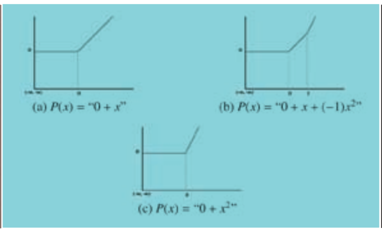
NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Obituary
Daniel Quillen

p. 19



Feature
Tropical Geometry

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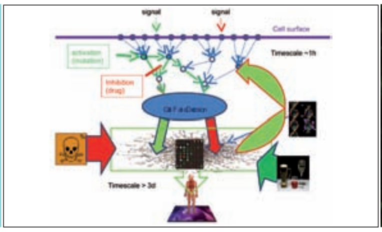

Feature
Mathematics in Health Care

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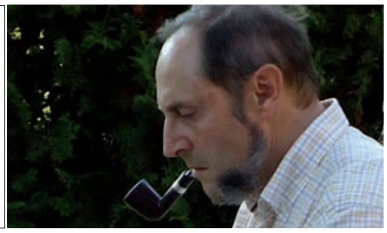

Interview
Tomas Recio

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March 2012
Issue 83
ISSN 1027-488X

European
Mathematical
Society

6th European Congress of Mathematics

July 2–7, 2012 | Kraków, Poland

PLENARY SPEAKERS

Adrian Constantin
UNIVERSITÄT WIEN, AUSTRIA

Camillo De Lellis
UNIVERSITÄT ZÜRICH, SWITZERLAND

Herbert Edelsbrunner
INSTITUTE OF SCIENCE AND TECHNOLOGY AUSTRIA

Mikhail L. Gromov
INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, FRANCE

Christopher Hacon
UNIVERSITY OF UTAH, USA

David Kazhdan
THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL

Tomasz Łuczak
ADAM MICKIEWICZ UNIVERSITY, POZNAŃ, POLAND

Sylvia Serfaty
UNIVERSITÉ PIERRE ET MARIE CURIE – PARIS 6, FRANCE

Saharon Shelah
THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL

Michel Talagrand
UNIVERSITÉ PIERRE ET MARIE CURIE – PARIS 6, FRANCE

EUROPEAN MATHEMATICAL SOCIETY

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6ECM Kraków 2012

European Mathematical Society

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EMS Executive Committee

EMS Agenda

2012

9–10 March
Meeting of the EMS Ethics Committee, Madrid, Spain
Arne Jensen: matarne@math.aau.dk

23–24 March
Meeting of ERCOM, Budapest, Hungary

31 March–1 April
Meeting of presidents of EMS member mathematical societies, Prague, Czech Republic
Stephen Huggett: s.huggett@plymouth.ac.uk

14 April
Meeting of the EMS Committee for Developing Countries, Limoges, France
Tsou Sheung Tsun: tsou@maths.ox.ac.uk

11–14 June
Operator Theory and Mathematical Physics (OTAMP), Barcelona, Spain
www.crm.cat/Activitats/Activitats/2011-2012/OTAMP/web-otamp

18–22 June
CIME-EMS Summer School in applied mathematics, Cetraro, Italy
php.math.unifi.it/users/cime

29–30 June
Executive Committee Meeting, Kraków, Poland
Stephen Huggett: s.huggett@plymouth.ac.uk

30 June–1 July
Council Meeting of European Mathematical Society, Kraków, Poland
www.euro-math-soc.eu

2–7 July
6th European Mathematical Congress, Kraków, Poland
www.6ecm.pl

6–8 July
Pre-world-congress Meeting of Young Researchers in Probability and Statistics 2012 (PWCYPS 2012), Istanbul, Turkey
http://pwc2012.ku.edu.tr/

23–27 July
17th Conference for Mathematics in Industry, ECMI 2012, Lund, Sweden
www.maths.lth.se/ecmi/ecmi2012.org

30 July–3 August
EMS-IAMP Summer School on Quantum Chaos, Erwin Schödinger Institute, Vienna

6–11 August
International Congress on Mathematical Physics, ICMP12 Aalborg, Denmark
www.icmp12.com

19–26 August
The Helsinki Summer School on Mathematical Ecology and Evolution 2012
wiki.helsinki.fi/display/huippu/mathbio2012

27–31 August
5th European Summer School in Financial Mathematics, Paris
www.cmap.polytechnique.fr/~euroschoolmathfi12

Zentralblatt MATH

Gert-Martin Greuel, Editor-in-Chief

If someone had told me two years ago that I would become editor of Zentralblatt MATH, I would not have believed them. Of course I have known Zentralblatt since my time in Göttingen, when I was working on my diploma thesis. Later in Bonn, after my PhD thesis, I reviewed articles for Zentralblatt for several years. I remained a regular user, first of the printed volumes and later of the online version, sometimes a critical one. Maybe that was the reason that I was asked to succeed Bernd Wegner, who had not only worked for 37 years as Editor-in-Chief but had also become the face of Zentralblatt MATH. During his term of office the age of electronic and online information started and it has not yet reached its peak. In fact, the development is extremely dynamic and nobody knows what the digital world will look like in, say, 10 or 20 years from now. In any case, Zentralblatt MATH is aware of the new challenges and ready to face them.

Zentralblatt as a community service

Zentralblatt MATH has three editorial institutions: the European Mathematical Society (EMS), FIZ Karlsruhe and the Heidelberger Akademie der Wissenschaften (Heidelberg Academy of Sciences). These are responsible for the content and for the running of the database (FIZ Karlsruhe). Springer-Verlag is the publisher, responsible for marketing, sales and invoicing and for the print version *Excerpts from Zentralblatt MATH*. As is well known, Springer is a commercial publisher and therefore many mathematicians believe that Zentralblatt makes a lot of money and that most of the profit goes to Springer.

However, I can say that this is definitely not the case. In fact, Springer has only a little share in Zentralblatt; the main partners are the non-profit organisations EMS, FIZ and the Heidelberg Academy. Mathematicians should be aware of this fact, which can also be seen from the rather generous offers:

- Free access for institutions of developing countries.
- Free access for every individual member of the EMS.
- Free access to the first three hits of any query for everybody.

Of course, Zentralblatt cannot be completely free, although most of us would perhaps wish it to be. Generating content and maintenance of the infrastructure, including the IT, is extremely costly. The Berlin office of Zentralblatt, for instance, has about 20 full-time employees who manage 120,000 items, drawn each year from more than 3,500 journals and 1,100 serials. Moreover, about 6,000 reviewers all over the world write short abstracts of published papers with some additional information. These contributions are the main content of Zentralblatt MATH and it shows that Zentralblatt is a service from the community for the community of mathematicians.

Do we need more than one review service?

Many mathematicians ask this question, in particular when their library suffers severe budget cuts. I think there are good reasons why we should have more than one. It is never a good idea to be dependent on just one monopolist because then:

- There will be no competition for low prices.
- There will be no competition for complete and high quality content.
- There will be no competition for improving the product.
- There will be no independent control of bibliometric data.

Comparing MathSciNet and ZBMATH, it is easy to see that both have advantages and disadvantages. Zentralblatt is proud to offer access to more than 3 million records and thus to be the largest and most comprehensive reference database in mathematics. It is also the longest running reference database, as it contains data that date back more than 150 years – without doubt a great treasure (see the very instructive articles by S. Göbel, "Glimpses into the history of Zentralblatt MATH" in *80 Years of Zentralblatt MATH* by O. Teschke, B. Wegner, D. Werner (Editors), Springer 2011, and the abridged version "80th anniversary of Zentralblatt Math" in the *EMS Newsletter*, September 2011).

It is perhaps worth mentioning that MathSciNet and ZBMATH work together in several fields, e.g. identifying plagiarism and further developing the MSC classification scheme for mathematics. This shows that competition and collaboration can go together for the benefit of the mathematical community.

The European Mathematical Society, one of the main editors of Zentralblatt MATH, promotes the development of all aspects of mathematics in Europe and Zentralblatt MATH contributes to these promotional activities. It is important that European mathematicians especially take advantage of this fact and also support Zentralblatt in the future. The EMS Council, including its president, as well as the collaborators of Zentralblatt, including myself, are fully dedicated to the aim of making Zentralblatt a continuing success story.

New role of the review services

When I was a student and later an assistant, I used to go to the library once a week to make my own excerpts from articles in my field on small tabs, or to copy the review from Zentralblatt or Math Reviews. For many years, this helped me to keep track of my area. Nowadays, mathematicians try to get the information online first before they go to the library, if at all.

The transformation of Zentralblatt into a reference database made online access possible and it is now used intensively by working mathematicians as a fast and reliable source of information. In many cases, the review of an article or of a book provides useful extra information. But besides this, the rather complete database allows an easy search for the most important publications in any area, specified by its MSC classification or related to given keywords. Facing an explosive growth in the number of publications, it is especially useful for young researchers to have well-prepared, selected and structured information, which differs from the greedy search engines. However, this is not obvious and it will be a challenge to convince young mathematicians to make even better use of the reviewing services.

Besides information about publications, the reference databases provide bibliometric data about individual authors through their author profiles and the use of these data is a growing trend. Although every mathematician knows that bibliometric data cannot substitute for peer reviewing, many mathematicians use these data as additional information. But this means that services providing such data have enormous influence and power.

In a sense, the author profiles from Zentralblatt MATH and MathSciNet have come to be used as a "rating agency" for mathematicians. Even if we do not like this, it is clear that we cannot stop this trend. But we have to be aware of it and stress its limitations.[1]

Completeness and reliability

The problem of completeness for reference databases has been addressed in a recent article by Bernd Wegner (see B. Wegner, "Completeness of reference databases, old-fashioned or not?", *EMS Newsletter*, June 2011). His statement "…complete reference services will very soon be the only integrating factor for the large variety of mathematical publications" is probably true. However, there are two questions concerning completeness and reliability: firstly, which articles have to be considered as mathematics and, secondly, which journals have a sufficiently high quality to be indexed?

Neither question is easy to answer nor can the answer be automated. For example, there are many new journals in mathematics appearing every year and several of them claim to be peer-reviewed but are nothing more than a business model.

This means that completeness must always be balanced by quality. Having in mind that reference databases are also used to rate the scientific quality of a person, we see that this is of utmost importance. Ensuring completeness and reliability is one of the major tasks, to which Zentralblatt MATH dedicates great care.

Perspectives

The future of reference databases like ZBMATH or MathSciNet is not at all clear. We have seen that they provide very useful and valuable information that cannot be obtained from other sources, at least not with the same completeness and reliability. Of course, other sources like Google or Google Scholar, which are free, also provide information about scientific publications and even bibliometric data. However, it is my impression that the information given there is often not reliable. On the other hand, nowadays most people, including mathematicians, have become accustomed to using services like Google every day. Hence, they try to find full texts of publications there, even if they could have access to the full text via a reference database.

In any case, I am convinced that for ZBMATH and MathSciNet to survive they will have to add unique new features and services. These have to be based on and designed for electronic use and must be accessible online through the internet.

There are already some ideas on how to improve Zentralblatt MATH. As a first step, we would like to get systematic feedback from our users about their wishes and expectations. A poll will be organised together with the EMS. We will also put in some extra effort into improving the author profile of ZBMATH.

A new and, we hope, useful service is the SMATH project. Here we are creating an open access database on mathematical software, which will be linked with the reviews in ZBMATH. It addresses not only users of Zentralblatt but also anyone interested in mathematical software. For a more detailed description of SMATH see the article "Building an Information Service for Mathematical Software – the SMATH Project" in this Newsletter.

Some other innovative projects for ZBMATH have just started. I would like to mention the DeliverMath project for enhanced and semi-automatic text analysis and the MathSearch project for indexing and searching mathematical formulas within ZBMATH. Other projects are planned. I think we can look forward to new and exciting developments within the next few years.

If you have any questions or suggestions, please contact me at greuel@zentralblatt-math.org.

[1] When bibliometric data are the only source to rank scientists, the results can be very surprising, not to say wrong (see O. Teschke, "Negligible Numbers", *EMS Newsletter*, December 2011). Moreover, depending on the data and the way that these are processed, the results can vary quite a lot (see O. Teschke, B. Wegner, "Author profiles at Zentralblatt MATH", *EMS Newsletter*, March 2011).

New Editor on the Newsletter

Eva-Maria Feichtner graduated in 1997 from TU Berlin, Germany, with a PhD thesis on the topology of arrangements and configuration spaces. After postdoctoral studies at MIT and the Institute for Advanced Study at Princeton, she held an assistant professorship at ETH Zurich, Switzerland. In 2005 she was awarded a research professorship by the Swiss National Science Foundation. After being appointed Professor of Geometry and Topology at the University of Stuttgart in 2006, she joined the University of Bremen in 2007, now holding the Chair of Algebra. In 2011 she co-founded the Institute for Algebra, Geometry, Topology and Their Applications (ALTA) at the University of Bremen.

Her research interests are in discrete structures that ubiquitously appear as core data in algebra, geometry and topology. Notably, the emerging field of tropical geometry provides for ample appearance of discrete data at crucial points.

Eva-Maria Feichtner has organised numerous research workshops and conferences over the years. In 2009 she was lead organiser of the programme on Tropical Geometry at MSRI, Berkeley.

Ten Years of Publishing by the EMS

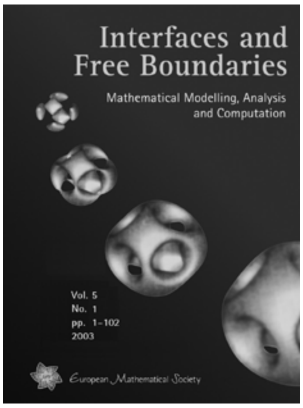
Thomas Hintermann (EMS Publishing House)

As so often happens with a long story, there is no telling how exactly it began. Like most people working in publishing, I have occasionally indulged in fantasies of founding my own company, speculating on how I would improve all the things I thought the others were not doing very well. By late 1999, I was ready to leave the security of my well established job and embark on the adventure of founding a new publishing house. In view of the rising antagonism between the large commercial publishers and the academic world it required no mental leap to identify the European Mathematical Society as a sensible candidate to work with. Sometime in 1998 I approached Bernd Wegner asking what the stance of the EMS would be towards such a plan. He immediately responded that the right person to talk to would be Rolf Jeltsch, most likely (but unbeknownst to me) the next president of the EMS and, by lucky coincidence, professor of ETH in Zürich, my hometown. A meeting was arranged and it quickly transpired that, within the EMS, the idea of an in-house publishing venture had already circulated. Naturally, there were large obstacles ahead but the handshake after this meeting was for me the beginning.

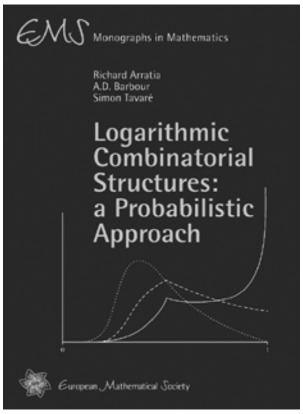
In the closing speech of the ECM in Barcelona in 2000, Rolf Jeltsch declared the foundation of a publishing house associated with the EMS as one of the goals of his presidency. By then, the contours of the enterprise had become somewhat clearer but the most crucial element, the financing, was still missing. In the next few months, some fundraising was undertaken and finally, with a loan of 300,000 Swiss Francs from the Huber-Kudlich-Stiftung in Zurich and the help of the ETH who provided some of the infrastructure, we decided to take the plunge into the cold water. The first working day was 1 September 2001 but the publishing house was officially registered in the chamber of commerce in Zurich in April 2002.[1] This, for lack of a better definition, now officially stands as the birthday of the EMS Publishing House.

The first months of the publishing house presented many challenges entirely different from those of an established publisher. There was no website, no logo, no telephone number and no bank account, not to mention typesetters, warehouse, printer or other necessary components of every publisher. The worst problem was perhaps that most prospective partners or providers I contacted asked to see a track record of the publishing house where absolutely none existed. Necessarily, there was much guessing and groping in the dark and, myself being the only employee, nobody versed in publishing matters to discuss ideas. What was needed most sorely was a first project on which to try out and hone the tools which had been tentatively put together. There were a few ideas but none of them fit the requirements of a publisher that we had in mind. Relief came in the form of *Interfaces and Free Boundaries*, a journal originally founded by Oxford University Press who, unwilling to continue it, sought to transfer it to another publisher. Although it would evidently lose money in strictly commercial terms, we were very happy to have it and it

[1] For the purposes of this article, the situation has been somewhat simplified. For more information on the legal structure of the EMS Publishing House, see the section “About Us” on our homepage (www. ems-ph.org).

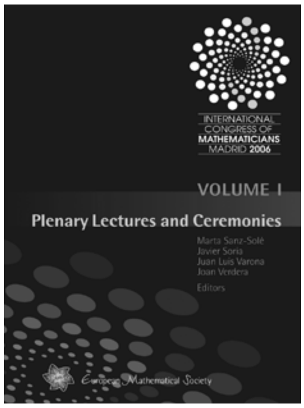


has remained a valued item on our list ever since. It unleashed many pent-up procedures and in a few months we made real what had been only plans for so long. The first print order was issued and the first invoices were written to the first subscribers, among many other firsts. So, *Interfaces Free Bound*. **5** (2003) became our first publication. One of the main obstacles was its online presentation, by then an absolute necessity for every serious mathematical journal. John Ewing of the American Mathematical Society was immediately ready to help us out and host the journal on our behalf. A somewhat improvised scheme was set up by which visitors to the journals homepage were led to a page hosted by the AMS which mimicked our design, and contents and access data were regularly exchanged. This lasted another two years until the EMS Publishing House homepage became whole again, establishing its own journals platform and assuming, in principle, the general form it has today.



By then, the first book had appeared (Richard Arratia, A. D. Barbour and Simon Tavaré, *Logarithmic Combinatorial Structures: a Probabilistic Approach*) and the *Journal of the European Mathematical Society* had been repatriated from Springer. It was clear, however, that we were still very far from a critical mass, and time – i.e. money – was slowly running out. Perhaps the largest step so far in the history of our young endeavour came in 2005. We were entrusted by the Swiss Mathematical Society to publish their journals *Commentarii Mathematici Helvetici* and *Elemente der Mathematik*, both well established academic journals. With its high reputation and wide dissemination, *Commentarii Mathematici Helvetici* brought our publishing house to the attention of many more scientists and libraries. More importantly, these journals enabled us to hire an experienced publisher Manfred Karbe, who considerably developed both the book and journal programmes in the following years.

The leadership of the EMS Publishing House by a team now gave the endeavour new vigour and opened up new possibilities of development. Another landmark was certainly the publication of the *Proceedings of the International Congress of Mathematicians in Madrid 2006*, a huge project for any mathematical publisher. Not counting the abstract booklets and the programme, it comprised almost 4400 pages in three volumes and was printed in 4000 copies. After many years of working in relative obscurity, imagine our joy at seeing thousands of mathematicians walking around the venue and crowding our booth, in their hands the programme booklet printed in familiar blue and yellow.



In the following years, more publications were added. Currently, we publish thirteen scholarly journals, the *Oberwolfach Reports* and the *EMS Newsletter*. At the time of this article appearing, the number of published book titles will have exceeded one hundred. On the other hand, we have also tried to keep a high standard of presentation and service. The website is continuously developed and endowed with new features such as usage statistics, archive download and RSS feed.

Right at the beginning of our publishing house, I was asked to propose a five year plan to the board of trustees – an impossible plan insofar as there was nothing to work or calculate with. Instead I presented a worst case scenario and an ideal case, leaving it to everybody‘s own judgement to choose a point of interpolation. Particularly in view of the consolidation process going on in the publishing world, it was unclear that it would be possible to build a new publisher from scratch at all, and under the circumstances there were obvious limitations to our most ambitious dreams and hopes. As it turned out, we have been very close to the ideal case, thanks to the support of many individuals and institutions, primarily the ETH Zürich and the Swiss Mathematical Society. Looking back on the last ten years we can say that the development could hardly have been better; however, predictions of the future are almost as hard as in the beginning. There are many fundamental changes going on in the publishing world and there are many challenges ahead for the trade in general and our small enterprise in particular. To survive we will need, even more than in the past, the support of the mathematical community. There are positive signs that a not-for-profit society publisher is welcomed by many as an alternative to commercialism but we are at a large disadvantage when it comes to creating the necessary income to keep the operation afloat. Every mathematician who endorses our policy is encouraged to publish in our publishing house and, most importantly, to support the acquisition of our publications by their home institution‘s library or to propose or support a licence of our publications in their home country.

We do not share the view of some that publishing is best done by the community itself. The collection and presentation of scientific results belongs in the hands of people who can do it effectively due to their specific training and experience and their ability to generate the income that is necessary to support the enterprise. However, commercialism should not determine or influence the way this process is being conducted. The EMS Publishing House, together with a number of other publishers with a similar philosophy, is here to ensure that this is not going to happen.

New Prize "EMS Monograph Award" by the EMS Publishing House

On the occasion of our tenth anniversary, we are happy to announce a new prize, open to all mathematicians. The **EMS Monograph Award** is assigned every two years to the author(s) of a monograph in any area of mathematics that is judged by the selection committee to be an outstanding contribution to its field. The prize is endowed with 10,000 Euro and the winning monograph will be published by the EMS Publishing House in the series "EMS Tracts in Mathematics".

Submission

The monograph must be original and unpublished, written in English and should not be submitted elsewhere until an editorial decision is rendered on the submission. The first award will be announced in 2014 (probably in the June Newsletter of the EMS); the deadline for submissions is 30 June 2013.
Monographs should preferably be typeset in TeX. Authors should send a pdf file of the manuscript by email and a hard copy together with a letter to:

European Mathematical Society Publishing House
ETH-Zentrum SEW A27
Scheuchzerstrasse 70
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Switzerland

E-mail: info@ems-ph.org

Scientific Committee

John Coates
Pierre Degond
Carlos Kenig
Jaroslav Nesetril
Michael Roeckner
Vladimir Turaev

6th European Congress of Mathematics

Second announcement

We are pleased to inform you about the progress in the organisation of the 6th European Congress of Mathematics, which will be held in Kraków (Poland), 2–7 July 2012.

Scientific programme

Plenary lectures (1 hour)

Adrian Constantin, *Some mathematical aspects of water waves*
Camillo De Lellis, *Dissipative solutions of the Euler equations*
Herbert Edelsbrunner, *Persistent homology and applications*
Mikhail Gromov, *In a search for a structure*
Christopher Hacon, *Classification of algebraic varieties*
David Kazhdan, *Representations of affine Kac-Moody groups over local and global fields*
Tomasz Łuczak, *Threshold behaviour of random discrete structures*
Sylvia Serfaty, *Renormalized energy, Abrikosov lattice, and log gases*
Saharon Shelah, *Classifying classes of structures in model theory*
Michel Talagrand, *Geometry of stochastic processes*

Invited lectures (45 minutes, in parallel)

Anton Alekseev, *Bernoulli numbers, Drinfeld associators and the Kashiwara-Vergne problem*
Kari Astala, *Holomorphic deformations, quasiconformal mappings and vector valued calculus of variations*
Jean Bertoin, *Coagulation with limited aggregations*
Serge Cantat, *The Cremona Group*
Vicent Caselles, *Exemplar-based image inpainting and applications*
Alessandra Celletti, *KAM theory: a journey from conservative to dissipative systems*
Pierre Colmez, *The p-adic Langlands program*
Alessio Corti, *Extremal Laurent polynomials and Fano manifolds*
Amadeu Delshams, *Irregular motion and global instability in Hamiltonian systems*
Hélène Esnault, *On flat bundles in characteristic 0 and $p>0$*
Alexandr A. Gaifullin, *Combinatorial realisation of cycles and small covers*
Isabelle Gallagher, *Remarks on global regularity for solutions to the incompressible Navier-Stokes equations*
Olle Häggström, *Why the empirical sciences need statistics so desperately*
Martin Hairer, *Solving the KPZ equation*
Nicholas J. Higham, *The matrix logarithm: from theory to computation*
Arieh Iserles, *Computing the Schrödinger equation with no fear of commutators*

Alexander S. Kechris, *Dynamics of non-Archimedean Polish groups*
Bernhard Keller, *Periodicity from the pentagon*
Sławomir Kołodziej, *Weak solutions to the complex Monge-Ampère equation*
Gady Kozma, *Phase transitions in self-interacting random walks*
Frank Merle, *On blow-up curves for semilinear wave equations*
Andrey E. Mironov, *Commuting higher rank ordinary differential operators*
David Nualart, *Stochastic calculus with respect to the fractional Brownian motion*
Alexander Olevskii, *Sampling, interpolation, translates*
Leonid Parnovski, *Multidimensional periodic and almost-periodic spectral problems: Bethe-Sommerfeld Conjecture and integrated density of states*
Florian Pop, *About covering spaces and numbers*
Igor Rodnianski, *TBA*
Zeev Rudnick, *Quantum chaos and number theory*
Benjamin Schlein, *Effective equations for quantum dynamics*
Andrew Stuart, *Probing probability measures in high dimensions*
Vladimír Sveřák, *On 2d incompressible Euler flows*
Piotr Śniady, *Combinatorics of asymptotic representation theory*
Stevo Todorčević, *Ramsey-theoretic analysis of the conditional structure of weakly-null sequences*

Mini-symposia (2 hours, in parallel)
(names of the organisers in parenthesis)
Absolute arithmetic and $\mathbb{F}_1$-geometry (*Koen Thas*)
Applied and Computational Algebraic Topology (*Martin Raussen*)
Arithmetic Geometry (*Wojciech Gajda, Samir Siksek*)
Bachelier Finance Society: Mathematical Finance (*Peter K. Friz*)
Braids and Configuration Spaces. (*Mario Salvetti*)
Computational Dynamics and Computer Assisted Proofs (*Warwick Tucker, Piotr Zgliczyński*)
Continuous Real Rational Functions and Related Topics (*Krzysztof Kurdyka*)
Differential Algebra and Galois Theory (*Zbigniew Hajto*)
Discrete Structures in Algebra, Geometry, Topology, and Computer Science (*Eva-Maria Feichtner, Dmitry Feichtner-Kozlov*
Fluid Dynamics (*Piotr Mucha, Agnieszka Świerczewska-Gwiazda*)
Geometric and Quantitative Rigidity (*Marta Lewicka*)
How Mathematics Illuminates Biology (*Marta Tyran-Kaminska, Michael C. Mackey*)
Hyperbolic Conservation Laws (*Piotr Gwiazda, Agnieszka Świerczewska-Gwiazda*)
Infinite-Dimensional Dynamical Systems with Time Delays (*Tibor Krisztin, Hans-Otto Walther*)
Implicitly Constituted Material Models: Modelling and Analysis (*Josef Malek, Endre Suli*)
Knot Theory and its Ramification (*Józef Przytycki*)
Matchbox Dynamics (*Krystyna Kuperberg*)
Optimal Stopping (*F. Thomas Bruss, Krzysztof Szajowski*)
Probabilistic Methods for Partial Differential Equations (*Dan Crisan*
Progress in Chemical Reaction Network Theory (*Carsten Wiuf, Elisenda Feliu*)
Semigroups of Operators: Theory and Applications (*Adam Bobrowski, Yuri Tomilov, Ralph Chill*)
Solutions to the Euler Equations of Incompressible Fluids. (*Xinyu He*)
Stochastic Models in Biosciences and Climatology (*Samy Tindel*)
25 Years of Quantum Groups: From Definition to Classification (*Alexander Stolin*)

Posters
Registered participants of the 6ECM will have the opportunity to present research posters. The deadline for submission of abstracts via the 6ECM website is 30 April 2012, and decisions concerning acceptance will be made before 19 May 2012. Abstracts submitted after 30 April 2012 will not be considered.

Round table/panel discussions (2 hours, in parallel)
Financing of mathematical research. *Moderator: Pavel Exner*
The role of emerging economies in the global development of mathematics. *Moderators: Andreas Griewank, Tsou Sheung Tsun*
Redressing the gender imbalance in mathematics: strategies and outcomes. *Moderator: Caroline Series*
What is expected from European learned societies? *Moderator: Marta Sanz-Solé*

Other activities include the Andrzej Pelczar Memorial Lecture (Lecturer: Maciej P. Wojtkowski), a special lecture devoted to the Centennary of Alan Turing (Lecturer: Philip Welch), a special lecture on the Mathematics of Planet Earth project (Lecturer: José-Francisco Rodrigues) and a presentation of the EMS Education Committee (Chair: Günther Törner).

Satellite activities

Satellite Thematic Sessions
organised by the 6ECM participants will be held on Sunday 1 July 2012 and Saturday 7 July 2012 (all day) and on the afternoons during the congress in the buildings of the AGH University, very close to the 6ECM venue. Proposals for the Satellite Thematic Sessions can be submitted until 30 April 2012 via a webform at the 6ECM website.

Women in Mathematics
Following tradition, just before the 6ECM on 1 July a meeting of European Women in Mathematics will be held at the 6ECM venue.

Satellite Conferences
There are many satellite conferences planned around the

time and in the geographical proximity of the 6ECM. A list of the satellite conferences can be found at the 6ECM website.

Promotion and Exhibitions
At the 6ECM there will be many opportunities to promote mathematical societies and other organisations, such as publishers and companies related to mathematical research and education (including software, mathematical games, etc.). Possible promotional opportunities include:
- an exhibition stand in the exhibition hall at the conference venue;
- distribution of promotional printed material by including them among the 6ECM materials that will be distributed to the participants;
- advertisements in the conference programme booklet, which will be distributed to the participants;
- display of company logos and/or advertisements on a screen in the main lobby of the conference venue;
- mention of companies as sponsors of the 6ECM on the 6ECM website (link „sponsors“) and in the conference programme booklet.

Everybody interested in promotional activities during the 6ECM is invited to contact the organisers at exhibition@6ecm.pl for more detailed information.

Social Programme
Social and cultural programmes, promoting informal contacts between congress participants and contact with the rich cultural heritage of Kraków, will be important components of the 6ECM. Included are a welcome reception on Monday 2 July, a conference dinner on Wednesday 4 June and guided sightseeing of Kraków. Excursions to the tourist attractions in the vicinity of Kraków will be offered at an extra charge for half days on Tuesday and Thursday and the whole day on Saturday.

The **Accompanying Persons Programme** includes the welcome reception, beverages and cookies during the breaks, the conference banquet, guided sightseeing in Kraków and assistance in arranging an individual sightseeing and cultural programme.

Registration and the Conference Fee

Participants of the 6ECM register by setting a personal account at the address: pay.ptm.org.pl, or via a link "Registration" from the 6ECM main website.

Fee	Until 31 March 2012	From 1 April 2012
Conference fee	1050 PLN	1250 PLN
EMS/PTM member conference fee	900 PLN	1050 PLN
Student conference fee	600 PLN	650 PLN
Accompanying person fee	600 PLN	600 PLN

Only payments in Polish currency (PLN) can be accepted! Our bank exchange rate is approximately 4.10 PLN for 1 € (as of February 2012). Note that recent exchange rates have been fluctuating considerably.

The EMS/PTM member conference fee applies to individual members of the European Mathematical Society and to members of the Polish Mathematical Society who paid their dues for the year 2011. The student conference fee applies to persons who are enrolled in a graduate (Master's or doctoral) programme in mathematics or a related field.

Accommodation
Participants of the 6ECM are kindly asked to book rooms directly at the hotels listed at the 6ECM website or via the reservation website of the Jagiellonian University which is accessible from the 6ECM website.

How to get to Kraków?
John Paul II International Airport Kraków – Balice is located 11 km from the city. It takes 20 minutes by shuttle train to get from the airport to the centre. There are several direct, regular connections to European airports, as well as some overseas connections. Low-fare airlines maintain connections to many European destinations.

Important forthcoming dates and deadlines
Here are the important deadlines. The 6ECM website will be systematically updated as soon as any new information becomes available.

01-04-2012	rise of registration fee
30-04-2012	deadline for poster submissions
30-04-2012	deadline for submission of proposals for Satellite Thematic Sessions
01-07-2012	arrival day
02-07-2012	opening of the 6ECM, beginning of the scientific programme
04-07-2012	conference banquet
06-07-2012	end of the scientific programme
07-07-2012	excursions and departure day

Contact
For more information and updates visit www.6ecm.pl or write to 6ecm@6ecm.pl.

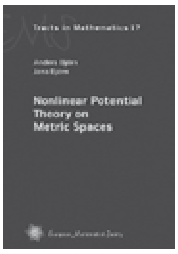
We are looking forward to seeing you in Kraków!

European Mathematical Society

Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.

EMS Tracts in Mathematics

This series includes advanced texts and monographs covering all fields in pure and applied mathematics. *Tracts* will give a reliable introduction and reference to special fields of current research. The books in the series will in most cases be authored monographs, although edited volumes may be published if appropriate. They are addressed to graduate students seeking access to research topics as well as to the experts in the field working at the frontier of research.



Anders Björn and Jana Björn (both Linköping University, Sweden)
Nonlinear Potential Theory on Metric Spaces (EMS Tracts in Mathematics, Vol. 17)

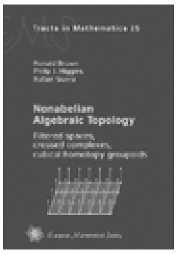
ISBN 978-3-03719-099-9. 2011. 415 pages. Hardcover. 17 x 24 cm. 64.00 Euro

The p-Laplace equation is the main prototype for nonlinear elliptic problems and forms a basis for various applications, such as injection moulding of plastics, nonlinear elasticity theory and image processing. Its solutions, called p-harmonic functions, have been studied in various contexts since the 1960s, first on Euclidean spaces and later on Riemannian manifolds, graphs and Heisenberg groups. Nonlinear potential theory of p-harmonic functions on metric spaces has been developing since the 1990s and generalizes and unites these earlier theories.
This monograph gives a unified treatment of the subject and covers most of the available results in the field, so far scattered over a large number of research papers. The aim is to serve both as an introduction to the area for an interested reader and as a reference text for an active researcher. The presentation is rather self-contained, but the reader is assumed to know measure theory and functional analysis.
Each chapter contains historical notes with relevant references and an extensive index is provided at the end of the book.

Marek Jarnicki (Jagiellonian University, Kraków, Poland) and Peter Pflug (University of Oldenburg, Germany)
Separately Analytic Functions (EMS Tracts in Mathematics, Vol. 16)

ISBN 978-3-03719-098-2. 2011. 306 pages. Hardcover. 17 x 24 cm. 58.00 Euro

The story of separately holomorphic functions began about 100 years ago. During the second half of the 19th century, it became known that a separately continuous function is not necessarily continuous as a function of all variables. At the beginning of the 20th century, the study of separately holomorphic functions started due to the fundamental work of Osgood and Hartogs.
This book provides the first self-contained and complete presentation of the study of separately holomorphic functions, starting from its birth up to current research. Most of the results presented have never been published before in book form. The text is divided into two parts. A more elementary one deals with separately holomorphic functions "without singularities", another addresses the situation of existing singularities. A discussion of the classical results related to separately holomorphic functions leads to the most fundamental result, the classical cross theorem as well as various extensions and generalizations to more complicated "crosses".



