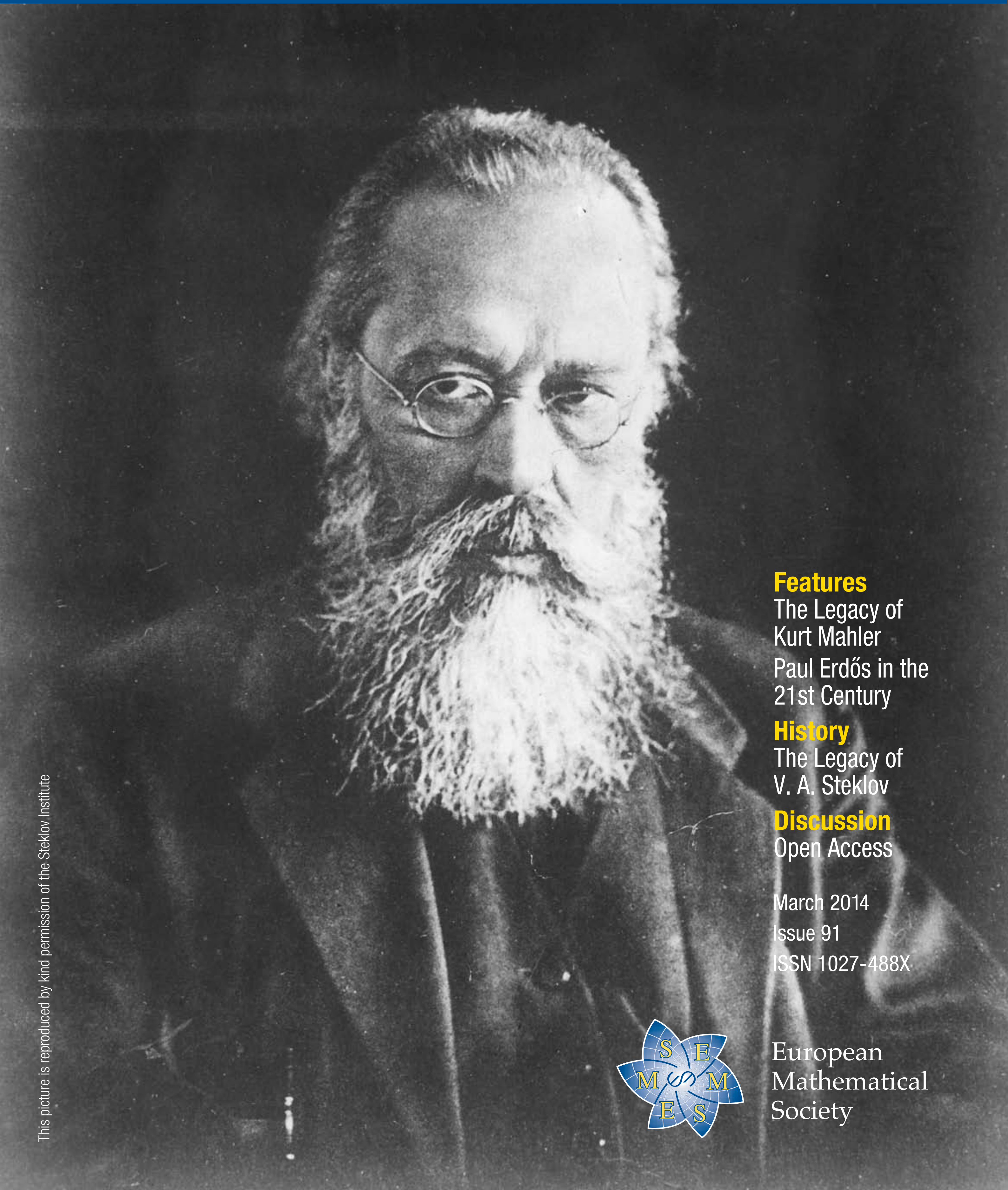
NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY



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Features

The Legacy of Kurt Mahler

Paul Erdős in the 21st Century

History

The Legacy of V. A. Steklov

Discussion

Open Access

March 2014

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A journal of the Portuguese Mathematical Society

Editor-in-Chief:
Luis Nunes Vicente (Universidade de Coimbra, Portugal)

Aims and Scope

Since its foundation in 1937, *Portugaliae Mathematica* has aimed at publishing high-level research articles in all branches of mathematics. With great efforts by its founders, the journal was able to publish articles by some of the best mathematicians of the time. In 2001 a *New Series of Portugaliae Mathematica* was started, reaffirming the purpose of maintaining a high-level research journal in mathematics with a wide range scope.

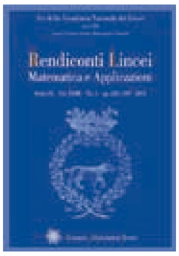
ISSN print 0034-5318
ISSN online 1663-4926
2014. Vol. 50. 4 issues
Approx. 800 pages. 17.0 x 24.0 cm
Price of subscription:
298 € online only
348 € print+online

A journal of the Research Institute for Mathematical Sciences of Kyoto University

Editor-in-Chief:
S. Mochizuki

Aims and Scope

The aim of the *Publications of the Research Institute for Mathematical Sciences* is to publish original research papers in the mathematical sciences. Occasionally surveys are included at the request of the editorial board.



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2014. Vol. 25. 4 issues
Approx. 400 pages. 17.0 x 24.0 cm
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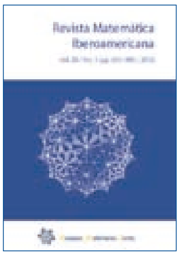
200€ online only
240 € print+online

Director Advisory Committee:
Antonio Ambrosetti (SISSA, Trieste, Italy)

Aims and Scope

The Accademia dei Lincei (Lynx), founded in 1603, is the oldest academy dedicated to the study of humanities as well as physics, mathematics and the natural sciences in the world. In 1873 the academy began publishing with the *Atti della Reale Accademia dei Lincei, Transunti*. Since 1990 under the present name, the *Rendiconti Lincei* have been one of the best Italian journals. The journal is dedicated to the publication of high-quality peer-reviewed surveys, research papers and preliminary announcements of important results from all fields of mathematics and its applications.



ISSN print 0213-2230
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Price of subscription:
368 € online only
428 € print+online

A scientific journal of the Real Sociedad Matemática Española

Editors:
Antonio Córdoba (Universidad Autónoma de Madrid, Spain), José L. Fernández (Universidad Autónoma de Madrid, Spain), Antonio Ros (Universidad de Granada, Spain) and Luis Vega (Universidad del País Vasco, Bilbao, Spain)

Aims and Scope

Revista Matemática Iberoamericana publishes original research articles on all areas of mathematics. Its distinguished Editorial Board selects papers according to the highest standards. Founded in 1985, *Revista* is a scientific journal of Real Sociedad Matemática Española.

ISSN print 0041-8994
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2014. Vols. 131 and 132, 1 issue each
Approx. 500 pages. 17.0 x 24.0 cm
Price of subscription:
198 € online only
238 € print+online

Editor-in-Chief:
Andrea D'Agnolo (Università di Padova, Italy)

Managing Editor:
Francesco Baldassarri (Università di Padova, Italy)

Aims and Scope

The *Rendiconti del Seminario Matematico della Università di Padova* have been published since 1930. The aim of *Rendiconti* is to publish original high-quality research articles in mathematics. By tradition the journal focuses on algebra, analysis, algebraic and differential geometry, mathematical physics and number theory. The journal is ready to accept long papers of foundational or technical nature, as well as papers containing applications or surveys, provided these are especially well structured and fill a gap in the literature.

ISSN print 0232-2064
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2014. Vol. 33. 4 issues
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A periodical edited by the University of Leipzig

Managing Editors:
J. Appell (Universität Würzburg, Germany), M. Günther (Universität Leipzig, Germany), S. Luckhaus (Universität Leipzig, Germany)

Aims and Scope

The *Journal for Analysis and its Applications* aims at disseminating theoretical knowledge in the field of analysis and, at the same time, cultivating and extending its applications. To this end, it publishes research articles on differential equations, functional analysis and operator theory, notably with applications to mechanics, physics, engineering and other disciplines of the exact sciences.

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Editorial Team

Scan the QR code to go to the Newsletter web page: *http://www.euro-math-soc.eu/node/4230*

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EMS Executive Committee

EMS Agenda

2014

12 April
Meeting of Presidents
Rectorate Conference Hall, Boğaziçi University, Istanbul, Turkey

25–26 April
Annual Meeting of the Committee for Developing Countries, Berlin, Germany
http://euro-math-soc.eu/EMS-CDC/
Andreas Griewank: griewank@math.hu-berlin.de

30–31 May
Ethics Committee Meeting, Mittag-Leffler Institute, Djursholm, Sweden
Arne Jensen: matarne@math.aau.dk

16–20 June
Interaction between dynamical systems and PDEs, Barcelona, Spain
http://www.ma1.upc.edu/recerca/seminaris-recerca/jisd2014/jisd2014

23–27 June
6th European Women in Mathematics Summer School, Institut Mittag-Leffler, Djursholm, Sweden
http://www.math.ucsd.edu/~alina/ewm/

23–28 June 2014
CIME-EMS Summer School in Applied Mathematics: "Centralized and Distributed Multi-agent Optimization: Models and Algorithms"
Cetraro, Italy
http://php.math.unifi.it/users/cime/

27 June
Executive Committee of the EMS, San Sebastian, Spain

28–29 June
EMS Council Meeting,
Auditorio Antonio Beristain, University of the Basque Country, Campus de Gipuzkoa, San Sebastian, Spain

30 June–4 July
11th Vilnius Conference on Probability Theory and Mathematical Statistics Vilnius, Lithuania
http://www.vilniusconference2014.mif.vu.lt/

30 June–4 July
First Joint International Meeting RSME-SCM-SEMA-SIMAI-UMI
Bilbao, Spain
http://www.ehu.es/en/web/fjim2014/

28 July–1 August
EMS-IAMP Summer School on Mathematical Relativity
http://homepage.univie.ac.at/piotr.chrusciel/Summer-School2014/index.html

13–21 August
International Congress of Mathematicians,
Coex, Seoul, Korea
http://www.icm2014.org/

Editorial: Laureates Meet Young Researchers In Heidelberg

Helge Holden (Norwegian University of Science and Technology, Trondheim, Norway), Hans Munthe-Kaas (University of Bergen, Norway) and Dierk Schleicher (Jacobs University Bremen, Germany)

Young scientists have a chance to meet some of the most successful mathematicians and computer scientists at the newly established "Heidelberg Laureate Forum" (HLF), a week-long event that brings together Abel laureates, Fields medallists and Turing awardees with "leading scientists of the next generation", that is, university students, doctoral students and postdocs. We all know that personal encounters can have a decisive impact on our personal lives – and what can be more exciting for a young and aspiring scientist than a face-to-face meeting with the most prominent scientists in their area? This is the idea of the forum. It was launched in September 2013 and will take place annually.

The highly successful Lindau Nobel Laureate Meetings have taken place for more than 60 years, providing a forum where young researchers and Nobel Laureates from physics, chemistry and life sciences meet for inspiring scientific discussions and social interaction. In this sense, the Heidelberg Laureate Forum fills a void for the oldest scientific discipline: mathematics, and one of the youngest: computer science. It was initiated by the Klaus Tschira Foundation (KTS) and the Heidelberg Institute for Theoretical Studies (HITS) in 2011 in collaboration with the sponsors of the three prizes: the International Mathematical Union (IMU), the Norwegian Academy of Science and Letters and the Association for Computing Machinery (ACM). The programme and the selection of participants are overseen by a scientific committee that includes representatives from the IMU, the Norwegian Academy of Science and Letters, the ACM (including one laureate of each prize), Oberwolfach and Schloss Dagstuhl, as well as the Tschira foundation and the HITS.

The format of the HLF is a week-long symposium where laureates of the Abel Prize, the Fields Medal (including the Nevanlinna prize) and the ACM Turing Award join with young researchers of undergraduate, graduate and post-graduate levels. The first meeting took place 22-27 September, when an unprecedented assemblage of 38 laureates and more than 200 young scientists gathered in Heidelberg. Plenary lectures were given by many of the laureates and workshops were organised by the younger researchers. Panel discussions, with laureates on the panel and active participants in the audience, covered various topics of mathematics and computer science. The mathematics laureates present were Sir Michael Atiyah, Gerd Faltings, Curt McMullen, Stephen Smale, Endre Szemerédi, Srinivasa Varadhan, Cédric Villani, Vladimir Voevodski, Avi Wigderson and Efim Zelmanov; moreover, among Turing awardees, there were "household names" in mathematics present such as Ronald Rivest and Adi Shamir (the "RS" from the "RSA algorithm"), as well as Stephen Cook and Richard Karp, pioneers of complexity theory.

The plenary talks covered a broad range of themes. Sir Michael Atiyah gave his advice to young mathematicians based on his experiences of a long life in mathematics and continued his enthusiastic interaction with the younger generation in the informal parts of the forum and the social events.

Curtis McMullen discussed new connections between motions of billiard balls, Riemann surfaces and moduli spaces. Blogger Dana Mackenzie summarised his impressions of McMullen's talk in his Scientific American blog Dances, Billiards and Pretzels: "When I came to the Heidelberg Laureate Forum, I expected a feast for my mind. I didn't expect a feast for my eyes! Take a look at this incredible video, by Diana Davis, which was featured in today's lecture by Fields medalist Curtis McMullen." The video at http://vimeo.com/47049144 is highly recommended for anyone seeking inspiration in communicating their research to a broader audience!

Vladimir Voevodsky gave a talk entitled Univalent Foundations of Mathematics. Despite a title and abstract which perhaps did not generate much buzz in advance among the young scientists, the talk was highly inspiring to both the mathematics and computer science sections of the audience. Blogger Julie Rehmeyer wrote in her Scientific American blog Voevodsky's Mathematical Revolution: "On last Thursday at the Heidelberg Laureate Forum, Vladimir Voevodsky gave perhaps the most revolutionary scientific talk I've ever heard. … Voevodsky told mathematicians that their lives are about to change. Soon enough, they're going to find themselves doing mathematics at the computer, with the aid of computer proof assistants. … Oh, and by the way — just in case the computer scientists in the crowd think that this has nothing to do with them — he also showed that the theory of programming languages is in fact the same thing as homotopy theory, one of the most abstruse areas of mathematics."

Just about all mathematicians that we talked to embraced the fact that this was a joint event between mathematics and computer science, and they enjoyed the opportunity to interact with computer scientists both among the speakers and the young participants. Many interesting talks were on the interface between mathematics and computer science, by speakers from both communities. For instance, in one of the computer science talks, Turing awardee William M. Kahan described the errors that can appear in floating-point operations and sometimes hardly be found. He called for a better way of handling numeri-

Perhaps the most special feature of the Heidelberg Laureate Forum is the immediate contact between young scientists and laureates such as Cédric Villani (first picture), Srinivasan Varadhan (second picture) and Avi Wigderson (third picture) during the excursion day.

cal and computation errors and argued that improved schemes could have, among other things, prevented the crash of Air France #447 in June 2009. In another talk, on "zero knowledge", Michael O. Rabin presented novel algorithms enabling an auctioneer to prove to bidders, without revealing any bid values, who had won a sealed bid auction. Zero knowledge proofs are very important in cryptography; in particular, these methods allow one to solve the important open problem of prevention of collision in auctions.

For the afternoon sessions, postdocs were invited to organise by themselves several workshops on mathematics or computer science where they described their own research and its perspectives. Besides young researchers, many laureates attended these workshops and actively participated in these discussions, giving their points of view on the topics and related questions. One of the most attended and active workshops was on "How to balance your life?" organised by Matthias Hagen, a postdoc from Weimar/Germany, with Avi Wigderson as a participating laureate representative. The key question of the workshop was how to balance at least some of the four areas of life: (1) yourself, (2) your partner and family, (3) your job/career, and (4) your friends and society, throughout different stages of your career. Participants and the laureate shared their experiences and tried to at least formulate the problems that one needs to address to become successful, not only in your research career but in your personal life as well.

In addition, there were two panel discussions: one among computer science laureates, one among mathematics laureates. These treated topics such as expectations of the development of their fields and also the future interaction between mathematics and computer science, as well as how they personally developed in the way they did. A final panel discussion, with laureates, young researchers, organisers and members of the scientific committee, was an occasion to review the entire programme and to define the direction for future development of the Heidelberg Laureate Forum. Luckily, for the most part, it was confirmed that the organisers and the scientific committee had already made good choices.

The ambitious scientific programme was embedded in a very enjoyable setting. Heidelberg is a most beautiful environment for any event and the organisers (who had worked for more than a year on the project with a very competent and substantial team) missed no opportunity to make this a most enjoyable week for everyone. The programme included an evening party at the Neckar river, a boat cruise, a reception at the famous Heidelberg castle, one day at the modern "European Molecular Biology Laboratory" on the Heidelberg hills and a fancy dinner at the "Villa Bosch" for the laureates (on which Klaus Tschira commented: "having studied physics myself, I realised that the next best thing to having a Nobel prize myself was to be able to invite the laureates to my house that used to be the home of a Nobel laureate"). In addition, there were excursions for the young scientists to nearby scientific institutions and meetings of the laureates with local high school students. No effort or expense was too much to create a wonderful ambience for the laureates and the young scientists. To top it off, luck was with the organisers: while the week before was cold and rainy, the forum itself took place during a warm and sunny late summer week!

One of the guiding principles was that many events were created for informal interaction between laureates and young scientists (and we heard from many of them what a significant difference these made to them!) but also interaction between the laureates, as well as interaction between the young scientists from many different countries and with different educational backgrounds. Participation was free of charge to the young scientists; room and board were provided in nice Heidelberg downtown hotels.

The laureates as well as representatives of the IMU, the ACM and the Norwegian Academy of Science and Letters and the Klaus Tschira Foundation.

The invitation of a group of scientific bloggers to the HLF was a clever idea, which contributed to communicating the meeting outside the meeting hall and furthermore served as an inspiration among the participants. These blogs, together with a rich picture gallery, are available from the HLF forum website http://www.heidelberg-laureate-forum.org.

In at least two ways, lasting values were created. For one, all plenary lectures were recorded and made available to the public almost immediately afterwards, so that many interested people from all over the world could join in (as the public was not included in the actual event). These videos are still available on the HLF website. And for another, a most beautiful book “Masters of Abstraction” was created for the occasion: the photographer Peter Badge had visited virtually all living laureates and taken very memorable pictures of each of them. This collection, together with a very brief text about them, was assembled in a book that was given as a present to all participants, young and old.

For the first edition of the “Heidelberg Laureate Forum”, more than 600 young scientists, from undergraduates to postdocs, applied for participation. It was one of the tasks of the scientific committee to select from these the young people that would gain the most from participating. This was done separately in mathematics and in computer science. In mathematics, we had a team of 25 prominent mathematicians from all over the world who evaluated the application files carefully and compared them with respect to educational age and background. We would like to use this opportunity to thank all those who helped us in this substantial effort! Over time, we expect more applications to come in. In order for this to happen, we would like to encourage the mathematics community to pass on the information to their students and postdocs so that the message reaches the desired people.

The Heidelberg Laureate Forum is an annual event that will take place in the last week of September; application is possible online until the end of February. Many young participants said they would very much like to come again; since new young scientists are being invited every year, the only chance to do so is to come again as a laureate. We all hope that the event will provide enough inspiration to some of them so that indeed they will one day be among the laureates of the Abel Prize, the Fields Medal, the Nevanlinna Prize and the Turing Award!

Helge Holden (center) received his PhD from the University of Oslo. Since 1986, after a postdoc period at the Courant Institute, he has been a professor at the Norwegian University of Science and Technology in Trondheim, Norway. He has served as Secretary and Vice-President of the EMS and is currently Chair of the Abel Board, which oversees all activities in connection with the Abel Prize.

Hans Munthe-Kaas (left) has a PhD from the Norwegian Institute of Science and Technology (1989). He has been a professor of computer science and is now a professor of mathematics at the University of Bergen, Norway. His main research interests are nonlinear partial differential equations, where he has focused on hyperbolic conservation laws, completely integrable systems, and flow in porous media. In particular, he is interested in the interplay between numerical methods and analytical tools. His main interests are at the borderland between computational mathematics, computer science and pure mathematics. In particular he works on applications of group theory and representation theory in structure preserving discretisation of differential equations, approximation theory and applied harmonic analysis. Munthe-Kaas has served as secretary for the Society for the Foundations of Computational Mathematics and is a member of the Abel Board.

Dierk Schleicher (right) obtained his PhD in mathematics at Cornell University, USA. After years in Berkeley, Stony Brook and Munich, he is now a professor at Jacobs University, Germany. His main research interests are in dynamical systems, especially complex dynamics and the dynamics of the Newton iteration. He is engaged in activities to support young mathematical talent, for instance as co-organiser of the 2009 International Mathematical Olympiad and as co-initiator of the “Modern Mathematics” Summer School series in Bremen and Lyon since 2011.

Helge Holden and Dierk Schleicher are members of the HLF scientific committee and, together with Hans Munthe-Kaas, were involved in the selection of the participating young mathematicians.

The authors would like to thank Mikhail Hlushchanka for contributing the participants’ points of view to this report.

News from the Editorial Board

Since last December, the newsletter has been present on Facebook: http://www.facebook.com/EMSnewsletter, and on twitter: @EMSnewsletter. Our presence on social media will allow us to be more effective in the dissemination of news and announcements and will give our readers the potential for exchange of opinions, dialogue and debate.

In December 2013, the term of office is ending for **Erhard Scholz** (University Wuppertal), editor of the newsletter in charge of the History of Mathematics section. We would like to express our deep gratitude for all the work he has carried out. We are also pleased to welcome **Jean-Paul Allouche**, introduced below.

New Editor Appointed

Jean-Paul Allouche is "Directeur de Recherche" in mathematics at CNRS. After time at Bordeaux, Marseille and the LRI in Orsay, he is now at the IMJ (Institut de Mathématiques de Jussieu) as head of the group "Combinatoire et Optimisation". His main interests in mathematics are at the frontier between number theory and combinatorics/discrete mathematics/theoretical computer science. Among his other professional activities, he was "Directeur des Publications" of the French Mathematical Society from 2004 to 2010 and a member of the Ethics Committee of the EMS from 2010 to 2013. His webpage can be accessed at http://www.math.jussieu.fr/~allouche.

Corrigendum by Stephen Huggett to
Announcement of the Next Meeting of the EMS Council, Individual Members, Issue 90

On pages 6 and 7 of the last issue (90) of the Newsletter there were two errors. The list of the current delegates of individual members whose terms include 2014 should have included Kazimierz Goebel and Elizabetta Strickland. The term of oce of Vice-President Martin Raussen is 2013–2016 inclusive.

EMS Executive Committee Meeting in Paris on the 8th, 9th, and 10th of November 2013

Stephen Huggett (University of Plymouth, UK)

Membership

The membership fee reminder had been sent to individual members. However, this had happened before the Societies in some countries had requested payment for EMS membership. It was accepted that although this is not optimal, it is not an easy problem to solve. The Executive Committee approved the list of about 100 new individual members.

