NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

June 2013
Issue 88
ISSN 1027-488X

European
Mathematical
Society

European Mathematical Society

New Prize "EMS Monograph Award" by the EMS Publishing House

On the occasion of our tenth anniversary, we are happy to announce a new prize, open to all mathematicians. The **EMS Monograph Award** is assigned every two years to the author(s) of a monograph in any area of mathematics that is judged by the selection committee to be an outstanding contribution to its field. The prize is endowed with 10,000 Euro and the winning monograph will be published by the EMS Publishing House in the series "EMS Tracts in Mathematics".

Submission

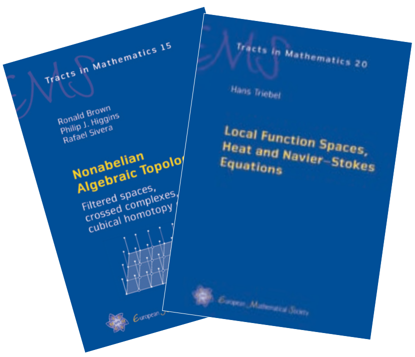
The monograph must be original and unpublished, written in English and should not be submitted elsewhere until an editorial decision is rendered on the submission. The first award will be announced in 2014 (probably in the June Newsletter of the EMS); the deadline for submissions is 30 June 2013. Monographs should preferably be typeset in TeX. Authors should send a pdf file of the manuscript by email and a hard copy together with a letter to:

European Mathematical Society Publishing House
ETH-Zentrum SEW A27, Scheuchzerstrasse 70, CH-8092 Zürich, Switzerland
E-mail: award@ems-ph.org

Scientific Committee

John Coates, Pierre Degond, Carlos Kenig, Jaroslav Nesetril, Michael Roeckner, Vladimir Turaev

EMS Tracts in Mathematics



Editorial Board:
Carlos E. Kenig (University of Chicago, USA)
Andrew Ranicki (University of Edinburgh, UK)
Michael Röckner (Universität Bielefeld, Germany, and Purdue University, USA)
Vladimir Turaev (Indiana University, Bloomington, USA)
Alexander Varchenko (University of North Carolina at Chapel Hill, USA)

This series includes advanced texts and monographs covering all fields in pure and applied mathematics. Tracts will give a reliable introduction and reference to special fields of current research. The books in the series will in most cases be authored monographs, although edited volumes may be published if appropriate. They are addressed to graduate students seeking access to research topics as well as to the experts in the field working at the frontier of research.

Most recent titles:

Vol. 20 Hans Triebel: *Local Function Spaces, Heat and Navier–Stokes Equations*
978-3-03719-123-1. 2013. 244 pages. 64.00 Euro

Vol. 19 Bogdan Bojarski, Vladimir Gutlyanskii, Olli Martio and Vladimir Ryazanov: *Infinitesimal Geometry of Quasiconformal and Bi-Lipschitz Mappings in the Plane*
978-3-03719-122-4. 2013. 216 pages. 58.00 Euro

Vol. 18 Erich Novak and Henryk Woźniakowski: *Tractability of Multivariate Problems. Volume III: Standard Information for Operators*
978-3-03719-116-3. 2012. 604 pages. 98.00 Euro

Vol. 17 Anders Björn and Jana Björn: *Nonlinear Potential Theory on Metric Spaces*
978-3-03719-099-9. 2011. 415 pages. 64.00 Euro

Vol. 16 Marek Jarnicki and Peter Pflug: *Separately Analytic Functions*
ISBN 978-3-03719-098-2. 2011. 306 pages. 58.00 Euro

Vol. 15 Ronald Brown, Philip J. Higgins and Rafael Sivera: *Nonabelian Algebraic Topology. Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids*
978-3-03719-083-8. 2011. 703 pages. 98.00 Euro

European Mathematical Society

Newsletter No. 88, June 2013

The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

ISSN 1027-488X

Published by the
EMS Publishing House
ETH-Zentrum SEW A27
CH-8092 Zürich, Switzerland.
homepage: www.ems-ph.org

EMS Executive Committee

President

Prof. Marta Sanz-Solé
(2011–2014)
University of Barcelona
Faculty of Mathematics
Gran Via de les Corts
Catalanes 585
E-08007 Barcelona, Spain
e-mail:ems-president@ub.edu

Vice-Presidents

Prof. Franco Brezzi
(2013–2016)
Istituto di Matematica Applicata
e Tecnologie Informatiche del
C.N.R.
via Ferrata 3
I-27100 Pavia, Italy
e-mail: brezzi@imati.cnr.it

Dr. Martin Raussen
(2011–2016)
Department of Mathematical
Sciences, Aalborg University
Fredrik Bajers Vej 7G
DK-9220 Aalborg Øst
Denmark
e-mail: raussen@math.aau.dk

Secretary

Dr. Stephen Huggett
(2007–2014)
School of Computing and
Mathematics
University of Plymouth
Plymouth PL4 8AA, UK
e-mail: s.huggett@plymouth.ac.uk

Treasurer

Prof. Jouko Väänänen
(2007–2014)
Department of Mathematics
and Statistics
Gustaf Hällströmin katu 2b
FIN-00014 University of Helsinki
Finland
e-mail: jouko.vaananen@helsinki.fi
and
Institute for Logic, Language
and Computation
University of Amsterdam
Plantage Muidergracht 24
NL-1018 TV Amsterdam
The Netherlands
e-mail: vaananen@science.uva.nl

Ordinary Members

Prof. Alice Fialowski
(2013–2016)
Eötvös Loránd University
Institute of Mathematics
Pázmány Péter sétány 1/C
H-1117 Budapest, Hungary
e-mail: fialowsk@cs.elte.hu

Prof. Gert-Martin Greuel
(2013–2016)
Mathematical Research
Institute Oberwolfach
Schwarzwaldstr. 9–11
(Lorenzenhof)
D-77709 Oberwolfach-Walke
Germany
e-mail: greuel@mfo.de

Prof. Laurence Halpern
(2013–2016)
Laboratoire Analyse, Géométrie
& Applications
UMR 7539 CNRS
Université Paris 13
F-93430 Villetaneuse, France
e-mail: halpern@math.univ-paris13.fr

Prof. Volker Mehrmann
(2011–2014)
Institut für Mathematik
TU Berlin MA 4–5
Strasse des 17. Juni 136
D-10623 Berlin, Germany
e-mail: mehrmann@math.TU-Berlin.DE

Prof. Armen Sergeev
(2013–2016)
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8
119991 Moscow, Russia
e-mail: sergeev@mi.ras.ru

EMS Secretariat

Ms. Terhi Hautala
Department of Mathematics
and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
FIN-00014 University of Helsinki
Finland
Tel: (+358)-9-191 51503
Fax: (+358)-9-191 51400
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dmitry Feichtner-Kozlov
FB3 Mathematik
University of Bremen
Postfach 330440
D-28334 Bremen, Germany
e-mail: dfk@math.uni-bremen.de

EMS Agenda

2013

8–12 July
EMS-IML Joint Summer School "The Bellman function technique in harmonic analysis"
http://www.mittag-leffler.se/summer2013/summerschools/bellman_function/

8–13 July
CIME-EMS Summer School in Applied Mathematics "Vector-valued Partial Differential Equations and Applications", Cetraro, Italy
http://php.math.unifi.it/users/cime/

20–25 July
29th European Meeting of Statisticians, Budapest, Hungary
www.ems2013.eu

26–30 August
EMS Summer School in Applied Mathematics "Sixth European Summer School in Financial Mathematics", Vienna, Austria
http://www.mat.univie.ac.at/~finance_hp/summer_school_Vienna_2013/

2–6 September
16th Conference of Women in Mathematics, Bonn, Germany
Lectures by the EMS Lecturer for 2013 Tamar Ziegler (Technion, Israel)
http://europeanwomeninmaths.org/activities/conference/16th-general-meeting-ewm

1–4 October
Meeting of the Education Committee, Moscow Center for Continuous Mathematical Education, Moscow, Russia
Guenter Toerner: guenter.toerner@uni-due.de

3–19 October
IMPAN-EMS Bedlewo School: "EMS School on Computational Aspects of Gene Regulation", Będlewo, Poland
http://bioputer.mimuw.edu.pl/school.php

12–13 October
EMS-PTM Joint Meeting "A. Mostowski Centenary", Warsaw, Poland

11–12 November
Meeting of the Publications Committee, Lake Bled, Slovenia
Bernard Teissier: teissier@math.jussieu.fr

23–24 November
Raising Public Awareness Committee meeting, Newcastle, UK
Ehrhard Behrends: behrends@mi.fu-berlin.de

2014

28–29 June
EMS Council Meeting, San Sebastian, Spain

2016

18–22 July
7th European Congress of Mathematics, Berlin, Germany

Editorial: EMS Facts and Policies on Journal Publishing

European Mathematical Foundation Board of Trustees

Introduction

In 2002 the EMS launched its Publishing House (EMS PH), dedicated to the publication of high quality peer-reviewed journals and books in all fields of mathematics. The plan arose from the necessity to provide a good service to mathematicians in the dissemination of mathematical knowledge, with the commitment of keeping the prices as low as is economically sustainable and maintaining a high standard in the editorial and publishing processes.

Today the EMS PH publishes fifteen journals, which are distributed by the traditional subscription model. This amounts to approximately 290 full packages, 2,000 individual subscriptions, 600 exchange copies, and 600 free copies.

The advent of electronic media is having a strong impact on this classical publishing model. We follow closely the activities of other publishers, particularly those of learned societies, and we are analyzing and discussing ideas that emerge as useful for mathematicians, librarians, and public funding agencies.

Recently, the EMS has appointed a Publications Committee whose job is to give advice to the Executive Committee on publications issues, to analyze, to promote debate, and to keep the mathematical community informed about new trends in publications. This allows us to take an active role in adopting new publishing models and to provide our membership with a forum for exchanging views and ideas.

The following summarises some of our current practices and views on journal publishing.

Public Access

One of the benefits of EMS membership is free online access to JEMS. We are currently considering extending this to other journals.

The EMS allows authors of their journals to post a pre-publication manuscript in any non-commercial environment, provided proper credit is given to the original source. On request, the EMS provides authors with the final peer-reviewed manuscript, which may be posted on his or her home institution's non-commercial repository.

Through ICTP's electronic Journal Delivery Service (http://ejds.ictp.it/ejds/) all articles from EMS journals are made available free of charge to mathematicians in countries whose economy does not allow them to purchase subscriptions.

Seven of the journals owned or hosted by the EMS are publicly accessible after a five year moving wall.

While recognizing the importance of open platforms such as arXiv or HAL, the EMS strongly supports evaluation by peer review, to ensure that high scientific quality is preserved as well as a defence against misconduct.

Pricing issues

Publishers are performing a crucial task in collecting, processing, disseminating and archiving scientific results. It is impossible to do this for free. Nevertheless, scientists are increasingly dissatisfied with commercial publishers, due mainly to pricing policies and bundling strategies.

The EMS journals are among the group of least expensive mathematical journals. The average page cost is quite similar to that of some journals published by other learned societies. For example, in 2012 the price per page of *JEMS* was €0.31, for *Commentarii Mathematici Helvetici* €0.38, and for the *Journal of Non-commutative Geometry* €0.36.

Bundling of journal subscriptions offers potential advantages to both publishers and librarians. However, it has helped to create a situation in which very large packages are practically forced on consortia, dictating market conditions and driving out many smaller and independent publishers. It should be noted that a large percentage of high-quality mathematical texts is published by small publishers. We would therefore like to encourage consortia to allocate a fair share of their funds to other publishers than those few dominating the market. National or even international consortia could possibly provide a solution to the current publishing problems. While reliably providing revenue to publishers, they in turn could generously handle access to their contents.

Ethical principles

The ethical behaviour of authors, editors, referees, and publishers (as viewed by the EMS) has been laid down in the EMS Code of Practice (http://www.euro-math-soc.eu/system/files/COP-approved.pdf). It is applied to all EMS journals and recommended for adoption by all others.

Open Access

The EMS endorses the general principle of allowing free reading access to scientific results, and declares that in all circumstances, the publishing of an article should remain independent of the economic situation of its authors.

We therefore do not support any publishing models in which the author is required to pay charges (APC). In particular, the EMS regards the so-called hybrid model (journals that publish APC articles along with "regular" articles) as unsuitable and potentially disadvantageous for libraries. It is interesting to note that, so far, the EMS PH has not received a single enquiry about APC publication.

Business Model

The income of the EMS PH comes solely from the sale of subscriptions – single or in packages – to libraries or consortia. However, we are open to considering other ways of funding, such as grants from public funding agencies that may complement or even replace current forms of revenue.

The traditional subscription model has served the community very well over a long time. The EMS believes that this model, based on fair and sound principles, should not be abandoned without other tested and reliable forms of publishing in place.

At present, the EMS has no fixed plans to create new journals where the costs are covered in a different way than by selling single subscriptions or through sale of packages. However, this may change, depending on the demand of the community and the way traditional business models evolve.

European Mathematical Foundation Board of Trustees:

Marta Sanz-Solé (University of Barcelona; Chair)
Nicola Bellomo (Polytechnic University of Turin)
Stephen Huggett (Plymouth University)
Emmanuel Kowalski (ETH Zürich)
Ari Laptev (KTH Stockholm and Imperial College London)
Christine Riedtmann (University of Bern)
Jouko Väänänen (University of Helsinki)

April, 2013

EMS Publications Committee

The EMS Executive Committee has recently set up a Publications Committee. This committee will take up relevant questions related to scientific publications and, in particular, mathematical publications. It will act as advisory group of the Executive Committee on publication matters and on publication strategies. The members of the committee are:

Chair: Bernard Teissier, Institut de Mathématique de Jussieu, Paris, France
Joan Elias, University of Barcelona, Spain
Rui Loja Fernandes, University of Illinois, USA
Timothy Gowers, University of Cambridge, UK
Pierangelo Marcati, University of l'Aquila, Italy
Tomaž Pisanski, University of Ljubljana, Slovenia

The Abel Prize Laureate 2013

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2013 to:

Pierre Deligne
(Institute for Advanced Study, Princeton, New Jersey, USA)

"for seminal contributions to algebraic geometry and for their transformative impact on number theory, representation theory, and related fields" (citing the Abel Committee).

Acknowledgement for the picture: Courtesy of Abelprisen. Photographer: Cliff Moore

W. K. Clifford Prize 2014 – Call for Nominations

The W. K. Clifford Prize is an international scientific prize for young researchers, which intends to encourage them to compete for excellence in theoretical and applied Clifford algebras, their analysis and geometry. The award consists of a written certificate, a one year online access to the Clifford algebra related journals, a book token worth €150 and a cash award of €1000. The laureate is also offered the opportunity to give the special *W. K. Clifford Prize Lecture* at University College London, where W. K. Clifford held the first Goldsmid Chair from 1871 until his untimely death in 1879.

The next W. K. Clifford Prize will be awarded at the *10th Conference on Clifford Algebras and Their Applications in Mathematical Physics* (ICCA10) at Tartu (Estonia) in 2014.

Send nominations to the Secretary at secretary@wkcliffordprize.org. Nominations are due by 30 September 2013.

For details see http://www.wkcliffordprize.org.

Thue 150 – Bordeaux, 30 September – 4 October 2013

Yuri Bilu, on behalf of the organising committee

The conference THUE 150 will take place in Bordeaux from 30 September to 4 October 2013.

The conference will be dedicated to the 150th anniversary of the birth of Axel Thue and the 105th anniversary of his seminal article *Über Annäherungswerte algebraischer Zahlen*, which determined the face of Diophantine approximations in the 20th century.

The conference will mainly address Diophantine geometry. However, other aspects of Thue-related mathematics (algebra, logic, etc.) may appear as well.

The list of invited speakers includes Jean-Paul Allouche, Daniel Bertrand, Yann Bugeaud, Pietro Corvaja, Jan-Hendrik Evertse, Kálmán Győry, Philipp Habegger, Aaron Levin, David Masser, Wolfgang Schmidt, Damien Roy, Martin Sombra, Michel Waldschmidt, Gisbert Wüstholz and Umberto Zannier.

Many leading experts in the field have already expressed their interest in participating. More information on the conference, including the list of participants, can be found on the conference website: http://www.math.u-bordeaux1.fr/~yuri/thue150/.

Anybody wishing to attend is welcome to contact the organisers. We do not charge any conference fee but we cannot contribute to participants' travel expenses (except the invited speakers).

We have some limited funds for sponsoring the local expenses of certain participants, like young researchers, researchers from developing countries and other participants who cannot afford to pay for their stay in Bordeaux.

Meeting the Nobel Peace Laureate Daw Aung San Suu Kyi

Mariolina Bartolini Bussi (Università di Modena e Reggio Emilia, Italy)

On 28 February 2013, I had the chance to meet the Nobel Peace laureate Daw Aung San Suu Kyi, in Naypyidaw, the new capital of the Republic of the Union of Myanmar (as the Republic of Burma is now called). She kindly agreed to welcome a small Italian delegation including two members of the staff of the University of Modena and Reggio Emilia (Dr Giuseppe Malpeli and me, see photograph) and two members of the Italian Parliament (the senator Albertina Soliani and deputy Sandra Zampa).

Daw Aung San Suu Kyi is considered the most famous political prisoner of the world after she spent 15 years under house arrest and continually refused to flinch from her determination to deliver peace and democracy to the people of Burma. Now she is a member of the Burmese Parliament and can move freely in the country and abroad and receive people. However, true reforms toward democracy in Burma are still only beginning.

The short trip to Burma (26 February to 7 March) was organised as a follow-up to the longstanding activity carried out by the Department of Education and Human Science of the University of Modena and Reggio Emilia (where I teach mathematics and mathematics education) for prospective teachers and educators: our students are introduced to the issue of cooperation with developing countries and with special programmes for approaches to intercultural mathematics education. After many years of cultural and voluntary service in Burma, during which he happened to meet several times and become a friend of Daw Aung San Suu Kyi, Dr Malpeli organised this short visit to discuss with the Burmese people the best ways to help their country to be reintegrated into the international community. I took part enthusiastically in this trip with two main aims:

- To offer to Daw Aung San Suu Kyi, on behalf of the Rector of the University of Modena and Reggio Emilia, the university seal and an invitation to visit our university, and to explore with her the possibility of organising student exchanges.
- To discuss with her ways of exploiting activities of the International Commission on Mathematical Instruction (ICMI) in order to promote the improvement of mathematics teaching and learning in Burma.

We all felt very grateful that she found time to meet our small delegation in her living room for nearly one hour, in the days before the Congress of her party, i.e. the National League for Democracy (NLD). The meeting was friendly, warm and cordial.

In this paper I shall report briefly on our mathematical chatter.

From the left, Mariolina Bartolini Bussi, Daw Aung San Suu Kyi and Giuseppe Malpeli

When I introduced the issue of mathematics education, she showed herself to be very interested. She talked about her early years, mentioning her father Aung San, who was a hero of Burmese independence for his efforts in bringing about the end of British colonial rule. He was killed in a 1947 conspiracy at only 32 when his daughter Suu Kyi was only two years old. She said that she and her father were very good at mathematics and liked the subject, perhaps surprisingly (in Western culture) as both had a humanistic education.

From Aung San's biography (authored by Aung San Suu Kyi), we know that he had a degree in humanities (with specialisations in English literature, modern history and political science) and was still studying to receive a second degree in law. However, he was very interested in mathematics. When I visited the Aung San Museum (opened in 2012 in Yangon), I found, in his personal library, some books of applied mathematics and a treatise on education.

Daw Aung San Suu Kyi enrolled in high school and college in India, where her mother was the Burmese ambassador for years. She first went to Oxford in 1964 to read politics, philosophy and economics at St. Hugh's College. During her first visit to India in November 2012 after the end of her house arrest, she mentioned the need to study mathematics when she decided to apply for Oxford. As she said: "In those days, you had to do Latin for the Oxford entrance, and I had no Latin, of course, because in Burma and India we are not taught Latin. So as an alternative, they said I should have to get an A level in Maths. When I tried to find tutors, they refused to take me, because they said I would not be able to make it in three months. And they did not want to be responsible for teaching a failure."[1] She was tutored by K. Rangaswamy, a political journalist in New Delhi, and later by a mathematics teacher. And she succeeded in achieving an A-level in three months!

During our meeting, I mentioned some ICMI activities to facilitate the transmission of information on all aspects of the theory and practice of contemporary mathematical education from an international perspective. Above all, I noted the ICMI Study no. 23 about early numeracy, i.e. whole numbers in primary mathematics, where I serve as co-chair (together with Sun Xuhua, from Macau University, China). Daw Aung San Suu Kyi immediately answered that it would be very important indeed to involve a Burmese voice in the study. She attaches importance to mathematics and agrees that teaching mathematics well in early childhood is the only way to counter the failure in mathematics with elder students.

The present situation of primary school education in Burma is not good. Some progress in the period 2006–2010 has been reported by UNICEF[2] but the survival rate at the end of primary school (5th grade), according to the available data of the government, is still only 70% and even less in the rural parts of the country, where ethnic conflicts are present. Mathematics (together with English) is the subject that is considered to be most difficult to teach and to learn.

In order to have a more concrete and effective discussion, Daw Aung San Suu Kyi put me in touch with an expert in education of the executive committee of the NLD, Dr Thein Lwin, a mathematics teacher who has been in Thailand for years, directing the Thinking Classroom Foundation.[3] This foundation is based in Chiang Mai (Thailand) and provides teacher training for Burmese teachers and education for Burmese refugees.

I met Dr Thein Lwin some days later, at the NLD base in Yangon. He mentioned some important activities for primary school mathematics that are going on in Burma and on the Burma-Thailand border for Burmese refugees, with networks for teacher training and after-school programmes to contrast the bad quality of governmental schools. I read some of his education papers, available on the website of the Thinking Classroom Foundation. I appreciated the thorough description of the cultural issues concerning minorities and refugees and the quality of educational interventions realised in spite of a lack of means and the political difficulties.

During this short cultural visit to Burma, we had the opportunity to meet several cultural and political groups, including a monastic school and a non-governmental organisation, to better understand the whole picture of the country where Daw Aung San Suu Kyi is still defending the values of liberty, love and humanity. She is really blending two cultures: on the one hand, she is at ease discussing Western culture and values; on the other hand she reflects deeply on Buddhism and on its implications for both private life and politics. This synthesis may explain well why she is both admired in the West as an icon of democracy and human rights and loved by her people as a person who can understand, formulate and defend the effects of general principles on the lives of everybody. Needless to say, we are now watching the Burmese process towards democracy with more and more trepidation.

Now, my aim is to start cooperation on mathematics education with Burmese colleagues and to share the richness of this experience with my Italian students. The existing activity in Burma is the unavoidable base for all respectful international cooperation programmes, in order to connect mathematics teaching and learning with the real world and context outside the classroom.

1 http://www.thehindu.com/opinion/op-ed/a-treasured-connection/article4091224.ece.

2 http://erc.undp.org/unwomen/resources/docs/gendereqaulity/UNICEF_Improving_Access_to_Quality_Basic_Education_in_Myanmar_2010.pdf.

3 www.thinkingclassroom.org.

Maria G. (Mariolina) Bartolini Bussi is a full professor of mathematics education at the University of Modena and Reggio Emilia and Chair of the University Programme for Pre-primary and Primary School Teachers. She was a member of the Executive Committee of the International Commission on Mathematica Instruction for two terms (2007–2012).

The Ferran Sunyer i Balaguer Prize 2013

fundació FERRAN SUNYER I BALAGUER
Institut d'Estudis Catalans

The Ferran Sunyer i Balaguer Prize 2013 winner was:

Professor Xavier Tolsa (Universitat Autònoma de Barcelona), for the work

Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory

Abstract: This book studies some of the striking advances that have occurred over the last two decades regarding analytic capacity and its relationship with rectifiability. The Cauchy transform plays a fundamental role in this area and it is one of the main themes of this book too. Another important topic discussed is so-called non-homogeneous Calderón-Zygmund theory, the development of which has been largely motivated by the problems arising in connection with analytic capacity.
The text contains full proofs of Vitushkin's Conjecture and of the semiadditivity of analytic capacity. Both were open problems until very recently. Other related questions, such as the relationship between rectifiability and the existence of principal values for the Cauchy transforms and other singular integrals, are also studied in the monograph.
This monograph will be published by Birkhäuser Verlag in the series *Progress in Mathematics.*

fundació FERRAN SUNYER I BALAGUER
Institut d'Estudis Catalans

Call for the

Ferran Sunyer i Balaguer Prize 2014

The prize will be awarded for a **mathematical monograph** of an expository nature presenting the latest developments in an active area of research in mathematics.

The prize consists of **15,000 euros** and the winning monograph will be published in **Birkhäuser Verlag**'s series "Progress in Mathematics".

DEADLINE FOR SUBMISSION:
2 December 2013
http://ffsb.iec.cat

The European launch of MPE 2013 in the Unesco Headquarters in Paris (5th March 2013)

Ehrhard Behrends (Freie Universität Berlin, Germany), chair of the rpa committee of the EMS

2013 is the year "Mathematics of Planet Earth 2013 (MPE2013)". On 5 March, the "official" launch of the European activities took place at the Headquarters of the UNESCO in Paris. The day started with an opening speech by Mrs Gretchen Kalonji (SC Assistant Director-General UNESCO) and this speech was followed by welcome addresses by François Hollande (President of France, read by Daniel Rondeau, Ambassador of France at UNESCO), Nuno Crato (Minister of Education and Science of the Government of Portugal), Jean Audouze (President of the French National Commission), Ingrid Daubechies (President of the International Mathematical Union), Wendelin Werner (Executive Committee member of the IMU and winner of the Fields Medal in 2006) and Marta Sanz-Solé (President of the European Mathematical Society).

Christiane Rousseau, the International Coordinator of MPE2013, then explained the "mission" of MPE2013:

- Encourage research to identify and address fundamental questions about our planet to which mathematics can contribute to a solution, including understanding Earth's climate and environment and addressing its sustainability.
- Encourage mathematics teachers at all levels to communicate issues related to Planet Earth through their instruction and curriculum development.
- Encourage mathematics students and beginning researchers to pursue research areas related to Planet Earth.
- Inform the public about roles that mathematics can play in addressing questions related to Planet Earth.

Later, Gert-Martin Greuel and Andreas Matt from Oberwolfach (Germany) presented the open source platform IMAGINARY (see imaginary.org) which plays an important role for MPE2013: the winning modules of the MPE2013 competition are available there. In this competition the participants were asked to submit "building blocks" that could be used by anyone with a desire to be active with projects popularising mathematics. There were about 30 submissions, the winners being chosen by an international jury in January in Providence, USA.

The chair of the jury, Ehrhard Behrends from Freie Universität Berlin, presented the winning modules and the associated teams.

Third prize: "How to predict the future of glaciers?" (some videos), by the team of Guilleaume Jouvet (France/Switzerland/Germany).

Second prize: "Dune Ash" , by the team of Tobias Malkmus (Germany). This is an interactive computer program to simulate the distribution of ashes after a volcanic eruption.

First prize: "Sphere of the Earth", by the team of Daniel Ramos (Spain). This module is also an interactive program, it shows that maps of the spherical surface of the earth on a flat plane must have distortions (see photo on the right).

mpe2013 day at the Unesco was continued in the afternoon with talks ("Utilizing the environment to manage HIV/AIDS", "Les modèles climatiques: aspects mathématiques, physiques et conceptuels") and a panel discussion ("What can mathematics do for the planet?").

Andrzej Mostowski, 1913–1975

Victor W. Marek (University of Kentucky, Lexington, USA) for the Organisers of the Meeting Mostowski100

Professor Andrzej Mostowski was born on 1 November 1913 in Lemberg (Austrian Empire), later known as Lwów (Poland) and now Lviv (Ukraine). As a child he moved to Warsaw (and his entire career is connected to this town). Mostowski started his studies of mathematics at Józef Piłsudski University (later known as Warsaw University) in 1931.

We mention all these facts to indicate the complex changes that Poland (and other geolocations in what is known as Eastern Europe) was subjected to during his life.

The Faculty of Mathematics where Mostowski studied provided an excellent roster of individuals that were contributing to widely understood foundations of mathematics (including logic, set theory and foundations-related aspects of topology, e.g. descriptive set theory). This group of scientists included K. Kuratowski, St. Leśniewski, A. Lindenbaum, J. Łukasiewicz, W. Sierpiński and A. Tarski. Not all these scientists were on the faculty (notably Tarski) but they collaborated, shared problems and regularly met at seminars. Poland in the 1930s did not have enough positions in its universities to provide employment to the great talent. Often no positions were available or available only in minor schools (viz. Tarski). Moreover, on par with the rest of Europe during the 1930s, Polish universities were subjects of violent nationalistic events. This political tumult and violent episodes with deeply divided student bodies and faculty contributed to the worsening political situation. Mostowski graduated in 1936 and soon left for Vienna and Zürich for further studies. The studies were supposed to prepare him for work as an "applied mathematician". Apparently the idea that one should really be an *applied* scientist is not that new. Anyway, Mostowski was in Vienna when Gödel presented his ideas on constructibility and thus consistency of the Axiom of Choice, and took classes in Zürich. While Mostowski found the lectures in Zürich fascinating (and the photo of Hermann Weyl was permanently displayed on the wall in Mostowski's office in Warsaw after World War II), Mostowski devoted his life to foundations of mathematics, especially set theory and logic, areas that do not promise immediate applicability.

After coming back from travels, Mostowski presented the doctoral dissertation "On independence of definition of finiteness in a system of logic". This work and subsequent work on the independence of the axiom of choice from the "ordering principle" introduced what is now called the "Fraenkel-Mostowski" method of independence proofs. Specifically, one adds to the universe of sets a collection of individuals (sometimes called urelements). Then one considers a subuniverse of the universe so obtained, by considering elements that are preserved by a suitable group (we provide only a very rough picture). The resulting class (for a suitably chosen filter of groups) is a model that establishes a desired independence result. Of course it only sounds simple; technical problems need to be properly treated, and this is what Mostowski did. Mostowski's dissertation (formally under the supervision of Kuratowski but in reality under Tarski) was defended in 1938.

World War II was especially savage (no better expression comes to mind) in Eastern Europe (see current studies of historians, especially T. Snyder) and had as an immediate result a loss of any opportunities for mathematicians in German-occupied Poland. Eventually, Mostowski supported family working as an accountant in a bitumic-paper factory.

Even though Nazis, motivated by their racial ideology, attempted to eradicate all education over elementary-school level, Polish mathematicians (that is, those that were not immediately killed), at great risk, created and taught an Underground University; Mostowski was one of the lecturers. At the same time, Mostowski was working on his habilitationschrift.

Mostowski collated his wartime results in a "big black notebook". During World War II, he was in the prime of his life; those were his late 20s and early 30s. But history intervened again; during the Warsaw uprising of August 1944, Mostowski (like the entire population of Warsaw) was exiled from the city (which was subsequently 95% destroyed). The big black notebook burned with the city. Later on, Mostowski would tell us, his students, about the results he got during the war; some, but not all, were later published. Many important results there included his work on what we now call "Kleene-Mostowski hierarchy", i.e. arithmetical hierarchy, results on the consequences of the axiom of constructibility on projective hierarchy and many others.

After World War II, Mostowski defended in 1945 his habilitation at Cracow Jagellonian University, again on results related to the Axiom of Choice. He soon returned to Warsaw and for the next 30 years – till his untimely death – was associated with Warsaw University, rising to the position of Professor of Mathematics. Eventually he was also elected a member of the Polish Academy of Sciences. Before the Stalinist regime cut scientific connections to the West, Mostowski spent the academic year 1948/49 at the Institute of Advanced Studies at Princeton, New Jersey. Fortunately, isolation from the world of science was shorter this time; Polish mathematics came back to world science in 1954.

Once established at Warsaw University, Mostowski regularly contributed to all areas of foundations of mathematics. His significant work contributed to recursion theory, undecidability (extending work of Gödel), model theory, set theory, second-order theories such as second-order arithmetic and theories of classes (Gödel-Bernays and Kelley-Morse), constructibility, algebraic methods in foundations of mathematics and so-called generalised quantifiers. This is not a place to sum up his spectacular achievements, so we will mention only the Ehrenfeucht-Mostowski Theorem on the existence of models with indiscernibles, and his introduction and studies of the quantifier "There is infinitely many...". The Ehrenfeucht-Mostowski Theorem is discussed in all handbooks of model theory; generalised quantifiers form a cornerstone of abstract model theory and have important applications in computer science.

Mostowski created an important centre of studies of foundations of mathematics; his Warsaw seminar not only educated researchers in his native Poland but was also a magnet for many scientists from all over the world, and especially Europe. Warsaw became a place where logicians of the East and West met and collaborated. During the times of the "Cold War", it was a unique place where logicians could, and indeed actually did, meet.

Until his death in 1975, Mostowski led a large centre of foundational research in Warsaw. Among his numerous PhD students were H. Rasiowa, R. Sikorski, A. Grzegorczyk and A. Ehrenfeucht, well-known mathematicians in their own right. Mostowski's foundations of mathematics seminar was visited by a veritable who's-who of researchers of the area.

It is hard to overestimate Mostowski's role in the heroic period of foundations of mathematics, culminating in the feverish period of research after Cohen's discovery of forcing. This period, starting in the 1930s, coincided with the activities of Professor Mostowski. He died aged 62, on 22 August 1975, in Vancouver, British Columbia, on his way to London, ONT, Congress of the Section of LMPS of the International Union of History and Philosophy of Sciences.

A meeting devoted to Mostowski's legacy will be held at Warsaw University, 13–15 October 2013. See the conference page at http://mostowski100.mimuw.edu.pl. The meeting is sponsored (among several other organisations) by the EMS.

Organisers of the meeting Mostowski100.

Across the Board: Meeting Point for European Societies

The Editorial Board has recently decided to create this new section as a focal point for Europeans to know each other better. Little did we know during our discussions that we would start it with news of a criminal fire that has destroyed the Città della Scienza, a scientific museum in Naples, which was the pride of the entire city (that has already been touched by social and economic difficulties). In Italy, this criminal act has caused great emotion and mobilisation of many colleagues, intellectuals and citizens, who are collaborating to rebuild the museum as quickly as possible and to establish the primacy of culture and civil society over illegality.

"Città della Scienza" in Naples Burned in a Fire

Ciro Ciliberto (Università di Roma Tor Vergata, Italy), President of the Unione Matematica Italiana

On 4 March 2013, the "Città della Scienza" (Science Centre) in Naples burned in a fire. The Science Centre was located in the Bagnoli area, on the coast, in the northern part of Naples, at the border with the city of Pozzuoli. The area occupied by the Science Centre was part of a huge industrial environment devoted to a steel plant which closed down in 1992. The whole area, of about 120 hectares, has been abandoned since then, with the only exception being the Science Centre, which has been a great driving force for the cultural and economic rebirth of the whole neighbourhood and a mark of distinction and excellence for the city of Naples. The Science Centre was a Science Museum, opened to the public in 1996 thanks to the collaboration between scientists and people of culture, the Idis Foundation, the managing agency of the centre, the Italian Ministry of University and Research, Local Institutions, etc. The Science Centre, visited by more than 300,000 people, especially schoolchildren, included a planetarium, a theatre (the Galilei 104 Theatre), a conference centre and a science store, and it hosted children's workshops, international activities, exhibitions, conferences, etc. These activities were highly recognised at an international level, with the partnership of UNICEF, UNESCO, ANMS (National Association of Natural Science Museums), ECSITE (European Network of Science Centres and Museums) and Hands on! Europe (Association of children's museums), etc.

The Science Centre was an attempt to create in Italy a centre of distinction for high level popularisation and diffusion of science, at a level comparable with the best international experiences in this field. Its location in Naples, in the south of Italy, a region which particularly suffers from economic depression, was also meaningful as a sign of commitment to cultural and social development. Its destruction is therefore a grave event for the Italian scientific community and for the entire society.

The Science Centre burned on a Monday night, the day in which the centre is closed. This is seen as a sign of malice. Hence, the suspicion that we are facing arson is very high. If so, one might be tempted to think that somebody, to be identified, wanted to destroy an institution which was important not only from a cultural but also from a social and economic viewpoint. This would be further evidence of savagery to which too often our society exposes us. Against it, as human beings, and citizens and scientists, we have to raise our voices and fight.

The Italian Mathematical Union has put the problem on its agenda, in order to study possible actions to encourage the reconstruction of the Science Centre and to show that the will of knowledge can be fostered even amongst the ruins.