Ronald Brown (Bangor University, UK), Philip J. Higgins (Durham University, UK) and Rafael Sivera (Universitat de València, Spain)
Nonabelian Algebraic Topology. Filtered spaces, crossed complexes, cubical homotopy groupoids (EMS Tracts in Mathematics, Vol. 15)

ISBN 978-3-03719-083-8. 2011. 703 pages. Hardcover. 17 x 24 cm. 98.00 Euro

The main theme of this book is that the use of filtered spaces rather than just topological spaces allows the development of basic algebraic topology in terms of higher homotopy groupoids; these algebraic structures better reflect the geometry of subdivision and composition than those commonly in use.
The structure of the book is intended to make it useful to a wide class of students and researchers for learning and evaluating these methods, primarily in algebraic topology but also in higher category theory and its applications in analogous areas of mathematics, physics and computer science. Part I explains the intuitions and theory in dimensions 1 and 2, with many figures and diagrams, and a detailed account of the theory of crossed modules. Part II develops the applications of crossed complexes. The engine driving these applications is the work of Part III on cubical ω-groupoids, their relations to crossed complexes, and their homotopically defined examples for filtered spaces. Part III also includes a chapter suggesting further directions and problems, and three appendices give accounts of some relevant aspects of category theory. Endnotes for each chapter give further history and references.

Also published in this series:

Vol. 14 Steffen Börm: *Efficient Numerical Methods for Non-local Operators. $\mathcal{H}^2$-Matrix Compression, Algorithms and Analysis*
ISBN 978-3-03719-091-3. 2010. 441 pages. 58.00 Euro

Vol. 13 Laurent Bessières, Gérard Besson, Michel Boileau, Sylvain Maillot and Joan Porti: *Geometrisation of 3-Manifolds*
ISBN 978-3-03719-082-1. 2010. 247 pages. 48.00 Euro

Vol. 12 Erich Novak and Henryk Woźniakowski: *Tractability of Multivariate Problems. Volume II: Standard Information for Functionals*
ISBN 978-3-03719-084-5. 2010. 675 pages. 98.00 Euro

Vol. 11 Hans Triebel: *Bases in Function Spaces, Sampling, Discrepancy, Numerical integration*
ISBN 978-3-03719-085-2. 2010. 305 pages. 58.00 Euro

Vol. 10 Vladimir Turaev: *Homotopy Quantum Field Theory*. With Appendices by Michael Müger and Alexis Virelizier
ISBN 978-3-03719-086-9. 2010. 290 pages. 58.00 Euro

Vol. 9 Gebhard Böckle and Richard Pink: *Cohomological Theory of Crystals over Function Fields*
ISBN 978-3-03719-074-6. 2009. 195 pages. 48.00 Euro

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ERC on its Fifth Birthday – And What Next?

Pavel Exner (Vice-President of the ERC)

One thing you can be sure about time is that it is flying fast. It seems only yesterday that I described on these pages the opening of the new founding scheme in European science with evaluation based on a single criterion, namely scientific excellence. Today, hardly any European mathematician would not know what the acronym ERC[1] means and many have enjoyed or will soon enjoy the benefit of its support. This group includes not only the principal recipients of the ERC grants but also members of their teams, mostly postdocs and PhD students, altogether amounting to many hundreds of people.

The fifth birthday, which the ERC celebrated at the end of February, is a good opportunity to look at the work done and, more importantly, at the challenges we are going to face in the future. The first thing to observe is that the two main ERC funding streams, the Starting and Advanced Grants, are running well; it seems they are generally regarded as a fair and efficient way to select and support high-quality projects in all fields of science including mathematics.

They are now augmented by two other programmes. A small one, called *Proof of Concept*, aims at results of existing ERC grants which have a commercial potential; it gives limited seed money to the author to bring the idea to a practical realisation. The mathematical imprint in the first two rounds of this programme is negligible so far and one can only hope that this will change in the future. The second one, called *Synergy Grants*, is only just opening – the first call closed on January 25 – and has for the moment the same air of risky new enterprise as the first Starting Grants had five years ago.

The programme aims at small groups (two to four) of principal investigators who want to realise a common project. It has to be said loud and clear that the idea has nothing to do with common European networking. First of all, each principal investigator in this scheme must be individually excellent at the level required for the other ERC grants. Secondly, their collaboration must bring a significant added value; the results are expected to be more than mechanical addition of what each of them can produce alone. And finally, it is not collaboration at a distance; the Synergy Grant holders have to commit themselves to spending core time working together.

Changes are also to be expected in our two basic funding streams. It may be noticed that over the five years the balance between the Starting and Advanced Grants, originally favouring the latter, is slowly tilting toward the former. It is certainly a welcome effect which does not come from a voluntary decision of the Scientific Council but rather has emerged as a fact of life. While the annual numbers of Advanced Grant applications stabilised in the region of 2000–2500, the numbers of Starting Grants are steadily growing, already approaching the 5000 mark in the 2011 call.

The only logical route to take is to split the two internal groups we have so far in the Starting Grants, divided by age into "starters" (2–7 years after PhD) and "consolidators" (7–12 years after PhD), into two separate streams in the near future. They will have their own application deadlines and sets of evaluation panels; what will be preserved will be the interviews at the second stage of the evaluation, which has proved to be a very efficient tool to select the best projects.

These are, so to say, technical upgrades of the ERC scheme but one has to think also about its very basis. At present the ERC is, as the "Ideas" part, a component of the Seventh Framework Programme (FP7) with about a 15% share of its budget. From the beginning of 2014, the FP7 is to be replaced by a successor programme, inventively called *Horizon 2010*, and the structure of this programme is crucial for the future fate of the ERC. The currently existing proposal assumes that the ERC budget will grow by 77%.

It may seem to be a big increase but one has to keep in mind that the present budget grows roughly linearly from 300 million euro in 2007 to about 1.7 billion in 2013. The proposed amount would thus keep the annual funding at almost the same level over the seven-year period from 2014. And if the number of applications keeps growing at the present pace, the success rate will naturally decrease and probably stay below 10%.

We are all aware that times are economically tough and no government is in the mood to increase expenses. Thinking realistically we have to tell ourselves that with the proposed budget the ERC can exist in the coming years, even if it means that grant competition will be even tougher than today. A further budget reduction, on the other hand, could put the very existence of the ERC at risk and, having said that, we have to recall that the budget proposal has still to pass along a long and thorny political road before being approved.

We believe, of course, that supporting research projects in a broad competition based on scientific excellence is the best investment in the future. If the scientific community, in the wide sense of the word, agrees to this principle, it has to pass this message to the politicians elected to represent them, with the hope that those illuminated men and women are able to think more than four years ahead.

[1] European Research Council

A Competition in Connection with "Mathematics of Planet Earth 2013": Modules for a Virtual Exhibition

Ehrhard Behrends (Berlin), Andreas Matt (Oberwolfach) and José Francisco Rodrigues (Lisbon)

Logo of the Mathematics of Planet Earth Initiative

During the 2010 International Congress of Mathematicians, held in Hyderabad, India, at the meeting of delegates of the International Mathematical Sciences Institutes, Christiane Rousseau (Montreal) presented an invitation to institutes and societies in mathematical sciences around the world: *Mathematics of the Planet Earth 2013* (see www.mpe2013.org). This initiative, first launched in the USA and Canada, now has many partners in Europe, including the European Mathematical Society, and around the world, obtaining the endorsement of the International Mathematical Union, the International Council of Applied and Industrial Mathematics and the International Commission of Mathematical Instruction.

MPE2013 is now a worldwide project that consists of holding a year of activities in 2013 under the theme of the role mathematics plays in issues concerning the Planet Earth. The idea is to present scientific events, research programmes and activities for the public, media and schools. As an opportunity for raising public awareness in mathematics, several activities are expected and are being organised. They may include public lectures, panel discussions, media or television programmes, exhibitions, articles in newspapers and magazines, posters, websites, school activities and projects, outreach to teachers, etc.

A Global Exhibition on Mathematics of Planet Earth was proposed by the CIM (Centro Internacional de Matemática) director at the occasion of the ERCOM meeting, held at the Mathematical Institute of Oberwolfach in April 2011. The concept is based on an Open Source Exhibition with modules that could be reproduced and utilised by many users around the world, from science centres and museums to schools. The realisation should not be centralised. It will instead be split among many partners around the world, possibly with collaborative networks of participants. The exhibition will have a virtual part as well as several material parts. Copies of the material parts could be recreated or travel around the world and the virtual modules could be available on the basis of Creative Commons licences. These licences allow, for example, others to remix, tweak and build upon the licensed work non-commercially, as long as they credit the authors and license their new creations under identical terms. In this way the modules can be distributed and used by many partners.

If possible, there will be a global opening coordinated at the same day in many countries in order to amplify the visibility of the mathematics.

The MPE2013 Museum and Exhibits Committee has now launched a competition of virtual modules that, in particular, aims to fulfil this purpose.

The competition of modules for a virtual exhibition

With this note the readers of the Newsletter of the EMS are invited to participate in the competition in connection with MPE2013. It concerns a new type of exhibition.

The competition will be open from January 2012 to 15 September 2012. The competition will be coordinated by the MPE2013 Museum and Exhibits Committee. The prize winners will be selected by an international jury nominated by MPE2013 and will be announced in October 2012. The judges' decision will be final. The first, second and third prize winners will receive, respectively, prizes of US$5,000, US$3,000 and US$2,000. The winning modules will occupy a prominent place on the website of the exhibition. Moreover, it is planned to show the modules of the overall winners in exhibitions and museums. Important museums in North America and in Europe have already agreed to prepare installations and organise exhibitions in 2013.

For further information, visit the website of the competition: http://www.mpe2013.org/competition.

The web infrastructure for the competition is provided by IMAGINARY, the open source platform for interactive math communication by the Mathematisches Forschungsinstitut Oberwolfach.

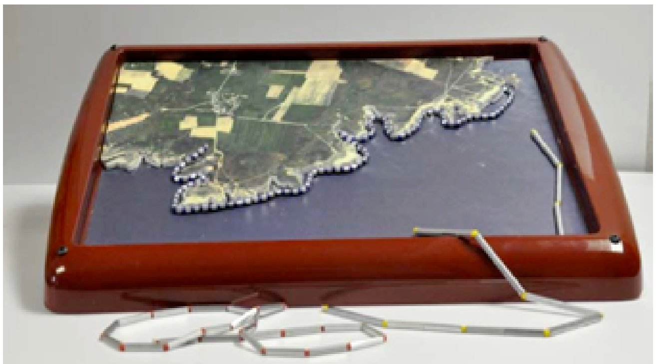
Examples of such modules

To stimulate imagination on the many domains where mathematics plays a crucial role in planetary issues the following (non exhaustive) four themes are proposed:

- *A planet to discover*: oceans, meteorology and climate, mantle processes, natural resources, celestial mechanics.
- *A planet supporting life*: ecology, biodiversity, evolution.
- *A planet organised by humans*: political, economic, social and financial systems, organisation of transport and communications networks, management of resources, energy.
- *A planet at risk*: climate change, sustainable development, epidemics, invasive species, natural disasters, risk analysis.

The typical modules submitted to this competition can be an idea for the construction of a physical exhibit, an interactive programme, a picture gallery with explanations or a film.

A physical module could be, for example, Fractal Coasts by Michel Darche and Mireille Chaleyat-Maurel, a polyester board with a coastline profile and several surveyor chains, an idea that has been realised at the international UNESCO supported exhibition EXPERIENCING MATHEMATICS (see http://www.mathex.org/MathExpo/ReadTheNature).



The "Fractal Coasts" module

Rocky coasts have a fractal character, unlike sandy coasts. To estimate the length of a coast, we can measure the length of an approximation of its contour with line segments of a given length: on the model, these segments are links of a chain. For a sandy coast, the length is approximately constant, when the segments are sufficiently short. But, for a rocky indented coast, the shorter the segments, the longer the length! In the limit the length is infinite. Users can test these properties by trying to measure the coasts using the different chains.

As another example one can find a five minute film at the homepage on periodic timetable optimisation of the Berlin subway by Rolf Möhring, Christian Liebchen and Sebastian Stiller.

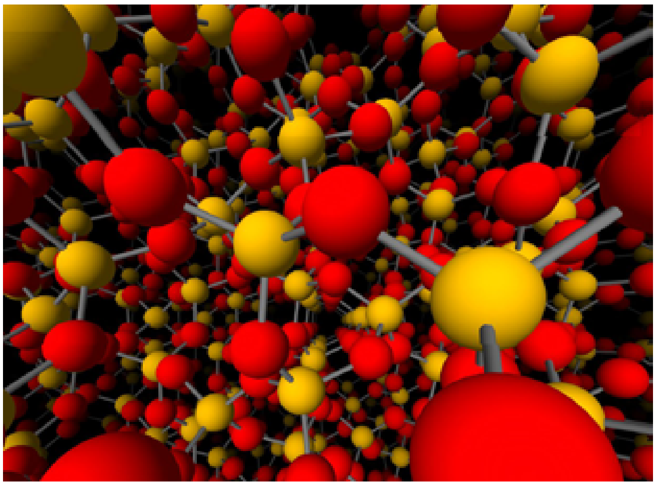
The Berlin Underground features nine lines that meet in 19 transfer stations. How can a periodic timetable be computed that minimises the total waiting time of all the passengers in this network, while respecting all safety matters? This is a highly complex task that has traditionally been handled manually by splitting it into subtasks.

The 2005 timetable for this network was computed by Matheon at the Institute of Mathematics at TU Berlin. The key was state-of-the-art combinatorial optimisation

Time table optimization

techniques. The public transport of Berlin is modelled as a graph with vertices at the stations, edges between them and constraints of different types. The optimal solution for its functioning (minimising the waiting time at stations and the number of trains required) is hidden somewhere in the set of all possible solutions respecting the constraints. But this set is much too large to be explored as a whole, even with the most powerful computers. Clever mathematical techniques produce powerful algorithms allowing the elimination of large portions of this set and concentration of the search into smaller subregions where the optimum lies.

The interactive programme Crystal Flight by Jeff Weeks serves as another example, this time for a virtual module to be used with a touch screen at an interactive station at an exhibition or museum. You can fly through a quartz, fluorite or diamond crystal and observe the mathematical structures of the crystal lattice by controlling a miniature spaceship. Using the speed control on the lower right side you can change the velocity of your spaceship.



Crystal Flight

Join this competition!

We invite you to participate in this competition and submit modules or ideas for modules to the competition. They should include instructions to realise the material module (physical module), installation instructions including technical requirements (interactive exhibit) and some accompanying scientific explanations for the public.

Everybody can be part of the competition: individuals or groups of individuals, institutions, schools or non-profit organisations. Contributions developed by profit organisations can be shared with us to be included in the open source exhibition but will not participate in the competition.

Some ideas of possible modules for the competition can be found in the document "31 ideas of modules for MPE2013" by Christiane Rousseau on the competition website. The ideas range from modules related to weather prevision, global positioning systems, cartography and population models to the management of resources, the economy of solidarity, percolation models and risk management tools.

Moreover, we would appreciate your help in making this competition known among your colleagues or anybody interested. Please spread the word! Together we can create a joint global and open exhibition to be reproduced in many countries. Let's celebrate mathematics and its many connections to our planet Earth.

References:
www.mpe2013.org/competition
www.mpe2013.org

ICIAM 2011

Some recollections from Chris Budd

If it wasn't enough to host the Winter Olympics in 2010, Vancouver in 2011 saw the influx of nearly 3000 mathematicians from over 70 countries to attend the 7th International Congress on Industrial and Applied Mathematics (ICIAM 2011). A splendid new conference centre

The conference centre at night, showing the grassy roof on top and the book exhibition within.

(originally built as the press centre for the Olympics) served as the perfect venue for displaying the power and breadth of the applications of mathematics. Topics as diverse as geometry, mechanics, the modelling of crowds, mathematical studies of cancer and oil spills, wavelet analysis, public health, epidemics, the economic crisis and climate change formed a rich diet of mathematics showing both the breadth and depth of its many applications. This filled over 500 mini-symposia, 700 lectures and 300 posters, which showed not only how mathematics can be applied but how studying applications leads to many new mathematical ideas and tools, as well as rapid developments in computational techniques. All of this demonstrated the twin truths that good mathematics has almost limitless applications and that to study these applications well requires the highest level of mathematical sophistication (with nearly every area of 'pure mathematics' featuring at some point in the meeting).

Nothing could illustrate this better then the life and work of Professor Jerry Marsden, whose fusion of dynamics with geometry and mechanics has led to profound advances in all of these areas. Jerry was co-chair of the Scientific Programme Committee for ICIAM 2011 and tragically died of cancer in September 2010. A very well conducted memorial lecture and reception in his honour (generously supported by Springer publishing), together with a series of excellent mini-symposia based on his work, formed a fitting centrepiece to ICIAM 2011.

The orca sculpture outside the conference centre.

Other highlights of the meeting were the opening ceremony, which saw some extraordinary dancing by a First Nation group, and the prizes and associated lectures. Those honoured for their work in mathematics and its applications included: Emmanuel Candes, Alexandre Chorin, Vladimir Rokhlin, James Sethian, Edward Lungu, David Keyes, Gunther Uhlmann, Ingrid Daubechies, Susanne Brenner, Beatrice Pelloni, Bjorn Engquist, Adam Oberman, Ian Frigaard and Michael Ward.

The meeting was also held in parallel with the Women in Mathematics Meeting. This organisation celebrated its 40^{th} anniversary with an excellent series of sessions including (in a particular highlight for me) a panel discussion by senior female academics on how to run a department. Excellent advice was given throughout!

The beautiful location, on the Vancouver waterfront and with fantastic views of the British Columbian coastal mountains, acted as both an inspiration for mathematical creativity and an excellent excuse to take time out from the lectures for animated networking and discussions (some of which took place on the hike up Grouse Mountain). Indeed, with so many delegates there is always the danger of overcrowding and being swamped by the sheer scale of the meeting. Often this has put me off attending very large meetings in the past but I was very pleased that this did not happen in Vancouver, a tribute both to the excellent conference centre (which simply seemed to absorb the delegates) and also the excellent organisation of the meeting. Students, established academics and industrialists all had the chance to meet as equals

Inside one of the main lecture rooms. There were several as large as this one!

to discuss ideas and progress in a diverse range of topics. The interactions were also greatly aided by an effective exhibition area (with good refreshment provision which seemed to avoid the queues which I have often encountered at similar meetings) combined with a well planned poster display.

Large meetings can suffer from a lack of clear information and guidance but this was avoided by having ICIAM TV constantly on tap, including highlights from the lectures and interviews with the lecturers. In fact, it was even one of the channels that you could view in many of the downtown hotels! Following the conference many of the interviews and lectures were summarised as short films on YouTube. As an example, have a look at the short film about Crowd and Traffic Dynamics which you can find at http://www.youtube.com/watch?v=SEUJuy7mY_w. The effective communication of mathematics was also a feature of the meeting with a series of popular talks for the general public combined with the highly engaging MathAmaze tournament demonstration by MITACS (which kept my son engaged for much of the meeting!).

The Jerry Marsden reception showing the view across the bay towards Stanley Park, the Lions Gate Bridge and the coastal mountains beyond.

ICIAM 2011 also acted as a focus for a large number of satellite meetings in and around Canada, including meetings in Banff and Toronto and on Vancouver Island. In fact there was so much activity that I ended up spending five weeks in Canada doing mathematics, whilst my family had a great Canadian holiday. I gather that many of the other participants also brought their families and had an equally good time.

It is a tribute to the organisers (including the executive committee Arvind Gupta Jim Crowley and Kenneth Jackson and the local committee comprising the MITACS team of Jo-Anne Rockwood (Chair), Josh Dobbs, Susan Lenio and Paul Zuurbier, together with teams from CAIMS and SIAM, as well as many others serving on countless committees) that ICIAM went so well and I would certainly like to record my thanks to them for a great meeting.

Good luck to Beijing who will be hosting ICIAM 2015.

Feza Gürsey International Summer School in Mathematical Physics

The third Feza Gürsey International Summer School in Mathematical Physics, entitled "Probabilistic Aspects of Contemporary Physics", will take place from 20 June to 6 July in the convivial environment offered by the Feza Gürsey Institute in Istanbul. The summer school has been named and supported as a satellite event to the Bernoulli Society's 8th World Congress in Probability and Statistics (www.worldcong2012.org), which will be held at Koç University in Istanbul, 9–14 July.

As suggested by its title, the summer school will have a considerable probabilistic flavour, covering research directions in the analysis of systems with infinitely many degrees of freedom, such as in classical and quantum statistical mechanics as well as quantum field theory. Each one of the four planned sets of lectures will strive to give a detailed account of the present state of their respective subjects in 12 to 16 hours. Time permitting, participants at the summer school will be encouraged to present their research work at informal seminars.

For full information as well as registration, please consult the website of the summer school at: www.fezagurseysummerschool.com

Encyclopedia of Mathematics: An Invitation to Advertise your Research Field

Ulf Rehmann (Bielefeld)

Are you working in a mathematical field which you feel deserves more attention in the professional scene because its development has been so substantial and rapid over the last, say, 20–30 years?

Do you want to make the main achievements of your field available to a broader community, possibly outside your close peer group, because you expect there ought to be some impact on other research areas?

Then there is now a great opportunity for you and your field. The EMS has undertaken the organisation of a wiki-based "Encyclopedia of Mathematics (EoM)" on the World Wide Web for the professional mathematician, with free access to everyone (http://www.encyclopediaofmath.org).

To bring your area to the general recognition of researchers or just to amend and complete the documentation of relevant areas of mathematics (but also in order to attract graduate and postdoc students) you may want to identify and/or update articles within the EoM representing your field (or create new articles to fill gaps) thereby adding references and links to actual research literature (as well as links to their review representations in MathSciNet and Zentralblatt für Mathematik by including their respective MR and Zbl numbers).

The basis of the EoM is the material from the classical *Encyclopaedia of Mathematics* (note the tiny spelling difference!), which was published in print during 1985–2001 by Reidel and Kluwer Academic Publishers, and since 2003 by Springer, within 10 volumes, together with three supplements, and which itself was an updated and annotated translation of the five volume *Soviet Mathematical Encyclopaedia* (*Matematicheskaya éntsiklopediya*), edited by I. M. Vinogradov, V. I. Bitjuckov and Ju. V. Prohorov and published during 1977–1985. This material contains more than 8,000 encyclopedic key entries of vibrant mathematics, equipped with substantial bibliographical references.

In 1997, an electronic version was produced and distributed on CD. Later, the CD material was freely but statically accessible on the Web.

Some time ago, Springer approached the EMS and proposed a collaboration. Under the condition that the EMS would organise future editions of the *Encyclopedia*, Springer would donate the material of the *Encyclopaedia* to the public as basic material and provide a server equipped with "Mediawiki" software in order to allow its further dynamic "wiki like" development under the standard Wikipedia licence (which is the so-called "Creative Commons Attribution Share-Alike License" – alternatively, and essentially equivalently, the "GNU Free Documentation License" can also be chosen).

The EMS considers this encyclopedic corpus to be a rich mathematical heritage which deserves maintenance and further development, and established an editorial board to monitor any changes to articles, with full scientific authority over alterations and deletions.

Of course there is already a surprisingly large amount of well based Wikipedia articles in mathematics, so one could ask why there should be another attempt. However, as the number of articles in EoM is quite substantial, its development is expected to yield a substantial contribution to the presentation of mathematics in the public domain. Moreover, the "Encyclopedia of Mathematics" is mainly addressed to the professional mathematician, including undergraduate and graduate students, and therefore contains relatively technical articles.

Also, it is, in contrast to general Wikipedia pages, based on the MathJax software (http://www.mathjax.org/) which allows arbitrary TeX or LaTeX code and which makes the writing and editing of articles much easier for the professional mathematician. The author of this article was surprised by the capabilities of the software. As an example, look at the page http://www.encyclopediaofmath.org/index.php/Help:Fancy_diagram with a rendering of the answer to one of the more demanding exercises from Knuth's TeX book. (Of course the code given there is plain TeX but it ought to be emphasised that MathJax understands the standard LaTeX macros and environments as well.)

Unfortunately, the old articles from the CD edition are not written in TeX. Formulas at that time were electronically represented by png images, which appear now on the (old) EoM web pages, since the TeX code used to produce them has been lost during the transitions of the material through the various publishers mentioned above.

But on the EoM server, software is offered to ease the transcription from the old style into a new and handy TeX version.

Also, the articles in the print volume were equipped with classification entries using the 1980 Mathematics Subject Classification (MSC) scheme, which did not make it into the online edition. However, software support is also given to equip the articles with actual 2010 MSC.

Please have a look into this interesting representative compendium of mathematics and consider contributing and helping to update all the vibrant mathematical fields here.

An Agreement of Cooperation between the Bernoulli Society for Mathematical Statistics and Probability and the European Mathematics Society

Ed Waymire (President of the Bernoulli Society), Marta Sanz-Solé (President of the EMS)

Professional societies in the mathematical sciences exist to provide an infrastructure and solidarity that supports research and educational agenda that involve many activities with typically artificial geographic boundaries. Certainly there is an abundance of evidence for the benefits of international collaborations at the scale of individual members. It also holds true that societies themselves can benefit in providing mutual support for initiatives by publicising events and pooling resources.

In this spirit of collaboration, the Bernoulli Society President Ed Waymire and the President of the European Mathematics Society Marta Sanz-Solé are pleased to announce a new formal agreement between the two societies aimed at strengthening support of the advancement of probability and mathematical statistics and their applications.

The details of this agreement, approved by the councils of both societies and positively endorsed by the Chair of the Bernoulli Society's European Regional Committee Laszlo Markus involves a pledge to organise joint scientific events, such as conferences, workshops, schools and special sessions pertaining to research in probability, statistics and their applications. This will include member discounts to registration fees at scientific events organised by either the Bernoulli Society or the European Mathematical Society. The term of the agreement is for a three year period, with the option for either of the societies to terminate this arrangement with a six month courtesy notice.

The first co-organised activity under this agreement is the joint support of the 2nd Young Statisticians Satellite Workshop to the 8th World Congress in Probability and Statistics in Istanbul this summer.

The Gold Medal "Guido Stampacchia" Prize

The Gold Medal "Guido Stampacchia" Prize is promoted by the International School of Mathematics "Guido Stampacchia" in collaboration with the Unione Matematica Italiana. It is assigned every three years to mathematicians not older than 35 who have made an outstanding contribution to variational analysis, the field where Guido Stampacchia made celebrated results.
The prize is assigned by an international committee and the winner is awarded at the international workshop "Variational Analysis and Applications" which takes place in Erice at the International School Guido Stampacchia.
The first winner was *Tristan Rivière* (ETH Zurich) in 2003, the second was *Giuseppe Mingione* (University of Parma) in 2006 and the third was *Camillo De Lellis* (University of Zurich).

Fourth edition of the Gold Medal "Guido Stampacchia" Prize
The Unione Matematica Italiana announces the fourth international competition for assigning a gold medal, to a researcher who is not older than 35 years on 31 December 2012 and who has done meaningful research in the field of variational analysis and its applications. Designations must be sent, before 31 March 2012, to:

Unione Matematica Italiana, Dipartimento di Matematica, Piazza di Porta San Donato 5, 40126, Bologna (Italy).

An international committee will evaluate the designations and assign the medal, which will be given on 15 May 2012 at the beginning of the International Conference "Variational Analysis and Applications" (in memory of G. Stampacchia and M. K. V. Murthy) to be held in Erice (Sicily) at the "E. Majorana" Foundation, 14–22 May 2012.

Late Style – Yuri I. Manin Looking Back on a Life in Mathematics

Agnes Handwerk and Harrie Willems
DVD-Video NTSC, 55 minutes, 2012
Series: Springer VideoMATH
ISBN 978-3-642-24482-7

Rewind

At the 2011 Annual Meeting of the DMV, a remarkable video documentary of the life of an internationally eminent mathematician was premiered. Its title was "Late Style – Yuri I. Manin Looking Back on a Life in Mathematics". The story it tells is not only a glimpse into the life of a leading scholar and teacher but a brief look into the political history of his time.

Yuri Manin and his wife Xenia at the ICM in Madrid, 2006.

Yuri Manin began his study of mathematics in 1953 at Moscow State University. In that same year, Stalin died and the long-reclusive Soviet Union cautiously began to open up to the West. A time of thaw had begun. Manin was then 16 years old and at a prime age to experience the "golden years of Moscow mathematics", characterised by the close cooperation between teachers and students. He defended his PhD thesis in 1960 under the supervision of Igor Shafarevich, accepted a permanent research position at the Steklov Institute and, after defending his second thesis, also became Professor of Algebra at Moscow State University at the age of 28. In the brief period of closer political relations between the East and West, Manin had, in 1967, a unique opportunity to visit the Institute of Hautes Etudes in Bures near Paris, where he collaborated with Alexander Grothendieck. After 1968, because of his outspoken independence, the Soviet authorities imposed restrictions on him. In particular he was no longer allowed to travel to the West; nevertheless, he managed to maintain communication with mathematicians such as Alexander Grothendieck, Jean-Pierre Serre, David Mumford and Pierre Cartier. Of no lesser importance, he put his energy into the education of his students, many of whom later became important mathematicians themselves.

This sensitive video document was created by Agnes Handwerk and Harrie Willems. Agnes Handwerk is a journalist and documentary filmmaker in Hamburg; she received the 2008 DMV Journalism Prize for a radio feature about Alexander Grothendieck. Harrie Willems is a documentary filmmaker in Amsterdam. In 2006, they collaborated on a documentary film about the short life and influential work of the mathematician Wolfgang Doeblin. Both films can be ordered as DVDs in Springer's Video Math series.

(German original published in "Mitteilungen der Deutschen Mathematiker-Vereinigung", no 3/2011.)

Pre-World-Congress Meeting of Young Researchers in Probability and Statistics 2012

Date: 6–8 July 2012
Venue: Koç University, Sariyer, Istanbul, Turkey

KOÇ ÜNİVERSİTESİ

As a satellite meeting to the 8th World Congress in Probability and Statistics (www.worldcong2012.org), the Pre-World-Congress meeting of Young researchers in Probability and Statistics 2012 (PWCYPS 2012) seeks to promote the active participation of statisticians and probabilists in graduate education or in their early careers at the epicentre of the 8th World Congress.

Information: Enes Ozel (enozel@ku.edu.tr)
Website: http://pwc2012.ku.edu.tr/

Daniel Quillen

Graeme Segal

Daniel Quillen

Daniel Quillen, who died on 30 April 2011 at the age of 70, was among the most creative and influential mathematicians of his time, transforming whole areas of the subject. He solved a number of famous and important problems, but his most valuable contributions came from finding new ways of looking at the central questions of mathematics and opening paths into previously inaccessible terrain.

He was born in Orange, New Jersey, the elder of two brothers. His father Charles Quillen was a chemical engineer who became a teacher in a vocational high school and his mother Emma (née Gray) was a secretary. His mother in particular was very ambitious for her sons, and sought out scholarships for Dan, which carried him first to Newark Academy, an excellent private secondary school, and then (a year before finishing high school) to Harvard, where after his undergraduate degree he became a graduate student working under Raoul Bott. His thesis was on overdetermined systems of linear partial differential equations. Immediately on completing his PhD in 1964 he obtained a post at MIT, where he stayed (though with a number of years away on leave, at the IHES and in Princeton, Bonn and Oxford), till he moved to Oxford.

He said that Bott – a large outgoing man universally beloved for his warmth and personal magnetism, outwardly quite the opposite of his shy and reticent student – was a crucial model for him, showing him that one did not have to be quick to be an outstanding mathematician. Unlike Bott, who made a performance of having everything explained to him many times over, Quillen did not seem at all slow to others, yet he saw himself as someone who had to think things out very slowly and carefully from first principles, and had to work hard for every scrap of progress he made. He was truly modest about his abilities – very charmingly so – though at the same time ambitious and driven. Bott was a universal mathematician, who made contributions to many different areas of the subject while always preserving the perspective of a geometer, and Quillen too never confined himself to a 'field'. His most famous achievements were in algebra, but he somehow came at algebra from the outside. He was interested in almost all of mathematics and in a lot of physics too: when his eldest daughter was studying physics at Harvard he carefully worked through the problem sheets she was given, and 20 years later he was doing the same when his youngest daughter was studying electrical engineering at Imperial College. It was a characteristic of his mathematics that he drew in ideas from very diverse areas to use for his own purposes. Throughout his life he kept a beautifully-written record of the mathematical thoughts he had each day,[1] and they form an extraordinary archive, covering a huge range of topics, often his own reworkings of papers he had read or lectures he had attended. One finds, for instance, that in 1972, in the middle of the section where he was working out his treatment of algebraic K-theory for categories with exact sequences, there is a long digression entitled "Education in statistical mechanics", which begins with a conventional account of ideal gases and Carnot cycles that one might find in an undergraduate physics course, and then moves through a more mathematical discussion of entropy in statistical mechanics into considering how one can perturb the Hamiltonian or the symplectic structure on the product of a large number of copies of a symplectic manifold. It ends, puzzlingly, "Possible idea to use: entropy and how it arises from the gamma replacement for factorials".

The second great mathematical influence on Quillen – as on many others of his generation – was the towering figure of Alexander Grothendieck. Grothendieck is famous for his mystical conviction that a mathematical problem will solve itself when, by sufficient humble attentiveness, one has found exactly its right context and formulation. However that may be, he opened up one of the most magical panoramas of modern mathematics, connecting number theory, algebra and geometry. Grothendieck's influence is most evident in Quillen's first lastingly famous work, his Springer Lecture Notes volume *Homotopical Algebra*, published in 1967, on a completely different subject from that of his thesis.