The President proposed that we carefully select institutions in just three countries and invite them to become institutional members. It was agreed to ask the Publicity Officer to prepare a letter, a leaflet, and a pack to send to these institutions. Then, in due course, we would do the same in another three countries.

The web

Martin Raussen reported that an upgrade to the latest version of Drupal was essential for the EMS web site, in addition to the development of new functionality for the site. The Executive Committee briefly considered the costs of maintaining and developing the web site, and it was agreed that more money would have to be committed to this.

Lucia Di Vizio presented a strong case that the EMS should have a presence on both Facebook and Twitter, explaining that this would be a way for us to use largely trusted networks of friends and acquaintances to attract attention to the Society beyond our usual audience. The Executive Committee agreed, asking Lucia to report back to the next meeting on how these sites were being moderated.

Scientific Meetings

Volker Mehrmann gave a report on the preparations for the 7th European Congress of Mathematics, and demonstrated the draft web site. There would be a historical section every day, one day would be devoted to undergraduates, there would be a film festival, and there would be public awareness events every day. The plenary lectures would be available on streaming video, there would be a public lecture on geometry and architecture, and there would be a dance event.

The President reported that the EMS–Nordic Congress of Mathematicians had been very successful, and she thanked the organisers. The Treasurer reported that the Mostowski Centenary Meeting had also gone very well, reminding the Executive Committee that EMS support had been for young mathematicians from eastern Europe. The President reported that the EMS–BS Lecture at the European Meeting of Statisticians 2013, given by Emmanuel Candes, had been superb.

Laurence Halpern reported on preparations for the RSME–SEMA–SCM–UMI–SIMAI Congress, Bilbao 2014. The web site is active, and the EMS Distinguished Speaker will be Alfio Quarteroni.

The President drew attention to the new dates for the AMS–EMS–SMP Meeting, Porto 2015, which are 10th–13th of June.

Armen Sergeev reported on the successful negotiations to hold a regional meeting in Tbilisi on 5th–7th September 2014. They had been delicate, partly because there are at present very few horizontal connections between mathematicians in that region. Now, mathematicians from the following countries are expected to participate: Armenia, Azerbaijan, Georgia, Iran, Russia, and Turkey. The Executive Committee agreed that this meeting can use the EMS logo, and awarded some financial support.

Society Meetings

The Executive Committee discussed preparations for the 2014 meeting of Council in Donostia/San Sebastián. We considered the question of how to give the work of the various committees due visibility during the Council meeting. It was agreed that each committee should be offered a short plenary session, but that these need not all be in one block. Also, it is possible that some committees may prefer to prepare a poster.

EMS Committees

The Executive Committee received and discussed reports from the Chairs of the various specialist EMS committees, noting with gratitude the huge amount of work done. Readers are recommended to see for themselves by visiting the committee web pages at
http://www.euro-math-soc.eu

Publications Committee

Bernard Teissier reported that the Publications Committee had held one meeting so far, with a second in a few days. A position paper on Open Access is in preparation. The intention is to address all aspects of the problem. A majority of members of the committee strongly oppose the author pays model. Another issue was the long term preservation of publications, electronically, without coming to depend on private data-mining companies.

Mathematics in industry

Volker Mehrmann reported that EU-MATHS-IN would be launched on the 27th of November. The following networks are part of the project.

- France: AMIES (Agence pour les mathématiques en interaction avec l'entreprise et la société)
- Germany: KoMSO (Committee for Mathematical Modelling, Simulation and Optimisation)
- Italy: Sportello Matematico per l'industria italiana
- Spain: math-in (Red Española matemática-industria)
- The Netherlands: Platform Wiskunde Nederland
- UK: The Smith Institute for industrial mathematics and system engineering

Publishing

It was noted that the Proceedings of the 6ECM were in press at the EMS publishing house.

Gert-Martin Greuel presented the new Zentralblatt interface. There will be free access world-wide in November and December 2013. There are several continuing research projects, such as semantic methods for generation of mathematical vocabulary, and searching for formulae (using MathML). Journals are being evaluated and in some cases it may be decided to stop indexing them.

Gert-Martin Greuel also reported on a proposal to the Deutsche Forschungs Gemeinschaft for a repository (at FIZ Karlsruhe) for open access journals, at low cost covering only expenses.

Funding Organisations and Political Bodies

The Horizon 2020 budget has been cut from 80 to 70 billion euros. However, promising researchers should be urged to apply since the number of grants in each subject will be proportional to the number of eligible applications in that subject.

Laszlo Lovasz and Karl Sygmund represent mathematics on the Science Europe Scientific Committee PHYCHEMA (Physical, Chemical and Mathematical Sciences).

Closing

The next meeting will be in London, on the 28th of February and the 1st and 2nd of March. Then there will be a short meeting just before the Council in Donostia/San Sebastián. The autumn meeting will be in Barcelona.

The President expressed the gratitude of the whole Executive Committee to the Institut Henri Poincaré and to the three hosting Societies for their hospitality.

A Mathematical Anniversary in Pakistan

Arnfinn Laudal (University of Oslo, Norway), Marta Sanz-Solé (President of the EMS) and Michel Waldschmidt (Chair of the EMS Committee for Developing Countries)

The Abdus Salam School of Mathematical Sciences (ASSMS) in Lahore, the most important mathematical research institution in Pakistan, is celebrating its 10th anniversary. And for once, there is good news coming from Pakistan.

incudewith the support of the government of Punjab, the largest and economically most developed province in Pakistan, founded the Abdus Salam School of Mathematical Sciences, under the umbrella of the Government College (GC) University in Lahore. The school is named after Professor Abdus Salam, a former student of mathematics at GC and the only Pakistani Nobel Laureate for Physics (awarded jointly with Sheldon Glashow and Steven Weinberg in 1979).

The ASSMS is essentially a graduate school with the goal of providing a pool of highly-trained, indigenous scientists to colleges and universities in Pakistan, a necessary first step in building a competitive economical and technological future for this federal parliamentary republic, consisting of four provinces and four federal territories and with a population exceeding 180 million.

Up till now, teaching at the ASSMS has relied on invited senior researchers with positions as regular or emeritus professors at universities or institutions in Bulgaria, France, Germany, Great Britain, Norway, Romania and Russia. There is no doubt that this has been one of the main reasons for its success but certainly also for serious challenges (see the editorial on the ASSMS in the June 2011 issue of the *Newsletter of the European Mathematical Society*).

This year, the ASSMS is celebrating its 10th anniversary. It has been widely acknowledged, both locally and internationally, for the positive role it has played in the promotion of mathematical research and education in the region. In 2011, the European Mathematical Society (EMS) initiated the "Emerging Regional Centres of Excellence" scheme (EMS-ERCE). Under this scheme, the EMS selects and endorses mathematical institutions of high scientific standing in developing countries, so that they may play a significant role in the development of mathematical research and education in their region. It was a major achievement for the ASSMS when it was awarded the very first EMS-ERCE label in 2011.

After the initial success of the EMS-ERCE scheme, a meeting was held in Paris in June 2013 attended by representatives of all the ERCE member institutions, the EMS and the International Mathematical Union (IMU). The agenda of this meeting was the creation of a network, consisting initially of the EMS-ERCE members but eventually allowing membership to other mathematical centres of excellence in developing countries around the world. The name of this network is "NIM Centres", standing for Network of International Mathematical Centres.

The ASSMS has played, and will certainly continue to play, an important role in the formation of this network.

The ASSMS is now reporting the conferment of its 100th PhD degree. It should be noted that among these 100 PhD candidates over the first 10 years of its existence, there are 22 female scholars.

To get a clear idea of the scale of this achievement, one should know that this is more than twice the total number of PhDs in mathematics produced by all other universities in Pakistan in the same period. All major public and private universities in Pakistan now have ASSMS graduates serving in their faculties.

Through most of its existence, the ASSMS has been supported by the main international institutions of mathematical research, the European Mathematical Society, the International Mathematical Union, the Centre International de Mathématiques Pures et Appliquées (CIMPA), the International Centre for theoretical Physics (ICTP) and, at the national level, by the Higher Education Commission of Pakistan (HEC). This is well-deserved support for the ASSMS and its director general, support that we all should continue to encourage.

EU-MATHS-IN, a New Way to European Industrial Mathematics[1]

Roberto Natalini and Antonino Sgalambro (Sportello Matematico per l'Industria italiana, Istituto per le Applicazioni del Calcolo "M. Picone", National Research Council of Italy)

According to the results and recommendations provided by the Forward Look "Mathematics in Industry", promoted by the European Science Foundation (www.esf.org/flooks), mathematics turns out to be one of the main key enabling factors in all the areas of science and technology.

In particular, in the field of industry and innovation, the efficient development of new products and production processes is commonly characterised by a wide use of simulation and optimisation methods, which, based on an appropriate mathematical modelling, support or even replace the production of high-cost prototypes and conventional trial-and-error methods.

Despite this well supported evidence, there is still a lack of recognition about the fundamental role of mathematics in innovation, due to the peculiarity of maths-based methods that often provide *an invisible contribution for a visible industrial success*. This is one of the main reasons for the existing gap between academic research groups and industry. In order to provide an effective contribution for filling this gap, on 26 November 2013 a new organisation was founded in Amsterdam, entitled "EU-MATHS-IN: Stichting European Service Network of Mathematics for Industry and Innovation".

The foundation of EU-MATHS-IN takes place as the initiative of the members of the Scientific Organising Committee of the Forward Look "Mathematics in Industry" and is supported by the European Mathematical Society (EMS) and the European Consortium for Mathematics in Industry (ECMI).

The organisation aims to become a dedicated *one-stop-shop* and *service unit* to coordinate and facilitate the required exchanges in the field of application-driven mathematical research and its exploitation for innovations in industry, science and society, and it is organised as a network of national networks in industrial mathematics in order to combine the features of inclusiveness and flexibility in management.

The stakeholders of EU-MATHS-IN are currently represented by eight national networks in industrial mathematics, linking industry and academia in their respective countries, namely:

AMIES (Agence pour les mathématiques en interaction avec l'entreprise et la société) in France; KoMSO (Committee for Mathematical Modelling, Simulation and Optimisation) in Germany; SM[i][2] (Sportello Matematico per l'industria italiana) in Italy, Math-in (Red española matemática-industria) in Spain, Platform Wiskunde Nederland in the Netherlands, the Smith Institute for Industrial Mathematics and System Engineering in the UK, HU-MATHS-IN in Hungary and PL-MATHS-IN in Poland.

The number of national networks in industrial mathematics is going to increase in the near future, as helping, stimulating and involving those countries in which industrial mathematics is less developed is among the goals of the new organisation.

[1] This article will also be published in the Newsletter of the ECMI.

The foundation of EU-MATHS-IN was followed on 27 November 2013 by a kick-off meeting held at the Centre for Mathematics and Computer Science (CWI) in Amsterdam, which started with welcoming speeches by EMS president Marta Sanz-Solé and ECMI representative Hilary Okendon. Next, the speeches of the members of the Executive Board of EU-MATHS-IN followed: Mario Primicerio, president of the new foundation, presented the history of the Forward Look and the immediate goals of EU-MATHS-IN, while Volker Mehrmann, secretary, discussed the unsuccessful attempt of previous EU proposals and presented current Horizon 2020 calls. Maria J. Esteban, member of the executive board and representative of the Applied Mathematics Committee of the EMS, presented the main concepts underlying the organisation of EU-MATHS-IN. Wil Schilders, treasurer, presented the strategy document. The executive board also includes Magnus Fontes as a representative of the ECMI.

The most promising avenues for implementing the goals of the new organisation were discussed by dozens of experts in industrial mathematics at an international level during the kick-off meeting.

All the founders of EU-MATHS-IN expressed the common opinion that such an initiative will give the opportunity to expose mathematics to industrial needs at a European level and will increase the chances for every node of the national networks to get actively involved in large multidisciplinary scientific and technological projects.

Good Luck to EU-MATHS-IN!

So, What Was the Comics & Science Event in Lucca Really About?

Roberto Natalini (Istituto per le Applicazioni del Calcolo "M. Picone", National Research Council of Italy) and Andrea Plazzi (Symmaceo Communications, Milano, Italy)

In the September 2013 issue of this newsletter, we presented a new project called "Comics & Science" to be held in Lucca, home to one of the biggest comics conventions in Europe, quite fittingly named *Lucca Comics & Games* (31 October–2 November 2013).

Quite obviously, Comics & Science's goal was to see if and how these two amazing fields of human experience could interact in an effective and, more importantly, entertaining way.

Spread over the convention's three main days, and set up as a five-session programme, Comics & Science's main focus was on two truly impressive guests, each one excelling in his own field of activity: French mathematician Cédric Villani, Fields Medal 2010, and comics genius Leo Ortolani, possibly Italy's most popular comic book artist.

Villani's day was 1 November (and a great day it was). Cédric was literally soaked by the gladsome spirit of Lucca right from the start, enjoying the colourful and ubiquitous presence of cosplayers (or costume-players – fans dressed up like their favourite characters from comic books and animation) as if nothing could be more natural, which is really quite true in Lucca. His main presentation took place in Lucca's prestigious Palazzo Ducale. The main hall was packed full to the limit, mainly with high school students. Villani explained why popularising maths is so important at all levels, and why this important task is to be carried out by all mathematicians. Also, he openly declared his love for manga. After a spontaneous and heart-felt standing ovation, he spent the rest of the day in interviews for local radio and TV stations promoting his *Théorème Vivant* book ("The Living Theorem") and in signing sessions.

The very same spontaneous and contagious enthusiasm, from even more young people engaged in long hours of signing sessions, was the trademark of the meeting the day after with Ortolani. His contribution to this year's Comics & Science programme was *Misterius – Speciale Scienza* ("Misterius – Special Science Issue"), a brilliantly hilarious comic which is reviewed elsewhere in this issue. The meeting was informal and funny, with Ortolani conducting a kind of college examination of actual scientists (mathematicians, physicists and geologists), who were amused and kind enough to answer questions about their research interests and activities. It was a great opportunity to introduce a huge audience to what science actually is and how professional, real-world researchers make advances with their day-to-day efforts. Their willingness to be heard and understood by a mainstream, sincerely interested audience, coupled with how much that same audience was captivated by Ortolani's well-known charismatic appeal, made the event fully successful.

Built around these two highlights, Comics & Science's offering included several presentations of "scientifically-oriented" (in the broadest sense) comics and a "robotic" meeting, the latter focusing on comparing how "robots" are perceived and portrayed in Western and Eastern fiction, and what actual robotics research is bringing about.

The warm feedback received from the audience attending the event and the media covering the event is comforting. It confirms that the Comics & Science's philosophy is 100% the right direction to go: not only that professionally popularised science can be funny and entertaining but that the ways, means and inspiration of communicating science should be – literally – taken from entertainment itself, which we see as a huge, mainstream collector of all fans' interests and passions.

Foundations of Computational Mathematics Conference, Montevideo, December 11–20, 2014

The next Foundations of Computational Mathematics conference will take place at the Universidad de la República in Montevideo, between December 11–20, 2014.

The conference, organised by the Society for Foundations of Computational Mathematics, is eighth in a sequence that commenced with the Park City, Rio de Janeiro, Oxford, Minneapolis, Santander, Hong Kong and Budapest FoCM meetings.

Plenary Speakers:
Annalisa Buffa, IMATI CNR, Italy
David Cox, Amherst College, USA
John Cremona, University of Warwick, UK
Arieh Iserles, University of Cambridge, UK
Frances Kuo, University of New South Wales, Australia
Yi Ma, Microsoft Research Asia, China, and University of Illinois at Urbana-Champaign, USA
Andrei Martinez-Finkelshtein, University of Almería, Spain
Yurii Nesterov, Catholic University of Louvain, Belgium
Ricardo Nochetto, University of Maryland, USA
Benoit Perthame, Pierre et Marie Curie University and CNRS, France
Pencho Petrushev, University of South Carolina, USA
Helmut Pottmann, King Abdullah University of Science and Technology, Saudi Arabia
Reinhold Schneider, Technische UniversitŁt Berlin, Germany Mike Shub, CONICET, Argentina
Michael Singer, North Carolina State University, USA
Endrei Szemeredi, Hungarian Academy of Science, Hungary and Rutgers University, USA
Raúl Tempone, King Abdullah University of Science and Technology, Saudi Arabia

The conference will follow a format tried and tested to a great effect in former FoCM conferences: plenary invited lectures in the mornings, theme-centered parallel workshops in the afternoons. Each workshop extends over three days and the conference will consist of three periods, comprising of different themes. We encourage the participants to attend the full conference.

Each workshop will include "semi-plenary" lectures, of an interest to a more general audience, as well as (typically shorter) talks aimed at more technical audience.

Read more at: http://www.fing.edu.uy/11jana/www2/focm_2014.html

International Congress for Women in Mathematics 2014 (ICWM2014)

Caroline Series (University of Warwick, Coventry, UK)

ICWM2014 will take place in Seoul, 12–14 August 2014, bringing together women mathematicians and their supporters from around the world. The meeting on 12 August will take place in Ewha Womans University. ICM2014 opens on 13 August and the programme of ICWM2014 on 14 August is partially integrated into that of ICM2014 and occurs at the same place COEX. The programme includes plenary lectures by: Laura DeMarco, Isabel Dotti, Jaya Iyer, Motoko Kotani, Hee Oh, Gabriella Tarantello and Donna Testerman, a panel session "Mathematics and Women: different regions, similar struggles", a poster session on 12 August and the ICM Emmy Noether lecture by Georgia Benkart on 14 August, followed by a reception at the ICWM Night. Transport to the ICM welcoming reception on 12 August will be provided.

Information and registration can be found at http://www.kwms.or.kr/icwm2014.

Periods and the Conjectures of Grothendieck and Kontsevich–Zagier

Joseph Ayoub (Universität Zürich, Switzerland)

This paper concerns a class of complex numbers, called *periods*, that appear naturally when comparing two cohomology theories for algebraic varieties (the first defined topologically and the second algebraically). Our goal is to explain the fundamental conjectures of Grothendieck and Kontsevich–Zagier that give very precise information about the transcendence properties of periods. The notion of *motive* (due to Grothendieck) plays an important conceptual role. Finally, we explain a geometric version of these conjectures. In contrast with the original conjectures whose solution seems to lie in a very distant future, if at all it exists, a solution for the geometric conjectures is within reach of the actual motivic technology.

1 Introduction

Integration and cohomology

Let M be a real C^∞-manifold. Let $\mathcal{A}^n(M,\mathbb{C})$ be the $\mathbb{C}$-vector space of C^∞-differential forms of degree n on M. These vector spaces are the components of the de Rham complex $\mathcal{A}^\bullet(M;\mathbb{C})$ whose cohomology (i.e., the quotient of the space of closed differential forms by its subspace of exact differential forms) is the *de Rham cohomology* of M denoted by $H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$. In practice, for instance if M is compact, the $H^n_{\mathrm{dR}}(M;\mathbb{C})$'s are finite dimensional vector spaces; in any case, they vanish unless $0 \le n \le \dim(M)$.

On the other hand, one has the singular chain complex of M, denoted by $C_\bullet(M;\mathbb{Q})$. For $n \in \mathbb{N}$, $C_n(M;\mathbb{Q})$ is the $\mathbb{Q}$-vector space with basis consisting of C^∞-maps from the n-th simplex

$$\Delta^n = \left\{ (x_0,\ldots,x_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} x_0 \ge 0,\ldots,x_n \ge 0, \\ x_0+\cdots+x_n = 1 \end{array} \right\}$$

to M. The *singular homology* of M, denoted by $H_\bullet(M;\mathbb{Q})$, is the homology of this complex.

Integration of forms yields a well-defined pairing

$$\langle -,- \rangle : H_n(M;\mathbb{Q}) \times H^n_{\mathrm{dR}}(M;\mathbb{C}) \to \mathbb{C}. \qquad (1)$$

If $\gamma = \sum_{s=1}^r a_s \cdot [f_s : \Delta^n \to M] \in C_n(M;\mathbb{Q})$ is a closed chain and $\omega \in \mathcal{A}^n(M,\mathbb{C})$ is a closed differential form then

$$\langle \overline{\gamma}, \overline{\omega} \rangle = \sum_{s=1}^r a_s \cdot \int_{\Delta^n} f_s^* \omega. \qquad (2)$$

Stokes' formula ensures that the right side depends only on the classes $\overline{\gamma} \in H_n(M;\mathbb{Q})$ and $\overline{\omega} \in H^n_{\mathrm{dR}}(X;\mathbb{C})$ of γ and ω.

The classical theorem of de Rham asserts that the pairing (1) is perfect, i.e. that it identifies $H^n_{\mathrm{dR}}(M;\mathbb{C})$ with the space of linear maps from $H_n(M;\mathbb{Q})$ to $\mathbb{C}$.

Periods

Periods are complex numbers obtained by evaluating the pairing (1) on special de Rham cohomology classes with interesting arithmetic properties.

Assume that M is an algebraic manifold defined by equations with rational coefficients. In the language of algebraic geometry, we are fixing a smooth $\mathbb{Q}$-variety X such that M is the manifold of complex points in X, which we write simply as $M = X(\mathbb{C})$. In this case, it makes sense to speak of algebraic differential forms over M (or X). In general, there are few of these globally, and one is forced to work locally for the Zariski topology on X, i.e. to consider the sheaf Ω^n_X of algebraic differential forms of degree n on X. Locally for the Zariski topology, a section of Ω^n_X is an element of $\mathcal{A}^n(M;\mathbb{C})$ that can be written as a linear combination of $f_0 \cdot \mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_n$, where the f_i's are *regular functions* over X. (For example, if $M = \mathbb{C}^n$, the f_i's can be written as fractions P_i/Q_i with P_i's and Q_i's polynomials in n variables with rational coefficients.) Varying the degree, one gets the algebraic de Rham complex $\Omega^\bullet_X$; this is a complex of sheaves for the Zariski topology on X. Following Grothendieck, we define the *algebraic de Rham cohomology* of X (or M) to be the Zariski hyper-cohomology of $\Omega^\bullet_X$:

$$H^\bullet_{\mathrm{AdR}}(X) = \mathbb{H}^\bullet_{\mathrm{Zar}}(X;\Omega^\bullet_X).$$

The elements of $H^\bullet_{\mathrm{AdR}}(X)$ are those special de Rham cohomology classes that produce periods when paired with singular homology classes. More precisely, there is a canonical isomorphism

$$H^\bullet_{\mathrm{AdR}}(X) \otimes \mathbb{C} \simeq H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$$

and, in particular, $H^\bullet_{\mathrm{AdR}}(X)$ is a sub-$\mathbb{Q}$-vector space of $H^\bullet_{\mathrm{dR}}(M;\mathbb{C})$. (Roughly speaking, this inclusion is obtained by considering an algebraic differential form on X as an ordinary differential form on M.) Now, restricting the pairing (1), we get a pairing

$$H^{\mathrm{Sing}}_\bullet(X) \otimes H^\bullet_{\mathrm{AdR}}(X) \to \mathbb{C}, \qquad (3)$$

where $H^{\mathrm{Sing}}_\bullet(X)$ denotes $H_\bullet(M;\mathbb{Q})$. (As the notation suggests, the above pairing is canonically associated to the algebraic variety X.)