Meanwhile, the Science Centre appeals to the goodwill of people for concrete help; see http://www.cittadellascienza.it/news/rebuilding-citta-della-scienza-lets-join-our-efforts/?lang=en.

Ciro Ciliberto teaches geometry at the University of Roma "Tor Vergata". His scientific interests are devoted to algebraic geometry.

Maths Jobs Forum in France: A Second Successful Maths Fair

Stéphane Cordier (Université d'Orléans, France), Maria J. Esteban (CNRS and Université Paris-Dauphine, France), Edwige Godlewski (Université Pierre et Marie Curie, Paris), France, Marie Postel (Université Pierre et Marie Curie, Paris, France) and Adeline Samson (Université Paris Descartes, France)

The second Maths Jobs Forum took place in Paris on 11 January 2013 at the Arts and Crafts School and Museum. As with the first forum, this one was organised by the Société Française de Statistique (SFdS), the Société de Mathématiques Appliquées et Industrielles (SMAI) and the Agence pour les Mathématiques en interaction avec l'Entreprise et la Société (AMIES).

The first forum caused a big surprise, since nobody expected so many participants; hence the premises were too small for the crowd that attended. This time a much larger and more suitable location was found, with more room for the industrial and institutional stands. Also, three large conference rooms were booked for presentations and debates taking place throughout the day.

This time 1400 people registered and more than 1250 actually attended the event, a 30% increase on last year, confirming the interest of students in such an event. The number of companies with stands also increased compared to last year, and many returned again, reflecting their interest in the event. In a very large and beautiful room (the Textile room), more than 60 stands welcomed students and young PhDs. Forty-three stands had been "rented" by companies and some big institutions, the remaining ones being occupied by mathematics departments that wanted to talk to students about their Master's and doctoral programmes and their expected open academic positions. Despite the very large room, there were significant queues and at some moments the corridors were simply too crowded.

About 100 employers were present. They had job offers of all kinds: permanent jobs, CDDs, internships, post-doctorates and PhDs in companies, etc. An interesting novelty this year was the presence of a large number of SMEs through the presence of the alliances France Digitale and Teratec. The large variety of economic sectors represented at the stands was an excellent advertisement for mathematics students. The main sectors represented were transportation, energy, telecommunications, the pharmaceutical industry, banks and insurance companies, as well as services, software development, numerical economy, etc.

The range of students who attended the forum was diverse; most were Master's and doctoral students but there were also university students at lower levels who wanted to get an idea about existing opportunities for the future. They came mainly from Paris and its suburbs but not exclusively since most provincial universities had sponsored some student delegates to attend the event.

In the three big conference rooms presentations were being made practically non-stop; some were repeated twice so that those who missed them in the morning could come later in the afternoon. The variety of presentations was impressive. In one of the rooms there were presentations about how to prepare yourself for a job in the industry, how to prepare a CV, how to prepare and conduct a job interview, etc. The forum had the collaboration of a company and an association specialising in placing students in companies so the advice given to the students was extremely professional. There was also a presentation about the job market in other countries in Europe and beyond and about the tools and websites available to look for these jobs. At the European level

the Jobs page of the EMS and several dedicated websites in various countries were advertised. In the two other rooms the presentations were more about various fields of mathematics and their presence in industrial applications and job offers (e.g. energy, environment, health, marketing and finance/insurance/economy). There was also a very interesting round table about how to increase the attractiveness of Master's programmes in mathematics for students and how to design them in a way that is easily readable for companies.

After the forum, a survey was carried out among the participants. The main objective was to understand if their expectations had been fulfilled. The main criticism was that the stands were sometimes so stormed that it was difficult to move. From the numerous and enthusiastic feedback it seems clear that this operation has very quickly become not only useful but actually indispensable. Discussions are underway to find the means to perpetuate the forum.

The organisation of this forum mobilised about 60 colleagues. No professional company was hired for organisational purposes. Apart from colleagues, staff of several Parisian mathematics departments and also the AMIES helped with the organisation, which looked extremely professional throughout.

A professional video maker was hired. She videotaped some of the events plus a large number of interviews. The movie is available at: https://mi2s.imag.fr/node/1143/.

The official website of the Maths Jobs Forum is http://smai.emath.fr/forum-emploi/.

Other useful related links:
http://www.agence-maths-entreprises.fr/
http://www.sfds.asso.fr/
http://smai.emath.fr/
http://www.francedigitale.org/
http://www.teratec.eu/

At the EMS/DMF Joint Mathematical Weekend, Aarhus University, 5–7 April 2013

Bjarne Toft (University of Southern Denmark, Odense, Denmark)

At the EMS/DMF Joint Mathematical Weekend, Aarhus University, 5–7 April 2013, the Danish Mathematical Society celebrated its founding, which took place 140 years ago.

The Danish Mathematical Society (DMF) was founded at a meeting on the evening of 8 October 1873. The initiative was taken by the professor of astronomy at Copenhagen University (KU, the only university in Denmark at the time) Thorvald Nikolai Thiele (1838–1910), today known as one of the founders of modern statistics. He was joined actively by the two mathematics professors at KU and the Polytechnic School Hieronymus Georg Zeuthen (1839–1920) and Julius Petersen (1839–1910), today known as pioneers in enumerative geometry/history of mathematics and economics/geometry/graph theory, respectively.

The weather played an unusual role on the day of the founding. According to a contemporary source, the highest, lowest and mean temperatures were (in that order) 16.8, 12.0 and 11.6 (!) degrees. During the EMS/DMF Joint Mathematical Weekend at Aarhus University, 5–7 April 2013, celebrating the founding, the weather participated again: after a long, cold and grey winter, the sun, all of a sudden, started shining from a blue sky and remained shining for the whole meeting; but as soon as the meeting was over at 13:30 on Sunday it became overcast and started to snow!

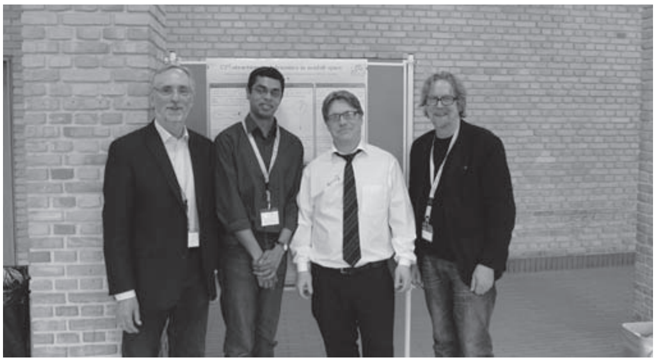
The meeting in Aarhus marks a high point in the recent history of the DMF. There are good reasons to thank the participants, the lecturers and organisers, the sponsors and the administrative staff (in particular the QGM-Centre in Aarhus) for a successful meeting. The scientific program consisted of a poster exhibition and 51 lectures, one of which was the EMS distinguished lecture by Jeremy Grey, four other plenary lecturers (Uffe Haagerup, Henri Berestycki, Herbert Edelsbrunner and Carsten Thomassen) and 46 lectures running in six parallel sections (Algebra and Number Theory, Algebraic Topology, History of Mathematics, Quantum and Riemannian Geometry, Partial Differential Equations and Applications, and Stochastics and Free Probability). The names of the lecturers, the titles of lectures and the abstracts may be found via DMF's homepage www.mathematics.dk. The prize for the best poster was given to Subhojoy Gupta (QGM, Aarhus University, Denmark) and DMF's poster prize was given to Vicent Gimeno & José Sotoca (Universitad Jaume, Castelló, Spain).

In connection with the EMS/DMF Joint Mathematical Weekend, a one-day *Meeting of Presidents* took place at Arhus University on 6 April, in which 41 presidents of

Participants

Meeting of Presidents

national mathematical societies in Europe and executive members of the EMS participated. This was also a success – a number of issues concerning mathematics in Europe were discussed under the efficient chairmanship of the President of the EMS Marta Sanz-Solé. It is clear that the national mathematical societies still play a very important role in spite of internationalisation – the EMS is not taking over but has an important role as publisher of books and its news magazine, and as a uniting umbrella and coordinator (for example, in relation to the EU and in creating unified ethical guidelines for mathematics in Europe).



A prize for the best poster was won by Subhojoy Gupta (QGM, Aarhus). This picture shows Vagn Lundsgaard Hansen from the scientific committee, Subhojoy Gupta, Helge Jensen from the software company ACTUA donating the prize of 2,000 DKK, and Bjarne Toft from the Danish Mathematical Society.

Finally, a mathematical exhibition IMAGINARY – through the eyes of mathematics for the general public opened during the meeting at the Steno Science Museum in Aarhus. The exhibition runs until the end of August 2013 and it is based mainly on the IMAGINARY material created at Oberwolfach in Germany. The Technical University of Denmark in Lyngby used 3D printing to create models and the QGM-Centre at Aarhus University, collaborating with the museum, arranged the beautiful exhibition.

The DMF is grateful to the EMS, Aarhus University and the QGM-Centre, to all outside sponsors and indeed to all participants, for making this a successful event.

Bjarne Toft [btoft@imada.sdu.dk] is an associate professor at the University of Southern Denmark in Odense, Denmark. He studied at Aarhus University and the University of London, obtaining a PhD in 1970. With T. R. Jensen, he is the author of a much cited book [Graph Coloring Problems, Wiley 1995], which contains a description of more than 200 open problems. Bjarne Toft was elected President of the Danish Mathematical Society in 2012.

Erratum

In the EMS Newsletter No. 84, in the obituary on Mikael Passare by Christer Kiselman, the caption to the photo on page 13 should read as follows:
Mikael (age 24), Jean François Colombeau, Leif Abrahamsson, and Urban Cegrell in November 1983 (Photo: Christer Kiselman)

The Nash Problem and its Solution

Camille Plénat (Aix Marseille Université, France) and Mark Spivakovsky (Institut de Mathématiques de Toulouse, France)

The goal of this paper is to give an historical overview of the Nash Problem of arcs in arbitrary dimension, as well as its affirmative solution in dimension two by J. Fernandez de Bobadilla and M. Pe Pereira and a negative solution in higher dimensions by T. de Fernex, S. Ishii and J. Kollár. This problem was stated by J. Nash around 1963 and has been an important subject of research in singularity theory.

1 Introduction

In this paper, $\Bbbk$ is an algebraically closed field of characteristic 0 (see Remark 1.8 below for the case of positive characteristic).

Resolution of singularities

Let X be a singular algebraic variety over $\Bbbk$ and $\pi : \tilde{X} \longrightarrow X$ a *divisorial* resolution of singularities of X (this means that π is bijective away from the closed set $\pi^{-1}(Sing\ X)$ – such morphisms are called **birational** – $\tilde{X}$ is a smooth variety and the **exceptional set** $E =: \pi^{-1}(Sing\ X)$ is a **divisor**, that is, it is of pure codimension one). Let

$$E = \bigcup_{i \in \Delta} E_i \tag{1}$$

be the decomposition of E into its irreducible components. The set E has two kinds of irreducible components: essential and inessential. Intuitively, an irreducible divisor is essential if it appears, as an irreducible divisor, on *every* divisorial resolution of X.

In general (that is, when dim $X \geqslant 3$) it is quite difficult to show that a given component is essential (see [31] for a discussion of this question as well as some sufficient conditions for essentiality and [3] and [16] for new criteria of essentiality). In dimension two there exists a unique *minimal* resolution $\tilde{X}$ of X (in the sense that any other resolution of X maps to $\tilde{X}$) and each irreducible exceptional divisor of $\tilde{X}$ is essential.

Example 1.1.
- *The first example is the following (see Ishii-Kollar [15]): Let X be defined by $xy - uv = 0$ in $\mathbb{C}^4$. The variety X is of dimension 3, with isolated singularity at 0. One can resolve it either by blowing up the point 0 (the map P_2 on the figure) or by blowing up the surface on X defined by $x = u = 0$ (the map P_1). In the first case, one obtains a divisorial resolution with one divisor; in the second case, the exceptional set E is a curve, hence of codimension 2, and so is not a divisor. Thus in dimension higher than 2, not all the resolutions are divisorial. The second resolution is an example of a* small resolution*, that is, a resolution in which every irreducible component of the exceptional set has codimension strictly greater than 1.*
- *The second example is the variety defined in $\mathbb{C}^4$ by*

$$x^2 + y^2 + z^2 + w^4 = 0.$$

This variety can be resolved by two blowing ups at 0, this resolution being divisorial. But there also exists a small resolution, given by only one blowing up the subvariety defined by $x - y = 0 = z - w^2$, which gives only one component for E. Thus one of the two divisors found in the first resolution is not essential.

The space of arcs of X

In order to study resolutions $\tilde{X}$ of X, J. Nash (around 1963, published in 1995 [25]) introduced the space X_∞^{sing} of *arcs* meeting the singular locus *Sing X*.

To give an idea of what the space X_∞^{sing} and its elements look like, let us first take $\Bbbk = \mathbb{C}$ and consider the space of all the germs of parametrized *analytic* curves, contained in the algebraic variety X over $\mathbb{C}$ and meeting the singular locus *Sing X*. For example, suppose X is an affine variety, defined in $\mathbb{C}^N$ by polynomial equations $f_1, \ldots, f_s$ in N variables. By definition, a parametrized analytic arc in X, meeting *Sing X*, is given by N convergent power series

$$\begin{cases} x_1(t) = a_{10} + a_{11}t + a_{12}t^2 + \ldots \\ x_2(t) = a_{20} + a_{21}t + a_{22}t^2 + \ldots \\ \quad\vdots \\ x_N(t) = a_{N0} + a_{N1}t + a_{N2}t^2 + \ldots \end{cases} \tag{2}$$

having the following properties:

(a) for all $j \in \{1, \ldots, s\}$, the convergent power series $f_j(x_1(t), \ldots, x_N(t))$ is identically zero as a power series in t,

(b) the point $(a_{10}, \ldots, a_{N0})$, which we refer to as *the origin* of the arc (2), belongs to *Sing X*.

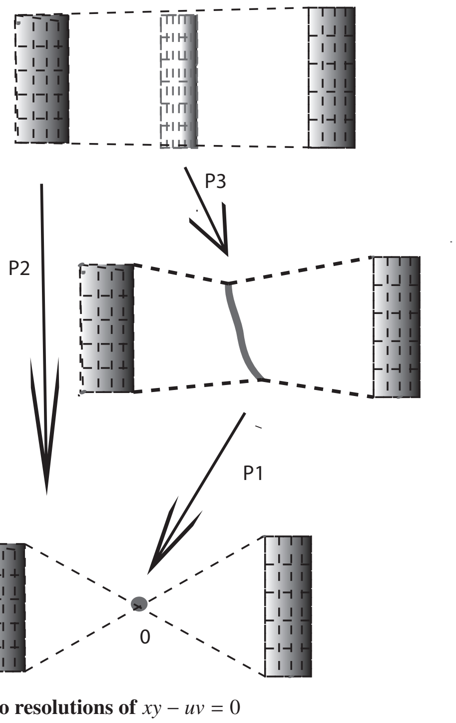


Figure 1. Two resolutions of $xy - uv = 0$

A specific example of this situation when X is the hypersurface in $\mathbb{C}^3$ defined by the equation $xy-z^{n+1} = 0$ is discussed in Example 1.3 below.

For the purposes of the Nash problem, it is natural to consider *formal* parametrized curves instead of analytic ones, that is, to drop the convergence assumption on the power series $x_1(t), \ldots, x_N(t)$ above. In algebraic language, we say that the power series (2) define a morphism from $Spec\ \mathbb{C}[[t]]$ to X such that the image of the closed point of $Spec\ \mathbb{C}[[t]]$ belongs to $Sing\ X$. Once the definition is expressed in algebraic language, it is natural to extend it to arbitrary algebraically closed fields $\mathbb{k}$ and to varieties X which are not necessarily affine:

Definition 1.2. *An **arc** is a $\mathbb{k}$-morphism from Spec $\mathbb{k}[[t]]$ to X.*

Let X_∞^{sing} be the set of arcs whose origin (that is, the image of the closed point) belongs to the singular locus of X.

The analogue of an arc in complex analysis is a test map from a small disc around the origin on the complex plane to X. We will also need to consider more general arcs, which are morphisms from $Spec\ K[[t]]$ to X, where K is a field extension of $\mathbb{k}$; they are called *K-arcs*.

Example 1.3. *Let us have a look at the singularity A_n given in $\mathbb{C}^3$ by the equation*

$$z^{n+1} = x.y.$$

It is the first example studied by J. Nash. It has an isolated singularity at 0.

An arc living on A_n and passing through 0 *is given by three formal power series*

$$\begin{cases} x(t) = a_1 t + a_2 t^2 + \ldots \\ y(t) = b_1 t + b_2 t^2 + \ldots \\ z(t) = c_1 t + c_2 t^2 + \ldots \end{cases}$$

whose coefficients are elements of $\mathbb{C}$ and such that $z(t)^{n+1} \equiv x(t).y(t)$. That last equation gives an infinity of equations on the coefficients of the arcs:

$$\begin{cases} a_1 b_1 = 0 \\ a_1 b_2 + a_2 b_1 = 0 \\ \vdots \\ a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_{n-1} b_1 \\ c_1^{n+1} = a_1.b_n + \ldots + b_1 a_n \\ \vdots \end{cases}$$

Let us denote the closed point (the origin) of $Spec\ \mathbb{k}[[t]]$ by 0 and the generic point by η.

An arc can be lifted to any resolution:

Lemma 1.4. *Let $f : \tilde{X} \to X$ be a resolution of singularities. Every arc $\alpha : Spec\ K[[t]] \to X$ such that $\alpha_x(\eta) \notin Sing(X)$ can be lifted uniquely to an arc $\tilde{\alpha} : Spec\ \mathbb{k}[[t]] \to \tilde{X}$.*

The proof comes from the fact that the resolution map π is proper. In other words, as the resolution of singularities is an isomorphism away from E, one can lift the arc without the origin, and then take the closure.

Remark: the closure of each lifted arc intersects at least one of the irreducible exceptional divisors; moreover, if an arc is general enough, its lifting intersects transversely one and only one irreducible exceptional divisor.

Let us fix a divisorial resolution of singularities $\tilde{X} \to X$ and let $E = \pi^{-1}(Sing\ X)$. Consider the decomposition (1) of E into irreducible components, as above. Let $\Delta' \subset \Delta$ denote the set which indexes the essential divisors.

M. Lejeune-Jalabert [19], inspired by Nash's original paper [25], proposed the following decomposition of the space X_∞^{sing}: for $i \in \Delta'$, let C_i be the set of arcs whose lifting in $\tilde{X}$ intersects the essential divisor E_i transversally but does not intersect any other exceptional divisor E_j. M. Lejeune-Jalabert shows that

$$X_\infty^{sing} = \bigcup_{i \in \Delta'} \overline{C_i} \qquad (3)$$

and the set $\overline{C_i}$ is an irreducible algebraic subvariety of the space of arcs.

The statement of the problem

Nash used the decomposition (3) to show that X_∞^{sing} has finitely many irreducible components, $F_1, \ldots, F_r$, called *families of arcs*, and defined the following map:

Definition 1.5 (Nash [25])**.** *Let*

$$\mathcal{N} : \{F_1, \ldots, F_r\} \to \{ \textit{ essential divisors } \}$$

be the map sending the family F_i to the exceptional divisor E_i such that the generic arc of F_i has lifting to the resolution passing through a general point of the component E_i.

He showed that this map, now called the *Nash map*, is injective. The celebrated *Nash problem*, posed in [25], is the question of whether the Nash map is surjective.

Since the $\overline{C_i}$ are irreducible, the families of arcs are among the $\overline{C_i}$'s. Moreover there are as many $\overline{C_i}$ as essential divisors E_i. Then the Nash problem reduces to showing that the $\overline{C_i}$, $i \in \Delta'$, are precisely the irreducible components of X_∞^{sing}, that is, to proving $card(\Delta')(card(\Delta') - 1)$ non-inclusions:

Problem 1.6. *Is it true that $\overline{C_i} \not\subset \overline{C_j}$ for all $i \neq j$?*

Example 1.7. *Let us keep our attention on the singularity A_n given in $\mathbb{C}^3$ by the equation*

$$z^{n+1} = x.y.$$

It has an isolated singularity at 0*. The exceptional divisor of the minimal resolution of A_n consists of n irreducible curves E_i, arranged in a chain, such that E_i intersects E_{i+1} transversely for $i \in \{1, \ldots, n-1\}$.*

One can show that the first n equations of Example 1.3 completely describe the n irreducible components of $A_{n,\infty}^{sing}$ and that the irreducible component F_i is given by the $a_1 = \ldots = a_{i-1} = b_1 = \ldots = b_{n-i} = 0$. A general element of F_k has the form:

$$\begin{cases} x(t) = a_k t^k + a_{k+1} t^{k+1} + \ldots \\ y(t) = b_{n+1-k} t^{n+1-k} + b_{n+2-k} t^{n+2-k} + \ldots \\ z(t) = c_1 t + c_2 t^2 + \ldots \end{cases}$$

with a_k, b_{n+1-k} and c_1 different from zero.

Remark 1.8. *All of the above definitions make sense also when char $\mathbb{k} > 0$, with the following modification. An arc* *family is said to be* good *if its general element is not entirely contained in Sing X. When char* $\mathbb{k} = 0$ *it is easy to show that all the arc families are good. Over fields of positive characteristic there may exist some bad families, and the Nash map is only defined on the set of good families. With this in mind, the Nash problem remains the same: is the Nash map, defined on the set of good families, surjective? See [36] for some recent work on the Nash problem in positive characteristic.*

Some partial answers in dimension 2

Before the work of Fernandez de Bobadilla – Pe Pereira, the Nash problem for surfaces has been answered affirmatively in the following special cases: for A_n singularities by Nash; for minimal surface singularities by A. Reguera [33] (with other proofs by J. Fernandez-Sanchez [6] and C. Plénat [28]); for sandwiched singularities by M. Lejeune-Jalabert and A. Reguera (see [20] and [34]); for toric varieties in all dimensions by S. Ishii and J. Kollar [15] (using earlier work of C. Bouvier and G. Gonzalez-Springberg ([1] and [2])); for a family of non-rational surface singularities by P. Popescu-Pampu and C. Plénat [30]; and for quotients of $\mathbb{C}^2$ by an action of finite group [26] by M. Pe Pereira in 2010 based on the work [4] of J. Fernandez de Bobadilla (other proofs for $\mathbf{D_n}$ in 2004 by Plénat [29] and for E_6 in 2010 by C. Plénat and M. Spivakovsky [32], with a method that works for some normal hypersurface singularities) and by M. Leyton-Alvarez (2011) for E_6 and E_7, by applying the method for the following classes of normal hypersurfaces in $\mathbb{C}^3$: hypersurfaces $S(p, h_q)$ given by the equation $z^p + h_q(x, y) = 0$, where h_q is a homogeneous polynomial of degree q without multiple factors, and $p \geqslant 2$, $q \geqslant 2$ are two relatively prime integers [22]. A. Reguera [36] gave an affirmative answer to the Nash problem for rational surface singularities simultaneously and independently from the work in [5].

See the bibliography for a (hopefully) complete list of references on the subject.

In 2011, J. Fernandez de Bobadilla and M. Pe Pereira [5] showed that the answer is positive for any surface singularity. The main aim of this paper is to give an idea of their proof. Before going further into the details, we need to recall some earlier results that lead to the final proof.

The rest of the paper is organised as followed: §2 is dedicated to the work preceding the paper [5]; in §3 an outline of the proof is given; and §4 contains a brief discussion of the Nash problem in dimension three and higher.

2 The wedge problem

The Wedge problem [17] . . .

In 1980, M. Lejeune-Jalabert proposed to look at the Nash problem from a new point of view. She formulated in [17] what is now called "the wedge problem", which is related to a "Curve Selection Lemma" in the space of arcs.

Let X be a singular algebraic variety over $\mathbb{k}$.

Let us first define *wedge*:

Definition 2.1. *Let K be a field extension of $\mathbb{k}$. A K-wedge on X is a $\mathbb{k}$-morphism*

$$\omega : Spec(K[[t, s]]) \to X$$

which maps the set $\{t = 0\}$ to Sing X.

The wedge ω induces two arcs on X as follows: a K-arc obtained by restricting ω to the set $\{s = 0\}$ (this arc is called the *special arc* of ω), and a $K((s))$-arc, obtained by restricting ω to the set $Spec(K[[t, s]]) \setminus \{s = 0\}$ (this arc is called the *general arc* of ω). We regard ω as a deformation of its special arc to its general arc or, alternatively, as an arc in the space of arcs X_∞^{sing}.

The wedge is said to be *centered* at an arc γ_0 if its special arc is γ_0.

Let $(X, 0)$ be a germ of a normal surface singularity and let $\pi : (\tilde{X}, E) \to (X, 0)$ be its minimal (and so divisorial) resolution, with $E = \bigcup E_j = \pi^{-1}(0)$. Let E_i, E_j be irreducible components of E (they are essential as X is a surface). Let C_i and C_j be as above. Then if $C_j \subset \overline{C_i}$, E_j is not in the image of the Nash map. If one had a Curve Selection lemma in the space of arcs X_∞^{sing}, the inclusion above would just mean that one has a $\mathbb{k}$-wedge with special arc in C_j and generic arc in C_i. Then the morphism ω would not lift to the resolution $\tilde{X}$ as it has an indeterminacy at 0.

M. Lejeune-Jalabert proposed the following problem:

Problem 2.2. *For all irreducible essential divisors of the minimal resolution, any $\mathbb{k}$-wedge centered at $\gamma_i \in C_i$ can be lifted to $\tilde{X}$.*

It is not trivial to generalise the classical Curve Selection Lemma to the case of infinite-dimensional varieties such as X_∞^{sing}. A. Reguera proved a Curve Selection Lemma for X_∞^{sing} thus establishing the equivalence between the Nash problem and the wedge problem. The wedges appearing in A. Reguera's theorem are K-wedges rather than $\mathbb{k}$-wedges, where K is an extension of $\mathbb{k}$ of infinite transcendence degree. In the following section we discuss this work of A. Reguera and its generalisations due to J. Fernandez de Bobadilla and A. Reguera – M. Lejeune-Jalabert, which reduce the Nash problem to the problem of lifting of $\mathbb{k}$-wedges to the minimal resolution.

. . . is equivalent to the Nash problem of arcs

In the paper [35], A. Reguera has shown that a positive answer to the wedge problem is equivalent to the surjectivity of the Nash map. She has also extended the wedge problem to all dimensions. Note that she does not assume the singular varieties to be normal. More precisely, she proves the following:

Theorem 2.3. *Let X be a singular variety.*
Let E_i be an essential divisor over X. Let γ_i be the generic point of $\overline{C_i}$ (the closure of the set of arcs lifting transversally to E_i) and $\mathbb{k}_i$ its residue field. The following are equivalent:
1. *E_i belongs to the image of the Nash map.*
2. *For any resolution of singularities $p : \tilde{X} \to X$ and for any field extension K of $\mathbb{k}_i$, any K-wedge whose special arc maps to γ_i, and whose generic arc maps to X_∞^{sing}, lifts to $\tilde{X}$.*
3. *There exists a resolution of singularities $p : \tilde{X} \to X$ satisfying the conclusion of* (2).

To prove this she needed a Curve Selection lemma for X_∞^{sing} for curves defined over K. This field is of infinite transcendence degree over $\mathbb{k}$, so it is quite difficult to work with. J. Fernandez de Bobadilla [4] and M. Lejeune-Jalabert with

A. Reguera [21] have shown, independently, that one may replace K by $\mathbb{k}$ in A. Reguera's theorem, provided that $\mathbb{k}$ is uncountable.

Let us cite some results from Fernandez de Bobadilla's paper. First, he gives the definition of wedges that realise an adjacency between two essential divisors:

Definition 2.4. *Let E_u and E_v be two essential divisors, and C_u and C_v the irreducible subvarieties of X_∞^{sing} associated to these divisors.*
A K-wedge realises an adjacency from E_u to E_v if its generic arc belongs to C_u and its special arc belongs to $\dot{C}_v$ (i.e. it is transverse to E_v in a general point of E_v).

Note that if such a wedge exists then C_v is not in the image of the Nash map. This statement can be interpreted as the easy part of the previous theorem of A. Reguera ($1 \implies 2$): a wedge realising the adjacency cannot be lifted to any resolution.

J. Fernandez de Bobadilla proves the following theorem:

Theorem 2.5. *Let $(X, 0)$ be a normal surface singularity defined over an uncountable algebraically closed field $\mathbb{k}$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor of a resolution. Then the following are equivalent:*
1. *The set C_v is in the Zariski closure of C_u, where E_u is another irreducible component of the exceptional divisor.*
2. *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists an algebraic $\mathbb{k}$-wedge realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*
3. *There exists a formal $\mathbb{k}$-wedge realising an adjacency from E_u to E_v.*
4. *Given any proper closed subset $\mathcal{Z} \subset \overline{C_u}$, there exists a finite morphism realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*

See [4] for the definition of finite morphism realising an adjacency from E_u to γ.

J. Fernandez de Bobadilla also proved in [4] that the Nash problem for surfaces is a topological problem. In other words, if the answer is affirmative for a certain normal surface singularity $(X, 0)$, it is also positive for any normal surface singularity, diffeomorphic to X:

Theorem 2.6. *The set of adjacencies between exceptional divisors of a normal surface singularity is a combinatorial property of the singularity: it only depends on the dual weighted graph of the minimal good resolution. In the complex analytic case this means that the set of adjacencies only depends on the topological type of the singularity and not on the complex structure.*

A sketch of the proof for normal dimension two singularities is the subject of the following section. We will need to first define what a geometric Milnor representative of an arc and a wedge are.

3 Solution of the Nash problem for surfaces

Theorem 3.1. *Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 and $(X, 0)$ a normal singular surface over $\mathbb{k}$.*
The Nash map associated to $(X, 0)$ is bijective.

In [4] (7.2 p. 163), J. Fernandez de Bobadilla shows that the families of arcs are stable under base change and so is the bijectivity of Nash map. This allows one to reduce the problem to the case of normal surface singularities over $\mathbb{C}$.

Let $(X, 0)$ be a normal surface singularity over $\mathbb{C}$. (The non-normal case can be reduced to the normal one).

The proof proceeds by contradiction.

Let $E = \bigcup_{i=0}^{n} E_i$ be the decomposition of E into irreducible components. Suppose there are two irreducible subvarieties of X_∞^{sing} $\overline{C_0}$ and $\overline{C_i}$ associated with two essential divisors E_0 and E_i of the minimal resolution such that $\overline{C_0} \subset \overline{C_i}$.

From now on, replace X by its underlying complex-analytic space. By abuse of notation, we will continue to denote this space by X. Let $\pi : \tilde{X} \to X$ be the minimal resolution of X.

For an analytic wedge $\alpha : (\mathbb{C}^2, 0) \to X$ we denote the generic arc by $\alpha(t, s) = \alpha_s(t)$ and the special arc $\alpha(t, 0)$ by $\gamma(t)$. Aiming for contradiction, we now consider an analytic wedge $\alpha : (\mathbb{C}^2, 0) \to X$ realising the adjacency from E_i to E_0, that is, a wedge such that the generic arc belongs to C_i and the special arc belongs to C_0.

J. Fernandez de Bobadilla and M. Pe Pereira define the notion of Milnor representative of arcs and wedges.

Let us call $X_{\varepsilon_0} = X \cap B_{\varepsilon_0}$ the Milnor representative of X. This means, by definition, that for all $0 < \varepsilon \leqslant \varepsilon_0$ the sphere S_ε is transverse to X and $X \cap S_\varepsilon$ is a closed subset of S_ε.

Consider the special arc $\gamma : (\mathbb{C}, 0) \to X_{\varepsilon_0}$. It is proved in [26] and [5] that there exists $\varepsilon \leqslant \varepsilon_0$ such that, restricted to X_ε, γ becomes a *Milnor arc*:

Definition 3.2. Milnor arc
A Milnor representative of γ is a map of the form

$$\gamma|_U : U \to X_\varepsilon$$

such that $\gamma|_U$ is a proper morphism, U is diffeomorphic to a closed disc, $\gamma^{-1}(\partial X_\varepsilon) = \partial U$ and the mapping $\gamma_{|U}$ is transverse to any sphere $S_{\varepsilon'}$ for $\varepsilon' \leqslant \varepsilon$. The radius ε is called a Milnor radius for γ.

Let $\gamma|_U \colon U \to X_\varepsilon$ be a Milnor Representative of γ.

For the disc D_δ of radius δ around the origin in the complex plane we will use the notation $\dot{D}_\delta = D_\delta \setminus \{0\}$.

We replace α by its restriction to $U \times D_\delta$, where δ is a small positive number, specified immediately below.

Milnor wedge. There exist $\delta > 0$ small enough, an open set $\mathcal{U} \subset U \times D_\delta$ and a map

$$\beta : U \times D_\delta \to X_\varepsilon \times D_\delta$$
$$(t, s) \to (\alpha_s(t), s)$$

such that the set $U_s = \mathcal{U} \cap \mathbb{C} \times \{s\}$ is diffeomorphic to a disc for all s and satisfying some other transversality and finiteness conditions, which we omit in order not to overburden the exposition with technical details.

Definition 3.3. *The map β restricted to $\mathcal{U}$ is a* Milnor representative *of the wedge α.*

Remark 3.4. *One has to prove that such a representative does exist, in particular that the set $\mathcal{U}$ can be taken to be differomorphic to a bidisk. See [26] or [5].*

These definitions of representatives are a key point in the proof of the theorem.

The main idea of the proof: Let $\alpha_s : U_s \to X_\varepsilon$ be a generic arc of the wedge. By construction, U_s is a disk and thus has Euler characteristic equal to one. The aim of the rest of the proof is to show that the Euler characteristic of U_s is bounded above by an expression less than or equal to 0, and thus get a contradiction.

Eliminating the indeterminacy of $\tilde{\alpha}$

Let $\tilde{\beta}$ be the meromorphic map defined as the composition of $\sigma^{-1} \circ \beta$ with $\sigma = (\pi, id\,|_{D_\delta})$:

$$\begin{array}{ccc} & & \tilde{X}_\varepsilon \times D_\delta \\ & \overset{\tilde{\beta}}{\nearrow} & \downarrow \sigma \\ \mathcal{U} & \xrightarrow{\beta} & X_\varepsilon \times D_\delta \end{array}$$

The indeterminacy locus of $\sigma^{-1} \circ \beta$ is of codimension 2. Thus we may assume that, shrinking the radius δ, if necessary, $(0, 0)$ is the only indeterminacy point of $\tilde{\beta}$.

Moreover there exists a unique meromorphic lifting $\tilde{\alpha}$ of α such that:

$$\begin{array}{ccc} Y & \hookrightarrow & \tilde{X}_\varepsilon \\ \downarrow & \overset{\tilde{\alpha}}{\nearrow} & \downarrow \pi \\ \mathcal{U} & \xrightarrow{\alpha} & X_\varepsilon \end{array}$$

Let Y be the analytic Zariski closure of $\sigma^{-1}(\beta(\mathcal{U})\backslash(\{0\} \times D_\delta))$ and let $Y_s = Y \cap (\tilde{X}_\varepsilon \times \{s\})$. The surface Y is reduced and is a Cartier divisor in the smooth threefold $\tilde{X} \times D_\delta$. One can prove the following [5]:

$$Y_s = \tilde{\alpha}_s(U_s) \;\; \forall s \in \dot{D}_\delta.$$

Remark 3.5. *Y_s can be thought of as the topological image of the lifting of the arc α_s on $\tilde{X}_\varepsilon$.*

Moreover, one has that:

Lemma 3.6. *The mapping $\tilde{\alpha}_s : U_s \to Y_s$ is the morphism of normalization of Y_s.*

To prove this, we use the following lemma:

Lemma 3.7. *The mapping $\alpha_s : U_s \to X_\varepsilon$ is one–to–one.*

Definition 3.8. Returns
Elements of the set $\alpha_s^{-1}(0)\backslash\{0\}$ are called returns. *Their images by α_s are 0 and by $\tilde{\alpha}_s$ points of the exceptional set E.*

As explained before, to obtain a contradiction we want to show that U_s has non-positive Euler characteristic. To do this, Fernandez de Bobadilla and Pe Pereira give an upper bound on $\chi(U_s)$ in terms of $\chi(Y_s)$, $\chi(Y_0)$ and the possible returns.

