Costa Rica, Paraguay, Guatemala, Nicaragua, Bolivia, Ecuador, Peru, Cuba, El Salvador and Brazil's Amazonia. These schools have been partially funded by the Centre International de Mathématiques Pures et Appliquées (CIMPA).

A few years after its creation, UMALCA decided to organise a Latin American Congress of Mathematicians (CLAM), responding to the need to have a major mathematical event in Latin America. The CLAM takes place every four years and represents the main mathematical meeting of the region covering all areas of mathematics. The first CLAM was held in Rio de Janeiro, Brazil, in 2000, the second in Cancún, Mexico, in 2004 and the third in Santiago de Chile in 2009. The 4th CLAM will take place in Córdoba, Argentina, 6–10 August 2012. The first announcement will be online soon on UMALCA's website (http://www.famaf.unc.edu.ar/clam2012/) and a webpage for the congress will be online soon.

Another successful programme of UMALCA is the series of research schools called ELAM. These are devoted to one research area and include tutorials and conferences. The Escuela Latinoamericana de Matemáticas actually existed before UMALCA's creation; the idea came out of discussions held during the 6th Colóquio Brasileiro de Matemática (Poços de Caldas, Brazil, 1967), after which an organising committee was appointed. The first ELAM Committee was formed by José Adem (Mexico), José de Barros Neto (Brazil) and Lluís Santaló (Argentina), and the first of these meetings was held in Rio de Janeiro in 1968, with courses by L. Schwartz, R. Seeley and F. Trèves. Since 1995, UMALCA has taken over the organisation of the ELAM research schools, the most recent of which, the 15th ELAM on non-commutative algebra and Lie theory, took place in Córdoba, Argentina, in May 2011. This ELAM was also a CIMPA Research School.

More information about UMALCA and its programmes can be obtained on its website http://www.umalca.org.

We have great expectations for the results of the cooperation between the EMS and UMALCA under this agreement. There are some well established joint research projects being developed by European and Latin American mathematicians that should receive further support, and new ones will certainly start under the framework of the cooperation agreement. For the near future, some activities such as workshops are being planned. The possibility of a joint EMS-UMALCA meeting is being considered, based on some previous experiences such as the EMS-SIAM-UMALCA meeting that took place in the Center for Mathematical Modeling in Santiago de Chile in 2005. Another way to foster cooperation could be with the creation of chairs in mathematical institutes, both in Europe and in Latin America, to host visiting mathematicians for periods from a month to, possibly, a year. A good experience in this sense is provided by the Lluís Santaló Visiting position of the Centre de Recerca Matemática of Barcelona, which is specifically addressed to mathematical researchers affiliated to Latin American institutions. This position was created to honour the memory of Santaló, a Catalan mathematician who based himself in Argentina where he became a reference for mathematical research in Latin America. We hope that the EMS-UMALCA cooperation agreement opens ways for this kind of cooperation mechanism and many others.

Carlos Di Prisco [cdiprisc@ivic.gob.ve] is Investigador Emérito of mathematics at Instituto Venezolano de Investigaciones Científicas (IVIC) and Secretary of the Executive Committee of UMALCA.

ICMP12

17th International Congress on Mathematical Physics, 6–12 August 2012, Aalborg, Denmark

Arne Jensen (Congress convenor)

The ICMP12 is the main congress of the International Association of Mathematical Physics (www.iamp.org) and is held every three years. It gathers a large group of mathematicians and physicists working at the interface between mathematics and physics.

Confirmed plenary speakers:
Dmitry Dolgopyat (University of Maryland), Philippe Di Francesco (CEA Saclay), Uffe Haagerup (University of Copenhagen), Klaus Hepp (ETH Zürich), Shu Nakamura (University of Tokyo), Jeremy Quastel (University of Toronto), Renato Renner (ETH Zürich), Wilhelm Schlag (University of Chicago), Benjamin Schlein (University of Bonn), Mu-Tao Wang (Columbia University), Simone Warzel (TU München), Avi Wigderson (Institute for Advanced Study, Princeton).

Topical sessions with confirmed session organisers:

Dynamical systems, classical and quantum; Kening Lu (Brigham Young University) and Rafael de la Llave (Georgia Institute of Technology).

Equilibrium and non-equilibrium statistical mechanics; Horia Cornean (Aalborg University) and Antti Kupiainen (University of Helsinki).

PDE and general relativity; Chang-Shou Lin (National Taiwan University) and Hans Ringström (KTH Stockholm).

Stochastic models and probability; Laszlo Erdős (LMU München) and Ofer Zeitouni (Weizmann Institute).

Operator algebras, exactly solvable models; Karl-Henning Rehren (Universität Göttingen) and Jean-Michel Maillet (ENS de Lyon).

Quantum mechanics and spectral theory; Rafael Benguria (P. Universidad Católica de Chile) and Jacob Schach Møller (University of Aarhus).

Quantum information and computation; Barbara Terhal (RWTH Aachen) and Michael Wolf (TU München).

Quantum many-body theory and condensed matter physics; Mathieu Lewin (Université de Cergy-Pontoise) and Marcel Griesemer (Universität Stuttgart).

Quantum field theory.

String theory and quantum gravity; Volker Schomerus (DESY) and Laurent Freidel (Perimeter Institute).

Reaching beyond: Mathematical physics in other fields.

Young Researcher Symposium: 3–4 August 2012, Aalborg, Denmark
Confirmed plenary speakers: Robert Seiringer (McGill University) and Thomas Spencer (Institute for Advanced Study, Princeton).

Information as of August 11, 2011.
Updated information at: www.icmp12.com
Inquiries: icmp12@math.aau.dk

EUROMATH 2011 Conference Creativity and Innovation from Early Age

A mathematics conference for students aged 12–18 Age

Gregory Makrides (President of the Cyprus Mathematical Society and President of the THALES Foundation)

The EUROMATH 2011 conference was organised this year in Athens, 30 March – 3 April 2011. The idea for the EUROMATH conference started in 2009 with EUROMATH running in parallel to the Cyprus student conference. Some 45 international students attended the conference in 2009 in Cyprus with some very interesting presentations. In 2010, the conference was organised in Austria with some 150 international students, while this year there were about 250 international students. The conference was organised by the Cyprus Mathematical Society and the THALES Foundation of Cyprus with the cooperation of the European Mathematical Society, the Hellenic Mathematical Society and the Department of Mathematics of the National and Kapodistrian University of Athens.

Dr Gregory Makrides delivering a welcoming address during the Opening Ceremony of EUROMATH 2011

Many national student conferences are organised in various European countries (including Cyprus, where we started the student conference in mathematics eight years ago) but a European level and a true international level student conference with so much success is today a unique event. The growth of this conference appears to be following a monotonically increasing function as within two years the number of papers presented by students has tripled, reaching 90 presentations. The 90 presentations involve some 200 students presenting in cooperation. It is with great pleasure that such an activity is found to be so attractive to students and we believe that this event will continue to grow. The ground gives the opportunity for students aged 12–18 to be creative in both the content of their presentations but also in the style of their presentations. Communicating through mathematics with fellow students of a similar age from all over the world helps young people to become science communicators, develop social skills, develop presentation skills, develop confidence and make new friends.

By the first day of the conference an abstract booklet is published which is expected to be transformed into an interesting conference proceedings for students and teachers for the years to come. In this year's event we had another new activity: the Math Poster Design Competition. With this, the students had the opportunity to express their interest in mathematics through a different talent: 'art', whether this be expressed by hand or using new technologies.

Each presentation could be the result of a mathematics project, a library study, etc. The presentation does not have to be original work but could be original in the presentation approach, the media used, etc. Students are asked to prepare an abstract with given specifications and they had 20 minutes for their presentation. Almost all students used PowerPoint presentations. We then ask the students to submit full papers, of up to 10 pages, which after some evaluation we include in electronic conference proceedings. In this way students may have a published paper from an early age. The whole process helps the students to develop research, presentation and writing skills, which are very useful for the development of young researchers. The conference, by bringing students from all over the world, also helps fight any xenophobia among European and international citizens, providing the ground for the European Dimension of Education and for cooperation between students from all over the world. We want to encourage cooperation among students from different countries in order to see joint presentations by students from different countries at future EUROMATH conferences. In order to encourage this we are considering developing a competition of best presentation of this type, i.e. presentations by mixed country joint work.

At this year's conference, we had several interdisciplinary presentations, like mathematics and other subjects: physics, music, sports, astronomy, informatics, robotics, economy, decision making, architecture, cryptography,

Prof. Dr. Ehrhard Behrends giving the Math Poster Design Competition prizes to the student winners on behalf of sponsors Munich RE, ERGO together with Dr Gregory Makrides, Chair of the EUROMATH 2011

etc. Applications of mathematics in real life and statistics were also presented. Among other mathematics topics the following were presented: geometry, special mathematics theorems, special mathematics properties, special mathematics problems, logic, number theory, game theory, combinatorics, history of mathematics, approximation methods, numerical methods, discrete mathematics, sequences and series, analysis, etc.

In the years to come, we plan to add more activities within the EUROMATH conference, such as the Mathematics Theatre Competition, a Competition on Teaching Mathematics using Cartoons and the FameScience Competition for Mathematics and Science Communication. These three rather new activities at a European level are also submitted under a proposal for European project funding, which will allow us to develop the infrastructure and provide support to students. The proposal is submitted with 25 partner organisations, with the Cyprus Mathematical Society as the coordinating organisation.

I have to say that this conference is very useful for teachers to attend; they have a lot to learn from the students. What I can assure all is that in the near future we will see some of these EUROMATH students on the list of top mathematicians of the world.

I see EUROMATH in the future as an event of 1000 students from all over the world. Becoming longer, maybe 4–5 days or even a full week, I see many parallel activities as mentioned above and more social activities and cultural events. I would like other disciplines to copy the idea and organise EUROPHYSICS or EUROCHEMISTRY or EUROBIOLOGY or EUROINFORMATICS. We could even consider a future event of EUROSCIENCE including all of the above. We are preparing EUROMATH 2012 to take place in Sofia, Bulgaria during 21–25 March 2012 and the EUROMATH 2013 to take place in either Romania or Sweden.

International Mathematical Summer School for Students

Martin Andler, Etienne Ghys, Victor Kleptsyn, Dierk Schleicher and Serge Tabachnikov

General presentation (goals, funding, organisation…)

This summer school is an introduction to top-level mathematical research topics for selected international students at the age of transition between high school and university. It is scheduled to be organised annually, alternating in location between Bremen, Germany, and Lyon, France.

This summer school is inspired by the summer school "Sovremennaya Matematika" (Contemporary Mathematics) that has been running successfully for 10 years in Dubna near Moscow for Russian students in the Russian language; other sources of inspiration are the summer schools at American universities and the Mathematics Advanced Study Semester run by Serge Tabachnikov at Penn State University. One of our goals is to bring these successful traditions into Western Europe (European Union member states and associated states) and at the same time to make them open to international participants. We hope that this initiative will help foster intra-European connections among the participants as well as develop the attractiveness of the European Research Area at the international level.

The scientific committee, chaired by Étienne Ghys (CNRS-ENS Lyon, France), included Frances Kirwan (University of Oxford, UK), Dierk Schleicher (Jacobs University, Germany), Alexei Sossinsky (Moscow University, Russia), Serge Tabachnikov (Penn State University, USA), Anatoliy Vershik (St. Petersburg State University, Russia), Wendelin Werner (Université Paris-Sud, France), Jean-Christophe Yoccoz (Collège de France), Don Zagier (Max Planck-Institute Bonn, Germany, and Collège de France) and Günter M. Ziegler (Freie Universität Berlin, Germany). The organising committee, chaired by Dierk Schleicher, included Anke Allner (University of Hamburg), Martin Andler (Université de Versailles-Saint-Quentin and Animath, France), Victor Kleptsyn (CNRS – Université de Rennes, France), Stephanie Schiemann (Deutsche Mathematiker-Vereinigung, Berlin, Germany) and Serge Tabachnikov (Penn State University, USA).

The project was funded by the Volkswagen Stiftung (for three sessions, in 2011, 2013 and 2015) and the Clay Mathematics Institute. It is part of the "European campus of excellence" project. It was endorsed by the European Mathematical Society, the Deutsche Mathematiker-Vereinigung and the French Animath.

Who applied? Who participated?

In spite of the rather late official announcement of the summer school (end of March 2011) and tight deadline to apply (4 May, with a few extensions granted), we received 166 applications, and finally accepted 110 students, out of whom 92 actually came. The ones who did not come declined for different reasons: personal issues, impossibility of getting a visa in time, insufficient financial support in view of the travel costs, calendar conflict with IMO preparation[1] or other competitions. The advertisement was sent out through various channels: International Mathematical Olympiad team leaders, websites like Artofproblemsolving, Mathlinks, American Mathematical Society, Animath, …

Of course, the information about the summer school did not reach all possible applicants. Nonetheless, students from 44 countries applied, and those who participated came from 30 countries. This sample gives an interesting insight into a large international group of talented students. The school was open to students between the end of the junior year (one year away from finishing high school) to the end of the second year of university. The selection committee turned down a few applicants who were beyond the 2nd year of university (but we did allow applicants who had finished high school more than 2 years ago, allowing for gap years and reorientations).

Altogether, the breakdown of those who attended is as follows: 22 students with two years of higher education, 22 students with one year, 29 who finished high school this year and 19 with another year to go, with ages ranging from 16 (with two younger) to 21, the median age being 18. In terms of nationalities, Germany, with 11 participants, and Russia, with 10, were the largest groups, then Italy and Romania (6), France (5), Austria, Paraguay, Spain, USA (4), Morocco, Peru, South Africa, South Korea, United Kingdom (3), Belarus, China,

1 The 2011 IMO took place 16–24 July; some countries had a last preparation camp during the summer school.

T. Tokieda with students

Czech Republic, Portugal, Sweden, Tajikistan, Ukraine (2), Albania, Belgium, Bulgaria, Croatia, Lithuania, Luxembourg, Mongolia, Venezuela, Vietnam (1). Most participants live in their country of origin but, not unexpectedly, a few study away from home in the UK or the US.

As one can see, this was a truly international event, with a mostly European participation, fully deserving the endorsement of the EMS.

Who lectured on what?

The main part of the school – though not the only important one! – was its lectures. They covered different topics and were of different difficulty but always of great interest.

Wendelin Werner's four-lecture course started with discrete random walks and harmonic functions on graphs and ended with the non-recurrence of 3D Brownian motion and an implication for real life: the big gradient of the electric potential at the sharp end of an electrode.

John Hubbard and Dierk Schleicher both spoke on complex dynamics. Hubbard gave an introductory course, visualising holomorphic dynamics and showing the connectedness of the Mandelbrot set. Schleicher's lectures were devoted to the dynamics of cubic polynomials, studying Newton's method for them, and the surprising stumbling upon regions where it doesn't work for some open initial domains (a discovery by Hubbard), and noticing the similarity between the effect observed in the phase and in the parameter space.

John Conway's series of lectures started with lexicographic codes, finding out that such codes form a vector space over natural numbers that become a field with some new, quite strange, addition and multiplication, before passing on to game theory, partisan games and surreal numbers, impartial games and nim addition and multiplication, only to discover that the addition and multiplication of nim-games are exactly the ones we've seen in the first lecture as the new "strange" operations on the natural numbers!

Tadashi Tokieda's outstanding lectures were on mathematics where it touches the real world – applied mathematics for real physics processes. Water waves, tsunamis and tides, estimating the power of a blast and stabilisation of the upper equilibrium for a rotating pendulum – all of this was blended into some magic of discovery, of constant brainwork and constant simplicity, simple arguments breaking through complicated questions.

Another beautiful, breathtaking walk of constant discovery, this time for pure algebra, came through the lectures by Don Zagier, who moved from the very origins of algebraic number theory to the Birch and Swinnerton–Dyer conjecture, one of the seven Millennium problems in the list by the Clay Institute.

Etienne Ghys' lectures were devoted to charts – in all their aspects, starting from their history, mentioning the Mercator projection, conformal maps and the Riemann and Poincaré–Coebe theorems, and ending with an engaging lecture on Tchebychev nets, arising from clothing questions.

Dmitry Fuchs and Serge Tabachnikov showed the beauty of seemingly almost-elementary questions: osculating curves, evolutes and involutes, cusps and the four-vertex theorem…

There were more and more: a course by Mario Bonk on Lipschitz functions, reaching the recent results in this domain; a lecture by Günter Ziegler, starting with a problem of cutting a convex polygon into polygonal parts of equal area and perimeter, in the study of which have appeared Voronoi diagrams, minimisation of functionals and equivariant cohomology; a course by Rostislav Matveev on elementary geometry, reaching Gauss-Bonnet's theorem; Martin Andler's course, passing from the planes distributions to the Heisenberg group and then to quantum mechanics; and two lectures of Victor Kleptsyn on lattices and codes. And all together this became 10 days of constant discovery, of finding hidden paths in the world of mathematics and revealing its beauty, creating the spirit of the school!

How it worked

One of the main goals of the school was to encourage interaction between the participants and the lecturers.

Everyone lived in the same building (one of the colleges of Jacobs University) and ate together in its cafeteria. A large lounge was available to the participants, equipped with table tennis, a billiard table, a variety of table games, etc. This room was always full of people, literally until midnight, when the lights went off; the organisers wanted the students to have enough time for a night's rest. Generous coffee breaks between the lectures also provided ample opportunity for interaction and so did the two excursions, to Bremen and to Island Wangerooge. In short, a typical picture was a lecturer surrounded by a group of students and engaged in conversation with them.

The topics of informal discussion with the students ranged from cultural (such as specifics of mathematical education in different countries or the relation between participating in mathematical Olympiads and doing mathematical research) to purely mathematical, often times starting with questions on the instructor's lecture but evolving to his own research interests and beyond. As a result, it happened a couple of times that the instructors changed the topics of their lectures "on popular demand" to address questions raised by the students; the organisers encouraged this kind of improvisation.

The participants of the school also interacted with graduate students of mathematics at Jacobs University. Every lecture had a dedicated teaching assistant, a graduate student who attended the talk and was available to the students for help afterwards.

With the participants coming from so many countries and speaking so many languages, the organisers feared that blending together could be problematic. This fear proved unfounded; from the first days of the school, one could see students from different countries talking to each other and engaging in various activities (for example, a soccer game against the instructors). The working language of the school was English and this did not appear to be a problem for the participants from non-English speaking countries (some help was available to those who had language difficulties).

All the lectures were recorded by a professional team and they will soon be available online. Information will be available on the school website at http://math.jacobs-university.de/summerschool/program/index.php.

What did the students think?

Only a few days after the end of the summer school, it is a bit early to have a full picture. Nonetheless, judging from the many obvious interactions during the summer school between students and faculty, and among students, and since then seeing the intense activity on Facebook, it is fair to say that the school was a big success. A few comments that we received are telling. While the mother of one of the younger students wrote to us telling us that her son came back "enthralled", another more senior student wrote: "This summer school was one of the best experiences I've ever had. (…) All the expectations I could possibly have had were triumphed by the reality of the school. I met fantastic people, full of intelligence and warmth, many of whom I had touching conversations with, that surely will impact my life in many ways. I think that it is safe to say, that this group of people you put together, has to be one of the most amazing ones possible. The professors all answered my question with full respect, and never once did I feel stupid for asking something. I had many pleasant discussions with lecturers about their topics, and they were always happy to share their knowledge, and treated you as their equal. (…) I have often felt very lonely in my life, and when I arrived here and got a chance to interact with other participants, that feeling vanished. I can not express how much happiness this has brought to me, and I have all intentions on continuing with mathematics, to be a part of this wonderful community. If I had any doubts before – they're all gone now, this is what I want to do."

D. Zagier with a student

Conclusion, future

Hopefully, the Bremen Summer School was the first of a long series of future European summer schools. We plan to organise a similar school every year, alternating between Bremen and Lyon. Funding for the next French summer schools has already been essentially secured thanks to the so-called Labex (French "grand emprunt"). It is now time for us to analyse this successful first attempt and to try to transform it into an even better success. One of the key questions that we will have to address concerns the selection of participants. Should we choose young students because they are already the best or do we aim at selecting students who could become the best after attending such a school?

Not an easy question! We'll do our best…

The Dissemination of Mathematics in Brazil: Searching for Talent among Schoolchildren

César Camacho (Director IMPA – Instituto Nacional de Matemática Pura e Aplicada)

Cocal dos Alves is a small city (5,600 inhabitants) in Piauí, one of the three states in Brazil with the lowest human development index. There are only two schools in Cocal dos Alves. They are no exception to the endemic problems that public schools suffer in the northeast region of Brazil: teachers with low salaries and the schools generally deprived of fundamental infrastructure for adequate teaching such as basic libraries, let alone computing facilities. Last year, four students from Cocal dos Alves won a gold medal in the Brazilian Mathematical Olympiad for Public Schools (OBMEP, www.obmep.org.br). They were among the 500 gold medallists of this national competition, which involved the participation of 19.6 million students – yes, the number of students that participate in this competition is counted in the millions and last year was the approximate equivalent to 10% of the Brazilian population. Indeed, this number of participants has been increasing annually since the first event in 2005.

OBMEP is organised by IMPA, a renowned research institution that, besides the practice of high level research in mathematics and the training of new researchers, has as part of its mission the dissemination of mathematics throughout the country at all levels. Last year more than 43,000 schools decided to join this competition.

The establishing of a partnership with the schools and their teachers has been of fundamental importance for the success of this initiative. For instance, approximately 120,000 teachers collaborate with the grading in the First Phase: a collection of 20 multiple-choice questions. These tests are offered to students at three different levels: Level 1 – 5th and 6th year of fundamental school (10–12 year old students); Level 2 – 7th and 8th year of fundamental school (13–15 year old students); Level 3 – secondary school students. For the grading of the tests the teachers use the set of solutions to the problems previously distributed by OBMEP to the schools. The role of each school is to select the students in the 5% bracket of best performance. These students are then invited to participate two months later in the Second Phase, a test with six mathematical problems that is held in 8,000 centres throughout the country. This time the students must provide a written explanation of the reasoning used to solve the problems. The grading is carried out by mathematicians, firstly on a regional level, in order to select the best 30,000 tests. These papers are taken to IMPA where they are graded and ordered. This yields the final results of the Olympiad, which is then announced. The 500 gold medals are distributed as follows: 200 to Level 1, 200 to Level 2 and 100 to Level 3. Then 900 silver medals and 1,800 bronze medals are distributed equally among the three levels. The 3,200 winners of these medals represent less than 0.02% of the total number of participants, thus forming an important bank of talent that has been enriched every year since 2005.

The tests of the First and Second Phases are prepared by a special committee formed by ten mathematicians with long experience in this kind of competition. The tests are thoroughly discussed and checked. Much of the success of this activity is due to the beautiful, interesting and challenging questions of the Olympiad, which can be appreciated on OBMEP's website. The logistical aspect of OBMEP is largely supported by a national network of about 70 university professors of mathematics, and this is fundamental for the success of the activity, as well as its vast national scope. They encourage the directors and teachers of the schools in their regions to participate. They are also responsible for the choice of the graders of the Second Phase and are ready to help with unexpected problems that may arise during the realisation of the tests. In fact, this network has extended others already existing that were working in national programmes linked in some way to IMPA or to the Brazilian Mathematical Society (SBM). This is the case, for instance, with the retraining of secondary school teachers and the Brazilian Olympiad of Mathematics (OBM), a longstanding activity which is devoted mainly to the selection of students that represent Brazil in International Olympiads. Thus, OBMEP is an activity very well integrated with the national mathematical community in their commitment to improve mathematical learning and the search for young, talented people. It is in this spirit that SBM supports OBMEP.

The distribution of medals has been quite uniform in geographical terms. Last year, the 3,200 medals were distributed among students from more than 800 cities. The number of cities in Brazil is 5,500 and last year 99% of them had at least one school participating in OBMEP.

OBMEP also offers the medallists a one-year training programme in mathematics. This is held on weekends in 170 centres distributed over the country according to the places of residence of the medallists. This programme is carried out with a different network consisting of more than 400 teachers.

Another programme associated with OBMEP is offered to any undergraduate student that at some time has won a medal in either OBMEP or OBM. This consists of a course of scientific initiation, for up to three years, followed by a two-year Masters course in mathematics, both offered by the Graduate Division of Mathematics Departments. This five-year activity is supported by fellowships from government agencies CNPq and CAPES. The goal of this programme is to offer any student that has shown capacity to win a medal in the Olympiads and who is enrolled in an undergraduate course the opportunity to complete a Masters course in mathematics simultaneously with his or her undergraduate degree. This programme has already supported more than 1,000 students in the best universities of Brazil.

All these activities are paid for by the Ministries of Education and of Science and Technology. The total per capita cost is less than one US dollar. This low cost is due essentially to the generosity of mathematicians and teachers that graciously participate in this project.

The performance of the students of Cocal dos Alves has been consistent through the years. Every year since 2005 they have had at least three medallists, and in 2010 they had 12. This indicates that this success is in fact a consequence of a training programme spontaneously established by local teachers aimed at preparing students for OBMEP.

OBMEP has established a natural path for good students who now see learning mathematics as a concrete access to the best universities in Brazil. There are also many cases of students for which success in OBMEP has brought to light their genuine talent, previously unnoticed by their teachers. For instance, this year half of the team that represents Brazil in the International Mathematical Olympiad is formed by students coming from public schools, an unprecedented event before the existence of OBMEP.

Many other success stories are due to the personal efforts of the students and their teachers who, in small cities of Brazil, are quietly studying mathematics using the texts distributed by OBMEP to every public school. No doubt this will have a strong impact in the future in the development of mathematics and science in Brazil.

Over the last 40 years, Brazil has adopted the strategy of developing scientific research through a heavy doctoral training programme in Brazil and abroad, mainly through government agencies for the support of science.

The country has established a solid scientific community based on scientific merit as a principle of selection and promotion. Nowadays, Brazil provides more than 10,000 PhDs a year in all areas of science. On the other hand, basic public education at large has been suffering from stagnation for a long time. Despite the considerable investments by the government, in the international evaluations on basic education Brazil still occupies one of the lowest ranks. Thus, OBMEP stands as an example of the beneficial influence that the well-established Brazilian scientific community can have on fundamental education. It is also expected that this beautiful initiative will be followed by special programmes devoted to the support of the intellectual development of these talented young people.

Paul Malliavin (10 September 1925–3 June 2010)

Anton Thalmaier (Université du Luxembourg)

Malliavin as a young boy of around 12 years

Paul Malliavin at the Hammamet conference in stochastic analysis, Tunesia, November 2009

On 3 June 2010, Paul Malliavin passed away at the American Hospital in Paris. Less than four weeks prior to his death many of his colleagues and friends came together at an international conference with 250 participants at the Chinese Academy of Science in Beijing, honouring him and his scientific work. Probably no one at this meeting anticipated that this would be the last opportunity to experience Paul Malliavin talking in public about mathematics. Malliavin seemed to be a timeless figure. Being in his 80s, his intellect was sharper than ever; his curiosity, passion and enthusiasm for mathematics was without limitation. His personality seemed to be untouchable by physical conditions; age could not bend or slow down this man. Still giving four talks within ten days in China, his health however deteriorated after returning to Paris. Despite suffering from pulmonary fibrosis for some years, his death came unexpectedly to everyone who knew him.

Born in 1925 in Neuilly-sur-Seine, Malliavin's way into mathematics was by no means straightforward. He had strong interests in other fields as well, including law, history and literature, which made the decision between law and mathematics a difficult choice; he began his university studies by taking courses in both fields. To say it in his own words: "*I was born into a family of intellectuals who were deeply involved in politics for several generations, either by writing books or by exercising political responsibilities at a national level in France. I have the highest respect for the fighting life of my parents, uncles, and grandparents; I have often seen their disillusions after fighting for carefully planned political proposals that were finally withdrawn. One of my reasons for choosing mathematics has been that as soon as truth is discovered, it enters immediately into reality.*" (From *Mathematicians: An Outer View of an Inner World*, 2009.)