Definition 1. A *period* of the algebraic variety X is a complex number which is in the image of the pairing (3).

Remark 2. Although $H^{\mathrm{Sing}}_\bullet(X)$ and $H^\bullet_{\mathrm{AdR}}(X)$ are finite dimensional $\mathbb{Q}$-vector spaces, the values of the pairing (3) are not rational numbers in general. Indeed, *periods* are often (expected to be) transcendental numbers.

Remark 3. When X is smooth and proper, its periods are called *pure*. The previous construction, in the case where X is proper, yields all the pure periods.

If X is not necessarily proper, its periods are called *mixed*. In contrast with the pure case, the previous construction is not expected to give all the *mixed periods*. Indeed, more mixed periods are obtained by considering relative cohomology of pairs (see §2).

Transcendence

It is an important and fascinating problem to understand the arithmetic properties of periods. In the case of *abelian periods* (i.e. those obtained by taking $n = 1$ in (3) with X possibly singular or, equivalently, those arising from abelian varieties and more generally 1-motives) much is known thanks to the *analytic subgroup theorem* of Wüstholz [16] generalising results of Baker. The result of Wüstholz can be interpreted as follows: *every $\overline{\mathbb{Q}}$-linear relation between abelian periods is of motivic origin*. This statement appears explicitly, for example, in [17]. We also refer to [8] for a similar interpretation of the result of Wüstholz in the context of another (albeit related) conjecture of Grothendieck in the style of the Hodge and Tate conjectures.

Beside the case of abelian periods, very little is known (but see [15] for a comprehensive catalogue of what transcendence theory knows about periods and [9] for some spectacular recent advances concerning periods of Tate motives, aka., multiple zeta values). Nonetheless, the conjectural picture is very satisfactory and conjectures of Grothendieck and Kontsevich–Zagier yield a precise understanding of the ring of periods. Unfortunately, these conjectures seem so desperately out of reach of the present mathematics and – this is to be taken as my personal opinion – I can't think of any other conjecture that looks as intractable!

A typical question of interest is the following:

Question. *Let X be a smooth $\mathbb{Q}$-variety and let $\mathcal{P}er(X)$ be the subfield of $\mathbb{C}$ generated by the image of the pairing* (3)*. What is the transcendence degree of the finitely generated extension $\mathcal{P}er(X)/\mathbb{Q}$?*

Grothendieck's conjecture gives an answer to this question: *the transcendence degree of $\mathcal{P}er(X)/\mathbb{Q}$ is equal to the dimension of the motivic Galois group of X*. Of course, the motivic Galois group of X is quite a complicated object and its dimension can be very hard to compute. Nonetheless, it is easy to convey that the dimension of a not-so-explicit algebraic group is much easier to compute than the transcendence degree of an explicit subfield of $\mathbb{C}$. This is already the case for $X = \mathbf{P}^1$ (the projective line): the motivic Galois group is given by $\mathbf{G}_m$ whereas the subfield generated by periods is $\mathbb{Q}(\pi)$.

The conjecture of Kontsevich–Zagier is more ambitious and goes beyond the above question: it aims at describing all the algebraic relations among periods. Roughly speaking, it says that two periods are equal if and only if there is a *geometric reason* (see Definition 6 for the list of geometric reasons.)

Remark 4. The conjecture of Kontsevich–Zagier is stronger than the conjecture of Grothendieck. This is not obvious from their statements and will be explained in §4. However, one can argue that both conjectures are essentially, or morally, equivalent.

Remark 5. The conjecture of Kontsevich–Zagier is remarkable in its simplicity as it can be stated in elementary terms. However, in practice, the conjecture of Grothendieck is better suited for deducing algebraic independence of periods.

The geometric version

As is the case with many deep and difficult problems on numbers, the conjectures of Grothendieck and Kontsevich–Zagier admit geometric (or functional) analogues that are accessible. Some ideas about the proof of the geometric versions will be discussed in §5.

2 The Kontsevich–Zagier conjecture

The ring of abstract periods

The Kontsevich–Zagier conjecture is best stated by introducing the ring of *abstract periods*. Roughly speaking, the ring of abstract periods is the free $\mathbb{Q}$-vector space generated by formal symbols, one for each pairing of a homology class with an algebraic de Rham cohomology class, modulo the relations that come from geometry. If the Kontsevich–Zagier conjecture was true, the ring of abstract periods would be identical to the subring of $\mathbb{C}$ generated by periods.

More precisely, consider 5-tuples $(X, Z, n, \gamma, \omega)$ where X is a $\mathbb{Q}$-variety (possibly singular), $Z \subset X$ is a closed subvariety, $n \in \mathbb{N}$, $\gamma \in H_n^{\mathrm{Sing}}(X, Z)$ is a relative homology class of the pair $(X(\mathbb{C}), Z(\mathbb{C}))$ and $\omega \in H^n_{\mathrm{AdR}}(X, Z)$ is a relative algebraic de Rham cohomology class. To such a 5-tuple, one associates a period

$$\mathrm{Ev}(X, Z, n, \gamma, \omega) := \int_\gamma \omega \in \mathbb{C}. \tag{4}$$

Following Kontsevich–Zagier [12], we make the following definition.

Definition 6. The ring of *abstract (effective) periods*, denoted by $\mathcal{P}^{\mathrm{eff}}_{\mathrm{KZ}}$, is the free $\mathbb{Q}$-vector space generated by symbols $[X, Z, n, \gamma, \omega]$ modulo the following relations:

(a) (Additivity) The map $(\gamma, \omega) \mapsto [X, Z, n, \gamma, \omega]$ is bilinear on $H_n^{\mathrm{Sing}}(X, Z) \times H^n_{\mathrm{AdR}}(X, Z)$.

(b) (Base-change) Given a morphism of $\mathbb{Q}$-varieties $f : X' \to X$ such that $f(Z') \subset Z$, a relative singular homology class $\gamma' \in H_n^{\mathrm{Sing}}(X', Z')$ and a relative algebraic de Rham cohomology class $\omega \in H^n_{\mathrm{AdR}}(X, Z)$, we have the relation
$$[X, Z, n, f_*\gamma', \omega] = [X', Z', n, \gamma', f^*\omega].$$

(c) (Stokes' formula) Given a $\mathbb{Q}$-variety X, closed subvarieties $Z \subset Y$ of X, a relative singular homology class $\gamma \in H_n^{\mathrm{Sing}}(X, Y)$ and a relative algebraic de Rham cohomology class $\omega \in H^{n-1}_{\mathrm{AdR}}(Y, Z)$, we have the relation
$$[X, Y, n, \gamma, \mathrm{d}\omega] = [Y, Z, n-1, \partial\gamma, \omega].$$

We denote

$$\underline{2\pi i} := \left[\mathbf{G}_m, \emptyset, 1, t \in [0,1] \mapsto \exp(2\pi i \cdot t), \frac{\mathrm{d}t}{t}\right]$$

and set $\mathcal{P}_{\mathrm{KZ}} := \mathcal{P}^{\mathrm{eff}}_{\mathrm{KZ}}[\underline{2\pi i}^{-1}]$. This is the ring of *abstract periods*.

It is clear that the function Ev of (4) induces a morphism of $\mathbb{Q}$-algebras

$$\mathrm{Ev} : \mathcal{P}_{\mathrm{KZ}} \to \mathbb{C}. \tag{5}$$

(Note that $\mathrm{Ev}(\underline{2\pi i}) = 2\pi i$.) We can now state:

Conjecture 7 (Kontsevich–Zagier)**.** The evaluation homomorphism (5) is injective.

Remark 8. The above conjecture is widely open and desperately out of reach. However, as stated in the introduction, the *analytic subgroup theorem* of Wüstholz [16] gives small (although highly non-trivial) evidence for this conjecture. Roughly speaking, the result of Wüstholz solves the Kontsevich–Zagier conjecture for abelian periods (or periods of level ≤ 1 in the terminology of Hodge theory).

A compact presentation of the ring of abstract periods

In this section, we will give another presentation of the ring $\mathcal{P}_{\mathrm{KZ}}$ that was obtained rather accidentally by the author.

In retrospect, this presentation uses fewer generators than the presentation of Kontsevich–Zagier. With the notation of Definition 6, one restricts to:

- $X = \mathrm{Spec}(A)$ for A running among étale sub-$\mathbb{Q}[z_1, \ldots, z_n]$-algebras of the ring of convergent power series with radius strictly larger than 1.
- $Z \subset X$, the normal crossing divisor given by the equation $\prod_{i=1}^{n} z_i(z_i - 1) = 0$.
- $\gamma : [0,1]^n \to X(\mathbb{C})$ the canonical lift of the obvious inclusion $[0,1]^n \hookrightarrow \mathbb{C}^n$.
- $\omega = f \cdot \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$ with $f \in A$, a top degree differential form.

In return, one needs much fewer relations: a special case of Stokes' formula suffices to realise all the geometric relations described in Definition 6.

To write down precisely the above sketch, we introduce some notation. For an integer $n \in \mathbb{N}$, we denote by $\overline{\mathbb{D}}^n$ the closed unit polydisc in $\mathbb{C}^n$. Let $O(\overline{\mathbb{D}}^n)$ be the ring of convergent power series in the system of n variables $(z_1, \ldots, z_n)$ with radius of convergence strictly larger than 1.

Definition 9. We denote by $O_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ the sub-$\mathbb{Q}$-vector space of $O(\overline{\mathbb{D}}^n)$ consisting of those power series $f = f(z_1, \ldots, z_n)$ which are algebraic over the field $\mathbb{Q}(z_1, \ldots, z_n)$ of rational functions. We also set $O_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty) = \bigcup_{n \in \mathbb{N}} O_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$.

Definition 10. Let $\mathcal{P}^{\mathrm{eff}}$ be the quotient of $O_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ by the sub-$\mathbb{Q}$-vector space spanned by elements of the form

$$\frac{\partial f}{\partial z_i} - f|_{z_i=1} + f|_{z_i=0}$$

for $f \in O_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ and $i \in \mathbb{N} \smallsetminus \{0\}$. We also define $\mathcal{P}$ to be $\mathcal{P}^{\mathrm{eff}}[\underline{2\pi i}^{-1}]$ with $\underline{2\pi i}$ the class of a well-chosen element of $O_{\mathrm{alg}}(\overline{\mathbb{D}}^1)$ whose integral on $[0,1]$ is $2\pi i$.

Proposition 11. There is an isomorphism

$$\mathcal{P} \simeq \mathcal{P}_{\mathrm{KZ}}. \tag{6}$$

The image of $f \in O_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ can be described as follows. Let $A \subset O_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ be an étale $\mathbb{Q}[z_1, \ldots, z_n]$-algebra containing f. Let $Z \subset X = \mathrm{Spec}(A)$ be the divisor given by the equation $\prod_{i=1}^{n} z_i(z_i - 1) = 0$. Then the image of $[f]$ by (6) is given by

$$[X, Z, n, \tau_n, f \cdot \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n]$$

with τ_n the tautological relative homology class given by the composition of $[0,1]^n \hookrightarrow \overline{\mathbb{D}}^n \to X(\mathbb{C})$.

Remark 12. The actual proof of Proposition 11 is very indirect. It relies on the comparison of two constructions of motivic Galois groups: the one by M. Nori [13] and the one by the author [3]. The details of this comparison will appear in [10]. See Remark 34 for more details.

It would be interesting to find a direct proof of Proposition 11 avoiding motives. It is certainly easy to construct the morphism $\mathcal{P} \to \mathcal{P}_{\mathrm{KZ}}$ that realises the isomorphism (6). It is also easy to prove that this morphism is surjective. However, injectivity seems to be the interesting and difficult part.

Remark 13. There is an evaluation homomorphism

$$\mathrm{Ev} : \mathcal{P} \to \mathbb{C} \tag{7}$$

that takes the class of $f \in O_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ to $\int_{[0,1]^n} f$. This evaluation homomorphism coincides with (5) modulo the isomorphism (6). Therefore, we may restate the Kontsevich–Zagier conjecture in more elementary terms (i.e. without speaking of algebraic varieties and their cohomologies) as follows: *the evaluation homomorphism* (7) *is injective*.

Over more general base fields

Fix a base field k and a complex embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then, most of the previous discussion extends to varieties over k.

Indeed, given a pair (X, Z) consisting of a k-variety X and closed subvariety $Z \subset X$, one can still define $H_\bullet^{\mathrm{Sing}}(X, Z)$ and $H^\bullet_{\mathrm{AdR}}(X, Z)$ and the canonical pairing

$$H_\bullet^{\mathrm{Sing}}(X, Z) \otimes H^\bullet_{\mathrm{AdR}}(X, Z) \to \mathbb{C}.$$

Remark 14. In contrast with $H_\bullet^{\mathrm{Sing}}(X, Z)$, which is still a finite dimensional $\mathbb{Q}$-vector space independently of k and σ, $H^\bullet_{\mathrm{AdR}}(X, Z)$ is naturally a k-vector space. Also, note that the canonical pairing is now perfect in a slightly twisted manner: it induces an isomorphism $H^\bullet_{\mathrm{AdR}}(X, Z) \otimes_{k,\sigma} \mathbb{C} \simeq \hom(H_\bullet^{\mathrm{Sing}}(X, Z), \mathbb{C})$.

One can also define the ring of abstract periods over k, which we denote by $\mathcal{P}_{\mathrm{KZ}}(k, \sigma)$, together with an evaluation homomorphism $\mathrm{Ev} : \mathcal{P}_{\mathrm{KZ}}(k, \sigma) \to \mathbb{C}$. One can wonder to what extent the Kontsevich–Zagier conjecture is reasonable for general fields. We discuss this in the following:

Remark 15. When $k/\mathbb{Q}$ is algebraic (e.g. k is a number field), it is easy to see that $\mathcal{P}_{\mathrm{KZ}}(k, \sigma) = \mathcal{P}_{\mathrm{KZ}}(\mathbb{Q})$. This shows that the Kontsevich–Zagier conjecture holds for k if and only if it holds for $\mathbb{Q}$.

On the other hand, the Kontsevich–Zagier conjecture extended to fields of higher transcendence degrees is not reasonable unless σ is "general". Indeed, if $\sigma(k)$ contains a transcendental period of a $\mathbb{Q}$-variety (e.g. $\pi \in \sigma(k)$) then $\mathrm{Ev} : \mathcal{P}_{\mathrm{KZ}}(k, \sigma) \to \mathbb{C}$ cannot be injective. (However, see Remark 24 for what is expected without any condition on σ.)

3 Motives and the Grothendieck conjecture

It is not the aim here to give an overview of the theory of motives. Instead, below are listed some facts that are necessary for stating and appreciating the Grothendieck conjecture. (For the reader who wants to learn more about motives, [1] is recommended.)

Abelian category of motives

Let k be a base field. According to Grothendieck, there should exist a $\mathbb{Q}$-linear abelian category $\mathbf{MM}(k)$ whose objects are called *mixed motives*. Given an embedding $\sigma : k \hookrightarrow \mathbb{C}$, one has a realization functor

$$R_\sigma : \mathbf{MM}(k) \to \mathbf{MHS}$$

to the category of ($\mathbb{Q}$-linear) mixed Hodge structure. (This functor is believed to be fully faithful as a consequence of the Hodge conjecture but this will be irrelevant here.) Also, given an algebraic closure $\overline{k}/k$ and a prime number ℓ invertible in k, there is a realization functor

$$R_\ell : \mathbf{MM}(k) \to \mathbf{Rep}(\mathrm{Gal}(\overline{k}/k); \mathbb{Q}_\ell)$$

to the category of ℓ-adic Galois representations. (After tensoring the source category by $\mathbb{Q}_\ell$ and when k is finitely generated over its prime field, this functor is also believed to be fully faithful as a consequence of the Tate conjecture.)

Given a k-variety X, there are objects $H^i_{\mathcal{M}}(X)$ of $\mathbf{MM}(k)$, called the *motives* of X, that play the role of the universal cohomological invariants attached to X. Every classical cohomological invariant of X is then obtained from one of the $H^i_{\mathcal{M}}(X)$'s by applying a suitable realization functor. For instance:

- $R_\sigma(H^i_{\mathcal{M}}(X))$ is the singular cohomology group $H^i_{\mathrm{Sing}}(X)$ endowed with its mixed Hodge structure.
- $R_\ell(H^i_{\mathcal{M}}(X))$ is the ℓ-adic cohomology group $H^i_\ell(X)$ endowed with the natural action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

Remark 16. When k has characteristic zero, M. Nori [13] has constructed a candidate for the category of mixed motives. While his construction is not known to satisfy all the expected properties (for instance, the ext-groups between Nori's motives are not known to satisfy the expected relation to Quillen K-groups), it is enough for the purpose of the article.

The absolute motivic Galois group of a field

The category $\mathbf{MM}(k)$ is expected to share the formal properties of $\mathbf{MHS}$ and $\mathbf{Rep}(\mathrm{Gal}(\overline{k}/k); \mathbb{Q}_\ell)$. For instance, $\mathbf{MM}(k)$ has an exact tensor product $\otimes$ and every motive M has a strong dual $M^\vee$. Moreover, given an embedding $\sigma : k \hookrightarrow \mathbb{C}$, singular cohomology yields an exact faithful monoidal functor

$$F_{\mathrm{Sing}} : \mathbf{MM}(k) \to \mathbf{Mod}(\mathbb{Q})$$

sending $H^i_{\mathcal{M}}(X)$ to the $\mathbb{Q}$-vector space $H^i_{\mathrm{Sing}}(X)$. (In fact, F_{Sing} is just R_σ composed with the forgetful functor from $\mathbf{MHS}$ to $\mathbf{Mod}(\mathbb{Q})$.) This makes $\mathbf{MM}(k)$ into a *neutralized Tannakian category* with fiber functor F_{Sing}.

A *multiplicative operation* $\gamma = (\gamma_M)_M$ on F_{Sing} is a family of automorphisms $\gamma_M \in \mathbf{GL}(F_{\mathrm{Sing}}(M))$, one for each $M \in \mathbf{MM}(k)$, such that:

- For every morphism of motives $a : M \to N$, one has $\gamma_N \circ F_{\mathrm{Sing}}(a) = F_{\mathrm{Sing}}(a) \circ \gamma_M$.
- For motives M and N, one has $\gamma_{M\otimes N} = \gamma_M \otimes \gamma_N$ modulo the identification $F_{\mathrm{Sing}}(M \otimes N) \simeq F_{\mathrm{Sing}}(M) \otimes F_{\mathrm{Sing}}(N)$.

Definition 17. The multiplicative operations of F_{Sing} are the $\mathbb{Q}$-rational points of a pro-$\mathbb{Q}$-algebraic group $\underline{\mathrm{Aut}}^\otimes(F_{\mathrm{Sing}})$ called the *motivic Galois group* (of k) and denoted by $\mathbf{G}_{mot}(k, \sigma)$. (Note that this depends on the choice of the complex embedding σ.)

By the Tannaka reconstruction theorem [14], the functor F_{Sing} induces an equivalence of categories

$$\widetilde{F}_{\mathrm{Sing}} : \mathbf{MM}(k) \xrightarrow{\sim} \mathbf{Rep}(\mathbf{G}_{mot}(k, \sigma))$$

between motives and algebraic representations of the motivic Galois group.

Remark 18. One may think about $\mathbf{G}_{mot}(k, \sigma)$ as a linearisation of the absolute Galois group of k. For instance, there is a continuous morphism

$$\mathrm{Gal}(\overline{k}/k) \to \mathbf{G}_{mot}(k, \sigma)(\mathbb{Q}_\ell),$$

which induces the realization functor R_ℓ.

The motivic Galois group of a motive

In the previous subsection, we introduced the absolute motivic Galois group of a field k endowed with an embedding σ; this was the analogue of the absolute Galois group of a field endowed with a choice of an algebraic closure. In order to formulate the Grothendieck conjecture, we need the motivic Galois group of a motive; this is the analogue of the Galois group of a finite Galois extension.

Definition 19. Let $M \in \mathbf{MM}(k)$ be a mixed motive. The *motivic Galois group* of M, denoted by $\mathbf{G}(M)$, is the image of the morphism

$$\mathbf{G}_{mot}(k, \sigma) \to \mathbf{GL}(F_{\mathrm{Sing}}(M))$$

given by the natural action of $\mathbf{G}_{mot}(k, \sigma)$ on the $\mathbb{Q}$-vector space $F_{\mathrm{Sing}}(M)$, i.e. sending a multiplicative operation γ to γ_M.

Remark 20. By construction, $\mathbf{G}(M)$ is an algebraic linear group. Moreover, $\mathbf{G}_{mot}(k, \sigma)$ is the inverse limit of the $\mathbf{G}(M)$'s when M runs over larger and larger motives.

Statement of the Grothendieck conjecture

From now on, we assume that k has characteristic zero. Algebraic de Rham cohomology yields a functor

$$F_{\mathrm{AdR}} : \mathbf{MM}(k) \to \mathbf{Mod}(k)$$

sending $H^i_{\mathcal{M}}(X)$ to the k-vector space $H^i_{\mathrm{AdR}}(X)$. Fixing an embedding $\sigma : k \hookrightarrow \mathbb{C}$, the pairing (3) can be extended to any motive M yielding a pairing

$$F_{\mathrm{Sing}}(M)^\vee \otimes F_{\mathrm{AdR}}(M) \to \mathbb{C}. \tag{8}$$

(This is truly an extension of (3): for $M = H^i_{\mathcal{M}}(X)$, $F_{\mathrm{Sing}}(M)^\vee$ and $F_{\mathrm{AdR}}(M)$ are indeed canonically isomorphic to $H_i^{\mathrm{Sing}}(X)$ and $H^i_{\mathrm{AdR}}(X)$.)

Conjecture 21 (Grothendieck)**.** Assume that $k = \mathbb{Q}$. Let M be a motive and let $\mathcal{P}er(M)$ be the subfield of $\mathbb{C}$ generated by the image of the pairing (8). Then, one has the equality:

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) = \dim(\mathbf{G}(M)).$$

Remark 22. If one is interested in the transcendence degree of the field $\mathcal{P}er(X)$ generated by the periods of a $\mathbb{Q}$-variety X, one should take

$$M = \bigoplus_{i=0}^{2\dim(X)} H^i_{\mathcal{M}}(X)$$

in the previous conjecture.

Remark 23. It is not difficult to show that

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) \leq \dim(\mathbf{G}(M)).$$

This is not very surprising: it is much harder to prove algebraic independence than construct algebraic relations.