End of the proof

The curve $Y_0 = Y \cap (\tilde{X}_\varepsilon \times \{0\})$ does not need to be reduced. It contains $Z_0 := \tilde{\alpha}_0(U_0)$ and a sum of the exceptional components E_i with suitable multiplicities. We express this situation by the equation $Y_0 = Z_0 + \sum a_i E_i$; the analytic space Y_0 is reduced along $Z_0\backslash E$.

A crucial point in the proof of Fernandez de Bobadilla–Pe Pereira is the fact that Y_s is a deformation of Y_0, and hence is numerically equivalent to it, that is, Y_s and Y_0 have the same intersection number with any compact curve in $\tilde{X}_\varepsilon$. We construct a tubular neighbourhood of E in the following way.

Define $\dot{E}_i = E_i\backslash Sing(Y_0^{red})$. Let $Sing(Y_0^{red}) = \{p_0, p_1, \ldots, p_m\}$, where $p_0 = Z_0 \cap E$. Let B_k be a small ball in $\tilde{X}$ centered at p_k. For $j \in \{0, \ldots, n\}$, let T_j be a tubular neighbourhood of E_j, small enough so that its intersection with each B_k is transverse. Let T_{n+1} be a tubular neighbourhood of Z_0, small enough so that its intersection with B_0 is transverse. Let

$$W_j = T_j \backslash \left(\bigcup_{k=0}^{m} B_k \right).$$

All the neighbourhoods are chosen so that

$$\chi(U_s) = \sum_{j=0}^{n+1} \chi(\tilde{\alpha}_s{}^{-1}(Y_s \cap W_j)) + \sum_{k=0}^{m} \chi(\tilde{\alpha}_s{}^{-1}(Y_s \cap B_k)). \quad (4)$$

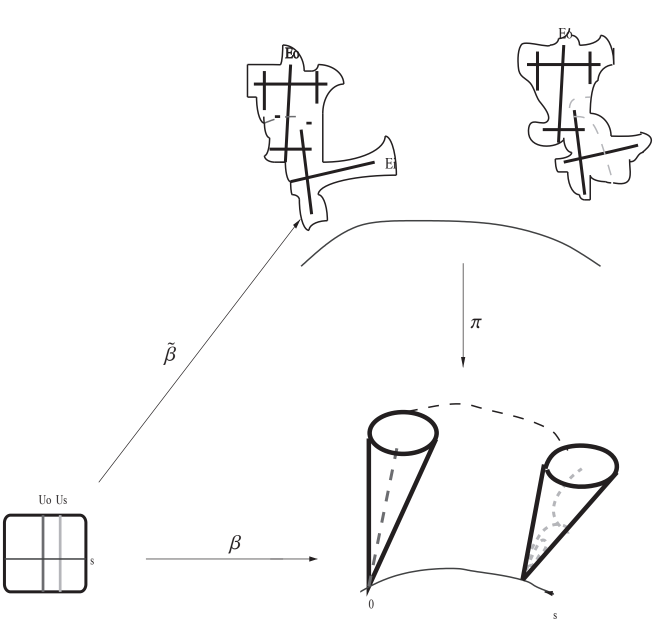


Figure 2. Wedge representative

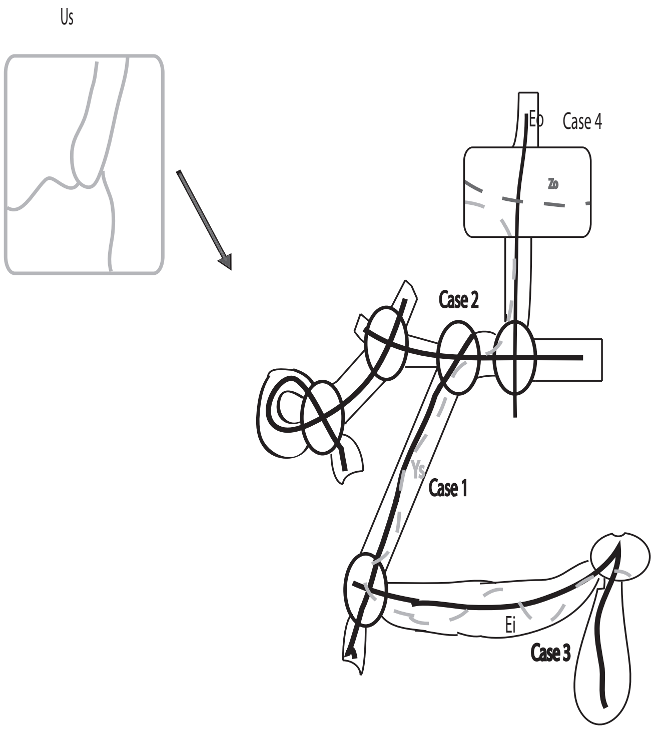


Figure 3. Normalization map

We do not need to count $\chi(Y_s \cap T_j \cap B_k)$ since by the assumed transversality each of these intersections is a finite union of circles and thus

$$\chi(Y_s \cap T_j \cap B_k) = 0. \tag{5}$$

It remains to bound above each summand on the right hand side of (4). Using topological techniques, the authors prove, under some extra assumptions, that

$$\chi(U_s) \leqslant \sum_i a_i(2 - 2g_i + E_i.E_i). \tag{6}$$

This last sum is less than or equal to 0 as each member is less than or equal to 0. This proves that the disc U_s has non-positive Euler characteristic, which gives the desired contradiction. In the general case, the authors obtain a more complicated version of the formula (6), which leads to the same final conclusion.

4 Higher dimensions

For singularities of higher dimensions, the Nash Problem stated as above is false, though a few positive results have been proved. In [15], S. Ishii and J. Kollar give an affirmative answer for toric varieties in all dimensions. Affirmative answers were given for a family of singularities in dimension higher than 2 by P. Popescu-Pampu and C. Plénat [31] and for another family by M. Leyton-Alvarez [22] (2011).

In [15], S. Ishii and J. Kollár give a counterexample to the Nash problem in dimension greater than or equal to 4: the hypersurface

$$x^3 + y^3 + z^3 + u^3 + w^6 = 0$$

which has a resolution with two irreducible exceptional components. These are essential, as one is the projectivization of the tangent cone at the singular point (hence it clearly corresponds to a Nash family) and the other one is not uniruled. Then the authors construct geometrically a wedge whose generic arc is in the Nash family and whose special arc is in the second family.

In May 2012, T. de Fernex gave a counterexample in dimension 3 ([3], 2012). The equation is

$$(x^2 + y^2 + z^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0. \tag{7}$$

In the algebraic setting, he can prove that the two exceptional components obtained after two blowing ups are essential. But as an analytic variety, the hypersurface obtained from (7) by blowing up the origin is locally isomorphic to the non-degenerate quadratic cone, hence it admits a small resolution; this implies that the second exceptional component is not essential, so the counterexample does not apply in the analytic category. Deforming the equation (7), de Fernex obtains a counterexample to the Nash problem in dimension 3, valid in both the algebraic and the analytic settings:

$$(x^2 + y^2)w + x^3 + y^3 + z^3 + w^5 + w^6 = 0.$$

An even more recent paper on the Nash problem is due to J. Kollár [16]. In this paper, J. Kollár gives a new family of counterexamples to the Nash problem in dimension 3, called $c\mathcal{A}_1$-type singularities:

$$x^2 + y^2 + z^2 + t^m = 0$$

with m odd, $m > 3$. These singularities are isolated and have only one Nash family but two of the exceptional components in the resolution are essential.

Moreover, Kollár formulates the Revised Nash problem, which we now explain.

Definition 4.1. *Let X be a variety over a field k, $k \subset K$ a field extension of k and $\phi : Spec\ K[[t]] \to X$ an arc such that $Supp\ \phi^{-1}(Sing(X)) = \{0\}$. A* sideways deformation *of ϕ is an extension of ϕ to a morphism $\Phi : Spec\ K[[t, s]] \to X$ such that $Supp\ \Phi^{-1}(Sing(X)) = \{(0, 0)\}$.*

Definition 4.2. *We say that X is* arcwise Nash-trivial *if every general arc in X_∞^{sing} has a sideways deformation.*

Definition 4.3. *Let X be a variety over k. A divisor over X is called* very essential *if the following holds. Let $p : Y \to X$ be a proper birational morphism such that Y is $\mathbb{Q}$-factorial and has only arcwise Nash-trivial singularities. Then $center_Y E$ is an irreducible component of $p^{-1}(Sing(X))$.*

In fact, in the three counterexamples above, the components corresponding to Nash families are given precisely by the unique very essential divisor. Imitating and conceptualising the proofs of non-essentiality appearing in the above counterexamples, one can show in full generality that divisors appearing in the image of the Nash map are always very essential. We are led to the following problem:

Problem 4.4. *Is the Nash map surjective onto the set of very essential divisors?*

Bibliography

[1] C. Bouvier, *Diviseurs essentiels, composantes essentielles des variétés toriques singulières,* Duke Math. J. **91** (1998), 609–620.

[2] C. Bouvier and G. Gonzalez-Sprinberg, *Système générateur minimal, diviseurs essentiels et G-désingularisations des variétés toriques,* Tohoku Math. J. (2) **47** (1995), 125–149.

[3] T. De Fernex, *Three-dimensional counter-examples to the Nash problem,* preprint. arXiv:1205.0603.

[4] J. Fernandez de Bobadilla, *Nash problem for surface singularities is a topological problem*, Adv. Math., **230**, iss. 1, (2012) pp. 131–176.

[5] J. Fernandez de Bobadilla and M. Pe Pereira, *The Nash problem for surfaces*, Ann. Math. **176** (2012), Issue 3, 2003–2029.

[6] J. Fernandez-Sanchez, *Equivalence of the Nash conjecture for primitive and sandwiched singularities*, Proc. Amer. Math. Soc. **133** (2005), 677–679.

[7] P. González Pérez *Toric embedded resolutions of quasi-ordinary hypersurface singularities,* Ann. Inst. Fourier (Grenoble) **53**, no. 6, (2003) 1819–1881.

[8] P. Gonzalez Perez and H. Cobo Pablos, *Arcs and jets on toric singularities and quasi-ordinary singularities*, Abstracts from the workshop held January 29–February 4, 2006. Convex and algebraic geometry. Oberwolfach Reports. Vol. **1** (2006), 302–304.

[9] P. González Pérez, *Bijectiveness of the Nash map for quasi-ordinary hypersurface singularities.* Intern. Math. Res. Notices, N°19, article ID rnm076 (2007).

[10] G. Gonzalez-Sprinberg and M. Lejeune-Jalabert, *Sur l'espace des courbes tracées sur une singularité*, Progress in Mathematics, **134** (1996), 9–32.

[11] G. Gonzalez-Sprinberg and M. Lejeune-Jalabert, *Families of Smooth Curves on Surface Singularities and Wedges*, Ann. Polon. Math., **67**, no. 2 (1997), 179–190.

[12] S. Ishii, *Arcs, valuations and the Nash map*, J. Reine Angew. Math. **588** (2005), 71–92.

[13] S. Ishii, *The local Nash problem on arc families of singularities*, Ann. Inst. Fourier, **56**, no. 4 (2006), 1207–1224.

[14] S. Ishii, *The arc space of a toric variety.* J. of Algebra **278** (2004), 666–683.

[15] S. Ishii and J. Kollár, *The Nash problem on arc families of singularities,* Duke Math. Journal **120**, no. 3 (2003), 601–620.

[16] J. Kollár, *Arc spaces of cA_1 singularities*, preprint, arXiv:1207.5036.

[17] M. Lejeune–Jalabert, *Arcs analytiques et résolution minimale des singularités des surfaces quasi-homogènes*, Séminaire sur les Singularités des Surfaces, Lecture Notes in Math. **777** (Springer-Verlag, 1980), 303–336.

[18] M. Lejeune–Jalabert, *Désingularisation explicite des surfaces quasi-homogènes dans $\mathbb{C}^3$*, Nova Acta Leopoldina, **NF 52**, Nr 240 (1981), 139–160.

[19] M. Lejeune–Jalabert, *Courbes tracées sur un germe d'hypersurface*, Amer. J. Math. **112** (1990), 525–568.

[20] M. Lejeune–Jalabert and A. Reguera, *Arcs and wedges on sandwiched surface singularities*, Amer. J. Math. **121** (1999), 1191–1213.

[21] M. Lejeune–Jalabert and A. Reguera, *Exceptional divisors which are not uniruled belong to the image of the Nash map*, Journal of the Institute of Mathematics of Jussieu, **11**, Issue02 (2012), 273–287.

[22] M. Leyton-Alvarez, *Résolution du problème des arcs de Nash pour une famille d'hypersurfaces quasi-rationnelles*, Annales de la Faculté des Sciences de Toulouse, Mathématiques, Sér. 6, **20** no. 3 (2011), 613–667.

[23] M. Leyton-Alvarez, *Une famille d'hypersurfaces quasi-rationnelles avec application de Nash bijective*, C. R., Math., Acad. Sci. Paris **349**, No. 5–6 (2011), 323–326.

[24] M. Morales, *Some numerical criteria for the Nash problem on arcs for surfaces*, Nagoya Math. J. **191** (2008), 1–19.

[25] J. F. Nash, *Arc structure of singularities,* Duke Math. J. **81** (1995), 31–38.

[26] M. Pe Pereira, *Nash Problem for quotient surface singularities*, preprint. arXiv:1011.3792.

[27] P. Petrov, *Nash problem for stable toric varieties,* Math. Nachr., **282**, iss. 11 (2009), pp. 1575–1583.

[28] C. Plénat, *A propos du problème des arcs de Nash,* Ann. Inst. Fourier (Grenoble) **55**, no. 5 (2005), 805–823.

[29] C. Plénat, *A solution to the Nash Problem for rational double points D_n (for n greater than 4)*, Annales de l'Institut Fourier, **58**, no. 6 (2008), 2249–2278.

[30] C. Plénat and P. Popescu-Pampu, *A class of non-rational surface singularities with bijective Nash map,* Bulletin de la SMF **134**, no. 3 (2006), 383–394.

[31] C. Plénat and P. Popescu-Pampu, *Families of higher dimensional germs with bijective Nash map*, Kodai Math. J. **31**, no. 2 (2008), 199–218.

[32] C. Plénat and M. Spivakovsky, *Nash Problem and the rational double point E_6*, Kodai Math. J., **35**, Number 1 (2012), pp. 173–213.

[33] A. Reguera, *Families of arcs on rational surface singularities*, Manuscripta Math **88**, 3 (1995), 321–333

[34] A. Reguera, Image of the Nash map in terms of wedges, C. R. Acad. Sci. Paris, Ser. I **338** (2004), 385–390.

[35] A. Reguera, *A curve selection lemma in space of arcs and the image of the Nash map*, Compositio Math, **142** (2006), 119–130.

[36] A. Reguera, *Arcs and wedges on rational surface singularities*, Journal of Algebra, **366** (2012), 126–164.

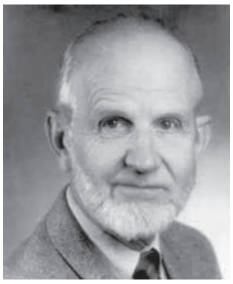
Mark Spivakovsky [spivakov@picard.ups-tlse.fr] obtained his PhD in 1985 from Harvard University under the direction of Heisuke Hironaka; his PhD dissertation deals with desingularization of surfaces by normalized Nash transformations. He was a junior fellow at Harvard from 1986 to 1989 and a professor at the University of Toronto from 1991 to 2000. Since 2000 he has held the position of Directeur de Recherche at CNRS and Institut de Mathématiques de Toulouse.

Camille Plénat [plenat@latp.univ-mrs.fr], former French student of M. Spivakovsky, obtained her PhD in 2004 on the Nash problem for D_n singularities. She has been maître de conférences at Aix Marseille Université since September 2005, in the LATP department.

A Tribute to Lars Hörmander

Nicolas Lerner (Université Pierre et Marie Curie, Paris VI, Paris, France)

Foreword



Lars Hörmander

Lars Hörmander died on 25 November 2012 at the age of 81. He was one of the most influential mathematicians of the 20th century. He played a fundamental role in the development of the analysis of partial differential equations for more than 40 years, displaying exceptional technical abilities combined with a broad and deep vision of the subject. His style of exposition was characterized by concision, precision and completeness.

He was awarded the Fields Medal in 1962, the Wolf Prize in 1988 and the Steele Prize in 2006. His monumental four-volume treatise, *The Analysis of Linear Partial Differential Operators*, is considered to be the ultimate reference on the topic of linear partial differential operators. He was a member of the Swedish Royal Academy since 1968, was elected as a member of the USA National Academy of Sciences in 1976 and served between 1987 and 1990 as a Vice-President of the International Mathematical Union.

Before the Fields Medal

Lars Hörmander was born in 1931, in southern Sweden, where his father was a teacher. He got his high school degree in 1948 and a Master's degree two years later at the age of 19 at the University of Lund, with M. Riesz as an advisor. He wrote a PhD thesis under the guidance of L. Gårding and the publication of that thesis *On the theory of general partial differential operators* [41] in *Acta Mathematematica* in 1955 can be considered as the starting point of a new era for partial differential equations.

Amongst other things, very general theorems of local existence were established, without using an analyticity hypothesis of the coefficients. L. Hörmander's arguments relied on a priori inequalities combined with abstract functional analytic arguments. Let us cite L. Gårding in [35], writing about a general linear PDE

$$P(x, D_x)u = f. \qquad (1)$$

> It was pointed out very emphatically by Hadamard that it is not natural to consider only analytic solutions and source functions f even if P has analytic coefficients. This reduces the interest of the Cauchy-Kowalevski theorem which says that (1) has locally analytic solutions if P and f are analytic. The Cauchy-Kowalevski theorem does not distinguish between classes of differential operators which have, in fact, very different properties such as the Laplace operator and the Wave operator.

L. Hörmander's filiation with J. Hadamard's work is clear. J. Hadamard (1865–1963) introduced the fruitful notion of well-posedness for a PDE problem: existence and uniqueness are important properties but, above all, continuous dependence of the solution with respect to the data should be emphasised as one of the most important properties for a PDE. After all, the data (boundary or Cauchy data, various quantities occurring in the equation) in a physics problem are known only approximately and even if the solution existed and was proven unique, this would be useless for actual computation or applications if minute changes of the data triggered huge changes in the solution. In fact, one should try to establish some *inequalities* controlling the size of the norms or semi-norms of the solution u in some functional space. The lack of well-posedness is linked to instability and is also a very interesting phenomenon to study. We can quote again at this point L. Gårding (op. cit.):

> When a problem about partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable inequality and much of our knowledge is, in fact, essentially contained in such inequalities.

L. Ehrenpreis [32] and B. Malgrange [93] had proven a general theorem on the existence of a fundamental solution for any constant coefficients PDE, and the work [42] by L. Hörmander provided another proof along with some improvement on the regularity properties, whereas [41] gave a characterization of hypoelliptic constant coefficients PDE, in terms of properties of the algebraic variety

$$\operatorname{char} P = \{\zeta \in \mathbb{C}^n, P(\zeta) = 0\}.$$

The operator $P(D)$ is hypoelliptic if and only if

$$|\zeta| \to \infty \text{ on } \operatorname{char} P \Longrightarrow |\operatorname{Im} \zeta| \to \infty.$$

Here hypoellipticity means $Pu \in C^\infty \Longrightarrow u \in C^\infty$. The characterization of hypoellipticity of the constant coefficient operator $P(D)$ by a simple algebraic property of the characteristic set is a tour de force, technically and conceptually: in the first place, nobody had conjectured such a result or even remotely suggested a link between the two properties, and next, the proof provided by L. Hörmander relies on a very subtle study of the characteristic set, requiring an extensive knowledge of real algebraic geometry.

In 1957, Hans Lewy made a stunning discovery [92]: the equation $\mathcal{L}u = f$ with

$$\mathcal{L} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3} \qquad (2)$$

does not have local solutions for most right-hand-sides f. The surprise came in particular from the fact that the operator $\mathcal{L}$ is a non-singular (i.e. non-vanishing) vector field with a very simple expression and also, as the Cauchy-Riemann operator on the boundary of a pseudo-convex domain, it is not a cooked-up example. L. Hörmander started working on the Lewy operator (2) with the goal of getting a general geometric understanding of a class of operators displaying the same defect of local solvability. The two papers [46], [45] published in 1960 achieved that goal. Taking a complex-valued homogeneous symbol $p(x, \xi)$, the existence of a point (x, ξ) in the cotangent bundle such that

$$p(x, \xi) = 0, \quad \{\bar{p}, p\}(x, \xi) \neq 0 \qquad (3)$$

ruins local solvability at x (here $\{\cdot,\cdot\}$ stands for the Poisson bracket). With this result, L. Hörmander gave a generalisation of the Lewy operator and provided a geometric explanation in invariant terms of that non-solvability phenomenon. We may note also that Condition (3) is somehow generically satisfied: considering a non-elliptic operator with a complex-valued principal symbol p, the symbol p will vanish somewhere and generically $\{\bar{p}, p\} \neq 0$ there, so that "most" non-elliptic operators with a complex-valued symbol are non-solvable.

A. Calderón's 1958 paper [21] on the uniqueness in the Cauchy problem was somehow the starting point for the renewal of singular integrals methods in local analysis. Calderón proved in [21] that an operator with real principal symbol with simple characteristics has the Cauchy uniqueness property; his method relied on a pseudodifferential factorisation of the operator which can be handled thanks to the simple characteristic assumption. It appears somewhat paradoxical that L. Hörmander, who later became one of the architects of pseudodifferential analysis, found a generalisation of Calderón's paper using only a local method, inventing a new notion to prove a Carleman estimate. He introduced in [44], [43] the notion of pseudo-convexity of a hypersurface with respect to an operator and was able to handle the case of tangent characteristics of order two.

In 1957, L. Hörmander was appointed as a professor at the University of Stockholm, where he was to stay until 1964, but he also spent some time in Stanford University as well as at the Institute for Advanced Study in Princeton.

In 1962, at the age of 31, L. Hörmander was awarded the Fields Medal. His impressive work on partial differential equations, in particular his characterization of hypoellipticity for constant coefficients and his geometrical explanation of the Lewy non-solvability phenomenon, were certainly very strong arguments for awarding him the medal. Also, his new point of view on PDE, which combined functional analysis with a priori inequalities, had led to very general results on large classes of equations, which had been out of reach in the early '50s. L. Hörmander wrote in the book *Fields Medallists' lectures* [6]:

> The 1962 ICM was held in Stockholm. In view of the small number of professors in Sweden at the time, it was inevitable that I should be rather heavily involved in the preparations, but it came as a complete surprise to me when I was informed that I would receive one of the Fields medals at the congress.

From the first PDE book to the four-volume treatise

L. Hörmander spent the summers 1960–61 at Stanford University as an invited professor, and took advantage of this time to honour the offer of the *Springer Grundlehren series* of publishing a book about PDE. It was done in 1963 with the publication of his first book, *Linear partial differential operators.* That book was a milestone in the study of PDE and a large portion of the mathematical public discovered L. Hörmander's exposition of recent progress in the area.

In the first place, the role of distribution theory was emphasised as the perfect tool for linear PDE. Although the notion of a weak solution for a PDE was already known to S. Sobolev and to the Russian school in the '30s, it is indeed L. Schwartz' definition of distributions which created the best perspective, combining abstract aspects of functional analysis with Fourier analysis. L. Hörmander had been familiar for quite a long time with Schwartz theory but he had noticed that many mathematicians, including his mentor M. Riesz, were rather negative (to say the least) about it. F. Treves in [121] tells the following anecdote: L. Schwartz visited Lund University in 1948 and gave a talk there on some elements of distribution theory. Having written on the board the integration by parts formula to explain the idea of a weak derivative, he was interrupted by M. Riesz saying "I hope you have found something else in your life". Also, M. Riesz claimed that the known examples of fundamental solutions of hypoelliptic PDE with constant coefficients were always locally integrable, so that distributions were useless for their study. A few years after his retirement, Hörmander went back to this problem and found an example in 14 dimensions of an hypoelliptic operator whose fundamental solution is not locally integrable (see [79]). During his thesis work, L. Hörmander managed to avoid explicit reference to Schwartz theory but in 1963 it was a different story and he chose to present Schwartz Distribution Theory as the basic functional analytic framework of his book. As a matter of fact, the first chapter of his book is devoted to a (dense) presentation of the theory, including the geometric version on manifolds.

Large segments of the book were devoted to constant coefficient operators but it also contained a great deal of the recent progress on uniqueness for the Cauchy problem, with Carleman estimates and a wide array of counterexamples due to A. Pliś and P. Cohen. Anyhow, the book soon became a standard text to be studied by anybody wishing to enter the PDE field.

L. Hörmander's career developed at a quick pace after the Fields Medal; he wrote in [75]:

> Some time after two summers (1960, 1961) at Stanford, I received an offer of a part time appointment as professor at Stanford University … I had barely arrived at Stanford when I received an offer to come to the Institute for Advanced Study as permanent member and professor. Although I had previously been determined not to leave Sweden, the opportunity to do research full time in a mathematically very active environment was hard to resist … I decided in the fall of 1963 to accept the offer from the IAS and resign from the universities of Stockholm and Stanford to take up a new position in Princeton in the fall of 1964.

Hypoellipticity

A. Kolmogorov introduced in 1934 the operator in $\mathbb{R}^3_{t,x,v}$

$$\mathcal{K} = \partial_t + v\partial_x - \partial_v^2, \tag{4}$$

to provide a model for Brownian motion in one dimension. That was L. Hörmander's starting point. He took up the study of general operators

$$\mathcal{H} = X_0 - \sum_{1\le j\le r} X_j^2, \tag{5}$$

where the $(X_j)_{0\le j\le r}$ are smooth real vector fields whose Lie algebra generates the tangent space at each point. The rank of the X_j and their iterated Poisson brackets is equal to the dimension of the ambient space (for $\mathcal{K}$, we have $X_0 = \partial_t + v\partial_x, X_1 = \partial_v, [X_1, X_0] = \partial_x$). These operators were proven in [53] to be hypoelliptic, i.e. such that $\operatorname{singsupp} u =$

singsupp$\mathcal{H}u$ for the C^∞ singular support. This Hörmander paper was the starting point of many studies, including numerous articles in probability theory, and the operators $\mathcal{H}$ soon became known as *Hörmander's sum of squares*. Their importance in probability came from the fact that these operators appeared as a generalisation of the heat equation where the diffusion term $\sum_{1\le j\le r} X_j^2$ was no longer elliptic but had, instead, some hypoelliptic behaviour.

Pseudodifferential Equations

The aforementioned article by A. Calderón on uniqueness for the Cauchy problem led to renewed interest in singular integrals and the notion of pseudodifferential operator along with a symbolic calculus was introduced in the '60s by several authors: J. J. Kohn and L. Nirenberg in [87] and A. Unterberger and J. Bokobza in [123]. L. Hörmander wrote in 1965 a synthetic account of the nascent pseudodifferential methods with the article [50].

Complex analysis

The now classical book *An introduction to complex analysis in several variables* [51] and the paper [49] provide a PDE point of view on the holomorphic functions of several variables: they are considered as solutions of a PDE, the $\overline{\partial}$ system, and that perspective along with L^2 estimates turned out to be very fruitful for their study. Here is an excerpt from the preface of the book:

> Two recent developments in the theory of partial differential equations have caused this book to be written. One is the theory of overdetermined systems of differential equations with constant coefficients, which depends very heavily on the theory of functions of several complex variables. The other is the solution of the so-called $\overline{\partial}$ Neumann problem, which has made possible a new approach to complex analysis through methods from the theory of partial differential equations. Solving the Cousin problems with such methods gives automatically certain bounds for the solution, which are not easily obtained with the classical methods, and results of this type are important for the applications to overdetermined systems of differential equations.

Inhomogeneous Cauchy-Riemann equations in a polydisc, power series, Reinhardt domains, domains of holomorphy, pseudo-convexity and plurisubharmonicity, and Runge domains are dealt with in the second chapter. Included are theorems due to Hartogs, Dolbeault-Grothendieck, Cartan [22], Cartan-Thullen [23], Bochner [10], Lewy [91], Oka [110], Serre [113] and Browder [19]. After a chapter on commutative Banach algebras, Chapter IV is devoted to existence and approximation theorems for solutions of the inhomogeneous Cauchy-Riemann equations in domains of holomorphy. The technique is to prove L^2 estimates involving weight functions. Next, L. Hörmander introduces the notion of Stein manifolds, which are modelled on the properties of domains of holomorphy in $\mathbb{C}^n$. The theorems on existence and approximations of solutions of the Cauchy-Riemann equations are extended to these manifolds and it is shown that a manifold is a Stein manifold if and only if it can be represented concretely as a closed submanifold of a space $\mathbb{C}^N$ of sufficiently high dimension. Analytic continuation and the Cousin problems are studied for Stein manifolds. These results are due to Cartan, Grauert, Bishop, Narasimhan and Oka. Chapter VI gives the Weierstrass preparation theorem and studies divisibility properties in the ring A_0 of germs of analytic functions. Submodules of A_0^p are studied along with K. Oka's theorem on the module of relations [109]. This is needed for the theory of coherent analytic sheaves, which is presented in the next and final chapter. There the study of the Cousin problems is extended to coherent analytic sheaves on Stein manifolds. A discussion of the theorem of Siu [114] on the Lelong numbers of plurisubharmonic functions is added. The L^2 techniques are essential in the proof and plurisubharmonic functions play such an important role that it is natural to discuss their main singularities.

Spectral Asymptotics

The article [54], which contains the first occurrence of Fourier Integral Operators, provides the best possible estimates for the remainder term in the asymptotic formula for the spectral function of an arbitrary elliptic (pseudo)differential operator. This is achieved by means of a complete description of the singularities of the Fourier transform of the spectral function for low frequencies.

In spite of this outstanding activity, L. Hörmander did not feel that comfortable at the IAS:

> It turned out that I found it hard to stand the demands on excellence that inevitably accompany the privilege of being an Institute professor. After two years of very hard work I felt that my results were not up to the level which could be expected. Doubting that I would be able to stand a lifetime of pressure, I started to toy with the idea of returning to Sweden when a regular professorship became vacant. An opportunity arose in 1967, and I decided to take it and return as professor in Lund from the fall term 1968.

So, in 1968, L. Hörmander had gone full circle and was back in Lund where he had started as an undergraduate in 1948. He was to remain there until his retirement, with interruptions for some visits, mainly in the US.

The microlocal revolution

The fact that singularities should be classified according to their spectrum was first recognised in the early '70s by three Japanese mathematicians: the Lecture Notes [112] by M. Sato, T. Kawai and M. Kashiwara set the basis for the analysis in the phase space and microlocalisation. The analytic wave-front-set was defined in algebraic terms and elliptic regularity as well as propagation theorems were proven in the analytic category. The paper [18] by J. Bros and D. Iagolnitzer gave a formulation of the analytic wave-front-set that was more friendly to analysts.

The definition of the C^∞ wave-front-set was given in Hörmander's [56] by means of pseudodifferential operators. The propagation-of-singularities theorem for real principal type operators (see, e.g., Hörmander's [57])certainly represents the apex of microlocal analysis. Since the 17th century with the works of Huygens and Newton, the mathematical formulation for propagation of linear waves lacked correct definitions. The wave-front-set provided the ideal framework: for P a real principal type operator with smooth coefficients (e.g., the wave equation) and u a function such that $Pu \in C^\infty$, WFu is invariant by the flow of the Hamiltonian vector field of the principal symbol of P. These results found new proofs via Hörmander's articles on Fourier Integral Operators [55] and

[28] (joint work with J. Duistermaat). It is interesting to quote at this point the introduction of [55] (the reference numbers are those of our reference list):

> The work of Egorov is actually an application of ideas from Maslov [94] who stated at the International Congress in Nice that his book actually contains the ideas attributed here to Egorov [30] and Arnold [4] as well as a more general and precise operator calculus than ours. Since the book is highly inaccessible and does not appear to be quite rigorous we can only pass this information on to the reader, adding a reference to the explanations of Maslov's work given by Buslaev [20]. In this context we should also mention that the "Maslov index" which plays an essential role in Chapters III and IV was already considered quite explicitly by J. Keller [85]. It expresses the classical observation in geometrical optics that a phase shift of $\pi/2$ takes place at a caustic. The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential operators. In fact, we only use the simplest expansions which occur in geometrical optics, and a wealth of other ideas remain to be investigated.

The introduction of the next article [28] begins with

> The purpose of this paper is to give applications of the operator theory developed in the first part. These concern the existence and regularity of solutions of
>
> $$Pu = f$$
>
> in a manifold X. In particular we construct and study parametrices for P; we consider the above equation under the assumption that P has a principal symbol p which is homogeneous of degree m and real.

Local Solvability

After Lewy's counterexample (2) and L. Hörmander's work on local solvability mentioned above, L. Nirenberg and F. Treves in 1970 ([104], [105], [106]), after a study of complex vector fields in [103] (see also the S. Mizohata paper [100]), introduced the so-called condition (Ψ) and provided strong arguments suggesting that this geometric condition should be equivalent to local solvability. The necessity of condition (Ψ) for local solvability of principal-type pseudodifferential equations was proved in two dimensions by R. Moyer in [101] and in general by L. Hörmander ([65]) in 1981.

The sufficiency of condition (Ψ) for local solvability of differential equations was proved by R. Beals and C. Fefferman ([9]) in 1973. They created a new type of pseudodifferential calculus, based on a Calderón-Zygmund decomposition, and were able to remove the analyticity assumption required by L. Nirenberg and F. Treves. The sufficiency of that geometric condition was proven in 1988 in two dimensions by N. Lerner's [88]. Much later, in 1994, L. Hörmander, in his survey article [74], went back to local solvability questions giving a generalisation of N. L.'s article [89]. In 2006, N. Dencker [26] proved that condition (Ψ) implies local solvability with loss of two derivatives.

More on pseudodifferential calculus

The outstanding results by R. Beals and C. Fefferman [9] on local solvability of differential equations were supplemented by L. Hörmander's paper [62] in which a propagation argument provides local existence of C^∞ solutions for C^∞ right-hand-sides. However, a most striking fact in R. Beals and C. Fefferman's proof was the essential use of a non-homogeneous pseudodifferential calculus which allowed a finer microlocalisation than what could be given by conic microlocalisation. The efficiency and refinement of the pseudodifferential machinery was such that the very structure of this tool attracted the attention of several mathematicians, among them R. Beals and C. Fefferman [8], R. Beals [7] and A. Unterberger [122]. L. Hörmander's 1979 paper [64], *The Weyl calculus of pseudodifferential operators*, represents an excellent synthesis of the main requirements for a pseudodifferential calculus to satisfy; that article was used by many authors in multiple circumstances and the combination of the symplectically invariant Weyl quantisation along with the datum of a metric on the phase space was proven to be a very efficient approach.

Writing the four-volume book, 1979–1984

On 25 March 1982, L. Hörmander received a *Doctorate Honoris Causa* from the Université Paris-sud at Orsay. The main scientific address was written by J.-M. Bony and J. Sjöstrand. The whole PDE community in Orsay and elsewhere was waiting for Hörmander's forthcoming book to appear in the *Springer Grundlehren series*. Three or four volumes, joint work or not, table of contents ... nothing was clear-cut at this moment and the expectations were high that the book would represent a landmark in the history of PDE. The first two volumes appeared in 1983.

First volume: Distribution Theory and Fourier Analysis

It is now a classical book of analysis and an excellent presentation of distribution theory. In particular, that introduction remains elementary and free from very abstract functional analytic arguments. In the notes of Chapter II, L. Hörmander writes:

> The topology in $C_0^\infty(X)$ is the inductive limit of the topology in $C_0^\infty(K)$ when the compact set K increases to X, so it is a $\mathscr{LF}$ topology. We have avoided this terminology in order not to encourage the once current misconception that familiarity with $\mathscr{LF}$ space is essential for the understanding of distribution theory.

As a result, this first volume is highly readable and represents a useful tool for teaching various elements of distribution theory. The organisation of the whole treatise is also quite impressive. For instance, Chapter I in this first volume contains a quite refined notion of partitions of unity, not to be used before Chapter XVIII in the third volume. Several mathematical gems can be found in this first volume: a new proof of the Schwartz kernel theorem in Chapter V, a proof of the Malgrange preparation theorem and an extensive study of the methods of stationary phase in Chapter VII. Self-containedness is also perfect: the very classical Gaussian integrals get computed explicitly and the three-page treatment of the Airy function in Chapter 7 is a model of concision and clarity.