Its historical context was the development over the previous few decades of the new field of "homological algebra": the art of assigning homotopy types – or, initially, homology groups – to many algebraic and combinatorial structures such as groups and algebras which at first sight have nothing space-like about them. Grothendieck's special contribution to this field was the invention (with his student Verdier) of the *derived category* in which any given abelian category – such as the modules for a given ring – can be embedded. The derived category is to the abelian category as the homotopy category is to the category of topological spaces. More strikingly, Grothendieck showed how to associate a homotopy type to an arbitrary commutative ring, and to an algebraic variety over any field, in a way which promised to prove Weil's conjectures (made in 1949) relating the number of points of algebraic varieties defined over finite fields to the topology of the corresponding varieties over the complex numbers. Quillen had made himself a master of the ideas of the Grothendieck school but at the same time had immersed himself in a different mathematical tradition, that of the MIT algebraic topologists, especially Daniel Kan, who was his third great influence. Kan was the apostle of simplicial methods: he proved that the homotopy theory of topological spaces can be studied by entirely combinatorial means. The homotopy category is obtained from the category of topological spaces by formally inverting maps which are homotopy equivalences, and Quillen realised that Kan had proved that the *same* category is obtained by inverting a class of maps in the category of simplicial sets. He asked himself when it makes sense to invert a class of morphisms in an arbitrary category and call the result a homotopy category. He saw that the key lay in the concepts of fibration and cofibration, the traditional tools of algebraic topology, and that these were the right context for the projective and injective resolutions of homological algebra – an injective module, for example, is the analogue of a simplicial set obeying the Kan condition. His book went on to develop a very complete abstract theory of homotopy. At the time it attracted little attention except from a small band of enthusiasts, but it proved very prescient; thirty years later the theory was being widely used, and it is central on the mathematical stage today. The book was severely abstract, with hardly any examples and no applications, but Quillen immediately went on to apply the ideas to develop a cohomology theory for commutative rings – now called 'André-Quillen cohomology' – and the associated theory of the cotangent complex and, after that, to show that the rational homotopy category can be modelled by differential graded Lie algebras or, equivalently, by commutative differential graded algebras.

None of his subsequent works has the same unmistakable Grothendieck flavour as this first book. Both Grothendieck and Quillen sought for what was absolutely fundamental in a problem but, where Grothendieck found the essence in generality, Quillen's guiding conviction was that to understand a mathematical phenomenon one must seek out its very simplest concrete manifestation. He felt he was not good with words, but his mathematical writings, produced by long agonised struggles to devise an account that others would understand, are models of lucid, accurate, concise prose, which, as Michael Atiyah has pointed out, are more reminiscent of Serre than of Grothendieck.

He spent the year 1968/9 as a Sloan Fellow at the IHES near Paris, where Grothendieck was based. The following year, spent at the Institute for Advanced Study in Princeton, was the most fertile of his life, and he produced a torrent of new work. Perhaps the most exciting item at the time was a proof of the Adams conjecture, which identifies – in terms of K-theory and its Adams operations – the direct summand in the stable homotopy groups of spheres which comes from the orthogonal groups. Quillen had already given an outline proof of this three years earlier, showing how it follows from the expected properties of Grothendieck's étale homotopy theory for algebraic varieties in characteristic p.[2] Meanwhile, however, he had been carefully studying the work of the algebraic topologists centred on Chicago, who had used ideas of infinite-loop-space theory to calculate the homology of many important classifying spaces. He now realised that the crucial idea of his first proof amounted to saying that the classifying spaces of the discrete group $GL_n(\bar{\mathbb{F}}_p)$ and of the Lie group $GL_n(\mathbb{C})$ have the same homology away from the prime p, and

Photograph taken at admission in Harvard

he saw how to prove this directly. (Here $\bar{\mathbb{F}}_p$ denotes the algebraic closure of the field with p elements.) This led straight to his development of algebraic K-theory, which is the achievement he is most remembered for; but before coming to that I shall mention a few other things.

First, the Adams conjecture was almost simultaneously proved by Dennis Sullivan, also using Grothendieck's theory but in a different way. While Quillen's proof led to algebraic K-theory, Sullivan's was part of a quite different programme, his determination of the structure of piecewise-linear and topological manifolds. This was just one of several places where Quillen's work intersected with Sullivan's, though they had different objects in view. Another was their independent development of rational homotopy theory, where Sullivan was motivated by explicit questions about the groups of homotopy equivalences of manifolds. Ib Madsen has remarked on the quirk of mathematical history that, a few years later, Becker and Gottlieb found a much more elementary proof of the Adams conjecture which did not use Grothendieck's theory; if this had happened earlier one may wonder when some active areas of current mathematics would have been invented.

At the ICM in Nice in 1970 Quillen described the theme of his previous year's work as the cohomology of finite groups. Besides the Adams conjecture and algebraic K-theory, another fertile line of development came out of this. Quillen had shown that the mod p cohomology of any compact group is controlled by the lattice of its elementary p-subgroups, proving, among other things, the Atiyah-Swan conjecture that the Krull dimension of the mod p cohomology ring is the maximal rank of an elementary p-subgroup, and calculating for the first time the cohomology rings of the spin groups. He was interested in using these ideas to obtain significant results in finite group theory, but quite soon he left the field to others.

Another achievement of this golden period concerned the complex cobordism ring and its relation to the theory of formal groups. This idea is the basis of most recent work in stable homotopy theory, beginning with the determination by Hopkins of the primes of the stable homotopy category, and the "chromatic" picture of the homotopy groups of spheres. Milnor's calculation of the complex cobordism ring in 1960 by means of the Adams spectral sequence had been one of the triumphs of algebraic topology. Quillen had been thinking about Grothendieck's theory of "motives" as a universal cohomology theory in algebraic geometry, and also about the

George Lusztig, Daniel Quillen, Graeme Segal and Michael Atiyah at IAS (Princeton), 1969

use Grothendieck had made of bundles of projective spaces in his earlier work on Chern classes and the Riemann-Roch theorem. He saw that complex cobordism had a similar universal role among those cohomology theories for smooth manifolds in which vector bundles have Chern classes, and he recognised that the fundamental invariant of such a theory is the formal group law which describes how the first Chern class of a line bundle behaves under the tensor product. He made the brilliant observation that the complex cobordism ring is the base of the universal formal group, and he succeeded in devising a completely new calculation of it, not using the Adams spectral sequence but appealing instead to the fundamental properties of the geometric power operations on manifolds. This work is yet another mélange of Grothendieck-style ideas with more concrete and traditional algebraic topology. After his one amazing paper on this subject he seems never to have returned to the area.

I shall not say much about Quillen's refoundation of algebraic K-theory here, as so much has been written about it elsewhere. As he explained it in 1969–70, one key starting point was the calculation of the homology of $BGL_\infty(\bar{\mathbb{F}}_p)$ and another was when he noticed that the known Pontrjagin ring of the union of the classifying spaces of the symmetric groups essentially coincided with the (also known) Pontrjagin ring of $\Omega^\infty S^\infty$, the infinite loop space of the infinite sphere. This led him to the idea that from a category with a suitable operation of 'sum' – such as the category of finite sets under disjoint union, or of modules over a ring under the direct sum – one can obtain a cohomology theory if, instead of forming the Grothendieck group from the semigroup of isomorphism-classes, one constructs in the homotopy category the group-completion of the topological semigroup which is the *space* of the category. The famous 'plus construction', which he used in his 1970 ICM talk, is a nice way to realise the group-completion concretely; it came from a suggestion of Sullivan, but I do not think it was the basic idea. Throughout his year in Princeton Quillen was making lightning progress understanding the homotopy theory of categories, which he had not thought much about before. He realised that he must find a homotopy version of the more general construction of Grothendieck groups in which the relations come from exact sequences rather than just from direct sums, and eventually he settled on the "Q-construction" as his preferred method of defining the space. The culmination of this work was the definitive treatment he wrote for the 1972 Seattle conference on algebraic K-theory. He published only one paper on algebraic K-theory after that: his proof in 1976 of Serre's conjecture that projective modules over polynomial rings are free. This came from reflecting deeply on what was already known about the question – especially the work of Horrocks – and seeing that, when brewed lovingly in the way Grothendieck advocated for opening nuts, the result fell out.

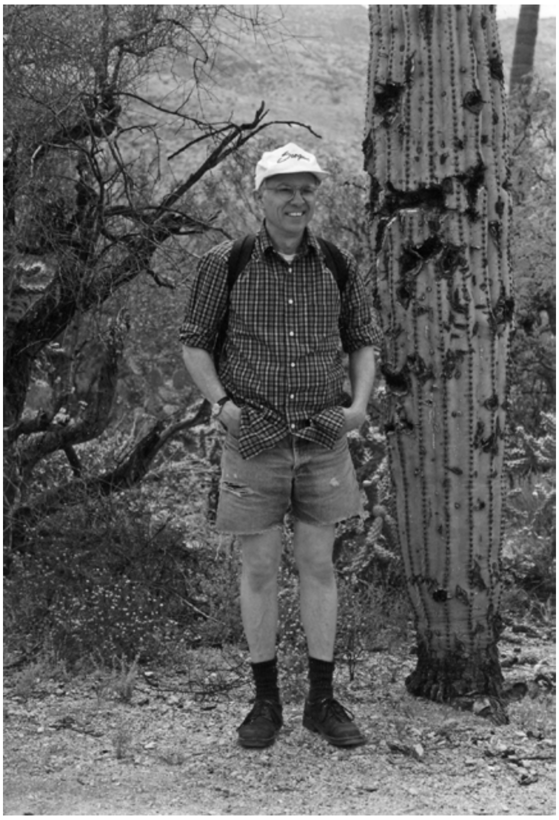
By 1978, when he was awarded a Fields Medal, Quillen's interests had shifted back towards global geometry and analysis. His notebooks of the years 1976-77 are mainly concerned with analysis: Sturm-Liouville theory for ordinary differential equations, scattering and inverse-scattering theory in one dimension, statistical mechanics, the theory of electric transmission lines, quantum and quantum-field-theoretical aspects of the same questions, and also orthogonal polynomials, Jacobi matrices and the de Branges theory of Hilbert spaces of entire functions. He gave a wonderful graduate course on these topics at MIT in 1977. He published nothing of this, however. He felt, I suppose, that he hadn't got any decisively new results. Nevertheless, I think one can say that a single circle of ideas connected with global analysis and index theory – an area extending to quantum field theory at one end and at the other end to algebraic K-theory through Connes's treatment of index theory by cyclic homology – held his interest in many different guises forever after. The very last graduate course he gave in Oxford (in the year 2000, I think) was on scattering theory for the discretised Dirac equation in two dimensions.

Early in 1982 he decided that Oxford was the place he wanted to be, attracted to it especially by the presence of Michael Atiyah. He spent the year 1982/3 on leave there, and in 1985 moved permanently from MIT to Oxford as Waynflete Professor. (The joke surged irresistibly around the mathematical world of a dean at MIT rushing to Dan with an offer to halve his salary.)

In the 1980s he made at the very least three outstanding contributions which will shape mathematics for a long time: the invention of the 'determinant line' of an elliptic partial differential operator as a tool in index theory, the concept of a 'superconnection' in differential geometry and analysis, and the Loday-Quillen theorem relating cyclic homology to algebraic K-theory.

The first of these came from thinking about the relation of index theory to anomalies in quantum field theory. Determinant lines were a familiar idea in algebraic geometry, and defining regularised determinants by means of zeta functions was standard in quantum field theory, and had been studied by mathematicians such as Ray and Singer. Nevertheless, the simple idea that any Fredholm operator has a determinant line in which its determinant lies, and that the role of the zeta function is to 'trivialise' the determinant line (i.e. identify it with the complex numbers) brought a new perspective to the subject.

'Superconnections' came from thinking about the index theorem for families of elliptic operators and also about Witten's ideas on supersymmetry in quantum theory. When one has a bundle whose fibres are compact Riemannian manifolds



there is a virtual vector bundle on the base which is the fibrewise index of the Dirac operators on each fibre. The index theorem for families gives a formula for the Chern character of this virtual vector bundle. Quillen's idea was to combine the formula expressing the index of a single Dirac operator D as the supertrace of the heat-kernel $\exp D^2$ with the identical-looking definition of the Chern character form of a connection in a finite-dimensional vector bundle as the fibrewise supertrace of $\exp D^2$, where D now denotes the covariant derivative of the connection, whose curvature D^2 is a matrix-valued 2-form. He aimed to prove the index theorem for families by applying this to the infinite-dimensional vector bundle formed by the spinor fields along the fibres, defining a superconnection D, with $\exp D^2$ of trace-class, by adding the fibrewise Dirac operator to the natural horizontal transport of spinor fields. Superconnections are now very widely used but, after the first short paper in which he gave the definition and announced his project, Quillen himself did not return to the index theorem for families, as Bismut published a proof of it the following year along Quillen's lines. Only two of his subsequent papers involved superconnections. One of them (jointly with his student Mathai) was extremely influential, though it dealt only with finite-dimensional bundles. It gave a beautiful account of the Thom class of a vector bundle in the language of supersymmetric quantum theory and has provided a basic tool in geometrical treatments of supersymmetric gauge theories.

The last phase of Quillen's work was mostly concerned with cyclic homology. He was attracted to this from several directions. On one side, cyclic cocycles had been invented as a tool in index theory, and "S-operator" of Connes is undoubtedly but mysteriously connected with Bott periodicity, whose role in general algebraic K-theory Quillen had constantly tried to understand. More straightforwardly, cyclic homology is the natural home of the Chern character for the algebraic K-theory of a general ring. Yet again, it seemed that cyclic theory ought somehow to fit into the framework of homotopical algebra of Quillen's first book. Connes was a virtuoso in developing cyclic cohomology by means of explicit cochain formulae but to someone of Quillen's background it was axiomatic that these formulae should not be the basis of the theory. In trying to find the "right" account of the subject he employed a variety of techniques, pursuing especially the algebraic behaviour of the differential forms on Grassmannians when pulled back by the Bott map. He had at least the one notable success already mentioned, when he proved a conjecture of Loday which, roughly, asserts that cyclic homology is to the Lie algebras of the general linear groups exactly as algebraic K-theory is to the general linear groups themselves.[3] In a paper written in 1989, dedicated to Grothendieck on his 60th birthday, he succeeded in giving a conceptual definition of cyclic homology, but still wrote that "a true Grothendieck understanding of cyclic homology remains a goal for the future". He produced quite a few more papers on the subject in the 1990s, most of them jointly with Cuntz, but on the whole I think he felt that, in T. S. Eliot's words, the end of all his exploring of Connes's work had been to arrive at where he started and know the place for the first time.

Outside mathematics his great love was music, especially the music of Bach. He always said that he met his wife Jean, whom he married before he was 21, when he was playing the triangle – and she the viola – in the Harvard orchestra. (She, however, says that he was the orchestra's librarian and occasional reserve trumpeter.) The triangle seems just the right instrument to go with his minimalist approach to mathematics. He delighted in 'figuring out' things about how music worked and in devising tiny compositions of 20 or 30 bars, but he was far too driven mathematically to let himself spend much time on music. He and Jean had two children before he completed his PhD, and went on to have six altogether. His family was his whole life apart from mathematics and, tongue-tied as he was, he never needed much encouragement from those he knew well to talk about his children's adventures and misadventures. Although his hair turned white in his 20s, he never lost the look or the manner of a teenager.

The last decade of his life was tragically blighted by steadily encroaching dementia. He is survived by his wife, his six children, 20 grandchildren and one great-grandchild.

Notes

1. Contradicting what he often said about his own slowness, he said that he needed to write these long careful accounts to slow himself down, as otherwise his thoughts rushed headlong onwards and ended in chaos and confusion.
2. This sketch proof was made complete a few years later in Friedlander's MIT thesis.
3. This theorem was proved independently and roughly simultaneously by Tsygan.

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Some aspects of tropical geometry

Erwan Brugallé

The goal of this text is to provide a quick overview of tropical geometry: what it is about and why it is useful, through some examples. After having defined basic objects in tropical geometry in Sections 1 and 2, we explain their relations to classical geometry via the notion of *amoebas* of algebraic curves in Section 3 and give some applications of tropical geometry to real and enumerative geometry in Section 4. We end this text with some further remarks in Section 5.

Only a few bibliographical references are given along the text and we tried each time to refer to the most elementary texts. For the interested reader who would like to deepen parts of this note, we refer for example to the general texts [Bru09], [RGST05], [BPS08], [IM], [Vir08], [Gat06], [IMS07], [Mik06] and references therein.

1 Tropical algebra

The first point of view on tropical geometry given here will be algebraic geometry built upon the *tropical semi-field*.

Tropical semi-field

The set of tropical numbers is defined by $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$, which we equip with the following tropical addition and multiplication (written within quotation marks):

$$\text{“}x + y\text{”} = \max(x, y) \qquad \text{“}x \times y\text{”} = x + y.$$

For example, we have the following identities:

$$\text{“}1 + 1\text{”} = 1, \qquad \text{“}1 + 2\text{”} = 2, \qquad \text{“}1 \times 2\text{”} = 3,$$
$$\text{“}1 \times (-2)\text{”} = -1, \qquad \text{“}(5 + 3)^2\text{”} = 10.$$

It follows immediately from the definition that tropical addition is *idempotent*, i.e. “$x + x$” $= x$ for any tropical number x. As a consequence, a tropical number x does not have an inverse for tropical addition except if $x = -\infty$ (the neutral element for “ + ”). However, this lack of additive inverse is the only axiom of a field which is not fulfilled by ($\mathbb{T}$, “ + ”, “ × ”). This is precisely the definition of a semi-field. Note that any $x \neq -\infty$ has an inverse for tropical multiplication and that “x^{-1}” $= -x$.

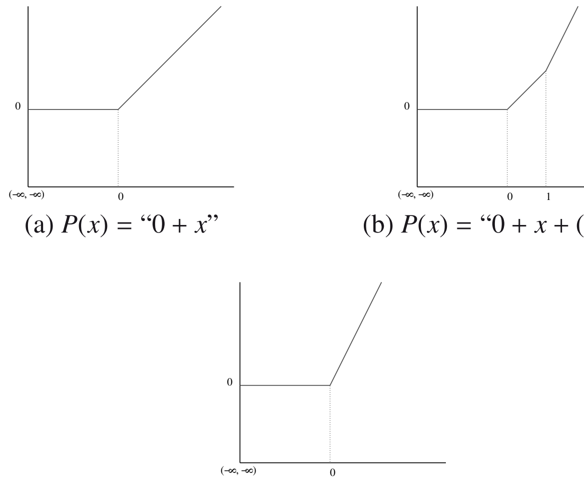
Tropical polynomials

As soon as addition and multiplication are defined, polynomials show up. By definition, a tropical polynomial function is of the form

$$P(x) = \text{“}\sum_{i=0}^{d} a_i x^i\text{”} = \max_i (a_i + ix), \text{ where } a_0, \ldots, a_d, x \in \mathbb{T}.$$

What is a tropical root of a tropical polynomial? Here we face a recurring problem in tropical mathematics: several equivalent definitions in classical mathematics may produce different tropical objects.

If we look at tropical numbers x_0 such that $P(x_0) = -\infty$, the answer doesn't bring so much information about P: since $P(x) \geq a_0$ for any $x \in \mathbb{T}$, there does not exist such a x_0 except in the case where $a_0 = -\infty$ and $x_0 = -\infty \ldots$



(a) $P(x) =$ “$0 + x$”

(b) $P(x) =$ “$0 + x + (-1)x^2$”

(c) $P(x) =$ “$0 + x^2$”

Figure 1. Graph of tropical polynomials

To bring out a meaningful notion of a tropical root, let us look instead at the graph of a tropical polynomial function (see Figure 1). A tropical polynomial $P(x)$ is a piecewise affine function and its graph has some *corner points* that will be our tropical roots. They are exactly the tropical numbers where $P(x)$ is not locally given by a monomial.

More precisely, we say that $x_0 \in \mathbb{T}$ is a tropical root of order k of $P(x) =$ “$\sum_{i=0}^{d} a_i x^i$” if there exist two indices i and j such that in a neighbourhood of x_0 in $\mathbb{T}$ we have $P(x) =$ “$a_i x^i + a_j x^j$” and $k = |i - j|$ is minimal. The next proposition shows that this notion of tropical root is equivalent to a more classical definition.

Proposition 1.1. *The tropical semi-field $\mathbb{T}$ is algebraically closed. Moreover, x_0 is a tropical root of order k of a tropical polynomial $P(x)$ if and only if there exists a tropical polynomial $Q(x)$ which does not have x_0 as a tropical root and such that*

$$P(x) = \text{“}(x + x_0)^k Q(x)\text{”} \quad \forall x \in \mathbb{T}.$$

For example we have the following factorisations (see Figure 1):

“$0+x+(-1)x^2$” = “$(-1)(x+0)(x+1)$” and “$0+x^2$” = “$(x+0)^2$”.

2 Tropical curves

Definition

Let us now turn to tropical polynomials in two variables and the tropical curves they define. Since it makes all definitions simpler, we restrict ourselves to tropical curves in $\mathbb{R}^2$ instead of $\mathbb{T}^2$.

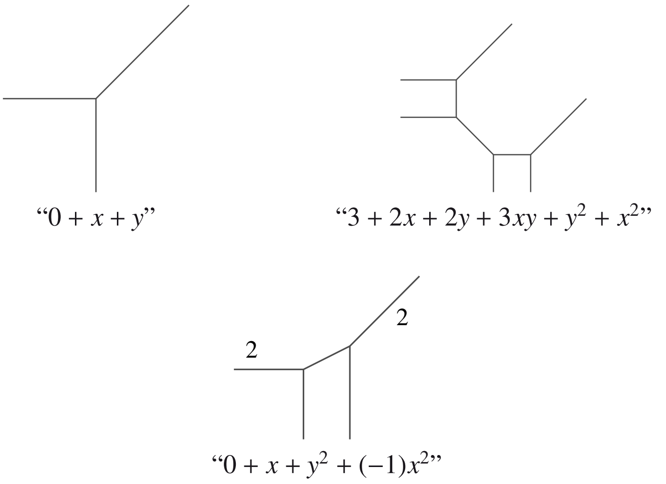


Figure 2. A tropical line and two tropical conics

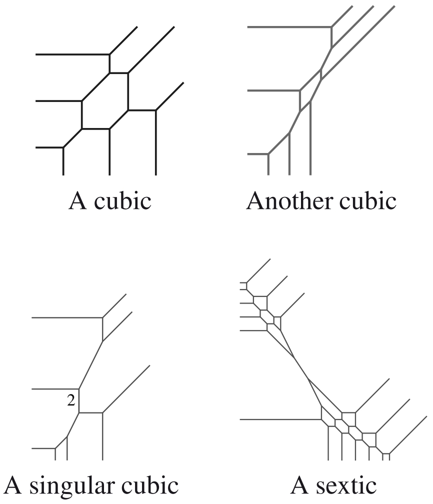


Figure 3. Tropical cubics and sextic

Similarly to the case of univariate polynomials, let us define the *corner set* $V(P) \subset \mathbb{R}^2$ of a tropical polynomial $P(x, y) =$ “$\sum a_{i,j}x^iy^j$” as the set where $P(x, y)$ is not locally given by a monomial. That is to say

$$V(P) = \{(x_0, y_0) \in \mathbb{R}^2 \mid \exists (i, j) \neq (k, l), \\ P(x_0, y_0) = \text{“}a_{i,j}x_0^iy_0^j\text{”} = \text{“}a_{k,l}x_0^ky_0^l\text{”}\}.$$

Since $P(x, y)$ is a piecewise affine function, the set $V(P)$ is a piecewise linear graph in $\mathbb{R}^2$ (from now on the word “graph” has to be understood in its graph theoretical sense). Any edge e of $V(P)$ is adjacent to exactly two connected components E_1 and E_2 of $\mathbb{R}^2 \setminus V(P)$. Let us say that the value of $P(x, y)$ is given by the monomial “$a_{i,j}x^iy^j$” on E_1 and by “$a_{k,l}x^ky^l$” on E_2. We define the *weight* of e as

$$w(e) = \gcd(|i - k|, |j - l|).$$

The weight of an edge might be seen as the 2-dimensional analogue of the order of a tropical root. The *tropical curve C defined by* $P(x, y)$ is the set $V(P)$ enhanced with this weight function on its edges. In the examples of tropical curves depicted in Figures 2 and 3, we specify the weight of an edge only when it is at least 2.

Balancing condition

There exists an equivalent definition of a plane tropical curve that one can formulate into combinatorial terms. Let Γ be a piecewise linear graph in $\mathbb{R}^2$ equipped with a weight function $w : \mathrm{Edge}(\Gamma) \to \mathbb{Z}_{>0}$ and whose edges admit an integral direction. Given an edge e of Γ adjacent to a vertex v, we may choose $\vec{u}_{v,e}$ the smallest (i.e. primitive) direction in $\mathbb{Z}^2$ of e pointing away from v. The graph Γ is called *balanced* if it satisfies the following *balancing condition* at any vertex v:

$$\sum_{e \text{ adjacent to } v} w(e)\vec{u}_{v,e} = 0.$$

Proposition 2.1 (Mikhalkin)**.** *Tropical curves in* $\mathbb{R}^2$ *correspond exactly to balanced graphs in* $\mathbb{R}^2$.

For example, the three primitive integral directions in the case of the vertex of a tropical line are $(-1, 0)$, $(0, -1)$ and $(1, 1)$, whose sum is indeed 0.

Bézout’s Theorem

Tropical curves share many properties with plane complex algebraic curves, i.e. subsets of $\mathbb{C}^2$ with equation $P(z, w) = 0$ where $P(z, w)$ is a polynomial with complex coefficients. For example, both classes of objects satisfy Bézout’s Theorem, the genus formula, etc. Here we focus on the former.

Classically, Bézout’s Theorem states that two plane complex algebraic curves of degree d and d' in general position have exactly dd' intersection points.

To prove an analogous statement for tropical curves, we first have to introduce the *multiplicity* of a tropical intersection point. Let C and C' be two tropical curves such that the set $C \cap C'$ does not contain any vertex of C or C'. Hence, a point $p \in C \cap C'$ lies on an edge e of C of weight w and on an edge e' of C' of weight w' (see Figure 4a). Let $\vec{u}_e$ be a primitive integral direction of e and $\vec{u}_{e'}$ be a primitive integral direction of e'. The multiplicity $m(p)$ of p is defined as the Euclidean area of the parallelogram spanned by the two vectors $w\vec{u}_e$ and $w'\vec{u}_{e'}$, i.e.

$$m(p) = ww'|\det(\vec{u}_e, \vec{u}_{e'})|.$$

The following tropical version of Bézout’s Theorem has a very elementary proof which requires only very basic mathematical knowledge.

Proposition 2.2 (Sturmfels)**.** *If C and C' are two tropical curves of degree d and d' in general position then* $\sum_{p \in C \cap C'} m(p) = dd'$.

For example, the tropical line and conic in Figure 4b have two intersection points, both of multiplicity 1, whereas the ones in Figure 4c have only one intersection point, of multiplicity 2.

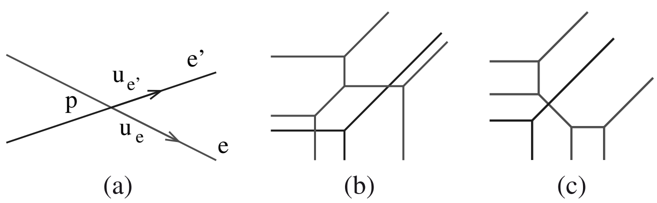


Figure 4. Tropical intersection

3 Link to classical geometry

The similarity between tropical and complex curves mentioned in the previous section is much more than an amusing coincidence. Tropical geometry has very deep connections and relations with classical algebraic geometry.

Maslov dequantization

Let us start by relating the tropical semi-field with a classical semi-field that we all know quite well: $(\mathbb{R}_{\geq 0}, +, \times)$. This process, studied by Maslov and his collaborators since the 90s, is known as the *Maslov dequantization of real numbers*.

Given a positive real number t, the bijection

$$\log_t = \frac{\log}{\log t} : \mathbb{R}_{\geq 0} \to \mathbb{T} = [-\infty; +\infty)$$

induces a semi-field structure on $\mathbb{T}$, where the two operations denoted by " $+_t$ " and " $\times_t$ " are given by

$$\text{“}x +_t y\text{”} = \log_t(t^x + t^y) \quad \text{and} \quad \text{“}x \times_t y\text{”} = \log_t(t^x t^y) = x + y.$$

We already have the appearance of classical addition as multiplication " $\times_t$ " on $\mathbb{T}$. By construction, all the semi-fields $(\mathbb{T}, \text{“} +_t \text{”}, \text{“} \times_t \text{”})$ are isomorphic to $(\mathbb{R}_+, +, \times)$. Moreover, we have the following simple inequalities:

$$\forall t > 0, \quad \max(x, y) \leq \text{“}x +_t y\text{”} \leq \max(x, y) + \log_t 2.$$

In particular, when t tends to infinity, the law " $+_t$ " converges to tropical addition! Hence, the tropical semi-field arises naturally as a degeneration of the classical semi-field $(\mathbb{R}_+, +, \times)$. From an alternative perspective, one can see the classical semi-field $(\mathbb{R}_+, +, \times)$ as a deformation of the tropical semi-field, which justifies the "dequantization" terminology.

Amoebas

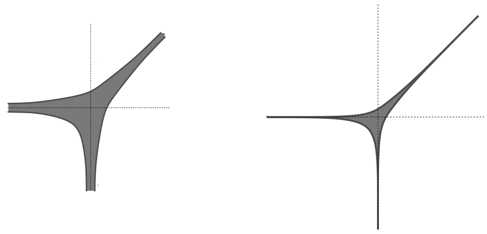
This dequantization process also applies to plane complex curves. For this, we need the following map

$$\begin{array}{rccc} \mathrm{Log}_t : & (\mathbb{C}^*)^2 & \longrightarrow & \mathbb{R}^2 \\ & (z, w) & \longmapsto & (\log_t |z|, \log_t |w|). \end{array}$$

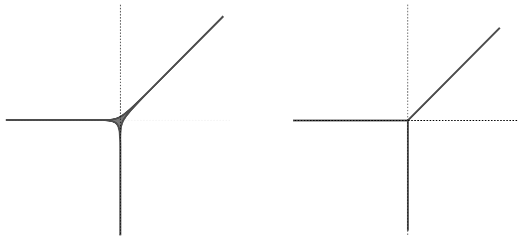
Given an algebraic curve in $(\mathbb{C}^*)^2$, its image under the map Log_t is called its *amoeba* (in base t). Let us look more closely at these amoebas with the help of a concrete example, namely the line L with equation $z + w + 1 = 0$ in $(\mathbb{C}^*)^2$. One can compute by hand that the amoeba of L is as depicted in Figure 5a. In particular, we see that it has three asymptotic directions: $(-1, 0)$, $(0, -1)$, and $(1, 1)$.

By the definition of $\log_t$, the amoeba of L in base t is a contraction by a factor $\log t$ of the amoeba of L in base e (see Figures 5b and 5c). Hence when t goes to $+\infty$, the whole amoeba is contracted to the origin and only the three asymptotic directions remain. In other words, what we see at the limit in Figure 5d is a tropical line!

Of course, the same strategy applied to any classical curve will produce a similar picture at the limit: the origin from which the asymptotic directions of the amoeba emerge. To get a more interesting limit, one should look not at amoebas in base t of a single complex curve but at the family of amoebas $(\mathrm{Log}_t(C_t))_{t>0}$, where $(C_t)_{t>0}$ is now *a family* of complex curves. If we do so then the limit becomes much richer. For example, we depict in Figure 6 the shape of the amoeba of the curve with equation $1 - z - w + t^2z^2 - tzw + t^2y^2 = 0$ for t large enough and its limit, which is ... a tropical conic.

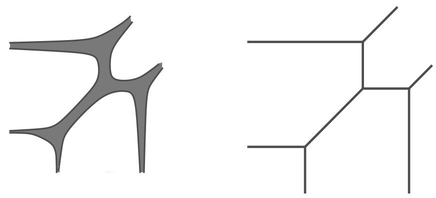


(a) $\mathrm{Log}(L)$ (b) $\mathrm{Log}_{t_1}(L)$



(c) $\mathrm{Log}_{t_2}(L)$ (d) $\lim_{t\to\infty} \mathrm{Log}_t(L)$

Figure 5. Dequantization of a line ($e < t_1 < t_2$)



$\mathrm{Log}_t(C_t)$ $\lim_{t\to\infty} \mathrm{Log}_t(C_t)$

Figure 6. $C_t : 1 - z - w + t^2z^2 - tzw + t^2y^2 = 0$

Theorem 3.1 (Mikhalkin, Rullgård). *Let $P_t(z, w) = \sum_{i,j} \alpha_{i,j}(t) z^i w^j$ be a polynomial whose coefficients are functions $\alpha_{i,j} : \mathbb{R} \to \mathbb{C}$ and suppose that $\alpha_{i,j}(t) \sim \gamma_{i,j} t^{a_{i,j}}$ when t goes to $+\infty$ with $\gamma_{i,j} \in \mathbb{C}^*$. If C_t denotes the curve in $(\mathbb{C}^*)^2$ defined by the polynomial $P_t(z, w)$ then the amoeba $\mathrm{Log}_t(C_t)$ converges to the tropical curve defined by the tropical polynomial $P_{trop}(x, y) = \text{“} \sum_{i,j} a_{i,j} x^i y^j \text{”}$.*

It remains for us to explain the relation between amoebas and weights of a tropical curve. Let $P_t(z, w)$ and $P'_t(z, w)$ be two families of complex polynomials, defining two families of complex algebraic curves $(C_t)_{t>0}$ and $(C'_t)_{t>0}$ respectively. As in Theorem 3.1, these two families of polynomials induce two tropical polynomials $P_{trop}(x, y)$ and $P'_{trop}(x, y)$, which in turn define two tropical curves C and C'.

Proposition 3.2 (Mikhalkin). *Let $p \in C \cap C'$ which is a vertex of neither C nor C'. Then the number of intersection points of C_t and C'_t whose image under Log_t converges to p is exactly equal to $m(p)$.*

It is worth remarking that the number of intersection points which converge to p *only depends on C and C'*, that is to say only on the order at infinity of the coefficients of $P_t(z, w)$ and $P'_t(z, w)$.

A combination of Theorem 3.1, Proposition 3.2 and the tropical version of Bézout's Theorem provides a proof of Bézout's Theorem for complex curves. In the next section, we explore two deeper applications of tropical geometry.

4 Examples of application

Combinatorial construction of real algebraic curves

Given a real polynomial $P(z, w)$, it is usually very difficult to compute the "picture" realised by the real algebraic curve with equation $P(z, w) = 0$ in $\mathbb{R}^2$. More generally, the problem of classifying all possible mutual arrangements of the connected components of a real algebraic curve of a fixed degree is a beautiful but extremely difficult question. This is the 16th problem that Hilbert posed in his famous list, and is still widely open. Up till now the complete answer is known only up to degree 7, and the method described below was one of the tools thanks to which Viro completed this classification.

Tropical geometry produces real algebraic curves whose arrangement of connected components can be recovered thanks to some elementary combinatorial rules. Despite its simplicity, this method produces real algebraic curves with very rich topology. For example, Itenberg disproved drastically in this way Ragsdale's conjecture posed one century ago (see [IV96])!