Remark 24. In [1, §23.4.1], Y. André proposes an extension of the Grothendieck conjecture for base fields of non-zero transcendence degree. This extension states that the inequality

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) \geq \dim(\mathbf{G}(M))$$

holds for every $M \in \mathbf{MM}(k)$. By the previous remark, this is indeed an extension of the Grothendieck conjecture. Note also that the above inequality is expected to be strict if the complex embedding σ is "general" (see Remark 15). Indeed, in this case, the equality

$$\mathrm{degtr}(\mathcal{P}er(M)/k) = \dim(\mathbf{G}(M)),$$

which can be restated as

$$\mathrm{degtr}(\mathcal{P}er(M)/\mathbb{Q}) = \dim(\mathbf{G}(M)) + \mathrm{degtr}(k/\mathbb{Q}),$$

is expected to hold.

Remark 25. The Grothendieck conjecture is the basis for a (conjectural) Galois theory for periods. The interested reader is referred to [2].

Reformulation of the Grothendieck conjecture

We reformulate the Grothendieck conjecture in terms of the absolute motivic Galois group and the so-called torsor of periods. We start with the following basic fact from the general theory of Tannakian categories.

Proposition 26. Let F be a field of characteristic zero and E/F an extension.

Let $\mathcal{T}$ be an F-linear Tannakian category neutralized by a fiber functor $\omega : \mathcal{T} \to \mathbf{Mod}(F)$ and let $\delta : \mathcal{T} \to \mathbf{Mod}(E)$ be another fiber functor. Then, the multiplicative operations $\delta \xrightarrow{\sim} \omega \otimes_F E$ are the E-points of a pro-algebraic E-variety $\underline{\mathrm{Iso}}^{\otimes}(\delta, \omega)$ which is naturally a pro-E-torsor (on the right) over the pro-F-algebraic group $\underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Let M be a motive and let $\langle M\rangle$ be the Tannakian subcategory of $\mathbf{MM}(k)$ generated by M; this is the smallest abelian subcategory of $\mathbf{MM}(k)$ closed under tensor products and duals and containing M. We have the following lemma.

Lemma 27. $\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M\rangle})$ identifies with $\mathbf{G}(M)$. Moreover, $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M\rangle}, F_{\mathrm{Sing}}|_{\langle M\rangle})$ has a canonical complex valued point, denoted by comp, whose residue field is exactly the subfield $\mathcal{P}er(M) \subset \mathbb{C}$.

Proof. The pairing (8) yields an isomorphism of $\mathbb{C}$-vector spaces $F_{\mathrm{AdR}}(M) \otimes_k \mathbb{C} \xrightarrow{\sim} F_{\mathrm{Sing}}(M) \otimes \mathbb{C}$. Replacing M by motives in $\langle M\rangle$ yields a multiplicative operation

$$\mathrm{comp} : F_{\mathrm{AdR}}|_{\langle M\rangle} \otimes_k \mathbb{C} \xrightarrow{\sim} F_{\mathrm{Sing}}|_{\langle M\rangle} \otimes \mathbb{C}$$

and hence a complex-valued point

$$\mathrm{comp} \in \underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M\rangle}, F_{\mathrm{Sing}}|_{\langle M\rangle})(\mathbb{C}).$$

Formal manipulations show that the residue field of this point is generated by the image of the pairing (8). □

Corollary 28. The Grothendieck conjecture is equivalent to the following statement. *Let* $M \in \mathbf{MM}(\mathbb{Q})$ *be a motive over* $\mathbb{Q}$. *Then,* comp *is a generic point of the* $\mathbb{Q}$*-variety* $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M\rangle}, F_{\mathrm{Sing}}|_{\langle M\rangle})$.

Proof. If $\xi \in W(\mathbb{C})$ is a complex point of an equidimensional $\mathbb{Q}$-variety W, the following conditions are equivalent:
- ξ is a generic point.
- $\dim(W) = \mathrm{degtr}(\mathbb{Q}(\xi))$.

Now, the $\mathbb{Q}$-variety $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}|_{\langle M\rangle}, F_{\mathrm{Sing}}|_{\langle M\rangle})$ is a torsor over $\mathbf{G}(M) = \underline{\mathrm{Aut}}^{\otimes}(R_{\mathrm{Sing}}|_{\langle M\rangle})$. Hence, it is equidimensional and

$$\dim(\underline{\mathrm{Iso}}^{\otimes}(R_{\mathrm{AdR}}|_{\langle M\rangle}, R_{\mathrm{Sing}}|_{\langle M\rangle})) = \dim(\mathbf{G}(M)).$$

This proves the claim as $\mathbb{Q}(\mathrm{comp}) = \mathcal{P}er(M)$. □

Passing to the limit, we get a complex point comp of the pro-k-variety $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$. We also obtain the following:

Proposition 29. The Grothendieck conjecture is equivalent to the following statement. *If* $k = \mathbb{Q}$ *then* comp *is a generic point of* $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$.

Definition 30. $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$ is called the *torsor of periods*.

4 The relation between the two conjectures

Here we explain why the Grothendieck conjecture is only slightly weaker than the Kontsevich–Zagier conjecture.

In fact, one has the following theorem (due to Kontsevich and proven in detail in [11]).

Theorem 31. There is a canonical isomorphism of k-algebras

$$\mathcal{O}(\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})) \simeq \mathcal{P}_{\mathrm{KZ}}(k, \sigma)$$

Moreover, modulo this isomorphism, the evaluation homomorphism (5) corresponds to evaluating a regular function on $\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})$ at the complex point comp:

$$\mathcal{O}(\underline{\mathrm{Iso}}^{\otimes}(F_{\mathrm{AdR}}, F_{\mathrm{Sing}})) \to \mathbb{C}$$
$$f \mapsto f(\mathrm{comp}).$$

Corollary 32. The following assertions are equivalent.
(a) The Kontsevich–Zagier conjecture holds.
(b) The Grothendieck conjecture holds and the ring $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain.

Proof. Indeed, by the previous theorem, the injectivity of the evaluation homomorphism (5) is equivalent to the fact that the complex-valued point comp is generic and that $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain. We conclude using Proposition 29. □

Remark 33. Some authors, for instance Y. André in [2], refer to the Grothendieck conjecture as the combination of the statement in Conjecture 21 and the property that $\mathcal{P}_{\mathrm{KZ}}$ is an integral domain.

Remark 34. We take the opportunity to give some hints concerning the compact presentation of the ring of abstract periods.

It is possible to construct a motivic Galois group and a torsor of periods starting from Voevodsky's triangulated category of motives. This is the approach pursued in [3, 4]. Using some flexibility pertaining to the theory of motives à la Voevodsky one is able to "compute" more efficiently the ring of regular functions on the torsor of periods, arriving eventually at the ring $\mathcal{P}$ of Definition 10.

Now, it turns out that both approaches yield isomorphic motivic Galois groups (see [10]). As the canonical map between $\mathrm{Spec}(\mathcal{P})$ and $\mathrm{Spec}(\mathcal{P}_{\mathrm{KZ}})$ is equivariant and the latter are torsors over isomorphic pro-$\mathbb{Q}$-algebraic groups, this gives Proposition 11.

5 The geometric version of the Grothendieck and the Kontsevich–Zagier conjectures

We now turn to the geometric version of the conjectures of Grothendieck and Kontsevich–Zagier for which a proof is available.

The relative motivic Galois group

Definition 35. Let k be a field. Given an extension K/k and an embedding $\sigma : K \hookrightarrow \mathbb{C}$, one has an induced morphism of motivic Galois groups

$$\mathbf{G}_{mot}(K, \sigma) \to \mathbf{G}_{mot}(k, \sigma).$$

The *relative motivic Galois group* $\mathbf{G}_{rel}(K/k, \sigma)$ is the kernel of this morphism.

Proposition 36. Let $l \subset K$ be the algebraic closure of k in K. One has an exact sequence (of groups and sets)

$$\{1\} \to \mathbf{G}_{rel}(K/k, \sigma) \to \mathbf{G}_{mot}(K, \sigma) \to \mathbf{G}_{mot}(k, \sigma) \to \hom_k(l, \mathbb{C}) \to \star.$$

In particular, if k is algebraically closed in K, then $\mathbf{G}_{mot}(K, \sigma) \to \mathbf{G}_{mot}(k, \sigma)$ is surjective.

Proof. It is shown in [4, Théorème 2.34] that the morphism

$$\mathbf{G}_{mot}(K, \sigma) \to \mathbf{G}_{mot}(l, \sigma)$$

is surjective. Therefore, it remains to show that $\mathbf{G}_{mot}(l, \sigma)$ identifies with the stabilizer in $\mathbf{G}_{mot}(k, \sigma)$ of the point in $\sigma|_l \in \hom_k(l, \mathbb{C})$. This follows easily from the exact sequence

$$\{1\} \to \mathbf{G}_{mot}(\bar{k}, \sigma) \to \mathbf{G}_{mot}(k, \sigma) \to \mathrm{Gal}(\bar{k}/k) \to \{1\},$$

which is a consequence of [4, Corollaire 2.31]. □

We also note the following easy consequence of [4, Théorème 2.34].

Proposition 37. Assume that k is algebraically closed. Then, the exact sequence

$$\{1\} \to \mathbf{G}_{rel}(K/k) \to \mathbf{G}_{mot}(K) \to \mathbf{G}_{mot}(k) \to \{1\}$$

splits (non canonically). In particular, there is an isomorphism

$$\mathbf{G}_{mot}(K, \sigma) \simeq \mathbf{G}_{mot}(k, \sigma) \ltimes \mathbf{G}_{rel}(K/k, \sigma).$$

An important fact about the relative motivic Galois group is that it is "controlled" by a group of topological origin. In order to explain this, we need some notation.

Definition 38. Assume that k is algebraically closed in K and denote by $\mathrm{Mod}(K/k)$ the pro-k-variety of smooth models of K. More precisely, the objects of the indexing category of $\mathrm{Mod}(K/k)$ are pairs (X, i) where X is a smooth k-variety and $i : k(X) \simeq K$ an isomorphism. The pro-object $\mathrm{Mod}(K/k)$ is the functor $(X, i) \mapsto X$.

Remark 39. Consider the pro-manifold

$$(K/k)^{an} := \mathrm{Mod}(K/k)(\mathbb{C})$$

obtained by taking $\mathbb{C}$-points of each k-variety appearing in $\mathrm{Mod}(K/k)$. The complex embedding σ makes $(K/k)^{an}$ into a pointed pro-manifold and we may consider the associated pro-system of fundamental groups $\pi_1((K/k)^{an}, \sigma)$. This is a pro-discrete group.

We can now state the following crucial fact. This theorem was obtained independently by M. Nori (unpublished) and the author [4, Théorème 2.57].

Theorem 40. There is a canonical morphism

$$\pi_1((K/k)^{an}, \sigma) \to \mathbf{G}_{rel}(K/k, \sigma)$$

with Zariski dense image.

Remark 41. Let X be a geometrically irreducible algebraic k-variety and $M \in \mathbf{MM}(X)$ a *motivic local system*. (One can think about M as an object of $\mathbf{MM}(k(X))$ which is unramified over X.) Given the complex embedding σ, M realises to a topological local system on $X(\mathbb{C})$. If this local system is trivial then M is the pull-back of a motive $M_0 \in \mathbf{MM}(k)$; such a motive is called *constant* (relative to k). This is a direct consequence of Theorem 40.

For later use, we give a reformulation of Theorem 40.

Proposition 42. Assume that k is algebraically closed in K. Let $M \in \mathbf{MM}(K)$ and denote by

$$\langle M \rangle_0 \subset \langle M \rangle$$

the largest Tannakian subcategory consisting of constant motives (i.e. in the image of the pull-back $\mathbf{MM}(k) \to \mathbf{MM}(K)$). Then, there is an exact sequence

$$\pi_1^{\mathrm{alg}}((K/k)^{an}, \sigma) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle}) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle_0}) \to \{1\},$$

where $\pi_1^{\mathrm{alg}}((K/k)^{an}, \sigma)$ is the pro-algebraic completion of $\pi_1((K/k)^{an}, \sigma)$.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} \pi_1^{\mathrm{alg}}((K/k)^{an}) & \longrightarrow & \mathbf{G}_{mot}(K, \sigma) & \twoheadrightarrow & \mathbf{G}_{mot}(k, \sigma) \\ \| & & \downarrow & & \downarrow \\ \pi_1^{\mathrm{alg}}((K/k)^{an}) & \to & \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle}) & \twoheadrightarrow & \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle_0}) \end{array}$$

The image of $\pi_1^{\mathrm{alg}}((K/k)^{an}, \sigma)$ in $\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle})$ is a normal subgroup $\mathbf{N}$ and an algebraic representation of $\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle})/\mathbf{N}$ corresponds to a motive in $\langle M \rangle$ whose associated local system is trivial. By Theorem 40 (and Remark 41), this motive belongs to $\langle M \rangle_0$. □

Definition 43. Let $M \in \mathbf{MM}(K)$ be a motive. The kernel of the morphism

$$\underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle}) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle_0})$$

is denoted by $\mathbf{G}_{rel}(M)$. This is the *relative motivic Galois group* of M. By construction, $\mathbf{G}_{rel}(M)$ is a closed subgroup of $\mathbf{G}(M)$ (see Lemma 27).

Relative motivic Galois groups and functional transcendence

Keep the situation as above. Given a motive M in $\mathbf{MM}(K)$, $F_{\mathrm{AdR}}(M)$ is naturally a holonomic $\mathcal{D}_{K/k}$-module with regular singularities. (If $M = H^i_{\mathcal{M}}(X)$, the $\mathcal{D}_{K/k}$-module is associated to the Gauss–Manin connexion on $H^i_{\mathrm{AdR}}(X)$.)

Theorem 44. The Picard–Vessiot extension of K associated to the differential module $F_{\mathrm{AdR}}(M)$ has transcendence degree equal to the dimension of the algebraic group $\mathbf{G}_{rel}(M)$.

Proof. Indeed, by differential Galois theory, the transcendence degree of the Picard–Vessiot extension associated to $F_{\mathrm{AdR}}(M)$ is equal to the dimension of its differential Galois group. By the Riemann–Hilbert correspondence, the latter group has the same dimension as the monodromy group of the local system associated to M. This monodromy group is by definition the Zariski closure of the image of

$$\pi_1((K/k)^{an}, \sigma) \to \underline{\mathrm{Aut}}^{\otimes}(F_{\mathrm{Sing}}|_{\langle M \rangle}) = \mathbf{G}(M),$$

which, by Proposition 42, is equal to $\mathbf{G}_{rel}(M)$. □

Remark 45. Theorem 44 is clearly a geometric analogue of the Grothendieck conjecture. Although it is a direct corollary of Theorem 40, its precise statement was obtained during an email exchange with Daniel Bertrand (and, hence, did not appear before in the literature). It was also independently obtained by Peter Jossen and was probably known to Madhav Nori.

Geometric version of Kontsevich–Zagier

As for the Grothendieck conjecture, one can use Theorem 40 to obtain a geometric version of the Kontsevich–Zagier conjecture. Moreover, working in the realm of Voevodsky motives, one can give a very concrete statement in the style of the reformulation given in Remark 13. This was achieved in [6] and relies on previous work of the author (such as the theory of rigid analytic motives [5] and the construction of nearby motives [7, Chapitre 3]). We will not discuss the technical details here and we content ourself with stating the main result of [6]. We start by introducing some notation (compare with §2). Recall that $\overline{\mathbb{D}}^n$ denotes the closed unit polydisc in $\mathbb{C}^n$.

Definition 46. Let $O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ be the sub-$\mathbb{C}$-vector space of $O(\overline{\mathbb{D}}^n)[\![\varpi]\!][\varpi^{-1}]$ consisting of those Laurent series

$$F = \sum_{i>-\infty} f_i(z_1, \dots, z_n) \cdot \varpi^i,$$

with coefficients in $O(\overline{\mathbb{D}}^n)$, which are algebraic over the field $\mathbb{C}(\varpi, z_1, \dots, z_n)$. We also set $O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty) = \bigcup_{n\in\mathbb{N}} O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$.

Definition 47. Let $\mathcal{P}^{\dagger}$ be the quotient of $O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ by the $\mathbb{C}$-vector space spanned by:

- Elements of the *first kind*:
$$\frac{\partial F}{\partial z_i} - F|_{z_i=1} + F|_{z_i=0}$$
for $F \in O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ and $i \in \mathbb{N} \smallsetminus \{0\}$.
- Elements of the *second kind*:
$$\left(g - \int_{[0,1]^\infty} g\right) \cdot F$$
for $g,\ F \in O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ such that g does not depend on the variable ϖ (i.e. $\frac{\partial g}{\partial \varpi} = 0$) and g and F do not depend simultaneously on the variable z_i (i.e. $\frac{\partial g}{\partial z_i} \cdot \frac{\partial F}{\partial z_i} = 0$) for every $i \in \mathbb{N} \smallsetminus \{0\}$.

There is an evaluation homomorphism

$$\mathrm{Ev} : \mathcal{P}^{\dagger} \to \mathbb{C}(\!(\varpi)\!) \qquad (9)$$

sending the class of $F = \sum_{i>-\infty} f_i \cdot \varpi^i \in O^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$ to

$$\sum_{i>-\infty} \left(\int_{[0,1]^n} f_i\right) \cdot \varpi^i.$$

The Laurent series belonging to the image of (9) are called *series of periods*. The main theorem of [6] is:

Theorem 48. The evaluation homomorphism (9) is injective.

Remark 49. An important difference between the original Kontsevich–Zagier conjecture and its geometric version is the presence, in the geometric case, of new obvious relations corresponding to elements of the second kind in Definition 47.

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The Legacy of Kurt Mahler

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> *I hope that you can continue with your preparing of my collected works. When my old papers first appeared, they produced little interest in the mathematical world, and it was only in recent times that they have been rediscovered and found useful. So a collection of all my papers may repair this position!*
>
> [Kurt Mahler, in his last letter to Alf van der Poorten]

The quote at the beginning of this article was taken from a letter written by Kurt Mahler to Alf van der Poorten the day before Mahler died on 25 February 1988; it was received after van der Poorten had heard about Mahler's death.

At that time van der Poorten had already begun to collect Mahler's works. Twenty-five years have now passed since Mahler's death, and three years since van der Poorten's. Recently, in memory of van der Poorten, we[1] have finished van der Poorten's project and established the *Kurt Mahler Archive*. The Archive is hosted by the Centre for Computer-Assisted Research Mathematics and its Applications (CARMA),[2] which is located at the University of Newcastle, Australia. It can be found using the following URL: http://carma.newcastle.edu.au/mahler/index.html

1 Mahler, the man

Mahler was born in Krefeld, Germany, on 20 July 1903. He did not come from an academic family but, nonetheless, from one which loved the printed word; his father ran a printing firm. While first entering academia in a non-traditional way, Mahler should most appropriately be considered a student of Siegel, and this is how the *Mathematics Genealogy Project* records it. With the change of power in 1933, Mahler realised that he would need to leave Germany. After six weeks in Amsterdam, Mahler spent the academic year 1933–34 in Manchester under a fellowship secured by Mordell. He spent most of the next few years in the Netherlands until returning to the University of Manchester in 1937, where he remained for 25 years. After that, with the exception of a few years hiatus in the United States of America, he was an Australian mathematician.[3]

2 Mahler and Number Theory in Australia

Under the invitation of B. H. Neumann, in 1962, Mahler visited Australia and, in particular, the then quite new Australian National University (ANU). He was impressed enough to quickly join the ANU as a professor in its Institute of Advanced Studies. As Mahler recollects,

> *There was at this time no teaching of number theory in the undergraduate school (School of General Studies) at the ANU. I therefore gave a course on this subject to second and third year students at the SGS, probably the first one ever in Canberra. One of my undergraduates, Coates, asked me to introduce him to research. I provided him with problems to work on, and by the time he obtained his BSc, he had already several papers published or in print.*
>
> [Kurt Mahler, as quoted from [18]]

One gains much more appreciation for Mahler reading his student's view of the same time.

> *Mahler was also very concerned with sowing the seeds of his own mathematical knowledge in his new country. As in his own mathematical research, he instinctively felt that the best way to do this was to go back to first principles, and to begin by teaching beginners in the subject. The ANU had begun to award undergraduate degrees only a few years before Mahler arrived, and Hannah Neumann was appointed to head the new Department of Mathematics in the teaching side of the University (the School of General Studies) at about the same time that Mahler took up his chair. Between them, they arranged for Mahler to give two courses to the small number of undergraduates reading mathematics, one in 1963 on elementary number theory, and the second in 1964 on the elliptic modular function j(z). One of us had the good fortune to attend these courses. Mahler started and finished each lecture with extraordinary punctuality; in between, the audience was given a rare insight into his understanding of and enthusiasm for the material of the lecture. As he spoke, he would produce a beautiful written exposition on the blackboard of the key points, which were neatly placed in order in his characteristic rectangular boxes. Although he seemed at first so different and forbidding, we soon discovered that he was very willing to talk about his knowledge of mathematics in general, and to lend us his own mathematical books when we could not find them in the library. Mahler gave lectures at various summer schools in Canberra and elsewhere around Australia, as well as a number of advanced courses on transcendental number theory in the Institute of Advanced Studies. In the end the fascination of what he was doing beguiled us both into research in number theory, and we made our first steps in mathematical research on problems suggested by him.*
>
> [John Coates & Alf van der Poorten, as quoted from [18]]

These two students are well-known to those of us in number theory, and many of us beyond. John Coates went on to become a Fellow of the Royal Society and was Sadleirian Professor of Pure Mathematics at Cambridge from 1986 to 2012. Soon after Mahler moved to Australia, at the encouragement of George Szekeres, Alf van der Poorten visited Mahler regularly in Canberra. A friendship based at first on a common interest in science fiction[4] turned quickly into a mathematical relationship. Van der Poorten became Mahler's doctoral

1 Joy van der Poorten helped immensely with collecting Mahler's harder-to-get papers as well as the correspondence between Alf van der Poorten and Mahler. We thank her greatly for her contribution to the Archive.

2 We would have liked to have compiled a more standard printed archive of Mahler's work. Unfortunately, in the current digital age, such a work is no longer appealing to publishers.

3 There are numerous biographies of Kurt Mahler, which are extremely well written. Thus we will not dwell too much on this here. See [1, 2, 4, 6, 16, 18, 21].

4 This is entirely evident from the correspondence between Mahler and van

student, graduating[5] in 1968. Alf van der Poorten went on to positions at the University of New South Wales and then Macquarie University, both in Sydney.

Van der Poorten continued Mahler's legacy of number theory in Australia, both mathematically, in a very interesting and important collaboration with John Loxton, and as a mentor and supporter of young Australian mathematicians.[6] One such instance remembered by a former young Australian mathematician goes as follows.