Second volume: Differential Operators with Constant Coefficients

L. Hörmander writes in the preface to this volume:

> This volume is an expanded version of Chapters III, IV, V and VII of my 1963 book … The main technical tool in this volume is the Fourier-Laplace transformation. More powerful methods for the study of operators with variable coefficients will be developed in Volume III. However, the constant coefficient theory has given the guidelines for all that work. Although the field is no longer very active – perhaps because of its advanced state of development – … the material presented here should not be neglected by the serious student who wants to get a balanced perspective of the theory …

The third and fourth volumes appeared two years later in 1985. L. Hörmander writes in the preface to these volumes:

> The first two volumes of this monograph can be regarded as an expansion of my book … published in the Grundlehren series in 1963. However, volumes III and IV are almost entirely new. In fact they are mainly devoted to the theory of linear differential operators as it has developed after 1963. Thus the main topics are pseudodifferential and Fourier integral operators with the underlying symplectic geometry.

Here the style of writing has drastically changed: these last two volumes are no longer intended for gifted graduate students; the targeted readership is obviously researchers already conversant with some technicalities of the subject.

Third volume: Pseudodifferential operators

Chapter XVII may be an exception to the above remark – although the technique of Carleman estimates is far from easy, the content of that chapter remains elementary as far as tools are concerned.

Chapter XVIII is concerned with pseudodifferential calculus: the 30-page presentation *Basic Calculus* is certainly an excellent introduction to the topic and L. Hörmander was cautious enough to give a separated treatment of the most classical case of pseudodifferential calculus, leaving aside the refinements for later sections in the same chapter. R. Melrose's totally characteristic calculus ([97]) and L. Boutet de Monvel's transmission condition ([15]) are given a detailed treatment in this chapter. The last sections are devoted to Weyl calculus as described in L. Hörmander [64] and results on new lower bounds by C. Fefferman and D. H. Phong [34] are also given a thorough treatment.

Chapter XIX deals with elliptic operators on a manifold without boundary and the index theorem. In the Notes of Chapter XVIII, L. Hörmander writes:

> It seems likely that it was the solution by Atiyah and Singer [5] of the index problem for elliptic operators which led to the revitalization of the theory of singular integral operators.

Chapter XX is entitled *Boundary Problems for Elliptic Differential Operators*. It reproduces at the beginning elements of Chapter X in [48] and takes into account the developments on the index problem for elliptic boundary problems given by L. Boutet de Monvel [15], [14] and G. Grubb [36].

Chapter XXI is a presentation of symplectic geometry and begins with a series of classical results. Next, one finds various sharp results on normal forms of smooth functions in a symplectic space, in particular the results of J. Duistermaat and J. Sjöstrand [29]. Also, this chapter is an important preparation for local solvability results of Chapter XXVI with the normal form given in the paper by L. Nirenberg and F. Treves [105]. Section 21.5 is devoted to the symplectic reduction of complex-valued quadratic forms and remains an excellent reference on the topic.

Chapter XXII is concerned with hypoelliptic operators: on the one hand, operators with a pseudodifferential parametrix, such as the hypoelliptic constant coefficient operators, and on the other hand generalisations of the Kolmogorov operators (5). Results on lower bounds for pseudodifferential operators due to A. Melin [95] are a key tool in this analysis. Results of L. Boutet de Monvel [16], J. Sjöstrand [115], L. Boutet de Monvel, A. Grigis and B. Helffer [17] are given.

Chapter XXIII deals with the classical topic of strictly hyperbolic equations and begins with the exposition of the classical energy method. The classical estimates are obtained for first order pseudodifferential operators and then a factorisation argument allows one to deal with higher order operators. Also, a version of the Lax-Mizohata theorem is given, which asserts the necessity of weak hyperbolicity for a weak version of well-posedness, following the work by V. Ivrii and V. Petkov [84].

The last chapter in volume 3 is Chapter XXIV, which is devoted to the mixed Dirichlet-Cauchy problem for second order operators. Singularities of solutions of the Dirichlet problem arriving at the boundary on a transversal bicharacteristic will leave again on the reflected bicharacteristic. The study of tangential bicharacteristics required a new analysis and attracted the attention of many mathematicians. Among these works are the papers by R. Melrose [96], M. Taylor [118], G. Eskin [33], V. Ivrii [83], R. Melrose and J. Sjöstrand [98], [99], K. Andersson and R. Melrose [3], J. Ralston [111] and J. Sjöstrand [116].

Volume 3 should not be left without paying attention to the two appendices, providing a self-contained description of classical results on distributions in an open manifold, as well as the exposition of some tools of differential geometry.

Fourth volume, Fourier integral operators

Chapter XXV is devoted to the theory of Fourier integral operators, including the case of complex phase. Although the propagation-of-singularities theorem for real principal type operators is already proven by pseudodifferential methods in a previous chapter (XXIII), the FIO method provides another proof.

Chapter XXVI deals with principal type operators. The real principal type case appears now as quite simple and the second section drives us into the much more complicated realm of complex-valued symbols. The necessity of condition (Ψ) for local solvability, taken from the already mentioned [65] and [101] is proven in Section 26.4. The last seven sections of this chapter are devoted to very precise propagation theorems for operators with complex symbols satisfying the stronger condition (P). The main ingredients used in the proof are the Malgrange preparation theorem, Egorov's theorem on conjugation of pseudodifferential operators by Fourier integral operators, Nirenberg-Treves estimates on degenerate Cauchy-Riemann equations [105], Beals-Fefferman non-homogeneous localization procedure [9] and Hörmander's propagation result [62].

Chapter XXVII is concerned with subelliptic operators. A pseudodifferential operator of order m is said to be subelliptic with a loss of δ derivatives whenever

$$Pu \in H^s_{loc} \Longrightarrow u \in H^{s+m-\delta}_{loc}. \qquad (6)$$

The elliptic case corresponds to $\delta = 0$, whereas the cases $\delta \in (0,1)$ are much more complicated to handle. The first complete proof for operators satisfying condition (P) was given by F. Treves in [120], using a coherent states method, and that proof is given in Section 27.3. Although it is far from an elementary proof, the simplifications allowed by condition (P) permit a rather compact exposition. The last three sections of that chapter are devoted to the much more involved case of subelliptic operators satisfying condition (Ψ) and one could say that the proof is extremely complicated. Let us cite L. Hörmander in [77]:

> For the scalar case, Egorov [31] found necessary and sufficient conditions for subellipticity with loss of δ derivatives ($\delta \in [0,1)$); the proof of sufficiency was completed in [63]. The results prove that the best δ is always of the form $k/(k+1)$ where k is a positive integer… A slight modification of the presentation of [63] is given in Chapter 27 of [70], but it is still very complicated technically. Another approach which covers also systems operating on scalars has been given by Nourrigat [107, 108] (see also the book [40] by Helffer and Nourrigat), but it is also far from simple so the study of subelliptic operators may not yet be in a final form.

Chapter XXVIII is entitled *Uniqueness for the Cauchy problem*. It appears as a natural sequel to Chapter VIII in the first book [48]. The Calderón uniqueness result along with uniqueness under a pseudoconvexity condition are given and the notion of principal normality is enlarged, using the Fefferman-Phong inequality [34]. However, pseudodifferential methods are greedy with derivatives so that the aforementioned chapter in [48] is not entirely included in this chapter. The last section of this chapter is devoted to a result on second order operators of real principal type essentially due to N. Lerner and L. Robbiano [90].

Chapter XXIX is entitled *Spectral Asymptotics*. This chapter is devoted to the asymptotic properties of the eigenvalues and the spectral function for self-adjoint elliptic operators. If P is a positive operator of order m, $P^{1/m}$ is a pseudodifferential operator with eigenvalue λ equal to those of P for λ^m. The corresponding unitary group $e^{itP^{1/m}}$ can be viewed as a Fourier integral operator. Here also L. Hörmander presents an excellent synthesis of many works on this topic: J. Chazarain [24], J. Duistermaat and V. Guillemin [27], V. Ivrii [82], V. Guillemin [37, 39, 38], Y. Colin de Verdière [25] and A. Weinstein [124].

The very last chapter is the 30th, *Long Range Scattering Theory*. It is devoted to the study of operators of type $P_0(D) + V(x,D)$ where P_0 is elliptic of order m and V is of order m so that $P_0(D) + V(x,D)$ is also elliptic and

$$V(x,\xi) = V_S(x,\xi) + V_L(x,\xi),$$

where the short range part V_S has coefficients decreasing as fast as an integrable function of $|x|$ and V_L satisfies some estimates similar to those satisfied by $(1+|\xi|)^m(1+|x|)^{-\varepsilon}$ for some $\varepsilon > 0$. Here also L. Hörmander gives an excellent synthesis of his work along with the works of many mathematicians, among them S. Agmon [1].

There is certainly no better conclusion to the review of this treatise than the citation for the 2006 Leroy P. Steele Prize, awarded to Lars Hörmander for mathematical exposition:

> In these four volumes, Hörmander describes the developments [of microlocal analysis] in a treatment that is seamless and self-contained. Moreover, the effort to make this treatment self-contained has inspired him to recast, in much more simple and accessible form, the approach to much of this material as it originally appeared in the literature. An example is the theory of Fourier integral operators, which was invented by him in two seminal papers in the early 1970s. (These get a completely new and much more elegant reworking in volume four.) In brief, these four volumes are far more than a compendium of random results. They are a profound and masterful rethinking of the whole subject of microlocal analysis. Hörmander's four volumes on partial differential operators have influenced a whole generation of mathematicians working in the broad area of microlocal analysis and its applications. In the history of mathematics one is hard-pressed to find any comparable "expository" work that covers so much material, and with such depth and understanding, of such a broad area of mathematics.

Intermission Mittag-Leffler 1984–1986 and back to Lund 1986

L. Hörmander spent the academic years 1984–86 as Director of the Mittag–Leffler Institute in Stockholm. He wrote about this:

> I had only accepted a two year appointment with a leave of absence from Lund since I suspected that the many administrative duties there would not agree very well with me. The hunch was right…

L. Hörmander was back at the university of Lund in the autumn of 1986.

Nonlinear hyperbolic equations

During three semesters in 1986–87, Hörmander gave some lectures on global existence or blowup for nonlinear hyperbolic equations. Ten years later, in 1996, the book *Lectures on Nonlinear Hyperbolic Differential Equations* [76] appeared in the Springer series *Mathématiques & Applications.*

Some classical topics on scalar first order equations are covered and revisited in the first chapters of the book. Chapter 5 concerns compensated compactness. The main tool is Young measures associated to an L^∞ bounded sequence of functions. The author uses them to prove "compensated compactness" theorems, generalising the "Murat-Tartar div-curl lemma" [102], [117]. Applications of these ideas to scalar or two-by-two systems are included.

The rest of the book is devoted entirely to nonlinear problems in several space variables. The first subject which is treated is the problem of long-time existence of small solutions for nonlinear wave or Klein-Gordon equations. L. Hörmander uses the original method of S. Klainerman [86]. It relies on a weighted L^∞ Sobolev estimate for a smooth function in terms of L^2 norms of $Z^I u$, where Z^I stands for an iterate of homogeneous vector fields tangent to the wave cone. The chapter closes with a proof of global existence in three space dimensions, when the nonlinearity satisfies the so-called "null condition", i.e. a compatibility relation between the nonlinear terms and the wave operator.

The last part of the book is concerned with the use of microlocal analysis in the study of nonlinear equations. Chapter 9 is devoted to the study of pseudodifferential operators lying in the "bad class" $S^0_{1,1}$ (such operators are not bounded on L^2). The starting point for the study of this class is due to G. Bourdaud [13], followed by [71]. L. Hörmander proves that a necessary and sufficient condition for such an operator $a(x, D)$ to be bounded on L^2 is that the partial Fourier transform of its symbol $\hat{a}(\xi, \eta)$ satisfies a convenient vanishing property along the diagonal $\xi + \eta = 0$. These operators form a subclass of $S^0_{1,1}$ for which he discusses composition, adjoints, microlocal ellipticity and Gårding's inequality. The results of Chapter 9 are applied in Chapter 10 to construct Bony's paradifferential calculus [11, 12]. One associates to a symbol $a(x, \xi)$, with limited regularity in x, a paradifferential operator and proves the basic theorems on symbolic calculus, as well as "Bony's paraproduct formula". Next, Bony's paralinearisation theorem is discussed: it asserts that if F is a smooth function and u belongs to $C^\rho(\rho > 0)$, $F(u)$ may be written as $Pu + Ru$, where P is a paradifferential operator with symbol $F'(u)$ and R is a ρ-regularising operator. This is used to prove microlocal elliptic regularity for solutions to nonlinear differential equations. The last chapter is devoted to propagation of microlocal singularities. After discussing propagation of singularities for solutions of linear pseudodifferential equations with symbols in the classes defined in Chapter 9, the author proves Bony's theorem on propagation of weak singularities for solutions to nonlinear equations. The proof relies on a reduction to a linear paradifferential equation, using the results of the preceding chapter.

Notions of convexity

L. Hörmander wrote in 1994 another book entitled *Notions of convexity* [73], published in the Birkhaüser Series *Progress in Mathematics.* The main goal of the book is to expose part of the thesis of J.-M. Trépreau [119] on the sufficiency of condition (Ψ) for local solvability in the analytic category. For microdifferential operators acting on microfunctions, the necessity of condition (Ψ) for microlocal solvability was proven by M. Sato, T. Kawai and M. Kashiwara in [112]. However the book's content clearly indicates a long approach to J.-M. Trépreau's result; the reader is invited first to a pleasant journey in the landscape of convexity and the first chapters of the book are elementary.

Students

L. Hörmander had the following PhD students:

· Germund Dahlquist, at Stockholm University, in 1958,
· Vidar Thomée, at Stockholm University, in 1959,
· Christer Kiselman, at Stockholm University, in 1966,
· Göran Björck, at Stockholm University, in 1966,
· Jan Boman, at Stockholm University, in 1967,
· Johannes Sjöstrand, at Lund University, in 1972,
· Anders Melin, at Lund University, in 1973,
· Lars Nysted, at Stockholm University, in 1973,
· Arne Enqvist, at Lund University, in 1974,
· Gudrun Gudmundsdottir, at Lund University, in 1975,
· Anders Källén, at Lund University, in 1979,
· Nils Dencker, at Lund University, in 1981,
· Ragnar Sigurdsson, at Lund University, in 1984,
· Hans Lindblad, at Lund University, in 1989,
· Pelle Pettersson at Lund University, in 1994.

Retirement in 1996

L. Hörmander retired in 1996 and became an emeritus professor. He was still very active, publishing about two or three research papers every year. His enthusiasm and interest for mathematics remained at a high level until the very end of his life.

Final comments

After this not-so-short review of Lars Hörmander's works, we see in the first place that he was instrumental in the mathematical setting of Fourier Integral Operators, (achieved in part with J. Duistermaat) and also in the elaboration of a comprehensive theory of pseudodifferential operators. Fourier Integral Operators had a long heuristic tradition, linked to quantum mechanics, but their mathematical theory is indeed a major lasting contribution of Lars Hörmander. He was also the first to study the now called *Hörmander's sum of squares* of vector fields and their hypoellipticity properties. These operators are important in probability theory and geometry but also gained renewed interest in the recent studies of regularisation properties for Boltzmann's equation and other nonlinear equations. Hörmander also played an essential role in the completion of the theory of subelliptic operators and there is no doubt that, without his relentless energy and talent, the clarification of this part of the theory would have taken many more years.

Lars Hörmander was also a great mathematical writer and a man of synthesis. The eight books published by Hörmander are reference books, all with a very personal perspective. Broad, dense and deep, these books are essentially self-contained and bring the reader up to state-of-the-art in several mathematical domains. The four-volume treatise on linear PDE, the book on several complex variables and the volume on non-linear hyperbolic equations are here to stay as outstanding contributions to mathematics.

Acknowledgements. The author is grateful to J.-M. Bony and J. Sjöstrand for several helpful comments on earlier versions of this manuscript.

Bibliography

[1] S. Agmon, *Some new results in spectral and scattering theory of differential operators on* $\mathbf{R}^n$, Séminaire Goulaouic-Schwartz (1978/1979), École Polytech., Palaiseau, 1979, pp. Exp. No. 2, 11. MR 557513 (81j:35091)
[2] S. Agmon and L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse Math. **30** (1976), 1–38. MR 0466902 (57 #6776)
[3] K. G. Andersson and R. B. Melrose, *The propagation of singularities along gliding rays*, Invent. Math. **41** (1977), no. 3, 197–232. MR 0494322 (58 #13221)
[4] V. I. Arnol'd, *On a characteristic class entering into conditions of quantization*, Funkcional. Anal. i Priložen. **1** (1967), 1–14. MR 0211415 (35 #2296)

[5] M. F. Atiyah and I. M. Singer, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. **69** (1963), 422–433. MR 0157392 (28 #626)

[6] Michael Atiyah and Daniel Iagolnitzer (eds.), *Fields Medallists' lectures*, World Scientific Series in 20th Century Mathematics, vol. 5, World Scientific Publishing Co. Inc., River Edge, NJ, 1997. MR 1622945 (99b:00010)

[7] R. Beals, *A general calculus of pseudodifferential operators*, Duke Math. J. **42** (1975), 1–42. MR 0367730 (51 #3972)

[8] R. Beals and C. Fefferman, *Classes of spatially inhomogeneous pseudodifferential operators*, Proc. Nat. Acad. Sci. U.S.A. **70** (1973), 1500–1501. MR 0338840 (49 #3604)

[9] ———, *On local solvability of linear partial differential equations*, Ann. of Math. (2) **97** (1973), 482–498. MR MR0352746 (50 #5233)

[10] Salomon Bochner and William Ted Martin, *Several Complex Variables*, Princeton Mathematical Series, vol. 10, Princeton University Press, Princeton, N. J., 1948. MR 0027863 (10,366a)

[11] Jean-Michel Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 2, 209–246. MR 631751 (84h:35177)

[12] ———, *Analyse microlocale des équations aux dérivées partielles non linéaires*, Microlocal analysis and applications (Montecatini Terme, 1989), Lecture Notes in Math., vol. 1495, Springer, Berlin, 1991, pp. 1–45. MR 1178555 (93k:35282)

[13] Gérard Bourdaud, *Une algèbre maximale d'opérateurs pseudo-différentiels*, Comm. Partial Differential Equations **13** (1988), no. 9, 1059–1083. MR 946282 (89g:47063)

[14] Louis Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord*, C. R. Acad. Sci. Paris **261** (1965), 4587–4589. MR 0188607 (32 #6043)

[15] ———, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord. I. La propriété de transmission*, J. Analyse Math. **17** (1966), 241–253. MR 0239254 (39 #611)

[16] ———, *Hypoelliptic operators with double characteristics and related pseudo-differential operators*, Comm. Pure Appl. Math. **27** (1974), 585–639. MR 0370271 (51 #6498)

[17] Louis Boutet de Monvel, Alain Grigis, and Bernard Helffer, *Parametrixes d'opérateurs pseudo-différentiels à caractéristiques multiples*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Soc. Math. France, Paris, 1976, pp. 93–121. Astérisque, No. 34–35. MR 0493005 (58 #12046)

[18] J. Bros and D. Iagolnitzer, *Tuboïdes et structure analytique des distributions. II. Support essentiel et structure analytique des distributions*, Séminaire Goulaouic-Lions-Schwartz 1974–1975: Équations aux dérivées partielles linéaires et non linéaires, Exp. No. 18, Centre Math., École Polytech., Paris, 1975, p. 34. MR 0399494 (53 #3338)

[19] Andrew Browder, *Cohomology of maximal ideal spaces*, Bull. Amer. Math. Soc. **67** (1961), 515–516. MR 0130580 (24 #A440)

[20] V. S. Buslaev, *The generating integral and the Maslov canonical operator in the WKB method*, Funkcional. Anal. i Priložen. **3** (1969), no. 3, 17–31. MR 0467854 (57 #7705)

[21] A.-P. Calderón, *Uniqueness in the Cauchy problem for partial differential equations.*, Amer. J. Math. **80** (1958), 16–36. MR 0104925 (21 #3675)

[22] Henri Cartan, *Sur les matrices holomorphes de n variables complexes*, J. Math. Pures Appl. **19** (1940), 1–26. MR 0001874 (1,312a)

[23] Henri Cartan and Peter Thullen, *Zur Theorie der Singularitäten der Funktionen mehrerer komplexen Veränderlichen*, Math. Ann. **106** (1932), no. 1, 617–647. MR 1512777

[24] J. Chazarain, *Formule de Poisson pour les variétés riemanniennes*, Invent. Math. **24** (1974), 65–82. MR 0343320 (49 #8062)

[25] Yves Colin de Verdière, *Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques*, Comment. Math. Helv. **54** (1979), no. 3, 508–522. MR 543346 (81a:58052)

[26] N. Dencker, *The resolution of the Nirenberg-Treves conjecture*, Ann. of Math. **163** (2006), 2, 405–444.

[27] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic geodesics*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973), Amer. Math. Soc., Providence, R. I., 1975, pp. 205–209. MR 0423438 (54 #11416)

[28] J. J. Duistermaat and L. Hörmander, *Fourier integral operators. II*, Acta Math. **128** (1972), no. 3-4, 183–269. MR 0388464 (52 #9300)

[29] J. J. Duistermaat and J. Sjöstrand, *A global construction for pseudo-differential operators with non-involutive characteristics*, Invent. Math. **20** (1973), 209–225. MR 0344942 (49 #9681)

[30] Ju. V. Egorov, *The canonical transformations of pseudodifferential operators*, Uspehi Mat. Nauk **24** (1969), no. 5 (149), 235–236. MR 0265748 (42 #657)

[31] ———, *Subelliptic operators*, Uspehi Mat. Nauk **30** (1975), no. 2(182), 57–114. MR 0410473 (53 #14222)

[32] Leon Ehrenpreis, *Solution of some problems of division. I. Division by a polynomial of derivation*, Amer. J. Math. **76** (1954), 883–903. MR 0068123 (16,834a)

[33] Gregory Eskin, *Parametrix and propagation of singularities for the interior mixed hyperbolic problem*, J. Analyse Math. **32** (1977), 17–62. MR 477491 (81e:35077)

[34] C. Fefferman and D. H. Phong, *On positivity of pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), no. 10, 4673–4674. MR 507931 (80b:47064)

[35] Lars Gårding, *Some trends and problems in linear partial differential equations*, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York, 1960, pp. 87–102. MR 0117434 (22 #8213)

[36] Gerd Grubb, *Problèmes aux limites pseudo-différentiels dépendant d'un paramètre*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 12, 581–583. MR 615453 (82f:35194)

[37] Victor Guillemin, *The Radon transform on Zoll surfaces*, Advances in Math. **22** (1976), no. 1, 85–119. MR 0426063 (54 #14009)

[38] ———, *Some spectral results for the Laplace operator with potential on the n-sphere*, Advances in Math. **27** (1978), no. 3, 273–286. MR 0478245 (57 #17730)

[39] ———, *Some classical theorems in spectral theory revisited*, Seminar on Singularities of Solutions of Linear Partial Differential Equations (Inst. Adv. Study, Princeton, N.J., 1977/78), Ann. of Math. Stud., vol. 91, Princeton Univ. Press, Princeton, N.J., 1979, pp. 219–259. MR 547021 (81b:58045)

[40] Bernard Helffer and Jean Nourrigat, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*, Progress in Mathematics, vol. 58, Birkhäuser Boston Inc., Boston, MA, 1985. MR 897103 (88i:35029)

[41] Lars Hörmander, *On the theory of general partial differential operators*, Acta Math. **94** (1955), 161–248. MR 0076151 (17,853d)

[42] ———, *Local and global properties of fundamental solutions*, Math. Scand. **5** (1957), 27–39. MR 0093636 (20 #159)

[43] ———, *On the uniqueness of the Cauchy problem*, Math.

Scand. **6** (1958), 213–225. MR 0104924 (21 #3674)

[44] ______, *On the uniqueness of the Cauchy problem. II*, Math. Scand. **7** (1959), 177–190. MR 0121569 (22 #12306)

[45] ______, *Differential equations without solutions*, Math. Ann. **140** (1960), 169–173. MR 0147765 (26 #5279)

[46] ______, *Differential operators of principal type*, Math. Ann. **140** (1960), 124–146. MR 0130574 (24 #A434)

[47] ______, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140. MR 0121655 (22 #12389)

[48] ______, *Linear partial differential operators*, Die Grundlehren der mathematischen Wissenschaften, Bd. 116, Academic Press Inc., Publishers, New York, 1963. MR 0161012 (28 #4221)

[49] ______, *L^2 estimates and existence theorems for the $\bar\partial$ operator*, Acta Math. **113** (1965), 89–152. MR 0179443 (31 #3691)

[50] ______, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517. MR 0180740 (31 #4970)

[51] ______, *An introduction to complex analysis in several variables*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966. MR 0203075 (34 #2933)

[52] ______, *Pseudo-differential operators and non-elliptic boundary problems*, Ann. of Math. (2) **83** (1966), 129–209. MR 0233064 (38 #1387)

[53] ______, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171. MR 0222474 (36 #5526)

[54] ______, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218. MR 0609014 (58 #29418)

[55] ______, *Fourier integral operators. I*, Acta Math. **127** (1971), no. 1-2, 79–183. MR 0388463 (52 #9299)

[56] ______, *Linear differential operators*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars, Paris, 1971, pp. 121–133. MR 0513000 (58 #23766)

[57] ______, *On the existence and the regularity of solutions of linear pseudo-differential equations*, Enseignement Math. (2) **17** (1971), 99–163. MR 0331124 (48 #9458)

[58] ______, *A class of hypoelliptic pseudodifferential operators with double characteristics*, Math. Ann. **217** (1975), no. 2, 165–188. MR 0377603 (51 #13774)

[59] ______, *Non-uniqueness for the Cauchy problem*, Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 36–72. Lecture Notes in Math., Vol. 459. MR 0419980 (54 #7997)

[60] ______, *The existence of wave operators in scattering theory*, Math. Z. **146** (1976), no. 1, 69–91. MR 0393884 (52 #14691)

[61] ______, *The Cauchy problem for differential equations with double characteristics*, J. Analyse Math. **32** (1977), 118–196. MR 0492751 (58 #11822)

[62] ______, *Propagation of singularities and semiglobal existence theorems for (pseudo)differential operators of principal type*, Ann. of Math. (2) **108** (1978), no. 3, 569–609. MR 512434 (81j:35110)

[63] ______, *Subelliptic operators*, Seminar on Singularities of Solutions of Linear Partial Differential Equations (Inst. Adv. Study, Princeton, N.J., 1977/78), Ann. of Math. Stud., vol. 91, Princeton Univ. Press, Princeton, N.J., 1979, pp. 127–208. MR 547019 (82e:35029)

[64] ______, *The Weyl calculus of pseudodifferential operators*, Comm. Pure Appl. Math. **32** (1979), no. 3, 360–444. MR 517939 (80j:47060)

[65] ______, *Pseudodifferential operators of principal type*, Singularities in boundary value problems (Proc. NATO Adv. Study Inst., Maratea, 1980), NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., vol. 65, Reidel, Dordrecht, 1981, pp. 69–96. MR MR617227 (83m:35003)

[66] ______, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1983, Distribution theory and Fourier analysis. MR 717035 (85g:35002a)

[67] ______, *The analysis of linear partial differential operators. II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 257, Springer-Verlag, Berlin, 1983, Differential operators with constant coefficients. MR 705278 (85g:35002b)

[68] ______, *L^2 estimates for Fourier integral operators with complex phase*, Ark. Mat. **21** (1983), no. 2, 283–307. MR 727350 (85h:47058)

[69] ______, *The analysis of linear partial differential operators. III*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 274, Springer-Verlag, Berlin, 1985, Pseudodifferential operators. MR 781536 (87d:35002a)

[70] ______, *The analysis of linear partial differential operators. IV*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 275, Springer-Verlag, Berlin, 1985, Fourier integral operators. MR 781537 (87d:35002b)

[71] ______, *Pseudo-differential operators of type* $1,1$, Comm. Partial Differential Equations **13** (1988), no. 9, 1085–1111. MR 946283 (89k:35260)

[72] ______, *Remarks on Holmgren's uniqueness theorem*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 5, 1223–1251. MR 1275197 (95b:35010)

[73] ______, *Notions of convexity*, Progress in Mathematics, vol. 127, Birkhäuser Boston Inc., Boston, MA, 1994. MR MR1301332 (95k:00002)

[74] ______, *On the solvability of pseudodifferential equations*, Structure of solutions of differential equations (Katata/Kyoto, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 183–213. MR 1445340 (98f:35166)

[75] ______, *Autobiography of Lars Hörmander*, Fields Medallists' lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, pp. 82–85. MR 1622936

[76] ______, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997. MR 1466700 (98e:35103)

[77] ______, *Looking forward from ICM 1962*, Fields Medallists' lectures, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, pp. 86–103. MR 1622899 (99d:01028)

[78] ______, *On the uniqueness of the Cauchy problem under partial analyticity assumptions*, Geometrical optics and related topics (Cortona, 1996), Progr. Nonlinear Differential Equations Appl., vol. 32, Birkhäuser Boston, Boston, MA, 1997, pp. 179–219. MR 2033496

[79] ______, *On local integrability of fundamental solutions*, Ark. Mat. **37** (1999), no. 1, 121–140. MR 1673428 (2000j:35042)

[80] ______, *The analysis of linear partial differential operators. III*, Classics in Mathematics, Springer, Berlin, 2007, Pseudo-differential operators, Reprint of the 1994 edition. MR 2304165 (2007k:35006)

[81] ______, *The analysis of linear partial differential operators.*

IV, Classics in Mathematics, Springer-Verlag, Berlin, 2009, Fourier integral operators, Reprint of the 1994 edition. MR 2512677 (2010e:35003)

[82] V. Ja. Ivriĭ, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 25–34. MR 575202 (82m:58057)

[83] ______, *Wave fronts of solutions of boundary value problems for a class of symmetric hyperbolic systems*, Sibirsk. Mat. Zh. **21** (1980), no. 4, 62–71, 236. MR 579879 (82a:35066)

[84] V. Ja. Ivriĭ and V. M. Petkov, *Necessary conditions for the correctness of the Cauchy problem for non-strictly hyperbolic equations*, Uspehi Mat. Nauk **29** (1974), no. 5(179), 3–70, Collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ (1901–1973), III. MR 0427843 (55 #873)

[85] Joseph B. Keller, *Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems.*, Ann. Physics **4** (1958), 180–188. MR 0099207 (20 #5650)

[86] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984), Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, pp. 293–326. MR 837683 (87h:35217)

[87] J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305. MR 0176362 (31 #636)

[88] Nicolas Lerner, *Sufficiency of condition* (ψ) *for local solvability in two dimensions*, Ann. of Math. (2) **128** (1988), no. 2, 243–258. MR MR960946 (90a:35242)

[89] ______, *Nonsolvability in* L^2 *for a first order operator satisfying condition* (ψ), Ann. of Math. (2) **139** (1994), no. 2, 363–393. MR 1274095 (95g:35222)

[90] Nicolas Lerner and Luc Robbiano, *Unicité de Cauchy pour des opérateurs de type principal*, J. Analyse Math. **44** (1984/85), 32–66. MR 801286 (86j:35005)

[91] Hans Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. of Math. (2) **64** (1956), 514–522. MR 0081952 (18,473b)

[92] ______, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155–158. MR 0088629 (19,551d)

[93] Bernard Malgrange, *Equations aux dérivées partielles à coefficients constants. I. Solution élémentaire*, C. R. Acad. Sci. Paris **237** (1953), 1620–1622. MR 0060112 (15,626f)

[94] V.P. Maslov, *Theory of perturbations and asymptotic methods*, Moskov Gos. Univ. Moscow (1965).

[95] Anders Melin, *Lower bounds for pseudo-differential operators*, Ark. Mat. **9** (1971), 117–140. MR 0328393 (48 #6735)

[96] R. B. Melrose, *Microlocal parametrices for diffractive boundary value problems*, Duke Math. J. **42** (1975), no. 4, 605–635. MR 0517101 (58 #24409)

[97] ______, *Transformation of boundary problems*, Acta Math. **147** (1981), no. 3-4, 149–236. MR 639039 (83f:58073)

[98] R. B. Melrose and J. Sjöstrand, *Singularities of boundary value problems. I*, Comm. Pure Appl. Math. **31** (1978), no. 5, 593–617. MR 0492794 (58 #11859)

[99] ______, *Singularities of boundary value problems. II*, Comm. Pure Appl. Math. **35** (1982), no. 2, 129–168. MR 644020 (83h:35120)

[100] S. Mizohata, *Solutions nulles et solutions non analytiques*, J. Math. Kyoto Univ. **1** (1961/1962), 271–302. MR MR0142873 (26 #440)

[101] R. D. Moyer, *Local solvability in two dimensions: necessary conditions for the principal type case*, Mimeographed manuscript, University of Kansas, 1978.

[102] François Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), no. 3, 489–507. MR 506997 (80h:46043a)

[103] L. Nirenberg and F. Treves, *Solvability of a first order linear partial differential equation*, Comm. Pure Appl. Math. **16** (1963), 331–351. MR MR0163045 (29 #348)

[104] ______, *On local solvability of linear partial differential equations. I. Necessary conditions*, Comm. Pure Appl. Math. **23** (1970), 1–38. MR MR0264470 (41 #9064a)

[105] ______, *On local solvability of linear partial differential equations. II. Sufficient conditions*, Comm. Pure Appl. Math. **23** (1970), 459–509. MR MR0264471 (41 #9064b)

[106] ______, *A correction to: "On local solvability of linear partial differential equations. II. Sufficient conditions" (Comm. Pure Appl. Math.* **23** *(1970), 459–509)*, Comm. Pure Appl. Math. **24** (1971), no. 2, 279–288. MR MR0435641 (55 #8599)

[107] J. Nourrigat, *Subelliptic systems*, Comm. Partial Differential Equations **15** (1990), no. 3, 341–405. MR 1044428 (91c:35040)

[108] ______, *Systèmes sous-elliptiques. II*, Invent. Math. **104** (1991), no. 2, 377–400. MR 1098615 (92f:35048)

[109] Kiyoshi Oka, *Sur les fonctions analytiques de plusieurs variables. VII. Sur quelques notions arithmétiques*, Bull. Soc. Math. France **78** (1950), 1–27. MR 0035831 (12,18a)

[110] ______, *Sur les fonctions analytiques de plusieurs variables*, Iwanami Shoten, Tokyo, 1961. MR 0132202 (24 #A2048)

[111] James V. Ralston, *Solutions of the wave equation with localized energy*, Comm. Pure Appl. Math. **22** (1969), 807–823. MR 0254433 (40 #7642)

[112] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), Springer, Berlin, 1973, pp. 265–529. Lecture Notes in Math., Vol. 287. MR MR0420735 (54 #8747)

[113] Jean-Pierre Serre, *Une propriété topologique des domaines de Runge*, Proc. Amer. Math. Soc. **6** (1955), 133–134. MR 0067488 (16,736c)

[114] Yum Tong Siu, *Analyticity of sets associated to Lelong numbers and the extension of closed positive currents*, Invent. Math. **27** (1974), 53–156. MR 0352516 (50 #5003)

[115] Johannes Sjöstrand, *Parametrices for pseudodifferential operators with multiple characteristics*, Ark. Mat. **12** (1974), 85–130. MR 0352749 (50 #5236)

[116] ______, *Propagation of analytic singularities for second order Dirichlet problems*, Comm. Partial Differential Equations **5** (1980), no. 1, 41–93. MR 556454 (81e:35031a)

[117] L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math., vol. 39, Pitman, Boston, Mass., 1979, pp. 136–212. MR 584398 (81m:35014)

[118] Michael E. Taylor, *Grazing rays and reflection of singularities of solutions to wave equations*, Comm. Pure Appl. Math. **29** (1976), no. 1, 1–38. MR 0397175 (53 #1035)

[119] J.-M. Trépreau, *Sur la résolubilité microlocale des opérateurs de type principal*, Conference on Partial Differential Equations (Saint Jean de Monts, 1982), Soc. Math. France, Paris, 1982, pp. Conf. No. 22, 10. MR MR672289 (84a:58079)

[120] François Trèves, *A new method of proof of the subelliptic estimates*, Comm. Pure Appl. Math. **24** (1971), 71–115. MR 0290201 (44 #7385)

[121] François Treves, Gilles Pisier, and Marc Yor, *Laurent Schwartz (1915–2002)*, Notices Amer. Math. Soc. **50** (2003), no. 9, 1072–1084. MR 2002753 (2004h:01034)

[122] André Unterberger, *Oscillateur harmonique et opérateurs pseudo-différentiels*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 3, xi, 201–221. MR 552965 (81m:58077)

[123] André Unterberger and Juliane Bokobza, *Les opérateurs de Calderon-Zygmund précisés*, C. R. Acad. Sci. Paris **259** (1964), 1612–1614. MR 0176360 (31 #635a)

[124] Alan Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J. **44** (1977), no. 4, 883–892. MR 0482878 (58 #2919)

The Newsletter thanks the Société Mathématique de France and the Société de Mathématiques Appliquées et Industrielles for the permission to reprint this article. The French original appeared in *Matapli*, issue 100, and in *Gazette des mathématiciens*, issue 137.