We now describe this method, known as *combinatorial patchworking*. Let $P_t(z, w) = \sum_{i,j} \alpha_{i,j}(t) z^i w^j$ be a polynomial whose coefficients are *real* valued functions $\alpha_{i,j} : \mathbb{R} \to \mathbb{R}$. As in Theorem 3.1, we suppose that $\alpha_{i,j}(t) \sim \gamma_{i,j} t^{a_{i,j}}$ at infinity, and we denote by C the tropical curve defined by $P_{trop}(x, y)$. For simplicity, we assume the following technical condition: all weights of C are equal to 1 and, given any vertex v of C and any two edges e and e' adjacent to v, the Euclidean area of the triangle spanned by $\vec{u}_{v,e}$ and $\vec{u}_{v,e'}$ is the minimum possible, i.e. $|\det(\vec{u}_{v,e}, \vec{u}_{v,e'})| = 1$. Such a tropical curve is said to be *non-singular*. This condition ensures that the complex algebraic curve defined by $P_t(z, w)$ in $(\mathbb{C}^*)^2$ is non-singular when t is large enough. All the tropical curves depicted in Figures 2 and 3 except the ones of Figures 2c and 3c are non-singular.

We first recover the real algebraic curve defined by $P_t(z, w)$ in the quadrant $(\mathbb{R}_{>0})^2$. Given an edge e of C adjacent to two connected components E_1 and E_2 of $\mathbb{R}^2 \setminus C$, we may assume that the value of $P_{trop}(x, y)$ is given by the monomial "$a_{i,j} x^i y^j$" on E_1 and by "$a_{k,l} x^k y^l$" on E_2. Let us erase e if the signs of $\gamma_{i,j}$ and $\gamma_{k,l}$ coincide.

We denote by $\mathbb{R}C$ the piecewise linear curve obtained in $\mathbb{R}^2$ after performing this operation to all edges of C. For example, starting with the tropical curve depicted in Figure 7a, we obtain the curve depicted in Figure 7b by choosing appropriate signs for the coefficients $\gamma_{i,j}$.

Theorem 4.1 (Viro)**.** *For t large enough, the real algebraic curve defined by $P_t(z, w)$ in $(\mathbb{R}_{>0})^2$ is isotopic to $\mathbb{R}C$.*

In other words, the mutual arrangement of the connected components of the real algebraic curve defined by $P_t(z, w)$ in the positive quadrant of $\mathbb{R}^2$ is given by $\mathbb{R}C$. By symmetry, one can of course deduce the real curve defined by $P_t(z, w)$ in the whole plane $\mathbb{R}^2$. For example, the signs we chose to obtain Figure 7b actually produce the real tropical curve depicted in Figure 7c, which attests the existence of a real algebraic sextic in $\mathbb{R}^2$ arranged as in Figure 7d. Such a curve was first constructed by Gudkov in the late 60s. An interesting piece of trivia is that Hilbert claimed in 1900 that such a curve could not exist!

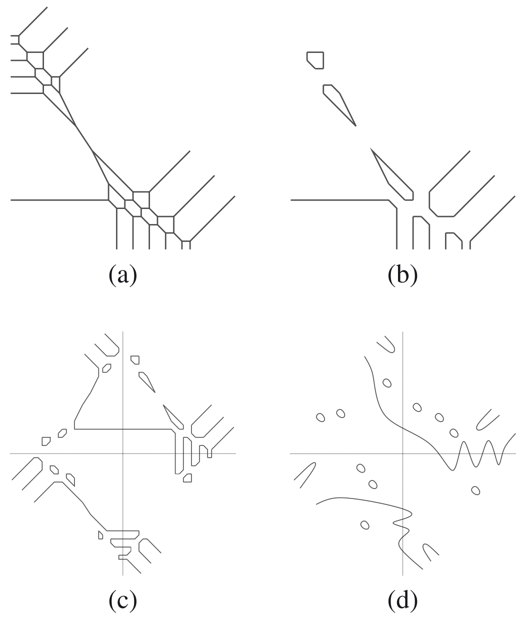


Figure 7. Gudkov curve

Enumerative geometry

Tropical geometry has also turned out to be very fruitful in enumerative geometry, the art of counting curves.

A double point of a complex algebraic curve C is a point where two branches of C intersect. The typical example of such a double point is the intersection point of two lines, and an irreducible complex algebraic curve of degree d in $\mathbb{C}^2$ has at most $\frac{(d-1)(d-2)}{2}$ double points. Given two integers $d \geq 1$ and $0 \leq r \leq \frac{(d-1)(d-2)}{2}$, a simple example of an enumerative problem is the following: how many irreducible complex algebraic curves of degree d with r double points pass through a generic configuration of $\frac{d(d+3)}{2} - r$ points?

Note that this number does not depend on the choice of the generic configuration of points (like the number of roots of a complex polynomial only depends on its degree and not on its coefficients). It is known as a *Gromov-Witten* invariant of the plane and we denote it by $N(d, r)$.

For example, $N(1, 0) = 1$ since there is a unique line passing through 2 points and, more generally, $N(d, 0) = 1$ since the solution curve is given by an invertible system of linear equations on its coefficients. The first non-trivial value is $N(3, 1) = 12$, i.e. there exist 12 cubic curves with one double point passing through 8 points.

There exist several methods to compute those numbers and one of them, suggested by Kontsevich, is via tropical geometry. Indeed, one can reformulate this classical enumerative problem into tropical terms and the answer turns out to be the same as in complex geometry. This is a deep and beautiful theorem by Mikhalkin ([Mik05]).

Theorem 4.2 (Mikhalkin)**.** *The number of irreducible tropical curves, counted with multiplicity, of degree d with r double points passing through a generic configuration of $\frac{d(d+3)}{2} - r$ points in $\mathbb{R}^2$ is equal to the corresponding number of complex curves.*

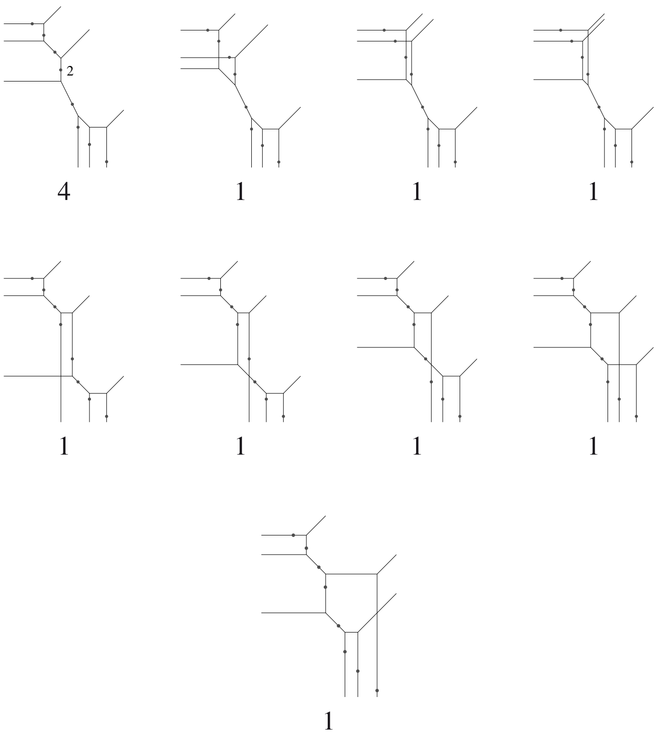


Figure 8. $N(3, 1) = 12$

We do not have space here to define a double point of a tropical curve or to specify the multiplicity of a tropical curve (see instead [Mik05] or [BPS08]). Let us just say that those definitions are completely combinatorial. As an example, we depict in Figure 8 all tropical cubics with one node passing through 8 points and we indicate in each case the multiplicity of the tropical curve. Summing up all those multiplicities, we find again $N(3, 1) = 12$.

It is worth mentioning that tropical geometry is also a very powerful tool in *real enumerative geometry*. Counting real algebraic curves is much more delicate than counting complex algebraic curves. One of the main advantages of tropical geometry is that it solves an enumerating problem by *exhibiting* all the solutions. In particular it allows one to count complex *and* real curves at the same time. For example, tropical geometry is a very useful tool to compute *Welschinger invariants*, a real analogue of Gromov-Witten invariants when r is maximum. For example, we can deduce from those Welschinger invariants and Figure 8 that there *always* exist *at least* eight real cubics with one double point passing through eight points in the real plane.

Theorem 4.2 has been the starting point of many developments in tropical enumerative geometry and its applications. For example, the problem of computing Gromov-Witten and Welschinger invariants can be turned into a purely combinatorial problem via the so-called *floor diagrams* (see [BPS08]).

5 Further in tropical geometry

Tropical varieties of higher (co)dimension

So far we only considered plane tropical curves. What about tropical varieties of higher dimension and codimension?

We have seen three equivalent definitions of a tropical curve:

(1) an algebraic one via tropical polynomials;
(2) a combinatorial one via balanced graphs;
(3) a geometric one via limits of amoebas.

All these three definitions can be generalised to arbitrary dimension. If they remain equivalent for tropical hypersurfaces of $\mathbb{R}^n$ (see [IMS07]), they produce different objects in higher codimension. For example, it is known in classical geometry that any cubic curve of genus 1 in $\mathbb{C}P^3$ is contained in a plane. However, there exist tropical cubics of genus 1 in the 3-space which are not contained in any tropical hyperplane (see [Mik05]). Such a pathological tropical curve cannot be a limit of amoebas of any family of spatial complex cubic curves.

The problem of determining which balanced polyhedral complexes are limits of amoebas is very important in tropical geometry and is still widely open.

Tropical projective spaces

The logarithm transforms multiplications to additions. As a consequence, any operation performed in complex algebraic geometry using only monomial maps translates mutatis mutandis in the tropical setting. In other words, tropical toric varieties can be constructed exactly as in complex geometry. Let us illustrate this with a classical construction: projective spaces.

The projective line $\mathbb{C}P^1$ may be obtained by taking two copies of $\mathbb{C}$, with coordinates z_1 and z_2, and gluing them along $\mathbb{C}^*$ via the identification $z_2 = z_1^{-1}$. Similarly, the projective plane $\mathbb{C}P^2$ can be constructed by taking three copies of $\mathbb{C}^2$, with coordinates (z_1, w_1), (z_2, w_2) and (z_3, w_3), and gluing them along $(\mathbb{C}^*)^2$ via the identifications

$$(z_2, w_2) = (z_1^{-1}, w_1) \text{ and } (z_3, w_3) = (z_1, w_1^{-1}).$$

Since "x^{-1}" $= -x$, the above constructions also yield the projective tropical line $\mathbb{T}P^1$ and plane $\mathbb{T}P^2$. In particular, we see that $\mathbb{T}P^1$ is a segment (Figure 9a) and $\mathbb{T}P^2$ is a triangle (Figure 9b). More generally, the projective space $\mathbb{T}P^n$ is a simplex of dimension n, each of its faces corresponding to a coordinate hyperplane.

For example, the tropical 3-space $\mathbb{T}P^3$ is a tetrahedron (see Figure 9c). Note that tropical projective spaces carry much more than just a topological structure: since all gluing maps are classical linear maps with integer coefficients, each open face of dimension p can be identified to $\mathbb{R}^p$ together with the lattice $\mathbb{Z}^p$ inside.

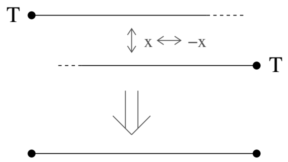
As usual, the space $\mathbb{R}^2 = (\mathbb{T}^*)^2$ embeds naturally into $\mathbb{T}P^2$ and any tropical curve in $\mathbb{R}^2$ has a closure in $\mathbb{T}P^2$. For example, we depict in Figure 9d the closure in $\mathbb{T}P^2$ of a tropical line in $\mathbb{R}^2$.

Tropical modifications

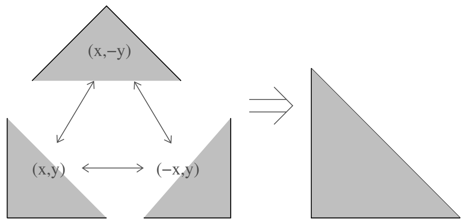
A new feature of tropical geometry comes out at that point: the polymorphism of tropical objects.

Over the complex numbers, there is no difference between an abstract projective line and a line in $\mathbb{C}P^2$; they are isomorphic. However, the tropical version of these two different situations produces two different objects: a segment in Figure 9a and a tripod in Figure 9d. It seems that we constructed two different tropical projective lines . . .

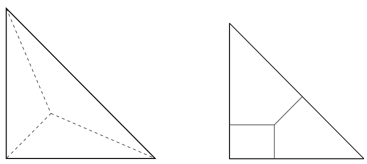
What does it mean? What is the relation between these two tropical manifestations of the same projective line? For-



(a) $\mathbb{T}P^1$



(b) $\mathbb{T}P^2$



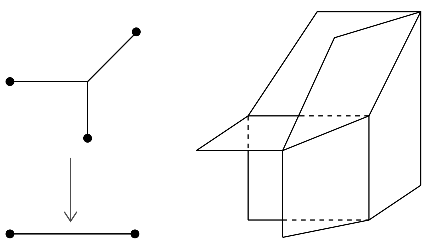
(c) $\mathbb{T}P^3$ (d) a line in $\mathbb{T}P^2$

Figure 9. Tropical projective spaces

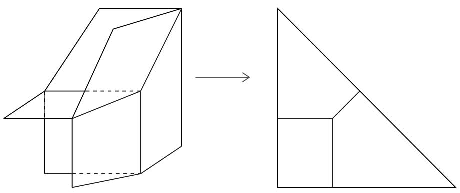
getting the second coordinate in $\mathbb{T}P^2$, the tropical line is projected to the horizontal side of the triangle. However, this side is nothing else but … the $\mathbb{T}P^1$ of Figure 9a (see Figure 10a). Hence, even if we constructed two different $\mathbb{T}P^1$, they are related by this projection of the tripod to the segment which contracts an edge of the tripod.

This phenomenon is not specific to the one dimensional case. For example, we depict in Figure 10b the tropical projective plane Π obtained by taking the closure in $\mathbb{T}P^3$ of a tropical plane in $\mathbb{R}^3$. Forgetting the third coordinate induces a projection from Π to the triangle $\mathbb{T}P^2$, which contracts 3 of the 6 faces of Π to a tropical line (see Figure 10c).

These two projections are examples of the so-called *tropical modifications*. Any tropical variety has infinitely many models as polyhedral complexes, and all these models are related by a sequence of tropical modifications. So the two tropical projective lines or planes that we constructed above are not two different $\mathbb{T}P^1$ or $\mathbb{T}P^2$ but two different *representatives* of $\mathbb{T}P^1$ and $\mathbb{T}P^2$.



(a) (b)



(c)

Figure 10. Tropical modifications

What is the significance of those infinitely many tropical representatives of the same variety? Given a family of complex algebraic varieties $X = (X_t)_{t>0}$, there is no canonical way of associating a tropical variety to X. In other words, X has no canonical *tropicalization*. All embeddings of (open subsets of) X as a family in $(\mathbb{C}^*)^n$ will produce as many different balanced polyhedral complexes in $\mathbb{R}^n$. However, all those tropical representatives are related by tropical modifications and the consideration of all those tropical modifications (or rather the inverse limit) is *the* tropicalization of X. This is a topological space, which is homeomorphic to another construction in algebraic geometry: the analytification in the Berkovich sense of X (see [Pay09]).

Hence, given some problem on X, one step in the tropical approach is to choose the simplest tropical representative of the tropicalization of X for which tropical geometry can actually help.

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Mathematics in Health Care – "Nice to Have" or Key for Future Success?

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1. Mathematics for industry – a scientific challenge or an intellectual playground?

When I joined an industry laboratory after finishing my PhD thesis in mathematics, the response of the faculty was almost binary. Some of the faculty members were proud of the students getting interesting jobs in industry, while for others they were lost sons who had decided to spend their lives with trivialities. Mathematicians often complain about their perception in industry, especially where they are compared to engineers, scientists and life scientists. In the last few years, various reports [1, 2] have been published describing today's situation of mathematics in industry and discussing the specific issues of mutual perception. Moreover, highlights of industrial applications as well as roadmaps toward the realisation of the benefits mathematics could provide to industry have been published [3]. Hence, we will not discuss here the situation of mathematicians in industry; having joined industry, most mathematicians, after a while, act very similar to other scientists and engineers, as problem-solvers using their specific skills. Instead, we will focus on the rising potential mathematics may have for future needs in industry and society with a special focus on health care.

In order to avoid misunderstandings we must concede that there are (and will always be) fundamental differences in doing mathematics in academia and in industry.

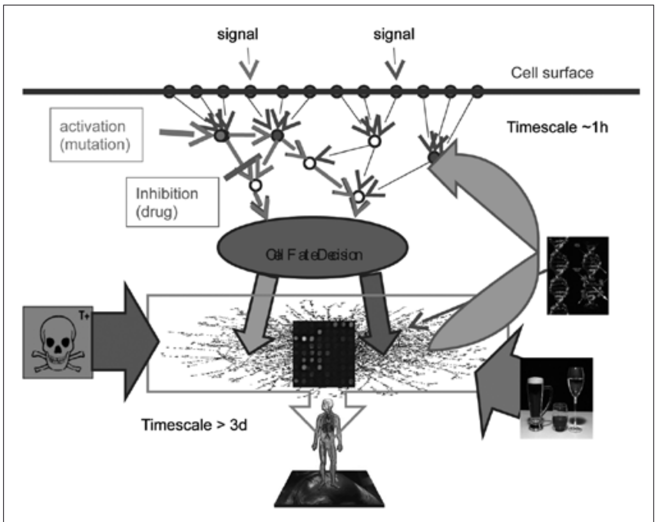

Fig. 1: Diseases and therapies interact in an extremely complex network which is affected by extrinsic factors as well, such as lifestyle or food. The networks are multi-scale networks covering a multitude of processes in closed loop mutual control, such as signaling, stress response and gene regulation and many more. The detailed mechanisms of the mutual control are not fully understood. Moreover, predictive models have to cover very heterogeneous time scales from hours for intracellular signaling to weeks and months for occurrence of resistance.

Research in industry is not driven by curiosity but by developing new products or technologies which either help to improve processes or offer new beneficial solutions to customer needs. As mathematics almost never has the role of a product by itself (the finance business being an exception), it can be seen as a typical "enabling technology", providing tools for product or process development. This fundamental basis of industrial (and applied) research results in different approaches to mathematics in industry:

Specific features of mathematics research in academia, such as restriction of methods to specified areas of mathematics, have no purpose in industry.

A priori settings for a theorem (or algorithm, etc.) are given by the problem to be solved. Hence it is almost useless to develop a mathematical theorem, algorithm or workflow if the a priori assumptions used to guarantee the proof do not cover the specific requirements of the problem. In contrast, if the algorithm covers the problem, an extension to a broader range of assumptions is almost never of interest and is sometimes even regarded as a waste of time and money. Therefore, investment of time in an extension must be justified by novel applications which will be covered by the extended algorithm.

More serious issues arise from the fact that the above-mentioned requirements are often not well specified. Hence a proof of the theorem or correctness of the algorithm is often not possible in the academic sense. I strictly object to the widespread opinion that in industry a proof is not recognised to be relevant. It is of fundamental relevance that the mathematics delivers correct results in any possible setting of the application. So a proof in its basic meaning is highly desirable in industry as well as in academia. However, because of either poorly defined specifications or due to the complexity of the problem, a proof covering all possible applications can often not be developed within an affordable timeframe.

From these issues, a fundamental challenge arises for industrial mathematics. In any case where a proof covering all possible applications cannot be found, industry needs efficient algorithms for testing the validity of a theorem or algorithm prior to any definite application (necessary for all online applications). If this is not feasible, at least efficient a posteriori tests on validity are indispensable for industrial applications. This holds even more for health care applications, where erroneous decisions can lead to fatal consequences.

A further obstacle for broad application of mathematics in industry is often the lack of reliable models. For system-simulation and optimisation, the availability of reliable quantitative models is apparently indispensable. A model means either a set of equations or an algorithm which allows the prediction of the quantitative outcome of the system given the input variables. It therefore represents the full system behaviour for all feasible combinations of the input variables. In principal, establishing models can be performed in three ways:

A model is called mechanistic if the underlying equations or algorithm represent the intrinsic system structure in a quantitative manner. Hence a mechanistic model exhibits the detailed mechanism of the system thereby allowing not only simulation and optimisation but fostering scientific understanding of the system as well.

In contrast, black box models do not represent any mechanistic details of the system. They are realised by approximation of the system behaviour using machine learning methods which allow the representation of the input-output relation of almost any system. They are sufficient for simulation and optimisation but do not provide any scientific insight into the inner structure of the system.

Grey box models provide an appropriate mixture between mechanistic and black box models [4].

All approaches are characterised by specific benefits and drawbacks. Industrial and, in particular, health care applications of mathematics suffer from a significant lack of models. Either the intrinsic mechanisms of the systems are not sufficiently understood to establish mechanistic models or the data demand to establish reliable black box models is too high to be affordable. So, for a broad range of applications, establishing reliable models is crucial. The challenge is the development of case specific combinations of inverse problems, statistics, pattern recognition and many other mathematical disciplines. Perfect examples are health care and pharmaceutical applications. This is based on a lucky coincidence of some key factors which have become pressing:

In the past, biology – and medicine even more – was well separated from mathematics (except statistics). Today, however, novel experimental techniques are going to produce a deluge of data which exhibit unprecedented complexity.

In ageing societies complex diseases become more and more dominant in clinics. They require more individualised and advanced therapies which must be based on the data. Moreover, modern therapies need to be optimised, as the increasing costs of health care systems must be affordable for the society in the future.

In the last decade tremendous investments have been made in systems biology but the participation of mathematics has been surprisingly low. Systems biology aims to establish quantitative models to simulate, understand and control living systems, ranging from relatively simple bacteria cells up to mammalian cells, malignant cells and organs. The tangible outcome for medicine, however, has been limited and far removed from the promises of the past. Summing up, the generic finding is that the overwhelming complexity of living systems, even for "simple" applications, is apparently much more relevant than expected.

Today, funding agencies are focusing on systems medicine where predictive models with clinical relevance shall be developed, based on the deluge of data.

It seems that mathematics can and must play a key role in tackling the abovementioned challenges. This will, however, provide scientific challenges for mathematical science and mathematicians must be aware of that. However, according to a survey published in the recent ESF-report on industrial mathematics [2], the discrepancy between the self-assessment of the mathematics community and the assessment from the outside on the impact of mathematics on the life sciences was significantly higher than in any other application area. This clearly shows that there are apparent gaps in mutual perception and future progress will therefore require significantly more intensive collaborations and mutual understanding, which will be a challenge in itself.

With this in mind, the further discussion here is focused on a more detailed specification of the needs of the health care systems, today and in the future, on an outline of the specific mathematical challenges in modelling living systems. Two application examples demonstrate the feasibility and the potential benefits of mathematical modelling in health care.

2. The challenges for the health care systems and the pharmaceutical industry

2.1. Challenges for mathematics in health care arising from industrial and clinical needs

The health care systems of Western ageing societies suffer from a continuously increasing frequency of complex diseases, such as cancers, metabolic syndrome, auto immune diseases and diseases of the central nervous system. In contrast to infectious diseases, all these diseases are characterised by a dysfunction of the biological regulation systems of the patient. They cannot be reduced to single root causes and we still lack a sound mechanistic understanding of even the un-diseased function of the relevant regulatory systems. Consequently, little progress has been seen in drug research and there are no "silver bullets" for cancer or Parkinson's in sight that would have similar efficacy as antibiotics still have in many microbial infections. In all complex systemic diseases the medical need is, however, very high.

Despite steadily increasing investments in drug research and development operations and the introduction of novel technology platforms like high throughput and high-content screening, the output of novel, effective drugs for complex diseases has not only been low but has shown a continuous downturn. As a direct consequence of a lack of R&D efficiency, the average investment per drug newly approved by the regulatory agencies already exceeds $1 billion. From project initiation to marketing authorisation, an average pharmaceutical R&D project today takes more than 10 years. Even worse, up to 83% of the drug candidates which are successful in pre-clinical tests fail in the clinical development phase where the drug candidate is tested on human volunteers and patients and a still significant proportion fails in the most expensive late pivotal trials.

High attrition rates in clinical development are an important contributor to the overall cost of novel drugs.

Our inability to predict these failures is, at least partially, caused by the lack of tools which allow the prediction of efficacy in humans based on laboratory and pre-clinical animal data. Most of the modern, targeted drugs show high efficacy for a limited percentage of patients, the so-called responders, while others, the non-responders, show almost no positive reaction to the drug. Unfortunately, the percentage of non-responders is usually higher than that of the responders even in approved marketed drugs, such that the health care systems suffer from ineffective therapies. Hence, future challenges for drug research and development will be:

- Optimal design of clinical studies and therapies by optimisation of the dosing strategies according to the patient's needs.
- Realisation of the "fail early" principle: find the efficacy of a novel drug in humans by means of laboratory and pre-clinical data.
- A priori characterisation and identification by means of diagnostic tools of responders and non-responders.

On the other side of the health care system, medical doctors in clinics are beginning to ask for mathematical help.

Typical responder ratios for modern targeted drugs in oncology are about 20%, which means that 80% of the patients do not benefit from the respective drug. Moreover, even in the same organ, tumours show a tremendous variability with significant impact on therapy. Hence it is common sense that a more precise characterisation of the disease is necessary to provide the optimal therapy to the individual patient. This approach, which is summed up by the concept of "Personalised Medicine", requires the availability of diagnostic tools which allow a proper characterisation of the disease. As the underlying mechanisms are mostly unknown, these tools will be based on the concept of "biomarkers". Biomarkers mean models which are based on surrogate parameters which can easily be assessed in patients. The respective output of the model must show a high correlation to the desired therapeutic effect, e.g. to the response to a drug. Because the modern high-throughput diagnostic tools such as sequencing, next-generation sequencing, mass spectroscopy and micro arrays allow the quantitative assessment of a hundred thousand parameters and more in each patient, providing information about a large set of diverse biological entities such as genotype, gene expression, proteomics and metabolomics, the discovery of precise diagnostic biomarkers was expected. However, more than 150,000 papers describing promising biomarkers have been published with only about 100 in clinical testing and five approved by the authorities. Most of these biomarkers use a single diagnostic parameter, while the success stories of complex biomarkers using models which are based on more parameters are extremely rare. So, despite the ever increasing deluge of data, neither drug research nor diagnostics have benefited sufficiently from the billions of invested euros.

Moreover, in ageing societies, patients do not only suffer from more complex diseases; many patients show multi-morbidity from a set of interacting diseases. Multi-morbidity forces doctors to administer many drugs to the patient, which can show unexpected cooperative effects and possibly even result in fatal consequences. Moreover, cooperative effects of multi-morbidity can significantly hamper therapies which may be effective for the isolated diseases. Hence, there is an increasing demand from the clinical side to analyse the large available data sets in order to characterise patients with respect to their disease status and optimal therapies.

While "in silico" models of high complexity are widely used in many engineering oriented industries, such as the aircraft and automotive industries where they have proven their economic benefits, they still play a marginal role in pharmaceutical research and development as well as in clinics. On the other hand, the ability to support drug development with in silico models would obviously have dramatic effects on the efficacy in the pharmaceutical industry. Possessing reliable model-based biomarkers for characterisation of disease and prediction of patient response to therapy would be equally important for the health care system.

2.2 Challenges for mathematics arising from systems theory of biological systems

From the abstract viewpoint of systems theory, establishing quantitative models suffers from unsolved challenges: if we define a model, either mechanistic or phenomenological, as a computational algorithm which allows the representation of the quantitative behaviour of a system in all its states and satisfies a set of priori defined constraints then the development of a model will require some a priori assumptions with respect to the behaviour of the system.

A system can be defined as a subset of the reality which is fully described by abstract states. It includes at least the subject under consideration, it is closed and its states can be properly defined. Then the "reality" can be split into a (hopefully small) system and its environment, which can be neglected in a first approximation step. However, splitting of a real life system into a "relevant" subsystem and a negligible environment is useful only if the closedness of the subsystem under well defined system states is guaranteed and if the states of the system can be clearly defined:

- "Closed" means that, under the a priori constraints on the environment, the entities of the system interact much more intensively with each other than with the environment. Then we can describe the behaviour of the system by describing its entities and neglect the environment up to small perturbations. This "closedness" requirement seems trivial and, of course, in most sciences with established "theoretical" components, e.g. in physics, the systems under consideration can be defined this way. However, in biology this issue is far from being trivial. If we aim at modelling the response of a patient to a therapy then the system should be the patient. However, since biological systems are open systems far from equilibrium (in the sense of thermodynamics), the interactions with the environment are strong and can cause qualitative rearrangements of the systems structure upon environmental changes. Therefore the behaviour of the system may show structural variations if the constraints on the environment are not properly chosen and controlled. Hence, under varying environmental conditions the behaviour of the system may not be comparable, leading to very different results of therapy. Therefore the standard procedure in systems modelling consists of:

- Splitting of a large but weakly coupled system with well defined environmental impacts (like a patient suffering from a tumour) into smaller subsystems (like tumour cells).
- Then modelling of the subsystems under a few special environmental settings.

This will result in poor predictivity for the clinical applications if the environmental settings of the model are not properly characterised or if they do not cover the full range of environmental conditions in the patient (which is almost never the case).

- The states of the system as well as of its environment must be properly defined. For systems with clinical relevance, i.e. human patients, it is a hopeless task to define the full system properly: we are not able to measure all molecular components of a patient. Therefore a model (not depending on its realisation) will always be restricted on an extremely small subset of parameters which are involved in the system's functionality. However, all results from clinical data analysis indicate that the parameters are strongly co-regulated such that the description of the system may be reduced to a (hopefully) small set of control variables. Hence, measuring all parameters will only lead to a tremendous amount of redundancy and a waste of time and money. However, the identification of all control parameters which allow a proper characterisation of the state of the biological system is by no means trivial and is, at least for most applications, an open problem.

These requirements strongly challenge systems biology applications in clinics.

Mostly, detailed models for isolated parts of the cells are developed. They benefit from well defined system states but they are often not closed with respect to the huge variations in environmental conditions which appear in clinical applications. The alternative large scale models describing complex biological interaction networks are probably more closed but they suffer from the data demand for model identification. The latter issue may only be overcome by identification of the biological co-regulation structures leading to a characterisation of the intrinsic redundancies.

Apparently, a proper characterisation of biological systems which are large enough to be clinically relevant can rarely be performed using a mechanistic understanding of the systems as in engineering sciences. If this is possible, as described in the example in Section 3.1, then there is a good chance of providing significant benefits. In most applications, however, proper system characterisation must be supported by pattern recognition using the abovementioned deluge of data. Applying statistical methods to these massive data sets, however, leads to a significant challenge.

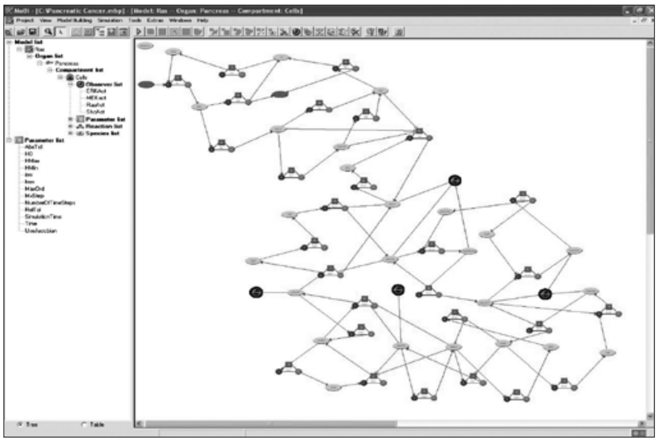

Fig. 2: On the molecular level mathematics is challenged by the complexity of the cellular networks (here a very small subset). Moreover, the intra-cellular networks are not closed subsystems; they can change their structure according to the environmental embeddings.

Because of the extreme dimensionality of the data (>20,000 parameters for each sample), all pattern recognition methods will lead to massive $n \leq p$ problems which are extremely ill-posed. Hence, stable feature selection requires sophisticated validation procedures. These, however, require either extensive Monte Carlo simulations, which can be extremely time-consuming, or proper test statistics must be developed. The drawback in biological systems is that the existing systems are never arbitrary: all biological entities have to cooperate in order to guarantee the survival of the entire system. Most random variations will never be observed in reality as the respective cells will not survive. Hence, all observed and relevant ensembles of biological systems will show a significant bias towards cooperative effects which must be taken into account in the test statistics [5].

Even worse, all existing biological systems are the product of billions of years of evolutionary optimisation. Hence, it cannot be expected that any distribution of relevant parameters will show random behaviour in an unbiased sense.

This short selection of open challenges for mathematics in health care shows that modelling of biological systems is by no means trivial or straightforward, explaining today's poor outcome of the investments into data. However, reliable solutions for these challenges will be the key for a quantum leap in treatment of complex diseases, which can be demonstrated by applications in the pharmaceutical R&D workflow, where the closedness of the relevant biological sub-system is given and the subsystem can be properly characterised. For these applications, successful modelling and simulation shows the high potential of in silico approaches already, which will be demonstrated by two examples from the systems biology group at Bayer Technology Services GmbH.

3. Application examples

3.1 Modelling and simulation of the blood coagulation system for clinical studies

Modelling and simulation of the complex processes involved in blood coagulation is a successful example. Blood coagulation is a subsystem with well defined interactions and well understood mechanisms. The external and internal triggers and the individual processes that take place in the blood vessels during coagulation are well understood and can be simulated by computer.

To this end, scientists use the "coagulation simulator", a computer model that virtually recreates the processes taking place in the blood vessels down to the smallest detail: the waterfall-like activation of biochemical coagulation factors, the cascades activating blood platelets (thrombocytes) and the interaction of both platelets and coagulation factors with blood fluid dynamics and vessel surfaces. The interest in blood coagulation and its simulation on the computer is driven by the wish to support a holistic understanding of the effects of one of Bayer's new drug products with the active ingredient rivaroxaban and support optimal clinical study designs. Specialists can recreate the most varied situations in our coagulation simulator and predict, for example, the consequences of drug dosing on the prevention of thromboses or the risk of bleeding.

The simulation results are used as an independent source of information that reduce the risk of unfavourable clinical trial outcomes due to inappropriate dosing. Comprehensive preliminary work was necessary for this. Specialists from various areas of clinical pharmacology, preclinical research and modelling developed the coagulation model. The existing knowledge about coagulation, experimental data and scientific assumptions went into hundreds of mathematical equations. In this way, a testable, standardised representation of all existing information has been created, capable of producing predictions that would not have been possible by experimental means or only with huge expenditure.