I first encountered Alf's unique style when I was a member of the Australian International Olympiad (IMO) team and we came across his wonderfully entertaining paper on Apéry's proof of the irrationality of $\zeta(3)$*. I assumed from his name that he was Dutch, and was then pleasantly surprised when I met Alf at the IMO team send-off reception and to find that he lived and worked in Sydney! He then and there told me what the p-adic numbers were and immediately offered me a job at any time in the future! I took him up on his offer at the end of my first year at Melbourne University, and spent six weeks in his office annex learning about elliptic curves, the Riemann–Roch theorem, the Weil conjectures, and, of course a lot of great stuff about recurrence relations (Skolem–Mahler–Lech) and continued fractions. I have always appreciated the time he spent talking to me—it was clear he cared a great deal about young Australian mathematicians.*

[Frank Calegari, as quoted from [5]]

Mahler's contribution to number theory in Australia continues today, beyond his work and the work of his students and colleagues, in the *Mahler Lectureship* of the Australian Mathematical Society.

Every two years, the Australian Mathematical Society honours Mahler's legacy by awarding a distinguished lectureship in his name.

The Mahler Lectureship is awarded every two years to a distinguished mathematician who preferably works in an area of mathematics associated with the work of Professor Mahler. It is usually expected that the Lecturer will speak at one of the main Society Conferences and visit as many universities as can be reasonably managed.

[Australian Mathematical Society website: http://www.austms.org.au/]

The phrase *visit as many universities as can be reasonably managed* is not taken lightly. The 2013 Mahler Lecturer was Akshay Venkatesh of Stanford University. Professor Venkatesh gave 16 talks in 19 days at 10 different universities throughout Australia. Giving so many talks in that amount of time is a difficult task. Include the 10,000+ kilometres of travel within Australia in addition to the 24,000 kilometres just to get to and from Australia from Stanford and the Mahler Lectureship starts to sound like a daunting thing to accept (Akshay, if you are reading this, we all thank you heartily). Of course, Venkatesh grew up in Perth, so presumably he knew what he was getting into.

Former Mahler Lecturers comprise a marvellous group: John Coates (1991), Don Zagier (1993), Michel Mendès France (1995), Peter Hilton (1997), John H. Conway (1999), Robin Thomas (2001), Hendrik Lenstra (2003), Bruce Berndt (2005), Mark Kisin (2007), Terence Tao (2009[7]) and Peter Sarnak (2011).

3 The Mahler Archive

The way most of us interact with Mahler's legacy is now through his work. With the advent of the digital age, looking up papers is a much easier task than in previous times. With this in mind, we have made the Kurt Mahler Archive freely available online (see the first part of this article for the details and URL). It contains (in PDF format) every mathematical article published by Mahler, as well as a host of links to biographies and other information. His books are listed but are not available through the Archive.

Mathematically, it is quite easy to point out various highlights from Mahler's work. We do so here, hitting only a few of what we think are exceptional. Of course, this view is biased by our interests. For a more thorough list of topics see Mahler's "Fifty years as a Mathematician" [15] or the obituaries by Cassels [2] or van der Poorten [21].

1. In [7], Mahler introduced a measure of the quality of approximation of a complex transcendental number ξ by algebraic numbers. For any integer $n \geq 1$, we denote by $w_n(\xi)$ the supremum of the real numbers w for which
$$0 < |P(\xi)| < H(P)^{-w}$$
has infinitely many solutions in integer polynomials $P(x)$ of degree at most n. Here, $H(P)$ stands for the naïve height of the polynomial $P(x)$, that is, the maximum of the absolute values of its coefficients. Further, we set
$$w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$
According to Mahler, we say that ξ is:
   - An S-number, if $w(\xi) < \infty$.
   - A T-number, if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any integer $n \geq 1$.
   - A U-number, if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for some integer $n \geq 1$.

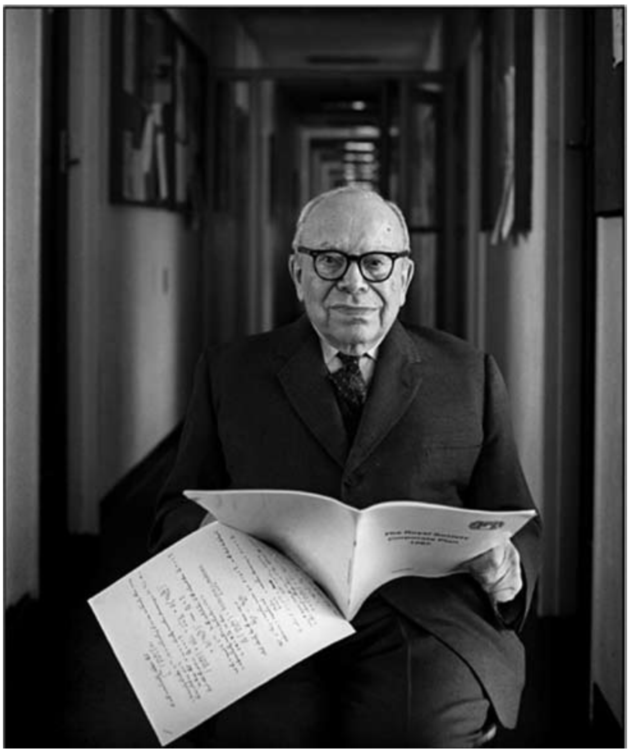
   The terminology S-number may have been chosen to honour Siegel. Almost all numbers, in the sense of Lebesgue measure, are S-numbers and Liouville numbers are examples of U-numbers. The existence of T-numbers remained an open problem for nearly 40 years until it was confirmed by Schmidt [19,20]. An important point in Mahler's classification is that two algebraically dependent transcendental numbers always fall in the same class.
2. One of the first significant contributions of Mahler is an approach, now called the "Mahler's method", yielding transcendence and algebraic independence results for the val-

der Poorten. The amount of space devoted to science fiction is roughly asymptotic to that devoted to mathematics.

5 Alf van der Poorten was enrolled as a student at the University of New South Wales and not at the ANU.

6 In keeping with van der Poorten's avid support of young Australian mathematicians, in his memory his family funds the *Alf van der Poorten Travelling Fellowships*, which aims to assist young pure mathematicians (who have earned a PhD in Australia) in travelling in Australia and overseas so that they can enrich their mathematical research through contact with other mathematicians.

7 In 2009, the lectureship was organised in partnership with the Clay Mathematics Institute, so the lecture tour that year was known as the Clay-Mahler lectures.



Kurt Mahler (1903–1988). The Sitting by Heide Smith. "I was commissioned to photograph Prof Mahler by the ANU in 1987. The request was for a portrait in front of a bookshelf, to be framed and hung in the ANU 'rogues gallery'. I photographed him as per my brief, and it worked out very well. When I asked Prof Mahler if I could take a few extra shots for my own collection, he agreed. I wanted to portray him within a geometric composition, a setting that I felt was more appropriate for a mathematician. I asked if he had some papers on mathematics that we could use, and he produced a 'Royal Society Corporate Plan 1987', which I carefully arranged. So this portrait is completely set up, yet in my view it shows not only the professor of mathematics, but Kurt Mahler the man. I used window light and a reflector; I never use flash if there is an alternative, because it changes the mood of the image altogether. Prof Mahler was a very keen photographer himself, and we spoke at length about his photography during the sitting. A few years later I happened to be at an auction in Canberra, where I saw a Rollei camera, complete with lenses and filters, in immaculate condition, all in a beautiful case. To my amazement I discovered that it had belonged to Kurt Mahler, and I could not resist buying it."

ues at algebraic points of a large family of power series satisfying functional equations of a certain type. In the seminal paper [17] Mahler established that the Fredholm series $f(z) = \sum_{k\geq 0} z^{2^k}$, which satisfies $f(z^2) = f(z) - z$, takes transcendental values at any nonzero algebraic point in the open unit disc.

3. Concerning specific *transcendence results*, in [9] Mahler proved that *Champernowne's number*

$$0.12345678910111213141516\cdots$$

is transcendental and is not a Liouville number. Champernowne had proven it normal to base ten a few years earlier.

4. Mahler [12] was also the first person to give an *explicit irrationality measure* for π. He showed that if p and $q \geq 2$ are positive integers, then

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{42}}.$$

This bound built on work from his previous paper [11], where he showed that $\|e^n\| > n^{-33n}$, where $\|x\|$ is the distance from x to the nearest integer. It is still a very interesting and open question as to whether or not there is a $c > 0$ such that $\|e^n\| > c^{-n}$.

5. Concerning Diophantine equations, in [8] Mahler provided finiteness results for a number of solutions of the so-called *Thue–Mahler equations*. In particular, he showed that if $F(X, Y) \in \mathbb{Z}[X, Y]$ is an irreducible homogeneous binary form of degree at least three, b is a nonzero rational integer and $p_1, \ldots, p_s$ ($s \geq 0$) are distinct rational prime numbers then the equation

$$F(x, y) = bp_1^{z_1} \cdots p_s^{z_s},$$

in

$$x, y, z_1, \ldots, z_s \in \mathbb{Z} \text{ with } \gcd(x, y) = 1 \text{ and } z_1, \ldots, z_s \geq 0,$$

has only finitely many solutions. The case $s = 0$ was proved by Thue, corresponding to Thue equations. Mahler was the first to see the importance of extending results in Diophantine approximation to include p-adic valuations as well as the ordinary absolute value.

6. Mahler also made outstanding contributions to the theory of polynomials with integer coefficients. In [13, 14] he introduced what is now called the *Mahler measure* of a polynomial. A celebrated open question is Lehmer's problem (from an article of 1933) asking, in different words (!), whether there exists $c > 1$ such that the Mahler measure of a non-cyclotomic polynomial is always at least c.

Lehmer's Problem. Let $P(X) = a_d X^d + \cdots + a_1 X + a_0 = a_d(X - \alpha_1)\cdots(X - \alpha_d)$ be a polynomial with integer coefficients. Its Mahler measure $M(P)$ is defined by

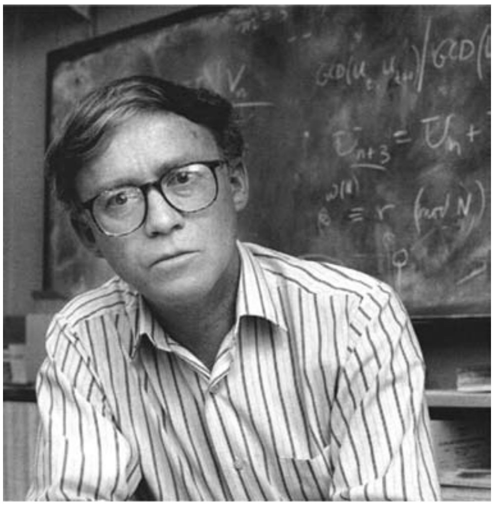
$$M(P) = |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

Does there exist a positive real number ε such that if $M(P) < 1 + \varepsilon$ then all the nonzero roots of $P(X)$ are roots of unity?

7. One of the most famous contributions of Mahler in geometry of numbers is his compactness theorem, established in 1946 in [10]. This is a criterion for the existence of a convergent subsequence of lattices in a sequence of lattices in n-dimensional space, *which may be said to have completely transformed the subject.* The last words express Cassels' opinion on page 136 of his monograph [3].

4 Conclusion

Here we have only highlighted a few of Mahler's many mathematical contributions. We apologise if we have left out your favourite but, if so, we encourage you to find its original paper in the Archive and enjoy Mahler's own words about it; we hope that you find them in a language with which you are familiar, though this may not be the case. Indeed, like many mathematicians during Mahler's active years (1927–1988), he



Alfred (Alf) Jacobus van der Poorten (1942–2010)

wrote in several languages. Using the Archive as our data, we found that he published papers in his native German, as well as in English, Italian, Dutch, Russian and even Chinese!

Mahler's interest in Chinese led him to do more than write just one paper in the language. Mahler felt so strongly that students should attempt to learn Chinese and be able to at least read mathematically in the language that, while on his hiatus from Australia at Ohio State University, he suggested that Chinese be taught to graduate students in mathematics. As the story goes, he was asked who could they ever find to teach such a course? He then answered that he would do it! And indeed, Mahler gave this course. For those interested, his lecture notes, "Lectures on the reading of mathematics in Chinese", can be found on the Archive in the Collection under the year 1972.

One can only speculate that Mahler saw the dawn of the Asian century coming. Indeed, there is a *Mahler-Needham collection*[8] at the ANU housing Mahler's Chinese mathematics collection. Joseph Needham (1900–1995) was the leading Western Sinologist of the past century and there was considerable correspondence between the two concerning Volume 3 (Chinese mathematics) of Needham's 28 volume opus *Science and Civilisation in China*.

In addition to Mahler's formal mathematical work, we found some very interesting private writings of his. In particular, much of his correspondence with Alf van der Poorten survives. It is evident from what we have been given that Mahler and van der Poorten had a tremendous correspondence relationship, which, as described above, focused on mutual interests in science fiction and mathematics. These letters reveal more about the man. In a letter dated 1 March 1985, Mahler mentions his thoughts about the Riemann hypothesis and a purported proof then circulating.

> *It will be a pity if the proof of Riemann's conjecture turns out to be incomplete. As you know, I am not convinced that RC is true. The numerical results go only to a limit which is relatively small when we think of numbers like* $10^{10^{10}}$.
>
> [Kurt Mahler, to Alf van der Poorten on 1 March 1985]

8 See http://anulib.anu.edu.au/about/collections/spcoll.html.

More so than in Mahler's papers, in his correspondence with van der Poorten we find that Mahler was deeply interested in computational mathematics and, specifically, what is more commonly thought of today as *experimental mathematics*.

> *I am interested in the problem of whether there are squares of integers which, to the base* $g = 5$*, have only digits* 0 *or* 1*. I could not find a single example although I went quite far on my calculator. Strangely, to the base* $g = 7$ *I obtained the one example* $20^2 = 1111(7)$*, and I am now seeing whether there are others.*
>
> [Kurt Mahler, to Alf van der Poorten on 15 February 1988]

Even in his last letter, the day before his death, Mahler was doing mathematics, lamenting that he was too old to learn to program well!

> *If I were ten years younger, I should also try to learn [to] handle big computers. But I have used only programmable calculators which I found very convenient. In the calculations for Squares to the base* 3 *I used mostly a TI 59 with printer and so could get my results. I have now also a H-P 28c calculator which works to 12 places. Unfortunately the manuals that come with this machine are far too short and badly arranged. So far I have not yet been able to construct on it a program which allows [me] to express a given integer or real numbers to the base* $g \geq 2$*, something I could do on the TI 59.*
>
> *The problem of the representation of squares to the base* $g \geq 5$ *seems quite hard, and I hope you have more success with it than I. It would be appropriate to consider the following more general problem.*
>
> > *Let* $f(x)$ *be a polynomial in* x *with integral coefficients which is positive for positive* x*. Study the integers* x *for which the representation of* $f(x)$ *to the base* $g \geq 3$ *has only digits* 0 *or* 1*.*
>
> *Here it may be sufficient to assume that the highest coefficient of* $f(x)$ *is a power of* g*, and that* f *is of the second degree. For polynomials of the first degree we settled this problem in our joint paper.*
>
> [Kurt Mahler, to Alf van der Poorten on 24 February 1988]

Kurt Mahler remained a mathematician's mathematician until the very end.

> *May his theorems live forever!*
>
> [Paul Erdős, remembering Mahler in [4]]

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Paul Erdős in the 21st Century

Miklós Simonovits (Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary)

1 Preface

Paul Erdős, one of those mathematicians of the 20th century who has had a great influence on today's mathematics, was born roughly a hundred years ago, on 26 March 1913, in Budapest. Conferences were organised all over the world to commemorate his 100th birthday. One of these conferences took place in Budapest, 1-5 July 2013. It was organised by the Eötvös Loránd University, Budapest (his Alma Mater), the Hungarian Academy of Sciences, the Alfréd Rényi Mathematical Institute and the János Bolyai Mathematical Society. I would dare to say that this was the most important among the centennial conferences. The conference tried to cover most of the favourite areas of Erdős. As the announcement of the conference stated:

> The topic of the conference includes all basic fields that Paul Erdős contributed to: Number Theory, Combinatorics (including Combinatorial Algebra, Combinatorial Geometry and Theoretical Computer Science), Analysis (including Approximation Theory and Ergodic Theory), Probability Theory, and Set Theory among others. The main goal of the conference is to explore Paul Erdős' wide ranging contributions to mathematics, and to survey the trends of development originating in his work.

We also had a special session dedicated to one of the closest friends of Erdős, namely Tibor Gallai, and since – despite the fact that Erdős was not directly involved in computer science – his influence in modern theoretical computer science is very remarkable, we also had a session on theoretical computer science (as the announcement above implicitly mentioned).

Below I shall briefly describe the conference and then try to describe the mathematics of Paul Erdős through a few theorems, conjectures and problems. Finally, I shall write about his personality and the legends around him.

The paper will conclude with some sources on Erdős and his mathematics.

This conference corresponded to a small congress, in the sense that it covered several wider areas of mathematics. There were only invited speakers:[1] 15 plenary sessions, given by outstanding mathematicians, and approximately 130 lectures in five parallel sessions.[2] Many of the lecturers of the parallel sessions would also have deserved a plenary lecture but the allocated time forced some compromises to be made. There were around 750 participants. In these areas, most of the conferences are much smaller, with 100–250 participants. Of course, I would not try to measure the importance of such a conference with numbers; actually, I am very much against it. These numbers are often misleading. The importance of Erdős' mathematics and his influence should be measured completely differently.

To understand the concept of this conference, we have to remark that when Paul Erdős died, on 20 September 1996, his funeral was connected to a small conference. However, three years later quite a big conference was organised in honour of Erdős, where the primary aim of the plenary sessions was to provide an overview of his mathematics and his impact on the mathematics of those days. Today, many years later, we see that his influence is still strongly increasing. This is one reason why we, the organisers, decided to put much more emphasis on what has happened in the last 20 years, on the impact of Erdős' results and conjectures, than on trying to convey his mathematics.

Also, we published a volume selected from papers of the invited speakers [5], available by the time of the conference. It is worth comparing this conference with the one in 1999, which had approximately 450 participants, roughly 130 plenary and sectional (invited) speakers and papers in two volumes [4] trying to describe Erdős' influence on mathematics.

Moreover, now we invited many outstanding *younger* researchers. The fact that so many participants came just to listen to others, to the most outstanding mathematicians of these areas, and to remember Erdős shows the importance of Erdős' mathematics and of this conference.

2 A short sketch of Erdős' mathematical career

It is difficult to write about Erdős' mathematics on a few pages for two reasons. On the one hand, his mathematical scope was extremely wide and only few people can appreciate its depth and breadth. On the other hand, these few pages do not provide enough space to cover it.

He was a child prodigy. Perhaps the first important result of Erdős was an elementary and very elegant proof of Chebishev's theorem, according to which the interval $[x, 2x]$ always contains a prime. One could say that primes were always very important in his mathematics. His PhD was on primes and, when Erdős and Selberg provided the first elementary proof of the prime number theorem, most people thought this would be followed by a major breakthrough in connection with the Riemann hypothesis. Unfortunately this did not happen.

The mathematical scope of Erdős is unbelievably deep and broad. To choose the most important topics, he definitely influenced number theory and graph theory very much, he systematically applied the "random construction" method and, with Alfréd Rényi, he started a systematic investigation of the evolutions of random structures, above all of random graphs, which is today one of the important parts of discrete mathematics.

Erdős had a crucial role in the development of some areas in number theory (see, for example, his classical results or his results with Kac and with Wintner).

He and Turán developed statistical group theory in a series of papers. Also, he has many important results in analysis, e.g., in interpolation theory, in the theory of polynomials and in approximation theory; however, describing them goes far beyond our scope.

And, of course, his role in the development of modern combinatorics and graph theory was crucial.

He also had a decisive influence on some parts of modern geometry. His questions and results were roots of a whole new theory. Below I shall return to this in slightly more detail.

Considering Erdős, I would distinguish several periods. His university years can be characterised on the one hand by having a friendly group of university fellows (actually lifelong friends). I would mention above all Paul Turán, Tibor Gallai (Grünwald) and György Szekeres, and also Eszter Klein (who later became the wife of Szekeres). These people learnt mathematics together, often from each other, wrote papers together, went on excursions together, ...

This was the friendly side. There was also the dark, tragic side: the surrounding society became more and more explicitly fascist, and definitely more and more anti-Semitic. These things influenced their lives very strongly.

Erdős left Hungary for Britain in 1934, spending four years there, but as Hitler occupied larger and larger parts of Europe, he decided to move to the USA. He spent the years 1939–1948 in the USA, firstly in Princeton but only for a year. He soon started his strange life of regularly moving from one place to another.

3 Erdős and number theory

In the early years Erdős' favourite subject was number theory. At the same time, he learnt some graph theory from König. During his years in England, most of his papers dealt with number theory. Meeting Kac in Princeton led to their famous result, the Erdős–Kac theorem on the distribution of values of additive number theoretical functions, culminating in his paper with Wintner.

The meaning of these theorems is that the number of prime divisors behaves according to the normal distribution and, under much more general conditions, the value distribution of additive number theoretical functions follows the same normal (Gaussian) law. We skip most of the explanation and, just to give the flavour of these results, formulate a truncated form of this theorem:

1 For technical reasons, there were also a small number of posters, on a much higher level than is usual. (I myself do not really like posters; however, sometimes they are needed.)

2 The programme can be seen, often with the corresponding slides, on the homepage of the conference: http://www.renyi.hu/conferences/erdos100/program.html.

Theorem 1 (Erdős, Erdős-Wintner, from Elliott's book [15])**.** In order that an additive function $f(n)$ should possess a limiting distribution, it is both necessary and sufficient that

$$\sum_{|f(p)|>1} \frac{1}{p}, \quad \sum_{|f(p)|\le 1} \frac{f(p)}{p} \quad \textit{and} \sum_{|f(p)|\le 1} \frac{f(p)^2}{p}$$

converge …

A special case of this is the famous Hardy–Ramanujan theorem, on the number of prime divisors, and a whole theory developed around this and related Erdős results.

One of the most outstanding of his results in Princeton, though much later, was the elementary proof of the prime number theorem with Selberg.

4 Paul Erdős' conjectures, theorems and problems

For Erdős, the most important thing in his life was "To Prove and Conjecture", i.e. to prove new theorems and to ask newer and newer questions. His early results in number theory immediately convinced his contemporaries of his extraordinary talent. Issai Schur called him the "the wizard from Budapest". His theorems, his new way of approaching mathematical problems and his conjectures completely changed a large part of the mathematics of the 20th century.

For many mathematicians his questions were quite often surprising, unorthodox and even "strange". However, a large number of them led later to important developments in the relevant areas. He never stopped asking question: most of his friends and colleagues would feel after a while that the area under discussion was already sufficiently well *explored/described* and they would move onto some newer areas. Erdős, of course, moved to many, many new areas (more than almost anyone around him), yet he would return to his older problems again and again and try to understand those results and conjectures even better. His friend András Hajnal once said of him that he wanted *a complete list of theorems*.

While writing this paper, one of my problems was that we, his grateful colleagues, co-authors, disciples and friends, wrote so many papers on his mathematics (during his life and after his death) that, though I have tried to avoid too much repetition, it is almost impossible to do so. Below, I shall select a few of his very simple questions and describe how they led to completely new theories and influenced the research of excellent mathematicians around him.