The early years: Malliavin as a harmonic analyst

Paul Malliavin finished his graduate studies in mathematics at Sorbonne University in Paris in 1946. He had the chance to take courses taught by the great masters of the French school of the beginning of the 20th century: Émile Borel for integration and Élie Cartan for geometry. He was deeply influenced by Jean Leray and Szolem Mandelbrojt, later his thesis advisor, both of whom had returned to France after the war. Szolem Mandelbrojt advised Malliavin to read his joint Comptes Rendus note with Norbert Wiener, which was devoted to the characterization of the set of zeros of some Laplace transforms. He asked him the question of what could be said about the set of real zeros of a holomorphic function in the right half-space satisfying a certain growth condition – a question which had its origin in this joint work. Malliavin detected in this problem of complex analysis of one variable a certain infinite dimensional non-linear duality, to which the Banach-Baire principle could be applied in order to prove the needed uniform estimate. He came back to Mandelbrojt with a complete and definitive answer to the question which resulted in his thesis published in *Acta Mathematica* and brought him, with the recommendation of Jean Leray, an invitation by Marston Morse to come as a postdoctoral fellow to the Princeton Institute for Advanced Study (IAS) in 1954–55. The IAS was at this time a unique gathering place of mathematicians from all over the world.

At Princeton he shared an office with Alberto Calderón for one year. Calderón, who had just finished his work with Zygmund on singular integrals, was renewing the theory of partial differential equations with the introduction of pseudo-differential operators. The contact with Calderón marked the beginning of a lifelong friendship, and opened up Malliavin's vision of Fourier analysis. Calderón showed him his forthcoming paper where he proved the localization of Littlewood-Paley theory for Fourier series of one variable, a method which Malliavin later used for his own work in collaboration with his wife.

In 1954 Arne Beurling, a visionary mathematician, had settled at the Institute for Advanced Study. From him Malliavin learned about the "spectral synthesis problem": *Is it true that in the normed ring of functions with absolutely convergent Fourier series, any closed ideal is the intersection of maximal ideals containing it?*" Four years later Malliavin noticed that an appropriate extension of the Wiener–Gelfand analytic symbolic calculus could be used to give a negative answer to the spectral synthesis problem on the real line. Malliavin's complete and definitive solution of the problem has been the subject of many lectures and related works; it brought him instant recognition. His final proof in 1959 that *spectral analysis fails for any non-discrete, locally compact, abelian group* made Malliavin's name famous. It nevertheless killed the field and marked the end of an era.

Beurling invited Malliavin again to the Institute in 1961, where he also met Lennart Carleson with whom he established another lifelong relation. The collaboration with Beurling turned out to be extremely fruitful; within one year they solved two hard open problems in analysis. Malliavin liked to tell anecdotes about Beurling's perfectionist style, not wanting to publish results which he thought were not yet in an ultimate and definitive form. As a consequence, their second joint *Acta Mathematica* paper did not appear until 1967, although the authors knew the results as early as 1961.

The later years: Malliavin as a probabilist

Malliavin's first steps into probability theory were anything but streamlined. Around the age of 40, Malliavin started working on functions of several complex variables. One of his objectives was a generalization of Blaschke's theorem from one to several complex variables. He realised that certain asymptotic estimates of the Green kernel near the boundary of the unit ball in several complex variables, along with subsequent results of Henri Skoda and Guennadi Henkin, would allow him to solve Blaschke's problem. Stuck in his effort to prove the necessary estimate, Malliavin had his first encounter with Itô's theory of stochastic differential equations. He had met Kiyosi Itô at the Institute for Advanced Study already in 1954, and had grasped from him the basics of Itô calculus. Concerning his problem, Malliavin observed that substituting Brownian motion associated to the natural Kähler metric on the unit ball into the corresponding Kähler potential and developing the resulting one-dimensional diffusion by means of Itô's formula leads to a process which, using easy geometric estimates, can be dominated by a diffusion on the real line, with a simple Sturm–Liouville type operator as generator. This "comparison lemma" published in 1972 did not only give the desired estimate; it turned out to be the first application of the later Ikeda–Watanabe comparison theory for stochastic differential equations. It also marks the starting point of Malliavin using probabilistic arguments in analysis and geometry, for which he would develop an unequalled mastership.

A turning point in Malliavin's career was Kiyosi Itô's talk at the ICM at Stockholm in 1962, where he showed that the Levi–Civita parallel transport of tensors on a Riemannian manifold can be done along the trajectories of Brownian motion. Taught by Élie Cartan, who had written two books on the method of moving frames, Malliavin immediately recognised the importance of this construction, which allows one to globalise the local Itô construction in the context of the bundle of orthonormal frames. This was the starting point of a new field: stochastic differential geometry, formed by mixing Élie Cartan's geometry on the frame bundle with Kiyosi Itô's theory of diffusion processes.

Malliavin saw right from the beginning what was later called "Malliavin's transfer principle", namely that every construction in differential geometry which can be done with smooth curves can also be done with paths of diffusions if the classical derivatives are interpreted in the sense of Stratonovich differentials. Since the tangent bundle of the orthonormal frame bundle over a Riemannian manifold is trivial (it is trivialised by the standard horizontal and vertical vector fields on the frame bundle), one can construct a diffusion associated to Bochner's horizontal Laplacian by solving a canonical stochastic differential equation on the frame bundle. The projection of this process down to the manifold then gives an intrinsic construction of Brownian motion associated to the Levi–Civita Laplacian. This elegant geometric method, developed by Eells–Elworthy and Malliavin, of constructing random processes on curved spaces by rolling the manifold along the paths of a flat Brownian motion in the tangent space, transfers the classical Cartan development of differentiable curves to the probabilistic world; it provides at the same time Brownian motion together with an intrinsic notion of parallel transport along its paths. Malliavin used the new method in 1975 for a probabilistic Feynman–Kac representation of the de Rham–Hodge semigroup on differential forms which allowed him to prove Bochner–Kodaira type vanishing theorems for the cohomology of the manifold.

At that time it already became clear that Brownian motion might serve as a tool to interpolate between the local and global geometry of a manifold: for small time Brownian motion is governed by the local geometry, while for large times it captures its global structure. Jean-Michel Bismut quickly absorbed the new ideas and used them later in his stochastic proof of the Atiyah-Singer index theorem for Dirac operators. Here one investigates the small time asymptotics of a certain deformed parallel transport in a Clifford bundle along Brownian loops. The local index density is then calculated as the expectation of the supertrace of this random holonomy under contraction of the Brownian loops to constant loops. The advantage of this method is that all relevant calculations can be done under the expectation at the level of random functionals; the evaluation of the supertrace is reduced to elementary linear algebra, and the so-called "fantastic cancellations" become fully transparent.

In the same way as a vector field on a manifold induces a flow, second order differential operators induce stochastic flows which however behave very irregularly in the time variable. In this sense, Brownian motion on a Riemannian manifold appears as the stochastic flow associated to the Laplace-Beltrami operator. In the 70s Malliavin became interested in the push forward of the underlying measure under such flows. Completely in the spirit of Wiener, he looked at these measures on path space as analytical objects, to which analytic methods should be applied. Wiener himself had recognised that his measure carries the same Hermitian structure as the standard Gaussian measure on the line, which led him to his famous spectral decomposition of the space of square-integrable functionals on Wiener space into subspaces of "homogeneous chaos". This decomposition can be seen as what quantum field theorists call the Fock space representation of the number operator.

Malliavin's goal was to develop a differential calculus on Wiener space which could be applied to functionals as general as those arising as solutions to Itô's stochastic differential equations. In infinite dimensions, like on path space, a function can be infinitely differentiable in the sense of Sobolev without being even well-defined at every point. Before Malliavin, differential analysis on Wiener space was mainly restricted to functions being differentiable in the classical sense of Fréchet. Based on results of R. H. Cameron and W. T. Martin, two students of Norbert Wiener, who had established quasi-invariance of the Wiener measure under translation by elements which are absolutely continuous with square integrable derivative, Malliavin chose a certain operator, known to probabilists as the Ornstein–Uhlenbeck operator, as the primary operation in his theory because it is self-adjoint and behaves well in calculations involving integration by parts. This was the starting point of a new kind of analysis in infinite dimensions which Malliavin called "Stochastic Calculus of Variations".

One of the first aims of Malliavin in this field was to give a purely probabilistic approach to Hörmander's famous hypoellipticity theorem which provides a condition for a partial differential operator, written as a sum of squares of vector fields, to be hypoelliptic. The question comes down to showing that the stochastic flow associated to this second order differential operator is sufficiently smooth and non-degenerate to guarantee that a certain induced heat kernel measure has a smooth density with respect to the Lebesgue measure. The required non-degeneracy condition can be expressed in terms of integrability conditions on the inverse determinant of the famous Malliavin matrix. This already marks a keystone of the new calculus.

Paul Malliavin discussing mathematics, Kent State University, 2008

Malliavin presented these ideas at the SDE Symposium in Kyoto 1976. His Japanese colleagues, particularly K. Itô and his students N. Ikeda and S. Watanabe, immediately recognised its potential and began to give it a formulation that has become standard. At the same time Dan Stroock gave a series of lectures in France on the new methods; he dubbed it "Malliavin Calculus", a term which soon became standard. The theory rapidly grew through numerous extensions, simplifications and alternative approaches. A crucial estimate which greatly simplified many calculations is due to P. A. Meyer. Over the years Malliavin calculus developed into a powerful machinery, with essential contributions from many other mathematicians like Bismut, Ikeda–Watanabe, Kusuoka–Stroock, Nualart–Zakai, Üstünel and Bouleau–Hirsch, to name just a few. A solid theory of Sobolev spaces on Wiener space was developed by Len Gross and Dan Stroock; integration by parts theorems for the measure induced by Brownian motion on path space of a manifold or by pinned Brownian motion on loop space were established by J.-M. Bismut, B. Driver, E.P. Hsu and P. Malliavin and his wife Marie-Paule. I. Shigekawa proved the Hodge decomposition on Wiener space, quasi-sure analysis was developed and an anticipative stochastic calculus was established in the 80s by Nualart–Pardoux–Zakai.

Terms like "Malliavin derivative", "Malliavin matrix" and "smooth in the sense of Malliavin" became standard vocabulary in graduate courses in probability and in conference talks. Currently more than 25 monographs on Malliavin calculus are available. Malliavin entered probability theory at the age of 45; in less than 15 years he had completely reshaped the field.

Around 2000 P.-L. Lions and his coworkers began to use methods from Malliavin calculus to stabilise the numerical computation of price sensitivities, so-called Greeks, in the theory of option pricing in finance. Malliavin was proud to see Malliavin calculus suddenly in the centre of such practical fields as finance; he even wrote a monograph "Stochastic Calculus of Variations in Mathematical Finance" to explain his point of view.

The aim of one of Malliavin's big projects over the last 12 years was the construction of natural measures on infinite dimensional spaces, with strong motivation from mathematical physics. Eminent examples are Brownian measures on the diffeomorphism group of the circle, on the space of univalent functions of the unit disc and on the space of Jordan curves in the complex plane. He understood that unitarizing measures for representations of Virasoro algebra can be approached as invariant measures of Brownian motion on the diffeomorphism group with a certain drift defined in terms of a Kähler potential.

Paul Malliavin in Uppsala, Sweden, 2005

Malliavin as a person

Malliavin never thought in terms of applied and pure mathematics, nor was he interested in formal generalizations; he aimed at concepts and ideas. For him mathematics was a unity and not divisible into different fields or branches. Whenever he recognised new promising ideas, even in the work of very young mathematicians or PhD students, he was extremely generous in offering his support. Many young mathematicians may have shared the potentially intimidating experience when, after a conference talk, Malliavin would come running behind them and shouting in a loud voice: "I need to talk to you…" However, such conversations usually turned out to be very encouraging and rewarding.

It was impossible to meet Malliavin without talking mathematics. When encountering him, his first question used to be: "What are you currently working on?" And then he would keep on asking questions until his curiosity was satisfied. Convinced of "the fundamental unity of mathematics", Paul Malliavin liked to characterize his career as one of "mathematical wandering", devoted to the establishing of relations between fields that seemed relatively unrelated. For him only ideas counted in mathematics and he would not start fighting with the necessary technical details before having understood a problem "from above" with a clear vision of what should be done.

The mathematical work of Paul Malliavin consists of about 200 research articles, and it would be foolish to try to go into details. He continued over the last years in a steady rhythm of publishing papers on themes as diverse as the Euler equation of deterministic incompressible fluid dynamics using tools of stochastic differential geometry, the Wiener measure on Jordan curves, unitarizing measures for a representation theory of Virasoro algebra, Stein's method for estimating the speed of convergence to Gaussian laws, numerical approximation schemes for stochastic differential equations and problems in mathematical finance, like the Fourier computation of volatilities for high frequency financial data.

To understand the person of Malliavin, one probably has to go back to his early childhood. Born as a single child into a very conservative environment – his mother couldn't have any more children after his birth – he kept a close and very emotional relationship to his parents all his life. Each year the family, together with numerous relatives, used to spend the summer months in a castle in the province of Auvergne. For birthdays of the small Paul, his grandfather ordered knights arriving on horses delivering the birthday presents. To inspire self-confidence in his grandson, the small boy had to receive the arriving delegations and the people from the village offering presents. It seems that this injection of self-confidence continued to have a lasting effect even 80 years later.

Collaborating with Malliavin has always been an exciting and challenging experience. When working on a specific problem and facing all the difficulties, one is often ready to give up, but not so Malliavin. The word "impossible" did not exist in his vocabulary. Armed with formidable technical skills, he liked such hopeless situations where he would finally turn things around by introducing new, unexpected ideas; he enjoyed it if the new approach turned out to work.

Malliavin still had many unfinished projects in mind and somehow during the last period of his life he felt that time was limited. Undeterred by technical difficulties, Malliavin pressed ahead even more than in his younger years. From his bed in hospital he still discussed mathematical projects with his collaborators. Some of his friends visiting him got worried by the alarms from the surrounding machines when he continuously lifted his breathing mask which disturbed him explaining mathematics. It is not known what the doctors in the hospital thought, when days before passing away he suggested transporting the machines necessary to prevent his lung from collapsing to his private home, as he was annoyed that without sitting at his home computer it was difficult for him to work properly.

His departure marks the end of an extraordinary career and leaves a huge gap in the community, or to say it with the words of Michèle Vergne, "*… the world without Malliavin is not quite the same*".

Einstein's 'Zurich Notebook' and the Genesis of General Relativity

Paulo Crawford

1 The long journey from special to general relativity[1]

In 1905, Einstein created special relativity (SR) – what he then called the Principle of Relativity – to reconcile the relativity of motion of inertial observers with the electromagnetic theory of James Clerk Maxwell (1831-1879). In November of 1915, Einstein came up with general relativity (GR) to reconcile gravity with the principles of SR and to extend the relativity of motion to include all observers. These are the main points we are remembering here.

Einstein himself, when he was preparing some notes for Erwin Freundlich's book (the first popular book on general relativity, published in 1916), divided the sequence of events of his search into three parts:

(a) In 1907, he had the basic idea for a generalized theory of relativity when he found a fundamental explanation for the equality of gravitational and inertial mass.
(b) In 1912, he finally recognised the non-Euclidean nature of the space-time metric and its physical determination by gravitation.
(c) In 1915, he arrived at the correct field equations for gravitation and the explanation of the anomalous precession of the perihelion of Mercury.

Let us start in September 1907, when Einstein agreed to write a review article on his 1905 special theory of relativity, commissioned by Johannes Stark for the Jahrbuch für Radioaktivität und Electronik (Yearbook of Radioactivity and Electronics). Einstein had only two months to write his Jahrbuch article, entitled "On the Relativity Principle and the Conclusions Drawn from it", in which he gave an excellent overview of the principle of relativity for electrodynamics, mechanics and thermodynamics. In a letter dated 1 November 1907, Einstein informed J. Stark that he had "so arranged the work that anyone could find his way with comparative ease into relativity theory and its applications so far". When the article was completed on 1 December 1907, the first four parts, devoted to an overview of the foundations of the principle of relativity, were followed by a fifth part, on nine pages, containing entirely new material. This last section, under the heading "The Relativity Principle and Gravitation", was committed to gravity. In it, Einstein argued that a satisfactory theory of gravity could not be achieved within the framework of special relativity and that a generalization of that theory was needed. Namely, once the special theory was limited to "inertial systems", that is, to reference frames in uniform, non-accelerated motion relative to one another, Einstein raised the question of whether the principle of relativity could be extended to accelerated motion. This question could be understood as asking whether the covariant group of the special theory of relativity could be extended beyond the Lorentz group. At the time, Einstein did not feel prepared to deal with that question. Eventually, he made a connection between the generalization to any system of reference on one hand and the relativistic treatment of gravity on the other. This was a great and lonely step forward, since other researchers, such as Henri Poincaré, Hermann Minkowski and Gustav Nordstrom to mention just a few, felt that a perfectly adequate theory of gravitation could be constructed simply by modifying Newton's theory of gravitation to meet the demands of special relativity. Einstein, in contrast, was looking for a generalization of the principle of relativity while working on "Relativity Principle and Gravitation". Suddenly, he found his way when looking for a fundamental explanation for the equality of inertial and gravitational mass. Then a thought occurred to him that he later described as "the happiest thought of my life" (Einstein 1922).

Starting from a "thought experiment" of a man falling freely from the roof of a house, Einstein realised that he would be able to treat gravitation within the framework of special relativity. He realised that for such an observer "there exists – at least in its immediate surroundings – no gravitational field". Everything happens as if the observer were at rest or in a state of uniform motion. In the last part of his 1907 article, Einstein deals with the "relativity principle and gravitation", where he tackles the problem of generalizing the principle of special relativity to accelerated frames. Using the principle of equivalence, which also states that there is no difference between a uniform gravitational field and a uniformly accelerated frame, it is possible to go from one to another. This approach proved fruitful and provided Einstein with a plan of attack for the next eight years in the battle that would lead him to produce a theory of gravitation compatible with SR.

Then, in 1911, shortly after his arrival in Prague, where he became a full professor at the local German University, Einstein published a paper in the Annalen der Physik entitled "On the Influence of Gravity on the Propagation of Light". As he explains at the beginning of the paper, he was regressing to a subject of 1907. Indeed, his problems were still the same as in 1907 and his methods were almost the same, and for that reason he obtained identical formulas. But some arguments, such as that of red shift, were new. However, he had come to realise that one of the most important consequences of that analysis was accessible to experimental test. In his own words, "according to the theory I am going to set forth, rays passing near the sun experience a deflection by its gravitational field, so that a fixed star appearing near the sun displays an apparent increase of its angular distance from the latter, which amounts to almost one second of arc".[2]

Einstein's search for general relativity spanned an eight year interval, 1907-1915, but it is fair to say that some periods were calm and some were more forceful. The moment the great development came about was sometime between the late summer of 1912, when Einstein moved from Prague to Zurich, and early 1913. Prior to his move to Zurich in August 1912, Einstein was already struggling with the puzzle of accommodating gravitation into his 1905 special theory of relativity. Then he saw it: the connection between gravity and non-Euclidean geometry. One could introduce the most general gravitational fields in the space-time of special relativity merely by curving its geometry. All this thought and knowledge accompanied him when he went back to his Alma Mater, Zurich Polytechnic.

Summarising, Einstein's approach was embodied in heuristic principles that guided his search from the beginning in 1907. The first and more lasting insight was the "Equivalence Principle", which states that gravitation and inertia are essentially the same. This insight implies that the class of global inertial frames singled out in special relativity can have no place in a relativistic theory of gravitation. In other words, Einstein was led to generalize the principle of relativity by requiring that the covariance group of his new theory of gravitation be larger than the Lorentz group. This led him on a long journey; in his first step, in his review of 1907, Einstein formulated the assumption of complete physical equivalence between a uniformly accelerated reference frame and a constant homogeneous gravitational field. That is, the principle of equivalence extends the covariance of special relativity beyond Lorentz covariance but not as far as general covariance. Only later did Einstein formulate a "Generalized Principle of Relativity" which would be satisfied if the field equation of the new theory could be shown to possess general covariance. But Einstein's story, appealing to this mathematical property of general covariance, is full of ups and downs.

The turning point in the history of Einstein's discovery of the gravitational field equations was in the early summer of 1912, when he realised the significance of the metric tensor and the general line element for a generalized theory of gravitation (Pais 1982, section 12b, and Stachel 1980). Then Einstein started to study properly the mathematics of Gaussian surface theory, apparently in collaboration with Grossmann,[3] becoming acquainted with Beltrami invariants. Grossman also discovered for Einstein the existence of the "absolute differential calculus" of Ricci and Levi-Civita (1901) that would enable Einstein to construct a generally covariant theory of gravitation. However, when Einstein and Grossmann published the results of their own research in early 1913, the theory of the resulting paper, commonly known as the "Entwurf" theory from the title of the paper (Einstein and Grossmann 1913), failed to comply with the generalized principle of relativity, since this theory offered a set of gravitational field equations that was not generally covariant.

Until the Autumn of 1915, Einstein continued to elaborate on and improve the "Entwurf" theory and explored many of its consequences. Already in 1913, Einstein and his friend Michele Besso had found that "Entwurf" equations did not account for the anomalous advance of the perihelion of Mercury, something that Einstein hoped to explain with his new theory of gravitation.[4] Although Einstein knew the failure of

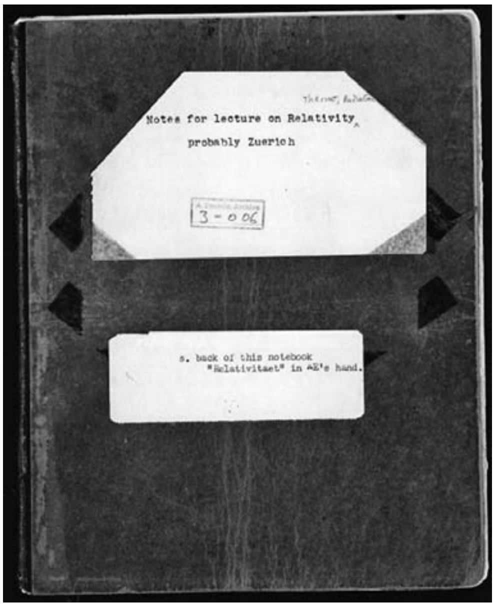

Figure 1. Einstein's Zurich Notebook (Einstein Archives Call Nr. 3-006)

the "Entwurf" theory to resolve the Mercury anomaly, he continued to hold on to this theory in spite of everything. How he overcame all obstacles and finally obtained in 25 November 1915 his final theory of gravitation will be explained in what follows.

In "Autobiographical Notes", Einstein points out that the importance of the equivalence principle in requiring a generalization of SR was clear to him in 1908 (actually it was in 1907). And he adds: "Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that co-ordinates must have an immediate physical meaning." (Einstein 1949, p.67). What we have discussed so far is not enough to illustrate this comment made by Einstein in 1949. But it was exactly the resolution of this puzzle that separated Einstein from the final theory, as can be discovered by reading the whole paper.

Einstein saw his work on general relativity as something quite unique in his life. He felt that if he had not created the special theory of relativity, someone else would have done so. His approach to a new theory of gravitation was entirely his own, carried through with considerable hard work and facing scepticism if not active opposition from physicists he respected, such as Max Planck and Max Abraham. He characterized his efforts on special relativity as mere child's play compared to what was needed to complete general relativity.

The task of a reconstruction of Einstein's building of the theory of general relativity has challenged several historians of science for a long time.[5] A major step forward in this venture is due to John Stachel's and John Norton's groundbreaking investigations.[6] A very important interpretative tool for understanding Einstein's search for the gravitational field equations is the so-called Einstein's Zurich Notebook,[7] a document written between Summer 1912 and Spring 1913, during his time in Zurich. It was in the course of preparing the editorial project of the Collected Papers of Albert Einstein that John Stachel first realised the importance of this manuscript for the reconstruction of Einstein's theory of gravitation (Stachel 1980). A little later, John Norton also published a comprehensive reconstruction of Einstein's discovery process (Norton 1984). Following these discoveries, a group of scholars including John Stachel, John D. Norton and Jürgen Renn undertook a systematic analysis of this notebook and revealed an unexpected result: Albert Einstein had written down, in 1912, an approximation to his final field equations of gravitation, which were derived by him three years later. He failed, however, to recognise the physical meaning of his mathematical results. In 1997, Jürgen Renn and Tilman Sauer have shown that the clarification reached by deciphering Einstein's research notes would have serious consequences for our understanding of the genesis of general relativity (Renn and Sauer 1999). According to them, the Zurich Notebook shows that in 1912-1913 "Einstein had already come within a hair's breadth of the final general theory of relativity". In any case, the period between 1913 and November 1915 should not be considered as a period of stagnation. It was, rather, a period during which Einstein arrived at a number of insights that created the prerequisites for his final triumph.

In what follows, we will recall the research carried out by these scholars[8] deciphering Einstein's notebook and the interplay between physics and mathematics ignited by the pursuing of the new theory of gravitation, the theory of general relativity.

2 Einstein's search for general relativity in the Zurich Notebook

The Zurich Notebook originally comprised 96 pages. The notebook has two front covers. Einstein wrote in it in both directions. On the first front cover, Helen Dukas (Einstein's secretary) typed the description of the notebook as "Notes for Lectures on Relativity ...". If we flip the notebook over, we find a second cover with the word "Relativität" in Einstein's handwriting. Eighty-four pages of this notebook contain calculations or short notes on various problems of physics, mainly on gravitation theory. According to Jürgen Renn and Tilman Sauer,[9] "most of the calculations are extremely sketchy, display a lot of false starts, and come with no explanatory text". Inside Relativity's cover are a few rough sketches and some recreational puzzles in mathematics. The page that faces it, however, contains serious physics. There we find Einstein recounting the elements of the four-dimensional approach to relativity and Minkowski's electrodynamics, starting with four space-time coordinates $(x, y, z, ict) = (x^1, x^2, x^3, x^4)$ and going on through scalars, four-vectors and six-vectors and their operations. Recall that it took some time for Einstein to embrace Minkowski's reformulation of special relativity in terms of a four-dimensional space-time manifold, which is a crucial instrument for the further development of a relativistic theory of gravitation. As late as July 1912, Einstein had not adopted the four-dimensional geometrical approach of Minkowski, even though a book using this approach had already been published (Laue 1911); apparently, Einstein became acquainted with Minkowski's formalism through Laue's book. All this changed with Einstein's move to Zurich in August 1912 where he began collaborating with his old mate Marcel Grossmann, now the chairman of the independent Section VIII of the Swiss Federal Technical University. Once there, Einstein was then

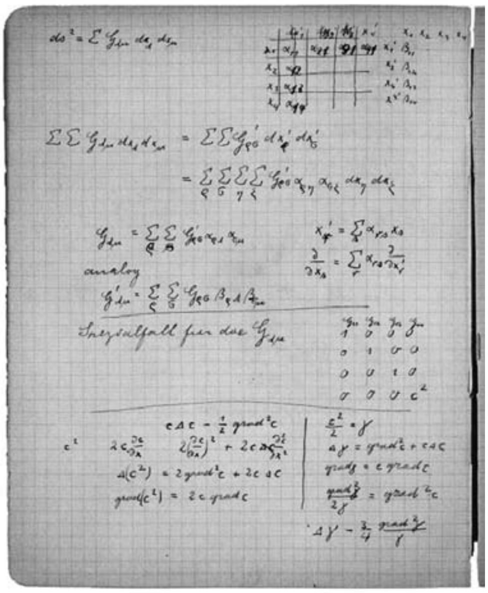

Figure 2. The "line element" written at the top of page 39L

introduced by Grossmann to the 'absolute differential calculus' of Ricci and Levi-Civita.