Nicolas Lerner is a professor at Université Pierre et Marie Curie (Paris VI). He has worked on microlocal analysis and recently wrote a book on the topic, Metrics on the phase space and non-selfadjoint pseudodifferential operators*, published by Birkhäuser. He was an invited section speaker at the 2002 ICM in Beijing.*

Homage to Srinivasa Ramanujan on the 125th Anniversary of his Birth

Krishnaswami Alladi (University of Florida, Gainesville, USA)

22 December 2012 was the 125th anniversary of the birth of the Indian mathematical genius Srinivasa Ramanujan. In connection with that event, the author describes several major developments in the world of Ramanujan related to various international conferences that were held to honour his legacy. – Editor

Introduction

In the galaxy of the greatest mathematicians in history, Srinivasa Ramanujan occupies a unique place. What sets Ramanujan apart from other mathematical luminaries is the manner in which he obtained his remarkable results, which have revealed surprising and deep connections between many areas of mathematics and have also found applications in physics and computer science.

Godfrey Harold Hardy of Cambridge University (Ramanujan's mentor) compared him to Euler and Jacobi for sheer manipulative ability. Hardy admired Ramanujan's genius and spectacular contributions and explained the significance of his discoveries in lectures and articles [5]. But Hardy could not fully comprehend the manifold consequences the work of the Indian genius would have. In the decades following Ramanujan's death, several leading researchers have studied Ramanujan's published papers and his notebooks closely and so we have now come to realise the vast implications of his discoveries, which have made a deep and lasting impact both within and outside of mathematics. 1n 1987, for his centennial, eminent mathematicians gathered in India at several conferences to pay homage to this singular genius. It was an appropriate time to reflect on his contributions, assess their impact and discuss directions for future work.

As anticipated during the centennial, Ramanujan's influence on mathematical research has continued to grow and not diminish with time. For example, in the last decade, there has been dramatic progress on some fundamental questions regarding the mock theta functions that Ramanujan discovered and communicated in his last letter to Hardy in 1920 shortly before his death. After the centennial, *Ramanujan's Notebooks* have been edited in a series of five volumes [4]. Likewise, *Ramanujan's Lost Notebook* is also being edited in five volumes [3] of which three have appeared. Also, his remarkable life story has been described in books [6] and movies. Conferences focusing on the impact of Ramanujan's work take place annually in his hometown in India and in the United States, and quite regularly in Europe as well. An international prize given to very young mathematicians for outstanding contributions in areas influenced by Ramanujan was created eight years ago. Thus in the 25 years since his centennial, there have been a variety of major developments in the world of Ramanujan.

Ramanujan's discoveries have continued to shape the growth of modern mathematics to such an extent that it was considered worthwhile to launch *The Ramanujan Journal*, devoted to all fields influenced by him. My proposal in 1996 to create this journal received widespread support from the international mathematical community ([2], article 24). This *Ramanujan Journal*, published by Springer since 2005, has tripled in size in just 15 years. This, by itself, testifies to the growing influence of Ramanujan's work on current mathematical research. The editorial board, consisting of 30 mathematicians of world repute, currently has about half a dozen eminent researchers from Europe. The Ramanujan Journal brought out a Special Volume in December 2012 to celebrate the 125th anniversary of his birth.

Also in connection with Ramanujan's 125th birth anniversary, several international conferences were held, all of them dealing with the latest research advances stemming from, or influenced by, his monumental work. In this article, I shall focus on some of these international conferences and describe major developments related to them. For an account of various other significant developments relating to Ramanujan in the quarter century since his centennial, I refer to my recent article [1] that appeared in the Notices of the American Mathematical Society.

Conference at the University of Florida

A three-day international conference "Ramanujan 125" was held at the University of Florida, Gainesville, 5–7 November 2012. Professors Frank Garvan of the University of Florida and Ae Ja Yee of Pennsylvania State University, were co-organisers of the conference with me. The conference attracted 70 active research mathematicians from around the world including more than a dozen from Europe.

The three greatest experts on Ramanujan's work – Professors George Andrews (Pennsylvania State University), Richard Askey (University of Wisconsin) and Bruce Berndt (University of Illinois) – all delivered invited talks at this conference. There were ten plenary lectures of one hour each and about 40 shorter research presentations. On the opening day, Professor Ken Ono in his plenary talk announced the most recent sensational progress relating to Ramanujan's work, namely the precise determination of a certain bounded term in Ramanujan's expression for mock theta functions.

Hardy had expressed the view that the real tragedy with Ramanujan's life was not his early death at the age of 32 but that, during his most formative years in India, Ramanujan did not receive proper guidance and so a significant proportion of his work was rediscovery. Hardy noted that in mathematics, more so than in other fields of science, the best work is done when one is very young. He cited Galois and Abel as two mathematical luminaries who died very young but had done their greatest work in their teens. Thus Hardy argued that if Ramanujan had lived longer, he might have done more mathematics but not work of higher quality. With our present understanding of Ramanujan's work, we feel that Hardy might have been wrong in this assessment. Ramanujan's work on mock theta functions and his other identities in the Lost Notebook that he did in India after his return from England, and in the few months before his death, are grander in design and greater in depth. Thus Ramanujan was definitely on the rise and one can only imagine what greater heights he would have scaled had he lived longer.

In his last letter to Hardy in January 1920, Ramanujan communicated his findings on what he called mock theta functions. These are functions which mimic the theta functions in the sense that their coefficients can be estimated with the same degree of precision as in the case of objects expressible in terms of theta functions. Ramanujan had obtained an asymptotic evaluation of these mock theta functions and in his letter observed that if certain well-behaved analytic expressions were subtracted from the mock theta functions, the resulting error is bounded. He also indicated the bounds in some instances. For many years, the exact connection between mock theta functions and the theory of theta functions and modular forms was not known, and this was one of the tantalising mysteries. In the last decade, Ken Ono (now at Emory), Kathrin Bringmann (now at Cologne) and their co-workers, developing fundamental ideas in a 2003 PhD thesis of Sander Zwegers written under the direction of Don Zagier in Bonn, have connected mock theta functions with harmonic Maass forms. This has provided a key to unlocking this mystery. During the Florida conference, Ono announced for the first time his recent joint work with Amanda Folsom (Yale University) and Robert Rhoades (Stanford University) in which they obtain a precise expression for the bounded error that Ramanujan indicated.

Other plenary talks included presentations by Robert Vaughan (Pennsylvania State University) on the Hardy-Ramanujan-Littlewood Circle Method, Dorian Goldfeld (Columbia University) on Ramanujan Sums, and Doron Zeilberger (Rutgers University) on Ramanujan as the greatest experimental mathematician. Two of the plenary talks were by Kannan Soundararajan (Stanford University) and Kathrin Bringmann (University of Cologne),

Group photograph of the participants of the Ramanujan 125 Conference at the University of Florida, Gainesville, 5–7 November 2012. Photograph courtesy of Ali Uncu

who are winners of the prestigious SASTRA Ramanujan Prize, about which there is a description in the next section. Two other plenary speakers from Europe were Christian Krattenthaler (University of Vienna) and Gerald Tenenbaum (University of Nancy).

The conference featured several impressive research presentations by graduate students. Of note was the talk by Michael Th. Rassias who is doing his PhD at ETH in Zurich under the direction of Professor Emmanuel Kowalski. Michael's precocity in mathematics has been demonstrated in mathematical olympiads in Europe, where he won gold medals, as well as at the International Mathematical Olympiad in Tokyo in 2003, where he won a silver medal as a high school student.

Inspired by work on Ramanujan being done at Penn State, Wisconsin and Illinois, I began a programme of research on Ramanujan-type identities at the University of Florida in 1987 upon joining the university. Our department is now recognised as one of the premier centres in the world on research related to Ramanujan, with two other number theorists Frank Garvan and Alexander Berkovich subsequently joining the department. The three of us have organised conferences on number theory annually in Florida with a focus on Ramanujan's work. But this conference was very special; it preceded a celebration of Ramanujan's 125th anniversary in India in December 2012 just as the conference at the University of Illinois in May 1987 preceded the Ramanujan Centennial Celebrations in India in December that year.

The refereed proceedings of the Florida conference will be published in the *Contemporary Mathematics* series of the American Mathematical Society.

Conference in Kumbakonam, Ramanujan's Hometown

I will start with a background about Ramanujan's home and hometown before providing a report of the conference.

Srinivasa Ramanujan's home is located in the town of Kumbakonam, in the State of Tamil Nadu, in South India. Ramanujan's father, a poor cloth merchant, worked in Kumbakonam and so Ramanujan grew up there. Ramanujan was born on 22 December 1887 in Erode, a nearby town, which was his mother's home. It is the tradition in India that, even though after marriage a woman will live in her husband's home, she will return to her parents' home to give birth to her child.

Krishnaswami Alladi and George Andrews in front of Srinivasa Ramanujan's home in Kumbakonam, South India.
Photograph courtesy of Krishnaswami Alladi

Kumbakonam is in the district of Tanjore, which is culturally very rich. There are more Hindu temples in Tanjore than anywhere else. Thus in this region steeped in culture, Ramanujan grew up and went to school. Ramanujan showed his mathematical talents at a very early age. He would get up in the middle of the night and write down formulae on a slate which he would copy into his notebooks in the morning. There is a legend that the Hindu Goddess Namagiri in the neighbouring town of Namakkal would come in his dreams and provide him these formulae. Hardy dismissed the Goddess of Namakkal story as mere fable but Hindus believe in divine dispensation for extraordinary human achievement. The Sarangapani temple is located a few hundred feet away from Ramanujan's home and Ramanujan's family offered daily prayers at this temple. The home had just one cot in a bedroom occupied by Ramanujan's parents. The home had several residents and all others including Ramanujan slept on the floor in the small courtyard. As a boy, Ramanujan would sit on the windowsill of the bedroom and do his "sums" as he watched the passers-by on the street. It was from this humble home that a thousand theorems emerged that influenced the development of several fields of mathematics.

Hardy said that Ramanujan returned to India from England with a reputation that transcended human jealousies. While Ramanujan was in England, Hardy made a great effort to get Ramanujan elected Fellow of the Royal Society (FRS) as well as Fellow of Trinity College of Cambridge University. These honours were bestowed on Ramanujan even though he did not possess a college degree. Yet, after Ramanujan died, in spite of the great fame that he brought to his motherland, nothing was done in India to support his widowed wife, nor any effort made to preserve and maintain his home in Kumbakonam. All this changed when SASTRA University purchased Ramanujan's home in 2003 and decided to maintain it as a museum.

SASTRA is a private university that started about 20 years ago in Tanjavur, the main town of the Tanjore district. Unlike public universities in India where most admissions are reserved for certain backward sections of society, entry into private universities like SASTRA is based on merit. SASTRA grew by leaps and bounds by recruiting good teachers and admitting top ranked students. The purchase of Ramanujan's home was a major event because it led to the involvement of academicians, students and university administrators in the preservation of Ramanujan's legacy.

In connection with the purchase of Ramanujan's home in 2003, SASTRA opened a branch campus in Kumbakonam that year and conducted an International Conference on Number Theory and Secure Communications during 20–22 December to coincide with Ramanujan's birthday. The conference was inaugurated by Dr Abdul Kalam, the President of India. I was invited to speak at this conference and to bring a team of mathematicians from abroad. During the conference, the participants suggested that SASTRA should hold such conferences every year in the month of December. I have helped SASTRA organise these conferences since 2003 and so have had the pleasure and honour of spending Ramanujan's birthday every year in his hometown, thereby making it an annual pilgrimage for me ([2], article 25). Each year about six to ten leading mathematicians from around the world give talks at these conferences.

On the first day of the second conference in 2004, the SASTRA Vice-Chancellor Professor R. Sethuraman told me just before the inauguration that he would like to set aside $10,000 each year for a recognition in the name of Ramanujan and asked for my suggestion. I said that it would be best to create an award called the SASTRA Ramanujan Prize to be given to a mathematician not exceeding the age of 32 for outstanding contributions to areas influenced by Ramanujan. The age limit of 32 was because Ramanujan had achieved so much in his brief life of 32 years. The Vice-Chancellor agreed to my suggestion and announced at the inauguration that the prize would be offered from 2005 (the very next year). He then asked me to serve as Chair of the Prize Committee. This is how I got involved with the prize ([2], article 26). The prize is now one of the most prestigious and coveted in the world. It is awarded every year in Kumbakonam during the SASTRA conference, except in 2012 where, for the 125th anniversary of Ramanujan, the prize was given in New Delhi, India's capital (for more on this, see the next section).

Since the SASTRA Prize in 2012 was to be given in Delhi, something unique and of significance had to take place at the SASTRA conference in Kumbakonam. The decision was made to bestow honorary doctorates to the Great Trinity of the Ramanujan world – Professors George Andrews, Richard Askey and Bruce Berndt. This grand convocation ceremony took place on 15 December 2012 during the second day of the SASTRA conference. We owe a debt of gratitude to these three pre-eminent experts:

(i) to George Andrews for explaining the significance of many of Ramanujan's identities, especially in the context of partitions, and for discovering Ramanujan's "Lost Notebook" and helping us understand hundreds of deep identities contained therein including those on the mock theta functions,
(ii) to Richard Askey for providing the broad picture of how Ramanujan's work fits in the world of Special Functions, and
(iii) to Bruce Berndt for editing Ramanujan's Notebook in five volumes [4] and Ramanujan's Lost Notebook along with George Andrews [3].

Thus, it was fitting to honour Professors Andrews, Askey and Berndt on Ramanujan's 125th anniversary in Ramanujan's hometown. After awarding the honorary doctorates, at the same ceremony, the Special Volume of the *Ramanujan Journal* for Ramanujan 125 was formally released. This volume, edited by me, George Andrews and Jonathan Borwein, has 26 important research contributions by more than 50 active researchers in the area of Ramanujan's mathematics. Also at the same ceremony, it was an honour for me that my book *Ramanujan's place in the world of mathematics* was released and the first copy handed to Professor Andrews.

The SASTRA Conference in 2012 had about a dozen main lectures by mathematicians of repute from Europe, USA and Australia, including three public lectures by the recipients of the honorary doctorates.

Conference in New Delhi, India's Capital

On 26 December 2011, at a highly publicised ceremony in Chennai (formerly Madras), India, the Prime Minister of India, Dr Manmohan Singh, declared the following year as a National Mathematics Year. A stamp of Ramanujan was released and a special scroll was presented to Robert Kanigel for his influential biography of Ramanujan [6] entitled *The man who knew infinity*. Also, a Collector's Edition was released of the photostat copies of *Ramanujan's Notebooks*, prepared by the Tata Institute of Fundamental Research, India. Throughout 2012 there were lectures all over India by eminent mathematicians from

The three recipients of the Honorary Doctorates in white robes – from left to right Professors Bruce Berndt, Richard Askey and George Andrews – at the SASTRA University Convocation in Kumbakonam, 15 December 2012. Also on the stage are Vice-Chancellor R. Sethuraman and Dean of Research S. Swaminathan, flanking Professor Andrews.
Photograph courtesy of Krishnaswami Alladi

Zhiwei Yun (MIT and Stanford) receiving the SASTRA Ramanujan Prize from the Minister of State Jitin Prasada of the Ministry for Human Resource Development, on 22 December 2012, in New Delhi, India. SASTRA Vice-Chancellor R. Sethuraman and Prize Committee Chair Krishnaswami Alladi look on.
Photograph courtesy of SASTRA University

several countries. The National Mathematics Year concluded with a six-day International Conference on the Legacy of Ramanujan at the University of Delhi during 17–22 December 2012. The conference featured 17 plenary lectures of one hour each and 24 invited talks of 40 minutes each. In addition, there were two public lectures in the evening by Professors Richard Askey and Bruce Berndt, as well as an hour lecture by the SASTRA Prize Winner. Of note was a talk by Professor Manjul Bhargava (Princeton University), winner of the First SASTRA Ramanujan Prize in 2005. He spoke about elliptic and hyper-elliptic curves and the number of rational points on such curves, providing an overview of some recent conjectures and progress on various important problems. Being a six-day conference with an impressive array of distinguished mathematicians, the topics covered at the conference included Ramanujan's tau function, identities from *Ramanujan's Lost Notebook*, modular forms and theta functions, partitions and q-hypergeometric functions, the circle method and additive number theory, automorphic functions, etc. – areas that have felt Ramanujan's magic touch and were gloriously transformed by him. On the final day of the conference, 22 December, the fitting finale was the award of the SASTRA Ramanujan Prize to Zhiwei Yun (MIT and Stanford) for far reaching contributions to several areas that lie at the interface of representation theory, algebraic geometry and number theory, including the Langlands Program. Ramanujan is a supreme example of monumental achievement in youth and the SASTRA Prize Ceremony was a fine way to conclude the conference as well as the National Year of Mathematics on Ramanujan's 125th birthday.

The refereed proceedings of the New Delhi conference will appear as a Special Volume published by the Ramanujan Mathematical Society.

In addition to these three conferences, there were two other Ramanujan 125 conferences. The first was at Mysore University, India, during 12–13 December 2012, just before the SASTRA conference. The second was a Special Session on Ramanujan 125 at the Annual Meeting of the American Mathematical Society in San Diego during 10–11 January 2013.

It is abundantly clear from what was presented at these conferences, and what has been published in the last quarter century, that Ramanujan's mathematics remains influential and youthful, thereby defying the passage of time! As Freeman Dyson remarked during the Ramanujan Centennial, we should be thankful to Ramanujan not only for discovering so much but also for giving others the pleasure of discovery by not revealing all his ideas.

References

1) K. Alladi, "Ramanujan's thriving legacy", in Srinivasa Ramanujan going strong at 125, parts I and II, Notices AMS, I: 59 (2012), 1522–1537; II: 60 (2013), 10–22.
2) K. Alladi, *Ramanujan's place in the world of mathematics – essays providing a comparative study*, Springer, New Delhi (2012), 177 pp.
3) G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook*, Parts I–V, Springer, New York, I:2005, II:2009, III:2012, IV and V: to appear.
4) B. C. Berndt, *Ramanujan's Notebooks*, Parts I–V, Springer, New York I: 1985, II: 1989, III: 1991, IV: 1994, V: 1998.
5) G. H. Hardy, *Ramanujan – Twelve Lectures on his Life and Work*, Cambridge University Press, Cambridge, 1940.
6) Robert Kanigel, *The Man Who Knew Infinity*, Charles Scribners, New York, 1991.

Krishnaswami Alladi is a professor of mathematics at the University of Florida, Gainesville, where he was Department Chairman during 1998–2008. He received his PhD from UCLA in 1978. His area of research is number theory. He is the Founder and Editor-in-Chief of The Ramanujan Journal *published by Springer. He helped create the SASTRA Ramanujan Prize given to very young mathematicians and has chaired the prize committee since its inception in 2005.*

The Mathematical Life of Shreeram Abhyankar

Steven Dale Cutkosky (University of Missouri, Columbia, USA)

Shreeram Abhyankar passed away on 2 November 2012 at the age of 82, in his home in West Lafayette, Indiana. He was at his desk, working on mathematics. He was active in research until the end of his life. Abhyankar made many fundamental contributions to algebra and algebraic geometry.

Shreeram Abhyankar was born on 22 July 1930 in Ujjain, India. His father was a professor of mathematics who encouraged his interest in the subject, teaching him the foundations while he was a child. After finishing high school in Gwalior, he attended the University of Bombay, graduating in 1951. One of his professors, Pesi Masani, had earned his PhD from Harvard under the direction of Garrett Birkhoff. Masani encouraged Abhyankar to pursue his graduate studies at Harvard.

On the ship to Boston he became ill, which delayed his arrival at Harvard until the break between the Fall and Spring semesters. When he first arrived at the mathematics department, almost no one was there, but by a great chance he met Oscar Zariski. Zariski advised Abhyankar on the classes that he should take, which included a course on projective geometry that Zariski himself would teach. Abhyankar often spoke fondly of this meeting and the subsequent development of his relationship with Zariski.

At this time the state of the art of resolution of singularities was Zariski's proof in [20] of resolution of singularities of three dimensional varieties over an algebraically closed field of characteristic zero. Zariski was very interested in the question of resolution of singularities of positive characteristic surfaces and mentioned this to Abhyankar as an important problem which was probably too difficult for a PhD. Abhyankar became fascinated with this problem and, after a great effort, solved it for his PhD, which he earned in 1955 from Harvard. The proof is published in his 1956 paper "Local Uniformization of Algebraic Surfaces over Ground Fields of Characteristic $p \neq 0$" [2].

Over the next decade, Abhyankar travelled widely and visited many institutions, including Columbia University, Erlangen, Münster and Kyoto. He was an associate professor at Cornell University and John Hopkins University. In 1967, Abhyankar became the Marshall Distinguished Professor of Mathematics at Purdue University. He had close connections with the University of Pune in India and founded a research institution in Pune called Bhaskaracharya Pratishthana, named after the great 12th century Indian mathematician Bhaskaracharya, who had lived in Ujjain.

In 1958, Abhyankar married his wife Yvonne, who survives him. They have two children, a son Hari and a daughter Kashi. They both earned PhDs in mathematics, from MIT and Berkeley, respectively.

Teaching was very important to Abhyankar. He was a charismatic man and a mesmerising speaker. He had 29 students complete their PhDs. A room was built onto his house, which is inside the Purdue campus, for the purpose of meeting with his students and collaborators. He held a regular seminar which met there. Students would often stay with him. His house was always the centre of a lot of activity. Yvonne managed all of this, making everyone welcome. I have very pleasant memories of visiting their home and discussing many things, mostly mathematics and Indian philosophy.

During the course of his life, Abhyankar worked on many different mathematical problems, all of which have a basis in algebra. Some of the problems he most focused on are resolution of singularities, especially in positive characteristic, ramification of finite mappings, affine algebraic geometry, Young tableaux and determinantal varieties, inverse Galois theory, the Jacobian problem and dicritical divisors. He had a deep interest in commutative algebra and often strove to prove results in the greatest possible generality. From the beginning he had a great love of polynomials and power series and his work is filled with ingenious manipulations and substitutions of polynomials.

His early works, mostly on resolution of singularities and the algebraic fundamental group, develop methods of algebraic number theory and general valuation theory within the context of algebraic geometry, especially the theory of ramification. These problems continued to interest him throughout his life.

In a series of papers, culminating in his book "Resolution of Singularities of Embedded Algebraic Surfaces" [4], which first appeared in 1966, Abhyankar gave a proof of embedded resolution of algebraic surfaces in all characteristics. Using this result, he proved that a resolution of singularities exists for a three dimensional algebraic variety over an algebraically closed field of characteristic p greater than 5. In [12], I give a simplified proof of this result. Recently, Cossart and Piltant [11] have succeeded in proving that resolutions of singularities exist for algebraic varieties of dimension three over fields of the remaining characteristics 2, 3 and 5. Their proof draws heavily on ideas from Abhyankar's papers. It is unknown if resolutions of singularities always exist for varieties of dimension greater than or equal to four and of positive characteristic.

Abhyankar found that in characteristic zero, hypersurfaces of maximal contact always exist for singularities. This allows a reduction of resolution of singularities to one dimension less and allows an inductive formulation of resolution. He found an explicit construction, which he called a Tschirnhaus transformation in honour of the 17th century mathematician. The transformation is a generalisation of the method of completing the square to solve quadratic equations. It is remarkable that this idea makes resolution of singularities possible. In the equation

$$f = z^d + a_1 z^{d-1} + \cdots + a_d,$$

where the a_i are polynomials or series in the variables $x_1, \ldots, x_n$ which vanish at the origin to order $\geq i$, make the substitution

$$\overline{z} = z + \frac{a_1}{d!}.$$

Then we have a new equation

$$f = \overline{z}^d + \overline{a}_2 z^{d-2} + \cdots + \overline{a}_d. \qquad (1)$$

Abhyankar showed that blowing up the most singular points of $f = 0$ makes the singularity better, except possibly at points which are on the transform of $\overline{z} = 0$. The transform of f continues to have the form (1) at these points. Points where the singularity is not better are always on the transform of $\overline{z} = 0$, no matter how many times you blow up. $\overline{z} = 0$ is called a hypersurface of maximal contact for $f = 0$.

Hironaka used the Tschirnhaus transformation as the starting point of his 1964 proof [16] of resolution of singularities of algebraic varieties of any dimension in characteristic zero. The Tschirnhaus transformation is the major part of Hironaka's proof which does not extend to characteristic $p > 0$. The transformation is not possible if p divides the degree d of f, as p times the identity is zero in characteristic p.

In his 1983 PhD thesis with Abhyankar, Narasimhan [18] gave an example showing that hypersurfaces of maximal contact do not generally exist in positive characteristic.

One of the classical approaches to resolution is Jung's 1908 discovery [17] that if a normal complex surface S is finite over a nonsingular surface and the branch divisor has only simple normal crossing singularities then S has only Abelian quotient singularities, and thus its singularities are very simple to resolve. Zariski suggested to Abhyankar that he extend this theorem to positive characteristic, as a possible means of proving resolution of surface singularities in positive characteristic. Abhyankar discovered that this theorem holds if the extension has only tame ramification (which always holds in characteristic zero) but fails if the ramification is not tame. This theorem is often called the Abhyankar-Jung theorem. It appears in his 1955 paper "On the ramification of algebraic functions" [1]. Abhyankar developed the theory of ramification in positive characteristic, defining the algebraic fundamental group and giving many important examples in his 1957 paper [3]. The finite quotients of the algebraic fundamental group of a normal variety are the Galois groups of unramified finite covers of the variety. Abhyankar made a very deep conjecture in this paper. In the introduction, he explains that his examples are part of "a general pattern which leads us to form a conjecture which roughly says that (ramification theory in characteristic $p \neq 0$) = (ramification theory in characteristic zero for the corresponding situation) + (the class of all quasi-p-groups)". The first part of this conjecture was proven by Grothendieck in 1959 [13], using his extension of the theory of algebraic fundamental groups to schemes with nilpotents. The proof is written up in Corollary XIII.2.12 [14]. The second part of the conjecture became known as Abhyankar's conjecture on the algebraic fundamental group. It was finally proven by Raynaud [19] and Harbater [15] in 1994.

In the 1970s, Abhyankar turned to affine geometry and proved with T. T. Moh the Ahyankar-Moh epimorphism theorem [9]. This theorem states that every embedding of a line in the plane extends to an automorphism of the plane. This result is valid over any algebraically closed field k of characteristic zero and a version of it is true in any characteristic. The natural generalisation of the problem to a hyperplane embedded in k^n is still open. The proof of the Abhyankar-Moh epimorphism theorem is based upon a development of the Tschirnhaus transformation given in their papers [10].

Abhyankar had a great interest in the Jacobian problem and made a fine analysis of this problem. Most of his last papers where on concepts underlying this problem, including [8] with Luengo and [7] with Heinzer on dicritical divisors occurring in the resolution of a pencil of curves.

Abhyankar always preferred concrete proofs involving an explicit algorithm which can be realised by a series of elementary algebraic manipulations. In the 1960s, driven by the successes of Grothendieck, algebraic geometry was dominated by a love of abstraction, which sometimes led to the consideration of arcane problems which cannot easily be motivated by classical mathematics. In recent years, the development of explicit algorithms has come to the fore, spurred on by the advancement of the computer. In this way, Abhyankar can be seen as being ahead of his time. Abhyankar called his approach "high school algebra", as a reaction to an emphasis on abstraction for its own sake. He expressed his views in his witty and barbed poem "Polynomials and Power Series", which was reprinted on page 783 of the proceedings of his 70th birthday conference [6]. A thoughtful summary of some of his work and thoughts can be found in an article which he wrote for the Bulletin of the AMS in 2000 [5].

Conferences have taken place in honour of Abhyankar's 60th, 70th and 80th birthdays. This summer, a conference was held at Purdue University in honour of Abhyankar's 82nd birthday, which is especially important as it corresponds to 1,000 full moons. I am very glad that I was able to attend all of these conferences. The conferences were tributes to his rich life, celebrated by his many students, collaborators and colleagues.

Bibliography

[1] S.S. Abhyankar, On the ramification of algebraic functions, Amer. J. Math. 77, (1955), 575–592.

[2] S.S. Abhyankar, Local uniformization of algebraic surfaces over ground fields of characteristic $p \neq 0$, Annals of Math. 63 (1956), 491–526.

[3] S.S. Abhyankar, Coverings of Algebraic Curves, Amer. J. Math. 79, (1957), 825–856.

[4] S.S. Abhyankar, Resolution of singularities of embedded algebraic surfaces, second edition, Springer Verlag, New York, Berlin, Heidelberg, 1998.

[5] Resolution of singularities and modular Galois theory, Bulletin of the AMS 38, (2002), 131–169.

[6] S.S. Abhyankar, Polynomials and Power Series, in Algebra, Arithmetic and Geometry, with Applications, Papers from Shreeram S. Abhyankar's 70th Birthday Conference, Bajaj, Christensen, Sundaram and Sathaye, editors, Springer Verlag, 2003.

[7] S.S. Abhyankar and W. Heinzer, Existence of dicritical divisors, Amer, J. Math. 134 (2012), 171–192.

[8] S.S. Abhyankar and I. Luengo, Algebraic theory of dicritical divisors, Amer. J. Math. 133 (2011), 1713–1732.
[9] S.S. Abhyankar and T.T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math. 276 (1975), 148–166.
[10] S.S. Abhyankar and T.T. Moh, Newton-Puiseux expansion and generalized Tschirnhausen transformations I and II, J. Reine Angew. Math. 260 (1973), 47–83; ibid. 261 (1973), 29–54.
[11] V. Cossart and O. Piltant, Resolution of singularities of threefolds in positive characteristic, I and II, Journal of Algebra 320 (2008), 1051–1082 and 321 (2009), 1836–1976.
[12] S.D. Cutkosky, Resolution of singularities for 3-folds in positive characteristic, Amer. J. Math. 131 (2009), 59–127.
[13] A. Grothendieck, Fondements de la Géométrie Algébrique, Séminarie Bourbaki, 1957–62, Séminaire 182 (1959).
[14] A. Grothendieck, Revêtements étale et groupe fondemental, Springer LNM 224 (1971).
[15] D. Harbater, Abhyankar's conjecture on Galois groups over curves, Inv. Math. 117 (1994), 1–25.
[16] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Annals of Math, 79 (1964), 109–326.
[17] H.W.E. Jung, Darstellung der functitonen eines algebraischen körpers zweier unabhängiger varänderlichen x, y in der umbhängiger einer stelle $x = a, y = b$, J. Reine. Angew. Math., 133, (1908), 289–314.
[18] R. Narasimhan, Hyperplanarity of the equimultiple locus, Proc. Amer. Math. Soc. 87 (1983), 403–408.
[19] M. Raynaud, Revêtements de la droite affine en caractéristique $p > 0$, Inv. Math. 116 (1994), 425–462.
[20] O. Zariski, Reduction of singularities of algebraic three-dimensional varieties, Ann. of Math. 45 (1944), 472–542.

Steven Dale Cutkosky [cutkoskys@missouri.edu] is a Curators' Professor of Mathematics at the University of Missouri, in Columbia, Missouri, USA. His research is in algebraic geometry and commutative algebra. He received his PhD in mathematics from Brandeis University in 1985.

Mathematics in Moscow: We Had a Great Epoque Once[1]

A. N. Parshin (Steklov Mathematical Institute, Moscow, Russia)

During the last year there has been a revival of the question "What is to become of us". I know of at least two, as they call them now, round tables where the state of our mathematics, its origins and prospects were discussed. One of them was held in December at the St. Petersburg Division of the Steklov Mathematical Institute, the other one in February, at the Philosophy Department of the Moscow State University.[2]

There is of course much to say on this subject, and I would like to break down my talk into three parts:

what was,
what is,
and what will be.

In the preceding talk, S. S. Demidov described to us what was possibly the most important part of Soviet mathematics: *schools*. A school is a community of individuals who work in the same branch of science, who are in close communication with each other, who have a leader, a teacher, amongst whom each generation passes on the torch to the next one, and all this forms one integral organism.

Everyone knows the school of N.N. Luzin from which every other school has arisen: the school of A.N. Kolmogorov, the school of I.M. Gelfand, the school of I.R. Shafarevich, the school of L.S. Pontryagin.

I will speak of what remains in my memory, of what is the closest to me. My examples will of course be quite arbitrary, my evaluations subjective. But this cannot be avoided, should one try to speak with sincerity, and it makes no sense to speak of such matters in any other way.

In the words of Eduard Limonov, indeed we have had a "great epoque". Its theater was *mech-mat*[3] of MGU[4], in the sixties and seventies of the twentieth century.[5] Back then I had just started my studies (together with Sergei Sergeevich). We entered *mech-mat* in 1959, and so it can be said that this epoque took place directly in front of my eyes.

Imagine room 16-10, occupied not by the Moscow Mathematical Society, which in those years filled the room entirely, but by the audience of the seminar on deformation theory of complex structures, which was dedicated to the study of the recently published papers of K. Kodaira and D.C. Spencer. These papers were on the frontier between complex analysis, theory of elliptic equations, and geometry. There were three people who had decided to study those papers: Evgenii Borisovich Dynkin, whose main interest was the theory of Markov processes and Lie group theory, Mikhail Mikhailovich Postnikov, one of the creators of algebraic topology, and Igor Rostislavovich Shafarevich, known for his works in algebraic number theory and Galois theory. All three of them were quite distant from the subject chosen, but they organized such a seminar nevertheless. And the room, even if not packed, was almost full. Nowadays such a thing cannot be imagined. At the time I was a second-year student and attended the seminar.

The Moscow Mathematical Society

Another memory of mine: in the same years, Shafarevich was creating our school of algebraic geometry. It began

[1] A talk at the Moscow House of Scientists on 19 March 2009. Among the other speakers, there were S. S. Demidov, A. V. Bulinskiy, V. M. Tikhomirov, M. I. Zelikin, A. M. Abramov, A. G. Sergeev, A. Ya. Khelemskiy. It was first published in *Istor.-Mat. Issled.* (2) 14 (49) (2011), 11–24. The present article is up to minor changes a translation of the Russian original by Ekaterina Pervova, Università di Pisa, Dipartimento di Matematica Applicata "Ulisse Dini", via Buonarroti 1/c, 56127 Pisa, Italy. The Newsletter expresses sincere thanks to Prof. Pervova. Thanks also go to Edwin F. Beschler (Boston) for his excellent advice in the final editing.

[2] After this talk I discovered a significant number of similar discussions. They can be found at the site www.polit.ru/science.

[3] Mechanics-Mathematical Faculty of Moscow State University.

[4] Moscow State University.