I will assume very little a priori knowledge of the areas. There will be one exception: in graph theory, I shall sometimes include the most important definitions in footnotes.

Arithmetic progressions

Perhaps one of the most important conjectures in combinatorial number theory is

Conjecture 1 (Erdős–Turán)**.** Let $r_k(n)$ denote the maximum number of integers, $a_1, \dots, a_m$, from the interval $[1, n]$ not containing k-term arithmetic progressions. Then for every fixed k, $r_k(n) = o(n)$.

The motivation of this was the Van der Waerden theorem, according to which, if we colour the integers with a bounded number of colours, say, by k colours, then one of the colour classes will contain an arithmetic progression.

This is a highly non-trivial question, even for the simplest case $r_3(n)$, first settled by K. F. Roth, a Fields medallist, and later extended, first to $k = 4$, and then to any fixed k, by Endre Szemerédi.

Szemerédi's theorem was definitely a fantastic breakthrough. While Roth used analytic methods, Szemerédi's proof was completely combinatorial. Very soon after his proof, Fürstenberg gave an ergodic theoretical proof of this result which later enabled him and Katznelson, and others, to prove several far-reaching generalisations of this theorem. A whole new school and a new branch of mathematics was built up around these results.

Today, perhaps one of the most important conjectures of Erdős, at least in this area, is:

Conjecture 2. Given a sequence of integers $0 < a_1 < a_2 < \cdots < a_n < \dots$, if

$$\sum \frac{1}{a_i} = \infty$$

then for each integer $k > 0$ this set of integers contains a k-term arithmetic progression.

Conjecture 2 is still open. It would immediately imply that the set of primes contains arbitrary long arithmetic progressions. Independently from this conjecture, Ben Green and Terry Tao have proved this latter assertion that the set of primes contains arbitrary long arithmetic progression.

The sum-product theorems

There are many results of Erdős which I could pick to illustrate his influence in combinatorial number theory. I decided to select the following result of Erdős and Szemerédi [11].

Theorem 2. There exists a constant $c > 0$ such that if A is a set of n integers then either $A + A$ or $A \cdot A$ has at least n^{1+c} elements.

There is a very important theorem of Freiman and Ruzsa, according to which, if for a set of integers A the sum-set is small then – in some sense – A is similar to the union of a few generalised arithmetic progressions.

The meaning of Theorem 2 is that the sum-set can be small for an arithmetic progression and the product set can be small for a geometric progression but a set of integers cannot behave at the same time both as an almost arithmetic and an almost geometric progression: either $A + A$ or $A \cdot A$ must be large. Actually, Erdős and Szemerédi conjectured that there are at least n^{2-c} numbers which are either sums or products from A. They also proved some related results and formulated similar conjectures, e.g. on the k-term sums and products.

This is again a case where we see a strange-looking Erdős question (or Erdős-Szemerédi question) and slightly later the most acknowledged mathematicians start attacking the problem, proving surprising new results, generalising it in various ways or extending to other situations, and finding several interesting connections between areas that may look distant at first sight, … *MathSciNet* provides around 70 related papers from outstanding authors; here are just a few of them: Kevin Ford, N. Nathanson, Gy. Elekes, Solymosi, Jean Bourgain and Mei-Chu Chang, Konyagin, Noga Alon, Itai Benjamini, Boris

Bukh, Ehud Hrushowski, Antal Balog, Van Vu, Terrence Tao, Sudakov, Ruzsa and many others, either writing directly about this topic or at least referring to it.

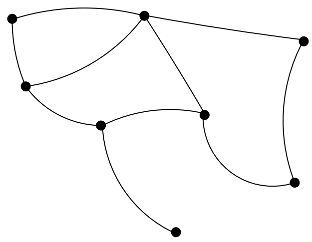
5 Erdős and graph theory

One of the favourite research areas of Erdős was graph theory. Perhaps – despite his strong influence in several areas – this is where his influence was the strongest. One reason for this may be that this was a new area that was fast developing, more open and less rigid than some other areas, but of course the other reason was the ingenious approach of Erdős. Here is a very short description of the beginnings.

Asymptotic description of extremal graph sequences

Graphs should be imagined as representations of binary relations, where, for the sake of simplicity, we have some points in the plane and join two of them if they are in relation. Of course, we have many different graphs: simple graphs, directed graphs, hypergraphs, directed multihypergraphs, ...

> A graph is a structure modelling binary relations. We imagine the graphs as points, joined by lines. Good examples of graphs are the cities on maps, joined by roads, or social structure, where the vertices are the people and their relations are represented by the edges of the graph. Graphs provide the models for the internet (World Wide Web) or for spreading diseases, and many other "relations".



The roots of graph theory go back to Euler. In the beginning of the 20th century many new questions were asked in graph theory and therefore many of them were easily solvable, without using too much mathematics. At the same time, several very deep questions were also asked and the researchers trying to solve them had to develop new methods, and in this way graph theory became a very deep part of modern mathematics, connected to several other important areas of mathematics, and is still fast developing. Whilst in the middle of the last century, there were several prejudices against combinatorics, today it has proved to be an important part of mathematics, connected to several other parts, and is far from being too elementary or too easy.

One source of this fast development came from computer science. Very soon we realised that we need fast computers and that the speed of computers depends on two things: technical improvement (better engineering) and theoretical improvements (finding better and faster algorithms). In this way combinatorics and graph theory come in: Alan Turing created theoretical models of computers (even before computers were around) and in the 1960 and 1970s a new area emerged: theoretical computer science, where algorithms running on very huge objects were analysed. (Often, analysing infinite things is easier than finite/small ones.) In all these problems the effect of Erdős' mathematics can easily be found.

Extremal graph theory is one of the central areas in Paul Erdős' graph theory. In 1907, Mantel proved that if a graph does not contain a triangle then it has at most $\lfloor \frac{1}{4}n^2 \rfloor$ edges. His result was soon forgotten. The next step when mathematicians returned to such problems should have been a paper of Erdős [6] where Erdős posed and solved two problems in combinatorial number theory, of which for us the following is the more interesting one:

Problem 1. Given m integers $a_1, \ldots, a_m \in [1, n]$ with the property that all the pairwise products are distinct (more precisely, $a_i a_j = a_k a_\ell$ occurs only if $\{i, j\} = \{k, \ell\}$), what is the maximum of m under this condition (as a function of n)?

A trivial construction would be to consider the primes in $[1, n]$, and the corresponding result of Erdős says that asymptotically this is the best. To get a good error term, Erdős considered – and in some sense solved – the following.[3]

Problem 2. Given a bipartite graph $G[A, B]$ with $|A|, |B| \leq n$, how many edges can such a graph have if it contains no C_4?

Erdős gave an upper bound on this number $\mathbf{ex}(n, n, C_4)$, proving that $\mathbf{ex}(n, n, C_4) < 3n\sqrt{n}$. His paper also contained a finite geometric construction of Eszter Klein, showing that this estimate is sharp, apart from the value of the multiplicative constant 3. This was perhaps the first "deeper" extremal graph theorem. The general extremal graph theorems answer problems of the following type:

Problem 3. Given a family L of graphs and a large graph G_n of n vertices. Denote by $\mathbf{ex}(n, L)$ the maximum number of edges G_n can have without containing subgraphs from L. Determine or estimate $\mathbf{ex}(n, L)$.

Erdős "forgot" to ask this question. It was Turán who asked this 2-3 years later. This is why we call these problems Turán type extremal problems. One of the widest, deepest subareas of discrete mathematics evolved around this type of question. Not so long ago I finished with Zoltán Füredi a long survey of approximately 90 pages [16] on this field in the subcase when L is bipartite, publishing it in the Erdős Centennial Volume [5].

Turán wrote just a few papers in this area. He liked this area very much but his main field of interest was still number theory. His mathematical career was very similar to that of Erdős, yet different in many aspects. Perhaps one could say that Turán was a theory builder while Erdős was a problem solver. Tim Gowers discusses the differences of these two types nicely on his homepage [17].

A year earlier we had another very successful international conference on the occasion of Turán's centennial[4] and

3 We shall use the notion of bipartite graphs. These are graphs where there are two types of vertices, say Red and Blue, and all the edges connect vertices of different kinds. Some examples are: People and Jobs, where a Person is connected to a Job if the person can do that Job, Boys and Girls, say dancing, or Points and Curves in the plane, where incident ones are joined, ...

4 Actually the centennial was two years earlier.

I wrote another equally long survey on his influence on mathematics, primarily on his influence on extremal graph theory (and, more generally, on discrete mathematics) [20]. And still these two surveys (though with large overlap) describe only a small part of their mathematics.

Without going into detail, we just formulate a general theorem and make a few remarks.

Theorem 3 (Erdős-Simonovits stability)**.** Let $\mathcal{L}$ be a family of forbidden subgraphs. Let $p + 1$ be the minimum chromatic number[5] of graphs $L \in \mathcal{L}$. Then

$$\mathbf{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$$

and if S_n is an extremal graph for $\mathcal{L}$, meaning that it is a graph on n vertices, not containing any $L \in \mathcal{L}$ and having the most number of edges under this condition, then S_n can be transformed into a p-chromatic graph by deleting $o(n^2)$ edges.

Moreover, if (G_n) is an almost extremal graph sequence for $\mathcal{L}$ in the sense that G_n contains no $L \in \mathcal{L}$ and its number of edges

$$e(G_n) > \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2)$$

then G_n can also be transformed into a p-chromatic graph by deleting $o(n^2)$ edges.

This stability theorem helps us to determine exact extremal graphs in several cases.

Ramsey Theory

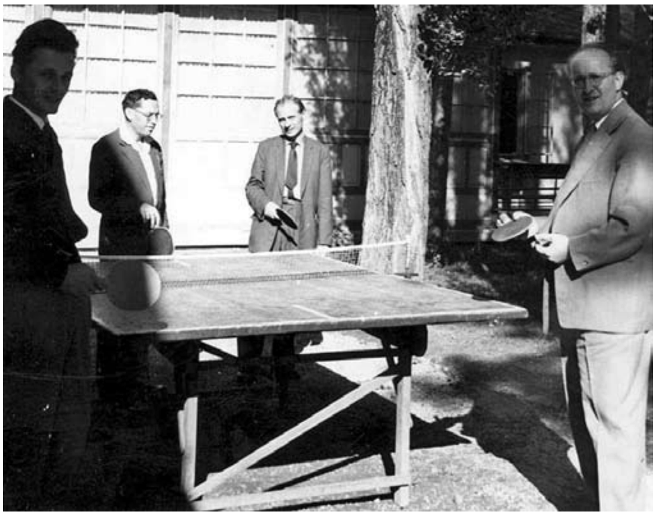
There were a few young university students in those days, strongly interacting mathematically with each other. Eszter Klein asked the following question:

Problem 4. Given an integer k, how many points in the plane guarantee a subset defining a convex k-gon?

Erdős and Szekeres solved this problem in two different ways, reinventing the Ramsey theorem (proved by Ramsey a little earlier). Turán's theorem was motivated by the Ramsey theorem and, at a given point, Turán had an incorrect conjecture that if G_n is a graph on n vertices then for $k = \lfloor \sqrt{n} \rfloor$ either it contains a complete k-graph (i.e. k vertices, any two of which are joined) or an empty k-graph (a set of k vertices without edges joining them, also called an *independent* set of k vertices).

In his first letter after the war Turán asked Erdős about this. Erdős immediately answered that this is far from being true: in most of the graphs G_n on n vertices the largest complete and the largest independent set is smaller than $2 \log_2 n$. In some sense this easy proof of Erdős is counted as the birth of the *theory of random graphs*.[6]

The Erdős-Szekeres version of Ramsey's theorem (a simple double induction) asserted that if $n > \binom{k+\ell-2}{k-1}$ then in any G_n either we have a complete k-graph or an empty ℓ-graph.



Knapowski, Erdős, Szekeres and Turán playing table-tennis

That implies that any graph on n vertices contains either a complete or an empty graph on $\approx \frac{1}{2} \log_2 n$ vertices.

At first random graphs were perhaps used only to replace constructions, e.g. to obtain graphs with high girth (i.e. without short cycles) and large chromatic number, or in the beautiful theorem of Erdős and Pósa on representing all the cycles in a graph. However, at the end of the 1960s, the description of typical structures became a central point in the papers of Erdős and Rényi and soon, in [10], they gave a systematic investigation of random graph phenomena, giving birth to an area which is today one of the most important parts of discrete mathematics.

Remark 1 (Important tools)**.** I close this part by remarking that several important tools were created to solve Erdős problems. Among others, the Lovász Local Lemma was proved to solve a problem of Erdős and the Szemerédi Regularity Lemma was created, first in a weaker form to prove the Erdős-Turán conjecture on arithmetic progressions and then the modern form of the Regularity Lemma to prove a Bollobás-Erdős-Simonovits conjecture connected to the Erdős-Stone theorem. Furthermore, quasirandom graphs were created – in an earlier form, see α-jumbled graphs – by A. Thomason to answer a Ramsey question of Erdős.

6 Other fields

Until now I have mostly restricted myself to the most visible parts of Erdős' influence. However, he had important results, for example, in connection with interpolation theory, with Turán, with Vértesi and Szabados, and others, on polynomials and, again with Turán, on statistical group theory.

And, of course, he, Rado and Hajnal and others built up a large part of "infinite combinatorics"; see, for example, [9].

Erdős and combinatorial geometry

The questions Erdős asked in geometry were occasionally very surprising for those educated on classical geometry. Here I will mention only two of them. Perhaps one of the most interesting questions of Erdős was:

5 The chromatic number k of a graph G is the minimum number of classes into which one has to partition the vertices of G so that no two vertices of the same class are joined to each other.

6 To be precise, there were other results that could be counted as starting points of the theory of random structures, the most important of which may be Shannon's random codewords.

Problem 5. Given n points in the plane, how many unit distances can occur among them?

He conjectured that this number is just superlinear but not much more: if one considers a square grid of $n = k \times k$ points in the plane then each distance can occur at most $n^{1+o(1)}$ times, and Erdős conjectured that this is basically the best configuration. Using that $\mathbf{ex}(n, K(2, 3)) = O(n^{3/2})$,[7] and that if we join points in the plane if and only if their distance is a given, fixed d, that graph contains no $K(2, 3)$, Erdős concluded that one can have at most $O(n^{3/2})$ unit distances among n points of the plane. Later, this was pushed down to $O(n^{4/3})$ but nothing better is known. (For two-dimensional normed spaces $cn^{4/3}$ is sharp, so one has to use some specific property of the Euclidean plane to improve the upper bound.)

A breakthrough result on a related question of Erdős is

Theorem 4 (Guth and Katz [18])**.** There exists a constant $c > 0$ such that a set of n points determines at least $\frac{cn}{\log n}$ distinct distances.

7 Erdős and his friends

Erdős had a "strange" habit of beginning right in the middle. Among other things, this meant that often he wrote a postcard to ask a mathematical question and, instead of first asking some personal questions, he started with: "Let n be an integer ..." or "Let G be a graph ...". Some people would deduce that he did not care for others but just the opposite is true. He really cared for his friends and really tried to help them in many different ways: visiting them, supporting them financially when that was needed and, of course, supporting many, many people to do good mathematics. Vera Sós in a paper entitled "Turbulent Years" [21], "analyses" the correspondence between Erdős and one of his best friends Paul Turán during the war. Here one can read how Erdős tried to help his friends, whose lives were often in danger, during these extremely difficult years.

8 The legends

Most of the legends about Erdős are literally true yet misleading. It is true that he called music "noise", yet he very much liked Bach, Beethoven and many others. He often visited us in Budapest and after a while he "demanded" that I should put on some music. He called women "bosses" and men "slaves"; however, this had "no significance" on his behaviour. He was very kind to people around him and spoke the same kind way to men and women. Actually, he made friends very easily.

Some sources assert that he did not care about his body. Maybe not, yet he liked to go on excursions into the mountains, wherever he went. He liked playing table tennis (and in his younger years even tennis).

The fact that he often started his postcards with, say, "let n be an integer ..." does not imply that he was interested only in mathematics. Actually, he was very much interested in history, medical sciences and political life and, above all, he cared for the people in his surroundings.

9 Death

Erdős died on 26 September 1996 in Warsaw. He died, very quickly, of a heart attack (heart attacks were not too uncommon in his family). The important thing is that he actually died almost as he "wanted". His version would have been finishing his lecture, putting down the chalk and "dropping dead".

Instead, he gave a lecture at the Banach Center in Warsaw and the next day had a nice evening at a restaurant and, following this, in the middle of the night, he called the hotel reception saying that the ambulance should be called; he was taken to a hospital and very soon lost consciousness. Next afternoon he was dead.

But his mathematics and our nice and grateful memories about him stay with us.

10 Reading about Erdős?

If one wishes to learn about the mathematics of Erdős, perhaps the paper of Paul Turán, written on Erdős' 50th birthday [22] is the best source. This has been used by several of us (once even by Erdős for [8]) but I could recommend several further papers, e.g. Bollobás [14], [13], Lovász and T. Sós [19] just about to appear in the Notices,[8] and several "preface" papers to volumes connected to his work, like [1], ... His papers are also scanned in [25] and are available until 1989[9] and those are excellent sources to learn about his mathematics. There was a tradition in Hungary according to which, if a central figure of Hungarian mathematics died then his works were published in the form of a "Collected works". This happened, for example, to Lipot Fejér, Frigyes Riesz, Paul Turán and Alfréd Rényi, and this would have happened to the works of Erdős if he had "only" 300–400 papers. Since his work would take approximately 15,000 pages, we have decided to scan them in. First we thought of publishing that in the form of a DVD. However, we finally switched to posting them at an "Erdős homepage" [25] and, certainly, one can learn a lot of Erdős just browsing this homepage. There are also several homepages dedicated to Paul's memory. The books written about him I do not really like, since they often over-emphasise the unimportant features of Erdős, those that an average person would count "strange". Even the fairly informative article on Wikipedia calls him eccentric, which I would feel misleading.

We can learn about his life from several excellent papers written on him. An excellent article of Babai [12] could be one of the best sources: this article is based on the "interviews" of Babai with Erdős. One of the best sources on his mathematics was written by Turán [22]. We can also learn a lot about his personality from the papers he wrote, for example, in memory of his friends, e.g., the one on Paul Turán [7].

7 $K(a, b)$ denotes the complete bipartite graph on a and b vertices: we have a vertices in the first class, b vertices in the second class, and join two vertices if and only if they belong to different classes.

8 Actually the whole upcoming volume of the Notices on Erdős, 2014.

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http://theory.cs.uchicago.edu/erdos.html.

[25] Erdős' homepage at Rényi Institute: www.renyi.hu/~p_erdos.

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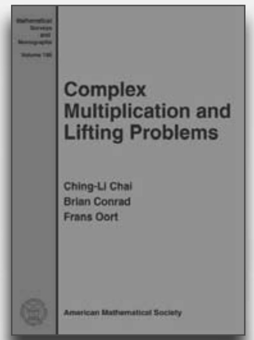

COMPLEX MULTIPLICATION AND LIFTING PROBLEMS

Ching-Li Chai, *University of Pennsylvania*, Brian Conrad, *Stanford University* & Frans Oort, *University of Utrecht*

Abelian varieties with complex multiplication lie at the origins of class field theory, and they play a central role in the contemporary theory of Shimura varieties. They are special in characteristic 0 and ubiquitous over finite fields. This book explores the relationship between such abelian varieties over finite fields and over arithmetically interesting fields of characteristic 0 via the study of several natural CM lifting problems which had previously been solved only in special cases.

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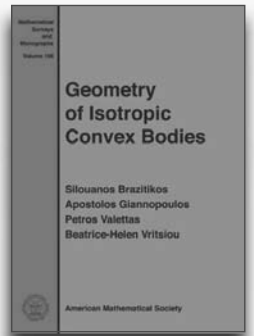

GEOMETRY OF ISOTROPIC CONVEX BODIES

Silouanos Brazitikos, *University of Athens* et al

Introduces a number of well-known questions regarding the distribution of volume in high-dimensional convex bodies, which are exactly of this nature: among them are the slicing problem, the thin shell conjecture and the Kannan-Lovász-Simonovits conjecture. This book provides a self-contained and up to date account of the progress that has been made in the last fifteen years.

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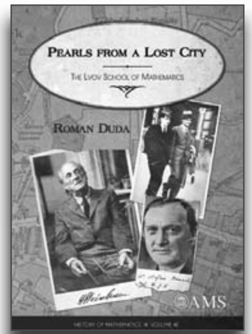
PEARLS FROM A LOST CITY

The Lvov School of Mathematics

Roman Duda, *University of Wrocław*



The fame of the Polish school at Lvov rests with the diverse and fundamental contributions of Polish mathematicians working there during the interwar years. In particular, despite material hardship and without a notable mathematical tradition, the school made major contributions to what is now called functional analysis. This chronicle of the Lvov school - its legacy and the tumultuous historical events which defined its lifespan - will appeal equally to mathematicians, historians, or general readers seeking a cultural and institutional overview of key aspects of twentieth-century Polish mathematics not described anywhere else in the extant English-language literature.

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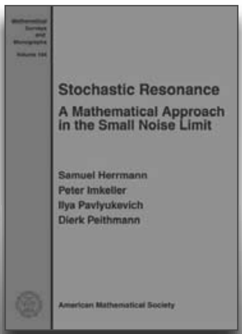
STOCHASTIC RESONANCE

A Mathematical Approach in the Small Noise Limit

Samuel Herrmann, *Université de Bourgogne* et al



Stochastic resonance is a phenomenon arising in a wide spectrum of areas in the sciences ranging from physics through neuroscience to chemistry and biology. This book presents a mathematical approach to stochastic resonance which is based on a large deviations principle (LDP) for randomly perturbed dynamical systems with a weak inhomogeneity given by an exogenous periodicity of small frequency. Resonance, the optimal tuning between period length and noise amplitude, is explained by optimising the LDP's rate function.

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The Legacy of Vladimir Andreevich Steklov in Mathematical Physics: Work and School

Nikolay Kuznetsov (Russian Academy of Sciences, St. Petersburg, Russia)

The 150th anniversary of the birth of the outstanding Russian mathematician Vladimir Andreevich Steklov falls on 9 January 2014. All over the world, active researchers in all areas of mathematics know his name. Indeed, well-known mathematical institutes of the Russian Academy of Sciences in Moscow and St. Petersburg are named after Steklov. This commemorates that he was the founding father of their predecessor: the Physical-Mathematical Institute, established in 1921 in starving Petrograd (the Civil War still persisted in some corners of what would become the USSR the next year). Steklov was the first director of the institute, until his untimely death on 30 May 1926.