But let's go back to the Zurich Notebook pages where Einstein was starting to deal with Minkowski's approach. A central element of Minkowski's geometrical representation of special relativity was the manifest invariance under linear, orthogonal transformations of the quantity

$$x^2 + y^2 + z^2 - (ct)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 ,$$

or in differential form

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 . \quad (1)$$

The development continues for 13 pages, recounting notions in electrodynamics and thermal physics. All of a sudden, without a word of warning of the new subject, we stumble on the basic notion of general relativity, the "line element" written at the top of page 39L, the first exploration of a metric theory. On that page, Einstein gathered the building blocks for his new theory of gravitation: the metric tensor and the gravitational equation of his second theory of static gravitational fields. This was quite possibly the first time Einstein had written down this expression. The coefficients $G_{\mu\nu}$ of what we now know as the "metric tensor" are written with an uppercase G. Einstein changed within a few pages to a lowercase g, which remained his standard notation from then on,

$$ds^2 = \sum_{\alpha,\beta=1}^{4} g_{\alpha\beta}dx^\alpha dx^\beta . \quad (2)$$

For Einstein at that time, the big project was to find how this quantity $g_{\mu\nu}$, the metric tensor, is generated by sources (masses or fields). Eventually, this would lead to the new gravitational field equation, that is, Einstein's analogue of Newton's inverse square law of gravity. The lower half of the page is clearly making rudimentary efforts in that direction. There Einstein chooses a "Spezialfall" – a special case – in which the coefficients of the metric tensor revert to the values of special relativity, excepting $G_{44} = -c^2$. The coefficients $G_{\mu\nu}$ enable us to compute the spatio-temporal interval ds^2 between events separated by infinitesimal coordinate differences dx^μ. If these coefficients assign spatio-temporal intervals that do not conform to a flat geometry then we have captured the full range of gravitational effects in the manner of Einstein's general theory.

Before we go any further in explaining the content of the notebook, let us recall the principal steps taken by Einstein in his path towards a new theory of gravitation. Then we will be able to establish the connection between those steps and the notebook's subject matter. As pointed out above, in 1907 when Einstein was still at the patent office in Bern, he had already discovered a practical way to deal with gravity and with accelerated observers. He realised then that the effects of acceleration were indistinguishable from the effects of gravity. Somehow, Einstein succeeded in unifying all kinds of motion. Uniform motion is indistinguishable from rest and acceleration is no different from being at rest in a gravitational field, at least locally. The crucial elements for that purpose were the Principle of Equivalence and the Generalized Principle of Relativity. Einstein saw a Generalized Principle of Relativity as guaranteeing the satisfaction of the Equivalence Principle (EP). Indeed, according to the EP, an arbitrarily accelerated frame of reference in Minkowski space-time can precisely be considered as being physically equivalent to an inertial frame provided a gravitational field is introduced which accounts for the inertial effects in the accelerated frame. As early as 1907, he had come to consider two possible physical consequences of the EP: the bending of light in a gravitational field and the gravitational red-shift.

Resuming our description of the Zurich Notebook where we left it, Einstein then employs the equivalence principle to interpret the matrix elements of eq. (2), $g_{\mu\nu}$, that had arisen with the introduction of arbitrary coordinates. In the special case of the EP, the transformation from (1) to (2) is from an inertial coordinate system to a uniformly accelerated coordinate system but where c is now a function of the spatial coordinates (x', y', z'). That is, (1) is now

$$ds^2 = -c^2(x', y', z')dt'^2 + dx'^2 + dy'^2 + dz'^2\,. \qquad (3)$$

According to the EP, the presence of a gravitational field is the only difference between the space-time (3) and that of special relativity (1). Consequently, based on the EP, Einstein was led to interpret the line element (3) as representing a static gravitational field with the coordinate dependent $c(x', y', z')$ of (3) representing the gravitational potential and the $g_{\mu\nu}$ of (2) representing a more general gravitational field.

The differential equation for the velocity of light c written on the bottom of page 39L may be recognised as the left side of a nonlinear generalization of the classical scalar Poisson equation for the field c representing the potential of the static gravitational field

$$c\Delta c - \frac{1}{2}(grad\ c)^2 = \kappa c^2 \sigma\,, \qquad (4)$$

which Einstein had published in March 1912 (Einstein 1912b). On the right side of equation (4), κ is a constant and σ denotes the field generating mass and (non-gravitational) energy density.

To set the differential equation (4) into a form allowing its interpretation as one particular component of a 10-component tensorial field equation for the metric tensor, which is a second order symmetric tensor, Einstein performs the transformation $c^2/2 = \gamma$. The transformed equation would thus represent a special limiting case, for static fields, of the general tensorial field equations. For Renn and Sauer (1999), this is a good example of the central question concerning all calculations of the Zurich Notebook coping with the problem of gravitation: *What is the appropriate differential expression $\Gamma_{\mu\nu}$ which is formed from the metric tensor and its first and second derivatives and which enter a field equation of the form*

$$\Gamma_{\mu\nu} = \kappa T_{\mu\nu}\,, \qquad (5)$$

with the stress-energy-tensor $T_{\mu\nu}$ of *matter (and energy) as the source term on the right hand side?*

However, at this point it is very obvious that Einstein has not yet used any of the techniques of the Ricci and Levi-Civita absolute differential calculus, now called "tensor calculus". Instead, he has used older methods due to Beltrami to see what invariant quantities can be formed from a scalar φ. He starts with simple questions. As a first attempt, he looks at the coordinate divergence of the metric tensor and asks "Ist dies invariant?" - "Is this invariant?". As the calculation that follows immediately shows in the notebook, it is not. How these quantities transform under a change of coordinates is clearly a major focus of his analysis. The analysis continues for three more pages and then we find one of the most fascinating pages.

Here Einstein finds several of the key notions of his new theory in a familiar ground: classical physics. Einstein rederives a standard result: if a mass is free to move inertially, except that it is constrained to move within a curved surface, what is the curve traced by the mass on the surface? It proves to be a geodesic of the surface, a curve of shortest distance. The result is very close to the central idea of Einstein's general theory of relativity, the final theory he obtains in 1915. The following table shows how close it comes.

The surface is defined by a scalar field f in space. Constant values of f, such as $f = 0$, pick out the surface. The equation of motion of the mass moving in the surface is just that its acceleration vector $(d^2x/dt^2, d^2y/dt^2, d^2z/dt^2)$ is proportional to the reaction force from the surface, which is proportional to the gradient of f $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ and orthogonal to the surface. That is,

$$m\frac{d^2x}{dt^2} \;=\; \frac{\lambda}{m}\frac{\partial f}{\partial x} = \lambda' \frac{\partial f}{\partial x},\ f = 0,$$

and so on ...

Table 1. Classical physics versus general relativity

Classical physics	General relativity
A mass moves freely in space, except that it is constrained to a 2-dimensional surface in 3-dimensional space.	A mass moves freely in space-time. That is, it is in free fall, so that gravity acts on it through the curvature of space-time.
Its spatial trajectory is a geodesic of the 2-dimensional surface. That is, it traces a curve of shortest length in the surface.	Its space-time trajectory is a geodesic of the space-time. That is, it traces a curve of extremal space-time interval in space-time.

Immediately below is a straightforward derivation where Einstein uses the variational principle. If a point mass in the surface obeys these equations of motion then the spatial length of the path traced on the surface $\int ds$ is extremal, in that it satisfies the condition $\delta \int ds = 0$. These computations proceed until we reach a page on which calculations from each side of the notebook meet. Flipping the notebook over and starting from the other side one can see a series of pages of computation in the statistical-thermal physics of heat radiation. After nine pages like this, Einstein starts a new heading "Gravitation" and we are finally deep into a considerable discussion of general relativity.

On this page Einstein sets up the equations for conservation of energy and momentum for continuous matter in general relativity. He starts with the equation of motion for a point mass – the geodesic equation – but now written in the form of an Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial H}{\partial \dot{x}} - \frac{\partial H}{\partial x} = 0, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{\partial \Phi}{\partial x}.$$

He then applies this to a cloud of non-interacting dust particles in free fall to arrive at what we now recognise as the condition of the vanishing of the covariant divergence of the symmetric stress-energy tensor $T_{\mu\nu}$,

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g_{\mu\alpha}T^{\mu\nu}\right) - \frac{1}{2}\sqrt{-g}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}T^{\mu\nu} = 0, \qquad (6)$$

which in modern notation is given by $T^{\nu}_{\alpha;\nu} = 0$. Einstein arrived at eq. (6) by the Autumn of 1912, when he was looking for a generalization of the special relativistic formulation of the conservation of energy and momentum as well as of the Newtonian law of motion for continuous matter in a gravitational field. Renn and Sauer (1999) call this requirement the "Conservation Principle", one of four heuristic requirements that Einstein took along to check each of the candidate field equations. The other three requirements were the equivalence principle, the generalized principle of relativity and the requirement of correspondence. According to this last requirement, the new theory should describe, under certain limiting conditions, the gravitational effects familiar from Newtonian physics. In other words, Einstein expected that the unknown gravitational field equation for the metric tensor would reduce to the Poisson equation for the scalar gravitational potential of classical theory. This explains equation (4) and all his attempts to find the appropriate left side for equation (5). Then, under some limiting conditions, the equation of motion of his new theory would yield Newton's second law. Therefore, Einstein assumed that the Newtonian limit of his theory could be obtained from the full equation with the limiting condition of weak static fields leading to a linearised field equation for the metric tensor, via a metric of the form (3). However, just looking at the notebook, there is good evidence that Einstein's knowledge of tensor calculus is still limited. He does not know or is not sure that the operator acting on $T_{\mu\nu}$ in this equation is a generally covariant operator. To check the operator, he replaces $T_{\mu\nu}$ by the tensor $g_{\mu\nu}$ and sees whether the result is zero or a four vector, as it should be if the operator is generally covariant. It proves to be zero and Einstein is pleased.

After this, the pages continue with increasingly elaborate attempts to form invariant quantities from the metric tensor, most probably with the intention of finding a generally covariant set of gravitational field equations. But still, at this point, the techniques of Ricci and Levi-Civita for producing invariant quantities from the derivatives of the metric tensor are absent. Most significantly, there is no sign of the fourth rank Riemann curvature tensor from which we now know the gravitational field equations are readily constructed.

3 The genesis of general relativity: a drama in three acts

Following John Stachel (2007), we may describe the genesis of general relativity as a drama in three acts:

(i) First act (1907). Einstein adopts the Equivalence Principle (EP), i.e. all bodies fall with the same acceleration in a gravitational field, as a criterion for building the theory; then he concludes that any scalar generalization of Newton's theory would not be adequate since it violates the Equivalence Principle. That is, Einstein adopts the EP as a chief criterion for the construction of his new theory of gravity.

(ii) Second act (1912). Einstein assumes the need for a curved space-time, through the metric tensor:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta. \qquad (7)$$

He regarded a non-flat generalization of the four-dimensional metric tensor of Minkowski space-time as the appropriate representation of the gravitational field.

(iii) Third act (1915). Einstein achieves the formulation of the covariant field equations. By 1913, he had convinced himself that generally covariant field equations are physically inadmissible since they cannot determine the metric field uniquely; he had been fighting with this for a couple of years. Eventually, in November 1915 he returns to general covariance and arrives at the final field equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G T_{\alpha\beta}. \qquad (8)$$

He finds that the spherically symmetric solution to these equations accounts for the anomaly in the perihelion of Mercury.

So far we have considered in some detail the first two acts. Let us take a close look at the third one: the problem of covariance of the gravitational field equations. In the 1915 paper (Einstein 1915c), as well as in the other papers of that period – including a paper from March 1918 (Einstein 1918), where Einstein returned to the question of the fundamental principles of general relativity – the equivalence principle is still understood as being included in the generalized principle of relativity, which Einstein believed is satisfied because of the general covariance of the new field equations. Going back to 1912 and to the Zurich Notebook we see, after the pages where Einstein is searching for invariants, that he finally finds the first reference to the Riemann tensor. Einstein writes at the head of the page the formula for the Riemann tensor, using the old "four-index symbol" notation (ik, lm) with the following entry: "Grossmann tensor fourth rank". This clearly suggests that Grossmann passed on the formula to Einstein, proving the often-told story that Einstein only learned of the methods of Ricci and Levi-Civita through his school friend, the mathematician Marcel Grossmann. When he was searching for methods that could accommodate arbitrary coordinate systems, Grossmann found Ricci and Levi-Civita's article of 1901 with the Riemann tensor and reported it to Einstein. In October 1912, Einstein wrote to Arnold Sommerfeld: "I am now working exclusively on the gravitation problem and I believe that I can overcome all difficulties with the help of a mathematician friend of mine here."[10]

In his attempt, with Marcel Grossmann, to find a gravitational field equation for the metric tensor, at the end of 1912 or at the beginning of 1913 Einstein came close to his final field equations when he considered the Ricci tensor, defined by the formula

$$R_{\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\mu}_{\alpha\mu}}{\partial x^{\beta}} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\lambda}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\lambda}, \tag{9}$$

where the objects

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)$$

represent the Christoffel symbols of second kind (these would be generalized, within a few years, to the concept of affine connection, by Levi-Civitá, Weyl and Cartan).

In the notebook, Einstein proceeds in essentially the modern way. He contracts the fourth rank Riemann tensor to produce the symmetrical, second rank Ricci tensor. Before going any further, Einstein inquires whether this new tensor would describe, under certain limiting conditions, the gravitational effects familiar from Newtonian physics. In other words, if it is to serve as a gravitation tensor it must reduce in the weak, static field to Newtonian form. This requires three of its four second-order derivative terms to vanish:

$$\sum_{k}\left(\frac{\partial^2 g_{kk}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{ik}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{mk}}{\partial x^k \partial x^i}\right) = 0, \tag{10}$$

as Einstein comments in the notebook. Discovering how to eliminate these three terms in the weak field limit becomes a major focus and major obstacle for Einstein in the pages to follow. Apparently this difficulty led Einstein to conclude that the Ricci tensor violated his requirement of correspondence mentioned above. The usual interpretation is that Grossmann failed to see how to extract a d'Alembertian operator from the linearised approximation to the Ricci tensor because of his inability to choose a suitable coordinate condition. Indeed, in the second "mathematical" part of the Einstein-Grossmann paper which was written by Grossmann, after defining the Riemann tensor Grossmann explains how one can give a covariant differential tensor of second rank and second order, the Ricci tensor, and states: "But it turns out that this tensor does *not* reduce, in the special case of an infinitely weak static gravitational field, to the expression $\Delta\varphi$."[11]

But, according to Stachel (1980), there is a better interpretation that depends on the existence of a physical misconception on the part of Einstein. Indeed, this misconception was responsible for his delay in finding a theory of gravitation capable of explaining the anomalous advance of Mercury, as we will see later. When he was working on the Einstein-Grossmann paper, Einstein took for granted, even before having the correct field equations, that he already knew the correct form of the metric tensor for a static gravitational field, based on his earlier work on the subject (Einstein 1912a, 1912b). On page 229 of Part I, he wrote the solution given by equation (3). It is easily shown that substituting this solution for the metric tensor into the formula for the Ricci tensor, neglecting all but the linear terms because we are dealing with an "infinitely weak" field, it follows from $R_{\mu\nu} = 0$ that g_{44} can depend at most linearly on the coordinates. Thus, it cannot possibly represent the gravitational potential of any (finite) distribution of matter, static or otherwise. No wonder Einstein gave up on the Ricci tensor at this point.

In the meantime, Grossmann had shown him how to derive a new candidate for the left side of the gravitational field equation

$$N_{\alpha\beta} = \frac{\partial \Gamma^{\mu}_{\alpha\beta}}{\partial x^{\mu}} - \Gamma^{\sigma}_{\alpha\mu}\Gamma^{\mu}_{\beta\sigma}. \tag{11}$$

This object is easily obtained from the Ricci tensor. One may subtract from the Ricci tensor a part that transforms tensorially under the restricted group of unimodular coordinate transformations which leave $g = det(g_{\mu\nu})$ invariant, that is,

$$\Gamma^{\mu}_{\alpha\mu\,;\beta} = \frac{\partial \Gamma^{\mu}_{\alpha\mu}}{\partial x^{\beta}} - \Gamma^{\sigma}_{\alpha\beta}\Gamma^{\mu}_{\sigma\mu}, \tag{12}$$

and take the remaining part. It is quite clear that under this restricted group

$$\Gamma^{\rho}_{\alpha\rho} = \frac{\partial(\ln g)}{\partial x^{\alpha}}$$

transforms as a vector and its covariant derivative transforms as a tensor. The "tensor" (11) was also reconsidered by Einstein in November 1915 (Einstein 1915a) and for this reason was called the "November tensor" by Renn and Sauer (1999). The analysis of the notebook has revealed that at the end of 1912 or the beginning of 1913 Einstein even happened to consider, in linearised form, the final equation of general relativity, equation (8) above, with on its left side what is now called the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{13}$$

and abandoned it as well because they could not find the appropriate static Newtonian limit[12] and therefore moved on to other candidate field equations.

Instead of pursuing this covariant approach, the notebook ends with a derivation of the left side of the so-called "Entwurf" field equations,

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right) - g^{\alpha\beta}g^{\lambda\rho}\frac{\partial g_{\lambda\mu}}{\partial x^{\alpha}}\frac{\partial g_{\rho\nu}}{\partial x^{\beta}} \\ - \frac{1}{2}\frac{\partial g_{\lambda\rho}}{\partial x^{\mu}}\frac{\partial g^{\lambda\rho}}{\partial x^{\nu}} \\ + \frac{1}{4} g_{\mu\nu} g^{\alpha\beta}\frac{g_{\lambda\rho}}{\partial\alpha}\frac{g^{\lambda\rho}}{\partial\beta}\,.\end{aligned} \tag{14}$$

This differential operator is covariant only under some restricted class of coordinate transformations, at least under linear transformations, but the precise transformational properties were unknown to Einstein and Grossmann when they published their paper (Einstein and Grossman 1913). Giving up general covariance could not have been an easy decision, mainly because Einstein was looking for generally covariant field equations from the start, and for him general covariance corresponded to the acceptability of arbitrary reference frames. In other words, this would constitute for Einstein an extension of the Principle of Relativity. On the other hand, Einstein saw the Principle of Relativity as guaranteeing the satisfaction of the Equivalence Principle as well. Taking all this into consideration, one should ask how Einstein found non-generally covariant gravitational field equations after all!

J. Stachel (1980) explains that Einstein based his search on the criteria that the field equations should: (i) generalize Poisson's equation for the Newtonian gravitational potential; (ii) be invariant at least under linear transformations (as he put it – manifest no *less* relativity than in the special relativity); (iii) include a gravitational stress-energy complex, built from the metric tensor and its first derivatives (by analogy with Maxwell's electromagnetic theory), that is, a tensor under linear transformations which enters the field equations as a source term in the same way as does the stress-energy tensor of ordinary (non-gravitational) matter or fields.

Based on these three requirements, Einstein was able to derive the set of gravitational field equations in Part I of the Einstein-Grossmann paper. He also gave a sketchy proof of how Newton's law of gravitation follows from the linear approximation of these equations for the static case. In particular, he stressed that the spatial metric remains flat in this approximation, as he was expecting. It seems that Einstein considered that the lack of general covariance could only be temporary. For example, in a letter to Paul Ehrenfest on 28 May 1913 he wrote:

> I am now inwardly convinced that I have found that which is correct, and, at the same time, that a murmur of disappointment will, of course, go through the rank of our colleagues when the work appears, which will be in a few weeks … The conviction to which I have slowly struggled through is that there are no preferred coordinate systems of any kind. However, I have only partially succeeded, even formally, in reaching this standpoint.

Conversely, by November 1913, Einstein had developed the "hole" argument against general covariance. He wrote to Ludwig Hopf on 2 November:

> I am now very content with the gravitation theory. The fact that the gravitational equations are not generally covariant, which a short time ago still disturbed me so much, has proved to be unavoidable; it is easily proved that a theory with generally covariant equations cannot exist if one demands that the field be mathematically completely determined by matter.[13]

The proof in question, alluded to in the letter, is the infamous 'hole' argument first published in the addendum to the "Entwurf" paper (Einstein and Grossmann 1913), signed by Einstein alone and not published in the original printing of the paper. Based on the letter to Hopf, quoted above, we can date the origin of the argument fairly closely. In his Vienna lecture of 23 September 1913 to the meeting of the Geselschaft Deutscher Naturforscher and Ärzte he stated:

> In the last few days I have found the proof that such a generally covariant solution cannot exist at all. (Einstein 1913, p. 1257, footnote 2).

Einstein repeats just about the same argument in two subsequent papers in 1914 and in several letters to friends and colleagues, and the core of his reasoning was complete by November 1913. In his clear formulation (Einstein 1914b), the hole argument proceeds as follows. Let there be a region of space-time H (the "hole"), an open subspace of a manifold M devoid of matter and energy, and a set of generally covariant field equations valid for the entire space-time manifold M, both inside and outside H. Given a coordinate system of the manifold, K, what happens physically in H is then completely determined by the solutions of the field equations, $g_{\mu\nu}$, the components of the metric tensor as functions of the coordinates x_ν. The totality of these functions will be represented by $G(x)$. Consider now a second coordinate system K' that coincides with K everywhere outside and on the boundary of H, and diverges from K within H but in such a way that the metric components $g'_{\mu\nu}$ referred to K', like $g_{\mu\nu}$ and their derivatives, are everywhere continuous. The totality of $g'_{\mu\nu}$ expressed in terms of the new coordinates x'_ν will also be represented by $G'(x')$. It is important to note, as Einstein did, that $G'(x')$ and $G(x)$ describe the same gravitational field. That is, they are two different mathematical representations of the same physical field. However, if we replace the coordinates x'_ν by the coordinates x_ν in the functions $g'_{\mu\nu}$ and represent them by $G'(x)$ then $G'(x)$ also describes a gravitational field with respect to K, which is different from the original gravitational field within the "hole" H. However, the two different solutions $G'(x)$ and $G(x)$, which are written in the same coordinate system, correspond to the same "reality" (the same sources and same boundary conditions). In summary, because generally covariant field equations admit non-equivalent solutions for events within H, such equations are not acceptable as an appropriate physical theory of gravitation. This is the "hole" argument against general covariance of the field equations. So, if we require that the course of events in the gravitational field be determined by the laws to be set up, we must therefore adopt a theory with restricted covariance properties. One could think that Einstein's argument was a sort of excuse to accommodate his "Entwurf" theory with limited covariant properties. But, indeed, at the end of the day, his argument was much deeper than that.

In trying to explain the line of reasoning behind Einstein's arguments, we kept as close as possible to the mathematical language and methods of his time. However, we must bear in mind that the modern terminology of differential geometry, which expresses geometrical concepts in coordinate-free language and distinguishes between coordinate transformations and (active) diffeomorphisms, was not available to Einstein. With modern terminology and methods one may more easily clarify these arguments.

Let's reconsider the "hole" argument from a more modern perspective (Roveli 2008). Assume the gravitational field equations are generally covariant. Consider a solution of these equations in which the gravitational field is g and there is a region H of the universe without matter: the "hole". Assume that inside H there is a point A where g is flat and a point B where g is not flat. Consider a smooth map $\phi : M \to M$ which reduces to the identity outside H, and such that $\phi(A) = B$, and let $\tilde{g} = \phi^* g$ be the pull-back of g under ϕ. The two fields g and $\tilde{g}$ have the same past and are both solutions of the field equations but have different properties at the point A. Therefore, the field equations do not determine the physics at the space-time point A. That is, they are not deterministic. However, we know that (classical) gravitational physics is deterministic. So, one must pick one of the following: (i) the field equations must not be generally covariant; or (ii) there is no meaning in talking about the physical space-time point A. The correct physical conclusion is the second one, that there is no meaning in referring to "the event A" without further specification.

By late 1915, after having returned to generally covariant field equations, Einstein introduces the point-coincidence argument, which maintains that a coordinatization of the manifold is itself not sufficient to determine an individuation of the points (events) of the manifold. Einstein then argues that the events of the space-time are implicitly defined and thus individuated only as points of intersection or coincidence of world lines. Then, as the coincidences are themselves determined by the metric, it is impossible to have two different sets of values of the functions $g_{\mu\nu}(x_\nu)$ assigned to *one and the same event* of the space-time manifold. Therefore, in regions where no matter is present, the points of a manifold are physically differentiated only by the properties that they inherit from the metric field. In general, a space-time manifold with metric field corresponds to a gravitational field; but a gravitational field corresponds to an equivalence class of manifolds with metric fields. In particular, any set of generally covariant field equations that has $G(x)$ as a solution in some empty region of space-time will also have $G'(x)$ as a solution in that region. $G(x)$ and $G'(x)$, together with all other mathematically distinct metric tensor fields that can be transformed into each other by being dragged along with an (active) diffeomorphism, form a equivalence class of solutions. But this equivalence class of mathematical distinct metric tensor fields corresponds to one physical solution to the field equations, that is, to one gravitational field.[14]

To clarify and give further support to the point-coincidence argument, Einstein repeatedly says: (i) only intersections of world lines are invariant under arbitrary, continuous coordinate transformations, the group of transformations under which the field equations themselves are also to be invariant; (ii) the observations by means of which we test the predictions of any physical theory consist, in principle, of just such coincidences.

When Einstein came back to generally covariant field equations with the tensor equation (13), what is now called the Einstein tensor, he realised that the weak field equation resulting from this tensor involves a metric with more than one variable component. Therefore, such a weak field equation cannot simply be reduced to the classical Poisson equation for one scalar potential, in contradiction to his first expectation. Nevertheless, the equation of motion for a test particle, the geodesic equation, reduces, in fact, under the mathematical conditions that correspond to Newtonian physics (weak static fields and low velocities), to the Newtonian equation of motion. Under these conditions only one component of the metric tensor, the component g_{44}, enters the equations of motion in first approximation. In this way Einstein was able to overcome the last stumbling block in the fulfilment of his program. On top of that, the fact that there exist other variable components of the metric (although they do not affect the equation of motion in the Newtonian limit) is quite significant since it is their existence that explains the perihelion shift of Mercury. After all, the Einstein tensor was compatible with his Correspondence Principle and the metric associated with the Newtonian limit also explained the perihelion advance of Mercury. What else could Einstein wish for?

> On the basis of the general theory of relativity ... space as opposed to 'what fills space' ... has no separate existence ... There is no such thing as an empty space, i.e., a space without [a gravitational] field ... Space-time does not claim existence on its own, but only as a structural quality of the field. (Einstein 1952)

These words were written late in Einstein's life and they synthesize his answer to a question that has its origin in 1913, when Einstein was searching for a field equation for gravity.

4 Conclusions

Summing up, in a schematic mode, all we have said before, one may say that Einstein was aware of the possibility of generally covariant field equations but he believed - wrongly, it turned out - that they could not possess the correct Newtonian limit. He then proposed a field equation covariant under linear coordinate transformations. To sustain his case against generally covariant field equations, Einstein conceived his "hole argument", which alleged to show that a satisfactory, generally covariant field equation couldn't exist. Eventually, Einstein came back to generally covariant equations for the gravitational field and found a way around his hole argument, late in 1915, through the point-coincidence argument, which led him to prove that (classical) gravitational physics is deterministic, although the same physical world can be described by different solutions of the equation of motion. From Einstein's discussions and arguments, one may conclude that in general relativity, general covariance is compatible with determinism only assuming that individual space-time events have no physical meaning by themselves; even the localization on the space-time manifold has no physical meaning, since the points of a manifold are differentiated only by the properties that they inherit from the metric field, i.e. a solution of the generally covariant field equations. Einstein's step toward a profoundly novel understanding of nature was accomplished through his arguments, which can be translated in a very short sentence: no metric, no space-time. Background space-time was eradicated from this new understanding of the world.

Notes

1. This is a new version of an article published in the Boletim da SPM, Número especial Mira Fernandes (2010), pp. 223–245. Reprinted with permission.
2. Einstein (1911).
3. The lectures of Professor Carl Friederich Geiser, which Einstein heard as a student at the ETH, had familiarised him with Gaussian theory of two-dimensional surfaces.
4. See Einstein to Conrad Habicht, 24 December 1907 (Klein et al. 1993, p.82): "At the moment I am working on the relativistic analysis of the law of gravitation by means of which I hope to explain the still unexplained secular changes in the perihelion of Mercury."
5. A very incomplete list of the older secondary literature certainly involves (Lanczos 1972), (Mehra 1974), (Earman an Glymour 1978) and (Pais 1982).
6. See (Stachel 1980), (Stachel 1982), (Stachel 1989) and (Norton 1984).
7. Einstein Archives Call Nr. 3-006.