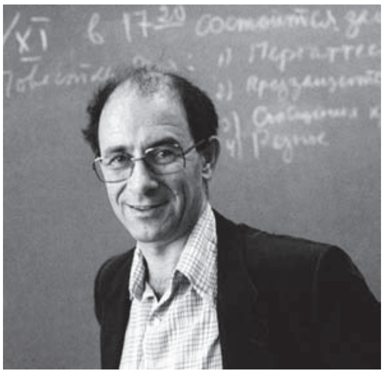
[5] If we limit ourselves to Moscow. Not less shining, and in some ways shining even more, was the mathematical life in Petersburg, at LOMI and at the mat-mech faculty of the University. We should also credit the Akademgorodok of Novosibirsk.

with the seminar of Shafarevich on the theory of algebraic surfaces. By that time algebraic curves were quite well understood, but the situation with surfaces was rather complicated. There was a theory of surfaces but only within the realm of the Italian school of algebraic geometry which had been created in the nineteenth century and which nobody understood. There was, for example, an Italian book by F. Severi on algebraic surfaces, written in a very difficult idiom, rather distant from how mathematical texts are written nowadays. Nevertheless it was read and studied. The seminar went on for two years. Then the *Proceedings of the Steklov Institute* published a book in which all chapters, dealing with different classes of algebraic surfaces, established future subjects relevant to our school of algebraic geometry. Indeed our school of algebraic geometry found its root in this seminar and developed into fruitful directions of our research.[6] The seminar of Shafarevich still continues today.

Igor Rostislavovich Shafarevich (Photo: Archive of P. Roquette, Heidelberg/MFO)

If we wish to recall later times, my memory offers me the seminar of V.I. Arnold. That was in the middle or in the end of the 1960s. The book of John Milnor on the Morse theory was about to come out and Milnor sent to Arnold its page proofs. One chapter of this book is an excellent course on Riemannian geometry (I don't know of a better exposition). Arnold broke the text down to pieces and distributed them amongst his students. The talks on this book went on for a year. How was it organized? Every speaker gave the necessary definitions (vector fields, indices of geodesics, etc.), carried out in detail all the calculations, wrote many formulae, … . Everyone listened and took notes. Five minutes to the end, Arnold got up, went to the blackboard, chose an empty corner and drew a careful picture. Look! Everybody looked, and there it was, everything clear without formulae! This was the science, this was the environment.

I remember Arnold during Gelfand's seminars, as Gelfand was explaining to him what a symplectic form is and how its geometry differs from the Euclidean geometry of quadratic forms. This was happening during a talk on a very different topic. The word "symplectic" had not been pronounced yet, and probably Arnold himself did not suspect that with the passage of time he would become one of the founders of symplectic geometry.



Vladimir Igorevich Arnold

Much can be said about Gelfand's seminar. I went there for several years, when I was a student. It was a rather surprising event, you never knew when it would start, what would happen there, or when it would end. I remember very well how in the beginning of the 1960s, whatever was the topic of the talk, Laplace transformations, differential equations, representation theory, Gelfand would always ask "what is a topological vector space?" (meaning a space of infinite dimension). Everybody would remain silent, and he would say, I think it's the category of finite-dimensional spaces. This remained firmly in my memory. Surprisingly, in algebra there recently appeared the notion of n-vector spaces, where 1-vector spaces are finite-dimensional spaces, whereas 2-vector spaces are categories of finite-dimensional spaces. Precisely the point that was tormenting Gelfand.

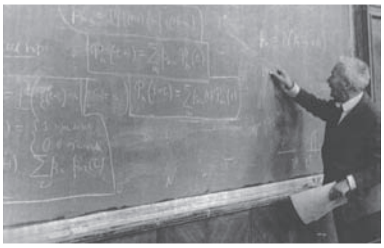
Israil Moiseevich Gelfand (Photo: K. Jacobs, Erlangen/MFO)

And such was the social setting that served as the background for a revolution in science. A flame burst out,

[6] For more details on this seminar see my article "Numbers as functions (the development of an idea in the Moscow school of algebraic geometry)" in the collection *Mathematical Events of the Twentieth Century*, PHASIS, Moscow 2003, Springer-Verlag, Berlin 2006.

engulfing, transforming and unifying almost completely, branches of mathematics such as algebraic geometry, algebraic and differential topology, complex analysis, dynamical systems, Lie algebras and groups, representation theory, differential geometry, automorphic functions and discrete groups, number theory in that part of it which had developed and was inspired by the influence of topology, and the said algebraic geometry. This, of course, was not the whole of mathematics but a very significant part of it nonetheless.

This involved a large circle of people, each working in their own subject, yet everybody was interested in everything. There wasn't the specialization so typical of the present times. For example, at the seminar of Shafarevich a talk on Diophantine equations could be followed by a talk on bounded domains on complex manifolds. Nevertheless, everyone listened with interest and tried to understand it all. This breadth of interest was not limited to mathematics but extended much further. In those days many so-called "pure" mathematicians were interested, not just in the applications, or in physics, but also in the humanities. Those years saw the appearance of papers by Kolmogorov on metrics in poetry, and Gelfand conducted, in addition to his big seminar, also a seminar on the physiology of the cell. From my student years I remember a seminar on descriptive linguistics. It took place in no other than room 01, on the ground floor of the main building.[7]



Andrey Nikolaevich Kolmogorov

Can this really be imagined, a seminar on linguistics at *mech-mat*? Who conducted it? Andrei Andreevich Markov, Vladimir Andreevich Uspenskii, both specialists in logic, and the then very young Andrei Anatol'evich Zaliznyak. He seemed to be at home in all languages either known or imaginable; if a question arose, he immediately said, in Turkish it is this, and in Swahili it is that. The seminar had a part dedicated to the study of Gleason's book[8] on descriptive linguistics. It contained exercises, which were assigned as homework. And every time when Andrei Andreevich asked 'Who completed the first one?', he himself, just as a *pioneer*, did not raise his hand first but put it up timidly, elbow on the desk.

And of course all of these activities were accompanied by a very rich cultural life, in the broadest sense: nature trips and concerts, visits to the music hall. At a good concert you could always see more than one familiar face from the *mech-mat*. This community was also bound together by the communality of political views, to a large degree of dissident nature. Even if there were some variations (which later on led to disagreements on a matter of principle), the attitude towards events such as Chekhoslovakia-68 or confinement to a psychiatric hospital of A. S. Esenin-Vol'pin was quite uniform. The reaction to the latter in the form of the letter of 99 became well known. The participation in it has complicated life (first of all, travels abroad) for many people.

There is much still to be remembered, but I would like not only to recall but also try to understand what it was that had brought about this remarkable uprising. I think that now, after all these years, such understanding is not only possible but also necessary.

Speaking of the breadth of interests and intense communication between experts in very different domains, I do not want to say that this was some particular privilege of the Soviet school. Something similar can be found in other countries as well, even if perhaps not to the same degree.[9] In the end of the Soviet epoque a great enterprise was conceived and to a large degree carried out, the Encyclopedia of the whole of contemporary mathematics. Initiated by R.V. Gamkrelidze, it was modelled on the German *Encyklopädie der mathematischen Wissenschaften*. It had a name that sounded rather boring, something like a subseries of one of the series edited by VINITI.[10] It was only later that Springer-Verlag began to publish it in a beautiful format and with its actual name.[11]

The intention was to explain the principal ideas from all areas of mathematics in the way they were thought of in our school, with numerous examples, motivations of all definitions, what comes from where and what serves for what. You could say, in a style as distant from that of Bourbaki as possible. Although I must say that I rather like Bourbaki, I grew up with those books. In general, it is possible, and necessary, to write about mathematics in different ways. Mathematics is a very diverse branch of science and has many different styles.

Reflecting now on the reasons for our uprising in those years, I came to a conclusion, which at first glance might seem unexpected and paradoxical. The mathematical community had a problem that it experienced in, I can say, a tortured way. This was the problem of isolation. Everybody knows that visits abroad, if they ever happened, were limited to a very few. Access to the literature was difficult to obtain. And not many mathematicians came to visit us.

I remember, in the 1960s, the notes of seminars at Harvard, printed on an IBM machine, with a red carton

7 Note by S. S. Demidov. No, it started on the 15th floor, but the audience was so large that it was transferred to that room.

8 H.A. Gleason, Jr., *An Introduction to Descriptive Linguistics*. Holt, Rinehart andWinston, NewYork 1955.

9 An excellent example is of course the Bourbaki seminar. From the postwar time on it has served as a precious source of information on the works in the above-mentioned areas of mathematics. To me it happened more than once that I arrived at an understanding of new results thanks to the fact that J.-P. Serre, or P. Cartier, or A. Borel, or somebody else explained them clearly and in detail in the proceedings of this seminar.

10 Institute for Scientific Information.

11 Nowadays all the Russian volumes are available at the site www.mathnet.ru.

cover. That machine had a kind of round ball with all the symbols, and you could type a paper with any kind of formulae. Such were the notes of Michael Artin on étale cohomology and those of the lectures of David Mumford on theta-functions. Whenever such a treasure appeared, there was someone who was its happy owner. All the others were limited to asking to look at it, touch it, borrow it for a night, etc. Nowadays, in a time of electronic archives and libraries, this seems absolutely strange.

At the end of the 1960s we had a visit from John Tate, from the US. It was during the time when algebraic K-theory was being created. He was brought to the Steklov, to the department of algebra, which was jammed; you couldn't find a place to sit down. Tate did not even say hello; he went straight to our well-worn blackboard and wrote down the definition of the K_2-group. So he spent an hour and a half explaining the definition, where what comes from and for what reason. The questions were pouring out. And only afterwards did the tension dissipate and a regular conversation take off; how is it going, etc.

I must say that, much later, at the beginning of the 1980s, Americans did contribute to our isolation. With the start of the war in Afghanistan, the American government prohibited the visits of American scientists to us. Then Mumford invented a remarkable solution. Whenever someone was in Europe, he would buy a tourist tour and travel to Moscow as if no barriers existed. And there were no problems.

According to Mumford, it was convenient and rather cheap. When he came to visit us in this way, there were of course problems such as how to get him into the institute, how to organize his seminar, etc. I can't really remember how we managed, but I do recall a meeting that lasted a whole day, in some apartment, a *khruschevka*[12] close to the institute. It was spring, we went out on the balcony. We were about five, all of us from Shafarevich's seminar. Mumford knew all of our papers, the names, but obviously not the faces. So it started, and you are…? Ah, "vector bundles", and you are "Fano varieties", and you, "K3 surfaces", and so on.

So we did have some contacts, and some information did reach us. Letters, just as now they take two weeks to be delivered, took the same two weeks back then… when they actually were delivered, of course. So what did bring about this unprecedented take-off of our school? Were there any reasons particular to us? I think that isolation played a significant role. In those days we suffered this impossibility or rarity of contacts. Nowadays my students, very young, have already spent several times more time abroad than I did in my whole life. In my time, even if they did let you out, until the last moment you never knew if you would really go or not. Nevertheless, now, after many years, looking back I think that the mathematical isolation was not just the obvious evil but also, to some extent, a benefit.

I would like to illustrate this thought with a comparison from biology. In the evolution of living beings, as understood by contemporary science, *isolation* plays a very substantial part in the appearance and development of new taxons. At the appearance of a new characteristic, isolation gives it an opportunity to take roots. A classic example in all textbooks is that of marsupials in Australia and South America.[13] It was precisely their being isolated that allowed the idea of "marsupiality", having arisen there, to flourish in parallel in many species. I feel that the same reasoning can be applied to the evolution of ideas in science.

I wouldn't want to regard isolation as the determining factor (later on I'll give examples of flourishing in those years of the cult of science, which was another significant factor), but it certainly was among the main ones.[14]

These are my thoughts on our past. Now here is what happened when the Soviet Union collapsed. The new epoque started in the 1990s. A huge number of people left. It is not true that they started leaving after 1991, when life became, to put it mildly, difficult. People started leaving as soon as it became possible. In 1986 there was a congress in Berkeley, and not many were allowed to go, but two years later you could go wherever you wanted. And people went, to all possible kinds of places. Nowadays almost all universities in the US, in England, in France, even in New Zealand, if not brimming with our people, contain quite a few of them. That is also our "import", not just oil and gas. To understand our future it is important to understand the composition of our diaspora, the reasons that generated it. The reasons were diverse, and this is not the time or place to discuss them in detail, but the attitude of those who left towards those who remained here, this attitude is worth discussing. This attitude was, and still is, of much variable nature. Here are two extreme examples. One extreme are those who kept their positions at Russian institutes. They come often, deliver seminars, sometimes hold lectures. I know of one mathematician who works in the US and who spent his whole sabbatical year in Moscow delivering a course for students.

The other extreme are people who left for good and who "couldn't care less". In that environment the prevailing point of view is that whoever can leave, should. Here are some characteristic phrases: "the best and the brightest are over there", "there are convertible mathematicians and nonconvertible ones". You understand who is where!

For me personally, it was a great shock that many of my friends and good acquaintances went to the West. Throughout the whole period of the 1990s, at very different places our Western colleagues kept asking me the same question, "Are you still in Moscow?" There was a very large circle of people for whom it was absurd that there is someone, seemingly normal, who remained --

[12] The very modest five-floor houses erected in Khroushev's time, end of 1950s and beginning of 1960s.

[13] See, for example, a survey by a notable paleontologist G.G. Simpson *Splendid Isolation: The Curious History of South American Mammals*, Yale University Press, New Haven and London, 1980.

[14] Recently a similar opinion about the ambivalent nature of isolation was expressed to me by V.M. Polterovich, one of our biggest experts in mathematical economics.

what does he do "there"?! Such attitude existed and to some extent remains still. Nevertheless, in 2000 the environment started gradually changing. The reasons for this were varied. One is that many of those who had left saw that not everything collapsed. Furthermore, the problems of science and scientific community exist both here and there, and while not identical, they do have a lot in common. In the last three or four years there formed, certainly not a stream, but a brook of people more and more oriented towards life here. People come to visit more often. The Steklov Institute opened 5 months positions which get filled more and more. There are even some who, having stayed in the West for ten years or more, decide to come back. This of course is not much but it does give some hope. Even more so because in the last years there are more and more young people who want to do mathematics for mathematics' sake and who are not anxious to leave for the West.

***

I pass now to what will become of us. I do not dare to speak of this at length, but still I would like to share some thoughts and doubts regarding our future. If we are talking about mathematics in itself, I see no reasons for doubt. Our science is developing with success. Theories of amazing beauty are being created. Problems of great importance are being solved. In the 1990s, Andrew Wiles proved Fermat's theorem; in the 2000s, G.Ya. Perelman proved the Poincaré conjecture. But the style of work is changing significantly.

Wiles announced his proof at Cambridge, in May of 1993. It was totally unexpected. No one had known anything in advance. It became known later that there was a text, but only particularly trusted people could have it, and they weren't even allowed to reveal such possession. After half a year a mistake was announced, then the mistake was fixed, and only later were there publications. But here is how it went with the work of Perelman, more than 10 years later. He wrote three pieces, with an about yearly interval, each of which was immediately placed on the arXiv, which can be accessed from anywhere in the world. What he proposed immediately implied the Poincaré conjecture, but he made no such claim. This fact was clear enough as it was. Later on several groups of mathematicians wrote detailed expositions of Perelman's proof and published them.

Let us return to the question of the future. What is obviously changing and will continue to change is the societal attitude towards mathematics and science in general. In the last ten to twenty years there has been a sharp growth of hostility towards science both in our and the Western society. This is entirely obvious. And it is especially visible in contrast with the attitude towards science that existed in the recent Soviet past. It is clear what the attitude towards science was among the intellectual layers of the population, but it also was most positive among the ruling class, including the topmost layers. The way to see the attitude of the chiefs is to look where they send their children to study.

It is clear that a large percentage of their sons and daughters did go to MGIMO[15] or to such institutes that prepared them for external trade. But a sizeable proportion went into the sciences. That included children of the highest-placed individuals, members of *Politbyuro*.[16] Here are well-known examples: the son of P.E. Shelest is a physicist, the grandson of A.N. Kosyghin is a mathematician, the daughter of V.V. Grishin is a division head at the Department of Philology at the MGU, the son of G.M. Malenkov is a biophysicist.[17] In those times doing science was prestigious. This was certainly due to the fantastic successes of science, to its applications. Let us recall the projects in atomic power and the flights into space. I remember, when Yuriy Gagarin was launched into orbit, there was a meeting with him at Red Square and we went there with the ranks of people from Moscow University. There were students, including me and Demidov, second-year students, but there was also S.P. Finikov, the eldest professor of MGU. We arrived at the entrance to Red Square on the right of the History Museum, only to hear an announcement that the affair was over. In a minute we would have seen Gagarin, and they simply told us that "it's over". We had managed to find a heavy wheeled carriage and brought it with us with a huge rocket on it, and among us there were some hot-headed Arab students. In the end we tried to storm the line of police officers. We certainly didn't succeed, but the enthusiasm was enormous.

How did things go in the 1990s? There is no need in this audience to tell this in detail. But here are a few stories (jokes), rather characteristic for that time.

> *"Weird, we stopped paying them their salary, but they still come, do who knows what, measure, count. Maybe we should charge an entrance fee?"*
>
> *"Our economy is just like the Turkish one, why should the science be any better (larger)?"*
>
> *(In the office of a high-ranking individual) "And why is it exactly that we should support you? Well, we did do the atomic project! And was that really necessary?"*

The official attitude has changed now but more in the direction of attention to the requirements of immediate and instant applications, technology outcomes. This was happening in Soviet times as well, but in those times there were science managers capable of explaining to the people in power that without the development of "pure" science no serious application can exist. Anyway, here is a fresh example of such a "bargaining" approach to science. Last fall "NG-Nauka" published an article the size of a whole page with a detailed account of what is wrong

[15] Moscow State Institute of International Relations.
[16] Political Bureau of the Central Committee of the Communist Party.
[17] One could also mention the children of N. S. Khruschev, the son of D. F. Ustinov, the daughters ofYu.V. Andropov and of G.V. Romanov, and perhaps many others. For today's ruling class this cannot be even imagined.

with science. It contained remarkable words of advice such as the ones that follow. "So, there is this *mech-mat* MGU and what do the scientists do over there?? Oh, there are of course some serious departments, but there are some horrible ones, totally unnecessary, such as the division of geometry or that of topology." That the author is an ignorant person, that is clear.[18] He does not have a clue that without the theory of elliptic curves (a branch of geometry) it is impossible to safely and quickly transfer money from one place to another. All the newest cryptography is based on that theory. Whole mathematical institutes survived the years of Eltsin by helping to perform financial transfers all over our boundless motherland.

In order to understand the change of mentality it is worthwhile to give one more example. Consider the ubiquitous mobile phone. As far as I understand, without a significant advance in solid state physics and in programming, the mobile phones would have been impossible. And well, they have taken over the world that no business can function without them. Yet we have heard no recognition on the part of the business community for at least those sciences that gave them an immediate contribution. On a similar occasion L.D. Faddeev wisely said that Faraday and Maxwell had paid for science for centuries to come.

In science itself there is an ongoing process of bureaucratisation. In whatever way we lived in Soviet times, we spent no time or effort on seeking grants; that just didn't happen. There were various meetings and *subbotniki*,[19] but they did not take away the mental energy. My attitude towards the system of grants is rather negative. And many share my opinion. In a recent interview of Yu.I. Manin, he said that one can do science on a budget. The interviewer immediately objected that this always leads to stagnation (the interview took place on a website where free market ideology is the sacred cow), which was followed by a very nice answer. But no, that did not happen (meaning in Soviet times). And I quite agree with this. Grants might make sense for routine activities, but in order to have Perelmans or even one-fourth of a Perelman they are only counterproductive.

The next tendency of this epoque, which changes science a lot, is of course our dependence on computers or, more precisely, *computer ideology*. Our Western colleagues have been complaining for a long time that this branch (computer science) takes away money and people from scientific endeavors. Moreover, in recent times universities have had the inclination to include mathematics in departments of computer science (for now with some sort of mixed name). But that is a superficial side of the question. There is something much more profound and disturbing. Computers have come to permeate our lives, and it has been long necessary to reflect on their true influence and where our dependence could take us.

I will begin with everyday mathematical life. I used to write my papers on a typewriter; sometimes my wife inserted formulae in the resulting text. Many others had to find a typist. The important thing was to do the research and write it down on paper. Then the papers were sent to a journal where they were refereed and sometimes the editors made significant corrections. Then came the process of typesetting, corrections, proofs, and finally printing. There was a long-lived and powerful infrastructure to carry out this process.

All that has now disappeared. His Majesty TEX replaced everything. This means that now the author is, all at the same time, a typist, a typesetter, a corrector, an editor, and all the rest. There is even no need for refereeing; one can just post the paper on the arXiv. This last circumstance has of course a very particular nature, and it should really be discussed separately. But the rest of the process means a huge expense in terms of time and effort. It is necessary to say that TEXis rather well done and has various advantages. But I have seen, in various mathematical centers, so many young people sitting day and night in front of the computer and typing their texts.[20] Where does the time come from for this chore? Precisely the time that could be used to attend a seminar on linguistics or that an expert in number theory could employ to go to a seminar on algebraic geometry. In the remote pre-computer period such time could have been found.

Let us now return to the doubtless advantage of the fact that, having prepared the text and placed it on the arXiv, we thus make it available to whomever, wherever on the very next day. To that, one can add the mention of huge electronic libraries containing all the journals one might want and an enormous number of books. I do not think that this is so important for the development of science. But, yes, in view of the enormous expansion of science one may look at this differently. Let us look back to our golden years. Pierre Deligne, during his residence in Bures-sur-Yvette, creating his remarkable theorems, simply wrote letters by hand. Even if he started in English, depending on the addressee, he later on may have passed to French ("I do have to think!"). These letters were sent to a few colleagues, then copied, and his work was spread in this way. Just as in the seventeenth century. Science was developing splendidly, and not having access to all of it in an arbitrary place on Earth did not prevent its development in the slightest.

I think that the fantastic availability of information that exists now has two aspects. One is the convenience of such freedom. I myself actively use the facilities offered by the Internet (even if I cannot, nor would want

[18] He writes in this manner about a number of "useless" scientific accomplishments, starting from the "impractical" (clearly he forgot what he had been taught in school) relativity theory of Einstein.

[19] Some public works done by scientists and many other people on Saturdays.

[20] One could object that editing a text on a computer (especially in TEX) is much easier than the old way with a typist. That is true, but this ease allows for writing more and more. The technology, simplifying the work, lures the person in with the opportunities it presents. In the end there is much less, not more, free time! Just think about the hours devoted to electronic mail.

to, consider myself an Internet person). But here is the other aspect: all that is just too easy and convenient, like the free cheese which you find you know where. In other words, this is a service that at some point, in some way, will have *to be paid for*. To clarify my idea, I'll give what to me seems a suitable example of an analogous situation, where the end result presented itself very quickly.

Perestrioka provided freedom of speech, and you can publish whatever you want. We all remember the half-million copy print runs of "Novyi Mir".[21] The system of publishing and distributing was excellently set up, and you only needed to abolish the censure. But this happy state of affairs did not last long, and the much-awaited freedom revealed its other side. What ended up dominating was not literature but rather something ready for the recycling bin. The number of print copies for specialized books, about 8–10 thousand, sometimes even 15, in the Soviet time, became something to forget about. Nowadays three thousand print copies is a paradise for a serious book on history, philosophy or letters, and I don't even speak about mathematics.

This historical example, very recent and from which everything can be seen with, so to say, a naked eye, suggests one possible fate of science: the growth of pseudoscience *inside* science itself (you don't need to look far to identify the obvious examples), the supremacy of bureaucracy and an ever greater formalization. Here is an explicit and specific example, the monopoly of Microsoft in the ranks of what should be a free market. The terrible Windows filled the world. It's true that there is no trace of them in Western scientific centers. There, they worked under Unix, now more often under Linux. But for us in Russia that will hardly happen.

The attempt to understand where this computer aggression will lead, requires of course a more serious, I'd say, philosophical analysis. Here are a few thoughts on the subject.

Let us start with considering how the work of a mathematician happens and the role of logic in such work. It consists of a clearly formulated sequence of operations. But it is actually more the end result of a mathematician's work that should be expressed in this way. What precedes it is intuitive sensations, vague images, even fantasies. The clear formulae appear later. This was stated by everybody, from Poincaré to Kolmogorov. These methods of work, the intuitive one and the logical one, can also be described by the terms, continuous and discrete. In a human being they are intertwinned, and it is not easy to separate them. There exists some interesting writing on the subject by Hermann Weyl. The continuous is a domain more of geometry with its drawings, and the discrete is that of algebra with its formulae.

Now if we look at the work of a computer then clearly the basis of it is a discrete, logical approach, which moreover is taken to its extremes. The continuous is completely foreign to a computer, which "digests" it with great difficulty, via digitalization (what a nasty word!) invented for this purpose.[22] And true computer fans perceive this. At the appearance of the mouse, they rejected it with disgust preferring to work on a keyboard. Computers take just one aspect of human activity and blow it out of proportion. The consequences of such development have proved to be unpredictable.[23]

To end my talk, I would like to say that science understood as the development of the ideas that it harbors possesses innumerable opportunities for further growth. But science as a social institution, and even more so as a bureaucratic structure, will undergo, and rather soon at that, radical changes. In particular, it will become much smaller. But as long as there are young people who want to do it without thinking of how much they could earn or who treats them in what way, we can feel at peace about our future.

[21] Translated as "NewWorld", a famous literary journal.

Comments on the talks by A.V. Bulinskii and V.M. Tikhomirov

I wanted to comment on the talk of A.V. Bulinskii but having heard Vladimir Mikhailovich, I cannot abstain from the comment with which I start. The subject is national traditions and schools in mathematics.[24]

Everybody of course has read the lectures of Felix Klein on the development of mathematics in the nineteenth century. Klein speaks in detail about the French school, the German school, the British one. He speaks about the philosophical traditions, which much influenced the development of science. All that happened and there is nothing to discuss. This is a classic from the history of science. Does there exist now some sort of melting pot inside of which the schools disappear? I do not think so!

From the stories told in the 1990s: at Harvard there is a Russian floor, the secretary there speaks Russian, people there are Russian. At some point one of our very well-known mathematicians held a lecture there, starting, naturally, in English. Somewhat later he took a look around the audience, saw that everybody was from Russia, and switched to Russian.

Speaking more seriously, as I already said, in the famous letters of Deligne you can find the following phrase: I now have to think and I switch to French. I admit that

[22] It is enough to compare the size of a text file (especially of a TeX file) with the size of a picture file. For the latter to be comparable with a good photo, it requires a disproportionally huge size.

[23] For some interesting remarks on this subject see the article of P. S. Krasnoschekov "Computerization… Let's be careful" in *Mathematical Events of the Twentieth Century*, loc. cit. (Let me say honestly that the recent tablet revolution is a move towards a more "continous" view of reality. Noted in December 2012.)

[24] In his talk Tikhomirov said that his "deep conviction that both the humankind and mathematics are united, that there is no Russian mathematics, <…>. The science belongs to all human beings and it already became unified <…>. The whole of humankind will not avoid perishing if it is not as united as mathematics can become".

the "everyday" language has a noticeable significance for the work of thought.

And for France we know of a high tradition of thought. In mathematics it was expressed in the second half of the twentieth century by the Bourbaki movement ideologically related to French structuralism.[25] This is a national French heritage, just like the paintings of Cézanne or the philosophy of Descartes. You may not like Bourbaki and consider its influence on mathematics to be destructive, but it is impossible to deny this historical fact. Now we see the process of levelling out, of a fusion into something much more integral. It cannot be disputed, it makes no sense to deny it, but it can be viewed in many different ways: i.e., considered as desirable, achievable, or the opposite of such. In my opinion, the process of annihilating schools in science that is happening in front of our eyes has without doubt a destructive nature. I do not know where it is going to lead in the end, but I do not exclude that in the future we will see a movement in the opposite direction: there will be a disintegration of the now-existing unity into independent, self-sufficient communities, not necessarily based on nationality. If you look at the history of the development of mathematics, you will see that there is no continuous straight vector of development. There are times of uprising and unification, and times of degradation and disintegration.

Regarding the presentation of Alexandre Vladimirovich, I want to say just one thing. I won't speak about the breads served in the morning at the current dormitories of École Normale or Trinity College that impressed him so much. Recall the 1920s and the 1930s, how people lived and how science was pursued! What breads! Read how Pontryagin travelled to lectures on the footstep of the tram. He, a blind man, ran after the tram on *Stromynka*[26] and jumped in.

But what is more important and is related to our future is this. We heard a number, 238 scientific journals on probability theory and mathematical statistics. Someone working in these areas, in order to keep current, must read these 238 journals. When will he or she find the time to just think? I'll give another example. There is an electronic archive[27] that contains publications in the main domains of physics, mathematics, and related sciences. Our famous astrophysicist Andrei Dmitrievich Linde (the creator of the theory of an inflating universe), who now works at Stanford, was telling an interviewer that he starts his morning with a perusal of this arXiv. Naturally, he looks at what is new in the astrophysics section. There arrive about fifty new texts that he should see and formulate an opinion about. Note that multiplying 50 by 360, we get about 15 thousand new texts per year. This abundance he must study in a year. This is not his main job, it's more like a part of a daily routine, like brushing teeth. And what is the result of this activity? About 100 interesting worthwhile papers that are worth paying attention to. Among this hundred, about ten deep and really new papers, they must not be just read but studied in detail. Well, among those ten there will be two or three truly outstanding papers. When does one do science living like this, and who can make such a selection? Nowadays science has an obvious excess of information. The problem is not what to read, but what not to read!

An impressive number of new journals exist (Springer-Verlag on its website has 22 thousand of them[28]), and they have sometimes been created for reasons other than scientific ones. Sometimes there is a fashionable subject, and "everybody" must work on it.[29] Sometimes there appears a group that publishes "their" people. The mathematical world is becoming more and more fragmented into groups, directions, cliques separated one from another by something almost like a Chinese wall.[30] In the huge flood of literature it is already impossible to find your way. Entering any Western library you see a wall of new additions, just to glance over it without looking through takes so much time.

Alexey Nikolaevich Parshin (born 1942) is head of the Department of Algebra and Number Theory at Steklov Mathematical Institute. His research area is number theory and algebraic geometry. Among his results, there is his proof that the Mordell conjecture is a consequence of a conjecture of his teacher Shafarevich. Parshin is a full member of the Russian Academy of Sciences. He received the Alexander von Humboldt Research Award in 1996, an honorary doctorate from University Paris-XIII, and was a plenary speaker at the ICM 2010.

[25] It is not surprising that AndréWeil could write a mathematical note on the structures of relationship inside primitive tribes discovered by Claude Lévi-Strauss.

[26] A university student dormitory in Moscow in the 1920s and 1930s.

[27] http://arxiv.org

[28] Now it has certainly more.

[29] The role of fashion in the science of the second half of the twentieth century has been more and more significant. I could give a number of examples of such power of fashion in the areas of mathematics close to me. Unfortunately, this phenomenon of scientific life has so far not been subjected to a detailed analysis.

[30] Dyson gave a splendid example of this in his famous article "Missed opportunities". I think that one could give many similar examples, even more grotesque ones.

Can Science Advocacy Make a Difference?

The campaign to protect the EU research budget from cuts and implications for the future of science in Europe

Wolfgang Eppenschwandtner (Initiative for Science in Europe, Heidelberg, Germany)

On 7–8 February 2013, the heads of state or government of the European Union convened to a decisive meeting to determine the budget priorities of the EU for the years 2014–2020. Strong pressure to cut on the overall EU budget came from net-payers, in particular from the UK, in the months ahead the summit. On the other hand, several countries, in particular France and the Eastern European countries, are known to be strong defenders of the large spending blocks of agriculture and cohesion. As a consequence, other budget headings, in particular the research budget of the European Union, were highly endangered as a target for substantial cuts.

Luckily, however, in the early morning hours of 8 February, a specific sentence was inserted in the agreement of the EU leaders: "[T]he funding for Horizon 2020 and ERASMUS for all programmes will represent a real growth compared to 2013 level."

How did that happen? What made EU leaders acquiesce to a move that has saved the research communities from stronger cuts?

We cannot look behind the scenes. But we know that never before has the European scientific community been as vocal as in the months before the recent high-level EU budget meetings. With very limited financial means, we were successful in creating media attention and in reaching out to the political communities. Careful strategic thinking is key for such a political campaign to be successful and to create impact and media attention. For good timing, it is important not to expend all your firepower before the final battle has even started. Most crucial was to identify the right moments for intervention before the summits.

The activities were coordinated by the Initiative for Science in Europe (ISE), an independent platform of learned societies and scientific organisations of which the European Mathematical Society is a member. The success would not have been possible without several individuals and organisations that worked with the ISE towards our common aim, notably the Young Academy of Europe (YAE), which was recently founded by ERC starting grant holders.

We were particularly glad to work with the ERC Scientific Council and the Nobel laureates and Fields Medallists in a first phase to rally the scientific community, build trust and, importantly, create media attention. A letter that was finally supported by 50 Nobel laureates and Fields Medallists was drafted to call EU leaders to recognise the importance of research in difficult times of crisis.

Tim Hunt (Nobel Laureate), Wolfgang Eppenschwandtner (Executive Coordinator, ISE), Maria Leptin (President, ISE), José Manuel Barroso (President, European Commission), Helga Nowotny (President, ERC), Jules Hoffmann (Nobel Laureate), Leif Schröder (YAE)

The letter gives a reminder that "[f]unding research at EU level is a catalyst to make better use of the resources we have and make national budgets more efficient and effective". It was placed in major European newspapers including the Financial Times, Le Monde, Frankfurter Allgemeine Zeitung and Corriere della Sera.

As a next step, the ISE launched the online petition "No-Cuts-on-Research.EU" to organise the broad movement of scientists and concerned citizens in support of the EU research budget. The response was initially most strong in the life sciences but more and more communities quickly joined; over the course of the following months, over 153,000 researchers and concerned citizens signed the petition. In fact, the petition played an important role in raising awareness in the research community and in stimulating actions at the national and European levels. Several organisations either joined the campaign or launched their own lobbying efforts and declarations.

With such a strong backing, we could approach the EU leaders and ask for a meeting to hand over the letter of the Nobel laureates and Fields Medallists and the list of signatories of the petition. Despite their busy schedules, a meeting in Brussels could be organised with President of the European Council Herman van Rompuy, President of the European Parliament Martin Schulz and President of the European Commission José Manuel Barroso.

Specific activities have been initiated, supported or triggered by the ISE and the YAE to contact the national governments of at least Germany, Austria, Sweden, the

Leif Schröder (YAE), Maria Leptin (President, ISE), Helga Nowotny (President, ERC), Herman van Rompuy (President, Council of the EU), Tim Hunt, Jules Hoffmann (Nobel Laureates)

Netherlands and the UK, but probably other countries as well. On the fringes of a reception in honour of Nobel laureate Serge Haroche, we succeeded in setting up a short meeting with the French president Hollande just four days before the decisive summit; the list of signatories was handed over to his scientific advisor.

Support could also be secured from the European Roundtable of Industrialists, which is a forum established by the CEOs of major international companies headquartered in Europe. It is remarkable that the leaders from European industry such as Siemens, Ericsson, BASF, Saint-Gobain, Nestlé, Shell and BP chose to express their support for the EU budget by means of a joint letter with the European Research Council (ERC). This is a strong signal that the value of frontier research is recognised by industry leaders along with applied and innovation programmes.

Now what are the next steps? The legal text still needs to be finalised and approved. It is still unclear how the divergence between commitments and payments in the EU budget will affect the exact amount that will actually be spent on research in the coming years. Decisions on the distribution of the budget within Horizon 2020 have not yet been taken. However, major shifts of spending priorities within Horizon 2020 in comparison to the commission proposal are not expected, given the time pressure. In order to avoid a funding gap, first calls of Horizon 2020 need to be issued by the end of 2013.

The result can be summarised as follows. The EU funding programme for research and innovation, Horizon 2020, will certainly be lower than what would be necessary to meet the ambitious innovation targets that the EU leaders have set themselves at various occasions. There will be no real paradigm change towards a more sustainable and future-oriented public spending at EU level. On the other hand, in these times of austerity in many countries and considering the very difficult political situation the EU is in at the moment, it cannot be undervalued that we could prevent strong cuts for the EU research budget.

The ERC will be able to consolidate its activities, although it will not be able to launch new or extended programmes. Funding for mobility and younger researchers within the now called Marie Skłodowska-Curie actions will be lower than in 2013 in the first calls of Horizon 2020, slowly rising in the following years. There will also be new opportunities, including for mathematics, with the FET (Future and Emerging Technologies) programme. Various efforts for simplifications and radical measures to reduce the time to grant will be taken. It will have to be proven in practice whether these measures will be effective in reducing red tape without negatively affecting the quality of selection and programme management.