The coagulation simulator is a neutral, safe instrument for the pharmacologists. Using its results, they can better decide how to design the next steps in the development of anti-thrombotic or haemophilia drug candidates. The simulations are used to determine, for example, the dosage at which a substance should be tested in clinical trials. This is a major advantage because gradually approaching the optimum dosage is usually a long and costly part of drug development. The simulation software allows Bayer's scientists to plan clinical trials more accurately and thus efficiently, saving a great deal of time in the launch of new drug products. The predictions increase the safety, efficacy and tolerability of the drug products for the patients. In the hospital tests, a dose that is not effective cannot be administered to patients for ethical reasons. That is another reason why it is advisable to determine the optimal dose range by means of simulation before the clinical tests even begin and then confirm its accuracy in the clinical trials. The interdisciplinary collaboration between the simulation specialists and the pharmaceutical and medical experts in clinical pharmacology is extremely successful. For example, in 2008 a Phase II trial on long-term prevention of circulatory disorders of the coronary heart vessels with some 1,600 patients confirmed that the dosage of rivaroxaban independently recommended on the basis of the coagulation simulator's results was exactly right.

3.2 Prediction of adverse drug reactions from genotype and physiological data

As already discussed, the development of tools, so-called predictive biomarkers, which allow one to predict whether a patient will respond to a therapy will be one of the challenges for the future. However, despite significant investments in biomarker research, so far the outcome is limited to applications where the decision for response or non-response can be associated to a single root cause, e.g. a mutation of a gene or the over-expression of a single protein. In complex diseases, the hunt for mono-causal biomarkers, however, has hardly ever been successful. Apparently, the response of a patient to the therapy often depends on the interaction of a variety of biological parameters. This requires the modelling of the interaction of physiological data, describing the overall status of a patient, and genomics data, describing the status of the patient on the genome level. As these interactions are largely unknown, data-driven methods have to be applied.

An illustrative example for the application of data-driven methods is the identification of biomarkers for the prediction of an adverse drug reaction (ADR) to statins, a highly effective class of cholesterol lowering drugs. In a clinical study, the genotype of a large set of genes of patients treated with statins was determined together with classical anthropometric and diagnostic physiological parameters. Genotypes were characterised by a long list of point mutations (SNP) on genes potentially relevant for rhabdomyolysis, a rapid breakdown of skeletal muscle that can occur under statin therapy. Potential ADRs were measured indirectly by a so-called surrogate marker indicating potential rhabdomyolysis. In the case of statins, the level of enzyme Creatinkinase (CK) in blood plasma is an appropriate surrogate marker. Since CK is an important enzyme in muscle cells, an increase in plasma indicates muscle tissue damage. Prediction of a high risk for CK elevation prior to the occurrence of the ADR would therefore allow an early adoption of therapies.

None of the attempts predicting the CK levels under statin therapy directly from the genotype data alone produced reliable results. Starting from standard statistical methods, more elaborate classifiers like support vector machines, neural networks or random forests have been applied but none of them could shift the sensitivity and selectivity of a potential diagnostic test to levels higher than around 70%, suggesting that a significant part of the information relevant for the ADR is not encoded in the genome. For an application in clinical practice the obtained power of diagnostic tests is simply insufficient.

Therefore, in a follow-up study, patient data from heterogeneous sources were combined. As the true biological factors are not known, the concept of surrogate markers had to be applied to the independent inputs for the classification, too. This means that data which are not crucial for the ADR but correlated with the root causes were used to indicate and quantify the mechanism. The correlations between biological surrogates are usually not bijective, so multivariate, nonlinear correlation models have to be identified in such an application. If, for example, a patient suffers from secondary liver damage with an impact on drug metabolism then this damage on the organ level will not be found in the genotype but it will negatively affect the patient's response to a drug in addition to the genotype impact. It may additionally shift the levels of other blood serum proteins which can be measured in blood tests, such that the liver damage as the root cause can be quantified indirectly by measuring the serum proteins.

The pure accumulation of heterogeneous data types, however, leads to an increase of the number of parameters to be included in the black box model. As the number p of SNPs plus physiological parameters is large compared to the number n of patients included in the study, identification of complex biomarkers is a typical $n << p$ problem. Therefore, a mere addition of parameters may not be necessarily a benefit for the classification problem.

To overcome this pitfall we used a technique which has been developed in the framework of process modelling. For systems with a priori known internal structure it has been shown that the effective dimensionality of the model identification problem can be reduced to the dimension of the largest functional node in the system when the system structure is explicitly implemented in the model identification algorithm. As the effective dimension of the input parameter space governs the amount of data required for the identification of black box models ("curse of dimensionality"), this structured hybrid modelling (SHM) approach can reduce the curse of dimensionality. In SHM, hierarchical networks with a fixed interaction structure can be represented by a directed graph. Each node represents an unknown, nonlinear function depending only on the variables which are represented by the directed edges into the node. Hence, if the structure of a hierarchical system can be properly represented by such a network then (under some restrictions of the structure of the system graph) the network model can be properly identified and the amount of data required for the identification can be reduced dramatically compared to unstructured model types. Therefore, the hierarchical network based SHM models allow an interpolation between unstructured models (like neural networks) and fully mechanistic models. The superior performance of SHM models for modelling hierarchically structured systems has been shown in various complex applications.

Although the exact interaction structure is not known, we started from a model structure where the inputs on genotype level, namely the SNPs, were inputs of the nodes associated with the genes. The outcome of the node was an effective activity which is formally a function of the SNP combination. Then we assumed that the products of two or more genes interact in the cells, such that the activity of the respective biological functionality can be quantified as a function of the activities of the respective genes. Subsequently the model can be established in terms of a directed functional graph. Although the computational effort for the identification of the optimal SHM structure was significantly higher than for pure black box models, the SHM approach allowed us to identify a model with increased stability. Moreover, the subsequent combination of physiological data and the structured "pharmacogenomics" model led to a sensitivity and selectivity of more than 90%.

4. Future trends

Recently, a Challenge Workshop "Mathematics for Health Care" was organised by the German Committee for Modelling, Simulation and Optimization (KoMSO) in Heidelberg [6]. Scientists from biological research, systems biology, the pharmaceutical and food industries, clinics and mathematics discussed and evaluated in detail the needs for mathematics arising from future trends in biology and clinics. In summary, a more intensive use of mathematics, combined with the development of problem-specific, novel mathematical algorithms and modelling methods, will be a key driver for future development in biology and medicine.

In detail, open challenges for novel mathematical development cover a broad range of applications as well as mathematical methodologies:

- Clinicians see a tremendously increasing need for novel, efficient methods for the analysis of highly multivariate patient data in order to predict the time course of the disease under therapy. Their data show an extremely heterogeneous structure. Genotype data must be combined with physiological parameters and overall diagnostic health parameters, thus linking together rather heterogeneous biological functionalities. Moreover, longitudinal studies must be linked with very large data sets which are available from patients who are monitored only at single time points. Despite significant investments in the past into clinical data analysis using machine learning approaches, the success stories are limited, indicating the existence of intrinsic, structural challenges in the unravelling of biological systems.
- There is a strong need for efficient model identification and validation approaches. Dynamic models, based on ODEs, are the common approach for systems biology. However, even if the system to be modelled is sufficiently closed, systems identification remains an open challenge. Mostly, systems are identified by minimisation of residues of simulation compared to experimental data. However, the autocorrelation of the residues is seldom properly analysed. Often the residues are reasonably low in L2 norm but the distribution of the residues along the trajectories show autocorrelations which hardly fit to white noise. This suggests that either the parameter fit is not sufficient or, even worse, that the assumed system structure does not sufficiently fit to biology, both possibilities probably leading to erroneous conclusions from the model predictions. Hence, tools for structural validation are indispensable for further success in more complex applications.

- Drug research will benefit significantly from efficient analysis of mechanisms on the cellular level, either for mechanisms of drugs or toxic action as well as reprogramming of cells. The cells respond to the disturbance by the drug or toxin, hence the signals arise from both the primary mode of action and the secondary cellular stress response. Therefore, identification of the mechanistic functional networks which are relevant for drug action requires the deconvolution of primary and secondary signals. Basically, two approaches which might allow the identification of functional networks are known today: assessment of the time evolution of the cellular parameters under drug action and, alternatively, the analysis of cellular response on combinatorial stress factors. Both approaches require a high degree of experimental investment and suffer from limitations in the size of the networks which can be identified [7, 8]. Moreover, the deconvolution of primary and secondary responses remains unsolved.

Although the challenges of complex diseases, namely the extreme complexity of the respective systems, may require novel mathematical approaches and frameworks, mathematics provides highly beneficial solutions for specific applications even today.

An application-oriented, well established modelling and simulation workflow must overcome the limitations of traditional statistical processes. The methods to be developed may integrate expert knowledge and public and published information with data from research, clinical and clinical research and development programmes. Consistent data preparation and mathematical formalisation ensure an impartial basis for decision-making.

In future, we expect that new disease models, based on an individual genetic and clinical diagnosis, could make more individually tailored treatment possible. Doctors today base their decisions about dosage of a chemotherapeutic agent on parameters like height, weight, age or sex of the patient. Models allowing individual simulation of the drug effect would make administration more precise, although this will not be possible for some time yet. However, the enormous flood of data from genome projects over the last few years has already compelled biologists into formal interdisciplinary co-operations with systems analysts and information technologists. The amount of genetic data in publicly accessible databases alone doubles every 18 months. In the worldwide physiome project (www.physiome.org) scientists are even trying to bring whole organs into virtual life. The computer power required is disproportionately high compared to the simulation of processes "only" in cells. Today it takes a supercomputer hour to simulate the effects of a single genetic mutation on the blueprint of an organ and ultimately to simulate a heartbeat but worldwide projects are working on improved, more efficient codes which will enable a broad range of further industrial and clinical applications. Future success will strongly depend on optimal communications and the interaction of mathematicians, biologists and clinicians. Novel mathematical approaches will be required, eventually integrating mathematical disciplines which are not involved in modelling and simulation today. It shows that we are still at the very beginning of the involvement of mathematics with pharmaceuticals, biomedicine and clinics.

> *Nature will throw out mighty problems, but they never reach the mathematician. He will sit in his ivory tower waiting for the enemy with an arsenal of strong weapons, but the enemy will never come to him. Nature does not offer her problems ready formulated. They must be dug out by pick and shovel, and he who will not soil his hands will never see them.*
>
> John L. Synge (1897–1995)

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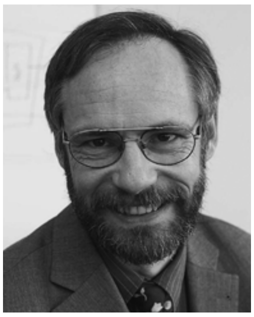

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His research is focused on data based modelling in material sciences, chemical engineering sciences and systems biology. His main ambition is the development of hybrid model structures combining artificial intelligence model components with mechanistic models and system structure. Today, the main application area is network re-engineering for complex biomedical systems, which enables the establishing of quantitative, predictive models for clinical therapies of complex diseases, as well as the optimisation and control of cellular engineering.

Interview with Tomás Recio

Marie-Françoise Roy (Rennes)

Tomas Recio, July 2010, at CADGME 2010 (Hluboka nad Vltavou, Chequia)

In which region of Spain were you born?
I was born in Oviedo, Asturias, 14 Dec 1949.

The region (Asturias) in the North of Spain is a narrow (around 60 km wide) strip of land between the Cantabric (Atlantic) Sea and the Cantabric Mountains (2000-2600 m altitude) – lots of rain, cool temperatures all year around at the coastal band, lots of snow in the nearby mountains, deep forests and gorges, wildlife: mountain goats, wild boar, deer, wolves, some bears. This is quite different from the typical image of Spain.

Asturias' economy in the 50s was based on coal mining, steel mills, shipyards, metal and chemical industries, and dairy production. With a lot of people of working class, it was a cradle for the left and trade union leaders, even under the Franco regime. And yet, many little mountain villages were still living on a subsistence economy and a traditional culture.

This land had a strong influence on my lasting hobbies: music, reading/writing, outdoor sports.

What about your family?
My father was a Latin teacher, then the director at the Oviedo Public High School for Boys. My mother, with a diploma as primary education teacher, never taught. Indeed, she gave birth to nine children; I am the oldest. The youngest is 16 years younger. My two parents were deeply religious and politically conservative.

When I finished high school, I was 16. My parents considered it was too early for me to go to the university and sent me to spend one year in the US, with an American Field Service scholarship http://www.afs.org/afs_or/home.

I had no idea of English; I had quite a few years of French but just three months of English in private lessons. The trip to and from the US was by ship. It was a deep cultural change for me, from a severely underdeveloped country – many small villages, for instance that of my grandparents, did not have a paved road, electricity, toilets or water facilities at home – to a blossoming, modern society. In Spain, in those days, it was difficult to phone someone not in the same town, so forget about calling someone abroad!

I started living on my own after that. It was a lasting experience, with a strong impact on my academic life later on.

A family picture circa 1966

I came back from the US one year later and started some preparatory courses at the University of Oviedo. I did not have a clear idea of what to do next, so I prepared myself for both sciences and engineering studies. I met there my future wife Isabel; we were both less than 18 years old. We decided to follow mathematics studies at the University of Madrid, where we started in the academic year 1968/69.

We married in 1973, just one year after completing the degree. We have two sons and five grandchildren.

What was it like studying mathematics in Spain at that time?
The University of Madrid, in 1968 and the following years, was a boiling pot. In the case of Spain, on top of the students' protest as in many other countries at that period, we had the extra ingredient of the revolt against Franco's regime. The police were for long periods of time present in the university hall; often, the police got into a classroom and asked the students to get out, if they considered the students were organising something illegal. Some students were obliged by the regime to live for some months at long distance from the university. We, their colleagues, used to visit them, with resources to help them keep in touch with the mathematical subjects corresponding to their future exams. Classes were interrupted – by the university authorities – quite often for long periods of time, due to the tense situation.

To give an idea of this climate, let me take an example from 1975. At that time both my wife and I were teaching, she at a high school, I as an assistant at the university. We had already one little son. Franco died one night and we three, without even consulting our bosses at work, packed our things and left next morning to Oviedo – where both our parents lived – for a period of time, just in case. Nobody called us from work to ask where we were: it was assumed everything would stop till further notice the day Franco died.

The tense political situation was not just students versus regime. It was also students versus students, students versus professors, professors versus students and professors versus professors. Here are two anecdotes – any resemblance to real events, being, of course, purely coincidental. You could be attending lectures on category theory and then some students would shout a political motto… and then the professor would call the nearby police and ask them to catch the "undisciplined" students… How to continue listening to the proof of Yoneda's Lemma? Or you could receive, at your student dormitory, a "visit" of a gang of colleagues (some from your own class) with the declared intention to beat you, since your position (could be related to continuing or discontinuing some strike, or the like) was considered an obstacle for their political agenda… And yet, years later, that professor who called the police would become your thesis advisor, or that student who "visited" you overnight would become your colleague, as a professor in the same faculty… This is what we call in Spain the "transition" to democracy.

On the mere academic side, I can say our mathematical studies were of quite a high level. A BSc degree in mathematics was obtained after at least five years of studies. Most of the subjects were compulsory and it was necessary to succeed in all. I recall having to read and to explain in the classroom excerpts of R. Narasimhan: Analysis on real and complex manifolds, as part of my homework during the third year, or parts of Grothendieck's EGA in my last year. Horvath's book on Topological Vector Spaces was the textbook for our compulsory undergraduate functional analysis course.

In general, self-education played a key role; the professors asked you to pass a substantially difficult, four hour long, written exam, including writing some kind of essay on different theoretical topics and solving some given problems, but that was all. You could learn all that by yourself rather than attending the – usually very abstract and rigorous – lectures and exercise classes. So I was reading different books – those that attracted my attention when browsing them in the library. But I also liked listening to a few professors, who showed me how to "feel doing" mathematics, for example: a very young and passionate – PhD student at the time – Jose Maria Montesinos, explaining – in Bourbaki style – the classification of modules over Principal Ideal Domains at a complementary seminar, attended by just three or four students, while our regular class included over 100 students; or the very elegant and global approach towards selected topics in mathematics of the senior professor German Ancochea. Let me mention too the dynamism of another PhD student Maria Jose Garbayo, lecturing on Atiyah-McDonald's Introduction to Commutative Algebra and the rare – in those days – algorithmic approach towards this discipline of Jose Luis Vicente Cordoba, who received his PhD in 1972.

Who are the people who influenced you most?

My thesis advisor Professor Pedro Abellanas, http://www.dma.fi.upm.es/mabellanas/pa/, was the most influential person at the beginning of my career. He was the only professor with a Chair and, practically, the only member with a permanent position at the very large Department of Algebra and Foundations of Mathematics of the University of Madrid, one of the most influential people in Spanish mathematics in those days, well connected to the education authorities and with a clear aim towards promoting his group to do research at international level, a quite extraordinary goal in Spain at that time.

I did under his direction my "tesina" to get the BSc degree, on 'Cartier's theory of divisors'. It was, simply, some 20 pages of my own words of what I was able to understand about this concept. Then Abellanas suggested to me – in 1972 (!) – to read some IHES Lecture Notes from Professor Lojasiewicz, a draft on subanalytic sets by Professor Hironaka, and to contribute to the study of real algebraic geometry objects defined by using Nash functions. The mere list of these topics, which quite a few years later become very important in real algebraic geometry, shows clearly his good intuition. He taught me another thing: hard work. He used to start his working day early in the morning and never ended it before 21h. Another invaluable gift I received from him was a kind of "crash course" on academic management issues, since I became very early –about the time of finishing my PhD – a kind of department secretary, dealing with all kinds of organisational and bureaucratic issues for a large number of people. Finally, I got from him a permanent interest towards mathematics education. Abellanas' department had the responsibility for all the courses concerning pre-service mathematics teacher training at the University of Madrid. Some biographical notes on Pedro Abellanas written by me appear at "Don Pedro Abellanas, cuarenta años de matemática española." La Gaceta de la RSME, vol. 4.1, 2001, pp. 119–133.

Don Pedro Abellanas, Chair of the Algebra Department, University of Madrid

Another influential person in my early career was Professor Donald W. Dubois from the University of New Mexico. I have described our absolutely unexpected first meeting in Madrid (1979) in a paper on Contemporary Mathematics, Vol 155, 1994 (*Proceedings of the RAG-SQUAD year*, Berkeley 1990/91): C. Andradas, T. Recio: "D.W. Dubois and the pioneer days of Real Algebraic Geometry".

I immediately profited from this contact with Professor Dubois and a few months later my students and I started long research stays at the University of New Mexico and a fruitful collaboration with Dubois and other colleagues in the USA, working on real geometry.

D. Dubois and the Spanish Armada, near Boulder, Co., 1983, attending the A.M.S.-NATO Summer Conference on Ordered Fields and Real Geometry. From left to right: Jesus Ruiz Sancho, Donald W. Dubois, Tomás Recio, Jose Manuel Gamboa Mutuberria, Maria Emilia Alonso Garcia, Carlos Andradas Heranz

Dubois was close to retirement, which he accomplished in 1985, but he had time to teach me, among many other things (such as the basics of real algebra, since my approach was till then essentially geometric) how to deal with the international community of researchers in a given topic, a community with its own rules and traditions, a world absolutely novel for me.

Please tell us about your academic career

Getting a permanent position: the Spanish way

I got a pre-doc fellowship for five years (Oct 71–Oct 76) because Abellanas required the Ministry of Education scholarships to enrol doctoral students in mathematics one year before they graduate, arguing that mathematics dissertations were more difficult and took longer than in other Sciences! My official working place was the Instituto Jorge Juan de Matematicas, on the Spanish Higher Council for Scientific Research (CSIC), but Abellanas was also the Chair of the Department of Algebra at the University of Madrid, so I was lecturing at the university in the morning.

After my PhD I was hired by the University of Madrid as a temporary assistant, teaching a lot of different subjects (commutative algebra, several complex variables, projective geometry, classical groups in mathematics education, etc.). Then, I got the opportunity to apply to some Oposiciones. The Oposiciones were formal, public exams with many different parts. Describing your CV was just one of them. Applying to a concrete Oposicion meant to submit a kind of dissertation (several hundred pages long) describing your own viewpoint concerning the particular subjects involved in the Oposicion, plus a collection of courses to teach it and the detailed content of each course: a long list of lessons one of which you would have, later on, to explain to the commission orally – and if the list was not long enough you would be refused beforehand as knowing little on the subject. It included, as well, solving in a couple of hours some problems posed by the commission on the specific subject of your Oposicion. Indeed, each Oposicion was bounded to particular curricula subjects and a university, e.g. "Assistant Professor of 'Geometry II and III' at the University of Oviedo". There was no regular calendar to apply to an Oposicion relatively close to your topic of expertise. Some years could elapse between one Oposicion and the next one, depending on vacancies occurring at some university in Spain.

These were the rules. In 1979 I succeeded getting a permanent assistant position as researcher in the Instituto Jorge Juan. There were just three other people with permanent positions at the institute, so it was a great success for me. But I had also applied to a different Oposicion – there was a difference of many months from the time one applied and the time the exam actually took place, so one had to apply simultaneously to various Oposiciones – as permanent assistant professor in algebraic geometry in Salamanca, and I won it too. I chose to pursue my career within the university system rather than staying at the Institute Jorge Juan. Indeed, life was awakening in the universities, with lots of young people wanting to learn and to do research. My wife had her work in Madrid and I managed to commute my position in Salamanca to a similar one at the University of Madrid. But not for long! Spain was blossoming at the end of the 70s and new universities were being established to cope with the baby boom generation from the Spanish economic push out in the 60s. And the new universities required new professors. So, about one year later (1980) I got a position as associate professor of algebra at the Universidad de Malaga. This was an important step forward in my career since the promotion to full professor was almost automatic. I was promoted one year later, to the University of Granada, as its Chair of Algebra.

Early years as professor

These years (79–81) were quite essential in my career also because I went to the University of New Mexico for some months, to a visiting position, and took with me a couple of young students from Madrid who wanted to do their dissertation on real algebraic geometry, with Don Dubois as advisor. Moreover, I got involved in my first specialized international meetings, a special AMS session in San Francisco (1981) and the meeting on Formes Quadratiques et Géométrie Algébrique Réelle organised in Rennes (1981). It was the starting point of the community of real algebraic geometers… and friends!

I applied in 1982 to a Chair of Algebra in the much smaller and novel Universidad de Cantabria. Indeed, Cantabria was very similar to my land of origin Asturias and close to it. So I moved to Santander with my whole family and am still staying there. My career at Santander is quite typical for a small university: the university structure is similar to the one of the big universities but fewer people have to take care of everything. You have many opportunities to develop your potential, if you want to accept the challenge. So I was soon involved in many interesting academic responsibilities (Vice-Rector of Research, Chair of the Teachers Training Centre, etc.).

At the end of the 80s I had already promoted eight different PhD dissertations: four from previous Madrid students, four from new ones in Cantabria. Some of these dissertations have been directed (totally or partially) by colleagues (D. Dubois from the University of New Mexico, A. Prestel from the University of Konstanz, and M. F. Roy and M. Coste from the University of Rennes) with whom I have sent these students to work for long periods of time. This "internationalization" approach to foster real geometry research in Spain was quite a qualitative jump from our previous experience.

Leading a research group

Around 1986 I started becoming involved in computational aspects in algebra and geometry and its applications, for instance, to robotics or CAD. It was, also, a way to enrol into my research group some colleagues in Cantabria who had a previous research background far from real geometry. We were all newcomers to these challenging new topics and collaboration was easier. Moreover, in the 80s there was, for the first time, money to support research projects, to buy computers, to attend and to organise meetings, to send students for research visits, to cooperate with local industries, to participate in European research projects.

So, for about 15 years, my career was essentially doing, managing and promoting research in real geometry, computer algebra and computational geometry. In the 90s I directed six new dissertations around these topics and started a series of Spanish regular meetings on computer algebra and computational geometry, meetings that have been regularly held in Spain since then. I was involved in the POSSO, RAAG, Model Theory and FRISCO European research projects, in the SAC network, in many different national basic research projects, in cooperation projects with industries (for robotics and CAD-CAM applications), etc.

My research group included younger colleagues from many different, larger universities such as Madrid, Malaga, Murcia, Barcelona, Vigo, etc. Along the years, diverse former constituents of our group established their own research teams. I keep excellent relations, both professional and personal, with them all.

Then you took responsibilities in mathematical education. How did that happen?

I used to go around lecturing at many teachers' in-service courses on the role of computer algebra and dynamic geometry packages in mathematics education at all levels. I attended the ICME meeting in Quebec (1992) and got involved in the ICME 96 in Seville, always around these topics. I also wrote a book (1998) on the subject for a Spanish editorial.

I also had early (1987) ERASMUS responsibility at my university and also a kind of "umbrella" to foster ERASMUS programmes expanding at mathematics faculties in other universities, such as Madrid and Sevilla. The impact of such programmes in Spain and particularly in Cantabria was quite large. Some of the students I was sending to foreign countries had never travelled outside of Cantabria before.

At the end of the 90s I got involved in the re-foundation of the Royal Spanish Mathematical Society (RSME). The society was practically non-existent since the political changes that happened in Spain at the end of the 70s and I was part of a group of colleagues taking the task of modernising it. Then I became, until 2006, the president of the RSME Commission for Mathematics Education and a member of the Spanish ICMI commission, chairing this ICMI commission from 2002 to 2007.

This decade (2000–2010) turned out to be a period in which mathematics education (and education in general) had a prominent role, not only because of my responsibilities on these commissions. In fact, Cantabria got – with many other Spanish regions – full competencies in education affairs around 1999. This means the regional government was fully responsible for the budget and the implementation of the educational system within the region. I was appointed in 1999 as the first president of the Consejo Escolar: all regional laws and norms related to education must be first discussed and informed at the Consejo Escolar, before being declared as laws. The Consejo is composed of about 50 people, including representatives from families, trade unions, municipalities, regional government, businessmen…

My task was, first, to articulate the procedures to accomplish the goals of the Consejo and, then, to discuss and to inform, during the next nine years (till 2008) all issues related to education in Cantabria, from very ideological issues concerning, for instance, Government support to private or charter versus public schools to the very down to earth problems on the conditions required for school buses or concerning the need to close some schools at remote villages for lack of students. It was a formidable challenge and I learned a lot. In particular, I would like to remark that I was appointed by a conservative government and then appointed again by a socialist government, so I had the opportunity to cooperate with authorities holding very different approaches to education.

What have been your main research contributions?

Let me state, beforehand, that I have worked with many co-authors. So "my" contributions are, actually, "our" contributions.

Real algebraic geometry

Algebraic geometry over algebraically closed fields is a well established field of research. The algebraic closed field where one looks for the solutions of polynomial equations simplifies matters: a univariate polynomial of degree n has always n roots. Of course, real solutions of polynomial equations have been studied for a long time but there was a lack of a systematic approach till the

70s, even though Hilbert included at least two problems (16th, 17th problems) closely related to this topic. Such a systematic approach required developing real algebra (related to ordered fields and, thus, to quadratic forms) and real geometry (related to differential/analytic geometry). Since there is, for instance, no simple relation between the degree of a univariate polynomial and number of real roots, decision procedures are important. Thus, logic (model theory, elimination of quantifiers, computability, etc.) play a relevant role in real geometry.

One of my contributions: determining that closed sets defined by equalities and inequalities of all kinds can also be described by using only relaxed inequalities.

Computer algebra and robotics

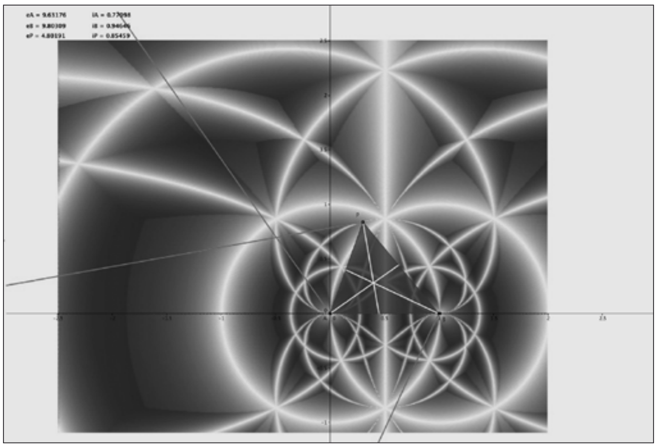
Since there is no theorem on the number of real roots as a sole function of the degree of a polynomial, one of my contributions to this subject was an algorithm, improving Sturm's theorem, using subresultants to cope with the problem of parametrizing Sturm's procedure.

Producing algorithms, one becomes interested in counting the cost (number, size of operations) of applying it, i.e. in complexity theory. The complexity of some algorithms in computational geometry cannot be improved because there is an intrinsic complexity to the problem they attempt to solve. And this complexity is related to the geometric complexity of an associated semi-algebraic set. I applied my knowledge of semi-algebraic sets to establishing some lower bounds, which have been only conjectured, for some computational geometry problems.

Robotics provides, in a natural way, a context for applications of computational real geometry. For years, roboticians were using kinematic equations describing robot configuration space. But such equations fall naturally into the real algebraic realm and deserved being studied with real geometry tools. In particular, this involved analysing the reality of some typical subset of the kinematic equations, describing the movement (i.e. the reality of orthogonal matrices). I had to revisit Weyl's 'The Classical Groups' from the "real" point of view. Relating "degenerate" robots, kinematic equations and functional decomposition of polynomials is also a contribution of mine.

Another natural context for real geometry applications is that of CAD, where parametric varieties are quite important. I have been working for many years on the problem of simplifying the coefficients of a parametrization. Imagine you are given a parametrization with complex coefficients of a real object. How to reparametrize it with real coefficients? How to display it, accurately, in the real plane or space? I have contributed mostly for curves. Now we are starting with surfaces, in my last accepted paper to appear in the journal Computer Aided Geometric Design (CAGD).

Finally, I have to mention my work on "automatic proving of geometry theorems", that is, providing algorithmic methods to prove elementary geometry statements. One of my contributions deals with "proving" false statements by finding algorithmically the missing hypotheses for the statement to become true. A concrete example is the finding of missing hypotheses for establishing a general statement of Steiner-Lehmus type (concerning the equality of lengths of three bisectors on a triangle).



Automatic computing of the locus of a triangle to verify equality of lengths of three (internal or external) bisectors at different vertices

From computer algebra to research in education

The need to foster the correct use of computers and graphing calculators in the classroom has attracted my interest to mathematics education since the 90s. More recently I have been involved in promoting the use of dynamic geometry in the classroom and, more generally, of the so-called dynamic mathematics software (such as GeoGebra), very adequate for an Inquire Based Mathematics Education. As usual, my interest for dynamic geometry was, first, a consequence of my work on automatic theorem proving (since some of my algorithms have been implemented in ad hoc dynamic geometry programs). The European projects Intergeo and Fibonacci and the Institute GeoGebra of Cantabria are some of the projects I have been very recently involved in.

Being for many years involved in the Mathematics Education Commission of Spain, I have dealt with a variety of problems and situations. One of them was the role of the competency approach to mathematics education, fostered by the PISA project. I have written several papers on these subjects, sometimes at international level but mostly for the Spanish community of mathematics educators. A specific mention should be made to my participation – appointed by the IMU – as a member of the committee in charge of the Klein Project, a very attractive joint ICMI/IMU initiative (see http://kleinproject.org/).

How many PhD students did you have?

According to the Mathematics Genealogy Project, I have had 15 students and 40 descendants.

I am happy to remark that five of my PhD students were female and 10 male. Seven of my former students are full professors (in Madrid, Hong-Kong and Santander) and five more are associate professors in Madrid, Santander and Oviedo, some in algebra and geometry, and some in computer science and mathematical education. Two of my former students did not follow a research career and one more is still starting it.

Among them, there is a former President of the Spanish Mathematical Society, a high officer of the Spanish Agency for Evaluation and Quality of the university system (ANECA), a leader of the FOCM association, and the mathematician who has recently disproved the Hirsch conjecture. I am equally proud of all of them. One of the things I have learned is that mathematical excellence is a complex mixture of many different kinds of intelligence…

Can you summarise the transformations mathematics in Spain went through during your academic career?

When I started my career (60s and 70s) and until the 80s there was no tradition at all, in this large and active research group, of publishing research results at an international level. Travelling outside Spain and Portugal was not easy for Spaniards and there was practically no financial support at all for research. We used to pay for photocopies, typing mathematical papers, travel expenses, etc. from our own pocket money.

Things have changed so completely! Now the Spanish community of mathematicians is comparable to that of any other first-world country. Spanish contribution to the development of mathematical knowledge is about the same "weight" as the weight of Spain regarding the international community, with respect to many other (economic, social, political) parameters.

My personal story is just an example of how Spain – or some other underdeveloped country – can advance, with the help of the international community, in the field of mathematical research. I like to make clear that the cooperation with so many international research groups of excellence has been essential in my career (and that of my students): Rennes I, Regensburg, Münster, Passau, Dortmund, Konstanz, Pisa, Genoa, Limoges, Paris VII, LAAS (Toulouse), Besançon, London (Ontario), RISC (Linz), New Mexico, Cornell, Berkeley, Baton Rouge (LSU), Hagen, Paderborn, Buenos Aires, Nice, Sophia-Antipolis, Austin (U. Texas), to mention just some groups with which I have had some bilateral cooperation agreement or where our group has visited for longer periods. European projects and the MEGA conference series have been essential in building up this community.

If you wanted to make some recommendations for the future?

At national level, I would dare to say that we need to improve a bit the quality of our research, but recalling that quality needs also quantity… On the other hand, Spain now shares the problems of most first-world countries. An important problem, in my opinion, is how to continue to attract young talent to mathematics. It is an old sociological problem. Richer people are less prone – generally speaking – to choose working as scientists as a profession. A partial solution to keep the world level of mathematical achievement is to attract young talent from other less developed nations. But Europe in general, and Spain in particular, has serious difficulties to organise this, while it is much simpler in Canada or the USA.

Right now my research group includes researchers and graduate students from Peru, from Niger, from Iran… But it is very difficult, from the bureaucratic point of view, in Europe in general, and Spain in particular, to deal with students coming from such diverse origins, while it is much simpler in Canada or the USA. Unfortunately, many research leaders will be tempted to give up if they have to devote such large bureaucratic efforts in attracting and keeping such students.

Marie-Françoise Roy [Marie-Francoise.Roy@univ-rennes1.fr] is a professor of mathematics in Rennes. She was President of the Societe Mathematique de France from 2004 to 2007. In 2007 she became scientific advisor for mathematics in sub-Saharan Africa at CIMPA/ICPAM. The first president of Femmes et Mathématiques from 1987 to 1989, she is currently the convenor of European Women in Mathematics. Her mathematical work is focused on real algebraic geometry and algorithms.

ICMI Column

Mariolina Bartolini Bussi

1. Symposium on Elementary Maths Teaching (SEMT)[1]

Charles University, the oldest university in the Czech Republic and in Central Europe, was founded by Charles IV in 1348. The Faculty of Education at Charles University in Prague has recently celebrated the 65th anniversary of its founding (1946). The faculty provides university-level pre- and in-service teacher education in the following fields: humanities, social sciences, art education, physical education, mathematics and natural sciences. The faculty is currently training approximately 4,200 students in both BA and MA study programmes, plus nearly 10,000 students being taught within a wide range of combined forms of study programmes, mainly as part of professional teacher development for both primary and secondary schools.