8. The definitive work on this subject is the four volume series edited by Jürgen Renn: *The Genesis of General Relativity*, 2007 Springer.
9. Renn and Sauer 1999. See also Janssen et al. (2007).
10. Klein et al. 1993, p. 505.
11. Einstein and Grossmann 1913, p. 257.
12. Carried out by Renn et al. at the Max Planck Institute for the History of Science.
13. Einstein Archives Call Nr. 13-290.
14. See Stachel (1993).

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Interview with Abel Laureate John Milnor

Martin Raussen (Aalborg, Denmark) and Christian Skau (Trondheim, Norway)

The Abel committee's citation

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2011 to John W. Milnor (Institute for Mathematical Sciences, Stony Brook University, New York) "for pioneering discoveries in topology, geometry and algebra".

All of Milnor's works display marks of great research: profound insights, vivid imagination, elements of surprise, and supreme beauty.

Milnor's discovery of exotic smooth spheres in seven dimensions was completely unexpected. It signaled the arrival of *differential topology* and an explosion of work by a generation of brilliant mathematicians; this explosion has lasted for decades and changed the landscape of mathematics. With Michel Kervaire, Milnor went on to give a complete inventory of all the distinct differentiable structures on spheres of all dimensions; in particular they showed that the 7-dimensional sphere carries exactly 28 distinct differentiable structures. They were among the first to identify the special nature of four-dimensional manifolds, foreshadowing fundamental developments in topology.

Milnor's disproof of the long-standing *Hauptvermutung* overturned expectations about combinatorial topology dating back to Poincaré. Milnor also discovered homeomorphic smooth manifolds with nonisomorphic tangent bundles, for which he developed the theory of microbundles. In three-manifold theory, he proved an elegant unique factorization theorem.

Outside topology, Milnor made significant contributions to differential geometry, algebra, and dynamical systems. In each area Milnor touched upon, his insights and approaches have had a profound impact on subsequent developments. His monograph on isolated hypersurface singularities is considered the single most influential work in singularity theory; it gave us the Milnor number and the Milnor fibration.

Topologists started to actively use Hopf algebras and coalgebras after the definitive work by Milnor and J. C. Moore. Milnor himself came up with new insights into the structure of the Steenrod algebra (of cohomology operations) using the theory of Hopf algebras. In algebraic K-theory, Milnor introduced the degree two functor; his celebrated conjecture about the functor — eventually proved by Voevodsky — spurred new directions in the study of motives in algebraic geometry. Milnor's introduction of the growth invariant of a group linked combinatorial group theory to geometry, prefiguring Gromov's theory of hyperbolic groups.

More recently, John Milnor turned his attention to dynamical systems in low dimensions. With Thurston, he pioneered "kneading theory" for interval maps, laying down the combinatorial foundations of interval dynamics, creating a focus of intense research for three decades. The Milnor–Thurston conjecture on entropy monotonicity prompted efforts to fully understand dynamics in the real quadratic family, bridging real and complex dynamics in a deep way and triggering exciting advances.

Milnor is a wonderfully gifted expositor of sophisticated mathematics. He has often tackled difficult, cutting-edge subjects, where no account in book form existed. Adding novel insights, he produced a stream of timely yet lasting works of masterly lucidity. Like an inspired musical composer who is also a charismatic performer, John Milnor is both a discoverer and an expositor.

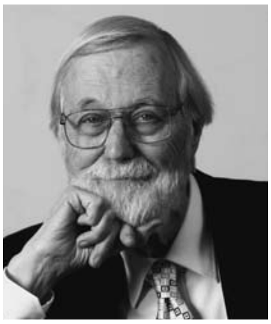
Professor John Milnor – on behalf of the Norwegian and Danish Mathematical Societies, we would like to congratulate you for being selected as the Abel Prize Laureate in 2011.

Thank you very much!

Student at Princeton University

What kindled your interest in mathematics and when did you discover that you had an extraordinary aptitude for mathematics?

I can place that quite clearly. The first time that I developed a particular interest in mathematics was as a freshman at Princeton University. I had been rather socially maladjusted and did not have too many friends but when I came to Princeton, I found myself very much at home in the atmosphere of the mathematics common room. People were chatting about mathematics, playing games and one could come by at any time and just relax. I found the lectures very interesting. I felt more at home there than I ever had before and I have stayed with mathematics ever since.



Abel Laureate John Milnor. Photo: Knut Falch.

You were named a Putnam Fellow as one of the top scorers of the Putnam competition in mathematics in 1949 and 1950. Did you like solving mathematics problems and puzzles?

I think I always approached mathematics as interesting problems to be solved so I certainly found that congenial.

John Milnor was interviewed by Martin Raussen and Christian Skau. Photo: Eirik Furu Baardsen.

Your first important paper was accepted already in 1949 and published in 1950 in the prestigious journal* Annals of Mathematics. *You were only 18 years of age at the time and this is rather exceptional. The title of the paper was "On the Total Curvature of Knots". Could you tell us how you got the idea for that paper?
I was taking a course in differential geometry under Albert Tucker. We learned that Werner Fenchel, and later Karol Borsuk, had proved the following statement: the total curvature of a closed curve in space is always at least 2π with equality only if the curve bounds a convex subset of some plane. Borsuk, a famous Polish topologist, had asked what one could say about total curvature if the curve was knotted? I thought about this for a few days and came up with a proof that the total curvature is always greater than 4π. (I think I did a poor job explaining the proof in the published paper but one has to learn how to explain mathematics.) The Hungarian mathematician István Fáry had produced a similar proof at more or less the same time; but this was still a wonderful introduction to mathematics.

That was quite an achievement! When you started your studies at Princeton in 1948 you met John Nash, three years your senior, who was a PhD student. John Nash is well-known through the book and movie 'A Beautiful Mind'. Did you have any interaction with him? And how was it to be a Princeton student?
As I said, I spent a great deal of time in the common room, and so did Nash. He was a very interesting character and full of ideas. He also used to wander in the corridors whistling things like Bach which I had never really heard before – a strange way to be introduced to classical music!

I saw quite a bit of him over those years and I also became interested in game theory in which he was an important contributor. He was a very interesting person.

You played Kriegspiel, Go and a game called Nash at Princeton?
That is true. Kriegspiel is a game of chess in which the two players are back to back and do not see each other's boards. There is a referee who tells whether the moves are legal or not. It is very easy for the referee to make a mistake and it often happened that we could not finish because he got confused. In that case we said that the referee won the game! It was a marvellous game.

The game of Go was also very popular there. My first professor Ralph Fox was an expert in Go. So I learned something of it from him and also from many other people who played. The game that we called Nash had actually been developed earlier in Denmark by Piet Hein but Nash invented it independently. This game, also called Hex, is based on topology. It is very interesting from a mathematical point of view. It is not hard to prove that the first player will always win if he plays correctly but there is no constructive proof. In fact, when you play, it often happens that the first player does not win.

You even published some papers on game theory with John Nash?
We often talked about game theory but there was only one joint paper. Together with C. Kalish and E. D. Nering, we carried out an experiment with a group of people playing a many-person game. This experiment convinced me that many-person game theory is not just a subject of mathematics. It is also about social interactions and things far beyond mathematics so I lost my enthusiasm for studying it mathematically.

One paper written on my own described a theoretical model for the game of Go. This was further developed by Olof Hanner, and much later by Berlekamp and Wolfe. (John Conway's construction of "surreal numbers" is closely related.)

Knot theory

You wrote your PhD thesis under the supervision of Ralph Fox; the title of the thesis was "Isotopy of Links". Did you get the idea to work on this topic yourself? And what was the impact of this work?
Fox was an expert in knot theory so I learned a great deal about knots and links from him. There were many people in the department then that were active in this area, although there were also other people at the department that considered it a low-class subject and not very interesting. I think it's strange that, although it wasn't considered a very central subject then, it's today a subject which is very much alive and active.

As one example, I often saw a quiet, Greek gentleman Christos Papakyriakopoulus around the common room but I never got to know him very well. I had no idea he was doing important work but Fox had managed to find money to support him for many years, while he did research more or less by himself. He finally succeeded in solving a very important problem in knot theory which, perhaps, was the beginning of a rebirth of the study of three dimensional manifolds as a serious part of mathematics. A paper in 1910 by Max Dehn had claimed to prove a simple property about knots. Essentially it said that if the fundamental group of the complement of a knot is cyclic then the knot can be un-knotted. This proof by Max Dehn had been accepted for almost 20 years until Hellmuth Kneser in 1929 pointed out there was a big gap in the argument. This remained a famous unsolved problem until 1957, when Papakyriakopoulus developed completely new methods and managed to give a proof of "Dehn's Lemma" and related theorems.

That was a big step in mathematics and an example of a case in which someone working in isolation made tremendous progress. There are relatively few examples of that. Andrew Wiles' proof of Fermat's last theorem is also an example of someone who had been working by himself and surprised everyone when he came up with the proof. Another example is Grigori Perelman in Russia who was working very much by himself and produced a proof of the Poincaré hypothesis. These are isolated examples. Usually mathematicians work in a much more social context, communicating ideas to each other. In fact, ideas often travel from country to country very rapidly. We are very fortunate that mathematics is usually totally divorced from political situations. Even at the height of the Cold War, we received information from the Soviet Union and people in the Soviet Union were eagerly reading papers from outside. Mathematics was much more open than most scientific subjects.

As a footnote to what you said: Max Dehn was a student of David Hilbert and he solved Hilbert's 3rd problem about three-dimensional polyhedra of equal volume, showing that you cannot always split them up into congruent polyhedra. No wonder people trusted his proof because of his name.

It's a cautionary tale because we tend to believe in mathematics that when something is proved, it stays proved. Cases like Dehn's Lemma, where a false proof was accepted for many years, are very rare.

Manifolds

For several years after your PhD your research concentrated on the theory of manifolds. Could you explain what a manifold is and why manifolds are important?

In low dimensions manifolds are things that are easily visualized. A curve in space is an example of a one-dimensional manifold; the surface of a sphere or of a doughnut are examples of two-dimensional manifolds. But for mathematicians the dimensions one and two are just the beginning; things get more interesting in higher dimensions. Also, for physicists manifolds are very important and it is essential for them to look at higher dimensional examples.

For example, suppose you study the motion of an airplane. To describe just the position takes three coordinates but then you want to describe what direction it is going in, the angle of its wings and so on. It takes three coordinates to describe the point in space where the plane is centred and three more coordinates to describe its orientation, so already you are in a six-dimensional space. As the plane is moving, you have a path in six-dimensional space and this is only the beginning of the theory. If you study the motion of the particles in a gas, there are enormously many particles bouncing around and each one has three coordinates describing its position and three coordinates describing its velocity, so a system of a thousand particles will have six thousand coordinates. Of course, much larger numbers occur; so mathematicians and physicists are used to working in large dimensional spaces.

The one result that made you immediately famous at age 25 was the discovery of different exotic structures on the seven-dimensional sphere. You exhibited smooth manifolds that are topologically equivalent to a seven-dimensional sphere but not smoothly equivalent, in a differentiable sense. Would you explain this result and also describe to us how you came up with the idea?

It was a complete accident, and certainly startled me. I had been working on a project of understanding different kinds of manifolds from a topological point of view. In particular, I was looking at some examples of seven-dimensional manifolds which were constructed by a simple and well understood construction. They were explicit smooth objects which I would have thought were well understood but looking at them from two different points of views, I seemed to find a complete contradiction. One argument showed that these manifolds were topological spheres and another very different argument showed that they couldn't be spheres.

Mathematicians get very unhappy when they have apparently good proofs of two contradictory statements. It's something that should never happen. The only way I could get out of this dilemma was by assuming there was an essential difference between the concept of a topological sphere (homeomorphic to the standard sphere) and the concept of a differentiable sphere (diffeomorphic to the standard sphere). This was something which hadn't been expected and I am not aware that anybody had explicitly asked the question; we just assumed the answer was obvious. For some purposes one assumed only the topology and for other purposes one assumed the differentiable structure; but no one had really considered the possibility that there was a real difference. This result awakened a great deal of interest and a need for further research to understand exactly what was going on.

You were certainly the driving force in this research area and you applied techniques both from differential geometry and topology, and also from algebraic topology, to shed new light on manifolds. It is probably fair to say that the work of European mathematicians, and especially French mathematicians like René Thom and Jean-Pierre Serre, who, by the way, received the first Abel Prize in 2003, made very fundamental contributions and made your approach possible. How did the collaboration over the Atlantic work at the time?

It was very easy to travel back and forth and I found French mathematicians very welcoming. I spent a great deal of time at the IHES near Paris. I hardly knew Serre (until much later) but I admired him tremendously, and still do. His work has had an enormous influence.

René Thom I got to know much better. He was really marvellous. He had an amazing ability to combine geometric arguments with hard algebraic topology to come up with very surprising conclusions. I was a great admirer of Thom and found he was also extremely friendly.

Building on the work of, among others, Frank Adams from Britain and Stephen Smale from the United States, you, together with the French mathematician Michel Kervaire, were able to complete, to a certain extent, the classification of exotic structures on spheres. There are still some open questions concerning the stable homotopy of spheres but at least up to those, we know what differentiable structures can be found on spheres.

That's true, except for very major difficulties in dimension four, and a few problems in high dimensions (notably, the still unsolved "Kervaire Problem" in dimension 126). There are very classical arguments that work in dimensions one and two. Dimension three is already much more difficult but the work of Bill Thurston and Grisha Perelman has more or less solved that problem. It was a tremendous surprise when we found, in the 60s, that high dimensions were easier to work with than low dimensions. Once you get to a high enough dimension, you have enough room to move around so that arguments become much simpler. In many cases, one can make such arguments work even in dimension five but dimension four is something else again and very difficult: neither high dimensional methods nor low dimensional methods work.

One seems to need much more hard pure analysis to work in dimensions three and four.

Well, yes and no. Michael Freedman first proved the topological Poincaré hypothesis in dimension four and that was the very opposite of analysis. It was completely by methods of using very wild topological structures with no differentiability. But the real breakthrough in understanding differential 4-manifolds was completely based on methods from mathematical physics: methods of gauge theory, and later Seiberg–Witten theory. Although motivated by mathematical physics, these tools turned out to be enormously useful in pure mathematics.

Terminology in manifold theory is graphic and down to earth. Some techniques are known as 'plumbing'. Also 'surgery' has become a real industry in mathematics and you have written a paper on 'killing', but of course just homotopy groups. May we ask to what extent you are responsible for this terminology?

To tell the truth, I'm not sure. I probably introduced the term 'surgery', meaning cutting up manifolds and gluing them together in a different way (the term 'spherical modification' is sometimes used for the same thing). Much later, the idea of quasi-conformal surgery has played an important role in holomorphic dynamics.

Simple graphic terminology can be very useful but there are some words that get used so much that one loses track of what they mean (and they may also change their meaning over the years). Words like 'regular' or 'smooth' are very dangerous. There are very many important concepts in mathematics and it is important to have a terminology which makes it clear exactly what you are talking about. The use of proper names can be very useful because there are so many possible proper names. An appropriate proper name attached to a concept often pins it down more clearly than the use of everyday words. Terminology is very important; it can have a very good influence if it's successfully used and can be very confusing if badly used.

Another surprising result from your hand was a counterexample to the so-called Hauptvermutung, the "main conjecture" in combinatorial topology, dating back to Steinitz and Tietze in 1908. It is concerned with triangulated manifolds or, more generally, triangulated spaces. Could you explain what you proved at the time?

One of the important developments in topology in the early part of the 20th century was the concept of homology, and later cohomology. In some form, they were already introduced in the 19th century but there was a real problem making precise definitions. To make sense of them, people started by cutting a topological space up into linear pieces called simplexes. It was relatively easy to prove that homology was well defined on that level, and well behaved if you cut the simplexes into smaller ones, so the natural conjecture was that you really were doing topology when you defined things this way. If two simplicial complexes were homeomorphic to each other then you should be able to cut them up in pieces that corresponded to each other. This was the first attempt to prove that homology was topologically invariant; but nobody could quite make it work. Soon they developed better methods and got around the problem. But the old problem of the Hauptvermutung, showing that you could always find isomorphic subdivisions, remained open.

I ran into an example where you could prove that it could not work. This was a rather pathological example, not about manifolds; but about ten years later, counterexamples were found even for nicely triangulated manifolds. A number of people worked on this but the ones who finally built a really satisfactory theory were Rob Kirby and my student Larry Siebenmann.

Over a long period of years after your thesis work, you published a paper almost every year, sometimes even several papers, that are known as landmark papers. They determined the direction of topology for many years ahead. This includes, apart from the themes we have already talked about, topics in knot theory, three dimensional manifolds, singularities of complex hypersurfaces, Milnor fibrations, Milnor numbers, complex cobordism and so on. There are also papers of a more algebraic flavour. Are there any particular papers or particular results you are most fond or proud of?

It's very hard for me to answer; I tend to concentrate on one subject at a time so that it takes some effort to remember precisely what I have done earlier.

Geometry, topology and algebra

Mathematics is traditionally divided into algebra, analysis and geometry/topology. It is probably fair to

say that your most spectacular results belong to geometry and topology. Can you tell us about your working style and your intuition? Do you think geometrically, so to say? Is visualization important for you?

Very important! I definitely have a visual mind so it's very hard for me to carry on a mathematical conversation without seeing anything written down.

On the other hand, it seems to be a general feature, at least when you move into higher dimensional topology, that real understanding arises when you find a suitable algebraic framework which allows you to formulate what you are thinking about.

We often think by analogies. We have pictures in small dimensions and must try to decide how much of the picture remains accurate in higher dimensions and how much has to change. This visualization is very different from just manipulating a string of symbols.

Certainly, you have worked very hard on algebraic aspects of topology and also algebraic questions on their own. While you developed manifold theory, you wrote, at the same time, papers on Steenrod algebras, Hopf algebras and so on. It seems to us that you have an algebraic mind as well?

One thing leads to another. If the answer to a purely topological problem clearly requires algebra then you are forced to learn some algebra. An example: in the study of manifolds one of the essential invariants – perhaps first studied by Henry Whitehead – was the quadratic form of a four dimensional manifold, or more generally a 4k-dimensional manifold. Trying to understand this, I had to look up the research on quadratic forms. I found this very difficult until I found a beautiful exposition by Jean-Pierre Serre which provided exactly what was needed. I then discovered that the theory of quadratic forms is an exciting field on its own. So just by following my nose, doing what came next, I started studying properties of quadratic forms. In these years, topological K-theory was also developed, for example by Michael Atiyah, and was very exciting. There were beginnings of algebraic analogues. Grothendieck was one of the first. Hyman Bass developed a theory of algebraic K-theory and I pursued that a bit further and discovered that there were relations between the theory of quadratic forms and algebraic K-theory. John Tate was very useful at that point, helping me work out how these things corresponded.

John Tate was last year's Abel Prize winner, by the way.

I made a very lucky guess at that point, conjecturing a general relationship between algebraic K-theory, quadratic forms and Galois cohomology. I had very limited evidence for this but it turned out to be true and much later was proved by Vladimir Voevodsky. It's very easy to make guesses but it feels very good when they turn out to be correct.

That's only one of the quite famous Milnor conjectures.

Well, I also had conjectures that turned out to be false.

Algebraic K-theory is a topic you already mentioned and we guess your interest in that came through Whitehead groups and Whitehead torsion related to K_1.

That is certainly true.

It is quite obvious that this is instrumental in the theory of non-simply connected manifolds through the s-cobordism theorem. That must have aroused your interest in general algebraic K-theory where you invented what is called Milnor K-theory today. Dan Quillen then came up with a competing or different version with a topological underpinning…

Topological K-theory worked in all dimensions, using Bott periodicity properties, so it seemed there should be a corresponding algebraic theory. Hyman Bass had worked out a complete theory for K_0 and K_1 and I found an algebraic version of K_2. Quillen, who died recently after a long illness, provided a satisfactory theory of K_n for all values of n. Quillen's K_2 was naturally isomorphic to my K_2, although our motivations and expositions were different. I did construct a rather ad hoc definition for the higher K_n. This was in no sense a substitute for the Quillen K-theory. However, it did turn out to be very useful for certain problems so it has kept a separate identity.

Giving rise to motivic cohomology, right?

Yes, but only in the sense that Voevodsky developed motivic cohomology in the process of proving conjectures which I had posed.

You introduced the concept of the growth function for a finitely presented group in a paper from 1968. Then you proved that the fundamental group of a negatively curved Riemannian manifold has exponential growth. This paved the way for a spectacular development in modern geometric group theory and eventually led to Gromov's hyperbolic group theory. Gromov, by the way, received the Abel Prize two years ago. Could you tell us why you found this concept so important?

I have been very much interested in the relation between the topology and the geometry of a manifold. Some classical theorems were well-known. For example, Preismann had proved that if the curvature of a complete manifold is strictly negative then any Abelian subgroup of the fundamental group must be cyclic. The growth function seemed to be a simple property of groups which would reflect the geometry in the fundamental group. I wasn't the first to notice this. Albert Schwarz in Russia had done some similar work before me but I was perhaps better known and got much more publicity for the concept.

I can bring in another former Abel Prize winner Jacques Tits, who proved what is now called the "Tits alternative" for finitely generated subgroups of algebraic groups. He proved that either there was a free subgroup or the group was virtually solvable. All the finitely generated groups I was able to construct had this property: either they contained a noncyclic free subgroup or else they contained a solvable subgroup of finite index. Such groups always have either polynomial growth or exponential growth. The problem of groups of intermediate growth remained unsolved for many years until Grigorchuk in Russia found examples of groups that had less than exponential growth but more than polynomial growth. It is always nice to ask interesting questions and find that people have interesting answers.

Dynamics

We jump in time to the last thirty years in which you have worked extensively on real and complex dynamics. Roughly speaking, this is the study of iterates of a continuous or holomorphic function and the associated orbits and stability behaviour. We are very interested to hear why you got interested in this area of mathematics?

I first got interested under the influence of Bill Thurston, who himself got interested from the work of Robert May in mathematical ecology. Consider an isolated population of insects where the numbers may vary from year to year. If there get to be too many of these insects then they use up their resources and start to die off but if they are very few they will grow exponentially. So the curve which describes next year's population as a function of this year's will have positive slope if the population is small and negative slope if the population gets too big. This led to the study of dynamical properties of such "unimodal" functions. When you look at one year after another, you get a very chaotic looking set of population data. Bill Thurston had gotten very interested in this problem and explained some of his ideas to me. As frequently happened in my interactions with Bill, I first was very dubious and found it difficult to believe what he was telling me. He had a hard time convincing me but finally we wrote a paper together explaining it.

This was a seminal paper. The first version of this paper dates from around 1977. The manuscript circulated for many years before it was published in the Springer Lecture Notes in 1988. You introduced a new basic invariant that you called the 'kneading matrix' and the associated 'kneading determinant'. You proved a marvellous theorem connecting the kneading determinant with the zeta function associated to the map, which counts the periodic orbits. Browsing through the paper it seems to us that it must have been a delight to write it up. Your enthusiasm shines through!

You said that the zeta function describes periodic orbits, which is true but it omits a great deal of history. Zeta functions were first made famous by Riemann's zeta function (actually first studied by Euler). Zeta functions are important in number theory but then people studying dynamics found that the same mathematical formalism was very useful for counting periodic orbits. The catalyst was André Weil who studied an analogue of the Riemann zeta function for curves over a finite field, constructed by counting periodic orbits of the Frobenius involution.

So there is a continuous history here from pure number theory, starting with Euler and Riemann, and then André Weil, to problems in dynamics in which one studies iterated mappings and counts how many periodic orbits there are. This is typical of something that makes mathematicians very happy: techniques that are invented in one subject turn out to be useful in a completely different subject.

You must have been surprised that the study of a continuous map from an interval into itself would lead to such deep results?

Well, it was certainly a very enjoyable subject.

Your work with Bill Thurston has been compared to Poincaré's work on circle diffeomorphisms 100 years earlier which led to the qualitative theory of dynamical systems and had a tremendous impact on the subject.

Use of computers in mathematics

This leads to another question. There is a journal called Experimental Mathematics. The first volume appeared in 1992 and the first article was written by you. It dealt with iterates of a cubic polynomial. The article included quite a lot of computer graphics. You later published several papers in this journal. What is your view on computers in mathematics?

I was fascinated by computers from the very beginning. At first one had to work with horrible punch cards. It was a great pain; but it has gotten easier and easier. Actually, the biggest impact of computers in mathematics has been just to make it easier to prepare manuscripts. I always have had a habit of rewriting over and over, so in the early days I drove the poor secretaries crazy. I would hand in messy longhand manuscripts. They would present a beautiful typescript. I would cross out this, change that and so on. It was very hard on them.

It has been so much easier since one can edit manuscripts on the computer.

Of course, computers also make it much easier to carry out numerical experiments. Such experiments are nothing new; Gauss carried out many numerical experiments but it was very difficult at his time. Now it's so much easier. In particular, in studying a difficult dynamical system it can be very helpful to run the system (or perhaps a simplified model of it) on a computer. Hopefully this will yield an accurate result. But it is dangerous. It is very hard to be sure that round-off errors by the computer, or other computing errors, haven't produced a result which is not at all accurate. It becomes a kind of art to understand what the computer can do and what the limitations are but it is enormously helpful. You can get a fast idea of what you can expect from a dynamical system and then try to prove something about it using the computer result as an indication of what to expect. At least, that's in the best case. There's also the other case where all you can do is to obtain the computer results and hope that they are accurate.

In a sense, this mathematical discipline resembles what the physicists do when they plan their experiments, and

when they draw conclusions from the results of their experiments…

There is also the intermediate stage of a computer assisted proof where (at least if you believe there are no mistakes in the computer program or no faults in the hardware) you have a complete proof.

But the assumption that there are no mistakes is a very important one. Enrico Bombieri had an experience with this. He was using a fancy new high-speed computer to make experiments in number theory. He found that in some cases the result just seemed wrong. He traced it back, and traced it back, and finally found that there was a wiring mistake in the hardware!

Do you have examples from your own experience where all experiments you have performed indicate that a certain conjecture must be true but you don't have a way to prove it in the end?

In my experience, computer experiments seldom indicate that something is definitely true. They often show only that any possible exception is very hard to find. If you verify a number theoretical property for numbers less than 10^{10}, who knows what would happen for 10^{11}. In dynamics, there may be examples where the behaviour changes very much as we go to higher dimensions. There is a fundamental dogma in dynamics, saying that we are not interested in events which happen with probability zero. But perhaps something happens with probability 10^{-10}. In that case, you will never see it on a computer.

Textbooks and expository articles

You have written several textbooks which are legendary in the sense that they are lucid and lead the reader quickly to the point, seemingly in the shortest possible way. The topics of your books deal with differential topology, algebraic K-theory, characteristic classes, quadratic forms and holomorphic dynamics. Your books are certainly enjoyable reading. Do you have a particular philosophy when you write mathematical textbooks?

I think most textbooks I have written have arisen because I have tried to understand a subject. I mentioned before that I have a very visual memory and the only way I can be convinced that I understand something is to write it down clearly enough so that I can really understand it. I think the clarity of writing, to the extent it exists, is because I am a slow learner and have to write down many details to be sure that I'm right, and then keep revising until the argument is clear.

Apart from your textbooks and your research contributions, you have written many superb expository and survey articles which are a delight to read for every mathematician, expert or non-expert.

Two questions come to mind. Do you enjoy writing articles of an historical survey type? You certainly have a knack for it. Do you think it is important that articles and books on mathematics of a popular and general nature are written by prominent mathematicians like yourself?