In the course of running the budget advocacy campaign, a number of challenges for the future became apparent, the most important three of which are:

1) We need to involve the society at large. To create a positive atmosphere for research investment in the long run, it will not be sufficient to try to influence policy makers by means of lobbying or by mobilising researchers: we need to go beyond the research and political communities.
   As a first initiative in that direction, the ISE launched together with a group of researchers and science communicators from Portugal the video contest “Invest in our Future – Invest in Science”. Three awards will be granted for the best video clips of up to two minutes which best convey the message that it is important for the future of Europe to invest in research. Submissions have reached us from all over the world. Please find all details about the contest on http://www.investinscience.eu and vote on Facebook for the best videos!
2) Science advocacy needs to operate strongly at the national and regional level, not only to protect the national research budgets but also to push for EU funds to be used for research.
   In fact, EU member states and regions manage large parts of the EU budget. As part of the regional policy of the EU, the so-called structural funds have in the past been used to improve the infrastructures of poorer regions in Europe. They shall, however, be more and more spent on research and innovation now in the new multiannual financial framework of 2014–20. Regions were called to develop strategies that incorporate research and innovation as priorities. In many cases, activities of the regions will have a narrow and short-term focus on business and job creation. But there have already been a number of very positive examples of regions spending money from structural funds for researcher training and research infrastructures; basic research could profit as well. That is perfectly possible as long as there are convincing and credible arguments that demonstrate the positive impact to the competitiveness of the region and job creation.
   The bulk of the structural funds will be channelled to the underdeveloped regions but even in Western Europe there will still be some limited money available, for example for inter-regional cooperation. Spending of these EU regional funds is highly decentralised;

therefore, little can be done at a European level to direct these funds towards research. Universities and research centres but also scientists and concerned citizens need to contact national and regional authorities to find out what strategy has been developed for their region or country and remind them of the importance of future-oriented spending of structural funds.

3) We need an adequately staffed and independent institution to monitor and analyse R&D budgets and proposals in Europe. Not all stakeholders and political actors have the time and capacity to analyse and digest the EU budget proposals in all their complexity. As a consequence, presentation is what matters.
The example of the Marie (Skłodowska-)Curie programme shows the pressing need for sound and serious analysis of budget numbers to take informed decisions. The commission proudly announced in their initial proposal a 21% increase for the Marie (Skłodowska-)Curie actions. A closer look, however, reveals that the share of Marie (Skłodowska-)Curie actions on the overall budget was planned to decrease from 9% to 7%. Also, the numbers are for the seven year lifetime of the financial frameworks of the European Union, i.e. the total sum for 2007–13 is presented in comparison with the total sum for 2014–20.
That is very misleading. Budget increases have already happened in the seven-year period of FP7 from 2007–13. It makes much more sense to compare 2013, the last year of FP7, with the development over the following years. That gives a completely different picture: the original commission proposal allowed stagnation at best for the coming years. As mentioned above, the final result will be a decrease in 2014 for the Marie Skłodowska-Curie programme and very moderate increases over the following years.

"No-Cuts-on-Research.EU" has been the largest petition for a science cause ever in Europe if not worldwide. Nonetheless, there was no more than two weeks for the preparation of the campaign. The financial and human resources we had at our disposal were very small compared to other advocacy campaigns. As mentioned earlier, the campaign was not adapted to the wider society beyond the research community. We were far from tapping the full potential.

What could be a next step? The "European citizen initiative" is a new and officially recognised EU instrument to collect support for proposed legislative action at EU level. If the initiative is successful with support of more than one million EU citizens the EU institutions are obliged to deliver an official response.

Open letters and petitions need not always target the highest political level; there are issues that will need to be solved by the actors of the science system themselves. For example, following a conference of the American Society of Cell Biology in San Francisco, a declaration to end the misuse of the impact factor has been supported by numerous societies and individuals. Find out how to join this initiative at http://www.i-se.org/researchassessment.

There are many more topics that need to be solved. Observers in science policy often get the impression that problems are known and have been discussed at numerous policy conferences – what is lacking is political will. That's when advocacy is needed for change to take place.

Wolfgang Eppenschwandtner is the Executive Coordinator of the Initiative for Science in Europe (ISE), an independent platform of European learned societies and scientific organisations to promote science in Europe.
Before joining the ISE, he worked at the Austrian Research Promotion Agency (FFG) in the Structural Programmes department as a programme manager for brainpower Austria (now called Talents/Researchers Career Grants). Prior to the FFG, he held a position of Policy Officer at Eurodoc, the European federation of young researchers. Eppenschwandtner is also a co-founder of the Austrian young researchers' network, doktorat.at. In the first half of 2007, he was a trainee at the Council of the European Union in Research and Industry Policy.
In 2006, Eppenschwandtner received a PhD in mathematics from Vienna University of Technology, where he worked in research projects on category theory and applications of algebra in logic and set theory.

Open Access – Who Pays?[1]

Tomaž Pisanski (University of Ljubljana, Slovenia)

Introduction

I come from an ancient time when there was no issue about money for the publishing of mathematics. Research was freely available to all of us; our library subscribed to the most important journals and kept the old issues. If I found a reference that I was interested in and this was not available in the library then I would write to the author and would receive a reprint by mail in a reasonable time. On rare occasions, I would use the inter-library loan system. In a balanced world, this meant only postage costs for each library. Of course, there were periodic budget cuts and we had to reduce the number of our subscriptions. But it was mathematicians who practically ran our library and we sacrificed part of our research money to keep the library running. Every time the economic situation improved we would buy the missing back-issues of the most expensive journals we had had to cancel. So the library paid.

When choosing a journal for my papers, I avoided the few (rare) journals that requested page charges. I would also use any opportunity to visit libraries in neighbouring countries that subscribed to journals not available in our library. This meant that I lived in a world where the author did not have to pay to broadcast new knowledge and the reader did not have to pay to get access to new knowledge. There was no discrimination concerning access to the contents of libraries among mathematicians. Clearly, it was our government that had to pay; it paid the library so that it could buy books and journals, and some governments gave money to mathematical societies to support their publications. Also, by exchanging our journal for the journals of other mathematical societies, we were able to save quite a lot of money and hence increase the number of journals available in our library.

There is a big difference between commercial publishers (often but not always expensive) whose profits go to their shareholders and learned societies whose profits go back to mathematics.

But then six things happened:

1. There was no serious attempt to curtail the apparent greed of some big commercial publishers, which now practically dictate library subscription policies.
2. The internet made electronic manuscripts in the form of preprints and reprints instantly available.
3. The spread of TeX and LaTeX shifted the preparation of the manuscript from the publisher to the author.
4. New 'bundling schemes' tied libraries to commercial publishers and effectively prevented newcomers from entering the mathematics publishing market.
5. Many learned societies are no longer willing to exchange their journals without charge. Their journals were either taken over by a commercial publisher or regarded by the societies as an important source of income.
6. Bibliometric indicators began to play an increasingly important role in the career progression of many mathematicians. The 'publish or perish' syndrome developed into a system in which only a minority of mathematicians had a secure career: recipients of major prizes and other first-class mathematicians with publications in the most prestigious journals. But this covers only a minority of research mathematicians; the remainder had to work harder, publish more papers or find a shortcut.

Three key questions about Open Access Policy

Electronic copies of papers began to supersede printed copies and the idea of open access became a welcome innovation. At the same time, 'garage publishing' became a viable and flourishing proposition — anyone can do it. But we love the imprimatur of our peers. Open access is a wonderful concept but there are three serious issues associated with it:

1. Who pays for it?
2. How can quality be assured?
3. How can fraud be prevented?

The EMS should be leading the way in Europe in addressing these issues. I will make some comments on them, in reverse order.

There are a large number of potentially vulnerable mathematicians who are targets of various predatory publishing houses who advertise instant success for a payment. Numerous electronic messages invite us to attend another conference or join the editorial board of another new, 'important' journal or submit an invited paper for another open access journal. In most cases, it is quite easy to spot and avoid predators. On the other hand, there are many scams set up to which a young and inexperienced mathematician could succumb. I believe that the EMS should advise its member societies and individual members about the dangers and bring to their attention various warnings available on the internet.

Next, mathematicians know how to ensure quality. A good set of editors and referees, adhering to the Code of Practice of the EMS, will produce valuable results in the form of important and interesting papers. One of the key arguments in favour of commercial publishers

[1] The author would like to thank Marston Conder, Garth Dales, Radu Gologan, Christine Jacob, Arne Jensen, Adolfo Quirós and Peter Šemrl for their valuable comments. However, the opinions expressed are my own.

is their role as impartial, multinational bodies that can ensure quality in an objective way. This argument can (but should not) disadvantage independent publishers and smaller learned societies, several of which have transferred their journals to the care of large commercial publishers. EMS should encourage its member societies to keep their journals independent while simultaneously opening their editorial boards to renowned European and international mathematicians.

'Open access' now has a variety of different meanings. Let us try to clarify some of these. In the publishing game, the main players can be identified as follows:

- Author.
- Reader, individual subscriber.
- Publisher.
- Editor, referee.
- Library.
- Government.
- Employer (university, department).

We may use the term 'open access' for any publishing policy in which the principle that 'the reader does not pay' is implemented. In the classical printed journal world, libraries would subscribe to journals and the reader would have free access to new mathematical knowledge. Mathematical libraries I knew never denied access to their holdings to interested readers outside the university. At present there are essentially two alternatives:

1. Open access – author pays.
2. Open access – reader does not pay, author does not pay (free access).

Before we describe possible implementations of 'free access', we would like to stress our strong rejection of the 'open access - author pays' model.

Open access - author pays

The 'author pays' model discriminates against poor mathematicians and is therefore unethical and totally unacceptable. It is natural that the author is the 'seller' and receives money for his work (from various sources), while the reader is the 'buyer' and may give money to obtain new information. In the majority of cases, the author is paid from public (government) sources. Just as it is not right for the author to be paid twice for the same work, it is completely illogical that the author (rather than the reader) should bear the costs of knowledge dissemination. Nowadays, dissemination of information over the internet is not very expensive and it would be logical that the government (as sponsors of libraries) and other public bodies should bear the costs of such dissemination.

Also, it is clear that the majority of authors are disadvantaged by the 'author pays' model. Only the 'rich and famous' will benefit from such a model. Young authors, who have no resources or do not have a well-resourced advisor or a university willing to pay, are at a big disadvantage.

The 'author pays' model has already produced several predatory publishing bodies, with an even greater number of journals taking authors' money. In order to succeed in their career, an author has to publish. On the other hand, the reader does not have to read. Instead of relying on the readers' market, the model exploits the 'publish or perish' paradigm.

This model opens up a serious possibility of a form of modern mathematical slavery. It encourages a situation in which holders of large paying grants are approached by young and unknown, but talented, mathematicians, who will offer their work in exchange for co-authorship.

This will make rich and famous mathematicians even more popular among young graduate students and postdocs, and exacerbate the 'brain-drain' in some constituencies, where young mathematician move from what they perceive to be a less favourable environment to a more favourable environment.

Authors will be tempted to use any kind of money to pay for open access, including their grant money, the money from their own pockets or travel money. A typical amount of travel money for a mathematician in southern, south-eastern or central Europe is about 2,000 to 3,000 euros per year. Under the 'author pays' model, this money could be completely spent on just one or two open access publications.

Tragically, some governments have already adopted an 'author pays – open access' policy. In the present economic climate, this has resulted in a direct attack on research funding, as, in many cases, money for publications will not be provided separately. Some governments provide funding to the universities for gold open access publishing. This leads to the question: "How are decisions on funding publication taken? By the head of the department or by some administrator (probably using bibliometric data to reach a 'fair' decision)?"

We believe that the EMS should vigorously oppose such an unethical and damaging policy and demand that governments should instead help reputable publishers and scientific societies so that they can offer free access journals and provide reliable access to old issues.

Free access

In the 'free access' model, neither the author nor the reader has to pay, but someone has to pay. In order to provide equal opportunities to mathematicians, who are already providing their services as authors, editors and referees, there is just one possibility: governments should subsidise publishers and libraries.

What can libraries do for existing high quality, free access journals? For journals that come in both electronic and printed version, libraries could subscribe to the printed version and thereby actively support the publisher.

What can the EMS do? The EMS should endorse all high quality, free access journals, especially those that are published by the EMS Publishing House or by any of its member societies or other corporate members. It should pay close attention to the new form of publications: the so-called epi-journals. However, this seemingly wonderful idea may also be prone to fraud.

What can publishers do? Even if they offer the 'author pays – open access' model, they should keep it as an option in a hybrid model. In particular, the decision of whether the author will pay and make the paper freely available should be taken only after the accepted paper is in its final form. Also, publishers should offer discounts to students, postdocs, retired people, unemployed people and authors from poor countries. Instead of taking copyright away from the author they should help the author fight plagiarism.

What can governments do? Governments should subsidise all publishers that offer free access journals. They should also encourage publication in free access journals and discourage publication in journals using the 'open access – author pays' model.

What can editors and referees do? They should refuse to work for journals in which there is no alternative to the 'author pays' model.

What can employers do? They should encourage their employees to use high quality, free access journals as a home for their work and discourage publication in any journal that does not uphold high ethical standards. In particular, they should stand up against the monopoly of the commercial publishers and blacklist predatory publishers and predatory journals.

What can authors do? They should opt for high quality, free access model journals and recommend them to their libraries. Using their grant money they should support such journals by subscribing to their printed-copy version.

The 'author pays' model is not the solution. It heavily discriminates against mathematicians from poor countries and creates more problems than it solves. The solution for publications in mathematics is quality-controlled free access.

Fatal misconception

Many mathematicians and decision makers believe that commercial publishers ensure quality of publications. Unfortunately, this is not the case. The quality of publications is ensured only by mathematicians who are involved in the publishing procedure as authors, referees and editors.

Mathematicians and other scientists have used an informal ranking system for making decisions. For some of them, ranking and comparing their colleagues and students is a favourite pastime activity. Everybody 'knows' that publishing an article in *Acta Mathematica* is harder and is therefore worth more than an article in an ordinary mathematical journal. When writing a letter of recommendation we are frequently asked to rank and compare the candidate against their peers. Commercial publishers, lately also predatory publishers, and other companies operating in the bibliometry business have simply made our implicit scales explicit. By a series of tricks they attained the impossible: they turned immeasurable quality of research into positive numbers. Anything can be compared against anything else. Everybody can check everybody. Agencies that give money for research love it. Bibliometrists learned our game and they are now setting the rules that most of us are willingly following. They are just feeding on our vanity. And we are happy to publish our research in journals that our libraries can hardly afford.

Tomaž Pisanski [Tomaz.Pisanski@fmf.uni-lj.si] is a professor of discrete and computational mathematics at the University of Ljubljana, Slovenia. He is a winner of a bronze medal at the IMO and a member of the Academia Europaea, the Engineering Academy of Slovenia and the International Academy of Mathematical Chemistry. He is one of the two Founding Editors and Editors-in-Chief of the free access, peer reviewed journal Ars Mathematica Contemporanea. His research interests include several areas of discrete mathematics ranging from chemical, topological and algebraic graph theory to discrete geometry. He is interested in historical, philosophical and ethical issues related to mathematical sciences.

The Danish Mathematical Society Turns 140

Vagn Lundsgaard Hansen (Technical University of Denmark, Kgs. Lyngby, Denmark) and Bjarne Toft (The University of Southern Denmark, Odense, DK)

On the evening of 8 October 1873, a group of 65 people attended a meeting in Copenhagen, called with the purpose of founding a society whose members could meet regularly for conversation, lectures, discussions and occasional social gatherings, to encourage a lively interaction for the benefit of the mathematical sciences and their practical applications. The new society was named *Matematisk Forening*. The name was changed to *Dansk Matematisk Forening* in 1952.

The idea to found the society was conceived by the actuarial scientist, mathematician and astronomer *Thorvald Nicolai Thiele* (1838–1910), who is now famous as a pioneer of statistics. There were nine founding members who called for the meeting, including the famous Danish mathematicians *Julius Petersen* (1839–1910), known for graph theory, the Petersen graph and a famous book on geometrical constructions, and *Hieronymus Georg Zeuthen* (1839–1920), known for his work in enumerative geometry and the history of mathematics. Among the founding members was also a young military officer *V. H. O. Madsen* (1844–1917), who taught mathematics at the Military Academy and later rose to the rank of general. In addition to the military and mathematics, Madsen had a political career and was secretary of defence in an important government 1901–1905, marking a shift in the political system in Denmark. He also served as president of the society 1903–1910.

The history of the Danish Mathematical Society is well documented in written material after 1905, while the history of the early years are based mostly on remembrances from members of the society collected around 1923 by *Christian Crone* (1851–1930), who was a board member of the society in the period 1880–1882. In the early years, the society held regular meetings in a restaurant in Copenhagen and archives of the society were apparently stored in a box in the cellar of the restaurant. For unknown reasons the box disappeared sometime in the 1890s.

Crone writes that Thiele was much appreciated by his students, although he was less easy in his interactions with other people than Zeuthen and Petersen. For a number of years, Thiele kept his house open for mathematicians on fixed evenings and it was here that the idea of creating a mathematical society was conceived. It is evident that Crone had a deep respect for Zeuthen and his ability for strong concentration within his research field of enumerative geometry, where he is one of the pioneers, and in his groundbreaking studies of Greek mathematics. Zeuthen was greatly appreciated by his many students due to his noble and gracious character. Petersen also made an impression on the young Crone, who summarises his view of Petersen with these somewhat ambiguous words: "Concerning Julius Petersen one can maybe say, that his strength was more the fertility of his endowment rather than his ability for strong concentration." But, nevertheless, Petersen gave brilliant lectures with clear expositions and many elegant details, to the undivided enjoyment of the participants. Petersen and Zeuthen were very different in their behaviour and attitudes, but mathematics in Denmark has never seen a better pair of colleagues – probably due to their common interest in mathematics, starting when they were boys of 10 living four houses apart in the town of Sorø.

In the years 1895–1903, the number of members of the society was about 40–50, showing a decline in the number of members in the period 1873–95 from the original 65 members signing up in 1873. In the years after 1903, the membership grew again and reached 95 in 1923. In 1951, the number of members was 119 and by the time of the 100 year anniversary in 1973 it had doubled to more than 200. This growth was stimulated by the new mathematical department that had opened at Aarhus University in 1954. The growth in membership was further enhanced by the mathematical departments that opened in Aalborg, Odense and Roskilde around 1970, so that today the Danish Mathematical Society has about 300 members of which around 70 are also members of the European Mathematical Society.

It is not possible to name the presidents of the society before 1892, partly due to the more informal organisation of the society in the early years and in particular due to the lost archives of the society from that period. Internationally known presidents of the society up to1951 include: J.L.W.V. Jensen (1859–1925), president 1892–1903, known for the famous *Jensen inequality*; and Harald Bohr (1887–1951), president 1926–29 and 1936–51, known for his theory of *almost periodic functions*. After the death of Harald Bohr, the society was restructured and fixed terms for presidents and other members of the board were introduced so that presidents could serve for only four years at a time. As mentioned earlier, the society also changed its name to *Dansk Matematisk Forening* in 1952. Of the presidents serving under the new terms let us mention: Børge Jessen (1907–93), president 1954–58, known for his contributions to *integration theory* and *subdivision of polyhedra*; and Werner Fenchel (1905–88), president 1958–62, known for his deep work in *differential geometry* and *convexity theory*.

Up to around 2000, the main activities of the society were regular evening meetings with an invited lecture followed by a gathering, where the participants and the lecturer would meet and share light refreshments with open sandwiches. Here you could discuss the lecture and what was going on in the various mathematical circles in Denmark. In this period of the life of the society, this functioned very well and was crucial, since it was the place where Danish mathematicians, mostly in the Copenhagen area, could meet each other on a regular basis. From the end of the 1980s it became more and more clear that evening meetings were no longer so popular, with many young parents among the members, a growing number of colloquia and mathematical seminars at all the major mathematical departments in Denmark and fast-growing specialisation in mathematics. It became difficult to attract people to the meetings.

Bodil Branner, president 1998–2002, made a substantial effort to transform the society into a mathematical society for all Danish mathematicians. To this end, she initiated a Danish newsletter *Matilde*, modelled on the EMS Newsletter, to provide a common reference point for the whole membership. She also tried to spread the activities of the society throughout Denmark, for example by having annual meetings at the various Danish universities in turn. In the beginning, all of this worked very well. But we have come to realise that it is difficult and expensive for a small country to run a nice, polished magazine. Lately, we have had difficulties in managing the editorial process of the production of Matilde. We hope, however, that we can succeed in getting our magazine running again in good shape. In the meantime, we enjoy the brilliant work done in both the editorial phase and in the production phase of the magnificent EMS Newsletter.

Bibliography

[1] C. Crone. *Matematisk Forening gennem 50 Aar*. Published by Matematisk Forening i København, Jul. Gjellerups Forlag, 1923.
[2] Steffen L. Lauritzen. *Thiele: pioneer in statistics*. Oxford University Press, 2002.
[3] F.D. Pedersen. *Dansk Matematisk Forening 1923-1973.* Published by Dansk Matematisk Forening, København, 1973.
[4] K. Ramskov. The Danish Mathematical Society through 125 Years. *Historia Mathematica*, 27: 223–242, 2000.

Vagn Lundsgaard Hansen [vlha@dtu.dk] is a professor emeritus at the Technical University of Denmark. He was the first Chairman of the EMS RPA Committee, 2000–2006. In 2012 he was named a Fellow of the American Mathematical Society (inaugural class). He served as President of the Danish Mathematical Society for the period 2008–2012.

Bjarne Toft [btoft@imada.sdu.dk] is an associate professor at the University of Southern Denmark in Odense, Denmark. He studied at Aarhus University and the University of London, obtaining a PhD in 1970. With T. R. Jensen he is the author of a much cited book [Graph Coloring Problems, Wiley 1995], which contains a description of more than 200 open problems. Bjarne Toft was elected President of the Danish Mathematical Society in 2012.

ICMI Column

Mariolina Bartolini Bussi (Università di Modena e Reggio Emilia, Italy)

The Volume of the 19th ICMI study

At the end of 2012, the Volume of the 19th ICMI study on "Proof and Proving in Mathematics Education" was published. This volume, edited by Gila Hanna and Michael de Villiers, was the outcome of the long process of realising the study: the official launch in 2007 with the appointment of the two co-chairs; the invitation of eight additional experts in the field of proof in mathematics education to serve on an International Program Committee (IPC); the organisation of two IPC meetings (in Essen and in Sèvres) to prepare the discussion document and to select the invited participants in the study; the organisation, in Taiwan, of the study conference, with additional invited scholars (Giuseppe Longo, Jonathan Borwein, Judit Grabiner and Frank Quinn) to deliver plenary talks on topics related to proof in mathematics; and the public presentation of the volume at ICME 12 in Seoul (South Korea). From this, it is evident that the intention was to take into account different cultural traditions. A panel was also organised at the study conference, with eminent experts (Karin Chemla, Wann-Sheng Horng and Men Keung Siu) to discuss proof as perceived in ancient Chinese mathematics writing.

The outcome is a very rich volume, accompanied by the proceedings of the study conference, freely available online from the ICMI website (http://www.mathunion.org/icmi/digital-library/icmi-study-conferences/). It is worthwhile mentioning that during the long process of the study, the publication of the proceedings of an additional conference on "Explanation and Proof" (Essen, 2006) took place, with most of the participants who took part in the ICMI study. Hence, in a few years, a number of relevant volumes about the teaching and learning of proof have been published. This coincidence is not trivial and bears witness the importance of proof in mathematics education.

One of the most significant tasks facing mathematics educators is to understand the role of mathematical reasoning and proving in mathematics teaching, so that its presence in instruction can be enhanced. This challenge has been given even greater importance by the assignment of a more prominent place of proof in mathematics curricula at all levels.

Along with this renewed emphasis, there has been an upsurge in research on the teaching and learning of proof at all grade levels, leading to a re-examination of the role of proof in the curriculum and of its relations to other forms of explanation, illustration and justification.

This book brings together a variety of viewpoints on issues such as:

- The potential role of reasoning and proof in deepening mathematical understanding in the classroom, as it does in mathematical practice.
- The developmental nature of mathematical reasoning and proof in teaching and learning from the earliest grades.
- The development of suitable curriculum materials and teacher education programmes to support the teaching of proof and proving.

The book considers proof and proving as complex but foundational in mathematics. Through the systematic examination of recent research, this volume offers new ideas aimed at enhancing the place of proof and proving in our classrooms.

The presence of scholars from the Far East and the conference location in Taiwan is a clear indication of the growing importance, acknowledged by the ICMI, of Eastern cultural tradition.

References

Hanna G., Jahnke H. N., Pulte H. (eds.) (2010), *Explanation and Proof in Mathematics. Philosophical and Educational Perspectives*, Springer.

Hanna G. & de Villiers M. (eds.) (2012), *Proof and Proving in Mathematics Education*. 19th ICMI Study, Springer.

Sociomathematical Norms: In Search of the Normative Aspects of Mathematical Discussions

Education Committee of the EMS

What are sociomathematical norms?

The notion of sociomathematical norms is important in mathematics education because it refers to what practices of participation and performance are regarded in a mathematics lesson as proper or correct, and even more importantly what practices are regarded as improper or incorrect and by whom. Attention to this notion over the last two decades has led research to acknowledge that, in all settings of teaching and learning mathematics, there are accepted and unaccepted ways of doing, reasoning, behaving, communicating…

Yackel, Rasmussen and King (2000) provide interesting examples of sociomathematical norms from undergraduate mathematics lessons in first-order differential equations. When looking at the norms that arise in the lessons, they see that the teacher in the classroom gives value to practices of explanation that are grounded in the explicit discussion of rates of changes, and when doing so he insists on the fact that only those students who first say they are certain about their reasoning are fostered to intervene. One question that appears is what practices of explanation are considered inappropriate in the context of that classroom. If the students whose reasoning is at an early stage of elaboration are not welcome to participate, how can their ideas be incorporated in the discussion? In this way, the interpretation of mathematical explanations as final, static, "secured" products is suggested, in opposition to mathematical explanations as in-construction and collective reasoning (see EMS Committee of Education, 2011, for a link to the solid finding on proof and explanation).

To gain a better sense of why the notion of sociomathematical norms can be considered a solid finding in mathematics education research, we give three more examples:

1) We can consider one classroom in which most mathematical practices are linked to procedural instruction with the processing of diverse algorithms. Meanwhile, in a second classroom most practices are linked to conceptual reasoning with the construction of diverse argumentation strategies. At first sight, the mathematical content and activity may be quite similar but, from the perspective of sociomathematical norms, the two classrooms are very differently constituted with respect to what becomes valuable, and it is quite reasonable to expect the contents of the students' learning to be different.

2) The students in a classroom are asked to collaborate in small groups for the resolution of open-ended problem situations and then to report their group decisions to the whole class. Meanwhile, the students in a second classroom are recommended, either implicitly or explicitly, to always ask for clarification from the teacher and act as individual participants with individual ideas. Again, while the mathematical content and the classroom activities may be generally the same, the two classrooms are very differently constituted with respect to the emphasis on letting the students gain autonomy from the teacher and become more active in the communication and negotiation of their mathematical thinking.
3) A teacher in one classroom teaches the meaning of concrete symbols by reference to the language of mathematics and illustrates how they are to be used by exemplifying formal mathematics on the board (e.g. $2 \notin Q$, which would accept the sentence "the square root of two is not rational"). A second teacher teaches the use of symbols by means of modelling them in situations of problem solving to represent ideas (see EMS Committee of Education, 2012c, for a link to the solid finding on modelling). What symbols are expected to be for would be differently interpreted, and learned, in the two settings. Also, how the use of such symbols influences understanding of mathematical concepts may come to be substantially different.

These examples may raise several questions and particularly point out the role of the teacher's knowledge in the establishment of norms (see examples in the solid finding on teachers' knowledge, EMS Committee of Education, 2012a). But what they all illustrate is the robustness of the notion of sociomathematical norms: different sociomathematical norms promote different mathematical learning opportunities and different ways of access to mathematical concepts. And, overall, mathematical learning opportunities and mathematical concepts are better approached through the establishment of certain sociomathematical norms in the context of teaching and learning, i.e. the norms that move actions in the direction of facilitating critical, autonomous and reflective learners.

The norms debate – a bit of history

In their seminal work, Yackel and Cobb (1996) coined the notion of sociomathematical norms to refer to the normative aspects of mathematical discussions that are specific to students' mathematical activity. In this way, they extended more general works on the ideas of obligations (Voigt, 1985), classroom social norms (Yackel, Cobb & Wood, 1991) and didactical contract (Brousseau, 1988), which was the topic in EMS Committee of Education, 2012b. Since then, the understanding of sociomathematical norms has been central to the interpretation of classroom interaction, students' mathematical activity and teachers' actions toward the creation of teaching and learning opportunities. Several mathematics lessons at different age levels have been analysed to illustrate how social and sociomathematical norms are manifested and developed in the interaction among participants. These works have led to knowledge on how meanings as to what counts as a quality, acceptable contribution are negotiated in contexts of mathematics teaching and learning. In brief, it is knowledge that applies to all forms of mathematics classrooms. Also, it applies to learning at university and thus to frameworks of mathematicians in their professional teaching practices.

Social norms are regulations that involve taken-as-shared meanings of what constitutes an appropriate contribution to a discussion. The term social implies being conjointly constructed by the participants in the classroom and does not bring up the specificity of the domain that is to be constructed. Most of the social norms either explicitly or implicitly wanted by mathematics teachers may be the same as those expected and modelled by language or science teachers. We can imagine all these teachers asking for cooperation in small group settings to complete activities. Such norms, therefore, do not necessarily inform about mathematical aspects of the activity. As said by Yackel (2001), however, the distinction between social norms and sociomathematical norms is subtle. For example, the understanding that students are expected to explain their solutions is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm.

Research on social and sociomathematical norms has been strongly associated to design experiments (Cobb et al., 2003) in classrooms. They consist of experimental teaching situations that may range from a few weeks to an entire school year or university term. One of the goals of the teaching experiments is to support the students' learning of mathematics by facilitating their progressive adjustment to certain seen-as-adequate norms through the orchestration of collaborative environments allowing negotiation of mathematical meanings and solutions. In brief, teaching experiments are scenarios for renegotiating concrete norms and gradually introducing new, more meaningful ways of doing and understanding mathematics.

Yackel (2001) reports a teaching experiment that took place for the whole school year in an elementary classroom. An instructional sequence was planned and developed to reinforce ways of reasoning about numerical facts. The students were systemically asked by the teacher (and she herself modelled this behaviour) to elaborate on explanations that described procedures. When adding quantities such as 13 and 12 to get 25, explanations such as: "One and 1 are 2, and 3 and 2 are 5," were challenged by remarks such as: "That's a 10 and that's another 10, and that's 20. And the answer is 25." By the end of the school year, some students took explanations as explicit objects of reflection and made comments like: "How can someone understand what you mean? They don't know what you're referring to." A norm had been established about taking the explanations as entities in and of themselves and commenting on their potential as acts of communication for the learning of mathematics.

McClain, Cobb and Gravemeijer (2000) describe a teaching experiment to support ways of reasoning about data. An instructional sequence was designed and implemented in a seventh-grade classroom with whole-class discussions primarily focused on the ways in which students organised data to develop arguments. The key sociomathematical norm that became established was that of explaining and justifying solutions in the context of the problem being explored. Consequently, students reconceptualised their understanding of what it means to know and do statistics as they compared and contrasted solutions. This is indeed a different approach to statistics in middle schools that allows students to engage in genuine problem solving that, in turn, supports the development of mathematical concepts.

Norms and beliefs – two sides of the coin

In EMS Committee of Education, 2013, the influence of beliefs in the teaching and learning of mathematics was presented as a solid finding. We now point to norms and beliefs as two sides of the same coin, in that they are closely related constructs. While sociomathematical norms is a sociological construct that helps to examine social issues, mathematical beliefs is a psychological construct that helps to examine the individual perspective. The relationship is clear in Yackel and Cobb (1996): *This paper sets forth a way of interpreting mathematics classrooms that aims to account for how students develop mathematical beliefs and values and, consequently, how they become intellectually autonomous in mathematics* (p. 458).

Given we are considering the mathematics classroom, we can take an approach similar to that adopted by the beliefs approach and recognise the importance of making norms visible enough so that all participants become aware of what the requirements in the context of teaching and learning are. Another parallelism may be established between classroom social norms and beliefs about our own role, others' roles and the general nature of mathematical activity. As with the case of social and sociomathematical norms, the distinction between social norms and certain beliefs is subtle. Yackel and Cobb (1996) refer to a student who changes her answer when the teacher asks the class if they agree with her. In the analysis of various discussions, it is revealed that the student interpreted the teacher's questions as indicating that she had made an error. This example points to differences in the understanding of the role and use of questions by the teacher. Planas and Gorgorió (2004) document a similar situation. When interviewed, the student says he does not want to introduce mathematical errors into the discussion and adds that it is easy for him to make errors because he is a learner. For him, the role and use of questions by the teacher is closely linked to beliefs on what a learner is in the mathematics classroom.

When investigating the teaching and learning of mathematics, various factors on influence appear: argumentation and proofs, professional development and teacher knowledge, reciprocal expectations between teacher and students, models and modelling, beliefs and orientations, etc. In this article, sociocultural aspects of the mathematics classroom have been raised. It has been argued that the study of norms, and the modification of some of them in the classroom interaction, gives us a way to make sense of and improve the conditions for the teaching and learning of mathematics. It has also been argued that the study of norms is linked to approaches in mathematics education that claim the need to emphasise collective practices of argumentation and reasoning. These are the practices that most mathematicians and mathematics educators consider to better represent the inward nature of mathematics.

Authorship

Even though certain authors have taken the lead in each article of this series, all publications in the series are published by the Education Committee of the EMS. The committee members are Tommy Dreyfus, Ghislaine Gueudet, Bernard Hodgson, Celia Hoyles, Konrad Krainer, Mogens Niss, Juha Oikonnen, Núria Planas, Despina Potari, Alexei Sossinsky, Ewa Swoboda, Günter Törner, Lieven Verschaffel and Rosetta Zan.

References

Brousseau, G. (1988). Le contrat didactique: Le milieu. *Recherches en Didactique des Mathématiques*, 9(3), 309–336.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in education research. *Educational Researcher*, 32(1), 9–13.

EMS Committee of Education (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *Newsletter of the European Mathematical Society*, 82, 50–53.

EMS Committee of Education (2012a). It is necessary that teachers are mathematically proficient, but is it sufficient? Solid findings in mathematics education on teacher knowledge. *Newsletter of the European Mathematical Society*, 83, 46–50.

EMS Committee of Education (2012b). What are the reciprocal expectations between teacher and students? Solid findings in mathematics education on didactical contract. *Newsletter of the European Mathematical Society*, 84, 53–55.

EMS Committee of Education (2012c). Models and modelling in mathematics education. Newsletter of the European Mathematical Society, 86, 49-52.

EMS Committee of Education (2013). Solid findings in mathematics education: Living with beliefs and orientations. Underestimated, nevertheless omnipresent, factors for mathematics teaching and learning. *Newsletter of the European Mathematical Society,* 87, 42–44.

McClain, K., Cobb, P., & Gravemeijer, K. (2000). Supporting students' ways of reasoning about data. In M. J. Burke & F. R. Curcio (Eds.), *Learning mathematics for a new century – 2000 Yearbook* (pp. 174–187). Reston, VA: NCTM.

Planas, N., & Gorgorió, N. (2004). Are different students expected to learn norms differently in the mathematics classroom? *Mathematics Education Research Journal*, 16(1), 19–40.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathématiques*, 6(1), 69–118.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Panhuizen-Heuvel (Ed.), *Proceedings of the Conference of the International Group for the PME* (vol. 1, pp. 1–9). Utrecht, the Netherlands: PME.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458–477.

Yackel, E., Rasmussen, C., & King, K. (2000). Social and sociomathematical norms in an advanced undergraduate mathematics course. *The Journal of Mathematical Behavior*, 19(3), 275–287.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education*, 22(5), 390–408.

ERME Column

Viviane Durand-Guerrier (University Montpellier 2, France) and Susanne Prediger (TU Dortmund University, Germany)

ERME, the European Society for Research in Mathematics Education, is a young society that was created in 1999 in Osnabrück (Germany) to support communication, cooperation and collaboration between European researchers.