The conference Symposium on Elementary Maths Teaching (SEMT) started in 1991 as a biannual conference, held in Prague, thanks to Jarmila Novotná and Michaela Kaslová from Charles University. It is focused on the teaching of mathematics to children between the ages of 5 and 12. Its genesis at the end of the '80s came from the common belief of researchers involved in elementary mathematics teaching and research in this domain that there was a lack of platform for discussing the items of interest in mathematics for this age range. All other international scientific events were not specialised to this age range and this resulted in a lack of space for discussing deeply specific questions related to it.

The 11th Symposium was held in August 2011.

Each SEMT focuses on one important topic of elementary mathematics teaching. The development from general topics towards more specific problems of elementary mathematics teaching can be easily recognised from the main topics of all SEMTs:

1991: The teaching of mathematics to elementary mathematics pupils.
1993: The changing face of elementary mathematics.
1995: Geometry and word problems for elementary mathematics.
1997: Assessment and evaluation.
1999: How the world of mathematics emerges from everyday experiences of children.
2001: What is meant by the competence and confidence of people involved in the teaching of elementary mathematics.
2003: Knowledge starts with pre-conceptions.
2005: Understanding the environment of the classroom.
2007: Approaches to teaching mathematics at the elementary level.
2009: The development of mathematical understanding.
2011: The mathematical knowledge needed for teaching in elementary schools.

SEMT brings together elementary teachers, student teachers and teacher educators and researchers from all parts of the world. This is reflected in both the participants attending and those giving lectures and workshops. Among the participants there are colleagues from most countries in Europe, the Middle East, Japan, Australia and America.

This presentation of SEMT gives the chance to announce the intention of ICMI to launch a Study on Primary School. It will be the first study focused on the teaching and learning of mathematics in early childhood. More details will be given in issues to come.

2. Task Design Study

The International Programme Committee (IPC) of the new ICMI Study (no. 22) on Task Design had a first meeting at Oxford University, 9–10 January, in order to prepare the discussion document for this study and also prepare the study conference that will take place in July 2013. The IPC is composed of Anne Watson (UK) (Co-chair), Minoru Ohtani (Japan) (Co-chair), Janet Ainley (UK), Michiel Doorman (Netherlands), Claire Margolinas (France), Carolyn Kieran (Canada), Glenda Lappan (US), Allen Leung (Hong Kong), Peter Sullivan (Australia), Janete Bolite Frant (Brazil), Yang Yudong (China), Jaime Carvalho e Silva (Portugal) (ICMI Secretary General ex officio) and Michèle Artique (ICMI liaison). The work has been progressing well and we expect the discussion document to be available in three months. More details will be given in issues to come.

3. ICME Conferences

In July 2012 the quadrennial International Conference of ICMI (ICME-12) will be held in Korea, chaired by Professor Sung Je Cho (Seoul National University). The third announcement is available at the website http://www.icme12.org/. The deadline for registration at a reduced price ($400) is 1 April 2012.

The next ICME (13) will be held in 1016 in Hamburg (Germany). ICMI hopes that the international mathematical education community will enthusiastically receive the invitation of German colleagues for 2016, so to make ICME-13 a huge international event that will advance studies, interest and support for mathematics education all over the world. The precise dates of ICME-13 will be announced later.

[1] I thank Jarmila Novotna for the information about SEMT. More details can be found in the *Mediterranean Journal for Research in Mathematics Education*, Volume 8, No. 1. Guest Editors: J. Novotná and D. PITTA-PANTAZI.

Two voluntary positions for working on ICMI data

ICMI Database Project

This is an opportunity for one or more researchers to continue the ICMI Database project. The aim is to build and update a database of the mathematics curricula all over the world.

For the first phase of this project we asked all ICMI representatives to send us a link to the webpage(s) of their country where anybody can find the official mathematics curricula at all levels of instruction (pre-primary, primary, elementary, middle, secondary, vocational, etc.).

We have for the moment received information on the following countries: Argentina, Austria, Bulgaria, Greece, Italy, Japan, Nepal, New Zealand, Portugal, South Africa, Switzerland, the Netherlands, Turkey, United States of America and United Kingdom. Information about the project and data: http://www1.mathunion.org/index.php?id=918.

This voluntary position would suit someone with an interest in mathematics education. The research would include collecting further data from new countries and update the existing links (if necessary). The researcher is invited to use this data to publish reports under his/her name, organise small seminars or, in the case of multiple researchers, form a research group or panel.

For both projects, enquiries should be directed to Lena Koch, IMU Secretariat, Berlin.
icmi.cdc.administrator@mathunion.org

Pipeline Researcher & Individual Country Data Collectors

This is an opportunity for one or more researchers to continue the IMU/ICMI Pipeline project. The aim is to collect and analyse comprehensive data on the flow of mathematical science students from school to university to workplace on a worldwide basis and study the issues associated with this flow.

Up to this point significant data has been collected for 12 countries from 1960 to the present and the first report was written in 2009. The data collected so far includes: School leavers' qualifications AND/OR cohort of students studying mathematics, Bachelor data (Mathematical Sciences & All Subjects), PhD data (Mathematical Sciences & All Subjects), Population data 20–24 year old, Destinations of mathematical science graduates by industry, Employment status of mathematical science graduates – employed / study / other, Number of graduates of senior secondary mathematics teacher training programmes.

This voluntary position would suit someone with an interest in mathematics education. The research would include collecting further data from new countries and completing the existing datasets.

The researcher is invited to publish further reports under his or her name, organise small seminars or, in the case of multiple researchers, form a research group or panel.

More information about the project:
http://www.mathunion.org/icmi/other-activities/pipeline-project/

ERME Column

João Pedro da Ponte

Sixth Yerme Summer School (YESS-6) in Faro, Portugal

http://yerme.eu/www.ese.ualg.pt/yess6

The Sixth YERME[1] Summer School (YESS 6) will take place next August (23–28) in Faro, Portugal, aiming to let young researchers from different countries meet and establish a friendly and cooperative style of work in the field of mathematics education research. YESS will let participants compare and integrate their preparation in mathematics education research in a peer discussion climate with the help of highly qualified and differently oriented experts. This event constitutes an opportunity to let young researchers present their ideas, theoretical difficulties, methodological problems and preliminary research results, in order to get suggestions from other participants and experts about possible developments, different perspectives, etc. and open the way to possible connections with nearby research projects and cooperation with researchers in other countries.

Participants include PhD students, post-doctoral researchers and Master's students in mathematics education and others entering mathematics education research from European countries and neighbouring countries, as well as from other counties as far as New Zealand and Thailand. The deadline for applications was 15 January 2012 and, for 70 places offered, there were 123 applicants, showing the interest raised by this ERME activity. There will be the possibility of some scholarships for participants depending on the financial situation. The participants will be invited to present a paper, according to the situation of their studies and research work. This may include comprehensive information concerning personal graduate studies and/or a research plan; a presentation of

[1] Young European Researchers in Mathematics Education

research work in progress (goals, theoretical framework, methodology); or a presentation of preliminary results (with essential information about their goals, theoretical framework and methodology).

The scientific staff comprises six leading experts in mathematics education invited to contribute to the summer school by giving lectures and taking part in the working groups. Moreover, discussion groups will be led by Paolo Boero, Ferdinando Arzarello, João Pedro da Ponte and other experts. The evaluation of the summer school will be done by Paolo Boero and João Pedro da Ponte.

According to the very successful experiences of the first five YESS events and the suggestions of the participants, the programme will include:

- Five 45 minute lectures given by the invited experts (with 30 minutes of additional discussions with the lecturer, prepared by small group discussions). Before the summer school, each expert will circulate (over the internet) a summary of their talk, pointing out relevant issues to be discussed.
- Ten 120 minute sessions in five parallel thematic working groups that will meet every day at least once. They will start from the participants' papers and will address the following "leading questions": How to choose research problems and research questions? How to refer to and use existing literature and frame our own research? How to construct an appropriate research methodology to frame our study? How to position theory in research conceptualisation and analysis? How to identify, present and interpret research results? How to disseminate research findings so that they have real impact in the schools and the society? Specific attention will be paid to each participant's own needs, regarding their own work.
- Two 90 minute discussion group sessions devoted to topics of common interest, possibly including the above "leading questions".
- Two 90 minute informal discussion group sessions devoted to issues that are relevant to the needs of individual YERME participants, self-organised by YERME.

The proposed topics for the 2012 summer school are:

- Teacher knowledge and practice; teacher education and professional development.
- Teaching and learning advanced mathematics.
- Information technologies in mathematics teaching and learning (software, internet, etc.).
- Cognitive and affective factors in learning and teaching mathematics.
- Theoretical perspectives, modelling and linguistic and representational aspects of teaching and learning mathematics.

The impulse for the realisation of the summer schools comes from the spontaneous aggregation of young researchers of different countries at the CERME-II (2001) and CERME-III (2003) conferences, in order to create a cooperative style of work among them and a support to the development of professional preparation and careers in the field of mathematics education. Former YERME summer schools took place in Klagenfurt (Austria, 2002), Podebrady (Czech Republic, 2004), Jyväskylä (Finland, 2006), Trabzon (Turkey, 2008) and Palermo (Italy, 2010).

The summer school will take place at the University of Algarve in Faro (http://www.ualg.pt/), Campus da Penha, located in the South of Portugal. The organising committee includes Ferdinando Arzarello and João Pedro da Ponte (ERME board representatives), Uffe Jankvist (YERME representative), Cláudia Canha Nunes and António Guerreiro (local group team representatives) and Paolo Boero (scientific coordinator). The local organisation is based at Instituto de Educação da Universidade de Lisboa and Escola Superior de Educação e Comunicação da Universidade do Algarve. The Instituto de Educação da Universidade de Lisboa has quite a dynamic PhD programme in mathematics education, with lines of research on teacher knowledge and practice, teacher development, teaching and learning algebra, numbers and statistics, use of technology in the classroom, students' assessment and history of mathematics education. It also has a Master's degree programme in mathematics education for in-service teachers and a pre-service teacher education programme for teachers of grades 7-12. The Escola Superior de Educação e Comunicação of the University of Algarve is dedicated to pre-service and in-service teacher education, especially for the early school years (preschool and grades 1-6).

It is Necessary that Teachers are Mathematically Proficient, but is it Sufficient? Solid Findings in Mathematics Education on Teacher Knowledge

Imagine a middle school mathematics class with 28 students solving algebra problems. Whereas student Andrea, although getting a lot of individual support, often comes up with $(a + b)^2 = a^2 + b^2$, Bettie is close to developing the quadratic formula identity by transforming $x^2 + px + q = 0$ into $(x + p/2)^2 - (p/2)^2 + q = 0$; Charly when dealing with $(a + b)^3$ develops an interest in Pascal's triangle whereas Diane has problems accepting that we can count with a's and b's and not just with natural numbers.

The class is taught by Marta, an experienced teacher with a deep background in content knowledge and in pedagogical content knowledge, and in diagnosing students' mathematical thinking in particular. Like many teachers, she deals with a diversity of students' pre-knowledge.

Her young colleague, Melvin, teaches the second mathematics class of the same grade at her school. He is also mathematically proficient but he has severe problems coping with the complexity of teaching. He mainly teaches from the front of the class which fits the needs of only a small number of his students. The other students would need different kinds of learning environments and specific support. Mina, another colleague, did not study mathematics at university but has to teach this subject due to a shortage of appropriately qualified teachers. She is an enthusiastic teacher but sometimes she is not able to support students' thinking because of her own lack of content knowledge; however, she is aware of this weakness and motivates students to self-study and learning with experienced others. Another teacher, Monte, has severe health problems and is close to burning out. His motivation has decreased over the years and he is frustrated about teaching in general and mathematics at this school in particular.

The school has no tradition of exchanging experiences among subject teachers. Thus the ideas, strengths, problems and strategies of Marta, Melvin, Mina, Monte and others remain individual. The principal does not really value mathematics; thus his interest in and support for mathematics teachers is limited. The mathematics colleagues never meet to share lesson plans or to discuss their understandings about what good mathematics teaching is and how the school might improve students' interest and competence in mathematics.

Nearly every teacher at this school goes their own way, with individual variations of knowledge and with idiosyncratic interpretations of the national curriculum. The teachers themselves do not seek collaboration and few of them have experienced the benefits of sharing educational opportunities and challenges in their own teacher education. Even if a new ambitious and mathematically competent teacher starts at this school, they may not be authoritative enough to change the "culture" of isolated teaching and the poor "general conditions" for mathematics teaching at this school.

The situation sketched above can be found in thousands of variations all over the world, influenced by specific cultural, geographical, historical, socio-economic and political circumstances. The story illustrates that content knowledge is at best a necessary, but far from being a sufficient, condition for "good teaching" (see, for example, Wilson, Cooney & Stinson, 2005). Teachers need more than mathematical knowledge; they also need to communicate with and learn from each other and to get adequate internal and external support for their task. Thus mathematics teaching, in addition to mathematical considerations, also needs to take into account a variety of individual, social and organisational aspects. In other words: *content* is important but so is *community* and *context*.

When reduced to a purely mathematical point of view, mathematics teaching seems to be well defined. However, when taking into account the individual, social and organisational aspects of schooling, the picture becomes more complex: not only do individual students' and teachers' knowledge and interests vary greatly but also the forms of professional communication, of teacher education and of the context of teachers' work at schools.

Research in mathematics education, in particular in mathematics teacher education, underlines the fact that *content, community* and *context* are decisive factors of mathematics teaching and teachers' learning (see, for example, Krainer, 2011). This is mirrored in a variety of research findings, from which some exemplary findings are sketched in the following.

Content: Mathematical knowledge for teaching

Knowing mathematics is naturally an essential prerequisite for teaching mathematics. However, efficient mathematics teaching, whoever the students are, also requires other kinds of knowledge and skills (Ball, Hill & Bass, 2005) and thus appropriate learning opportunities. This insight is reflected by the introduction of different types of teachers' knowledge in mathematics education research. The most prominent typology describing mathematics teachers' knowledge goes back to Shulman (1987), who differentiated six different types, from which Content Knowledge (CK), Pedagogical Content Knowledge (PCK) (meaning the specific knowledge that is needed for teaching mathematics) and Pedagogical Knowledge (PK) are the most important ones. However, teachers' knowledge can also be regarded as knowledge about learning and teaching processes, assessment, evaluation methods and classroom management; other foci are expressed by the notions of attention-based knowledge or knowledge of mathematics for teaching. A recent overview is given in the first volume of the *International*

Handbook of Mathematics Teacher Education (Sullivan & Wood, 2008).

Many studies and surveys – in several domains of teacher education – indicate the importance of Pedagogical Content Knowledge. In a recent German study (COACTIV project – see, for example, Baumert et al., 2010) the researchers show that mathematics teachers from Gymnasium (who traditionally get a strong CK background in teacher education) outscore teachers from other secondary-level school types on CK but also on PCK. Gymnasium teachers also exhibit a higher degree of cognitive connectedness between the two knowledge categories. Also within the COACTIV project, tests of secondary mathematics teachers' CK and PCK have been developed and implemented in a sample of teachers whose classes participated in the PISA 2003/04 longitudinal assessment. The study shows that students taught by teachers with a high PCK showed better PISA results than those of other students. In particular, they showed within one school year a greater increase in achievement. Reasons given were that teachers with a high PCK design their teaching so that the students are more actively cognitively engaged. This was shown, in particular, through the analysis of tasks the teachers used in the classroom. The effect seems to be specific for PCK since a similar correlation between teachers' CK and students' achievement was not found. This was confirmed in further analyses: PCK has "greater predictive power for student progress and is decisive for the quality of instruction" (Baumert et al., 2010, p. 164). Knowing mathematics is important but not enough to teach mathematics effectively.

Good teachers also have insights into the mathematics content their students will learn that allows them to identify misconceptions, set up lessons in which students make connections and provide students with questions and prompts that help them access important concepts. Studies show that supporting student teachers' reflections of their own thinking processes leads them to understand better and improve their instructional practices. Meaningful mathematics-related activities and reflections on them support prospective and practising teachers' growth (see, for example, Even & Ball, 2009). Such activities can include solving mathematical problems themselves, choosing appropriate content for the preparation of a lesson, investigating students' mathematical learning or observing and reflecting on one's own or other colleagues' teaching.

This principle holds in particular in Japanese lesson study (see, for example, Hart, Alston & Murata, 2011), a mode of teacher education whose efficiency has been established in studies in Japan and elsewhere. A typical *lesson study* process contains 4–6 steps, having a *study lesson* as its focus: 1) Collaboratively planning the study lesson; 2) Seeing the study lesson in action; 3) Discussing the study lesson; 4) Revising the lesson (optional); 5) Teaching the new version of the lesson (optional); and 6) Sharing reflections about the new versions of the lesson. The vast majority of elementary schools and many middle schools in Japan conduct lesson study in all subjects. Many schools solicit the support of an external advisor (most often instructional superintendents, sometimes experienced teachers on leave or university staff). Some schools even produce written reports about their work. For example, in the early 1990s the National Institute for Educational Research compiled over 4,000 reports written by teachers every year.

Similar approaches are carried out in other Asian countries, for example within the Keli project in China (see, for example, Huang & Bao, 2006), where exemplary lessons are developed and disseminated at the school and district level. It is assumed that the success of Asian countries in international comparative achievement studies like PISA and TIMSS can partly be explained by this focus on planning, observing, reflecting and revising lessons in a joint effort. This leads to the factor "*community*".

Community: Collaboration among mathematics teachers

Research on "successful" schools shows that such schools are more likely to have teachers who have continual substantive interactions and that inter-staff relations are seen as an important dimension of school quality (Reynolds, Creemers, Stringfield, Teddlie & Schaffer, 2002). The latter study illustrates examples of potentially useful practices, of which the first (illustrated by a US researcher who reflects on observations in other countries) relates to teacher collaboration and community building (p. 281): "Seeing excellent instruction in an Asian context, one can appreciate the lesson, but also understand that the lesson did not arrive magically. It was planned, often in conjunction with an entire grade-level-team (or, for a first-year teacher, with a master teacher) in the teachers' shared office and work area." Such a collaborative habit needs to be fostered from the very beginning of pre-service teacher education. This statement holds for primary and secondary teachers but also for university teachers (see, for example, Nardi, 2008).

One key characteristic of Japanese lesson study is that it is "*collaborative*" (Murata in Hart, Alston & Murata, 2011). She indicates that novice teachers who experience the lesson with experienced teachers are apprenticed into the profession through participation. This community aspect goes beyond the idea of collaboration among individual persons. It is about a way of further developing a profession (by engaging novices into serious academic activity and thus fostering identity building). Community building is not confined to teachers but also extends to students: the vision of Japanese teaching is organising collective thinking, focusing on student presentations and discussions. The succession of single work, group work, plenary discussion in classrooms and teachers' comments helps to balance individual and social learning (Krainer, 2011).

Also in the Chinese Keli project (Huang & Bao, 2006) collaboration plays a decisive role. In this project, a "community" consisting of experts, teachers and researchers is formed and the teachers improve their teaching action and upgrade their professional theory through unfolding the Keli process in cooperation with the members of the community.

Several studies worldwide indicate the positive impact of teachers' collegial learning. For example, Jackson and Bruegmann (2009) use longitudinal elementary school teacher and student data for documenting that students have larger test score gains when their teachers experience improvements in the observable characteristics of their colleagues. They show that teachers' students have larger achievement gains in mathematics and reading when they have more effective colleagues.

The Learning Communities in Mathematics (LCM) research and development project in Norway (see, for example, Jaworski, 2008) brought together teachers and didacticians (the term that the team preferred to use for the teacher educators) to work together as both practitioners and researchers. It involved a team of 14 didacticians working with eight primary and secondary schools. Among others, Jaworski (2008, p. 326) highlights that "teachers suddenly came to see, through their study of students' thinking and activity in algebra, how they could explore in their school environment ways to develop teaching and learning; didacticians saw the nature of a task that could lead to teachers' effective recognition of the nature of school goals for students' development and learning in mathematics".

At a university in Finland, new ways of teaching a lecture course and of giving peer support to beginning mathematics students doubled (at least) the number of students passing certain important courses compared to the earlier prevailing level (Oikkonen, 2009). Several studies indicate the new possibilities opened by digital networking (see, for example, Gueudet, 2010).

The importance of the "community" factor is mirrored by a stronger emphasis on sociological and socio-cultural aspects of teaching and teacher education. Concepts like didactical contract (which will be dealt with in the next EMS Newsletter), inquiry community, institutional constraints, negotiation of meanings and norms, organisational development, sharing of knowledge and systemic learning are used that go beyond cognitive views on learning. Lerman and Tsatsaroni (2004) report that between the time periods 1990–1995 and 1996–2001 the percentage of papers in selected journals and proceedings in mathematics education that draw on sociological and socio-cultural theories increased from 3% to about 10%. Two of the four volumes of the first *International Handbook of Mathematics Teacher Education* (Krainer & Wood, 2008; Jaworski & Wood, 2008) deal intensively with the collaboration among teachers or teacher educators and between teachers and teacher educators. When mathematics teachers (and teacher educators) share experiences, ideas, beliefs, competences, challenges and needs, they not only learn themselves but also learn to support others' learning. The processes include working in small teams, communities of practice and loosely-coupled networks. Mathematics teachers need to build identities specialising in students' mathematical learning through collaborative reflection.

Collaboration among mathematics teachers needs adequate general conditions and support of teachers' work. This leads to the next factor, the "*context*".

Context: General conditions and support of mathematics teachers' work

General conditions that are conducive to successfully supporting mathematics teachers include offering learning opportunities where both the factors of content and community are regarded as essential. In particular concerning the work with practising teachers, the context (resources, structures, commitment, etc.) plays an important role. In addition, the influence of principals and other stakeholders of the educational system is indicated by several studies.

In Japanese lesson study, knowledgeable others from outside the lesson study group are sometimes invited, to present observations or links to research or theories, for example. These experts are paid with the help of small external grants. It is common that teachers are internally supported by principals and get release time by hiring substitute teachers. Lesson study groups are supported by townships, boards of education, the ministry, etc. This financial investment is an expression of societal trust in teachers and of believing in school-based professional development. It makes sense to speak of a lesson study "culture" that is conducive to teachers' work. The importance of the context factor is increasingly stressed in studies worldwide. For example, Adler (2000) – in particular looking at the situation in South Africa – worked out the need for conceptualising resources as a theme for mathematics teacher education (e.g. classrooms and learning tools in poor and rich regions are quite different). Nickerson and Moriarty (2005) describe an urban school initiative aimed at teachers' professional development with the goal of increasing teachers' mathematics content knowledge and helping them improve in practice. The research shows that general social and organisational conditions like mathematics teachers' relationships with the school administration and other teachers or the presence of a teacher leader are relevant for (the further development of) good mathematics teaching at schools. Kazemi (2008) stresses the importance of engaging parents as intellectual and social resources. She indicates studies with interventions that have aimed to work with families around mathematics as a complex problem solving discipline and that led to significant increases in families' feelings of empowerment. Cobb and Smith (2008) report on a district development project as a means of improving mathematics teaching and learning at a large scale. Here, research in mathematics education meets with research in educational policy and leadership where institutional settings of schools and districts come to the fore. Also, when investigating the sustainable effects of a professional development programme, context issues like the principal's understanding of leadership or personnel fluctuation play a decisive role.

What do we learn from that? What can be done?

That mathematics teachers need a high level of Content Knowledge (CK) is necessary but not sufficient. There is clear research evidence that other kinds of knowledge are also important; in particular, Pedagogical Content

Knowledge (PCK) is decisive. Furthermore, teachers need to have high social competencies, in particular for two reasons: in order to support students' learning (as individuals, groups and whole classrooms) and in order to learn from other colleagues and other experts. Mathematics teachers need to understand themselves as learning communities that adopt new research findings, share their experiences and discuss ways of improving with others, etc. It is a decisive feature of a (scientific) community to share new knowledge and experiences; this also holds true for the teaching profession. Otherwise, teachers' efforts to improve remain limited to their own classrooms and the wheel is invented again and again. Mathematics teachers also need to understand that communication and collaboration are essential in order to improve not only the teaching of mathematics of one of their classes but of a whole school, or a whole district or even a nation. This needs them to bring in their ideas, interests, knowledge and visions but also their doubts and open problems. Teachers need to reflect on how to improve mathematics teaching at their school, on how to convince the principal to buy a new mathematics education journal or new mathematical software or on how to improve the curriculum. This means to engage in organisational activities, to reflect on the context of mathematics teachers' work at their school and on how to overcome financial and personal restrictions, etc.

Where is the place that mathematics teachers learn all this?

Of course, in the first phase of teacher education, there needs to be a special emphasis on CK, and also on PCK and PK. However, in order to cope with the collaborative nature of the teaching profession, collaboration should also get a clear focus in teacher education. Teachers teach as they have been taught. Therefore, mathematics teacher educators need to regard themselves and act as role models. Also, at least from time to time, they should provide learning opportunities to reflect on the context under which the student teachers learn (e.g. What is the "didactical contract" between the teacher educator and the class? What have mathematics and physics in common or where do they differ, for example, regarding their contribution to education?). Of course, some of these issues can also (and partially better) be learned when teachers have finished their studies and work in practice, and in collaboration with colleagues in their departments or in other forms of professional development. However, it would mean to miss a chance to focus mainly on CK in the first phase of teacher education. Vice versa, professional development should also deal with mathematical issues from time to time. Since not all mathematical fields can be covered in university studies (in particular in primary education), it is important that teachers learn strategies for learning the mathematics they will need.

Mathematics teacher education is a challenge. Teachers need teacher educators who work with them in the same innovative way teacher educators are expecting teachers to teach. These teacher educators need to collaborate with other colleagues like they expect teachers to do. Teacher educators need to evaluate and improve their courses. They need to gather teachers' interests and pre-knowledge because this increases the likelihood that teachers realise the power of having sufficient information about their learners. The list of needed actions and reflections by teacher educators could be easily extended. The stronger this culture of reflection and evaluation becomes established at universities, the stronger it develops at schools (and at school administrations). It is important to investigate the learning of mathematics students and mathematics teachers; however, we also need to put more emphasis on exploring and understanding our influence on mathematics teachers' learning. This kind of teacher education research is a key to supporting the development of mathematical thinking, both of teachers and students.

This paper started with the insufficient situation of mathematics teaching at Marta's school. How could this situation be turned around according to the findings described in this paper? Marta starts to carry out innovations in her algebra class. The number of students loving mathematics increases; others at least begin to lose fear and to construct meaning in thinking mathematically. Together with her colleagues Melvin, Mina and Monte, Marta shares experiences and ideas continuously. Together, they form a kind of learning community, supported by a mathematics teacher educator at the regional university. In collaboration with mathematicians and mathematics educators, mathematical contests, exhibitions and family clubs are established. Student teachers from the university come to this school and collaborate with experienced teachers. Mathematics begins to play an important role at this school. The principal supports the work of his students and teachers and he makes it visible in the district. Representatives of the mathematics teaching staff are invited to speak about their progress at other schools and districts. Considerable research is done with regard to mathematical teaching and learning in the context of this school district which is fed back to teachers, administrators and educational authorities. This movement also influences teacher learning in other subjects. To some extent, the idea of improving mathematics teaching and learning finally turns into a culture of innovation in all subjects. Isn't that a nice (educational) kind of "generalisation"?

Authorship

Even though certain authors have taken the lead in each article of this series, all publications in the series are published by the Education Committee of the European Mathematical Society. The committee members are Ferdinando Arzarello, Tommy Dreyfus, Ghislaine Gueudet, Celia Hoyles, Konrad Krainer, Mogens Niss, Jarmila Novotná, Juha Oikonnen, Núria Planas, Despina Potari, Alexei Sossinsky, Peter Sullivan, Günter Törner and Lieven Verschaffel.

Additional information

A slightly expanded version of this article with a more complete list of references may be found at http://www.euro-math-soc.eu/comm-education2.html.

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Building an Information Service for Mathematical Software – the SMATH Project

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We report about a project to create, with the help of Zentralblatt MATH, a comprehensive open access database SMATH that will provide broad coverage of mathematical software and, at the same time, will guarantee high quality and relevance of the referenced software packages.

Present Situation and Final Goal

Scientific publications are no longer the only form of mathematical knowledge. Over the past three decades, mathematical software has become an increasingly important tool for mathematical education and research. It is often an indispensable tool in many active areas of current mathematical research fields like numerics and statistics. But mathematical software is also used in some parts of pure mathematics, to check hypotheses, to construct counter-examples and to illustrate and sometimes even prove new mathematical results. Moreover, since mathematical software is operational knowledge, it is the most important link between mathematics and its applications.

For mathematical publications, comprehensive online collections and information services are available, as well as established classification systems and standard citation schemes. Unfortunately, the same is not true for mathematical software, where the information situation is much less complete. While there are many specialised online collections of mathematical software, a unified and commonly accepted approach to information retrieval is still missing. The SMATH project aims to fill this gap.

There are three main differences between SMATH and existing software collections or portals. SMATH:

- intends to be fairly complete;
- links the software with the publications using it;
- links the publication with the used software;

where the links are realised at the moment with the help of the publication database Zentralblatt MATH.

Difficulties Encountered

Nowadays, every student of mathematics or a related discipline is trained to become familiar with computer algebra systems and numerical software packages like Maple[1], Mathematica[2] and MATLAB[3] but there are many more (in fact several thousand), mostly specialised, software systems and packages in use. The specific requirements of potential software users differ very much depending on their individual background and the character of the problem to be solved. However, certain basic questions constantly arise in almost any context:

- How to find the suitable software for a given mathematical problem?
- Which mathematical methods does the software employ for solving that problem?
- How about the quality of the software?
- For which problem classes was the software used successfully in the past and by whom?
- What are the licence terms for a certain software package? Is the source code available, i.e. "open source" or not?
- Which programming languages are used?
- Which interfaces to other systems does the software provide?
- Is the software compatible with a certain operating system?

A detailed description of a software package necessarily has to be much more complex than is the case with an article. In addition to describing the goals and areas of application of the software, one has to specify technical parameters (such as programming languages, operating systems, interfaces, etc.) and licence terms or terms of use, as well as information about the current version. Finding all these metadata for a certain package can be very involved, since information about software is typically inhomogeneous and spread across different media.

In many cases, part of the required information can be found in certain special articles. The ideal case, however, is to have an informative and up-to-date website which is carefully maintained by the software developers and at any time reflects the current state of the software. Often the amount of information available depends heavily on the size and the development status of a software package. For many small and highly specialised mathematical software libraries, there is no such thing as a describing website.

Another problem is the dynamic character of any software development process. Unlike bibliographical metadata, information about software is subject to frequent changes during the whole software life cycle. Not only the technical parameters but also the areas of application of software can change, as can the software authors (i.e. the development team). These considerations may explain why there is still no commonly accepted and widely used standard for the description of mathematical software resources, even though there is no lack of theoretical work or standardisation attempts.

The SMATH Project

The SMATH project is scheduled for three years (2011–2013). It is directed by the second author of this article and

[1] http://www.maplesoft.com/products/maple/.
[2] http://www.wolfram.com/mathematica/.
[3] http://www.mathworks.de/products/matlab/index.html.

jointly carried out by Mathematisches Forschungsinstitut Oberwolfach (MFO)[4] and Fachinformationszentrum Karlsruhe (FIZ)[5]. MFO is already developing a database of selected high quality software packages, called ORMS, while FIZ is responsible for developing and hosting the publication database Zentralblatt MATH. Further institutions with proven expertise in the area of mathematical software are supporting the project: Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)[6], the DFG Research Center MATHEON[7], the Weierstraß-Institute for Applied Analysis and Stochastics (WIAS)[8] and the Felix-Klein-Zentrum für Mathematik (FKZ)[9] in Kaiserslautern.

Identification and detailed description of as many mathematical software packages as possible is an essential project goal. But SMATH goes one step further aiming to ensure that for any software package referenced in SMATH, all mathematical publications referring to that software will be identified and referenced, too.

There are essentially two categories of such publications. The first category consists of articles written by the main software developers with a view to presenting the software to the public ("developer's article", see Figure 1).

[MATRIX] Zbl 0902.14040 MPG SFX
Greuel, G.-M.; Pfister, G.; Schönemann, H.
SINGULAR: A computer algebra system for singularity theory, algebraic geometry and commutative algebra. (English)
Comput. Sci. J. Mold. 5, No.1, 3-9 (1997).

SINGULAR, as pointed out by the authors is a computer algebra system designed especially for commutative algebra, algebraic geometry and singular theory. For almost all computations SINGULAR requires a base ring. Computations are possible over such base rings as: polynomial rings, localizations of polynomial rings, quotient of such rings, exterior algebras, Weyl algebras and tensor products. The main algorithms are very general standard Gröbner basis algorithms, and include algorithms for the usual ideal theoretic operations, for different syzygy problems, for computing invariants of singularities and semiuniversal deformations of isolated singularities. SINGULAR contains several libraries to which the user may add more procedures.
Reviewer: Aigli Papantonopoulou (Ewing)

MSC 2010
14Q99 Computational aspects of algebraic geometry
13P99 Computational aspects of commutative algebra
68W30 Symbolic computation and algebraic computation

Keywords
SINGULAR

Figure 1. Example of a "developer's article" on the SINGULAR software (source: Zentralblatt MATH).

Such articles typically contain a more or less detailed description of certain software. The second category comprises articles written on a certain mathematical topic by an author who used an existing software package to obtain or illustrate his results and who therefore cites the software as an auxiliary tool ("user's article", see Figure 2).

At the moment we mainly use the database of Zentralblatt MATH for finding articles referring to given software but an extension to other data sources would of course be possible. The use of Zentralblatt has three main advantages:

[MATRIX] Zbl 0932.13021 MPG SFX
De Jong, Theo
An algorithm for computing the integral closure. (English)
J. Symb. Comput. 26, No.3, 273-277 (1998).

Finding efficient algorithms for computing the integral closure of a commutative noetherian ring R is one of the big challenges of computational algebra. The paper under review contains an algorithm that is based on a result by Grauert and Remmert, which has as a consequence the fact that, if $I \subset R$ is a radical ideal which contains a non-zero divisor, then R is normal if and only if $R = \mathrm{Hom}\ R\ (I,I)$. Several known criteria for normality used in existing algorithms are special cases of this. The algorithm terminates if the normalization is finitely generated as an R-module, e.g., for affine rings. It has been implemented in MACAULAY and SINGULAR.
Reviewer: Reinhard Laubenbacher and Abdul Jarrah (New Mexico)

MSC 2010
13P99 Computational aspects of commutative algebra
13B22 Integral closure; integrally closed rings, related rings
13E05 Noetherian rings and modules
68W30 Symbolic computation and algebraic computation
13-04 Machine computation, programs (commutative algebra)

Keywords
reduced noetherian ring; computing the integral closure; algorithm; MACAULAY; SINGULAR

Figure 2. Example of a "user's article" on the SINGULAR software (source: Zentralblatt MATH).