The answer to your first question is certainly yes. Mathematics has a rich and interesting history. The answer to the second question is surely no. I don't care who writes an article or a book. The issue is: is it clearly written, correct and useful.

Are you interested in the history of mathematics also – following how ideas develop?

I certainly enjoy trying to track down just when and how the ideas that I work with originated. This is, of course, a very special kind of history, which may concentrate on obscure ideas which turned out to be important, while ignoring ideas which seemed much more important at the time. History to most scientists is the history of the ideas that worked. One tends to be rather bored by ideas that didn't work. A more complete history would describe how ideas develop and would be interested in the false leads also. In this sense, the history I would write is very biased, trying to find out where the important ideas we have today came from – who first discovered them. I find *that* an interesting subject. It can be very difficult to understand old papers because terminology changes. For example, if an article written 100 years ago describes a function as being 'regular', it is hard to find out precisely what this means. It is always important to have definitions which are clearly written down so that, even if the terminology does change, people can still understand what you were saying.

Is it also important to communicate that to a wider mathematics audience?

It is important to communicate what mathematics is and does to a wide audience. However, my own expositions have always been directed to readers who already have a strong interest in mathematics. In practice, I tend to write about what interests me, in the hope that others will also be interested.

Academic work places

You started your career at Princeton University and you were on the staff for many years. After some intermediate stages in Los Angeles and at MIT, you went back to Princeton but now to the Institute for Advanced Study. Can you compare the Institute and the University and the connections between them?

They are alike in some ways. They have close connections; people go back and forth all the time. The big difference is that at the university you have continual contact with students, both in teaching and with the graduate students, and there is a fair amount of continuity since the students stay around, at least for a few years. The institute is much more peaceful, with more opportunity for work and more idyllic circumstances, but there is a continually rotating population, so almost before you get to know people, the year is over and they move on. So it's unsatisfactory in that way. But they are both wonderful institutions and I was very happy at both.

In the late 80s you left for Stony Brook, to the State University of New York, where you got in contact with students again, as an academic teacher.

Yes, that was certainly one strong motivation. I felt that the institute was a wonderful place to spend some years but for me it was, perhaps, not a good place to spend my life. I was too isolated, in a way. I think the contact with young people and students and having more continuity was important to me so I was happy to find a good position in Stony Brook.

There were also domestic reasons: my wife was at Stony Brook and commuting back and forth, which worked very well until our son got old enough to talk. Then he started complaining loudly about it.

A colleague of mine and I had an interview with Atle Selberg in Princeton in 2005. He told us, incidentally, that he thought Milnor would never move from the institute because his office was so messy that just to clean it up would take a tremendous effort. But you moved in the end…

I don't know if the office ever got cleaned up. I think it was moved into boxes and stored in our garage.

Development of mathematics

Are there any mathematicians that you have met personally during your lifetime who have made a special, deep impression on you?

There are many, of course. There were certainly the professors at Princeton. Ralph Fox, Norman Steenrod and Emil Artin all made a strong impression on me. Henry Whitehead, I remember, invited a group of very young topologists to Oxford. This was a wonderful experience for me when I was young. I mentioned René Thom. More recently Adrien Douady was a very important influence. He was an amazing person, always full of life and willing to talk about any mathematical subject. If you had a question and emailed him, you would always get an answer back within a day or so. These are the names that occur most prominently to me.

When we observe mathematics as a whole, it has changed during your lifetime. Mathematics has periods in which internal development is predominant and other periods where a lot of momentum comes more from other disciplines, like physics. What period are we in currently? What influences from the outside are important now and how would you judge future developments?

I think the big mystery is how the relation between mathematics and biology will develop.

You mentioned ecology as an example.

Yes, but that was a discussion of a very simplified mathematical model. It's clear that most biological problems are so complex that you can never make a total mathematical model. This is part of the general problem in applied mathematics; most things that occur in the real world are very complicated. The art is to realise what the essential variables are, in order to construct a simplified model that can still say something about the actual more complex situation. There has recently been tremendous success in the understanding of large data sets (also in statistical analysis). This is not a kind of mathematics I have ever done but, nevertheless, it's very important. The question of what kind of mathematics will be useful in biology is still up in the air, I think.

Work style

You have proved many results that are described as breakthroughs by mathematicians all around. May we ask you to recall some of the instances when an idea struck you that all of a sudden solved a problem you had been working on? Did that rather occur when you had been working on it very intensely or did it often happen in a relaxed atmosphere?

Here is one scenario. After a lot of studying and worrying about a question, one night you go to sleep wondering what the answer is. When you wake up in the morning, you know the answer. That really can happen. The other more common possibility is that you sit at the desk working and finally something works out. Mathematical conversations are definitely very important. Talking to people, reading other people's work and getting suggestions are usually very essential.

Talking, very often, makes ideas more clear.

Yes, in both directions. If you are explaining something to someone else, it helps you understand it better. And certainly, if someone is explaining something to you, it can be very important.

Is the way you do mathematics today any different from how you did mathematics when you were 30 or 40?

Probably, yes.

How many hours per day do you work on mathematics?

I don't know. I work a few hours in the morning, take a nap and then work a few hours in the afternoon. But it varies. When I was younger I probably worked longer hours.

Do you subscribe to Hardy when he said that mathematics is a young man's game? You seem to be a counterexample!

What can you say? Whatever age, do the best you can!

In an article around 15 years ago, you described several areas in mathematics that you first had judged as of minor interest but which later on turned out to be fundamental to solve problems that you had been working on yourself. I think Michael Freedman's work was one of the examples you mentioned. Do you have more examples and is there a general moral?

I think that one of the joys about mathematics is that it doesn't take an enormous grant and an enormous machine to carry it out. One person working alone can still make a big contribution. There are many possible ap-

Abel Lectures and Science Lecture
25 May 2011

Abel Lectures:
John Milnor, “Spheres”
Curtis McMullen, “Manifolds, topology and dynamics”
Michael Hopkins, “Bernoulli numbers, homotopy groups, and Milnor”

Science Lecture:
Etienne Ghys, “A guided tour of the seventh dimension”

Abel Lectures 2011. From left: Curtis McMullen, Etienne Ghys, John Milnor and Michael Hopkins. Photo: Eirik Furu Baardsen.

proaches to most questions so I think it’s a big mistake to have everything concentrated in a few areas. The idea of having many people working independently is actually very useful because it may be that the good idea comes from a totally unexpected direction. This has happened often. I am very much of the opinion that mathematics should not be directed from above. People must be able to follow their own ideas.

This leads to a natural question: what is mathematics to you? What is the best part of being a mathematician?
It is trying to understand things, trying to explain them to yourself and to others, to interchange ideas and watch how other people develop new ideas. There is so much going on that no one person can understand all of it; but you can admire other people’s work even if you don’t follow it in detail. I find it an exciting world to be in.

What’s the worst part of being a mathematician, if there is any? Is competition part of it?
Competition can be very unpleasant if there are several people fighting for the same goal, especially if they don’t like each other. If the pressure is too great and if the reward for being the successful one is too large, it distorts the situation. I think, in general, most mathematicians have a fair attitude. If two different groups produce more or less the same results more or less at the same time, one gives credit to everyone. I think it’s unfortunate to put too much emphasis on priority. On the other hand, if one person gets an idea and other people claim credit for it, that becomes very unpleasant. I think the situation in mathematics is much milder than in other fields, like biology where competition seems to be much more ferocious.

Do you have the same interest in mathematics now as you had when you were young?
I think so, yes.

Prizes

You received the Fields Medal back in 1962, particularly for your work on manifolds. This happened in Stockholm at the International Congress and you were only 31 years old. The Fields Medal is the most important prize given to mathematicians, at least to those under the age of 40. The Abel Prize is relatively new and allows us to honour mathematicians regardless of age. Receiving the Fields Medal almost 50 years ago, do you remember what you felt at the time? How did receiving the Fields Medal influence your academic career?
Well, as you say, it was very important. It was a recognition and I was certainly honoured by it. It was a marvellous experience going to Stockholm and receiving it. The primary motive is to understand mathematics and to work out ideas. It’s gratifying to receive such honours but I am not sure it had a direct effect.

Did you feel any extra pressure when you wrote papers after you received the Fields Medal?
No, I think I continued more or less as before.

You have won a lot of prizes throughout your career: the Fields Medal, the Wolf Prize and the three Steele Prizes given by the American Mathematical Society. And now you will receive the Abel Prize. What do you feel about getting this prize on top of all the other distinctions you have gotten already?
It is surely the most important one. It is always nice to be recognised for what you have done; but this is an especially gratifying occasion.

What do you generally feel about prizes to scientists as a means of raising public awareness?
It is certainly very successful at that. I’m not sure I like getting so much attention but it doesn’t do me much harm. If this is a way of bringing attention to mathematics, I’m all in favour. The danger of large prizes is that they will lead to the situations I described in biology. The competition can become so intense, it becomes poisonous; but I hope that will never happen in mathematics.

Personal interests

Having talked about mathematics all the time, may we finish this interview by asking about other things you are interested in: your hobbies, etc?
I suppose I like to relax by reading science-fiction or other silly novels. I certainly used to love mountain climbing,

although I was never an expert. I have also enjoyed skiing. Again I was not an expert but it was something I enjoyed doing… I didn't manage it this winter but I hope I will be able to take up skiing again.

What about literature or music?
I enjoy music but I don't have a refined musical ear or a talent for it. I certainly enjoy reading although, as I said, I tend to read non-serious things for relaxation more than trying to read serious things. I find that working on mathematics is hard enough without trying to be an expert in everything else.

We would like to thank you very much for this most interesting interview. This is, of course, on the behalf of the two of us but also on behalf of the Danish, Norwegian and the European Mathematical Societies. Thank you very much!

Speech by the EMS President at the Abel Prize Banquet

Marta Sanz-Solé (EMS President)

Your Majesty, Abel Prize Laureate, Honourable Minister of Education and Research, distinguished guests, ladies and gentlemen.

It is a great privilege and pleasure for me to give this address, in my role as President of the European Mathematical Society, to honour and to congratulate Professor John Willard Milnor, the recipient of the Abel Prize of the year 2011. This gives me also an exceptional opportunity to praise and to thank the Norwegian Government for having established the Niels Henrik Abel Memorial Fund and the Abel Prize.

By establishing such a prestigious award, the Norwegian Government has turned an old desire of a large part of the mathematical community into a reality. The Abel Prize, which acknowledges scientific contributions of exceptional depth and of the highest significance in mathematics, now fills the gap left by the Nobel Prize in the recognition of one of the more ancient and fundamental sciences.

Within its short existence of just nine years, the Abel Prize is seen as one of the most prestigious awards in mathematics. Brilliant mathematicians from different fields have seen their works rewarded. The Prize is a tribute to the extraordinary minds that have contributed significantly to the progress of the discipline and, therefore, to the progress of science and culture. In fact, since the origin of science as a human activity, mathematics has held a privileged position in the core of knowledge.

Your Majesty, it is extremely laudable that your government has achieved the political consensus to create the Abel Prize. Its creation helps maintain the awareness of the international community of Norway as a learned nation, and exhibits worldwide your generous commitment to the fostering of knowledge.

It is also a sign of the intellectual strength and wise vision of Norwegian politicians to consider raising the status of mathematics in society and stimulating the interest of young people and children in mathematics as two of the main contributions and reasons for the existence of the award. Today, some of you have learnt for the first time about the exceptional discoveries by Professor Milnor in topology, geometry and algebra. We have heard and read highly appreciative appraisals of his work: it contains profound insights, vivid imagination, elements of surprise, supreme beauty. It is possible that there is a certain feeling of disorientation among those in the audience who are not mathematicians.

Since our childhood, we have associated geometry with something very concrete: figures that can be drawn and constructed – for example, a rectangle that represents the contour of the house where we live, a sphere that represents the planet Earth that hosts us. But we are much less prepared to grasp the meaning of the exotic smooth spheres in seven dimensions that Professor Milnor has discovered. This leaves many of us perplexed and speechless.

Let me go further and mention that, to the eyes of a topologist (a specialist in the mathematical field of topology) like Professor Milnor, there is no single difference between the distorted and melted timepieces of the Catalan painter Salvador Dalí and the classical and harmonious Santos watch, a masterpiece by Cartier. Quite bewildering! Our curiosity, but also incomprehension, will certainly increase.

By the way, it is remarkable that Dalí, my compatriot, has bequeathed such beautiful illustrations of homeomorphisms that all can appreciate and admire.

The general public's lack of understanding of the job of a mathematician is more than a stereotype; it is a frequent and real fact. This happens even in learned circles. In my opinion, two reasons might be that mathematicians often do not use words as material substratum in their process of thinking, and record their thoughts in an esoteric notation. Both of these raise the bar to comprehension.

However, those who understand mathematicians and mathematics, even if partially (and there are not few of them), praise our beloved discipline. Obviously, we feel very pleased and flattered by this.

Recently, the Minister of Education of my country, the philosopher and professor of metaphysics Ángel Gabilondo, when addressing an audience of mathematicians, said that, in his opinion, mathematics is the central axis of culture; this is by its ability to simplify and to codify what cannot be grasped, by its ability to challenge chance and to struggle with the logic of paradoxes. This seems to me an exact perception of the role and characteristics of this science.

But what is the job of a mathematician about?

According to Noam Chomsky, our ignorance can be divided into problems and mysteries. If we accept this statement then one can say that mathematicians contribute to overcoming ignorance. With our discoveries, we solve problems and by inventing theories, we resolve mysteries.

It is not now the right time to go into the vivid philosophical debate about whether we discover or invent when we do mathematics. Personally, I place myself at the side of those who think that the two are interwoven. In any case, most will agree that these intellectual processes are very close. Both involve a large dose of imagination. Presumably, most mathematicians would agree that mathematical discovery is a natural consequence of a process of logic and systematic arguing and that, as such, there is no need for a Muse to transport our minds to a state of irrational perception. Perhaps this detracts a bit of glamour from the profession. There is no need to wait for a Beatrice to lead us to Paradise as she did for Dante.

Nevertheless, there are many similarities between the intricate process of mathematical discovery and artistic creation. It all starts with a long period of preliminary work, sometimes at the level of the unconscious. In this step, a huge number of ideas and arguments combine themselves through a tricky exercise. Many of these combinations turn out to be useless or inappropriate. After a selection procedure, the most robust prevail. The French mathematician Jacques Hadamard says that such a choice is imperatively driven by a sense of beauty. In my view, it is steered by something even more immaterial: by the talent and the creativity of the mind that finds the path.

Then, suddenly and unexpectedly, like a thunderbolt, illumination invades the mind. This is a very difficult moment to catch and to describe accurately because of its instantaneous high intensity. It is a critical point. After this, thoughts retreat and become more linear, in order to carry out their materialization. This is done with words, with signs, with any of the forms of mathematical language, and with a toolbox to check, to compute, to make the results precise, to make them useful and available to others. Today, we are honouring John Willard Milnor, a discoverer and an inventor of the highest calibre, someone who has fully experienced the intricate steps of the creative process, someone whose talent has produced exceptional results, a mathematician who has generously shaped his thoughts with an extremely clear and elegant language, making them available to our disposal and for our pleasure.

Abdus Salam School of Mathematical Sciences, Lahore, Pakistan

Background and history

The Abdus Salam School of Mathematical Sciences (ASSMS) was established in 2003 by the Punjab Provincial Government, under the aegis of the Government College University, Lahore, to serve as a Centre of Excellence for Advanced Studies and Research in Mathematics. Nobel Laureate Abdus Salam graduated from Government College Lahore and later taught mathematics there. The school was named in his honour.

It is, however, fair to say that even though the Government College in Lahore, and several other universities and research institutions, had offered general, and in some cases advanced, mathematics courses, the level of mathematics education and research in Pakistan was, in 2003, rather poor. Consequently, the newly established mathematical school had little national experience to build on. The established board of governors therefore called Professor A.D. Raza Choudary home from the USA and offered him the position of Director General of ASSMS.

Dr Choudary made the important decision to look for teachers for his new school in the international market. He had spent several years as a PhD student, and postdoc, in Romania and Germany before leaving for the USA and managed to call to his school a large group of very competent European mathematicians, on regular one-term and semester contracts. With these teachers and a small but pleasant building in the central area of Lahore the school started functioning in December 2003.

Objectives

ASSMS is a doctoral school and its main objective is to provide the developing nation of Pakistan with competent young mathematicians to serve in its institutions of higher education and research. At the same time, ASSMS has been given the responsibility of promoting mathematics in schools/colleges and providing professional enhancement training to the faculty at universities all over the country.

ASSMS organises a large number of seminars, colloquia, research schools, intensive courses and lecture series open to Pakistani faculty and researchers. ASSMS also organises a number of events for students of high schools and elementary schools in Pakistan to encourage students to excel in mathematics.

Location, internal structure and modus operandi

Today ASSMS is housed in a large, newly renovated building, consisting of a conference hall with a seating capacity of 300, an 80–90 person committee room for meetings, 10 class rooms and 35 research offices for professors and postdoctoral fellows. There are 26 permanent positions in education and research and some 30 foreign researchers with various visiting positions, taking care of most of the PhD education and contributing a great deal of research in many areas.

Building of the ASSMS.

The school has a small but rather effective library with 5000 books and 76 journals, available in hard copy. The library also provides study space for about 100 graduate students. Internet access is available throughout the premises of the school, including two computer labs with a seating capacity of 50.

There is a nice garden in the centre of the building. It functions as a very popular meeting place for faculty and students. During Autumn, Winter and Spring, seminars often take place outdoors, in the midst of beautiful flower arrangements and in the shadow of large oriental trees.

Students are carefully selected. A first screening is carried out through a written test, which generally takes place in April or early May. The written test date is advertised in the major newspapers of Pakistan as well as on several websites such as the website of the Higher Education Commission (HEC) Pakistan, a website which is most frequently visited by the young science graduates in Pakistan. The admission process and dates are also advertised on the website of the ASSMS itself. Over the past three years, about 350–450 students have participated each year in the written test and about 50 students were selected for interview with an international board of examiners. On the basis of these interviews 17–23 students were admitted onto the PhD programme every year.

Through the selection process and also due to very attractive stipends (by Pakistani standards), ASSMS succeeds in attracting the very best students from universities all over Pakistan onto its PhD programme in mathematics. However, due to a general weakness in the university education system in Pakistan, even the best students usually have a poor background in several areas of basic university mathematics. ASSMS therefore requires an intensive two-year course for all the students admitted onto its PhD programme. This two-year course is comparable to the course of an international M.S.

The first year is dedicated to basic university mathematics courses, which are taught mostly by European mathematicians. During the second year, students take more advanced and optional courses in diverse areas of mathematics. At the end of their second year, students choose their research area and get associated with one of the research groups at ASSMS.

Presently, 29 women are full-time PhD students at ASSMS. So far 63 students, and among them nine female students, have finished their PhDs. Most of them are now serving at different universities in Pakistan.

Students at work.

Postdoctoral programme

Taking advantage of the frequent visits of well-established foreign mathematicians, ASSMS also offers several postdoctoral fellowships every year. These fellowships are generally given to young researchers from different countries in the region. Sometimes, upon the recommendation of a foreign visiting professor, postdoctoral fellowships are also given to young researchers from academically developed countries. So far 28 postdoctoral positions, for periods of six months to two years, have been granted to young researchers from Indonesia, Nepal, Uzbekistan, Georgia, Romania, Italy and Russia.

Guest houses for foreign visiting researchers

In an effort to welcome well-established, foreign visiting mathematicians, ASSMS has acquired several guest houses with a total capacity of 47 persons. Upon the recommendation of our foreign visiting faculty, we keep on improving the quality of these guest houses. Breakfast and dinner is served at the guest houses to all the foreign visiting faculty and lunch is served at the school. The school also provides transport to all the foreign visiting faculty as well as primary medical care.

Areas of research

ASSMS has six main research groups.

The geometry group has nine members, including regular faculty and regularly visiting researchers. The group has six PhD students working in different areas of geometry including algebraic geometry, algebraic topology and differential geometry. In addition, eight local faculty members from different universities of Pakistan are also part of this research group.

In the past four years, the group, together with their PhD students, has produced 75 research papers accepted for publication in international, refereed journals; most of them are ISI journals.

The geometry group has organised nine Intensive Courses Open to Public, and four International Exposures (schools and workshops).

The algebra group is composed of 13 researchers, including regular and visiting faculty. At the moment the group has 21 PhD students working in diverse branches of algebra including commutative algebra, algebraic geometry and singularity theory, rings and modules, homological algebra, algebraic systems and computer algebra, Groebner bases over rings, primary decomposition of modules, Sagbi bases for local rings, signature of surface singularities and classification of hypersurface singularities in characteristic $p>0$.

Occasionally some young researchers from the countries in the region join this research group as postdoc fellows. In addition, 10 local faculty members from different universities of Pakistan are also part of this research group.

In the past four years, the faculty, the postdoc fellows and the PhD students together have written 137 research papers accepted for publication. Most of them were accepted in ISI journals.

The group has organised three Intensive Courses Open to Public and are planning two International Schools and Workshops for next year.

The analyses group has 11 members including regular and visiting faculty. The group has 29 PhD students and is involved in research areas including convex analysis, operator theory, approximation theory, differential equations, dynamical systems, harmonic analysis and differential inclusions. Several postdoc fellows from the region are also associated members. In addition, six local faculty members from different universities of Pakistan are part of this research group.

In the past four years, the group has produced 265 research articles accepted for publication, mostly in ISI journals.

The group has organised five Intensive Courses Open to Public and four International Exposures (schools and workshops).

The discrete mathematics group has eight members, including regular and visiting researchers. It has nine PhD students. Their research work is in graph theory, combinatorics and discrete geometry. At the moment, two postdoc fellows are working in these areas. In addition, 12 local faculty members from different universities of Pakistan are part of this research group.

In the past four years, the group has produced 197 publications and most of these are in ISI journals.

The group has given two Intensive Courses Open to Public and three International Schools and Workshops.

The applied mathematics group has seven members, including regular and visiting researchers. There are 11 PhD students working in this group. The applied mathematics group pursues research in areas of quantum statistical mechanics, fluid dynamics, control theory and numerical analysis. In addition, nine local faculty members from different universities of Pakistan are part of this research group.

During the last four years, the faculty and PhD students at ASSMS have produced 142 research papers in areas of applied mathematics. A large majority of the papers were accepted for publication in ISI journals.

The stochastic processes group has two members, regular and visiting.

This group has four PhD students and one postdoc fellow. The PhD students are pursuing their research in areas of stochastic processes, financial mathematics and simulation problems. In addition, two local faculty members from different universities of Pakistan are part of this research group.

In the past three years, faculty and students together have had 24 research papers accepted for publication in reputed international journals.

Publication of Lecture Series

ASSMS has recently started a programme to publish lecture notes of the lectures delivered by famous mathematicians. In 2009–2010, the school published lecture notes of the lectures delivered by A.A. Kirillov, J. Bernstein, V.I. Arnold and P. Grozman. These lecture notes turned out to be quite useful for the PhD students and young researchers in Pakistan and in the region.

National and international relations and recognitions

The ASSMS has formal relations to some 30 research and university institutions throughout the world and has, in particular, partnerships with The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy, and the Centre International de Mathématiques Pures et Appliquées (CIMPA), France.

The ASSMS has hosted seven international conferences, in particular five very successful conferences in the series "World Conference on 21st Century Mathematics". In collaboration with CIMPA, ASSMS has also hosted two international schools.

ASSMS is particularly proud to be able to attach the label ERCE to its name. On the recommendation of the Committee for Developing Countries (CDC) of the European Mathematical Society, the first "Emerging Regional Centre of Excellence of the European Mathematical Society" (ERCE label) has been attributed to the Abdus Salam School of Mathematical Sciences.

Financial situation and budget

The ASSMS is funded by the Federal Government and by the State of Punjab.

The Federal Government, through its Higher Education and Research Commission (HEC), and the provincial government last year allocated Pak Rs. 91.43 million. This does not include the honoraria or salaries of foreign visiting or regular faculty. This is paid directly by HEC to the foreign professors. This year the Punjab Government has reduced the financial support to ZERO. The result is that, as of today, ASSMS is financially completely dependent upon HEC.

Budget

a) Salaries of Gazetted and Non-Gazetted Staff = Pak Rs. 16.194 million.
b) Operating Expenses = Pak Rs. 60.073 million.
c) Commodities and Services = Pak Rs. 14.414 million.
d) Purchase of durable goods = Pak Rs. 0.505 million.
e) Others = Pak Rs. 0.244 million.

Basic information

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ICMI Column

Teaching Statistics in School Mathematics, Challenges for Teaching and Teacher Education, A joint ICMI/IASE study (18th ICMI study).

Since the mid-1980s, the International Commission on Mathematical Instruction (ICMI, www.mathunion.org/ICMI/) has investigated issues of particular significance to the theory or practice of mathematics education by organising specific ICMI studies on these themes.

The 18th study in this series has been organised in collaboration with the International Association for Statistical Education (IASE, www.stat.auckland.ac.nz/~iase/) and addresses some of the most important aspects of the teaching of statistics in schools by focusing on the education and professional development of teachers for teaching statistics. The study included an IASE Roundtable Conference and is fully reported in the Proceedings of the Study Conference (www.ugr.es/~icmi/iase_study/) and in the study book now published in the ICMI study series by Springer.

The main conclusions from research and exemplary practice reported and discussed in this study are as follows:

Teaching statistics at school level. Although the teaching of statistics in secondary schools has a long tradition, in recent years many countries have also included statistics in the primary curriculum. In addition, more attention has been paid to developing statistical thinking in students across all levels of education.

Teachers' attitudes, beliefs and knowledge. At the school level, statistics is usually taught within the mathematics curriculum by teachers who may or may not be specifically trained to teach statistics. Most teachers acknowledge the practical importance of statistics and are willing to give more relevance to the teaching of statistics. However, many mathematics teachers do not consider themselves well prepared to teach statistics nor face their students' difficulties. Research summarised in the study has shown a variety of difficulties and misconceptions of prospective teachers with respect to fundamental statistical ideas. There is little research related to teachers' statistical pedagogical content knowledge and what is available suggests that this knowledge is weak.

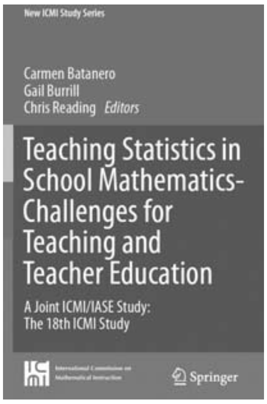
Current training of teachers. Few current teacher training programmes adequately educate teachers for teaching statistics at any school level. Few prospective secondary school teachers receive specific pedagogical preparation in statistical thinking. The situation is even more challenging for primary school teachers, since few of them receive any training in statistics. The study also involved sharing and analysing different experiences and initiatives in teacher education for teaching statistics. The following recommendations were produced.

Empowering teachers to teach statistics. There is a continuing need for finding approaches for preparing teachers that promote teachers' statistical literacy and reasoning, that engage teachers with real data and statistical investigations, and that connect teacher education to their teaching practice and the reality of their classrooms.

Collaboration in teacher education. Because of the nature of statistics and its key roles in all aspects of an information society, the statistics education of teachers could benefit from the support given by national statistical offices and statistical associations, which in many countries are increasingly involved in producing materials and organising initiatives to help increase statistical literacy of all citizens, with particular focus on education.

Relevance of research in statistics education. The rapid development of statistics and statistics education implies that further research in statistics education is needed. The analyses, research and case studies reported in the study provide a rich starting point for such research.

The study has been presented at the Conferencia Interamericana de Educación Matemática (CIAEM) conference in Recife, Brazil, June 2011, and will be released in August 2011.