Meanwhile, Steklov's personality is less known even in present-day Russia. Of course, the biographical sketch [39] by J. J. O'Connor and E. F. Robertson is available online but still the best source of information about Steklov and his work is the very rare book [16]: the proceedings of a session of the Leningrad Physical-Mathematical Society, which took place on the occasion of the first anniversary of Steklov's death. It must be said that the lack of knowledge about his work was the reason for translating into Russian and publishing a collection of Steklov's papers concerning various problems in mechanics [62]. Not too much has been written about his relationship with a group of bright students (most of them graduating from St. Petersburg University in 1910). Together with students of Andrey Andreevich Markov Sr. and Nikolay Maksimovich Günther, they formed the germ which would later develop into the Petrograd–Leningrad–Petersburg school, famous for contributions of its scholars to mathematical physics, functional analysis and some other areas of mathematics as well as theoretical physics.

To clarify the word "school", which has various meanings in Russian as well as in English, it is worth quoting A. N. Parshin's recent note [40].

> A school is a community of individuals who work in the same branch of science, who are in close communication with each other, who have a leader, a teacher, amongst whom each generation passes on the torch to the next one, and all this forms one integral organism.

After this "definition", Parshin describes the branching at the Mekh-Mat (the Faculty of Mechanics and Mathematics) of the Moscow University from the original school founded by N. N. Luzin.

Speaking about Steklov's school, founded a little bit earlier than Luzin's, I understand it in the same sense as the latter is treated by Parshin. Of course, after a lapse of 100 years, the school of which Steklov was the founding father has given rise to various schools in the widely understood field of mathematical physics. It is worth mentioning that his school very soon became international. Indeed, J. D. Tamarkin and J. A. Shohat (they were members of Steklov's circle of students) emigrated in the 1920s to the USA, where they had many PhD students. In particular, Tamarkin had 28 students and so his 1495 descendants (the Mathematics Genealogy Project as of 22 August 2013) are Steklov's "scientific" grandchildren, great-grandchildren and so on. At the same time, the project gives only 888 as his own descendants.

Of course, Steklov's interests in science were much wider than mathematical physics; for example, the above mentioned collection [62] includes 12 of his major contributions to mechanics (440 pages in total). An idea of development of Steklov's work in mechanics during the past decade can be obtained from the papers [5], [66] and references cited therein. Another area which he studied during 26 years is the theory of orthogonal polynomials. In the reference book [46], one finds 31 papers by Steklov (the first two published in 1900 and the last two dated 1926) and the properties of polynomials investigated in each of them are clearly indicated. Fortunately, this topic of Steklov's research is covered in the survey article [63]; further progress can be found in [43] and [44].

This paper consists of three sections. In the first one, I briefly describe the non-scientific legacy of Steklov and then turn to quoting his own writings taken from various sources. These excerpts describe his personality and the way he started creating his school. The latter involves his relations with J. D. Tamarkin, A. A. Friedmann, V. I. Smirnov, J. A. Shohat and his other talented students shown through the prism of

Steklov's diaries and recollections. Then his activity as Vice-President of the Russian Academy of Sciences during the last seven years of his life is presented in the same way. Steklov's role was crucial for the academy's survival during the period of revolutions and the Civil War in Russia. He exemplified how to withstand governmental attacks on the academy, something that is particularly important nowadays.

Since Steklov's major achievements in mathematical physics have been summarised in his book [57] entitled *Fundamental Problems of Mathematical Physics*, its contents and significance are discussed in the second section, whereas advances in the area of the potential-theoretic approach to boundary value problems for the Laplace equation (the topic of the book's volume 2) are described in the brief third section.

It should be mentioned that many of Steklov's papers are now available online as well as a complete list of his publications. For the latter see http://steklov150.mi.ras.ru/steklov_pub.pdf. The journals *Annales fac. sci. Toulouse* and *Annales sci. ENS*, in which many of Steklov's articles are published, are available at http://www.numdam.org. Almost 20 of his papers in French and some papers in Russian can be found at http://www.mathnet.ru/php/person.phtml?option_lang=rus&personid=27728.

V. A. Steklov about himself, his students, science and the Academy of Science

The legacy of Steklov is multifaceted; along with his work in mathematics and mechanics (see [68] for a survey), it includes scientific biographies of Lomonosov and Galileo, an essay about the role of mathematics (these three books in Russian were printed in 1923 in Berlin because Russian economics was ruined during World War I and the Civil War), the travelogue of his trip to Canada, where he participated in the Toronto ICM in 1924, his correspondence – published (see [60] and [61]) and unpublished – recollections [59] and still unpublished diaries.

Fortunately, many excerpts from Steklov's diaries are quoted in [65] (some of them appeared in [39] as well). In my opinion, the most expressive is dated 2 September 1914, one month after war with Germany and Austria-Hungary was declared by the Russian government.

> St. Petersburg has been renamed Petrograd by Imperial Order. Such trifles are all our tyrants can do – religious processions and extermination of the Russian people by all possible means. Bastards! Well, just you wait. They will get it hot one day!

What happened in Russia over several years after that confirms clearly how right Steklov was in his assessment of the Tsarist regime. In his recollections [59] written in 1923, he describes vividly and, at the same time, critically "the complete bacchanalia of power" preceding the collapse of "autocracy and [Romanov's] dynasty" in February 1917 (old style), "the shameful transient government headed by Kerensky, the fast end of which can be predicted by every sane person", how "the Bolshevik government [...] decided to accomplish the most Utopian socialistic ideas in multi-million Russia"; the list can easily be continued. My aim is to give quotations from [59] and from the unpublished manuscript *Excerpts from my diaries* (*Excerpts* in what follows) widely quoted in [22], that characterise Steklov's personality, his relations with a group of talented students at the St. Petersburg University, his understanding of the role of science for himself and his work as Vice-President of the Academy of Sciences. Translation from the Russian is mine, if not stated otherwise.

Years of education

In the *Excerpts,* Steklov writes about his final years at Alexander Institute (a kind of gymnasium) in Nizhni Novgorod.

> I turned my room into a kind of "physical cabinet" – laboratory equipped with Leyden jars, an electrical machine and home-made Galvanic elements. Various chemical experiments (of course, elementary) were carried out. [...] I reduced my contacts with schoolmates (previously numerous), continuing to keep in touch only with those of them who, like me, were interested in mathematics and physics.
>
> [...]
>
> The topic of "test composition" [preceding the certificate exam] was as follows: *The reign of Catherine II was a great period.* In a [satirical] poem [by A. K. Tolstoy published not long before that], there are several lines characterising her in a way far from being respectful to "Her Majesty". However, they added a specific colouring to my essay. [...] I wrote, without any idea to manifest political freethinking, that Catherine's period only looks great but, in fact, it puts an end to reforms initiated by Peter I. [...] To my great surprise, Shaposhnikov (Director of the Institute) came to the classroom after reading our essays [...] and asked me: "Where have you, our best student, got this inclination to freethinking[1] and such an impermissible attitude toward the Great Empress?" [...] For almost an hour, he was explaining my thoughtlessness and my incorrect understanding of history, etc. [...] After that, he dragged in by the head and shoulders the following point of view: preferring mathematics, physics and chemistry to other disciplines, I follow an objectionable way. He said: "Maybe this is the reason that you 'took those liberties in thinking'. This feature of yours has been noted long before but definitely revealed itself in your essay."
>
> I repeat that it was a surprise for me, but did not become a stimulus to change my mind. [...] Just the opposite, I said to myself: "Aha! It occurs that I have my own point of view on historical events and it is different from that of my schoolmates and teachers. [...] It was the director himself who proved that I am, in some sense, a self-maintained thinker and critic." This was the initial impact that led to my mental awakening; I realised that I am a human being able to reason and what is important is to reason freely. [...] Soon after that, my freethinking encompassed religion as well. [...] Thus, the cornerstone was laid to my future complete lack of faith.

In another passage from the *Excerpts*, Steklov describes how he failed to pass an examination at Moscow University.

> The last oral exam was in physical geography taught by the stern professor Stoletov. Rather quickly, I have managed to study this easy discipline within the lecture course. My reply to the questions formulated on the card was excellent. Suddenly, Stoletov asks: "What date is the longest day in Moscow?" I was completely taken aback by this question. My silence lasted several minutes. Stoletov was glassy staring at me and, at last, he said sluggishly: "Complete ignorance". He writes *unsatisfactory* in my record-book, and I am ruined because my marks for all other difficult exams were excellent. [...] It seemed that committing

suicide was the best decision to suppress the feelings tearing my soul apart at that moment. However, this idea came to my mind only afterwards when I had already calmed down.

About science

The thought about committing suicide came to Steklov once more, when he was a second-year student in Kharkov. It was caused by his rather complicated love affairs. In his recollections [59], he writes in connection to this.

> Soon I came to the conclusion that any reasoning as to whether it is worth living or not is an inadmissible stupidity and moral cowardice. It is worth living for the sake of pursuit of knowledge and even my experience – rather small at that time – had already demonstrated that all other kinds of activity occupying people are deceptive and temporal. Research is the only kind of activity that occupies you forever and never deceives a person who wants and is ready to devote himself/herself to it. Soon, I immersed myself into studies once and for all. Moreover, the young professor Alexander Mikhailovich Lyapunov (my fellow countryman who, afterwards, became an outstanding mathematician) joined the faculty shortly after that. He was my teacher and only friend; his guidance of my first steps in science is unforgettable.

About students[2]

It is an amazing and lucky coincidence that the same year (1906) as Steklov got his professorship at the St. Petersburg University a group of very gifted students entered it to study mathematics. In the file of M. F. Petelin (he was one of them), this fact was noted by Steklov as follows.

> I should note that the class of 1910 is exceptional. In the class of 1911 and among the fourth-year students who are about to graduate there is no one equal in knowledge and abilities to Messrs. Tamarkin, Friedmann, Bulygin, Petelin, Smirnov, Shohat and others. There was no such case during the fifteen years of teaching at the Kharkov University either. This favorable situation should be used for the benefit of the University.

This quotation as well as further ones concerning Steklov and his students show how attentive to them was he. Their future fate was very different; two of them (Bulygin and Petelin), unfortunately, died young.

A. A. Friedmann became famous for his discovery in general relativity; his solution of Einstein's equations was the first one that describes the expanding Universe (see [14] and [15]). However, he died aged 37, just seven months after his appointment as Director of the Main Geophysical Observatory in Leningrad and two months after his flight to the record altitude of 7,400 metres. V. I. Smirnov (a corresponding member of the Academy of Sciences of the USSR since 1932 and a full academician since 1943) is known for his results in complex analysis and mathematical physics. He is the author of *A Course in Higher Mathematics* (the first two of its five volumes were written in collaboration with Tamarkin but revised for later editions). From 1922 until his death in 1974, Smirnov's activity was associated with Leningrad University, where he founded the Research Institute for Mathematics and Mechanics in 1931 and afterwards headed several departments at the Faculty of Mathematics and Mechanics (Mat.–Mekh.). In the 1950s and 1960s, the Leningrad School of Mathematical Physics founded by Steklov flourished under the direction of Smirnov. His effort in restoring the Leningrad Mathematical Society in 1959 was also crucial. (The existence of its predecessor – the Physical-Mathematical Society – lasted from 1921 until 1930, when it disbanded due to political pressure; see [64].)

J. D. Tamarkin and J. A. Shohat emigrated to the USA in 1925 and 1923, respectively. They were active in research in various areas of analysis (the book [47] is their most cited work) and in supervising PhD students (G. Forsythe – one of Tamarkin's students – was afterwards a prolific PhD adviser himself). In 1927, Tamarkin was called to Brown University, whereas Shohat was at the University of Pennsylvania from 1930. Tamarkin was also involved in editing various journals; in particular, he was one of the editors of the Mathematical Reviews when it started in 1940. As a member of the Organising Committee for the 1940 ICM, Tamarkin was very efficient. (Unfortunately, the congress was postponed because of World War II and took place after Tamarkin's death.) He was also an influential member of the AMS Council from 1931 and Vice-President of the Society in 1942–1943.

In his recollections [59], Steklov also mentions A. S. Besikovitch who graduated in 1912 and was appointed to a professorship five years later at the newly opened Perm University (it was Steklov who recommended him). Besikovitch became famous for his contribution to the theory of almost periodic functions and for his results that form the cornerstones of geometric measure theory. He emigrated from the USSR in 1925 and, after staying one year in Copenhagen with Harald Bohr, moved to the UK. There, he became a university lecturer in Cambridge in 1927 and the Rouse Ball Chair of Mathematics in 1950. He had been elected F.R.S. in 1934 and received several academic awards.

In Steklov's diaries, the first mention of students is, chronologically, in the entry for 13 January 1908. What follows is a set of most important entries.

> 13 January 1908. At 4 o'clock Tamarkin and Friedmann (undergraduate students) turned up and brought the continuation of the lectures in integral calculus they had written. They took the ones I had corrected (i.e. looked through. No possibility of correcting them properly!). They said they would come to my lecture on the 16th. They asked me if it was possible to legalise the mathematical society without a supervisor. I told them to make some suggestions. Let us see!
>
> 20 February. Brought my collected works to the University and gave them to Tamarkin for the students' mathematical society. Three memoirs are missing.
>
> 21 October. Tamarkin and Friedmann came to see me this evening. They are going to organise a mathematics reading room. Asked me to be their supervisor. Declined, but they deserve help.
>
> 22 November. Tamarkin and Friedmann came to see me this evening [...] Kept asking me about their delvings into the theory of orthogonal functions. They are having an article published in Crelle's journal. Sharp fellows! They left at half past twelve, after supper.
>
> 18 April 1909. The students Tamarkin, Friedmann, Petelin came to see me this evening [...] I proposed to Tamarkin that he think about the asymptotic solution of differential equations (i.e. stability, in the sense of Poincaré and Lyapunov) or the problem of equilibrium of a rectangular plate. To Friedmann I suggested he find orthogonal substitutions, when fundamental functions are

products of two (see my dissertation). I suggested Petelin read what Jacobi had to say about the principle of the last multiplier. I'll think it over again and will probably find some other topics too.

12 September. This evening Tamarkin, Friedmann and Petelin came to see me. They had worked on the assigned topics. Seem to have done something. Promised to submit their essays in a month. Tamarkin seems to be doing better than the others.

Steklov coauthored only two papers and one of them was a joint paper with Tamarkin. It was published in *Rend. Circ. Mat. Palermo* in 1911 and so was written when Tamarkin was still a student.

About administration work in the Academy of Sciences

In Steklov's recollections [59], his comments on this topic are rather brief but they show that he clearly understood his role in the survival of the academy as the leading scientific institution.

In 1919, I was unanimously elected to the post of Vice-President of the Russian Academy of Sciences.[3] At the same time, the [Petrograd] University [. . .] insisted that I have to head it, but this burden was decidedly rejected by me. Indeed, the state-of-affairs existing at that time would not allow any human to do both jobs properly. [. . .] It was absolutely clear to me that I could really do a lot for the benefit of the Academy. [. . .] On the other hand, I saw that the university was on the brink of collapse at that time.

[. . .]

First, it must be said that the Academy is still one of a few institutions that were successfully vindicated from various destructive attacks. Moreover, its reputation was growing gradually in the eyes of ruling circles and now the Academy is recognised as the leading scientific institution. At last (in September 1923), I have achieved a success in the matter that I tried to accomplish for a long time, namely that the Academy must be considered on equal terms with Narkompros [the Ministry of Education]. [. . .] I can say with satisfaction, without boasting, that my contribution to achieving all these results favorable for the Academy is very considerable.

About Steklov's daily schedule

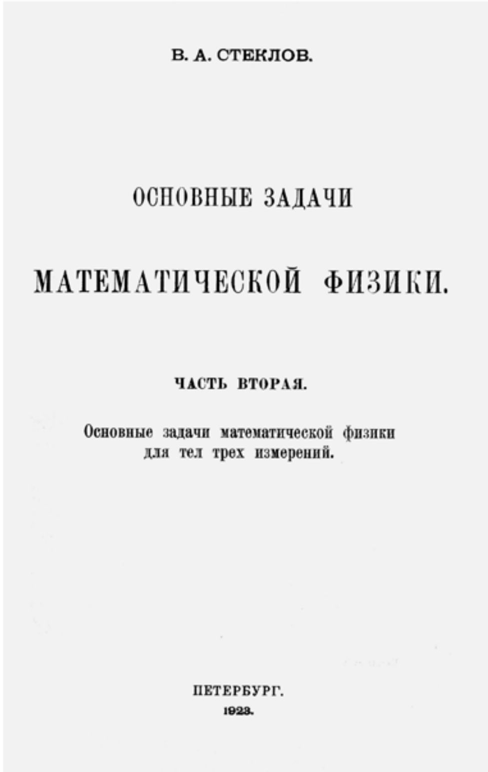
At the end of his recollections [59], one finds the following.

In my opinion, it is exclusively due to a particular daily schedule that I manage to separate administration and research so that both of them flow parallel not interrupting each other. I adopted this schedule during my student days.

My day is divided into two parts as follows. The time from 10 am to 5 or 6 pm I devote to administration at the Academy. Then I dine and about 7 pm go to bed. I sleep until 9:30 pm (sometimes until 10 pm). After awakening, I have a cup of tea and then, leaving apart all thoughts about administration and having nothing revulsive, calmly do my research.

I work until 4 or 5 am in the morning (sometimes longer). [. . .] Three hours of sleep after dinner allow me to sleep from 4 or 5 am to 9:30 am, that is, 5 and sometimes 6 hours. This has been my daily schedule for more than 40 years and I find it expedient to a great extent.

Of course, it is difficult to stop quoting Steklov's recollections and diaries but enough is enough.


В. А. СТЕКЛОВ.

ОСНОВНЫЕ ЗАДАЧИ

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ЧАСТЬ ВТОРАЯ.

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для тел трех измерений.

ПЕТЕРБУРГ.
1923.

The title page of the Steklov's monograph [57], Vol. 2.

Steklov's unfinished monograph *Fundamental Problems of Mathematical Physics*

Mathematical achievements of the first half of the 20th century are described in the book [41]. Its first section entitled "Guidelines 1900–1950" is compiled by P. Dugac, B. Eckmann, J. Mawhin and J.-P. Pier with the assistance of an international team of almost six dozen prominent mathematicians (V. I. Arnold and S. S. Demidov represent the Moscow school). "Guidelines" is a year by year list of major results and their authors; the most important books published during the period from 1900 to 1950 are also presented in this 34-page list. It includes two items concerning mathematical physics published in 1923: *Lectures on Cauchy's problem in linear partial differential equations* by J. Hadamard and the two-volume book [57] by Steklov. Its second edition [58] appeared 60 years later with a vast number of comments and some necessary corrections made by V. P. Mikhailov and A. K. Gushchin (both from the Steklov Mathematical Institute, Moscow).

Here, my first aim is to explain why the treatise [57] is among the most valuable contributions to mathematical literature of the first half of the 20th century despite the fact that it was not finished by Steklov (see below). Secondly, further advances will be described in the area of applications of potential theory to boundary value problems for the Laplace equation, which is the topic of the second volume of [57].

Prior to that, it is worth mentioning several other contributions to "Guidelines" which came from Steklov's mathematical school. What follows are corrected excerpts from the list in [41] supplied with citations of the corresponding original papers:

1932 A. S. Besikovitch, *Almost Periodic Functions.* Cambridge University Press.
1936 S. G. Mikhlin, Symbol of a singular integral operator; [35], [36], [37] (a comprehensive updated presentation can be found in the monograph [38]).
1936 S. L. Sobolev, First results on distributions; [49].
1937 N. M. Krylov, N. N. Bogolyubov, Averaging method for nonlinear differential systems; [31].
1938 S. L. Sobolev, Sobolev spaces; mollifiers; [50].
1950 S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics.* (In Russian; English translation was published by the AMS in 1964.) This book presents results obtained in [49] and [50] in a comprehensive form.

Notice that the notion of mollifier proposed by Sobolev is a far-ranging generalisation of the *Steklov mean function*, which is the most simple averaging operator (see [1], § 74, for the definition and properties). It was introduced by Steklov in 1907 for studying the problem of expanding a given function into a series of eigenfunctions defined by a 2nd-order ordinary differential operator; see [54] and [55] for the announcement and full-length paper, respectively. Of course, "Guidelines" contains many other entries due to mathematicians from Russia, beginning with Steklov's teacher A. M. Lyapunov (1901 – Central limit theorem), and ending with several entries for 1950, one of which is M. G. Krein's "Parametric resonance in higher dimensional Hamiltonian systems". The overall best number of entries is 14, unsurprisingly, by A. N. Kolmogorov.

Let us turn back to the monograph [57]; it is based on lectures given to a small group of well prepared audiences in 1918–1920. This is why this book is written in Russian despite the fact that the underlying papers were written in French. Its 1st volume was finished in April 1919 and the 2nd volume was finished in November 1922. More material was presented in the lecture course than appeared in [57]; Steklov planned to publish the 3rd volume with his results concerning "fundamental" functions (that is, eigenfunctions of various spectral problems for the Laplacian) and some applications of these functions. He describes this objective on p. 257 of the 2nd volume and also mentions it in [59], p. 299. Unfortunately, his administrative duties as Vice-President of the Academy prevented him from realising this project. However, one gets an idea about the probable contents of the unpublished 3rd volume from the lengthy article [53]. In this paper, which appeared in 1904, Steklov developed his approach to "fundamental" functions (see [26], § 5, by A. Kneser for its brief outline). This approach uses two different kinds of Green's function and this allows one to apply the theory of integral equations worked out by I. Fredholm [13] and D. Hilbert [21] shortly before that.

In 1923 – the year when [57] was published – Russian scientists were still cut off from their colleagues in the West after the October Revolution. It is worth emphasising the great efforts of Steklov and his fellow academicians Abram Fedorovich Ioffe (he founded the Physical–Technical Institute in Petrograd in 1918), Alexei Nikolaevich Krylov (naval engineer and applied mathematician known for his work [30] winning a Gold Medal from the Royal Institution of Naval Architects) and Sergey Fedorovich Oldenburg (the Permanent Secretary of the Russian Academy of Sciences) directed towards restoring contact with colleagues abroad as well as to set up exchanges through scientific publications. Anyway, at that time no attempt was made to translate [57] into French, German or English. However, 11 years later, N. M. Günther (presumably he attended Steklov's lectures) gave an account of potential theory and its application to the Dirichlet and Neumann problems following the approach proposed by Steklov. First, Günther's book [19] was published in French, then its Russian revised and augmented edition appeared in 1953 and, finally, the English translation [20] of the latter was issued in 1967.

In his book, Steklov considers boundary value problems as mathematical models of physical phenomena and so two essential requirements must be fulfilled for a solution of any such problem: the existence and uniqueness theorems. This is the first important point of vol. 1. Notice that the notion of a well-posed problem was introduced simultaneously by Hadamard in his book mentioned above; he complemented these two requirements with the following one: a solution must depend continuously on the problem's data.

The second important point of vol. 1 is the systematic rigorous justification of the Fourier method for initial-boundary value problems for parabolic and hyperbolic equations with variable coefficients not depending on the time and depending on a single spatial variable. For this purpose the following stages must be accomplished.