ERME cannot be dissociated from the European conference CERME held every two years in European countries. The conferences are organised in order to promote communication, cooperation and collaboration, aiming to find a balance between scientific requirements and an inclusive policy and attitude. Special attention in these activities is given to young researchers and researchers who work rather alone due to local circumstances. In order to reach these aims, CERME adopted a specific organisation with thematic working groups in which researchers have sufficient time to really get to know each other, share and discuss their research and engage in deep scholarly debate. These thematic working groups are mostly stabilised through the successive CERMEs, favouring the development of European communities of research in the main domains of mathematics education research. The next conference CERME 9 will be held in Prague, Czech Republic, 4–8 February 2015.

Being highly attentive to the diversity and representation of all European countries in ERME and CERME, the society organises financial support for researchers from under-represented European countries and those countries in which research in mathematics education is strongly connected to European communities (e.g. non-European Mediterranean countries). This year, financial support allowed 26 researchers to attend the CERME conferences.

From the very beginning, ERME has supported and encouraged young researchers by means of summer schools (YESS: Youth ERME Summer School), which are held every two years, alternatively with CERME, and by the YERME day that precedes the CERME conferences. The next YESS will take place in Kassel, Germany, 4–11 August 2014.

Over the last three years, further steps have been taken to increase the involvement of young researchers in ERME responsibilities. Firstly, since CERME 8 (Antalya, Turkey) in February 2013, two young researchers have been participating in the International Programme Committee of CERME as members. Secondly, the General Assembly in Antalya approved a modification of bylaws stating that the ERME Board (formally the Executive Committee) should also comprise two representatives of the Young Researchers.

The actual members of the ERME Board are Viviane Durand-Guerrier (France, President), Susanne Prediger (Germany, Vice-President), Nada Vondrova (Czech Republic, Secretary), Markku Hannula (Finland, Treasurer), Therese Dooley (Ireland), Uffe Thomas Jankvist (Denmark), João Pedro da Ponte (Portugal), Cristina Sabena (Italy), Carl Winslow (Denmark) and, as representatives of the Young Researchers, Miguel Ribeiro (Portugal) and Susanne Schnell (Germany).

Over the last four years, Ferdinando Arzarello, as former ERME President, and the Board have continuously pushed the development of the society. After the stabilisation of the constitution and bylaws, the next step was to engage collaboration with European societies in the field of mathematics education and with mathematicians through the European Mathematical Society, which had been initiated by Barbara Jaworski. This collaboration between the EMS and ERME is expressed by these regular ERME columns in the EMS Newsletter, by the presentation of selected solid findings in mathematics education research in the EMS Newsletter and by the substantial work done in order to rank the European journals in mathematics education (http://www.ems-ph.org/journals/newsletter/pdf/2012-12-86.pdf).

Also, due to the efforts of the former president Ferdinando Arzarello, ERME is now an Affiliated Association of the ICMI (International Commission of Mathematical Instruction), which officially represents the world community engaged in mathematics education. Along with the nomination of Ferdinando Arzarello as the new President of the ICMI and the venue of ICME 13 in Hamburg (Germany) in 2016, this offers opportunities for ERME to become more widespread and promote European research traditions and specificities in the worldwide community. One of the next tasks for ERME will be to improve the visibility of research in mathematics education to the European administrative and political bodies.

Website of ERME:
http://www.mathematik.uni-dortmund.de/~erme/

Viviane Durand-Guerrier has been the President of ERME since February 2013.
Susanne Prediger has been the Vice-President of ERME since February 2013.

An Invitation to the New zbMATH Interface

Helena Mihaljević-Brandt and Olaf Teschke (both FIZ Karlsruhe, Berlin, Germany)

This summer, zbMATH is launching its new website, accessible at http://zbmath.org, with the main aim of providing an intuitive and user-friendly information service with extended functionalities for the mathematical community.

A regular user of zbMATH might wonder: Is there really a need to replace an already established interface that was thoroughly overhauled just three years ago? Wouldn't it just be sufficient to add some new features if needed and rather concentrate on content improvement?

The way scientists use information infrastructure has certainly changed. Instead of the search for a single review or article it has become more important to answer multifaceted questions pertaining to authors, networks, topics or formulas. Furthermore, the structure of the manifold information that recently became available in the database, like author information[1], citations[2], mathematical software[3] and full text repositories[4], with their various connections and interrelations, has pushed the zbMATH interface to its limits. With the ongoing demands from the community to offer detailed and easily accessible information, a new framework became necessary with the potential to incorporate the results of recent and future developments.

Hence the answer is yes! Indeed, we do need a new interface as a fundament for current and future developments. We were encouraged in this engagement not least by the 2012 user survey, which gave us valuable feedback from the community and whose results we strongly aim to incorporate into our services.

Structure and main features

The underlying core of the web facility remains: a powerful search engine that allows for various logical combinations of queries in many indexed fields.

However, in our experience, a typical user can rarely exploit the entire complexity of the search engine. For instance, would you know how to find the author with the highest number of publications in the *Annals of Mathematics* or *Inventiones Mathematicae*? Or would you know how to search for a hot topic in Russian mathematics during the '70s? In the new zbMATH interface, information of this kind becomes easily accessible by using the new filtering function, which provides the user with an instant and interactive refinement of the results with respect to authors, journals, publication years and subject classification (see Figure 1).

Fig. 1. The new filtering function allows the formulation of complex queries and easy refinement of the search results.

The search for specific information related to, for example, authors, journals or documents is now organised in separate tabs. On the level of results, one then has the possibility of switching views in the sense of the content structure. The composition of the various displays was modelled on the author profiles that were introduced in 2011 on the old website[1]: the profile of a certain mathematician displays her most frequent co-authors and the number of articles they wrote together. Clicking on a co-author's name leads the user to the respective author profile, while the displayed number of their common publications links to the corresponding joint documents. Analogous navigation is provided for journals, publication years and mathematical subjects. In addition, there is the possibility of adding further information, such as pictures. This logic is now reflected in all search facets, giving an intuitive and natural site navigation.

Naturally, a relaunch of this kind comes with many changes regarding functional details and design that are better explored than described. As a general principle, the interface concentrates on key visual functions, while text parts (which have grown inevitably over the years) have been reduced. Additional information is now often available on mouse-over functions or scroll-down menus, which have lightened up the site significantly.

The interface is now easily usable by mobile devices, since a large part of the site navigation and search

---

1 "Author profiles in zbMATH", *EMS Newsletter* 79, pp. 43–44.
2 "Time lag in mathematical references", *EMS Newsletter* 86, pp. 54–55
3 "The Software Information Service swMATH – release of the first online prototype", *EMS Newsletter* 87, pp. 48–50.
4 EuDML: The Prototype and Further Development", *EMS Newsletter* 85, pp. 57–58.

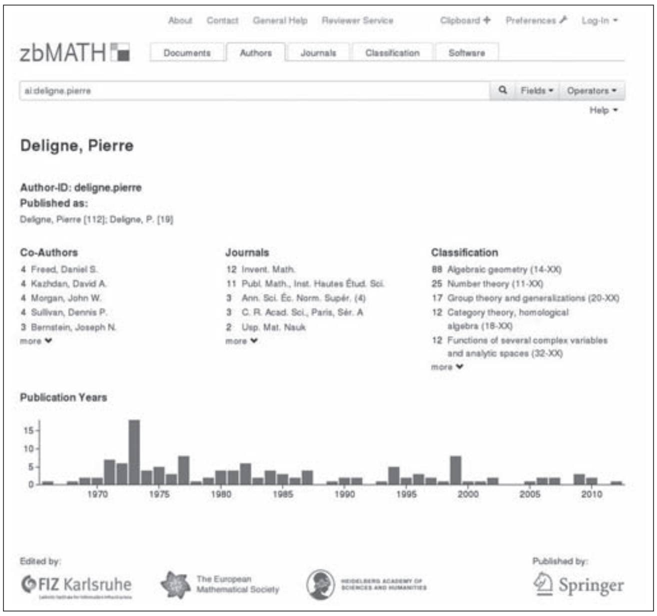


Fig. 2. New author profiles are interlinked with various search facets

specification is available via clicking (instead of typing only). Also, with the conversion of the retrieval system to HTML5, it was possible to circumvent the bugs of Internet Explorer (IE) when presenting MathML content on XML pages.[5] This problem (which has not been solved even in recent IE versions) had the annoying consequence that the old zbMATH interface still had to be maintained, and non-MathML browsers had to be directed there. With the new interface, such a split is no longer necessary. For those zbMATH users whose browsers are not MathML compatible (this number has currently dropped below 10%), there will be the additional opportunity to enable formula display by switching to (the less powerful) MathJax instead of MathML.

[5] "Zentralblatt MathMLlized", *EMS Newsletter* 76, pp. 55–57.

Further developments

Features like this also indicate the main directions of further development: customised preferences and the integration of further components. Within the preferences the user will be able to choose between many more options than just the display format or the number of results; for instance, it will also be possible to pick a certain sorting option or to decide on the exactness of the search query.

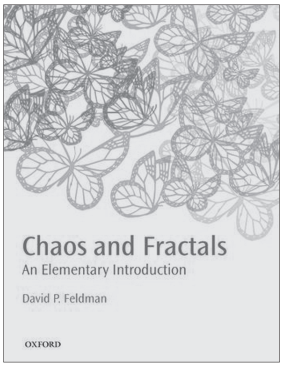
As a result of ongoing projects, further facets will be available and integrated into the scheme, most notably citation information and profiles as well as a formula index. The latter will allow standardised formula searching and browsing which goes beyond the already existing TeX search in zbMATH.

Of course, an interface is, above all, an interactive platform whose improvement can be best supported by the experience of our users. We are grateful for any comments or suggestions, which can be submitted to editor@zentralblatt-math.org.

Helena Mihaljević-Brandt was born in 1982 in Sarajevo and is currently working on the editorial board of zbMATH. She studied mathematics in Göttingen and obtained her PhD from the University of Liverpool in the field of complex dynamics.

Olaf Teschke [teschke@zblmath.fiz-karlsruhe.de] is member of the Editorial Board of the EMS Newsletter, responsible for the Zentralblatt Column.

Book Reviews



David P. Feldman

Chaos and fractals. An elementary introduction

Oxford: Oxford University Press
xxi, 408 p.
ISBN 978-0-19-956643-3/hbk; 978-0-19-956644-0/pbk

Reviewer: Thomas B. Ward

The Newsletter thanks Zentralblatt MATH and Thomas B. Ward for the permission to republish this review, originally appearing as Zbl 1252.37001.

This is an interesting and unconventional textbook aimed at introducing students with modest mathematical background to dynamical systems (in North American terminology one might optimistically pitch it at exceptionally energetic students whose major is not in the sciences but who have completed mathematical courses prior to but not including 'calculus'). This locates the main topic – dynamical systems, chaos and insight into the way that simple mathematical models can exhibit complex dynamical behaviour – much lower down in the mathematical tree than is usual. The preface opens with a well-known quotation from May's influential Nature paper of 1976 arguing that the way in which nonlinear models of the simplest kind lead inevitably to chaotic phenomena should be taught both early (prior to 'calculus') and widely (in politics, economics and so on). This book is one attempt to flesh that out and it is a largely successful one. The book is split into seven main parts: an introduction to discrete dynamical systems viewed as iterations of a map, some basic tools for graphing iterates, some early models and a short detour into the history of debates about determinism and the origins of our belief in mathematical models for the physical universe; 'chaos' as observed initially via the logistic equation, Lyapunov exponents, sensitive dependence on initial conditions, bifurcation diagrams and some statistical language; fractals arising in this context and some fractal geometry, box counting dimension and more probabilistic language; some complex dynamics and an introduction to the Julia and Fatou set and the idea of parameter spaces, the Mandelbrot set in particular; a quick overview of higher-dimensional and other kinds of systems, notably differential equations and cellular automata; a short narrative overview trying to draw conclusions about the pervasive presence of chaotic phenomena and fractal geometry; and finally three appendices on algebra, statistics and further reading. There are many ways to approach material like this, all having advantages and disadvantages. At one extreme the course might demand a background in calculus, analysis, measure theory, functional analysis, complex variables and so on, with the ability to study dynamical systems with a heavy-laden mathematical toolbox. This would dramatically narrow the number of students exposed to these beautiful and important ideas but enormously increase the scope of what could be covered. At the other extreme one might abandon all hope of rigour and present some of these topics essentially as observed mysterious phenomena, generating beautiful and multicoloured pictures – a deeply flawed pedagogical approach which is peculiarly tempting in complex dynamics and cellular automata. This book chooses a more difficult but ultimately more fruitful path. As far as possible the minimal prerequisites are not exceeded (there are flagged exercises requiring a little more) – though getting through all the material would require real perseverance and some innate mathematical ability. Where mathematical concepts are skipped over (and, of course, a great many concepts and proofs need to be) this is said so clearly and some intuition about what is meant by more sophisticated notions like ergodicity is provided. The fact that this material has been honed by real classes comes across clearly – the examples and explanations are invariably carefully thought out and clear. There are also some unconventional asides of great potential value. The third appendix, for instance, is not just a list of additional sources. It also explains what the peer-reviewed literature is and why it matters, as well as some of the shortcomings and possible pitfalls in the literature. There is also a practical guide to actually accessing some of the literature – if you are not located inside the mathematics department of a research-intense university, how do you lay your hands on a research article? For the right audience and instructor, this is a wonderful book. With considerable effort on both sides it can take a wide audience with modest mathematics to a reasonable understanding of what is behind much of the complex phenomena seen in modern mathematical models of the physical universe.

Tom Ward is Pro-Vice-Chancellor of Education and a professor of mathematics at Durham University. He works, when time permits, on dynamical systems of algebraic origin. He is the author of several monographs, including "Heights of polynomials and entropy in algebraic dynamics" with Graham Everest and "Ergodic theory with a view towards Number Theory" with Manfred Einsiedler.

Felix Hausdorff

Main Features of Set Theory
(Grundzüge der Mengenlehre. Mit 53 Figuren im Text.) (German)
Leipzig: Veit & Comp. 473 p. (1914).

Reviewer: Oliver Deiser

The Newsletter thanks Zentralblatt MATH and Oliver Deiser for the permission to republish this review, originally appeared as Zbl 1175.01034.

F. Hausdorff's 1914 treatise on set theory is one of the great books of mathematics. A list of its eminent qualities might begin with readability, clearness, conciseness, liveliness, ingenuity, wittiness, diversity, comprehensiveness and nonchalance. The command of language is that of a brilliant writer. Through a classical approach shines an idiosyncratic modernity. Marvellous presentations of textbook material are complemented with countless new ideas, which turned out to be seminal not only for set theory but also for topology and measure theory.

The German term "Grundzüge" can be translated as "main features" or "outlines". It denotes a broad treatment of a subject which might stop at a certain level of complexity but which gives a full picture of what is considered to be characteristic. Hausdorff speaks of "Hauptsachen der Mengenlehre" (main issues of set theory) in his foreword and addresses a wide audience consisting of all "who possess some abstraction of thinking". "Grundzüge" is definitely not to be read as "Grundlagen" (foundations) and thus the title already points at Hausdorff's understanding of set theory, which is explained in the first chapter of the book:

> *"Die Mengenlehre ist das Fundament der gesamten Mathematik. Über das Fundament dieses Fundamentes ist eine vollkommene Einigung noch nicht erzielt worden. Den Versuch, den Prozeß der uferlosen Mengenbildung durch geeignete Forderungen einzuschränken, hat E. Zermelo unternommen. Da indessen diese äußerst scharfsinnigen Untersuchungen noch*

nicht als abgeschlossen gelten können und da eine Einführung des Anfängers in die Mengenlehre auf diesem Wege mit großen Schwierigkeiten verbunden sein dürfte, so wollen wir hier den naiven Mengenbegriff zulassen, dabei aber tatsächlich die Beschränkungen innehalten, die den Weg zu jenem Paradoxon abschneiden."

(Set theory is the foundation of all mathematics. A complete agreement about the foundation of this foundation has not yet been reached. The attempt to delimitate the process of the boundless formation of sets by adequate postulates has been undertaken by E. Zermelo. But since this keen-witted analysis cannot be presumed to be completed and since an introduction of the beginner into set theory along these lines should be linked with major difficulties, we want to allow the naive notion of a set here, but in doing so we in fact keep to the limitations which cut off the way to that paradox.)

So the teacher is aware of the paradoxes of naive set theory but nevertheless teaches naive set theory. Hausdorff's book marks the beginning of what has been done ever since: beginners are not confronted with a – by now well-understood – axiomatic system; they are taught naive set theory with a hint at Russell's paradox. In 1914, when Zermelo's first axiomatic system of 1908, including his axiom of choice, was still being discussed controversially, completed and made precise, Hausdorff's attitude is of crucial importance. While the foundations of the new foundation of mathematics had to be clarified and disseminated, an outstanding mathematician was there writing a dauntless 476 page book about set theory, substantially advancing the subject and its impact for all of mathematics. The effect was stabilising. No one reading the book is left with the impression that set theory is something vague or inconsistent. The book is about the fascinating mathematics of infinity. After reading it, one might be eager to see how this rich theory can be given a proper foundation. Then Zermelo's system and its extensions by Abraham Fraenkel and others naturally supply the theory presented in Hausdorff's book with axioms. Thus Hausdorff, not interested in axiomatics himself, helped to promote axiomatic set theory.

Hausdorff's achievement appears even greater when we look at the treatises on set theory written before 1914. Cantor presented his theory in two lengthy journal articles in 1895 and 1897, and these remained the main sources of knowledge for a long time. Besides these there were, among others, Arthur Schoenflies's *Entwicklung von der Lehre von den Punktmannigfaltigkeiten* [1, 2, 3], Gerhard Hessenberg's *Grundbegriffe der Mengenlehre* [4] and William and Grace Chisholm Young's *The theory of sets of points* [5]. Commendable as they are, they now look of only historic value when compared to Hausdorff's book.

The book emerged from research and teaching in equal measure. It is very likely that Hausdorff met Cantor regularly in Leipzig and Halle before the turn of the century. In 1901, Hausdorff gave a course on set theory to three students in Leipzig. His first set theoretic publication was a note on cardinal arithmetic in 1904. In 1905, he wrote a review of Russell's influential *The principles of Mathematics*. Between 1906 and 1909 he wrote a series of highly original papers extending Cantor's systematic analysis of well-orderings to the more general theory of linear orderings. The *Mathematische Annalen* paper of 1908 has an unusual length of 70 pages and its introduction hints at a book about the subject. Between 1909 and 1914 Hausdorff published mainly non-mathematical writings under the pseudonym Paul Mongré, which he had used since 1897. It is Paul Mongré who is behind the remarkable eloquence of the *Grundzüge*. Concerning teaching, Hausdorff lectured on set theory in 1910 and 1912 at Bonn. In 1912 he began to write the book, which appeared in April 1914 at von Veit in Leipzig. The reception was slow, partially due to the First World War. But then the book was very well received by the mathematicians of the next generation, many from Poland and Russia, among them Pavel Alexandrov, Stefan Banach, Kazimierz Kuratowski, Wacław Sierpinski, Hugo Steinhaus, Alfred Tarski, Andrei Tikhonov, Stanisław Ulam and Paul Urysohn. Hausdorff once wrote to Alexandrov that "my star indeed rises in the east". An in-depth review of the book was written by Henry Blumberg for the *Bulletin of the American Mathematical Society* in 1920. Blumberg is full of admiration and praise: "It would be difficult to name a volume in any field of mathematics that surpasses the Grundzüge in clearness and precision." This is no longer true; using mathematical logic, the preciseness of the *Grundzüge* can easily be surpassed. But unless foundational matters are at stake, Hausdorff's level of preciseness is perfectly balanced and still an – if not the – ideal. And also in other respects there is basically nothing to complain about. Blumberg only notes that "little is left to the reader's imagination" and that there could have been "a more emphatic message" but he adds that "such remonstrance would be like quarreling with Beethoven for having written symphonies instead of operas".

The first of the ten chapters of the book introduces the approach of naive set theory and then defines all basic set theoretical operations with sets and systems of sets. Notably, Hausdorff studies, in modern terminology, σ-rings, lattices of sets and the ring generated by a lattice. In the second chapter functions are defined in the now standard way as certain sets of ordered pairs. Hausdorff notes parenthetically that an ordered pair (a,b) could be defined as {{a,1},{b,2}}. (Today, C. Kuratowski's more intrinsic definition {{a},{a,b}} [6] is preferred but Hausdorff's definition is a good example of the many small gems appearing in the book.) The rest of the chapter is devoted to operations with functions, including general Cartesian products.

Chapters 3–6 deal with the main themes of Cantor's set theory: cardinals and powers (Chapter 3), ordered sets and order types (Chapter 4), well-ordered sets and ordinal numbers (Chapter 5) and relations between ordered and well-ordered sets (Chapter 6). Hausdorff's fondness of ordered sets becomes apparent and, indeed, in his foreword he admits that this material is dealt with relatively broadly. His attitude to foundations is particularly important for these chapters. He assigns unspecified symbols to sets such that two sets M and N get the same symbol if and only if they are equipollent. The symbols are then called cardinals and the symbol of M is the cardinality of the set M. Order-types of linear orderings are introduced in the same way. From a modern point of view, Hausdorff does not define cardinals and order-types (which is a nontrivial task working in an axiomatic system) but he stresses the important properties of his symbols and achieves a rich mathematical theory, which can, a posteriori, be equipped with a formal definition. Hausdorff would not regard this last step as important, in contrast to John von Neumann, who gave the first formal definition of an ordinal number in 1923. The same applies to transfinite recursion, which Hausdorff takes for granted but which von Neumann proves.

Chapters 7–10 turn to "applications" of set theory and they have been of enormous impact. The first three of the four chapters give a detailed and comprehensive introduction to "point sets in general spaces" (Chapter 7), "point sets in special spaces" (Chapter 8) and "mappings or functions" (Chapter 9), spanning almost 200 pages. The presented concepts include neighbourhood, topological space, boundary of a set, compact set, relative topology, connectedness, density, separability, first and second countability, metric space, complete space, Euclidean space, continuous function, dimension and convergence of a sequence of functions. One might ask what was known before, what had to be systemised and what is completely new. But condensed to one sentence, the three chapters are the birth of modern set-theoretic topology. Moreover, they also contain important advances in descriptive set theory: Hausdorff continues the study of definable sets of reals that had begun with Cantor's analysis of closed sets. Transfinite hierarchies figure prominently here and, in particular, the book introduces the transfinite Borel hierarchy of sets which stratifies the σ-algebra generated by all open sets. Hausdorff would prove in 1916 that every Borel set is countable or of the cardinality of the continuum. In an appendix of the Grundzüge a partial result is established using methods apt to prove the later general theorem.

The final chapter of the book is devoted to measure theory and integration. Hausdorff rediscovers G. Vitali's now famous example of a non-measurable set [7]. He presents the Peano-Jordan content and Lebesgue's measure and integration theory for the Euclidean spaces. In the appendix to Chapter 10 we find Hausdorff's first presentation of his paradoxical decomposition of the sphere. It was later generalised by Banach and Tarski to what is now known as the Banach-Tarski-Paradox and it was the root of von Neumann's theory of amenable groups. Moreover, Hausdorff proves in the appendix that every content on a lattice of sets can be uniquely extended to the generated ring. This theorem, overlooked and reproved by several mathematicians, forms the basis of a measure theory different from Carathéodory's, as it has been advanced by Heinz König.

In 1923, the *Grundzüge* was out of print and Hausdorff was asked for a new edition by Walter de Gruyter Press in Berlin, who had bought von Veit after the war. The book would appear in the series "Göschens Lehrbücherei" and the extent was, according to the guidelines of the series, limited to 320 pages. Thus Hausdorff had to rewrite the book. It appeared in 1927 with the title *Mengenlehre. Zweite, neubearbeitete Auflage* [8]. Hausdorff streamlined the discussion of basic set theoretical notions and omitted most of order theory. He also sacrificed Lebesgue's measure and integration theory "because there is no lack of other presentations". The severe truncation that will "perhaps be more regretted" was the concentration on metric instead of topological spaces. Concerning foundations, Hausdorff's interests did not change: "I could not, now as then, convince myself to a discussion about paradoxes and foundational criticism." At least the references contain E. Zermelo's 1908 paper [9] and A. Fraenkel's second edition of his *Einleitung in die Mengenlehre*" [10]. Mockeries like "we have to leave it to philosophy to fathom the 'true being' of cardinal numbers" are still funny but in view of John von Neumann's work they also begin to look out of date, misguiding readers about the necessity and possibility of a proper mathematical definition.

But the second edition also contains a lot of new material, as it comprises a clear, thorough and up-to-date account of descriptive set theory. Borel sets are studied in great detail, as are the more general Suslin or analytic sets. Hausdorff presents his own contributions, together with the more recent work of Pavel Alexandrov, Henri Lebesgue, Nikolai Lusin, Wacław Sierpiński, Mikhail Suslin and others. Hausdorff's approach is general and establishes the results not only for the reals but for Borel and Suslin sets in Polish spaces, which are the basic structure of descriptive set theory today.

Hausdorff published a slightly extended third edition of *Mengenlehre* in 1935 [11]. A Russian translation, edited by Alexandrov and Kolmogorov, appeared in 1937. It tries to merge the advantages of the different editions by presenting what might be called a modernised compilation of the books. An English translation of the third edition appeared in 1957 [12] and was reprinted many times. Recently, the *Grundzüge* of 1914 as well as the *Mengenlehre* of 1927 and its additions of 1935 were photomechanically reprinted, annotated and commented in volumes II and III of Hausdorff's collected works [13, 14]. These volumes are an invitation to read and compare Hausdorff's books in their original form, with the help of essays providing historical and mathematical background.

List of references

1. Schoenflies, A. Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. (German) *Deutsche Math. Ver.* 8, No. 2, 1–250 (1900).
2. Schoenflies, A. *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten.* II. Teil. (German) Leipzig: B. G. Teubner. X + 331 pp. (1908).
3. Schoenflies, A. *Entwicklung der Mengenlehre und ihrer Anwendungen. Umarbeitung des im VIII. Bande der Jahresberichte der*

Deutschen Mathematiker-Vereinigung erstatteten Berichts, gemeinsam mit Hans Hahn herausgegeben von Arthur Schoenflies. Erste Hälfte. Allgemeine Theorie der unendlichen Mengen und Theorie der Punktmengen von A. Schoenflies. (German) Leipzig und Berlin: Teubner. XI + 389 pp. (1913).
4. Hessenberg, G. *Grundbegriffe der Mengenlehre*. (German) Göttingen: Vandenhoeck & Ruprecht. VIII + 220 pp. (1906).
5. Young, W. H.; Young, G. Ch. *The theory of sets of points*. (English) Cambridge: University Press. XII + 316 pp. (1906).
6. Kuratowski, C. Sur la méthode d'inversion dans l'Analysis Situs. (French) *Fundamenta math*. 4, 151–163 (1923).
7. Vitali, G. *Sul problema della misura dei gruppi di punti di una retta. Nota*. (Italian) Bologna: Gamberini e Parmeggiani. 5 pp. (1905).
8. Hausdorff, F. *Mengenlehre*. Zweite, neubearbeitete Auflage. (German) 285 pp. with 12 Fig. Berlin, Walter de Gruyter & Co. (Göschens Lehrbücherei Gruppe I Band 7) (1927).
9. Zermelo, E. Untersuchungen über die Grundlagen der Mengenlehre. I. (German) *Math. Ann*. 65, 261–281 (1908).
10. Fraenkel, A. *Einleitung in die Mengenlehre. Eine elementare Einführung in das Reich des Unendlichgroßen*. Zweite erweiterte Auflage. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Band IX.). (German) Berlin: J. Springer. IX + 251 pp. (1923).
11. Hausdorff, Felix. *Set theory*. 3rd ed. (*Mengenlehre*. 3. Aufl.) (English) Göschens Lehrbücherei Gruppe 1, Bd. 7. Berlin, Leipzig: Walter de Gruyter 307 pp. (1935).
12. Hausdorff, Felix. *Set theory*. Translated from the German by John R. Aumann et al. (English) New York: Chelsea Publishing Company. 352 pp. $6.00 (1957).
13. Hausdorff, Felix. *Collected works*. Vol. II: F*oundations of set theory*. (*Gesammelte Werke*. Band II: *Grundzüge der Mengenlehre*. Herausgegeben von E. Brieskorn, S. D. Chatterji, M. Epple, U. Felgner, H. Herrlich, M. Hušek, V. Kanovei, P. Koepke, G. Preuß, W. Purkert und E. Scholz.) (German) Berlin: Springer. xviii, 883 pp. EUR 99.95 (D); SFR 155.00 (2002).
14. Hausdorff, Felix. *Collected works*. Vol. III: *Set theory* (1927, 1935). *Descriptive set theory and topology*. Edited by U. Felgner, H. Herrlich, M. Hušek, V. Kanovei, P. Koepke, G. Preuß, W. Purkert und E. Scholz. (*Gesammelte Werke*. Band III: *Mengenlehre* (1927, 1935). *Deskriptive Mengenlehre und Topologie*.) (German) Berlin: Springer (ISBN 978-3-540-76806-7). xxii, 1005 pp. EUR 99.95; SFR 163.00 (2008).

Oliver Deiser teaches mathematics at the School of Education and the Center of Mathematics at the Technical University of Munich. His interests include the foundation of mathematics and its history as well as the teaching of mathematics, especially the transition from school to university mathematics. He is the author of several textbooks on set theory and real analysis.

Personal Column

Please send information on mathematical awards and deaths to Mădălina Păcurar [madalina.pacurar@econ.ubbcluj.ro]

Awards

Francisco Santos (University of Cantabria in Santander, Spain) has received a **Humboldt Research Prize** and will spend the associated research period in Germany at Freie Universität Berlin.

The former EMS President **Ari Laptev** was elected as a member of the Swedish Academy of Sciences.

Pierre Deligne (Institute for Advanced Study, Princeton, New Jersey, USA) receives the **Abel Prize** 2013.

The **BBVA Foundation Frontiers of Knowledge Award in the Basic Sciences** 2013 is awarded to **Ingrid Daubechies** (Duke University, US) and **David Mumford** (Brown University, US).

Uffe Haagerup (University of Copenhagen, Denmark) receives this year's **European Latsis Prize**.

The **Ferran Sunyer i Balaguer Prize** for this year goes to **Xavier Tolsa** (ICREA, UAB, Spain).

John Baez (University of California Riverside, US) and **John Huerta** (Instituto Superior Técnico in Lisbon, Portugal) received the 2013 **AMS Conant Prize**.

László Lovász (Eötvös Loránd University in Budapest, Hungary) and **Balázs Szegedy** (University of Toronto, Canada) have been awarded the 2012 **AMS Fulkerson Prize**.

Michael Larsen (Indiana University, US) and **Richard Pink** (The Swiss Federal Institute of Technology, Zurich, Switzerland) receive the 2013 **AMS E.H. Moore Research Article Prize**.

Jeremy Kahn (Brown University, US) and **Vladimir Markovic** (California Institute of Technology, US) received the 2012 **Clay Research Award**.

The 2012 **SMAI-Natixis Priz** has been awarded to **Nizar Touzi** (École Polytechnique, France).

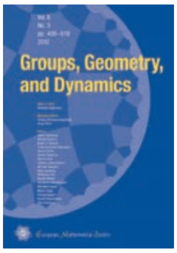
Cristóbal Bertoglio (Technische Universität München, Germany) receives the 2013 **SMAI-GAMNI PhD Award**.

Deaths

We regret to announce the deaths of:

Erik Balslev (11 January 2013, Denmark)
David Chillag (29 July 2012, Israel)
Lars Hörmander (25 November 2012, Sweden)
Eugenio Merino (25 December 2012, Spain)
Michel Las Vergnas (19 January 2013, France)

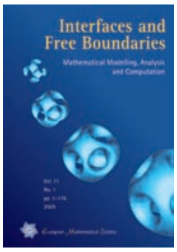
Journals published by the European Mathematical Society



ISSN print 1661-7207
ISSN online 1661-7215
2013. Vol. 7. 4 issues
Approx. 1000 pages. 17.0 x 24.0 cm
Price of subscription:
288 € online only
348 € print+online

Editor-in-Chief:
Rostislav Grigorchuk (Texas A&M University, College Station, USA; Steklov Institute of Mathematics, Moscow, Russia)

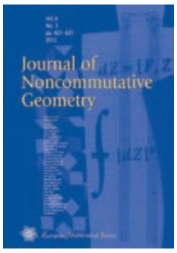
Aims and Scope
Groups, Geometry, and Dynamics is devoted to publication of research articles that focus on groups or group actions as well as articles in other areas of mathematics in which groups or group actions are used as a main tool. The journal covers all topics of modern group theory with preference given to geometric, asymptotic and combinatorial group theory, dynamics of group actions, probabilistic and analytical methods, interaction with ergodic theory and operator algebras, and other related fields.



ISSN print 1463-9963
ISSN online 1463-9971
2013. Vol. 15. 4 issues
Approx. 500 pages. 17.0 x 24.0 cm
Price of subscription:
390 € online only
430 € print+online

Editors-in-Chief:
José Francisco Rodrigues (Universidade de Lisboa, Portugal)
Charles M. Elliott (University of Warwick, Coventry, UK)
Robert V. Kohn (New York University, USA)

Aims and Scope
Interfaces and Free Boundaries is dedicated to the mathematical modelling, analysis and computation of interfaces and free boundary problems in all areas where such phenomena are pertinent. The journal aims to be a forum where mathematical analysis, partial differential equations, modelling, scientific computing and the various applications which involve mathematical modelling meet.



ISSN print 1661-6952
ISSN online 1661-6960
2013. Vol. 7. 4 issues
Approx. 1200 pages. 17.0 x 24.0 cm
Price of subscription:
298 € online only
358 € print+online

Editor-in-Chief:
Alain Connes (Collège de France, Paris, France)

Aims and Scope
The *Journal of Noncommutative Geometry* covers the noncommutative world in all its aspects. It is devoted to publication of research articles which represent major advances in the area of noncommutative geometry and its applications to other fields of mathematics and theoretical physics. Topics covered include in particular: Hochschild and cyclic cohomology; K-theory and index theory; measure theory and topology of noncommutative spaces, operator algebras; spectral geometry of noncommutative spaces; noncommutative algebraic geometry; Hopf algebras and quantum groups; foliations, groupoids, stacks, gerbes.

ISSN print 1664-039X
ISSN online 1664-0403
2013. Vol. 3. 4 issues
Approx. 400 pages. 17.0 x 24.0 cm
Price of subscription:
198 € online only
238 € print+online

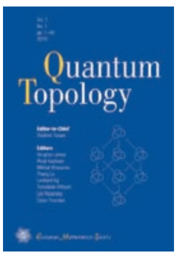
Managing Editor:
E. Brian Davies (King's College, London, UK)
Deputy Chief Editor:
Ari Laptev (Imperial College, London, UK)

Aims and Scope
The *Journal of Spectral Theory* is devoted to the publication of research articles that focus on spectral theory and its many areas of application.

ISSN print 1435-9855
ISSN online 1435-9863
2013. Vol. 15. 6 issues
Approx. 1800 pages. 17.0 x 24.0 cm
Price of subscription:
548 € online only
638 € print+online

Editor-in-Chief:
Haim Brezis (Université Pierre et Marie Curie, Paris, France; Rutgers University, USA; and Technion, Haifa, Israel)

Aims and Scope
The *Journal of the European Mathematical Society* is the official journal of the EMS. The Society, founded in 1990, works at promoting joint scientific efforts between the many different structures that characterize European mathematics. JEMS publishes research articles in all active areas of pure and applied mathematics.



ISSN print 1663-487X
ISSN online 1664-073X
2013. Vol. 4. 4 issues
Approx. 400 pages. 17.0 x 24.0 cm
Price of subscription:
198 € online only
238 € print+online

Editor-in-Chief:
Vladimir Turaev (Indiana University, Bloomington, USA)

Aims and Scope
Quantum Topology is dedicated to publishing original research articles, short communications, and surveys in quantum topology and related areas of mathematics. Topics covered include: low-dimensional topology; knot theory; Jones polynomial and Khovanov homology; topological quantum field theory; quantum groups and hopf algebras; mapping class groups and Teichmüller space categorification; braid groups and braided categories; fusion categories; subfactors and planar algebras; contact and symplectic topology; topological methods in physics.

European Mathematical Society Publishing House
Seminar for Applied Mathematics, ETH-Zentrum SEW A27
Scheuchzerstrasse 70
CH-8092 Zürich, Switzerland
subscriptions@ems-ph.org
www.ems-ph.org