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35th International Conference on the Psychology of Mathematics Education (10–15 July, Ankara, Turkey); http://www.arber.com.tr/pme35.org/index.php/home

The International Group on the Psychology of Mathematics Education (PME) is one of the affiliated study groups of the ICMI. The 35th PME was held at the Ankara campus of the Middle East Technical University, chaired by Prof. Dr. Behiye Ubuz. About 550 participants from 45 countries were there. Besides 79 Turkish mathematics educators, the geographical position of Anatolya fostered the participation of people from different continents (Europe, Asia and Africa). Many participants from European countries were there:

UK (37), Portugal (33), Germany (28), Italy (16), Greece (13), Finland (12), Spain (12), Netherlands (10), Sweden (7), Cyprus (4), Denmark (4), Norway (4), Austria (3), Czech Republic (3), France (3), Ireland (2), Romania (2), Russia (2), Serbia (1) and Switzerland (1). Moreover, there were 10 researchers from Israel, 61 from the US and 18 from Canada, 20 from Australia and 3 from New Zealand, more than 100 from the Far East (Hong Kong – China, Taiwan, Thailand, Japan, South Korea, Malaysia, India), 37 from Latin America (Brazil, Colombia, Mexico) and several others from non affluent countries, e.g. Iran (7), South Africa (4), Saudi Arabia, Bahrain, Ethiopia and the United Arab Emirates.

The scientific programme was rich and interesting.

Four plenary addresses were offered:
- Prof. Dr. Janet Ainley, University of Leicester, UK: *Developing Purposeful Mathematical Thinking: A Curious Tale of Apple Trees*.
  Reactor: Teresa Rojano, Centre for Research and Advanced Studies (Cinvestav) IPN, Mexico.
- Prof. Dr. Ali Doğanaksoy, Middle East Technical University, Turkey: *Morals of an Anecdote as Starting Point of a Lecture in Mathematics.*
- Prof. Dr. Brian Doig, Deakin University, Australia: *Children's Informal Reasoning: Concerns and Contradictions.*
- Prof. Dr. Konrad Krainer, Alpen-Adria-Universität Klagenfurt, Austria: *Teachers as Stakeholders in Mathematics Education Research.*
  Reactor: Minoru Ohtani, Kanazawa University, Japan.

A plenary panel was held on *Supporting the Development of Mathematical Thinking* (Prof. Dr. Olive Chapman, University of Calgary, Canada; Prof. Dr. Gabriele Kaiser, University of Hamburg, Germany; Prof. Dr. Uri Leron, Technion-Israel Institute of Technology, Israel; Prof. Dr. Frederick K.S. Leung, The University of Hong Kong, Hong Kong; Prof. Dr. Carolyn Maher, Rutgers-The State University of New Jersey, USA).

Moreover, there were **two research fora** on *Researching the Nature and Use of Tasks and Experiences for Effective Mathematics Teacher Education* (coordinated by Peter Sullivan and Orit Zaslavsky) and *Problem Posing in Mathematics Learning and Teaching: A Research Agenda* (coordinated by Florence Mihaela Singer, Nerida Ellerton, Jinfa Cai and Eddie Leung).

The following activities completed the offering:
161 research reports, nearly 200 short orals, nearly 100 poster presentations, 8 discussion groups and 5 working sessions.

An interesting national presentation was given by Turkey, with the coordination of Prof. Behiye Ubuz, about the doctoral programmes in mathematics education, mathematical thinking research and mathematics teacher education in Turkey.

Turkish mathematics educators gave the impression of a lively and competent community in the field.

"Solid findings" in mathematics education

Introduction

This series of articles is about the learning and teaching of mathematics. It has been prepared by the Education Committee of the European Mathematics Society (http://www.euro-math-soc.eu/comm-education.html). As part of its mission, this committee has decided to provide brief syntheses of research on topics of international importance. Each article aims to summarise an interesting and important "solid finding" of research on mathematics learning and teaching. As will be elaborated below, by solid finding we mean findings that:

- result from trustworthy, disciplined inquiry, thus being sound and convincing in shedding light on the question(s) they set out to answer.
- are generally recognised as important contributions that have significantly influenced and/or may significantly influence the research field.
- can be applied to circumstances and/or domains beyond those involved in this particular research.
- can be summarised in a brief and comprehensible way to an interested but critical audience of non-specialists (especially mathematicians and mathematics teachers).

The articles are, in an important way, different from (extensive and specialised) state-of-the-art papers and handbook chapters that review a particular topic in mathematics education, such as early childhood mathematics education, mathematical modelling and equity in mathematics education. Besides typically being (much) longer, most of these publications try to look at what seem to be the most important fundamental and unresolved questions, challenges and opportunities now facing research and development in mathematics education, to propose new conceptualizations of or perspectives on research problems and to suggest possible research programs to move the field forward. The primary goal of our summaries is rather to try to answer in a clear, straightforward and non-controversial way what we already do know from research about this topic or aspect of the teaching and learning of mathematics, what our research community learnt from approaching this theme with a lens that is different from how it was seen before and what this may tell us about how to improve the teaching and learning of mathematics.

Our audience

Our primary audience is, first of all, the group of mathematicians and researchers from adjacent disciplinary fields who are also directly or indirectly confronted with mathematics, mathematical thinking and mathematics education, such as psychologists, sociologists and anthropologists. Other important audiences are teacher educators, curriculum developers, policy-makers and test developers, as well as other people from outside the field who want to understand what mathematics education research is all about and what its relevance for those outside the immediate community might be.

For each article we will try to identify a significant and robust phenomenon, describe and explain it by making use of well-known and highly valued theoretical analysis and empirical research and then say something about what one might be able to do about this phenomenon from a mathematics educational point of view.

The overall message of our set of "solid findings" will not be that research in mathematics education has yielded unquestionable explanations and straightforward solutions of these phenomena. Instead, the main message is rather that things are more complex than one might think; but also that, particularly for the phenomena that have been selected as topics for our book, the community of mathematics educators has realised a serious and widely recognised breakthrough in identifying the complex nature of these phenomena, in understanding their impact on mathematics education and in suggesting potential solutions.

Mathematics education as a research field

People from outside our research community may be surprised that we consider mathematics education as an independent discipline. Against the background of the radical changes in psychology from the late 1950s onward, research in mathematics education has started to emerge as a field of study in its own right. Bishop (1992) proposed that there were three identifiable traditions reflecting the state of research in mathematics education around 1970: the empirical-scientist tradition, the pedagogue tradition and the scholastic-philosopher tradition. To a considerable extent, research in mathematics education was conducted within the empirical-scientific tradition, relying heavily on psychological theory and methodology. However, the limited perspective of such research, wherein all aspects of interest are treated as "variables" that can be defined and measured and wherein teaching is taken as "treatment" and learning as "effect", has always been open to criticism from the perspectives of the other two traditions. Relying on a deep knowledge of mathematics and/or a rich experience of how to teach it, leading to well-elaborated and strongly held views on the nature of mathematics and/or how it should be taught, scholars from these two other traditions reacted especially against the assumption that a theory of mathematics education could be derived from a domain-independent theory of cognition, learning or education. Increasing interactions between researchers and scholars working within different perspectives led to the first intimations of the idea that mathematics education could be delineated as a field of study in its own right while retaining strong links with other disciplines (Vergnaud, 1982). Of these other disciplines, psychology and mathematics itself remained, of course, pre-eminent but, increasingly, interest is being taken in the work of sociologists, linguists, anthropologists and historians, reflecting the increasing recognition of mathematics education's "situatedness" in social, cultural and historical contexts (De Corte, Greer & Verschaffel, 1996).

Important contributions to (and, at the same time, external signs of) the emergence of an identifiable community of mathematics education researchers were: firstly, the organisation of the first International Congress of Mathematical Education (ICME) in 1969 and the International Group for the Psychology of Mathematics Education (PME) in 1977; secondly, the formation of important research institutes in mathematics education in many countries (such as the Shell Centre for Mathematics Education at Nottingham University, the Freudenthal Institute at Utrecht University and the Institut für Didaktik der Mathematik in Bielefeld, or the Department of Mathematics Education at the University of Georgia in Athens); and, finally, the establishing of several domain-specific journals (such as *Educational Studies in Mathematics, For the Learning of Mathematics, International Journal for Science and Mathematics Learning, Journal of Mathematical Behavior, Journal of Mathematics Teacher Education, Journal for Research in Mathematics Education, Mathematical Thinking and Learning, Recherches en Didactique des Mathematiques* and *Zeitschrift für Didaktik der Mathematik)* and book series (e.g. Kluwer's *Mathematics Education Library)*. Over the last few decades, the task of self-definition of mathematics education as a research field on its own has largely been accomplished (Sierpinska & Kilpatrick, 1997).

So, there now exists a recognisable body of research within an identifiable community of mathematics education researchers – which means not only that this research is conducted within the realm of mathematical cognition, learning and teaching but also that the specific and unique nature of the mathematics domain has been seriously taken into account in all aspects of the work: framing research questions, choosing a mode of investigation, designing instruments, collecting data, interpreting results and suggesting implications (Grouws, 1992, p. ix). Certainly not all studies that are classified as mathematics education research give the mathematics involved the same attention nor is each study equally sensitive to each of the aspects of investigation mentioned. However, in our series, we will primarily look for "solid findings" that result from research that has put the specificity and integrity of the domain at the centre of the work rather than simply applying models and theories from other fields.

Criteria for "solid findings"

Our choice of solid findings was necessarily eclectic. Major criteria for the choice of solid findings were those that have been put forward by Schoenfeld in his chapter on "Method" in Lester's *Second Handbook of Research on Mathematics Teaching and Learning*. According to Schoenfeld (2007), good research in mathematics education must be examined along three somewhat independent dimensions.

The first – trustworthiness – refers to the degree of believability of the claims made in a single piece of research or a group of studies about a particular topic. There are various criteria for the trustworthiness of (empirical) research: its descriptive and explanatory power, its possibility to predict and falsify, its rigour and specificity, its replicability and whether the research makes use of multiple sources of evidence (so-called triangulation). Of course, not every criterion from the above list is relevant for every phenomenon to be studied and for every piece of research but trustworthiness is anyhow an essential criterion for good research. However, it is not enough. A study may be trustworthy but trivial, along one or both of the other dimensions of generality or importance.

Generality (or scope) refers to the question: how widely does this finding, this idea, this theory apply? In this respect, Schoenfeld distinguishes between claimed, implied, potential and warranted generality as ways to think about the scope or generality of a piece of research. So, although researchers mostly tend to study learning within clearly specified content domains, age groups and cultural and educational settings, some pieces of research have offered a number of important constructs for thinking more broadly about mathematical thinking and learning that extend beyond the bounds of the individual studies in which the constructs were expounded. A complicating factor in this respect is the cultural dependency of findings and recommendations concerning mathematical thinking, learning and teaching: what is working in China or Finland might fail in the U.S. or Japan. Where the mathematical statement $2 * 2 = 4$ may be true everywhere, this is not the case in mathematics education.

Thirdly, there is the importance criterion, which addresses the question: does it matter? What is the (actual or potential) contribution of this piece of research to theory and practice, and how important is this contribution? Of course, importance is to a large extent a value judgment. As in any other field of study, beliefs about what is essential and what is peripheral are not static but change over decades, reflecting trends both within and beyond the discipline.

From our perspective, we would like to emphasise that Schoenfeld's three major criteria for the solidity of a finding do not necessarily (and maybe not even primarily) rely on one particularly important and general piece of research but also, and even preferably, rely on a research line consisting of a larger set of related studies that together yield such a "solid finding". So, the term "solid" also includes an aspect of "robustness" in the sense that the finding should be repeatedly observed or confirmed in many studies reporting the same or similar results leading to the same (general) conclusions.

To these three criteria proposed by Schoenfeld (2007), we have added a fourth, which has to do with the specific aims of our series and its primary audience, namely the need for addressing phenomena, findings and insights that (a) have the potential to attract or surprise mathematicians and mathematics teachers and to be considered by them as (directly) useful to their teaching practice, and (b) can be clearly described and illustrated in a very brief way without reliance on technical language that is incomprehensible to people from outside the field.

In summary, in our search for "solid findings", we have looked for and ultimately selected lines of research that have converged on a particular and clear point and fit well within a larger line of "disciplined inquiry", thus being sound and convincing in shedding light on the questions they set out to answer. Moreover, the solid finding should be generalizable in that it can be applied to circumstances and/or domains beyond those studies themselves and it should address issues that do matter both to people inside and outside the community. Finally, it should be summarisable in a brief and comprehensible way to an audience of non-specialists.

A framework for conceiving and presenting "solid findings"

In the description of each solid finding, an attempt will be made to address its mathematical-epistemological, its psychological, its didactical and its institutional dimensions. Moreover, we will follow the same overall structure in describing these different solid findings, addressing the following five (sets of) questions:

(1) What is the *genesis* of the *issue or problem* being addressed by the solid finding? What is the motivation to consider the issue or problem? Why is it relevant and to whom?

(2) What is the *genesis* of the *solid finding* as an answer to the issue or problem addressed? What exactly is the finding? What is its nature (e.g. is it essentially a theoretical construct, an empirical or experimental result, an aggregation of several subfindings?). What were the first seminal publications?

(3) What makes the solid finding *solid*, and what is the *evidence* for its solidity? What is the nature and state of this evidence (e.g. theoretical, empirical, experiential, a mixture)? What are the conditions for the solid finding to hold, and hence what are its limitations? This includes cultural, societal, institutional and organisational aspects. The key references for the solid finding will be given here.

(4) What is the actual or potential *impact* of the solid finding on mathematics education practice (e.g. on curriculum design, teaching, assessment, teacher education)? For whom is the solid finding significant and in what ways?

(5) What are the *open questions* around and beyond the solid finding? What is not yet known in relation to the solid finding and are there promising and accessible ways that could provide new knowledge and insight in this respect?

Authorship

Even though certain authors have taken the lead in each article, all publications in the series are published by the Education Committee of the European Mathematics Society, with all committee members listed inside each publication.

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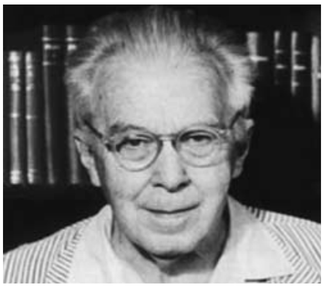
80th anniversary of *Zentralblatt für Mathematik*

Glimpses into the history of Zentralblatt MATH[1]

Silke Göbel

Formation and early successful years

In 1931, the first volume of *Zentralblatt für Mathematik und ihre Grenzgebiete* was published. It presented the bibliographic data of all recently published mathematical articles and books, together with reviews by mathematicians from all over the world. Zentralblatt became the second comprehensive review journal for mathematics in Germany; the *Jahrbuch über die Fortschritte der Mathematik*, established in 1868, was the first of its kind worldwide. The mathematicians Richard Courant (Göttingen), Otto Neugebauer (Göttingen), Harald Bohr (Copenhagen, brother of Niels Bohr) and Ferdinand Springer (publisher) had taken the initiative for the foundation of a new mathematical reviewing journal.



Otto Neugebauer

Richard Courant

They had known each other for many years; on many occasions, like conferences, private meetings and walking tours, they had thought about improving the working environment and the communication infrastructure for the mathematical community. World War I, and the amount of mathematical literature that grew explosively during the 1920s, had led to a large backlog in the Jahrbuch's editorial work. Furthermore, the Jahrbuch first collected the reviews and sorted them in subject chapters before publishing the entire annual collection. While Jahrbuch maintained the principles "completeness" and "classification of all articles of one year", Zentralblatt counted on "promptness" and "internationality". Springer intended to enlarge his publishing house after already running the well-established journals *Mathematische Annalen* (founded in 1868) and *Mathematische Zeitschrift* (founded in 1918).

Zentralblatt's first editorial board consisted of Pavel S. Alexandrov, Julius Bartels, Wilhelm Blaschke, Richard Courant, Hans Hahn, Godfrey H. Hardy, Friedrich Hund, Gaston M. Julia, Oliver Kellogg, Hans Kienle, Tullio Levi-Civita, Rolf H. Nevanlinna, Hans Thierring and Bartel L. van der Waerden. Otto Neugebauer became the first editor-in-chief of Zentralblatt. The first editorial office was on the premises of the Springer publishing house in Berlin. Zentralblatt was published several times per year. An issue was published as soon as sufficiently many reviews were available, resulting in a frequency of three to four weeks. The reviews were written in English, French or Italian as well as in German. Distinguished researchers from various countries belonged to the large group of reviewers. Even though the political and economic

Harald Bohr

Ferdinand Springer

situation was not easy, Zentralblatt became a successful journal with 18 volumes being published between 1931 and 1938.

The Years 1933–1945

Shortly after the Nazis assumed power, Zentralblatt experienced a critical change. On 7 April 1933, the *Gesetz zur Wiederherstellung des Berufsbeamtentums* was enacted which banned Jews and political enemies from holding jobs as civil servants. Even those who were formally excluded from the ban like Richard Courant, who had served at the front in the First World War, were not protected from the hatred of the Nazis. On 26 April, a call to dismiss Courant, Neugebauer, Landau, Bernays and Noether appeared in a local newspaper. Shortly after that, Courant escaped to Cambridge (UK) and later moved to New York.

Otto Neugebauer too was considered an intolerable person, not only because he had been a member of the *Sozialdemokratische Partei Deutschlands* (SPD) for a short period in his young days but, in particular, because he had spent some time in Leningrad studying Babyloni-

[1] The complete version of this survey will be published in the book *80 Years of Zentralblatt MATH. 80 Footprints of Distinguished Mathematicians in Zentralblatt*, edited by Olaf Teschke, Bernd Wegner and Dirk Werner, Berlin: Springer Verlag (2011). More original sources and references can be found therein.

an scriptures. Harald Bohr arranged a professorship for Neugebauer in Copenhagen, which Neugebauer took up in January 1934. With the support of his wife Grete Bruck and Ferdinand Springer, he took the editorial office of Zentralblatt with him to Copenhagen and continued his work as editor-in-chief from there.

In 1938, a breach occurred between Zentralblatt and its founding fathers. Wilhelm Blaschke, a member of the editorial board, had complained about the increasing number of anglophone reviews. Furthermore, the Jewish editor Tullio Levi-Civita had been removed from the editorial board without Neugebauer's knowledge. Filled with indignation, Neugebauer resigned from the post as editor-in-chief. His colleagues Bohr, Hardy, Courant, Tamarkin and Veblen followed.

Neugebauer left Europe and emigrated to the USA. He was appointed professor at Brown University, Rhode Island, through the intervention of Roland G. D. Richardson (Secretary of the American Mathematical Society). Shortly after, in 1940, Neugebauer and Richardson founded *Mathematical Reviews*, an American-based mathematical reviewing journal modelled on Zentralblatt.

For a short time, publishing only one volume, Egon Ullrich from Gießen headed Zentralblatt. The *Preußische Akademie der Wissenschaften* in Berlin and the *Deutsche Mathematiker-Vereinigung* took over the management of Zentralblatt in 1939. This was initiated by Ludwig Bieberbach, who was the chairman of the science division of the Prussian Academy, professor of mathematics at Berlin University and an active member of the Nazi party NSDAP. Harald Geppert, a mathematician and also an NSDAP member, was nominated as editor-in-chief of both Jahrbuch and Zentralblatt, while Bieberbach was appointed the supervising editor. Until 1945, the editorial offices of the two journals continued working independently of each other but sometimes shared information, scientific literature and reviews. At the end of World War II, Harald Geppert committed suicide.

The attempts of Bieberbach and Geppert to organise the entire editorial work consistent with the Nazi ideology were not too successful. Dismissals of Jewish staff members were not prevented; however, articles published by Jews or emigrants were still reviewed in Jahrbuch and Zentralblatt. For instance, almost all of the works of Wolfgang Döblin (later Vincent Doblin), who with his family had escaped from the Nazis to France in 1933, were reviewed in Zentralblatt. The same is true, for example, for articles of Courant, Einstein and Rademacher. Hermann Weyl, who had emigrated from Germany to the USA, commented in 1948:

> *It is true that even during the war, the Jahrbuch continued reviewing the papers of foreign and Jewish mathematicians in an objective and decentmanner.*

But there was a growing pressure to dismiss Jewish editors and reviewers. A typical example is a letter of Bieberbach at the beginning of 1938 to Helmut Grunsky, editor-in-chief of Jahrbuch until 1939:

> *Die Durchsicht der bisher erschienenen Hefte des Jahrgangs 1936 […] zeigt eine grosse Zahl an Juden vergebener Referate. […] Sie laufen jedenfalls Gefahr, dass Ihr Handeln als mangelnder politischer Instinkt ausgelegt werde.*[2]

Restart in 1947

After the end of the war, Berlin evolved as a divided city. West Berlin consisted of the sectors of the Western Allies; the Soviet sector formed East Berlin. The Soviet administration took the responsibility for the Prussian Academy of Sciences. It was reopened in 1946 with the new name *Deutsche Akademie der Wissenschaften*. Already in 1947 the 39^{th} volume of Zentralblatt was published by mutual agreement with Springer-Verlag. The publication of the Jahrbuch was discontinued when its publishing house Walter de Gruyter and the Academy did not come to an agreement on a new contract.

Hermann Ludwig Schmid from the University of Berlin became the new editor-in-chief of Zentralblatt. Zentralblatt's editorial board revived their contacts with former colleagues, inviting many of them to work again as editors or reviewers for Zentralblatt:

Hermann Ludwig Schmid

Zentralblatt staff 1965

2 Searching the first issues of the 1936 volume […] reveals a large number of articles assigned to Jewish reviewers. […] You are running the risk of being assessed as acting with a lack of political instinct.

> *"Aber schon allein die Tatsache, dass das Zentralblatt als erstes zu einer internationalen mathematischen Zusammenarbeit nach dem Kriegsende in Europa auffordert, ist so wichtig und hoffnungsvoll, dass man sich freuen kann, in dieser Arbeit teilzunehmen.*
> *Ihr sehr ergebener Bela v. Sz. Nagy (Szeged, 30 März 1948)."*[3]

Most of the papers documenting Zentralblatt's history were lost during World War II and the post-war period. However, some letters and records of 1947 until 1963 are today preserved in the office of Zentralblatt in Berlin. Others are to be found in the *Berlin-Brandenburgischen Akademie*, at the publishing house Springer in Heidelberg and other countries.

Zentralblatt – a cooperation between the two Germanys

In 1952, Hermann Ludwig Schmid, Zentralblatt's editor-in-chief at the time, was appointed professor at the University of Würzburg. He continued his editorial work from there until he suddenly died in 1956. Erika Pannwitz, who worked as an editor for both Jahrbuch (1930–1940) and Zentralblatt (since 1947) became his successor.

The construction of the Berlin Wall started in August 1961. The barrier cut off West-Berlin from the eastern part of the city, causing many complications for Zentralblatt and its editors. While the editorial office was located in Adlershof (East Berlin) on the premises of the Deutsche Akademie der Wissenschaften, Zentralblatt's publishing house Springer and about half of its staff members, including Erika Pannwitz, were located in the western districts. To make the communication and exchange of material between Zentralblatt's office and Springer possible, Pannwitz and two other members of staff were given special permits by the Volkspolizei (national police of the GDR) to enter East Berlin without strict control.

The split up of Zentralblatt became official with the cooperation agreement between the Deutsche Akademie der Wissenschaften and the *Heidelberger Akademie der Wissenschaften* in 1965. The academies agreed to continue Zentralblatt with the editing duties to be shared equally, in terms of both workload and technical level, by both Berlin offices, and with the printing and distribution to be done by the Springer publishing house. Walter Romberg was in charge of the eastern editorial board, while Erika Pannwitz continued as editor-in-chief of the western office.

It is a striking fact that this German-German cooperation continued successfully until 1977. Despite the remarkably complicated political situation, Zentralblatt recovered from the drawbacks and re-established a leading position in mathematical reviewing. Some effects of World War II remained noticeable for a long time, like economic problems, the backlog of uncompleted reports, which was not cleared before the 70s, and the need to catch up with the Mathematical Reviews. Erika Pannwitz retired in 1969 as editor-in-chief but she continued working as a section editor. Ulrich Güntzer from *Freie Universität* Berlin became her successor.

Walter Romberg in the centre with colleagues in the eastern part of Berlin

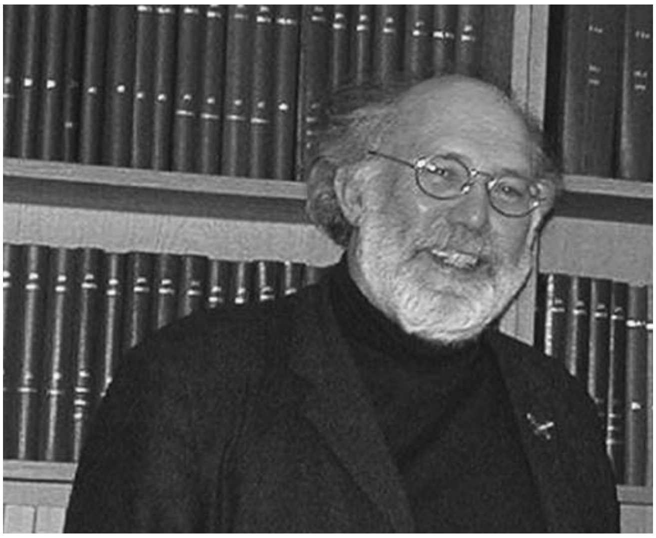
Erika Pannwitz in the centre with colleagues in the western part of Berlin

Transforming Zentralblatt into a reference database

In the 70s, the annual production of mathematical papers had reached a level which could no longer be handled by manual work. At the same time, computer science made big advances in database theory. Güntzer entered several partnerships with other scientific institutions, like *Chemie-Information, Großrechenzentrum für die Wissenschaft and Technische Universität* Berlin in order to benefit from their computing facilities for the editorial work at Zentralblatt. Bernd Wegner from TU Berlin, who became Güntzer's successor in 1974, further developed the ideas of technically advancing the editorial work. More than 35 years later, Wegner is still the editor-in-chief.

[3] "But the very fact that Zentralblatt is the first after the war in Europe to call for an international mathematical collaborations is so important and promising that one has to look forward to being part of this.
Yours faithfully Bela v. Sz. Nagy (Szeged, March 30, 1948"

Ulrich Güntzer



Bernd Wegner

In the 70s, the Federal Republic of Germany developed a plan to reorganise all information and documentation activities in the country. As a consequence, the *Akademie der Wissenschaften der DDR* (until 1972 Deutsche Akademie der Wissenschaften) withdrew from the cooperation contract with the Heidelberg Academy and consequently refused any further collaboration. Moreover, all reviewers from the GDR had to quit their services for Zentralblatt. For about two years, Zentralblatt was solely run by the Heidelberger Akademie and Springer.

In 1979, the Federal Republic of Germany established the *Fachinformationszentrum Energie, Physik, Mathematik* (today: FIZ Karlsruhe; Leibniz Institute for Information Infrastructure) in Karlsruhe. The Zentralblatt office was incorporated as a subsidiary, *Department Berlin*, while the Heidelberg Academy remained responsible for the content and Springer continued to publish the journal and was responsible for printing, marketing and distribution.

Between 1982 and 1985, discussions took place between the Heidelberg Academy, FIZ Karlsruhe and Springer on one side and the American Mathematical Society (AMS) on the other about a potential merger of Zentralblatt and Mathematical Reviews. Even though the negotiations failed, both review journals continue to cooperate. For example, their cooperation has led to the improvement of the Mathematics Subject Classification scheme (MSC). During the 60s, the AMS developed a mathematical classification scheme called "Mathematical Offprint Service" (MOS) which was soon adopted by Mathematical Reviews and somewhat later by Zentralblatt as well. The MOS scheme was revised by editors of both journals on an irregular basis. By the initiative of Bernd Wegner, in 1980, both parties concluded an agreement concerning the joint maintenance of the classification scheme, under the new name "Mathematics Subject Classification" (MSC). The agreement regulates the rules for future revisions. Since then, the MSC has been updated every ten years, the last time in 2010.

With the support of FIZ Karlsruhe, the first release of Zentralblatt as a searchable database was established in 1989 and made accessible to the public through the database provider STN International.