- First of all, the database will be updated continuously and automatically whenever new software or a new version is cited in an article indexed by Zentralblatt. Since all indexed articles are peer-reviewed, this is sufficient to guarantee the quality of the software.
- Every user who is searching for information about some software is linked to all articles indexed in Zentralblatt MATH where this software is used. These articles provide valuable information about the software which is available nowhere else.
- Everyone who reads a review in Zentralblatt has immediate access to basic information about the software mentioned via a link to SMATH.

It is an interesting and useful observation that we can use the articles in Zentralblatt MATH to identify a large number of additional software packages. To this end, we developed and implemented heuristic tools for the automatic identification of software-relevant articles in Zentralblatt MATH and elsewhere. Examples of such automatically identified articles are:

- *"KNITRO: an integrated package for nonlinear optimization."*
- *"PDETool: A multi-formalism modelling tool for discrete-event systems based on SDES description."*
- *"QuBE++: An efficient QBF solver."*

The key ingredient of the heuristics is the fact that articles describing software often exhibit a characteristic title structure. The titles of such articles often contain software-specific keywords (e.g. "software", "package", "solver", etc.), as well as the name of the software package. The characteristic spelling of many software names and their position within the title (mostly at the beginning) can be used for automatic extraction. So far we have identified about 3,500 software packages by means of heuristic analysis tools.

Further Schedule

In the first year the SMATH team defined a provisional metadata model for mathematical software and has set up the general framework and the technical infrastructure for SMATH. A focus of the second year is on further filling and consolidating the database. To this end, the publication-based heuristics approach will be combined with the input and specific know-how of the cooperating institutes; they will add relevant software from their fields of expertise to SMATH. For this purpose they will develop tools for the semantic analysis of software websites in order to reduce the manual effort to obtain software metadata. Another focus of the current project year is on the design and implementation of a functional and ergonomic web user interface for future SMATH users.

The first preliminary production version of SMATH is scheduled for mid-2012.

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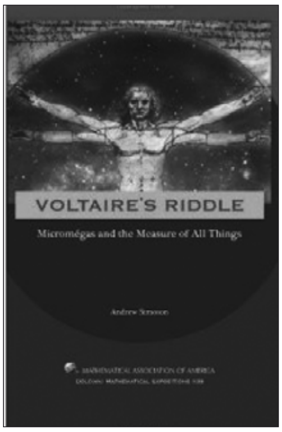
4 http://www.mfo.de/.
5 http://www.fiz-karlsruhe.de.
6 http://www.zib.de/.
7 http://www.matheon.de/.
8 http://www.wias-berlin.de.
9 http://www.felix-klein-zentrum.de/.

Book Reviews



Andrew J. Simoson

Voltaire's riddle. Micromégas and the measure of all things

Volume 39 of *Dolciani Mathematical Expositions*, Mathematical Association of America, 2010
(xvii+377 p.), hard cover,
ISBN 978-0-88385-345-0.

Reviewer: Adhemar Bultheel

The *Dolciani series* of MAA is a quality label since books are selected 'for their lucid expository style and stimulating content' both for undergraduates and for more advanced mathematicians. Simoson presents here his second volume in the series. The first was *Hesiod's Anvil* (2007), which followed a similar style to this one. He received the MAA *Chauvenet* Prize in 2007, awarded to 'an author of an outstanding expository article on a mathematical topic', for his paper on *The gravity of Hades*.

Voltaire (1694–1778) was a thoroughbred product of the Age of Enlightenment. He is a well known poet, playwright and philosopher but it may be less known that he also relates to mathematics. He was responsible for the French translation of Newton's Principia[1] and it is said that he invented the story about Newton and his apple. However, after pursuing mathematics for several years, he was rated as only mediocre and that is why he decided to go into writing wholeheartedly.

So what mathematics do we find in Voltaire's riddle? Let me start with the *Micromégas*. That is a story by Voltaire which is considered to be one of the first science fiction stories ever written. *Micromégas* is a huge giant from a planet of the star Sirius, who travels to Saturn where he meets a 'dwarf' who is still a giant by human standards. They both arrive on Earth, where they meet the French expedition returning from the arctic where they measured the length of three angular degrees on the Earth's surface. The latter is an historical fact. A similar expedition did measurements near the equator to settle the question of whether the Earth was flattened at the poles or at the equator, which was a big issue in Voltaire's days. Voltaire uses this to give a satirical account of human society. Anyway, the story ends abruptly when the giant gives a book to the humans containing the answers to everything, but it turned out that the book was empty. What is the meaning of this? That is Voltaire's riddle.

An annotated translation of the *Micromégas* forms the first chapter of this book. Some possible answers to the riddle are given in the last chapter. There are ten chapters in between dealing mainly with mathematical topics that are related to Voltaire. Every chapter is preceded by a 'vignette', which is a short interludium explaining something about Voltaire and his time, evoking a motive for the mathematics to follow. Let me quickly give some vague ideas.

Vignette 2 tells about giants that appear in literature, while Chapter 2 is about measuring from the very large to the very small and self-similarity. This opens the road to cosmology and fractals. The next vignette tells about Voltaire's rebellious nature and how he ended up in the Bastille. A parallel is seen in *A. Square*, the inhabitant of E. A. Abbott's *Flatland* who is also imprisoned for his controversial ideas. But this involves true mathematics because several models are considered for *Flatland* and two-dimensional gravitational models are worked out. Consider Flatland as an infinite rectangle of finite width with all mass concentrated on a line parallel to the baseline; what would be the path of a thrown ball?

Then the story is told about Voltaire trying very hard to become a mathematician, which culminated, but also ended, with his *Elements of Newton's Philosophy* (*Eléments de la philosophie de Newton*, 1738). It helped much in the popularisation of Newton's work in France but he received, however, some poor reviews from people in the French *Académie*, so Voltaire gave up mathematics, although he continued to support Émilie du Châtelet in translating Newton's work. The corresponding chapter is about Newton solving the equations of motion and two case studies: finding out the springtime period of a planet (the number of days between the winter and the summer solstices) and the voyage of *Micromégas* (in fact, the orbit of a comet in the solar system).

More loosely connected is the story of Voltaire becoming rich thanks to the state lottery. La Condaminne, one of Voltaire's teachers and also a member of the expedition that measured the arclength near the equator, detected a flaw in the lottery system. The trick was to buy as many low-cost tickets as possible. As a result they won six months in a row before the flaw in the system was detected. As a consequence the syndicate that bought the tickets had become very rich indeed. The vignette also has a discussion of the mathematical background of the Earth being flattened at the poles, as Newton predicted, or at the equator, as the French Academy tended to believe, following the arguments of Descartes and propagated by Cassini in Newton's days. The Academy proposed to send an expedition to the Caribbean and another one to Lapland to measure one degree of arc-length along the Earth's surface. If Newton was right then it should be longer near the poles than near the equator. Among the team going north were Voltaire's teachers Maupertuis and Clairaut. La Condaminne was among the team sent to Ecuador. The arctic team finished the job within a year but the Caribbean team needed eight years. The outcome

[1] Voltaire was a lover of Émilie du Châtelet and they had common teachers, like Maupertuis and Clairaut who Voltaire hired due to his wealth after finding a winning strategy with the lottery. Émilie du Châtelet actually did the translation.

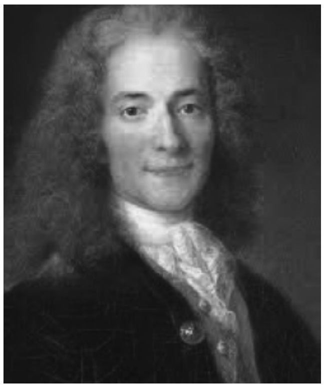
was that Newton was right but the discussion went on for some time. Fifty years later Delambre and Méchain did something similar. They measured a meridian from Dunkirk to Barcelona by a similar triangulation.[2] That resulted in the definition of the meter.

A short history and a remark of Voltaire about astrologers is an incentive to devote a chapter to the mechanics explaining the precession of the Earth's axis. The next vignette is about Voltaire's love life and romances but the link with the chapter attached, which is 'about a romantic family of curves', is a bit weak. Nevertheless, the discussion about trochoids and hypocycloids and how Dürer used them in his art is interesting enough. Again a link with mechanics is made via Johann Bernouilli's riddle: which path should a point in a gravitational field follow to move from A to B in the shortest possible time?

The next couple of chapters are also related to some historical mechanical riddles. The first is related to *Hesiod's anvil*, the author's previous book, where the question is raised about what would happen if something falls into a hole through the centre of the earth (Maupertuis was intrigued by this problem). Hesiot is a Greek poet (circa 700 BC) who claimed that it would take about nine days for an anvil to fall from the Earth's surface to the Underworld. Here, Isaac Newton's claim is that a freely falling pebble will describe an ellipse with respect to the stars. However, much more complicated situations are also considered and then the curves of the previous chapter and more three-dimensional curves show up when the path has to be described, taking into account the movement of the Earth within the solar system and the place on the Earth where the hole is dug. The other chapter is about the 'man in the moon': what path one will follow when tracing a moving object, for example a rocket starting on earth and always pointing to the moon. This chapter about pursuit curves is a version of a paper that was published before and for which Simoson received the George Pólya Award in 2008, another prize given for 'articles of expository excellence published in the *College Mathematics Journal*'.

The next chapter is about π, which is the number with the longest history. Here Simoson links it with Voltaire's theological ideas and how π shows up in the Bible or more generally in the Jewish tradition. A central theme is a discussion of the form and the dimension of Solomon's sea, a bronze basin, described in *I Kings 7:23*, where π is implicitly defined as 3. The penultimate chapter is a style break because it has no mathematics but is, instead, a story about Pantagruel's (a giant invented by Rabelais) expedition to the moon. This was one of the adventures of Pantagruel announced by Rabelais. However, he never got around to writing the story so Simoson has picked up the tale. It raises the question of how one can measure something that is beyond the reach of technology at that particular moment, in this case the composition of the moon with the technology that Rabelais had available. It is impossible of course. The only thing one can do is

A. J. Simoson



Voltaire

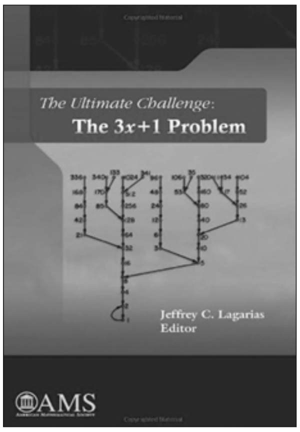
dream and make a story out of one's imagination. I will not reveal the possible answers for Voltaire's riddle proposed by Simoson but this review at least explains a bit of the riddle of how Voltaire can be linked to mathematics. The mathematics involved is not always simple. A good knowledge of (vector) calculus and linear algebra as well as differential equations is a minimal prerequisite.

Each chapter ends with a list of exercises, some of which have further comments in an appendix. But let me stress that it is not only the mathematics that are well covered; the historical facts are abundant and detailed. The book is very well illustrated with a lot of pictures and graphs, the notes of the *Micromégas* story are very extensive and there is also an appendix with a list of historical (and fictional) figures that feature in the book, from King Solomon to Stephen Hawking and from Dante to Napoleon (well, most of them, since, for example, Hesiod is missing). Simoson has created a book that is different from any other type I have read. The mathematics is not always the easiest but the mathematics is well seasoned and always pushing the reader one step further than they might have been ready for. There are a lot of juicy and creamy little facts and small-talk from the history beyond the history books. There is literature and fine arts and often there is the unexpected, like pyrolithic graphite, Jonathan Swift and George Orwell featuring on the same page. That keeps you reading on and on.

Adhemar Bultheel [Adhemar.Bultheel@cs.kuleuven.be] is emeritus professor at the Computer Science Department of the K. U. Leuven Belgium. He got a PhD in mathematics in 1979. His professional interest is in numerical analysis, rational approximation, orthogonal functions and structured linear algebra problems. He has been Vice-president (1999–2002) and President (2002–2005) of the Belgian Mathematical Society.

This review was originally published in Issue 85 of the Newsletter of the Belgian Mathematical Society and the National Committee of Mathematics, 15 November 2011. Reprinted with permission.

[2] See the book review about Daniel Kellman's *Het meten van de wereld* (*Measuring the World*) in the BMS-NCM Newsletter, Issue 58 (2006).



The Ultimate Challenge: The 3*x*+1 Problem

Edited by:
Jeffrey C. Lagarias

AMS, 2010
344 pp; hardcover
ISBN: 978-0-8218-4940-8

Reviewer: Vicente Muñoz

The $3x+1$ problem, also known as the Collatz problem, concerns the following, seemingly innocent, arithmetic procedure on the natural numbers: for a positive integer x, let $T(x) = x/2$ if x is even and let $T(x) = 3x+1$ if x is odd. The $3x+1$ problem asks whether, starting from any positive integer and applying T iteratively, we will eventually reach the number 1. Despite its simple appearance, this problem is unsolved. Actually, it is believed to be extraordinarily difficult to solve.

Jeffrey Lagarias is one of the world experts in this problem and has followed its developments for more than 25 years. This book is a compendium of selected articles on the problem which provides the reader with an awareness of the difficulties of the problem, what the known results are and how far one may expect to get with the techniques that have been used so far. It is basic reading material for someone who wants to try out this problem before expending time on it (though it is quite likely that, after reading the book, one will conclude that the wise decision is not to put much effort into the conjecture).

The origins of the problem are not completely clear. It is generally attributed to Lothar Collatz, who circulated it orally at the International Congress of Mathematicians in Cambridge, US, in 1950. He came to it by his study of the graphs associated to iterative procedures on the integer numbers. Another claimant to have originated the conjecture is Bryan Thwaites, who asserts he came up with the problem in 1952. The $3x+1$ problem has received many names in the literature: Collatz conjecture, Syracuse problem, Hasse's algorithm, Kakutani's problem and Ulam's problem.

Despite the simplicity of its formulation, many mathematicians simply think that mathematics is not ready for finding a solution. Paul Erdös said: "Mathematics is not yet ripe for such a question," and Richard Guy puts it in his list: "Don't try to solve these problems."

The first papers on the $3x+1$ problem date from the 60s. Since then, more than 300 papers have appeared in print. This book is an extensive account of the work carried out in relation to it. It is divided into five parts. In Part I there are two papers, written by J. Lagarias himself, which are devoted to a general overview of the problem and its current status. Most of the comments in this introduction are discussed in later chapters in more detail. The Collatz map T defines a dynamical system in the positive integers. As such, the orbit of a point can have three possible behaviours:

1) To hit 1 and then enter into the cycle 4, 2, 1.
2) To enter into a different cycle.
3) To escape to infinity.

The conjecture says that only possibility (1) occurs. A heuristic probability argument predicts that iterations of T end up decreasing the initial number, giving some support to the conjecture, which is however very weak.

Part II contains three survey papers. The first one, written by M. Chamberland, reports the progress in some directions up to 2007, emphasising viewing T as a dynamical system. For instance, if we have a cycle as in 2 above then writing $T^k(x) = x$, we get a diophantine equation, from which it can be proved that the cycle has length at least 272,500,658 (this question is related to the diophantine approximation of $\log_2 3$). It is also relevant to measure the set of integers which eventually land on 1 after iterations of T. For this, it is necessary that for any x, an iteration satisfies $T^k(x) < x$. It is proved that the set of integers satisfying this property is of full density. However, this is not enough even to prove that the set of seeds satisfying (1) above is of full density. The best result in this direction is that such a set is of logarithmic asymptotic measure 0.84 (i.e. $T^{0.84}$ numbers in the interval $[1,T]$ satisfy the conjecture for T large). Finally, M. Chamberland discusses extensions of the Collatz map to larger spaces, like the integers, the rational numbers, the 2-adic numbers and the complex plane.

The second paper in Part II, by K. Matthews, analyses the problem via Markov chain models. The $3x+1$ map has a natural generalization given by dynamical systems of the form $S(x) = (r_i\, x + d_i)/d$, for $x \equiv \mathrm{i} \pmod d$, where $d \mid (r_i i + d_i)$. Many of these maps have ergodic properties when extended to the d-adic integers $\mathbf{Z}_d$. However, this is not of much use, as $\mathbf{Z}$ has measure zero in $\mathbf{Z}_d$. The third paper, by M. Margenstern and P. Michel, surveys relations of the map T with the theory of computation and mathematical logic. In 1972, John Conway exhibited a generalization of the function T (of the type mentioned above), which was only slightly more complicated, giving rise to an undecidable computational problem. So one cannot decide if the iterations will end in a particular number simply because this iteration simulates a universal machine. The range of iterative maps for which undecidability happens does not cover the $3x+1$ function. Of course, if the Collatz conjecture is true, it should not be so but it is hard to explain why the Collatz problem is not undecidable when similar ones are.

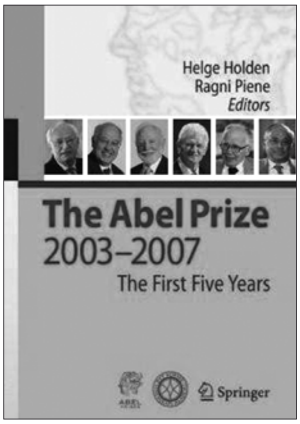
Part III presents two papers on mathematical modelling and empirical results. The first paper, by A. Kontorovich and J. Lagarias, gives stochastic models of the $3x+1$ iteration function and, for comparison, on iterates of the $5x+1$ function. This second function is expected to have iterations which grow indefinitely (but not for a single one has this yet been proved). Probabilistic models for these functions are proposed (assuming that the iterates of a given seed distribute uniformly in congruence classes). These models are checked against empirical computations (for a large range of numbers) and turn out to give good predictions. However, this is far from helping a general proof. The second paper, by T. Olivera e Silva, gives a collection of results on the $3x$+1 conjecture checked numerically, notably that the $3x$+1 conjecture holds for all $x < 20 \cdot 2^{58}$. The paper discusses efficient algorithms that can be implemented in computers to test the conjecture for large seeds.

Part IV reprints six published papers on the Collatz problem of historical interest. These papers are short and easy to read and most of them are hard to obtain from the original source. The editor provides a commentary after each of them. The first paper, by H. Coxeter, is from 1971 and it is the earliest published paper that explicitly states the $3x$+1 problem. The second one, published in 1972 by J. Conway, is very short and announces the undecidability result alluded to above. This is complemented with the sixth paper, published in 1987, in which J. Conway develops a universal programming language for arithmetic problems based on iterative processes very similar to the Collatz function. This language receives the name of FRACTRAN. The third paper, by C. Everett and published in 1977, gives a short proof that almost all positive integers have an iterate $T^k(x) < x$. In the fourth paper, R. Guy gives a list of four problems for which (he considers) one should not spend time on, and one of them is the Collatz conjecture. Finally, the fifth article, published in Chinese in 1986, is the only one on this problem written by L. Collatz himself. Here we see the trees associated to the map T which attracted the attention of the author.

Part V contains 75 pages commenting on all papers published about the problem until 1999. The author keeps an annotated bibliography from 2000 onwards in arxiv.org/abs/math/0608208.

Some of the papers in this book contain typographical errors, which could have been avoided. But most of the papers have very nice content. Of course, for some of the results mentioned in the survey papers, one has to address the original sources to read the proofs. However, the purpose of the book is completely fulfilled: the reader gets a good and quite accurate impression of the difficulty of the problem. All in all, one would say that, unless you definitely have a new idea of how to attack this problem or you are prepared to be happy with only very partial results, you'd better put this problem aside for the time being.

Vicente Muñoz
Universidad Complutense de Madrid, Spain



The Abel Prize 2003–2007
The first five years

Helge Holden,
Ragni Piene (editors)

Springer
ISBN 978-3-642-01372-0

Reviewer: Ulf Persson

Famously, at least to mathematicians, there is no Nobel Prize in mathematics. The reasons for this have been a source of wild speculation. But usually in life, the most likely explanation is often the most prosaic one, to which Arild Stubhaug also alludes in his foreword to this book, namely that Alfred Nobel was a very practical man and mathematics was clearly not a practical subject to him. This is even stated in the statutes of the prize, which calls for the invention and discovery of the past year that has benefited mankind the most. Very few prizes that have been awarded have actually fulfilled that aim, the X-rays of Roentgen being perhaps one of the few exceptions and, incidentally, a very favourable exception, which put the Nobel Prize in the public mind from the very start.

It is worth noting, as Stubhaug likewise does in his foreword, that efforts to set up a substantial prize in mathematics were already underway as the first centennial of Abel's birth was approaching. King Oscar II was considered, rightly or wrongly, as being a patron of mathematics, as Mittag-Leffler had already been able to attach his name (and funds) to an international prize (which, as the educated reader will already know, was not entirely uncontroversially given to Poincaré), and serious hopes were attached to him. In fact Stubhaug goes as far as to suggest that by splitting up the Union between Sweden and Norway in 1905 the eventual establishment of the Abel Prize was postponed for almost a century. But it is not easy to set up a new prize. Some things you have control over, others not. No-one could at the onset have predicted the spectacular success of the Nobel Prize but even within the first decade Mittag-Leffler worried that a mathematical prize would have been overshadowed by the Nobel Prize. In the 1920s the Canadian Mathematician Fields instigated the medals, which would subsequently bear his name and which, incidentally, were awarded for the first time in Oslo in 1936. The medal carries little money and the prize is mostly unknown outside mathematics (although of course in recent years it has been mentioned in commercial movies and the case of Perelman may at least have attached some notoriety to it) but it remains the ultimate accolade for a (young) mathematician, the announcement of which carries a lot of excitement. With the recently established Abel Prize it is different. There is already a pride of many extremely worthy senior mathematicians any one of whom would be eminently suitable to be so rewarded, regardless of whether or not they have already been appropriately rewarded in the past. Thus the announcement of an Abel Prize winner is more in the nature of a confirmation of greatness than a bestowing of the same. In particular you do not expect the unexpected. Of course as the tradition evolves this might eventually change.

The book under review is a valiant effort to put the Abel Prize on the map and one surmises from the title that the ambition is to produce such a volume every five years. The book is introduced by an historical survey of the prize by Arild Stubhaug, a survey which has already been alluded to. Then follows potted autobiographies of each laureate, although the editors of the book were apparently not able to induce Serre to submit one; instead, there is a bilingually presented interview with the man who so far has been the youngest ever to receive the Fields Medal, an interview with a title which in its optimism borders on hubris – Mon premier demi-siècle au Collège de France.[1]

In addition to the autobiographies there are authoritative presentations of the works of the prize winners, along with exhaustive bibliographies and formal curricula vitae. On the human interest level, the autobiographies should be most intriguing, while from the professional and educational view, the emphasis should be on the mathematical surveys. As to autobiographies one should treat that of Serre's first, as it is not really an autobiography, although it does contain some personal titbits (and one learns that he wrote his book on linear representations on groups to assist his wife who was a specialist in quantum chemistry). Instead, it provides a vehicle for Serre's *obiter dicta*. The punchline is that in mathematics truth is absolute truth, and that this might be the reason for the unpopularity of mathematics with the general public.

The real autobiographies provided are not literary masterpieces and were supposedly never intended as such. As usual, they suffer from the blandness invariably involved in the depiction of a successful life; only in the uncertainty of the formative years, when things could have tipped one way or another, is there drama to be found. Yet there are, nevertheless, some intriguing glimpses, such as the evocation of the Indian childhood of Varadhan and Carleson's confession that as a young man he had a craving for classical literature and, for that matter, the case of the young schoolboy Atiyah evading his bullies by helping them out with their homework. As Simone Weil (the sister of an obvious candidate for the prize, had it been established 20 years earlier) observed, it is boring to read about happiness but absolutely wonderful to live it. Could it be that the lack of drama merely indicates spectacular success? Still, every autobiography contains some words of wisdom; one thing which sticks particularly in mind is Carleson's remark that as a young man he had a rather superficial and romantic notion of mathematics and that it was easy for him to handle problems related to texts and thus pass the exams. But is there really anything wrong with that? This is a commonsense observation that educators should take note of, as there is so much high sounding rhetoric that pupils should be creative, at the expense of mastering the basics. After all, the latter is more often than not an absolute prerequisite for the former, which is the responsibility of the individual and not the educational institution.

The mathematical presentations vary widely. To write a survey of the work of Serre, with the author still alive looking over your shoulder, would if anything be a daunting proposition. Pilar Bayer solves it by simply giving an exhaustive catalogue of Serre's achievements. Such a thing could of course be very useful as a reference but it is hardly anything that turns the pages. The most innovative approach is by Tom Körner, who makes a valiant attempt to convey the flavour of Carleson's work, concentrating on the pointwise convergence of Fourier series of L^2 functions (the so-called Lusin's conjecture). Körner does, on one hand, presuppose almost no preparatory knowledge; on the other hand, he also provides some rather technical expositions, for which a true neophyte would probably lack the necessary stomach, even after multiple readings. Nevertheless, his attempt is charming and worth if not emulating directly at least pursuing and developing in other contexts. This seemingly ridiculous inconsistency is, in fact, an effective educational tool because it is always very instructive to pretend that you know less than what you ostensibly do know. Nigel Hitchin makes the sensible (not to say obvious) choice to restrict himself to the Atiyah-Singer Index theorem and, from an instructive point of view (not to mention impeccability), this is arguably the most effective of all the surveys.

Finally, each winner has their list of publications presented. It would be tempting in current bureaucratic spirit to rank them according to their number of publications. But such a mischievous project is best left as an exercise for the reader.

1 The English translation is even more provocative. The French only suggests two 'demi-siècles', while the English intimates an unending suite of fifty-year periods.

Solved and Unsolved Problems

Themistocles M. Rassias (Athens, Greece)

Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not.
Godfrey Harold Hardy (1877–1947),
A Mathematician's Apology,
Cambridge University Press, 1940

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

91. Let

$$S = \left\{ \sum_{n=1}^{\infty} \frac{x_n}{3^n} : x_n = 0 \quad \text{or} \quad 1 \right\}.$$

Show that S has Lebesgue measure 0 and find the set $S + S$.

(*Wing-Sum Cheung, Department of Mathematics, University of Hong Kong, Hong Kong*)

92. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$ and $f : X \to \mathbb{R}$ be a measurable function with $|f| < 1$. Show that the limit

$$\lim_{n\to\infty} \int_X \sum_{k=0}^{n} f^k d\mu$$

either exists in $\mathbb{R}$ or equals $+\infty$.

(*Wing-Sum Cheung, Department of Mathematics, University of Hong Kong, Hong Kong*)

93. Find all functions $f : (0, \infty) \to (0, \infty)$ which are differentiable at $x = 1$ and satisfy the property that

$$f(f(x)) = x^2 \quad \text{for every} \quad x \in (0, \infty).$$

(*Dorin Andrica, Babes-Bolyai University of Cluj-Napoca, Romania*)

94. (1) Let $f : [0, \infty) \to \mathbb{R}$ be a convex differentiable function with $f(0) = 0$.
Prove that

$$\int_0^x f(t)dt \le \frac{x^2}{2} f'(x) \text{ for all } x \in [0, \infty).$$

(2) Find all differentiable functions $f : [0, \infty) \to \mathbb{R}$ for which equality holds in the above inequality.

(*Dorin Andrica, Babes-Bolyai University of Cluj-Napoca, and Mihai Piticari, National College "Dragos Voda" Campulung Moldovenesc, Romania*)

95. Let $k \ge 2$ and j be such that $0 \le j \le k$. Assume that T_j is the multiple series

$$\sum_{n_1,n_2,\ldots,n_k=1}^{\infty} n_1 n_2 \cdots n_j (n_1 + n_2 + \cdots + n_k - \zeta(2) - \zeta(3) - \cdots - \zeta(n_1 + n_2 + \cdots + n_k)),$$

where the product $n_1 \cdots n_j$ disappears when $j = 0$. Prove that

$$T_j = \sum_{m=0}^{j} \binom{j}{m} \zeta(k + 1 + j - m),$$

where ζ denotes the Riemann zeta function.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Romania*)

96. A function $f : (0, 1) \to (0, +\infty)$ possesses the following property:

$$\frac{f(x)}{f(y)} \le 1 + \frac{x}{y} \text{ for every } x, y \in (0, 1).$$

Prove the existence of a finite limit $\lim_{x\to 0} f(x)$.

(*Vladimir Protasov, Department of Mechanics and Mathematics, Moscow State University, Russia*)

II Two new open problems

97*. Does there exist a constant C that does not depend on the dimension d and such that the following assertion holds? Let f be a function on the unit cube

$$I_d = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \,\middle|\, 0 \le x_i \le 1,\ i = 1, \ldots, d \right\}$$

such that for every $a, b \in I_d$ the restriction of f to the segment $[a, b]$ can be approximated by an affine function $\varphi_{a,b}$ with the precision 1, i.e.

$$\sup_{x\in[a,b]} |f(x) - \varphi_{a,b}(x)| \le 1.$$

Then there exists an affine function $\varphi : I_d \to \mathbb{R}$ such that

$$\sup_{x\in I_d} |f(x) - \varphi(x)| \le C.$$

Comment. This problem can be formulated not only for a cube in $\mathbb{R}^d$ but also for an arbitrary convex set in $\mathbb{R}^d$. It is known that for Euclidean balls the answer is affirmative with $C < 18$, whereas for simplices and for cross-polytopes the constant C grows with the dimension as $\log d$. For cubes the problem is still open.

(*Vladimir Protasov, Department of Mechanics and Mathematics, Moscow State University, Russia*)

98*. Let $A_n = \left(a_{i,j}\right)_{1\le i,j\le n}$ be the square matrix with real entries

$$a_{i,j} = \sum_{k=1}^{j} (-1)^k \cos^{2i}\left(\frac{k}{j+1} \cdot \frac{\pi}{2}\right).$$

Prove that

$$\det A_n = (-1)^n \frac{n!}{2^{n^2}}.$$

(*Mircea Merca "Constantin Istrati" Technical College, Câmpina, Romania*)

III Solutions

83. Prove that

$$\sum_{n=2}^{+\infty} f(\zeta(n)) = 1\,,$$

where $f(x) = x - \lfloor x \rfloor$ denotes the fractional part of $x \in \mathbb{R}$ and $\zeta(s)$ is the Riemann zeta function.

(Hari M. Srivastava, Department of Mathematics and Statistics, University of Victoria, Canada)

Solution by the proposer. For $n \geq 2$, by the definition of the Riemann zeta function, it is evident that

$$1 < \zeta(n) \leq \zeta(2) = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Therefore, for $n \geq 2$ we have

$$0 < \zeta(n) - 1 < 1$$

and, hence, for $n \geq 2$

$$\zeta(n) - 1 = f(\zeta(n)).$$

Thus, we obtain

$$\begin{aligned}\sum_{n=2}^{+\infty} f(\zeta(n)) &= \sum_{n=2}^{+\infty} (\zeta(n) - 1)\\ &= \sum_{n=2}^{+\infty}\sum_{\lambda=2}^{+\infty} \frac{1}{\lambda^n}\\ &= \sum_{\lambda=2}^{+\infty}\sum_{n=2}^{+\infty} \left(\frac{1}{\lambda}\right)^n\\ &= \sum_{\lambda=2}^{+\infty}\left(\left(1-\frac{1}{\lambda}\right)^{-1} - 1 - \frac{1}{\lambda}\right)\\ &= \sum_{\lambda=2}^{+\infty}\left(\frac{1}{\lambda-1} - \frac{1}{\lambda}\right)\\ &= 1.\end{aligned}$$

Also solved by W. Fensch (Karlsruhe, Germany), Ovidiu Furdui (Technical University of Cluj-Napoca, Romania) and José C. Petronilho (Department of Mathematics, University of Coibra, Portugal).

84. A continuous function $f : [0, 1] \to \mathbb{R}$ possesses the following property: every point $x \in [0, 1]$ is either a point of local minimum or a point of local maximum for f. Is it true that f is identically a constant?

(Alexander Kochurov, Department of Mechanics and Mathematics, Moscow State University, Russia)

Solution by the proposer. **Answer.** Yes.

Solution. If f is not identically a constant, there are points $a, b \in [0, 1]$ such that $f(a) < f(b)$. Without loss of generality it can be assumed $a < b$. Let us show that for every $y \in (f(a), f(b))$ the level set $\{t \in [0, 1] \mid f(t) = y\}$ contains an open interval. Taking an arbitrary rational point $q(y)$ from that interval, we obtain an injective map $q : (f(a), f(b)) \to \mathbb{Q}$, which is a contradiction because the interval $(f(a), f(b))$ is an uncountable set.

For each $y \in (f(a), f(b))$ the set of points $x \in [a, b]$ such that $f(t) \leq y$ for all $t \in [a, x]$ is nonempty (it contains a), and this is a segment. Denote this segment as $[a, r]$, where $r = r(y)$. Clearly, $a < r < b$ and $f(r) = y$. By the assumption, r is either a point of local minimum or a point of local maximum for f. If r is a point of a local maximum then there is $\varepsilon > 0$ such that $f(t) \leq f(r) = y$, whenever $t \leq r + \varepsilon$, which contradicts the choice of r. Hence, r is a point of local minimum. In this case for every $t \in (r - \varepsilon, r)$ we have $f(t) \geq f(r) = y$. On the other hand, $f(t) \leq y$ because $t \in [a, r]$. Therefore, $f(t) = y$ for all $t \in (r - \varepsilon, r)$, which completes the proof.

Also solved by Kari Ylinen (Department of Mathematics, University of Turku, Finland).

Remark. P. T. Krasopoulos (Athens, Greece) as well as M. Vermeulen (the Netherlands) informed me that Ehrhard Behrends (Germany), Stefan Geschke (USA) and Tomasz Natkaniec (Poland) have provided a more general solution of the problem in their article "Functions for which all points are a local minimum or maximum", Real Analysis Exchange 33(2) (2007/2008), 1–4.

85. Find all functions $f : \mathbb{R}_+ \to \mathbb{R}$ that satisfy the functional equation

$$f(pr, qs) + f(ps, qr) = (r + s)f(p, q) + (p + q)f(r, s) \qquad (1)$$

for all $p, q, r, s \in \mathbb{R}_+$. Here $\mathbb{R}_+$ is the set of positive real numbers.

(Prasanna K. Sahoo, Department of Mathematics, University of Louisville, USA)

Solution by the proposer. The general solution of the functional equation (1) is

$$f(p, q) = p\,[L_1(q) - L_2(p)] + q\,[L_1(p) - L_2(q)], \qquad (2)$$

where $L_1, L_2 : \mathbb{R}_+ \to \mathbb{R}$ are logarithmic functions.