With the fall of the wall in 1989, the political circumstances changed again. Die Akademie der Wissenschaften der DDR was reorganised and some former members of the East Berlin Zentralblatt editorial office resumed their work for Zentralblatt. The Heidelberg Academy, FIZ Karlsruhe and Springer continued as editorial institutions and publisher.

The Europeanisation of Zentralblatt and recent developments

The revolutionary development in information technology and computer science, in particular the invention of the CD-ROM and the World Wide Web, enormously helped to improve Zentralblatt's services. The first release of the database as an offline version on CD-ROM called CompactMATH was published in 1990. TeX, the ingenious typesetting system written by Donald Knuth to typeset complex mathematical formulae, was introduced in 1992. The transition of the Zentralblatt database to a service accessible through the internet was accomplished in 1996; it was named MATH and later ZBMATH.

One of the main agendas for the future is to establish Zentralblatt as a large European infrastructure for mathematics and to enhance the Europeanisation of Zentralblatt. French support of the Zentralblatt enterprise through *Cellule MathDoc* (Grenoble) has been active since 1997, improving the distribution in France and, even more importantly, developing the search software, allowing more convenient access to Zentralblatt via the internet.

The European Mathematical Society was invited and agreed to become involved in Zentralblatt as an additional editorial institution.

Some streamlining in the editorial workflow was achieved by means of the close cooperation of Zentralblatt with a number of mathematical institutions through the *LIMES* project (*Large Infrastructure in Mathematics & Enhanced Services*). The project was funded by the European Union. Currently, FIZ Karlsruhe/Zentralblatt are further strengthening the European scientific infrastructures as partners of the *European Digital Mathematics Library* project (EuDML), which started in 2010.

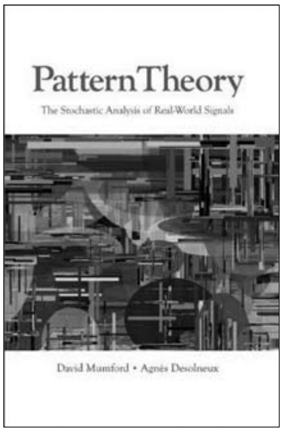
Since the beginning of the new century there have been several refinements of the database. Many documents contain links to digital libraries (via DOI and others) so it is possible to find the complete text of the articles through Zentralblatt online. Another important feature is MathML, which allows the displaying of mathematical symbols and formulas on the screen. Recently, an author database was launched.

In 2004, Zentralblatt MATH incorporated the Jahrbuch data as an extension. All Jahrbuch data 1868–1942 were digitized in the common framework of the ERAM-Project (*Electronic Research Archive for Mathematics*). Moreover, the complete bibliographic data of the "*Journal für die reine und angewandte Mathematik*", also known as Crelle's Journal, were added from its first issue of 1826. This makes ZBMATH the unique source of mathematical information from 1826 to the present.

Until 2009, Zentralblatt was distributed in both electronic format and print, resulting in 25 volumes of 600 pages each per year and about 3 million documents in the database. The print service, obviously somewhat out-of-date, was discontinued in 2010. Instead, Zentralblatt offers its new print service "*Excerpts from Zentralblatt MATH*". In contrast to the previous print edition, not all reviews from the database are included. Another new feature is the "*Looking Back*" section that allows a fresh look at classical mathematical works through contemporary reviews.

To finish this short survey about the history of Zentralblatt, we want to mention that the EMS recently established the "Otto Neugebauer Prize" in the section "History of Mathematics". The prize of 5000 euro will be first awarded in 2012 at the *European Congress of Mathematics in Kraków*.

Book Reviews



David Mumford
Agnès Desolneux

Pattern Theory
The Stochastic Analysis of
Real World Signals

AK Peters, Ltd, 2010
ISBN 978-1-56881-579-4

Yves Meyer

Pattern Theory: The Stochastic Analysis of Real World Signals by David Mumford and Agnès Desolneux is a masterpiece. It is one of the best books I have ever read.

Pattern Theory is given an indirect definition by the authors: "*By and large, the patterns in the signal received by our senses are correctly learned by infants, at least to the level required to reconstruct the various types of objects in our world, their properties and to communicate with adults. This is a marvelous fact and pattern theory attempts to understand why this is so*." It raises the issue of relating *Pattern Theory* to the functioning of the human brain. This problem will be addressed later on.

Instances of *real world signals* that are studied by the authors are the poem *Eugene Onyegin* by Pushkin, eight novels by Mark Twain, an oboe playing Winter 711 by Jan Beran, DNA sequences, a painting by Paolo Uccello, human faces and some car licence plates. In each example, the goal is to extract meaningful patterns. In the case of Eugene Onyegin, the story began with the mathematician Andreï Markov. He created what we now call Markov chains to characterize Pushkin's style. *In a digital world, style is mathematical*, as Dan Rockmore observed (quoted by Michelle Sipics in [8]). Mumford and Desolneux produced a concatenated string consisting of eight novels by Mark Twain, strung together with all punctuation and spacing removed. Then they proposed an algorithm that recovers the punctuation and the spacing by computing word boundaries from this string. This problem is not as artificial as it looks. The authors remind us that *in many written languages such as ancient Greek, word boundaries were not marked*. In the case of DNA sequences, one is trying to extract exons. One wishes to write the musical score from a recording of the oboe. In the painting "Presentazione della Vergine al tempio" by Paolo Uccello, one would like to delineate the salient contours. In many instances the challenge is to emulate human perception. In the case of licence plates, one tries

Presentazione della Vergine al tempio by Paolo Uccello

to read them in bad lighting conditions and from different perspectives.

Physicists claim that they can explain the surrounding world and predict the future. But they cannot predict what I am writing now. They cannot explain the beauty of a poem. They cannot tell why Las Meninas is my favourite painting. The authors adopt a completely distinct attitude. They do not try to explain the surrounding world. They offer us some tools for describing it. But a complete and exact description of the world is impossible. Modelling is a solution. One has to focus on some salient features and try to extract patterns. One finds some hidden structure that will eventually simplify the description and serve as a guide for constructing an appropriate model. This is the authors' goal. Natural images are strongly structured since they reflect the geometrical organisation that exists in nature. The authors do not use Euclidean geometry or the laws of physics to compute the geometrical patterns that can be found in most natural images. This would be an impossible task. Instead they teach us how to model these images. An alternative approach towards a better characterization of salient patterns consists in using non-parametric statistical methods. A simple well known example consists in checking the proportion of a commonly used word in all plays supposedly written by Shakespeare in order to decide whether their author is the same or not. Under the impulsion of R. Johnson, I. Daubechies and D. Rockmore, several wavelet-based techniques were tested in the Van Gogh Project: The challenge consists in classifying Van Gogh's painting according to the different periods that characterize the artist work or in separating them from fakes. One of the most promising method (proposed by P. Abry, S. Jaffard and H. Wendt, cf. [1]) is based on multifractal analysis techniques, and consists in computing, at several scales, space averages of quantities derived form wavelet coefficients, referred to as wavelet Leaders and in characterizing their power-law behaviors with respect to the scales. The exponents thus derived supply parameters on which classification is based, and remarkably allow to classify correctly a large fraction of the painting involved in the challenge by the Van Gogh Museum.

The authors' most important message is that the inherent variability and complexity of real world signals imposes a stochastic modelling. In principle, a model should be learned from the data and validated by sampling. Learning a model from data would be the subject of another book. Let us elaborate on validation by sampling, which is one of the authors' main messages. Experts are needed to validate. Working on music signals requires a good practical knowledge of music. It is not a surprise that some of the best experts in audio signals are also good musicians. Here is a striking example. Pierre Boulez wanted to draw a precise frontier between a synthetic sound and a human voice. He challenged Xavier Rodet to synthesize the Queen of the Night's grand aria from Mozart's Magic Flute. The result was not a copy of a human voice; it involved the creation of a purely numerical voice. Rodet decided to model audio signals as linear combinations of elementary waveforms [5], [7]. A waveform is a sinusoidal signal multiplied by a window. The role of the window is to model the attack and the decay of the performed note. The frequency of the sinusoid is the frequency of the musical note. The first solution proposed by Rodet was fine from the point of view of Fourier analysis but the reconstructed voice sounded completely artificial. Pierre Boulez complained that this synthesized voice was *plastic junk*. This synthesized voice was a sampling of the waveform model designed by Rodet. *The samples from many models that are used in practice are absurd oversimplifications of real signals*, as Mumford and Desolneux stress. Rodet improved on this model. He found that a human voice does not jump immediately onto the correct note. There are some transient fluctuations around the correct frequency. These transient fluctuations, which last a fraction of a second, should be carefully modelled. Then the synthesized voice no longer sounded plastic. Here comes the great news. In the chapter entitled *Music and Piecewise Gaussian Models*, Mumford and Desolneux are telling us the same story. They agree with Pierre Boulez and Xavier Rodet. Randomness should be incorporated in the model. The synthesized audio signal sounds like plastic junk unless it fluctuates around the frequencies indicated on the score of the music. This example illustrates the top-down stage where the synthesized signal is compared to the input signal. As the authors say: "*What needs to be checked is whether the input signal agrees with the synthesized signal to within normal tolerances, or whether the residual is so great that the input signal has not been correctly or fully analyzed … This framework uses signal synthesis in an essential way, and this requirement for feedback gives an intriguing relation to the known properties of mammalian cortex architecture*."

Here the authors indirectly give us some hints about the processing that is achieved by the human brain. One cannot draw a precise frontier between signal or image processing and the cognitive sciences. Every advance in image processing provides us with new clues about the functioning of our own brain. In a lecture at the Universidad Autonoma de Madrid, given in 1997, David H. Hubel provided us with a vision of the future of science [3]. He said that: "*Understanding the human brain is one of the greatest challenges that scientists face. It is surely the most complex machine in the known universe. It has commonly been thought that understanding the brain is a hopeless quest, since the main tool we have to work with is our own brain. I have never seen any merit in this logic.*"

What singles out this outstanding book is an extremely original subject development. Each chapter begins with a motivating problem. The reader is not handed a recipe; instead, he feels as if he were discussing with the authors and elaborating a model for the problem that is being addressed. The reader is eventually convinced that the scientific tools that emerge from the discussion are the most appropriate. The proposed models are not learned from the data. As in the Pierre Boulez case, the models are eventually improved by sampling and confrontation with the input signal. On the other hand, the authors do not show any methodological prejudice in the choice of a given model. Instead, they often confess the limitations or drawbacks of a proposed model. I*n Music and Piecewise Gaussian Models* (Chapter 2) they acknowledge that: "*The results, however, are not very convincing – they give a totally atonal, unrhythmic "music", but we can certainly construct various models for music of increasing sophistication that sound better*."

This book is so exciting. It is a detective fiction. It is an inquiry into "real world signals". In contrast to most detective stories, the beauty of the style is exceptional and meets the standards of the best writers. Art and beauty are present everywhere in this marvellous book. The reader is invited to admire an inscribed slab from the palace of *Sargon II* in Dur-Sharrukin, Khorsabad. They will be enthralled by the beauty of *Paolina Borghese*, which illustrates Chapter 5. *The Hunt by Night* by Paolo Uccello is a fascinating painting that illustrates the scaling properties of images. Winter 711 by Jan Beran was a true discovery.

The Hunt by Night by Paolo Uccello

Jackson Pollock's painting entitled *convergence* is compared to samples of a random wavelets model.

By the last page of a detective fiction we know everything. Here it is quite the opposite. The mysterious art of modelling is more and more enthralling as one continues reading this fascinating book. There are surprises everywhere. For example, the reader will be rewarded by learning on page 314 how flexible templates can be used to smoothly deform the image of a giraffe into that of a hippopotamus.

Convergence by Jackson Pollock

The overall organisation of the book is also marvellous. It is a crescendo. Each chapter is motivated by one of the "real world signals" of the above list. The mathematics used in processing this particular signal will show up again in the following chapters and be applied to other signals. Chapter 4 offers a large and beautiful scientific panorama. The deep connection between Ising models (used to describe the magnetization of a lattice of iron) and image segmentation is unveiled here with unprecedented clarity. Although I had already heard some pieces of this story in talks by Robert Azencott, I was still captivated.

The last chapter, entitled *Natural Scenes and their Multiscale Analysis*, is enthralling. It opens one of the most exciting scientific programmes in cognitive sciences. As David Field claims: *"The main thrust of this paper* ([2]) *is that images from the natural environment should not be presumed to be random patterns. Such images show a number of consistent statistical properties. In this paper we suggest that a knowledge of these statistics can lead to a better understanding of why the mammalian visual system codes information as it does.*"

In [6], D. Field and B. Olshausen write: "*We show that a learning algorithm that attempts to find sparse linear codes for natural scenes will develop a complete family of localized, oriented, bandpass receptive fields, similar to those found in the primary visual cortex. The resulting sparse image code provides a more efficient representation for later stages of processing because it possesses a higher degree of statistical independence among its outputs.*"

We are back to the fascinating relation between image processing and the processing achieved by the primary visual cortex. Did evolution lead to an optimal processing by the primary visual cortex? The optimality is defined by sparseness in [6]. As the authors observe in [4] *the principle of sparse coding could be expressed by the property that a given neuron is activated only rarely*. Field and Olshausen applied an independent component analysis to a collection of natural scenes images and discovered that the basic functions that emerged are the patterns to which the neurons discovered by Hubel and Wiesel maximally respond.

This book is ideal for a graduate course. It does not require extensive prerequisites in mathematics or signal processing. The necessary mathematical facts are explained with much care and clarity. Some appealing exercises conclude each chapter. Valuable software is also provided. As is stressed by the authors, stochastic models are crucially needed for taking into account the variability of "real world signals". Thus, the reader is supposed to have followed a basic course in probability theory and should have, at least, caught the spirit of stochastic modelling. But what is technically needed is recalled in the book.

The authors are leaders in signal and image processing and this book is based on their extremely innovative research. Reading this book is like entering David Mumford's office and beginning a friendly and informal scientific discussion with him and Agnès. That is a good approximation to paradise.

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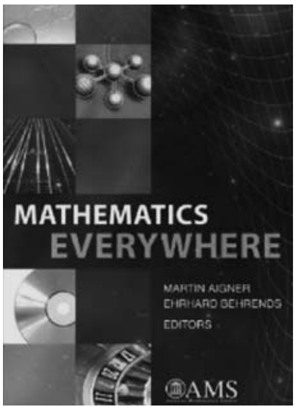

Mathematics Everywhere

Martin Aigner and Ehrhard Behrends (eds.)

AMS, 2010
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Adhemar Bultheel

The Urania Society in Berlin has a history that goes back to Alexander von Humboldt who gave in 1827/1828 public "cosmos lectures" intended for a general public. Urania became a formal society in 1888 with in its statutes the paradigm of "Spreading the knowledge, achievements, and joy of (the 'new') Sciences". Today it has over 2000 members and is one of the oldest and largest non-profit societies residing in Berlin.

Urania, Berlin

Besides a successful film-festival, and many other activities, one of its initiatives is, as it was at the start, still to organize generally understandable lectures concerning current questions of nature and *Geisteswissenschaften*. By 1990, the lectures treated all kinds of subjects but "there wasn't a single one dealing with mathematics". The classical false premises were used: that mathematics are "too abstract, too dry, and too hard" for the layman. In a world where "mathematics are everywhere", it was decided that this should change. By 2000 some fifty lectures had been discussing mathematical topics, and although they were about mathematics (and that includes also the equations and formulas!) they were presented in a lively way, explaining sometimes difficult mathematical topics using a step by step approach and illustrating them in an environment of every-day life or presenting them in a story-telling format or a gaming situation. The original German version of this book was entitled *Alles Mathematik* and appeared in 2000 (Vieweg). It contained a selection of elaborated texts of some of the lectures that were given in the Urania initiative. In subsequent second (2002) and third (2008) German editions new lectures were added.

Kepler

Fermat

Poincaré

This English translation contains 21 chapters, each one written by well known researchers. They are organized in three groups: 'Case Studies', 'Current Topics', and 'The Central Theme'. There is a 'Prologue' by a science journalist (G. von Randow) and an 'Epilogue' by a mathematician-philosopher (Ph. J. Davis).

The prologue has an author that is obviously convinced of the "joy of mathematics", and that mathematics gains a booming popularity in party-conversations. It is the reviewer's experience though that this is still the privilege of a happy-few enthusiasts, and that in most cases it is still a no-go zone if you want to socialize with non-mathematicians.

The epilogue is an interesting read. It gives excerpts of a lecture given in 1998 at the International Mathematical Congress in Berlin and discusses "The prospects for mathematics in a multi-media civilization" but the message is broader than just the multi-media aspects. Twelve years later, it is quite interesting to (re-)read it and see how much of the content has been realized and how much has faded away.

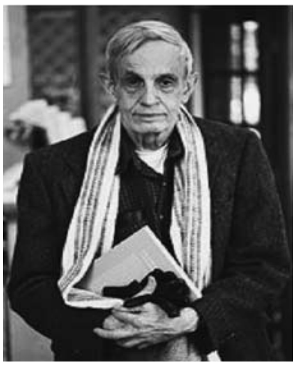
The other chapters, being written by different authors, have different styles and lengths and they also differ in the amount of the mathematical details. In the group of "case studies" we find some topics that are somewhat predictable like the encoding of CDs, different aspects of image processing in medical applications, shortest path and other graph theoretical problems and their applications. But there are also some chapters that I didn't encounter before as being popularizing math topics like Turing instability and spontaneous pattern forming phenomena in nonlinear dynamical systems. The nice thing about this chapter is that the reader is brought a long way by an analogy with the love-life of Romeo and Juliet and their twin-siblings Roberto and Julietta. Similarly, computer tomography is introduced using the game of battleship (where one has to find out blindly where the opponent has placed his battleships on a grid) while the chapter eventually becomes involved in nanotechnology. The chapter about "intelligent materials" stays at the surface of mathematics, and so does the chapter on reflections of hinged mirrors, spherical mirrors and hyperbolic geometry, but the latter of course can be very nicely illustrated. Being the written summary of lively presentations, it is clear that all chapters have ample occasion of visualizing illustrations.

The group of chapters on "current topics" aren't too much different. There is one on the role of mathematics in the financial markets, and this isn't inspired by the recent global crisis, but treats things such as the role of stochastics, arbitrage, and the Black–Scholes formula.

Fischer Black

Myron Scholes

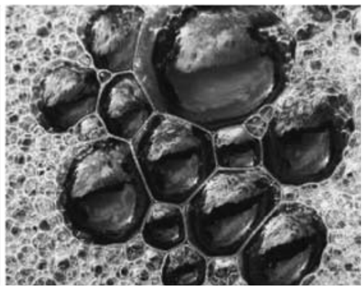
The next chapter deals with electronic money. After all, coins and bank notes are just symbols that do not have a value as such. Similarly electronic money is just a string of bits. So the problem is to distinguish between strings that are just information and others that have economic value just like money, which brings the reader to the subject of cryptography. The huge computational challenge lying in the simulation of the global dynamical system that leads to climate change is another such topic that fits into a decor of a changing world leading to catastrophic effects in an ever faster succession. Another chapter is about sphere packing, which is a more entertaining subject, but yet has lead to a new and deep mathematical machinery in a sequence of efforts to solve Kepler's conjecture. If spheres are packed like we see them piled up on display at fruit markets, then the density of the space covered by the spheres is $\pi/\sqrt{18}$. Kepler conjectured that one cannot do better. The chapter is dealing with the history of this conjecture and even formulates theorems and uses formulas. It brings us up to the computer proof of Hales in 1998. Other mathematical problems with a long history are dealt with in a chapter on Fermat's last theorem and one about the Nash equilibrium. The remaining chapter in this group is a short one on quantum computing.



John Forbes Nash

The latter ties up with the first of the next five chapters that are classified in the group "central theme". It explains how huge prime numbers, or rather the factorization of huge numbers into its prime factors, forms the heart of our current cryptosystems, but if ever we succeed in getting a quantum computer to work, then we break down the bounding walls of current computability and hence another basis for cryptography will have to be invented. The next two chap-

Sphere packing



Soap bubbles

ters, although they have serous applications and involve some good mathematics, will probably be perceived by a broader audience as being related to mathematical recreation. The first gives some insight into knot theory. This is often what is needed to design or solve some three-dimensional puzzles. Also the other chapter on the geometry and physics of soap bubbles is a fun-subject for many. Not so remote from these subjects is the much more fundamental subject of the structure of space and the Poincaré conjecture (every closed simply connected three-dimensional space is topologically equivalent to a three-dimensional sphere). More generally it gives the complete classification of *all* three-dimensional spaces as elaborated by Thurston, Hamilton and Perelman, and this depends on the theory of heat diffusion. This is a relatively long chapter which dives a bit deeper into the mathematics. The final chapter in this group is about the roots and applications of probability, which entered mathematics at a rather late stage of its evolution.

A most entertaining read including some nontrivial mathematics intended for a broad audience but most enjoyable for mathematicians as well.

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Solved and Unsolved Problems

Themistocles M. Rassias (Athens, Greece)

If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?
David Hilbert (1862–1943)

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

83. Prove that

$$\sum_{n=2}^{+\infty} f(\zeta(n)) = 1\,,$$

where $f(x) = x - \lfloor x \rfloor$ denotes the fractional part of $x \in \mathbb{R}$ and $\zeta(s)$ is the Riemann zeta function.

(Hari M. Srivastava, Department of Mathematics and Statistics, University of Victoria, Canada)

84. A continuous function $f : [0,1] \to \mathbb{R}$ possesses the following property: every point $x \in [0,1]$ is either a point of local minimum or a point of local maximum for f. Is it true that f is identically a constant?

(Alexander Kochurov, Department of Mechanics and Mathematics, Moscow State University, Russia)

85. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}$ that satisfy the functional equation

$$f(pr, qs) + f(ps, qr) = (r+s)f(p,q) + (p+q)f(r,s)$$

for all $p, q, r, s \in \mathbb{R}_+$. Here $\mathbb{R}_+$ is the set of positive real numbers.

(Prasanna K. Sahoo, Department of Mathematics, University of Louisville, USA)

86. Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the functional equation

$$\begin{aligned} f(u+x, v+y) + f(u-x, v) + f(u, v-y) \\ = f(u-x, v-y) + f(u+x, v) + f(u, v+y) \end{aligned}$$

for all $x, y, u, v \in \mathbb{R}$.

(Prasanna K. Sahoo, Department of Mathematics, University of Louisville, USA)

87. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers m, M with $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, prove that the following inequality holds:

$$\begin{aligned} &\left|\left\langle \left[f(A) - \left(\frac{1}{M-m} \int_m^M f(s)\,ds \right) 1_H \right] x, y \right\rangle\right| \\ &\le \frac{1}{M-m} \bigvee_m^M (f) \max_{t \in [m,M]} \Big[(M-t) \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} \\ &\quad + (t-m) \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} \Big] \le \|x\| \|y\| \bigvee_m^M (f) \end{aligned}$$

for any $x, y \in H$, where $\bigvee_m^M (f)$ denotes the total variation of f on $[m, M]$ and 1_H is the identity operator on H.

(Sever S. Dragomir, Mathematics, School of Engineering and Science, Victoria University, Australia)

88. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers m, M with $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, prove that the following inequality holds:

$$\begin{aligned} &\left|\left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A) \right] x, y \right\rangle\right| \\ &\le \sup_{t \in [m,M]} \left[\frac{t-m}{M-m} \bigvee_m^t (\langle E_{(\cdot)} x, y\rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y\rangle) \right] \bigvee_m^M (f) \\ &\le \bigvee_m^M (\langle E_{(\cdot)} x, y\rangle) \bigvee_m^M (f) \le \|x\| \|y\| \bigvee_m^M (f) \end{aligned}$$

for any $x, y \in H$, where $\bigvee_m^M (f)$ denotes the total variation of f on $[m, M]$ and 1_H is the identity operator on H.

(Sever S. Dragomir, Mathematics, School of Engineering and Science, Victoria University, Australia)

II Two new open problems

89*. **Conjecture**. Let n and p be two positive integers such that $n \equiv p \pmod 2$.

(a) The identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k cos^p \left(\frac{k\pi}{n} \right) = \frac{1}{2}$$

is true if and only if $n > p$.

(b) The inequality

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k cos^p \left(\frac{k\pi}{n} \right) > \frac{1}{2}$$

is true if and only if $n \le p$.

(Mircea Merca, "Constantin Istrati" Technical College, Câmpina, Romania)

90*. For an even natural number n denote by Q_n the set of algebraic polynomials of degree n that have real coefficients, have all roots in the disc

$$\{z \in \mathbb{C} \mid |z| \le 1\}$$

and have no real roots. For every n find

$$\inf_{p \in Q_n} \left\| \frac{p'}{p} \right\|_2$$

where

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 d\, x \right)^{1/2}$$

is the $L_2(\mathbb{R})$-norm.

(Vladimir Protasov, Department of Mechanics and Mathematics, Moscow State University, Russia)

III Solutions

75. Prove that for any integer $k \ge 1$ the equation

$$x_1^2 + x_2^2 + \cdots + x_{2k+1}^2 + 1 = x_{2k+2}^2$$

has infinitely many solutions in positive integers.

(Dorin Andrica, "Babes-Bolyai" University, Cluj-Napoca, Romania)

Solution by the proposer. Note that for any real number m the following relations hold:

$$\begin{aligned} &m^{2(2k)} + m^{2(2k-1)} + \cdots + m^2 + 1 \\ &\quad = \frac{(m^2)^{2k+1} - 1}{m^2 - 1} \\ &\quad = \frac{(m^{2k+1} - 1)(m^{2k+1} + 1)}{m^2 - 1} \\ &\quad = (m^{2k} + m^{2k-1} + \cdots + m + 1)(m^{2k} - m^{2k-1} + \cdots - m + 1) \\ &\quad = (m^{2k} + m^{2k-2} + \cdots + m^2 + 1)^2 - (m^{2k-1} + m^{2k-3} + \cdots + m)^2 . \end{aligned}$$

We obtain

$$\begin{aligned} m^{2(2k)} + m^{2(2k-1)} + \cdots + m^2 + (m^{2k-1} + m^{2k-3} + \cdots + m)^2 + 1 \\ = (m^{2k} + m^{2k-2} + \cdots + m^2 + 1)^2 , \end{aligned}$$

that is

$$\begin{aligned} m^2 + (m^2)^2 + \cdots + (m^{2k-1})^2 + (m^{2k})^2 + (m^{2k-1} + m^{2k-3} + \cdots + m)^2 + 1 \\ = (m^{2k} + m^{2k-2} + \cdots + m^2 + 1)^2 . \end{aligned}$$

An infinite family of solutions in positive integers to our equation is given by

$$\begin{gathered} x_1 = m, x_2 = m^2, \cdots , x_{2k-1} = m^{2k-1}, x_{2k} = m^{2k}, \\ x_{2k+1} = m^{2k-1} + m^{2k-3} + \cdots + m, \\ x_{2k+2} = m^{2k} + m^{2k-2} + \cdots + m^2 + 1 , \end{gathered}$$

where m is an arbitrary positive integer.

Remarks.

(1) The property in the problem shows that for any integer $k \ge 1$, on the quadratic hypersurface defined by the equation

$$u_1^2 + u_2^2 + \cdots + u_{2k+1}^2 + 1 = u_{2k+2}^2$$

in the Euclidean space $\mathbb{R}^{2k+2}$, there are infinitely many integer points, i.e. points having integer coordinates.

(2) The equation in the problem is different to the extended Pythagorean equation

$$x_1^2 + x_2^2 + \cdots + x_k^2 = x_{k+1}^2 ,$$

which can be completely solved in the set of integers. In this respect we refer to the new book of T. Andreescu, D. Andrica and I. Cucurezeanu [*An Introduction to Diophantine Equations. A Problem-Based Approach*, Birkhäuser, 2010, pp. 81–82]. An interesting problem is to find all positive integer solutions to the equation in our problem.

Also solved by Mihály Bencze (Brasov, Romania), W. Fensch (Karlsruhe, Germany) and Jerzy Malopolski (Department of Agricultural Engineering and Informatics, Kraków, Poland).

76. Solve the equation

$$\lfloor 3x - 2 \rfloor - \lfloor 2x - 1 \rfloor = 2x - 6 , \; x \in \mathbb{R} .$$

(Elias Karakitsos, Sparta, Greece)

Solution by the proposer. Set $\lfloor 3x - 2 \rfloor = a$, where $a \in \mathbb{Z}$. Then, it follows that there exists ϑ_1, with $0 \le \vartheta_1 < 1$, such that

$$3x - 2 - \vartheta_1 = a$$

or

$$x = \frac{a + \vartheta_1 + 2}{3} .$$

Set $\lfloor 2x - 1 \rfloor = b$, where $b \in \mathbb{Z}$. Then, it follows that there exists ϑ_2, with $0 \le \vartheta_2 < 1$, such that

$$2x - 1 - \vartheta_2 = b$$

or

$$x = \frac{b + \vartheta_2 + 1}{2} .$$

Therefore,

$$a - b = 2x - 6$$

or

$$x = \frac{a - b + 6}{2} .$$

Thus,

$$\frac{a + \vartheta_1 + 2}{3} = \frac{b + \vartheta_2 + 1}{2} = \frac{a - b + 6}{2} .$$

Hence,

$$2a + 2\vartheta_1 + 4 = 3a - 3b + 18$$

or

$$\vartheta_1 = \frac{a - 3b + 14}{2} .$$

Similarly

$$\frac{b + \vartheta_2 + 1}{2} = \frac{a - b + 6}{2}$$

or

$$b + \vartheta_2 + 1 = a - b + 6$$

or

$$\vartheta_2 = a - 2b + 5 .$$

Since $0 \le \vartheta_1 < 1$, we obtain

$$0 \le \frac{a - 3b + 14}{2} < 1$$

or

$$-14 \le a - 3b < -12 .$$

Thus

$$a - 3b = -14 \text{ or } a - 3b = -13 .$$

Furthermore, since $0 \le \vartheta_2 < 1$, we obtain

$$0 \le a - 2b + 5 < 1$$

or

$$-5 \le a - 2b < -4$$

or

$$a - 2b = -5\,.$$

By solving the systems of equations

$$a - 3b = -14$$
$$a - 2b = -5$$

and

$$a - 3b = -13$$
$$a - 2b = -5$$

it follows that $a = 13$, $b = 9$ and $a = 11$, $b = 8$, respectively.

The above solutions are acceptable, since a and b are integer numbers.

- For $a = 13$, $b = 9$ the solution of the initial equation is

$$x = \frac{a - b + 6}{2} = \frac{10}{2} = 5.$$

- For $a = 11$, $b = 8$ the solution of the initial equation is

$$x = \frac{a - b + 6}{2} = \frac{9}{2} = 4.5.$$

Therefore, the real solutions of the equation are the numbers 4.5 and 5.

Also solved by Mihály Bencze (Brasov, Romania), W. Fensch (Karlsruhe, Germany), Alberto Facchini (University of Padova, Italy), P. T. Krasopoulos (Athens, Greece) and Said El Aidi.

77. Find all positive integers n with the following property: there are two divisors a and b of the number n such that $a^2 + b^2 + 1$ is a multiple of n.

(Vladimir Protasov, Department of Mechanics and Mathematics, Moscow State University, Russia)

Solution by the proposer. Without loss of generality it can be assumed that $a \ge b > 0$. If a and b have a common divisor d with $d \ge 2$ then $a^2 + b^2 + 1$ is a multiple of d, hence 1 is a multiple of d, which is impossible. Thus, a and b are co-prime. This yields that as n is divisible by a and by b, it is divisible by ab. Whence,

$$a^2 + b^2 + 1 \;=\; p\,ab \qquad (1)$$

for some natural number p. In the case $a = b$ we have $2a^2 + 1$ is a multiple of a, therefore $a = 1$. Consequently, $(a, b) = (1, 1)$ and $p = 3$. Assume now $a > b$. Equation (1) considered as a quadratic equation in a has a positive integer solution; by Viéte's laws, its second solution

$$a' = pb - a = \frac{b^2 + 1}{a}$$

is also a positive integer. Moreover,

$$a' = \frac{b^2 + 1}{a} \le \frac{b^2 + 1}{b + 1} \le b < a.$$

Thus, if equation (1) has a solution (a, b) with $a > b$ then it also has another solution (b, a') with a strictly smaller sum of numbers. Applying the same argument to this new solution, then to the next solution, etc., we eventually arrive at a pair (a_0, b_0) that cannot be further reduced. Hence,

$$a_0 = b_0 = 1,$$

and thus $p = 3$.

Thus, starting with an arbitrary solution one can descend to the pair $(1, 1)$. Therefore, all possible solutions (a, b) of (1) are contained in the infinite chain $(1, 1) \to (2, 1) \to (5, 2) \to (13, 5) \to (34, 13) \to \ldots$. The transfer to the next pair is by the rule: $(x, y) \to (3x - y, x)$. It can easily be observed and then proved rigorously by induction that the kth pair (the pair $(1, 1)$ has number 0) consists of numbers (u_{2k+1}, u_{2k-1}), where $\{u_i\}_{i=1}^{\infty}$ is the Fibonacci sequence. Finally, since n is a multiple of ab and is a divisor of $3ab$, it follows that either $n = ab$ or $n = 3ab$. Thus,

$$n = u_{2k-1}u_{2k+1} \text{ and } n = 3u_{2k-1}u_{2k+1},\; k \in \mathbb{N},$$

where $\{u_i\}_{i=1}^{\infty}$ is the sequence of Fibonacci numbers:

$$u_1 = 1, u_2 = 1, u_{i+1} = u_i + u_{i-1}, \text{ where } i \ge 2.$$

Also solved by Mihály Bencze (Brasov, Romania), W. Fensch (Karlsruhe, Germany) and Baku Qafqaz University Problem Group.

78. Let n be a nonnegative integer. Find the closed form of the sums

$$S_1(n) = \sum_{k=0}^{n} \left\lfloor \frac{k^2}{12} \right\rfloor \quad \text{and} \quad S_2(n) = \sum_{k=0}^{n} \left[\frac{k^2}{12} \right],$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x and $[x]$ denotes the nearest integer to x, i.e. $[x] = \left\lfloor x + \frac{1}{2} \right\rfloor$.

(Mircea Merca, "Constantin Istrati" Technical College, Câmpina, Romania)

Solution by the proposer. Let

$$s(n) = \sum_{i=0}^{5} \left\lfloor \frac{(n-i)^2}{12} \right\rfloor.$$

Considering that for every nonnegative integer number k we have

$$\left[\frac{k^2}{12} \right] = \begin{cases} \left\lfloor \frac{k^2}{12} \right\rfloor + 1 & \text{if } k - 12\left\lfloor \frac{k}{12} \right\rfloor \in \{3, 9\} \\ \left\lfloor \frac{k^2}{12} \right\rfloor & \text{otherwise} \end{cases}$$

it follows that

$$\sum_{i=0}^{5} \left[\frac{(n-i)^2}{12} \right] = 1 + s(n)\,.$$

After some simple algebraic calculations made in Maple (computer system algebra) we obtain:

$$s(6k) = 18k^2 - 15k + 3 = \frac{(6k-2)(6k-3)}{2},$$
$$s(6k+1) = 18k^2 - 9k + 1 = \frac{(6k-1)(6k-2)}{2},$$
$$s(6k+2) = 18k^2 - 3k = \frac{6k(6k-1)}{2},$$
$$s(6k+3) = 18k^2 + 3k = \frac{6k(6k+1)}{2},$$
$$s(6k+4) = 18k^2 + 9k + 1 = \frac{(6k+1)(6k+2)}{2},$$
$$s(6k+5) = 18k^2 + 15k + 3 = \frac{(6k+2)(6k+3)}{2}.$$

Therefore we deduce that $s(n) = \binom{n-2}{2}$. The relations

$$S_1(n) = s(n) + S_1(n-6)\,, \quad S_2(n) = 1 + s(n) + S_2(n-6)$$

allow us to obtain

$$S_1(n) = \sum_{k=0}^{\lfloor \frac{n}{6} \rfloor - 1} s(n-6k) + S_1\left(n - 6\left\lfloor \frac{n}{6} \right\rfloor\right),$$

$$S_2(n) = \left\lfloor \frac{n}{6} \right\rfloor + \sum_{k=0}^{\lfloor \frac{n}{6} \rfloor - 1} s(n-6k) + S_2\left(n - 6\left\lfloor \frac{n}{6} \right\rfloor\right).$$

Taking into account

$$S_1\left(n - 6\left\lfloor \frac{n}{6} \right\rfloor\right) = \left(\left[\frac{n}{6}\right] - \left\lfloor \frac{n}{6} \right\rfloor\right) s\left(n - 6\left\lfloor \frac{n}{6} \right\rfloor\right),$$
$$S_2\left(n - 6\left\lfloor \frac{n}{6} \right\rfloor\right) = \left(\left[\frac{n}{6}\right] - \left\lfloor \frac{n}{6} \right\rfloor\right) + s\left(n - 6\left\lfloor \frac{n}{6} \right\rfloor\right)$$

we deduce that

$$S_1(n) = \sum_{k=0}^{[\frac{n}{6}]-1} s(n-6k), \quad S_2(n) = \left[\frac{n}{6}\right] + S_1(n).$$

Let (a_k), (b_k) and (c_k) be real number sequences so that

$$\sum_{i=0}^{k} s(n-6i) = a_k n^2 - b_k n + c_k.$$

The following recurrent relations are obtained:

$$a_{k+1} = a_k + \frac{1}{2}, \qquad a_0 = \frac{1}{2},$$
$$b_{k+1} = b_k + 6k + \frac{17}{2}, \qquad b_0 = \frac{5}{2},$$
$$c_{k+1} = c_k + 18k^2 + 51k + 36, \qquad c_0 = 3.$$

We solve these recurrent relations in Maple and we obtain:

$$a_k = \frac{1}{2}(k+1),$$
$$b_k = \frac{1}{2}(k+1)(6(k+1)-1),$$
$$c_k = \frac{1}{2}(k+1)(12(k+1)^2 - 3(k+1) - 3).$$

Thus we derive:

$$S_1(n) = \frac{1}{2}\left[\frac{n}{6}\right]\left(n^2 - \left(6\left[\frac{n}{6}\right] - 1\right)n + 12\left[\frac{n}{6}\right]^2 - 3\left[\frac{n}{6}\right] - 3\right).$$

Using Maple again we obtain:

$$S_1(6k) = \frac{1}{36}(6k)^3 + \frac{1}{24}(6k)^2 - \frac{1}{4}(6k),$$
$$S_1(6k+1) = \frac{1}{36}(6k+1)^3 + \frac{1}{24}(6k+1)^2 - \frac{1}{4}(6k+1) + \frac{13}{72},$$
$$S_1(6k+2) = \frac{1}{36}(6k+2)^3 + \frac{1}{24}(6k+2)^2 - \frac{1}{4}(6k+2) + \frac{1}{9},$$
$$S_1(6k+3) = \frac{1}{36}(6k+3)^3 + \frac{1}{24}(6k+3)^2 - \frac{1}{4}(6k+3) - \frac{3}{8},$$
$$S_1(6k+4) = \frac{1}{36}(6k+4)^3 + \frac{1}{24}(6k+4)^2 - \frac{1}{4}(6k+4) - \frac{4}{9},$$
$$S_1(6k+5) = \frac{1}{36}(6k+5)^3 + \frac{1}{24}(6k+5)^2 - \frac{1}{4}(6k+5) - \frac{19}{72}.$$

Therefore we make the following deduction

$$S_1(n) = \frac{1}{36}n^3 + \frac{1}{24}n^2 - \frac{1}{4}n + r_1(n), \quad |r_1(n)| < \frac{1}{2}$$

and

$$S_2(n) = \frac{1}{36}n^3 + \frac{1}{24}n^2 - \frac{1}{12}n + r_2(n), \quad r_2(n) = r_1(n) - \frac{n}{6} + \left[\frac{n}{6}\right].$$

We will demonstrate that $|r_2(n)| < \frac{1}{2}$. We have

$$r_2(6k) = r_1(6k) = 0,$$
$$r_2(6k+1) = r_1(6k+1) - \frac{1}{6} = \frac{13}{72} - \frac{1}{6} = \frac{1}{72},$$
$$r_2(6k+2) = r_1(6k+2) - \frac{2}{6} = \frac{1}{9} - \frac{1}{3} = -\frac{2}{9},$$
$$r_2(6k+3) = r_1(6k+3) - \frac{3}{6} + 1 = -\frac{3}{8} + \frac{1}{2} = \frac{1}{8},$$
$$r_2(6k+4) = r_1(6k+4) - \frac{4}{6} + 1 = -\frac{4}{9} + \frac{1}{3} = -\frac{1}{9},$$
$$r_2(6k+5) = r_1(6k+5) - \frac{5}{6} + 1 = -\frac{19}{72} + \frac{1}{6} = -\frac{7}{72}.$$

Because $S_1(n)$ and $S_2(n)$ are nonnegative integer numbers we deduce that

$$S_1(n) = [S_1(n) - r_1(n)] = \left[\frac{1}{36}n^3 + \frac{1}{24}n^2 - \frac{1}{4}n\right]$$

and

$$S_2(n) = [S_2(n) - r_2(n)] = \left[\frac{1}{36}n^3 + \frac{1}{24}n^2 - \frac{1}{12}n\right].$$

Also solved by S. E. Louridas (Athens, Greece)

79. Let $m, s \geq 2$ be even integers. Compute

$$\prod_{\substack{k=1 \\ k \not\equiv 0 (mod\ m)}}^{sm-1} \cos\frac{k\pi}{sm}.$$

(Dorin Andrica, "Babes-Bolyai" University, Cluj-Napoca, Romania)

Solution by the proposer. We have

$$x^{(s-1)m} + \cdots + x^{2m} + x^m + 1 = \frac{x^{sm}-1}{x^m-1} = \prod_{\substack{k=1 \\ k \not\equiv 0 \pmod m}}^{sm-1} \left(x - \cos\frac{2k\pi}{sm} - i\sin\frac{2k\pi}{sm}\right).$$

For $x = -1$ we get

$$s = \prod_{\substack{k=1 \\ k \not\equiv 0 \pmod m}}^{sm-1} \left(-1 - \cos\frac{2k\pi}{sm} - i\sin\frac{2k\pi}{sm}\right)$$
$$= \prod_{\substack{k=1 \\ k \not\equiv 0 \pmod m}}^{sm-1} (-1)\left(1 + \cos\frac{2k\pi}{sm} + i\sin\frac{2k\pi}{sm}\right)$$
$$= \prod_{\substack{k=1 \\ k \not\equiv 0 \pmod m}}^{sm-1} (-1)\left(2\cos^2\frac{k\pi}{sm} + 2i\sin\frac{k\pi}{sm}\cos\frac{k\pi}{sm}\right)$$
$$= \prod_{\substack{k=1 \\ k \not\equiv 0 \pmod m}}^{sm-1} (-1)2\cos\frac{k\pi}{sm}\left(\cos\frac{k\pi}{sm} + i\sin\frac{k\pi}{sm}\right).$$

There are $s-1$ multiples of m between 1 and $sm-1$, hence the number of factors in the above product is

$$sm - 1 - (s-1) = s(m-1).$$

The sum of all integers between 1 and $sm-1$ that are not a multiple of m is

$$\begin{aligned}(1+2+\cdots+(sm-1))-(m+2m+\cdots+(s-1)m)\\ &= \frac{1}{2}(sm-1)sm-\frac{1}{2}m(s-1)s\\ &= \frac{1}{2}sms(m-1).\end{aligned}$$

Replacing, we obtain

$$s = 2^{s(m-1)}(-1)^{\frac{1}{2}s(m-1)} \prod_{\substack{k=1\\ k\not\equiv 0 \pmod m}}^{sm-1} \cos\frac{k\pi}{sm}.$$

Therefore

$$\prod_{\substack{k=1\\ k\not\equiv 0 \pmod m}}^{sm-1} \cos\frac{k\pi}{sm} = \frac{(-1)^{\frac{1}{2}s(m-1)}s}{2^{s(m-1)}}.$$

Also solved by Mihály Bencze (Brasov, Romania).

80. Let the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ be defined (for $\Re(s)>1$) by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \text{and}$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \qquad (a\neq 0,-1,-2,\cdots), \qquad (2)$$

respectively, and (for $\Re(s)\le 1;\ s\neq 1$) by their meromorphic continuations.
In the usual notation, let $B_n(x)$ and $E_n(x)$ denote, respectively, the classical Bernoulli and Euler polynomials of degree n in x, defined by the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\,\frac{t^n}{n!} \qquad (|t|<2\pi)$$

and

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\,\frac{t^n}{n!} \qquad (|t|<\pi).$$

It is proposed to show that the values of the Bernoulli polynomials $B_n(\mathrm{x})$ at rational arguments are given, in terms of the Hurwitz zeta function $\zeta(s,a)$, by

$$B_{2n-1}\left(\frac{p}{q}\right)=(-1)^n\frac{2(2n-1)!}{(2\pi q)^{2n-1}}\sum_{j=1}^{q}\zeta\left(2n-1,\frac{j}{q}\right)\sin\left(\frac{2j\pi p}{q}\right)$$
$$(n\in\mathbb{N}\setminus\{1\};\mathbb{N}:=\{1,2,3,\cdots\};\ p\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ q\in\mathbb{N};\ 0\le p\le q)$$

and

$$B_{2n}\left(\frac{p}{q}\right)=(-1)^{n-1}\frac{2(2n)!}{(2\pi q)^{2n}}\sum_{j=1}^{q}\zeta\left(2n,\frac{j}{q}\right)\cos\left(\frac{2j\pi p}{q}\right)$$
$$(n\in\mathbb{N};\ p\in\mathbb{N}_0;\ q\in\mathbb{N};\ 0\le p\le q)\,.$$

Similarly, the values of the Euler polynomials $E_n(\mathrm{x})$ at rational arguments are given, in terms of the Hurwitz zeta function $\zeta(s,a)$, by

$$E_{2n-1}\left(\frac{p}{q}\right)=(-1)^n\frac{4(2n-1)!}{(2\pi q)^{2n}}\sum_{j=1}^{q}\zeta\left(2n,\frac{2j-1}{2q}\right)\cos\left(\frac{(2j-1)\pi p}{q}\right)$$
$$(n\in\mathbb{N};\ p\in\mathbb{N}_0;\ q\in\mathbb{N};\ 0\le p\le q)$$

and

$$E_{2n}\left(\frac{p}{q}\right)=(-1)^n\frac{4(2n)!}{(2\pi q)^{2n+1}}\sum_{j=1}^{q}\zeta\left(2n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)\pi p}{q}\right)$$
$$(n\in\mathbb{N};\ p\in\mathbb{N}_0;\ q\in\mathbb{N};\ 0\le p\le q)\,.$$

(Djurdje Cvijović, Atomic Physics Laboratory, Belgrade, Republic of Serbia, and H. M. Srivastava, Department of Mathematics and Statistics, University of Victoria, Canada)

A Sketch of the Proof: In order to derive the proposed formulas for the Bernoulli polynomials $B_n(x)$, it suffices to recall the known result regarding the Fourier series expansion of $B_n(x)$ (see [5, p. 27] and [8, p. 65]).

$$B_n(x)=-\frac{2n!}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}\cos\left(2k\pi x-\frac{n\pi}{2}\right),$$
$$(n\in\mathbb{N}\setminus\{1\}\ \text{ and }\ 0\le x\le 1;\ n=1\ \text{ and }\ 0<x<1)\,,$$

and to keep the following elementary series identity in mind:

$$\sum_{k=1}^{\infty}f(k)=\sum_{j=1}^{q}\sum_{k=0}^{\infty}f(qk+j), \qquad (q\in\mathbb{N}).$$

It remains now to show that, in view of the following known relationship (see [5, p. 29] and [8, p. 65]):

$$E_n(x)=\frac{2}{n+1}\left[B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right],$$

the proposed formulas for the Euler polynomials $E_n(x)$ are simple consequences of the formulas for the Bernoulli polynomials $B_n(x)$.

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Remark. Problem 70 was also solved by John N. Lillington, (Wareham, UK).

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.

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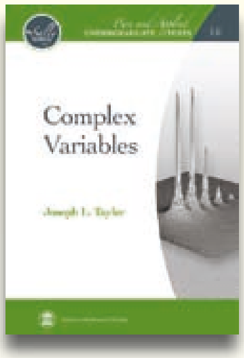

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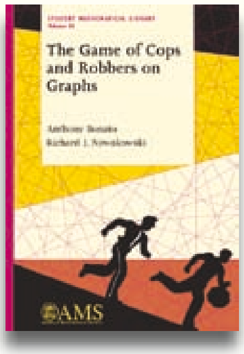

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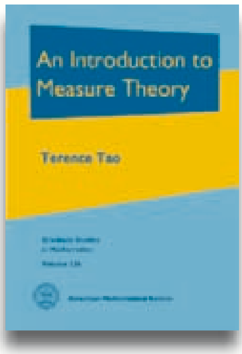

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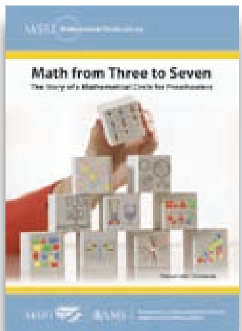

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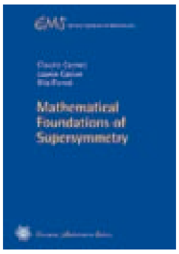

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ISBN 978-3-03719-097-5. 2011. 301 pages. Softcover. 17 x 24 cm. 42.00 Euro

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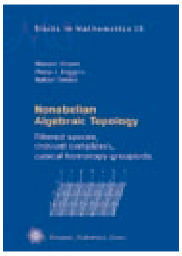
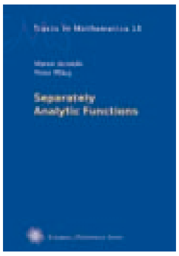
The purpose of the book is to lay down the foundations of the subject, providing the reader with a comprehensive introduction to the language and techniques, with a special attention to giving detailed proofs and many clarifying examples. It is aimed ideally at a second year graduate student. After the first three introductory chapters, the text divides into two parts: the theory of smooth supermanifolds and Lie supergroups, including the Frobenius theorem, and the theory of algebraic superschemes and supergroups. There are three appendices, the first introducing Lie superalgebras and representations of classical Lie superalgebras, the second collecting some relevant facts on categories, sheafification of functors and commutative algebra, and the third explaining the notion of Fréchet space in the super context.



Ronald Brown (Bangor University, UK), Philip J. Higgins (Durham University, UK) and Rafael Sivera (Universitat de València, Spain)
Nonabelian Algebraic Topology. Filtered spaces, crossed complexes, cubical homotopy groupoids (EMS Tracts in Mathematics, Vol. 15)

ISBN 978-3-03719-083-8. 2011. 703 pages. Hardcover. 17 x 24 cm. 98.00 Euro

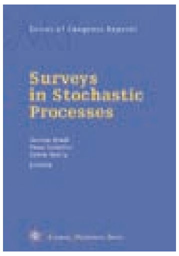
The main theme of this book is that the use of filtered spaces rather than just topological spaces allows the development of basic algebraic topology in terms of higher homotopy groupoids; these algebraic structures better reflect the geometry of subdivision and composition than those commonly in use.
The structure of the book is intended to make it useful to a wide class of students and researchers for learning and evaluating these methods, primarily in algebraic topology but also in higher category theory and its applications in analogous areas of mathematics, physics and computer science. Part I explains the intuitions and theory in dimensions 1 and 2, with many figures and diagrams, and a detailed account of the theory of crossed modules. Part II develops the applications of crossed complexes. The engine driving these applications is the work of Part III on cubical ω-groupoids, their relations to crossed complexes, and their homotopically defined examples for filtered spaces. Part III also includes a chapter suggesting further directions and problems, and three appendices give accounts of some relevant aspects of category theory. Endnotes for each chapter give further history and references.



Marek Jarnicki (Jagiellonian University, Kraków, Poland) and Peter Pflug (University of Oldenburg, Germany)
Separately Analytic Functions (EMS Tracts in Mathematics, Vol. 16)

ISBN 978-3-03719-098-2. 2011. 306 pages. Hardcover. 17 x 24 cm. 58.00 Euro

The story of separately holomorphic functions began about 100 years ago. During the second half of the 19th century, it became known that a separately continuous function is not necessarily continuous as a function of all variables. At the beginning of the 20th century, the study of separately holomorphic functions started due to the fundamental work of Osgood and Hartogs.
This book provides the first self-contained and complete presentation of the study of separately holomorphic functions, starting from its birth up to current research. Most of the results presented have never been published before in book form. The text is divided into two parts. A more elementary one deals with separately holomorphic functions "without singularities", another addresses the situation of existing singularities. A discussion of the classical results related to separately holomorphic functions leads to the most fundamental result, the classical cross theorem as well as various extensions and generalizations to more complicated "crosses".



Jochen Blath (Technische Universität Berlin, Germany), Sylvie Rœlly (Universität Potsdam, Germany) and Peter Imkeller (Humboldt-Universität zu Berlin, Germany), Editors
Surveys in Stochastic Processes (EMS Series of Congress Reports)

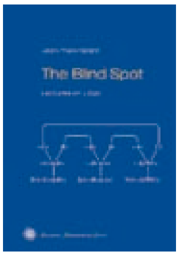
ISBN 978-3-03719-072-2. 2011. 264 pages. Hardcover. 17 x 24 cm. 78.00 Euro

The 33rd Bernoulli Society Conference on "Stochastic Processes and Their Applications" was held in Berlin from July 27 to July 31, 2009. It brought together more than 600 researchers from 49 countries, who communicated recent progress in the mathematical research related to stochastic processes, with applications ranging from biology to statistical mechanics, finance and climatology.
The present volume collects survey articles highlighting new trends and focal points in the area written by plenary speakers of the conference, all of them outstanding international experts. A particular aim of this collection is to inspire young scientists in setting up research goals within the wide scope of fields represented in this volume.

Andrzej Skowroński (Nicolaus Copernicus University, Toruń, Poland) and Kunio Yamagata (Tokyo University of Agriculture and Technology, Japan), Editors
Representations of Algebras and Related Topics (EMS Series of Congress Reports)

ISBN 978-3-03719-101-9. 2011. 744 pages. Hardcover. 17 x 24 cm. 98.00 Euro

This book is concerned with recent trends in the representation theory of algebras and its exciting interaction with geometry, topology, commutative algebra, Lie algebras, combinatorics, quantum algebras, and theoretical physics. The collection of articles, written by leading researchers in the field, is conceived as a sort of handbook providing easy access to the present state of knowledge and stimulating further development.
The book consists of thirteen self-contained expository survey and research articles and is addressed to researchers and graduate students in algebra as well as a broader mathematical community. They contain a large number of examples and open problems and give new perspectives for research in the field.



Jean-Yves Girard (CNRS, Marseille, France)
The Blind Spot. Lectures on Logic

ISBN 978-3-03719-088-3. 2011. 552 pages. Hardcover. 16.5 x 23.5 cm. 68.00 Euro

These lectures on logic, more specifically proof theory, are basically intended for postgraduate students and researchers in logic.
The question at stake is the nature of mathematical knowledge and the difference between a question and an answer, i.e., the implicit and the explicit. The problem is delicate mathematically and philosophically as well: the relation between a question and its answer is a sort of equality where one side is "more equal than the other", and one thus discovers essentialist blind spots.
Starting with Gödel's paradox (1931) – so to speak, the incompleteness of answers with respect to questions – the book proceeds with paradigms inherited from Gentzen's cut-elimination (1935). Various settings are studied: sequent calculus, natural deduction, lambda calculi, category-theoretic composition, up to geometry of interaction (GoI), all devoted to explicitation, which eventually amounts to inverting an operator in a von Neumann algebra.