Recall that a function $L : \mathbb{R}_+ \to \mathbb{R}$ is a logarithmic function if and only if it satisfies $L(xy) = L(x) + L(y)$ for all $x, y \in \mathbb{R}_+$.

It is easy to verify that f given by (2) satisfies (1). It is left to show that (2) is the only solution of (1).

Obviously, $f = 0$ is a solution of (1) and is of the form (2). We now suppose $f \neq 0$. Setting $r = s = \lambda$ in (1), we get

$$f(\lambda p, \lambda q) = \lambda\, f(p, q) + \lambda(p + q)\,\ell(\lambda), \qquad (3)$$

where

$$\ell(\lambda) := (2\lambda)^{-1} f(\lambda, \lambda).$$

Next, we show that ℓ is logarithmic. Replacing λ by $\lambda_1\lambda_2$ in (3), we get

$$f(\lambda_1\lambda_2 p, \lambda_1\lambda_2 q) = \lambda_1\lambda_2 f(p, q) + \lambda_1\lambda_2(p + q)\ell(\lambda_1\lambda_2). \qquad (4)$$

Using (3) twice, $f(\lambda_1\lambda_2 p, \lambda_1\lambda_2 q)$ can be written as

$$\begin{aligned} f(\lambda_1\lambda_2 p, \lambda_1\lambda_2 q) &= \lambda_1 f(\lambda_2 p, \lambda_2 q) + \lambda_1\lambda_2(p + q)\ell(\lambda_1) \qquad (5)\\ &= \lambda_1\,[\,\lambda_2 f(p, q) + \lambda_2(p + q)\ell(\lambda_2)] + \lambda_1\lambda_2(p + q)\ell(\lambda_1).\end{aligned}$$

Comparing (4) and (5), we see that

$$\ell(\lambda_1\lambda_2) = \ell(\lambda_1) + \ell(\lambda_2),$$

that is, ℓ is logarithmic.

Set $p = q$, $r = s$ in (1) to get

$$f(pr, pr) = r f(p, p) + p f(r, r). \qquad (6)$$

From this it follows that

$$L(p) := \frac{1}{p} f(p, p) \qquad (7)$$

is logarithmic on $\mathbb{R}_+$. Setting $q = s = 1$ in (1), we get

$$f(p,r) = (1+r)g(p) + (1+p)g(r) - g(pr), \tag{8}$$

where $g(p) := f(1,p)$. Note that (1) implies $f(1,1) = 0$ and f is symmetric. Now (7) and (8) give

$$g(p^2) = 2(1+p)\,g(p) - L(p), \qquad p \in \mathbb{R}_+. \tag{9}$$

With $p = r$, $q = s$, (1) yields

$$f(p^2,q^2) + f(pq,pq) = 2(p+q)\,f(p,q). \tag{10}$$

Putting (7), (8) and (9) into (10), we have

$$\begin{aligned}(1-p)(1-q)[2g(pq) - L(pq)]\\ &= (1-q)(1-pq)[2g(q) - L(q)]\\ &\quad + (1-q)(1-pq)[2g(p) - L(p)].\end{aligned}$$

Defining

$$2L_2(x) := \begin{cases}(1-x)^{-1}\,[(2g(x) - L(x)], & x \neq 1\\ 0, & x = 1,\end{cases}$$

we get from this definition and $g(1) = 0$ that

$$g(p) = \frac{1}{2}\,L(p) + (1-p)L_2(p), \qquad p \in \mathbb{R}_+,$$

and from the above equation, $L_2(pq) = L_2(p) + L_2(q)$ follows whenever $p \neq 1$, $q \neq 1$ and $pq \neq 1$. The function L_2 evidently satisfies $L_2(pq) = L_2(p) + L_2(q)$ when $p = 1$ or $q = 1$. To check that this equation is also true for the case $p \neq 1$ but $pq = 1$, we have to show that $L_2(p^{-1}) = -L_2(p)$ for $p \neq 1$. The latter is equivalent to

$$\left(1 + p^{-1}\right) f(p,\ p) + 2\,p\,f\left(1,\ q^{-1}\right) = 2f(1,\ p),$$

which can be obtained by putting $q = p$, $r = 1$ and $s = \frac{1}{p}$ in (1). Thus, L_2 is logarithmic on $\mathbb{R}_+$. Now using (8), we get (2), where $L_1 = \frac{1}{2}L + L_2$.

Also solved by Mihály Bencze (Brasov, Romania), W. Fensch (Karlsruhe, Germany) and Sotirios E. Louridas (Athens, Greece).

86. Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the functional equation

$$\begin{aligned}f(u+x,v+y) + f(u-x,v) + f(u,v-y)\\ = f(u-x,v-y) + f(u+x,v) + f(u,v+y)\end{aligned} \tag{11}$$

for all $x,y,u,v \in \mathbb{R}$.

(Prasanna K. Sahoo, Department of Mathematics, University of Louisville, USA)

Solution by the proposer. The general solution of the functional equation (11) is

$$f(x,y) = B(x,y) + \phi(x) + \psi(y), \tag{12}$$

where $B : \mathbb{R}^2 \to \mathbb{R}$ is a biadditive function and $\phi,\psi : \mathbb{R} \to \mathbb{R}$ are arbitrary functions.

Recall that a function $A : \mathbb{R} \to \mathbb{R}$ is an additive function if and only if it satisfies $A(x+y) = A(x) + A(y)$ for all $x,y \in A$. A function $B : \mathbb{R}^2 \to \mathbb{R}$ is said to be biadditive if and only if it is additive in each variable.

It is easy to check that (12) satisfies the functional equation (11). It is left to show that (12) is the only solution of (11). For a fixed v and y, we define

$$\ell(x) := f(x,v+y), \quad m(x) := f(x,v), \quad n(x) := f(x,v-y). \tag{13}$$

Then by (13), we see that (11) reduces to

$$g_1(u+x) + g_2(u-x) = g_3(u), \tag{14}$$

where

$$\begin{cases} g_1(u) := \ell(u) - m(u)\\ g_2(u) := m(u) - n(u)\\ g_3(u) := \ell(u) - n(u).\end{cases} \tag{15}$$

Letting $x = 0$ in (14), we have

$$g_3(u) = g_1(u) + g_2(u) \tag{16}$$

for all $u \in \mathbb{R}$. In view of (16), (14) yields

$$g_1(u+x) + g_2(u-x) = g_1(u) + g_2(u) \tag{17}$$

for all $x,u \in \mathbb{R}$. Replacing x by $-x$ in the last equation and then using the last equation, we see that

$$g_1(u+x) - g_2(u+x) = g_1(u-x) - g_2(u-x)$$

for all $x,u \in \mathbb{R}$. Hence

$$g_2(x) = g_1(x) + c_o, \tag{18}$$

where c_o is a constant. Putting (18) into (17) and defining

$$G(x) := g_1(x) - g_1(0), \tag{19}$$

we have

$$G(u+x) + G(u-x) = 2G(u) \tag{20}$$

for all $x,u \in \mathbb{R}$. Note that $G(0) = 0$. Letting $u = 0$ in (20), we conclude that G is an odd function in $\mathbb{R}$. Interchanging u with x in (20) and using the fact that G is an odd function, we get

$$G(u+x) - G(u-x) = 2G(x) \tag{21}$$

for all $u,x \in \mathbb{R}$. Adding (20) and (21), we have

$$G(u+x) = G(u) + G(x) \tag{22}$$

for all $u,x \in \mathbb{R}$. Hence $G(x) = A(x)$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive function. Now using (16), (18) and (19), we get

$$g_1(u) = A(u)+\psi, \quad g_2(u) = A(u)+b, \quad g_3(u) = 2A(u)+\psi+b, \tag{23}$$

where $\psi = g_1(0)$ and $b = g_1(0) + c_0$ are constants. From (15) and (23), we have

$$\begin{cases} \ell(u) - m(u) = A(u) + \psi\\ m(u) - n(u) = A(u) + b\\ \ell(u) - n(u) = 2A(u) + \psi + b.\end{cases} \tag{24}$$

From (13) and (24), we get

$$f(x,y+v) - f(x,v) = A(x) + \psi,$$

and then letting $v = 0$ in the last equation, we obtain

$$f(x,y) - f(x,0) = A(x) + \psi.$$

Because of the definition (13), the function $A(x)$ and the constants ψ and b are all dependent only on y (since $v = 0$). Hence we have

$$f(x,y) - f(x,0) = A(x,y) + \psi(y)$$

which is

$$f(x,y) = A(x,y) + \phi(x) + \psi(y), \tag{25}$$

where $A : \mathbb{R}^2 \to \mathbb{R}$ is additive in the first variable and $\phi(x) := f(x, 0)$. Substituting (25) into (11) and then letting $u = 0$ and simplifying, we obtain

$$A(x, v + y) + A(x, v - y) = 2A(x, v) \tag{26}$$

for all $x, y, v \in \mathbb{R}$. For a fixed x, (26) is an equation of the form (14) (with $g_1 = g_2 = g_3 = A(x, \cdot)$) and thus, from (23), we have $A(x, y) = B(x, y) + C(x)$, where B is additive in the second variable y. Since B is additive in the second variable, we have

$$A(x, y_1 + y_2) = A(x, y_1) + A(x, y_2) - C(x).$$

Using the additivity of A in the first variable, we see that C is additive and hence B is additive in the first variable also. Thus B is biadditive and f is of the form (12). Letting this form into (11), we see that ϕ and ψ are arbitrary functions as asserted.

Also solved by Mihály Bencze (Brasov, Romania) and Sotirios E. Louridas (Athens, Greece).

87. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers m, M with $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, prove that the following inequality holds:

$$\begin{aligned}&\left|\left\langle\left[f(A) - \left(\frac{1}{M-m}\int_m^M f(s)\,ds\right)1_H\right]x, y\right\rangle\right|\\ &\quad\le \frac{1}{M-m}\bigvee_m^M(f)\max_{t\in[m,M]}\Big[(M-t)\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2}\\ &\quad+(t-m)\langle(1_H - E_t)x, x\rangle^{1/2}\langle(1_H - E_t)y, y\rangle^{1/2}\Big] \le \|x\|\,\|y\|\bigvee_m^M(f)\end{aligned} \tag{27}$$

for any $x, y \in H$, where $\bigvee_m^M(f)$ denotes the total variation of f on $[m, M]$ and 1_H is the identity operator on H.

(Sever S. Dragomir, Mathematics, School of Engineering and Science, Victoria University, Australia)

Solution by the proposer. Assume that $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$. Then under the assumptions of the theorem for A and $\{E_\lambda\}_\lambda$, we have the following representation

$$\begin{aligned}&\langle x, y\rangle\frac{1}{M-m}\int_m^M f(s)ds - \langle f(A)x, y\rangle\\ &= \frac{1}{M-m}\int_{m-0}^M \Big\langle[(M-t)E_t + (t-m)(E_t - 1_H)]x, y\Big\rangle df(t)\end{aligned} \tag{28}$$

for any $x, y \in H$.

Indeed, integrating by parts in the Riemann-Stieltjes integral and using the well known spectral representation theorem, which states:

$$\langle f(A)x, y\rangle = \int_{m-0}^M f(t)d\langle E_t x, y\rangle \text{ for any } x, y \in H, \text{ we have}$$

$$\begin{aligned}&\frac{1}{M-m}\int_{m-0}^M \langle[(M-t)E_t + (t-m)(E_t - 1_H)]x, y\rangle\, df(t)\\ &= \int_{m-0}^M \left(\langle E_t x, y\rangle - \frac{t-m}{M-m}\langle x, y\rangle\right) df(t)\\ &= \left(\langle E_t x, y\rangle - \frac{t-m}{M-m}\langle x, y\rangle\right) f(t)\Big|_{m-0}^M\\ &\quad - \int_{m-0}^M f(t)\, d\left(\langle E_t x, y\rangle - \frac{t-m}{M-m}\langle x, y\rangle\right)\\ &= -\int_{m-0}^M f(t)\, d\langle E_t x, y\rangle + \langle x, y\rangle\frac{1}{M-m}\int_m^M f(t)\,dt\\ &= \langle x, y\rangle\frac{1}{M-m}\int_m^M f(t)\,dt - \langle f(A)x, y\rangle\end{aligned}$$

for any $x, y \in H$ and the equality (28) is proved.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Utilising this property we have from (28) that

$$\begin{aligned}&\left|\langle x, y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)x, y\rangle\right|\\ &\le \frac{1}{M-m}\max_{t\in[m,M]}|\langle[(M-t)E_t + (t-m)(E_t - 1_H)]x, y\rangle|\bigvee_m^M(f)\end{aligned} \tag{29}$$

for any $x, y \in H$.

Now observe that

$$\begin{aligned}&|\langle[(M-t)E_t + (t-m)(E_t - 1_H)]x, y\rangle|\\ &= |(M-t)\langle E_t x, y\rangle + (t-m)\langle(E_t - 1_H)x, y\rangle|\\ &\le (M-t)|\langle E_t x, y\rangle| + (t-m)|\langle(E_t - 1_H)x, y\rangle|\end{aligned} \tag{30}$$

for any $x, y \in H$ and $t \in [m, M]$.

If P is a nonnegative operator on H, i.e. $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalisation of the Schwarz inequality in H:

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\langle Py, y\rangle \tag{31}$$

for any $x, y \in H$.

On applying the inequality (31) we have

$$\begin{aligned}&(M-t)|\langle E_t x, y\rangle| + (t-m)|\langle(E_t - 1_H)x, y\rangle|\\ &\le (M-t)\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2}\\ &\quad + (t-m)\langle(1_H - E_t)x, x\rangle^{1/2}\langle(1_H - E_t)y, y\rangle^{1/2}\\ &\le \max\{M-t, t-m\}\\ &\times\left[\langle E_t x, x\rangle^{1/2}\langle E_t y, y\rangle^{1/2} + \langle(1_H - E_t)x, x\rangle^{1/2}\langle(1_H - E_t)y, y\rangle^{1/2}\right]\\ &\le \max\{M-t, t-m\}\\ &\times[\langle E_s x, x\rangle + \langle(1_H - E_s)x, x\rangle]^{1/2}[\langle E_s y, y\rangle + \langle(1_H - E_s)y, y\rangle]^{1/2}\\ &= \max\{M-t, t-m\}\|x\|\,\|y\|,\end{aligned} \tag{32}$$

where for the last inequality we used the elementary fact

$$a_1 b_1 + a_2 b_2 \le \left(a_1^2 + a_2^2\right)^{1/2}\left(b_1^2 + b_2^2\right)^{1/2}, \tag{33}$$

which holds for a_1, b_1, a_2, b_2 positive real numbers.

Utilising the inequalities (29), (30) and (32) we deduce the desired result (27).

88. Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers m, M with $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, prove that the following inequality holds:

$$\begin{aligned}&\left|\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right]x, y\right\rangle\right| \\ &\le \sup_{t\in[m,M]}\left[\frac{t-m}{M-m}\bigvee_m^t(\langle E_{(\cdot)}x,y\rangle) + \frac{M-t}{M-m}\bigvee_t^M(\langle E_{(\cdot)}x,y\rangle)\right]\bigvee_m^M(f) \\ &\le \bigvee_m^M(\langle E_{(\cdot)}x,y\rangle)\bigvee_m^M(f) \le \|x\|\,\|y\|\bigvee_m^M(f) \qquad (34)\end{aligned}$$

for any $x, y \in H$, where $\bigvee_m^M(f)$ denotes the total variation of f on $[m, M]$ and 1_H is the identity operator on H.

(Sever S. Dragomir, Mathematics, School of Engineering and Science, Victoria University, Australia)

Solution by the proposer. First we prove the following identity:

$$\begin{aligned}&\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right]x, y\right\rangle \qquad (35)\\ &= \int_{m-0}^M \langle E_t x, y\rangle\, df(t) - \frac{f(M)-f(m)}{M-m}\int_{m-0}^M \langle E_t x, y\rangle\, dt \\ &= \int_{m-0}^M \left[\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^M \langle E_s x, y\rangle\, ds\right] df(t)\end{aligned}$$

for any $x, y \in H$.

Integrating by parts and utilising the well known spectral representation theorem, we have

$$\begin{aligned}\int_{m-0}^M \langle E_t x, y\rangle\, df(t) &= f(M)\langle x, y\rangle - \int_{m-0}^M f(t)\, d\langle E_t x, y\rangle \\ &= f(M)\langle x, y\rangle - \langle f(A)x, y\rangle\end{aligned}$$

and

$$\int_{m-0}^M \langle E_t x, y\rangle\, dt = M\langle x, y\rangle - \langle Ax, y\rangle$$

for any $x, y \in H$.

Therefore

$$\begin{aligned}&\int_{m-0}^M \langle E_t x, y\rangle\, df(t) - \frac{f(M)-f(m)}{M-m}\int_{m-0}^M \langle E_t x, y\rangle\, dt \\ &= f(M)\langle x, y\rangle - \langle f(A)x, y\rangle - \frac{f(M)-f(m)}{M-m}(M\langle x, y\rangle - \langle Ax, y\rangle) \\ &= \frac{1}{M-m}[f(m)(M\langle x, y\rangle - \langle Ax, y\rangle) + f(M)(\langle Ax, y\rangle - m\langle x, y\rangle)] \\ &\quad - \langle f(A)x, y\rangle\end{aligned}$$

for any $x, y \in H$, which proves the first equality in (35).

It is well known that if $p : [a, b] \to \mathbb{C}$ is a bounded function, $v : [a, b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists then the following inequality holds

$$\left|\int_a^b p(t)\, dv(t)\right| \le \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \qquad (36)$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Applying this property to the equality (35), we have

$$\begin{aligned}&\left|\left\langle\left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m}\right]x, y\right\rangle - \langle f(A)x, y\rangle\right| \qquad (37)\\ &\le \sup_{t\in[m,M]}\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^M \langle E_s x, y\rangle\, ds\right|\bigvee_m^M(f)\end{aligned}$$

for any $x, y \in H$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$\begin{aligned}&\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^M \langle E_s x, y\rangle\, ds \qquad (38)\\ &= \frac{1}{M-m}\left[\int_{m-0}^t (s-m)\, d\langle E_s x, y\rangle + \int_t^M (s-M)\, d\langle E_s x, y\rangle\right],\end{aligned}$$

which holds for any $t \in [m, M]$ and for any $x, y \in H$.

Since the function $v(s) := \langle E_s x, y\rangle$ is of bounded variation on $[m, M]$ for any $x, y \in H$, then on applying the inequality (36) once more, we get

$$\begin{aligned}&\left|\langle E_t x, y\rangle - \frac{1}{M-m}\int_{m-0}^M \langle E_s x, y\rangle\, ds\right| \qquad (39)\\ &\le \frac{1}{M-m}\left[\left|\int_{m-0}^t (s-m)\, d\langle E_s x, y\rangle\right| + \left|\int_t^M (s-M)\, d\langle E_s x, y\rangle\right|\right] \\ &\le \frac{t-m}{M-m}\bigvee_m^t(\langle E_{(\cdot)}x, y\rangle) + \frac{M-t}{M-m}\bigvee_t^M(\langle E_{(\cdot)}x, y\rangle),\end{aligned}$$

which holds for any $t \in [m, M]$ and for any $x, y \in H$.

Now, taking the supremum in (39) and taking into account that

$$\bigvee_m^t(\langle E_{(\cdot)}x, y\rangle), \bigvee_t^M(\langle E_{(\cdot)}x, y\rangle) \le \bigvee_m^M(\langle E_{(\cdot)}x, y\rangle)$$

for any $t \in [m, M]$ and for any $x, y \in H$, we deduce the first and the second inequality in (34).

If P is a nonnegative operator on H, i.e. $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalisation of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\langle Py, y\rangle \qquad (40)$$

for any $x, y \in H$.

Further, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$ then we have by Schwarz's inequality for nonnegative operators that

$$\begin{aligned}\bigvee_m^M(\langle E_{(\cdot)}x, y\rangle) &= \sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, y\right\rangle\right|\right\} \\ &\le \sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y, y\right\rangle^{1/2}\right]\right\} := I.\end{aligned}$$

By the Cauchy–Buniakovski–Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}I &\le \sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y, y\right\rangle\right]^{1/2}\right\} \\ &\le \sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)x, x\right\rangle\right]^{1/2}\sup_d\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}} - E_{t_i}\right)y, y\right\rangle\right]^{1/2}\right\} \\ &= \left[\bigvee_m^M(\langle E_{(\cdot)}x, x\rangle)\right]^{1/2}\left[\bigvee_m^M(\langle E_{(\cdot)}y, y\rangle)\right]^{1/2} = \|x\|\,\|y\|\end{aligned}$$

for any $x, y \in H$. These prove the last part of (34).

Remark. Problem 75 was also solved by David Angell (School of Mathematics and Statistics, University of New South Wales, Sydney, Australia). Problem 76 was also solved by Gerald A. Heuer (Concordia College, Minnesota, USA). Problems 76 and 78 were also solved by John N. Lillington (Wareham, UK).

Problems 75, 76, 77, 78 and 79 were solved by Mihai Cipu (The Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania).

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR 15780, Athens, Greece, and by email to trassias@math.ntua.gr.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Real Analysis*.

Letters to the Editor

David Ruelle (IHES, Paris)

Dear Yves Meyer,

I was very pleased to read the text of your EMS interview as published in the *IAMP News Bulletin* of July 2011, especially what you said about the French Grandes Ecoles. I am, as you are, in favour of the elitist system of the Ecole Normale and the Ecole Polytechnique but against the lifelong privileges given to their students. The latter means that there is lifelong discrimination against those who did not graduate from one of the Ecoles. Amusingly (or sadly) many left-wing French intellectuals, who are ready to fight every sort of discrimination, make an exception for that one.

When I came to France in 1964 (after working in Belgium, Switzerland and the US), the IHES was not yet called IHÉS and it was outside of the French system. Discussions about people to invite were based on scientific quality at the international level; only later would one hear things like "he is one of the best mathematicians from the Ecole Normale in the last ten years". After my arrival at the IHES, my new colleague Louis Michel took time to tell me who in the French mathematics and physics community was from the Ecole Normale and who was from the Ecole Polytechnique (Michel became furious when I got it wrong). I was neither Normale nor Polytechnique and, after a while, I came to realise that this was important for many French people I interacted with (there were some visible exceptions like René Thom, who couldn't care less, and also, curiously, Jacques-Louis Lions). Later, in the physics section of the Académie des Sciences, I was surrounded by two blocs: Saclay-Polytechnique headed by Anatole Abragam and Normale-Ulm headed by Jean Brossel. At one point, Jacques Friedel made a list of those awarded the "grand prix" in physics, showing a fair equilibrium between the two blocs. But, as Friedel pointed out, very little was left to people outside the blocs…

Discrimination against various human groups is widespread and often has tragic consequences. The discrimination in France against those who are not from the Grandes Ecoles may be seen as a rather mild affair and one is tempted to simply shrug it off. One shouldn't. I think that Alexander Grothendieck's tragic story is strongly related to the fact that he was not from the Ecole Normale. If he had been, the IHÉS Director Léon Motchane would have been much more prudent in dealing with him and he might not have left. Or after he left the IHÉS, probably more effort would have been made by the community to find a suitable position for him.

Thank you again then for publishing your interview. This gave me a chance to repeat things that are fairly well-known but deserve to be discussed rather than just being silently accepted.

David Ruelle [ruelle@ihes.fr]

Letter originally published in IAMP News Bulletin, October 2011. Reprinted with permission.

Response to "Do Theorems Admit Exceptions?"

David Wells

According to the authors of "Do Theorems Admit Exceptions?" [The EMS Education Committee, *EMS Newsletter*, Dec. 2011],

> "Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences as well as everyday experience; *the concept of formal proof is completely outside mainstream thinking.*" [page 51, RH column]

This claim is at best misleading and at worst, false. Counterexamples are to be found among analyses within popular abstract games such as chess and Go, or Hex and Nine Men's Morris, and many much simpler abstract games; within popular abstract puzzles such as Lucas's Tower of Hanoi or Euler's Bridges of Königsberg or the Leaping Frogs puzzle & many other well-known 'investigations'; and (of course) in the wide cross-over genre of mathematical recreations such as pentominoes and knight tours which combine elements of abstract games with elements of puzzles. By tackling puzzles and games, and those intermediate recreations, which to pupils appear (at first) to be non-mathematical (mathematicians know better, of course) pupils can create their own totally convincing arguments (proofs) of various forms and hence develop their conceptions of proof in more-or-less familiar surroundings. Therefore it is not true that,

> "Teachers of mathematics at all levels … thus require students to acquire a new, non-natural basis of belief when they ask them to prove."

Puzzles and games are *natural* to all pupils from a young age, and suitably simple puzzles and games are easy to find, and many have been published. By making this simple link between mathematics and mathematical proof and games & puzzles, as is done in the book *Mathematics and Abstract Games: an Intimate Connection* (Wells 2007: Rain Press), now retitled *Games and Mathematics: an Intimate Connection* (Cambridge UP, forthcoming 2012) the central concepts of mathematical argument & proof can be firmly embedded in familiar situations, from a young age, and much of their claimed difficulty simply disappears.

David Wells [davidggwells@yahoo.co.uk]

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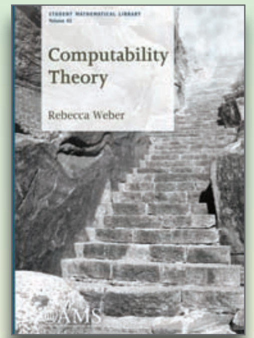

COMPUTABILITY THEORY

Rebecca Weber, *Dartmouth College*

What can we compute - even with unlimited resources? Is everything within reach? Or are computations necessarily drastically limited, not just in practice, but theoretically? These questions are at the heart of computability theory. The goal of this book is to give the reader a firm grounding in the fundamentals of computability theory and an overview of currently active areas of research, such as reverse mathematics and algorithmic randomness. Turing machines and partial recursive functions are explored in detail, and vital tools and concepts including coding, uniformity, and diagonalisation are described explicitly.

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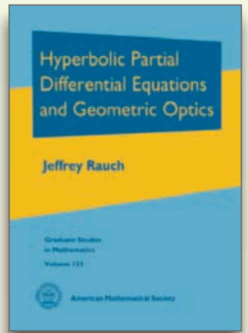

HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS AND GEOMETRIC OPTICS

Jeffrey Rauch, *University of Michigan*

Introduces graduate students and researchers in mathematics and the sciences to the multifaceted subject of the equations of hyperbolic type, which are used, in particular, to describe propagation of waves at finite speed. Among the topics carefully presented in this book are nonlinear geometric optics, the asymptotic analysis of short wavelength solutions, and nonlinear interaction of such waves. Studied in detail are the damping of waves, resonance, dispersive decay, and solutions to the compressible Euler equations with dense oscillations created by resonant interactions. Many fundamental results are presented for the first time in a textbook format. In addition to dense oscillations, these include the treatment of precise speed of propagation and the existence and stability questions for the three wave interaction equations.

Graduate Studies in Mathematics, Vol. 133

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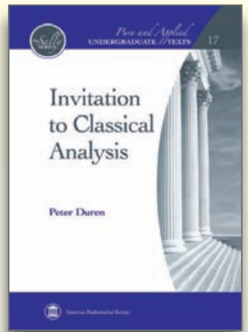

INVITATION TO CLASSICAL ANALYSIS

Peter Duren, *University of Michigan*

Gives a rigorous treatment of selected topics in classical analysis, with many applications and examples. The exposition is at the undergraduate level, building on basic principles of advanced calculus without appeal to more sophisticated techniques of complex analysis and Lebesgue integration. Among the topics covered are Fourier series and integrals, approximation theory, Stirling's formula, the gamma function, Bernoulli numbers and polynomials, the Riemann zeta function, Tauberian theorems, elliptic integrals, ramifications of the Cantor set, and a theoretical discussion of differential equations including power series solutions at regular singular points, Bessel functions, hypergeometric functions, and Sturm comparison theory. Preliminary chapters offer rapid reviews of basic principles and further background material such as infinite products and commonly applied inequalities.

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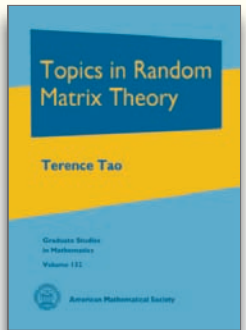

TOPICS IN RANDOM MATRIX THEORY

Terence Tao, *University of California*

The field of random matrix theory has seen an explosion of activity in recent years, with connections to many areas of mathematics and physics. However, this makes the current state of the field almost too large to survey in a single book. This text focuses on one specific sector of the field, namely the spectral distribution of random Wigner matrix ensembles (such as the Gaussian Unitary Ensemble), as well as iid matrix ensembles. The text is largely self-contained and starts with a review of relevant aspects of probability theory and linear algebra. With over 200 exercises, the book is suitable as an introductory text for beginning graduate students seeking to enter the field.

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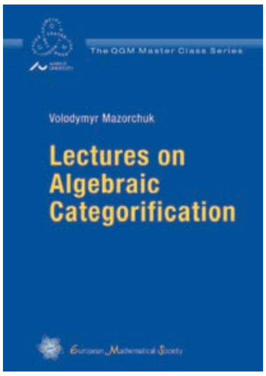
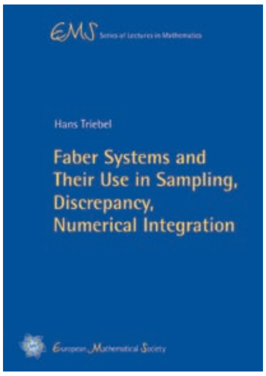


Volodymyr Mazorchuk (Uppsala University, Sweden)
Lectures on Algebraic Categorification (The QGM Master Class Series)

ISBN 978-3-03719-108-8. 2012. 110 pages. Softcover. 17 x 24 cm. 28.00 Euro

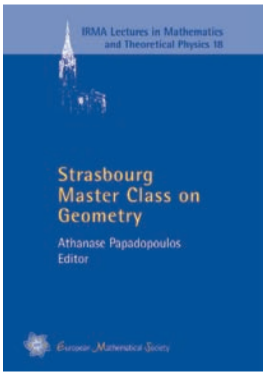
The term "categorification" was introduced by Louis Crane in 1995 and refers to the process of replacing set-theoretic notions by the corresponding categorytheoretic analogues. This text mostly concentrates on algebraical aspects of the theory, presented in the historical perspective, but also contains several topological applications, in particular, an algebraic (or, more precisely, representation-theoretical) approach to categorification. It consists of fifteen sections corresponding to fifteen one-hour lectures given during a Master Class at Aarhus University, Denmark in October 2010. There are some exercises collected at the end of the text and a rather extensive list of references. Video recordings of all (but one) lectures are available from the Master Class website.
The book provides an introductory overview of the subject rather than a fully detailed monograph. Emphasis is on definitions, examples and formulations of the results. Most proofs are either briefly outlined or omitted. However, complete proofs can be found by tracking references. It is assumed that the reader is familiar with the basics of category theory, representation theory, topology and Lie algebra.



Hans Triebel (University of Jena, Germany)
Faber Systems and Their Use in Sampling, Discrepancy, Numerical Integration (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-107-1. 2012. 115 pages. Softcover. 17 x 24 cm. 28.00 Euro

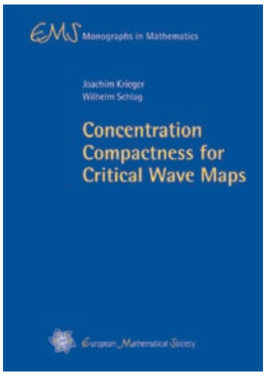
This book deals first with Haar bases, Faber bases and Faber frames for weighted function spaces on the real line and the plane. It extends results in the author's book Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration (EMS, 2010) from un-weighted spaces (preferably in cubes) to weighted spaces.
The obtained assertions are used to study sampling and numerical integration in weighted spaces on the real line and weighted spaces with dominating mixed smoothness in the plane. A short chapter deals with the discrepancy for spaces on intervals.
This book is addressed to graduate students and mathematicians having a working knowledge of basic elements of function spaces and approximation theory.



Athanase Papadopoulos (IRMA, Strasbourg, France)
Strasbourg Master Class on Geometry (IRMA Lectures in Mathematics and Theoretical Physics Vol. 18)

ISBN 978-3-03719-105-7. 2012. 461 pages. Hardcover. 17 x 24 cm. 48.00 Euro

This book contains carefully revised and expanded versions of eight courses that were presented at the University of Strasbourg, during two geometry master classes, in 2008 and 2009. The aim of the master classes was to give to fifth-year students and PhD students in mathematics the opportunity to learn new topics that lead directly to the current research in geometry and topology. The courses were held by leading experts. The subjects treated include hyperbolic geometry, three-manifold topology, representation theory of fundamental groups of surfaces and of three-manifolds, dynamics on the hyperbolic plane with applications to number theory, Riemann surfaces, Teichmüller theory, Lie groups and asymptotic geometry.
The text is addressed to students and mathematicians who wish to learn the subject. It can also be used as a reference book and as a textbook for short courses on geometry.



Joachim Krieger (EPFL Lausanne, Switzerland) and Wilhelm Schlag (University of Chicago, USA))
Concentration Compactness for Critical Wave Maps (EMS Monographs in Mathematics)

ISBN 978-3-03719-106-4. 2012. 490 pages. Hardcover. 16.5 x 23.5 cm. 88.00 Euro

Wave maps are the simplest wave equations taking their values in a Riemannian manifold (M,g). Their Lagrangian is the same as for the scalar equation, the only difference being that lengths are measured with respect to the metric g. By Noether's theorem, symmetries of the Lagrangian imply conservation laws for wave maps, such as conservation of energy.
In coordinates, wave maps are given by a system of semilinear wave equations. Over the past 20 years important methods have emerged which address the problem of local and global wellposedness of this system. Due to weak dispersive effects, wave maps defined on Minkowski spaces of low dimensions, such as $\mathbb{R}^{2+1}_{t,x}$, present particular technical difficulties. This class of wave maps has the additional important feature of being energy critical, which refers to the fact that the energy scales exactly like the equation.
Around 2000 Daniel Tataru and Terence Tao, building on earlier work of Klainerman–Machedon, proved that smooth data of small energy lead to global smooth solutions for wave maps from 2+1 dimensions into target manifolds satisfying some natural conditions. In contrast, for large data, singularities may occur in finite time for $M=\mathbb{S}^2$ as target. This monograph establishes that for $\mathbb{H}^2$ as target the wave map evolution of any smooth data exists globally as a smooth function.
While we restrict ourselves to the hyperbolic plane as target the implementation of the concentration-compactness method, the most challenging piece of this exposition, yields more detailed information on the solution. This monograph will be of interest to experts in nonlinear dispersive equations, in particular to those working on geometric evolution equations.