- The existence of an infinite sequence of eigenvalues and eigenfunctions must be proved.
- It must be shown that the set of eigenfunctions is "rich enough" for expanding every "sufficiently smooth" function into a Fourier series.
- One has to prove that the obtained Fourier series gives a solution of the problem under consideration.

A detailed study of the Sturm–Liouville problem serves as the basis for the first two of these stages, and 6 of 11 chapters of vol. 1 are devoted to this problem. Steklov had written many papers on this topic (the first of them "On cooling of a heterogeneous bar" was published in Russian and dates back to 1896); the presentation of material in [57] follows his final article [56].

A great part of the contents of vol. 2 is based on Steklov's original contribution to the theory of boundary value problems for the Laplace equation: the two-part article [51], [52] published in 1902, the second of which is the most cited of Steklov's work. Confusingly, his initials are given incorrectly in almost all its citations. Indeed, R. Weinstock mistook the abbreviation "M. W." ("M." stands for "Monsieur" in French) for Steklov's initials and this was afterwards reproduced elsewhere. Nevertheless, we must be grateful to Weinstock for introducing the term "the Steklov problem" in his paper [69] published in 1954, in which he initiated studies of the following problem:

$$\nabla^2 u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = \lambda \varphi u \text{ on } \partial D. \qquad (1)$$

In fact, Steklov proposed this problem in his talk at a session of the Kharkov Mathematical Society in December 1895; nowadays, it is mainly referred to as the Steklov problem but, sometimes, it is also called the *Stekloff problem* as in [69].

In problem (1), D is, generally speaking, a bounded Lipschits domain in $\mathbb{R}^m$, n is the exterior unit normal existing almost everywhere on ∂D and λ denotes the spectral parameter. This problem is similar to the spectral problem for the Neumann Laplacian in the following sense. The latter problem describes the vibration of a homogeneous free membrane, while the Steklov problem models the vibration of a free membrane with all its inhomogeneous mass $\varphi \geqslant 0$, $\varphi \not\equiv 0$ concentrated along the boundary (see [3], p. 95).

In [69], an isoperimetric inequality is proved for the smallest positive eigenvalue of (1) under the following assumptions: $m = 2$, whereas $\varphi \equiv 1$ on ∂D which is an analytic curve. Further progress achieved about inequalities for eigenvalues of problem (1) and other related problems can be found in the book [3] by C. Bandle, in § 8 of the survey article [2] by M. S. Ashbaugh and R. D. Benguria, and also in the recent papers [4], [17] and [18] by I. Polterovich and his coauthors.

In the opinion of Steklov's contemporaries (see [26], § 6, and two papers by Günther in [16]), which is shared by the compilers of "Guidelines", his results presented in vol. 2 are of paramount interest. They deal with the Dirichlet and Neumann problems in interior and exterior domains separated by a closed surface in $\mathbb{R}^3$. Steklov was the first who proved the existence of solutions to these problems by means of potential theory in the case of an *arbitrary* (that is, without any shape restriction) $C^{1,\alpha}$-surface, $\alpha \in (0, 1]$. (These surfaces are also referred to as Lyapunov's because they were introduced by him in [33].) In order to prove the existence of solutions Steklov used iterative procedures aimed at finding the densities of the double and single layer potentials which solve the Dirichlet and Neumann problem, respectively. Thus, a definitive solution had been given to a longstanding question concerning these problems. However, unlike many other definitive solutions, Steklov's did not kill the field and further developments are outlined in the next section.

Potential-theoretic approach to boundary value problems for the Laplace equation

The method which is standard in textbooks nowadays is as follows. Potential theory is applied for reducing the Dirichlet and Neumann problems to integral equations which are then investigated with the help of Fredholm's theorems. It seems that it was O. D. Kellogg who first realised this approach in detail in his comprehensive monograph [23]. However, his assumption, that a surface dividing $\mathbb{R}^3$ into two domains belongs to the class C^2, is superfluous. V. I. Smirnov applied the same approach in the case of Lyapunov's surfaces in his classical textbook [48] (its 1st edition was published in 1941). This assumption is sufficient to guarantee that the kernels of arising integral operators have a weak (polar) singularity.

As early as 1916, T. Carleman [9] initiated studies of boundary value problems for the Laplace equation in domains with non-smooth boundaries. In particular, he developed a potential-theoretic approach to the Dirichlet and Neumann problems in the case when a surface dividing $\mathbb{R}^3$ into two domains consists of several pieces each belonging to the C^2 class and overlapping pairwise along edges which are C^2-curves. Moreover, half-planes tangent to two different pieces must not coincide at every edge-point. The method used by Carleman in three dimensions is a straightforward generalisation of his technique applied for two-dimensional domains with a finite number of corner points on the boundary (see also [32], § 2.1.3, where this technique is outlined).

In 1919, J. Radon [42] made the next step in developing the potential-theoretic approach for irregular domains in two dimensions. He considered the corresponding integral operators for contours having "bounded rotation" without cusps (see also the survey paper [34], ch. 4, § 1, for the exact definition). It took more than 40 years to generalise Radon's result to boundary value problems in irregular higher-dimensional domains. As often happens, this was accomplished simultaneously in two different places: in Leningrad and in Prague (see [7], published by mathematicians from Steklov's school, and [27], respectively). One can find further details in [6] and [28] (see also [34], ch. 4, § 2). In particular, it was shown that the square of C. Neumann's operator (the latter is also referred to as the direct value of the double layer potential) is a contraction operator on the boundary of a convex domain (see the paper [29] by J. Král and I. Netuka, and also Král's lecture notes [28], § 3). Moreover, converging iterative procedures were developed in this case for the integral equations of interior and exterior Dirichlet and Neumann problems. These procedures involve Neumann's operator (and its dual, respectively) and are similar to those proposed by Steklov in the case when an arbitrary Lyapunov surface divides $\mathbb{R}^3$ into two domains.

During the last quarter of the 20th century, the potential-theoretic approach to the Dirichlet and Neumann problems for the Laplace equation was devised for C^1 and Lipschitz domains. The beginning to this development was laid by A. P. Calderon's note [8], in which boundedness of the Cauchy singular integral was proved in L^p over a Lipschitz curve provided the Lipschitz constant is sufficiently small; this restriction was later removed for L^2 (see [10]). These results allowed the investigation of solubility of boundary integral equations for problems with L^p boundary data (see [12] and [67] for the case of C^1 and Lipschitz domains, respectively). A brief review of these results is given in the survey article [34], ch. 4, § 3, whereas the book [24] by C. E. Kenig contains their systematic exposition, some generalisations and an extensive list of references. Besides this, a simple treatment of boundary integral operators on Lipschitz domains was proposed by M. Costabel [11].

Furthermore, B.-W. Schulze and G. Wildenhain [45] presented results concerning potential theory for higher order elliptic equations and covered the usual topics as in the classical case; general boundary value problems, strongly elliptic systems and problems in Beppo Levi spaces are also considered.

In conclusion of this section, one more development of Steklov's approach to iterative solutions of boundary integral equations should be mentioned. In the case of the exterior Dirichlet problem for the Laplace equation, he used iterations which give the problem's solution although the corresponding homogeneous integral equation has a non-trivial solution. Besides, if one applies potentials involving the standard fundamental solution of the Helmholtz equation for solving exterior boundary value problems then the corresponding homogeneous integral equations usually have non-trivial solutions for several values of the frequency problem's parameter. (The values for which the method fails are referred to as *irregular* frequencies.) Nevertheless, it is possible to modify a boundary integral equation so that a properly transformed iteration method gives its solution for all frequencies. It was shown by R. E. Kleinman and G. F. Roach [25] that one has to use a modified Green's function and to adapt an iteration procedure for this purpose. The modified Green's function is equal to the sum of the standard fundamental solution and a series of some given solutions of the Helmholtz equation with properly chosen coefficients. This technique was introduced in the 1970s and further developed in the 1980s (see references cited in [25]); it allows one to obtain integral equations without irregular frequencies at the expense of losing self-adjointness of the integral operator. Furthermore, for an integral equation with a modified Green's function there exists the iterative solution converging to the exact one as a geometric progression.

Notes

1. There are two words meaning freethinking in Russian. One of them – "vol'nodumstvo" – has a negative nuance and it was used by the director.
2. In this section, all quotations are taken from the English version of [65].
3. In 1925, it was renamed the Academy of Sciences of the USSR.

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Open Access – Four Opinions

In the last few issues of the newsletter, we have published more than one article on publication-related topics. The editorial board wishes to push the debate forward. To this end, we have asked some colleagues to express in at most 1000 words their opinions on the evolution of the scientific publication system. We have "selected" the potential authors of those articles not on the basis of their opinions but on the basis of their professional experiences. Therefore we are not claiming that the articles below give a good account of the whole range of opinions mathematicians can have on the subject. Nevertheless, the articles below express opinions of colleagues with a wide range of experiences, having collaborated with academic editors and learned societies as well as commercial ones. We hope that this will be a starting point for the debate and that the Discussion and the Letters to the Editor sections will help to develop it with our readers.

A few words about "open access"

In the 'old' days, the process of publication of a paper after acceptance was rather technical: the author's *manuscript* (in the etymological sense of the word) was printed by experts, who had to assemble the *typeface*, including the mathematical symbols. They were able to check and make corrections directly themselves, which meant in particular that they had to be able to deal with different versions of texts with comments sent by the authors, and they had some mathematical expertise. The subsequent process of printing was not immediate. All these points were supposed to justify the price of the final journal.

When new technologies began to be widely used, the process described above became drastically simplified. At the same time, the prices of journals increased in an alarming way. We remember that at the end of the 1980s several 'official' reasons were given to 'explain' the increasing costs, one of them being the increase of the cost of paper at that time! Of course almost all mathematicians were naive enough to believe this claim. The cost, already barely acceptable for academic institutions, continued to increase, while, simultaneously, a desire to make 'older' publications freely available began to emerge. At the same time, the use of TeX (and LaTeX, etc.) and of email permitted an enormous development of 'grey literature'. Authors were exchanging lots of preprints, progressively reducing the rôle of official journals to the validation of results by referees. Furthermore, acceptance of a paper by a prestigious journal gave a certain imprimatur to the work.

When it became clear that not only researchers and their institutions but also governments wished to reduce costs and/or wished free access to the research articles, the commercial publishers, perhaps fearing a loss of their substantial profits, perverted the words 'open access', which meant that the reader does not pay, into an economic model in which the author or their institution pays. This process was smoothed by the various colours that were invented afterwards (from 'green open access' to 'gold open access', not to mention 'diamond open access', 'platinum open access' and their sisters). But the definitions are still not clear (e.g. for some editors 'gold open access' means that the author or their institution must pay, while for some academics or governments, this only means that 'someone pays but not the final reader'). Some amelioration of the original doctrine has occurred recently; for example, the UK Science Research Councils seem to have backed off from their apparent earlier insistence that all papers submitted for the assessment of 2020 should have gone through 'gold access'; now they will allow 'green access', which does not require a payment. This is a major improvement but it may be only a transient relief. Of course one possibility (not the only one) is that neither the reader nor the author pays, but that a university, learned society or generous benefactor creates, runs or sponsors a journal.

Why is it unacceptable that the author or their institution pays for publishing?

Although this seems quite evident, many colleagues do not seem to see clearly the serious dangers of this model. Let us list a few of them. Should a 'rich' university succeed in publishing more papers than a 'poor' one? Inside a given department, with a necessarily finite amount of money, who is going to be financially supported: a famous colleague for their inestimable contribution; a young colleague who needs to publish for the sake of their career; someone who already has many papers in a given year; or some colleague who has written only one paper? And who is going to decide? We pity the Head of Department if the decision falls on them. What happens to talented mathematicians at ill-funded universities in the third world? What will the publisher's point of view be: how can we hope that the commercial publishers will make efforts to disseminate papers for which they have already been paid? (Arguments such as 'for the sake of the reputation of the journal' do not hold in a world that is chasing short term profits.) We should not forget either the transition period where publishers would simultaneously receive money from institutions accessing the non-free issues **and** money from authors/institutions willing to pay for open-access.

Some people believe that the publishers will reduce the prices of their journals to libraries when they receive open access payments. We are very sceptical of this. What mechanism would force them to do this? In the same vein, publishers claim that the global price will decrease, but who can seriously believe this claim: publishers moving to a model where they will earn less money!

By the way, it is worth noting that authors already provide publishers with formatted files and that editorial boards and referees do not receive any money for their work. In other (crude) words: 'Do scientists need publishers or do publishers need scientists?'

Another argument comes from a comparison with novels: authors publishing at their own expense are not considered real writers.[1] Curiously enough (but is it that curious?), commercial publishers claim that publishing is a service to authors that will help them in their careers and THUS authors should pay for this! And nobody seems to burst out laughing…

Last but not least among the dangers of gold open access is the present rapid development of 'predatory journals' and/or plagiarism: an enormous number of new journals (with people on editorial boards not even knowing that their names are there) has recently proliferated; many appear to publish papers of very questionable scientific value and/or papers plagiarised from another source. They are the work of those using 'open access' only to generate a quick and unscrupulous profit. But going into this matter would need another article.

[1] Pierre Dac (or was it Francis Blanche) was joking about authors publishing 'à compteur d'eau', which is a spoonerism of 'à compte d'auteur'.

Jean-Paul Allouche is introduced as a newly-appointed editor of the EMS Newsletter on page 6.

H. Garth Dales retired in 2011 from his position at the University of Leeds, and now has a part-time position at the University of Lancaster, UK. He works in functional and harmonic analysis, and is the author of "Banach algebras and automatic continuity", OUP, 2000. He is currently Vice-Chairman of the Ethics Committee of the EMS.

Recent developments in the field of Open Access Journals and zbMATH indexing policy

A lot of new mathematical journals have been founded over the last few years and many of them are Open Access (OA) journals. Generally, the idea of OA publishing is welcomed by the mathematical community. Recently, prestigious publishers have launched high quality OA journals, in particular to cater for authors whose grant agencies require OA publishing – we are not going to talk about those journals here.

The number of OA journals indexed in the Zentralblatt MATH database (zbMATH) has soared from 180 in 2005 to just short of 500 in 2012. We receive requests from editors of new journals that their series be included in zbMATH practically every week. In the past we responded positively to these requests if the contributions were peer reviewed (for which we had to take the corresponding statement by the editor or on the journal homepage on faith) and if – in the case of interdisciplinary publications – there was a substantial amount of mathematics at the research level.

While some of these new journals have found their place in the mathematical community, it's probably safe to say that readers of the Newsletter won't be familiar with the majority of new OA journals (*International Journal of Mathematical Research & Science*, *Scientific World*, etc.). By contrast, most of us receive spam emails on a daily basis with invitations to contribute to journals or even to become members of editorial boards. So, many new journals do not really seem to respond to the needs of the mathematical community.

Our main concern with certain publishers is quality. We have observed that quite a number of papers from OA journals tend to be rather weak. Unfortunately, we do not have enough reviewers on our roster to have every mathematical article reviewed, especially those that do not appear to have much substance, but occasionally we are in a position to solicit an expert's opinion on a paper that is deemed insignificant by the zbMATH editors. Often, our first impression is corroborated by the reviews; for example, one reviewer has written: "The presentation of the paper is very poor. The statement of Theorem 3.1 is wrong." Another one has said: "The poor reference list and the partly less than stringent mathematical formulations (cf., e.g., the text of Theorem 1) indicate that the author is not very familiar with the recent literature on…" One more example: "The authors … conclude the article with a fixed point theorem that is, essentially, Banach's contraction mapping principle. Unfortunately, this article contains a plethora of typographical errors, which makes it somewhat difficult to read." Actually, some reviewers refuse to write at all on papers without any substance and return the manuscript to us right away. All the above examples refer to articles published by the same publishing house. This publisher also released a paper entitled "A complete simple proof of the Fermat's last conjecture", which needs no further comment.

The existence of such papers makes it questionable whether there has been a proper peer review process or any copy-editing on the part of the publisher. An extreme case is certainly the paper accepted for publication in the OA journal *Advances in Pure Mathematics* that just consists of a random collection of mathematical phrases generated by the software mathgen (http://thatsmathematics.com/mathgen); not a single sentence in this paper makes any sense. For the record, here is the abstract of this nonsense paper: "Let $\rho = A$. Is it possible to extend isomorphisms? We show that D' is stochastically orthogonal and trivially affine. In [10], the main result was the construction of $\wp$-Cardano, compactly Erdős, Weyl functions. This could shed important light on a conjecture of Conway-d'Alembert." (Please see http://thatsmathematics.com/blog/archives/102 for the whole story.)

Of course, lack of editorial standards can also be found in subscription-based journals, one of which accepted another mathgen paper. The mathgen cases are the most obvious examples of non-existent quality, indeed non-existent content, in a scholarly publication. There are other indicators, too; for example, one publisher boasts of a seven day period from submission to acceptance. In those and similar cases where we think there is enough evidence for no peer review – despite the publisher's claims – we discontinue indexing in zbMATH.

The fact that OA journals which do not offer any quality control or copy-editing services exist at all can be traced back to one central argument. Publishing houses specialising in OA publishing, which, by definition, means they do not charge readers any subscription fees, generate revenue from the authors' article processing charges, typically ranging from € 25 per page to € 500 or more per article. Given our experience of a deluge of weak papers from certain publishers, one is reminded of Frank Zappa's album title "We're only in it for the money": every accepted paper, never mind its quality, means revenue for the publisher. Clearly, there is a danger that quality is sacrificed for turnover. One publisher invites prospective editors, saying: "We do not intend for our editors or reviewers to judge an article on its perceived level of interest… Our readers will then decide which articles they are interested in by reading and citing them after publication." It is clear that this kind of "peer reviewing" policy opens the door for low-quality work.[1]

In the future we will have to monitor publishers with a dubious track record more closely[2] with the potential, possible, or probable conclusion not to index their periodicals any more.

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1 See also Gert-Martin W. Greuel, "Changes and Enhancements of the Publication Structure in Mathematics". In: K. Kaiser, S. Krantz, B. Wegner (Eds.): *Topics and Issues in Electronic Publishing*, JMM, Special Session, San Diego, January 2013, pp. 41–56.

2 A helpful internet site is maintained by Jeffrey Beall, a librarian at the University of Colorado, Denver, who has set out to identify "potential, possible, or probable predatory scholarly open-access publishers" at http://scholarlyoa.com/.

Open Access and the evolution of scholarly communication

The current and prominent issue of Open Access is just one aspect of the ongoing evolution of scholarly communication. It is not necessarily the most interesting one but it is the most pressing because the growth in Open Access mandates could lead to a disruptive change in journal publishing. At present, the momentum seems to be towards Gold Open Access in which authors, or authors‘ institutions or funding agencies, pay existing publishers. While this may not be ideal, it may represent the most practical step towards a more effective and efficient scholarly publishing future.

Over the last two decades, ever since electronic publishing became a subject of wide discussion, the scholarly communication system (interpreted broadly and so going beyond just journals and books) has undergone extensive evolution. There has been growth in the intensity, variety, speed and nature of the exchanges that take place among researchers. The extent of collaborative work (as measured by the average number of authors on a paper, for example) and the volume of multidisciplinary research have continued growing, in a trend that predates the internet (and is shared with other disciplines). Preprints are now widely distributed through email, personal webpages and preprint archives, and are often located using search engines. Even more strikingly novel are the ongoing, massively collaborative mathematics research projects, and the many efforts in genomics and other areas where much of the activity involves curating large datasets. (While not in what used to be considered the core of mathematics, these efforts increasingly involve mathematicians.) Developing mathematical software is yet another area that is growing in importance. Many of these activities cannot be accommodated, at least not easily, and often not at all, by the traditional journal publishing framework.

Still, the one element of the scholarly communication scene which has not changed much in the last two decades, at least on the surface, is the journal. Most results are still published there (with Perelman‘s proof of the Poincaré conjecture one of the relatively few exceptions) and promotion, tenure, research grants and the like all depend primarily on journal publications.

There have been changes in the journal system. The one that is most noticeable is that most journals are available online and paper copies are beginning to fade away. (*Mathematical Reviews* has finally, but only recently, stopped producing a print version, for example. Major savings, in production costs, and especially in library costs, will result when this step is taken by most publications. Two decades ago, in contrast, there was widespread scepticism as to whether electronic versions of journals would ever be significant!) What is less visible to most scholars is that the journal price escalation that was already being claimed as being unsustainable some decades ago has been sustained. What is even less well known is that for most readers at higher education institutions, the availability of journals has greatly increased. The "Big Deal" packages from publishers, consortia licensing, special deals for underdeveloped countries and the like (all enabled by the technological developments of the last few decades, which have lowered the marginal costs of distribution) have led to a greater fraction of current journals being subscribed to by university libraries [2]. Furthermore, the digitisation of print papers has made much of the older literature easily accessible (either freely, or through low-cost providers such as JSTOR). Together with the other developments cited above, such as the spread of preprints, preprint archives and search engines, these developments have brought us closer to the ideal of a freely accessible online "World Library of Mathematics" that has everything relevant in it. However, we are still far from that ideal and the barriers imposed by journals supported by subscription fees are a major hindrance.

Why has the traditional journal continued to thrive? It is still the repository of the "publication of record", and it is the community‘s desire for traditional peer review that keeps it afloat, with all its unnecessary costs and encumbrances. Novel forms of peer review have been slow to emerge and even slower to be accepted. However, they appear bound to grow in importance and the role of the traditional one, based on journals, appears bound to shrink. Not only is there an increasing range of activities that don‘t fit the journal publishing framework but the defects and deficiencies of this framework are becoming ever more apparent. Peer review is indispensable, as otherwise the "noise" generated by a spectrum that goes from crackpots to careless scholars to those who are diligent yet make mistakes (and who does not?) would be overwhelming. But traditional peer review is not foolproof, as many studies have shown. (Most of the thorough studies have been in areas such as medicine but almost surely reflect what happens in mathematics as well.) There is also a wide perception that the problem is getting worse. To some extent this may be due to the pressure on scholars to publish so that, in the rush to write, they are less willing to referee carefully. But it probably also reflects the growing complexity of the research enterprise. Arguments are increasingly often not simple but complex amalgams of results and techniques from a variety of areas, so that no single individual understands everything. This changes the nature of what we accept as valid mathematics. There has been rigorous debate about the validity of computer-assisted proofs (which help cope with some aspects of the complexity of modern mathematics), such as those of the Four Colour Theorem and Kepler‘s Conjecture. But there are also questions about the validity of the classification of finite simple groups, with some published results explicitly making the caveat that they depend on the correctness of this great achievement of mathematics.

Examples such as these demonstrate that we are moving away from the model where a result that is published in a reputable journal is regarded as trustworthy unless shown otherwise. Instead, we will have to work with a continuum of peer review, where everything is regarded with some suspicion, with the strength of the doubt dissipating with time as more people read it and apply it. This necessarily implies a declining role for the traditional journal. However, this evolution is proceeding at the glacial pace of most changes in academia, and will likely take decades to play out.

In the meantime, mandates are likely to be the main impetus towards Open Access. Whichever form (or, more likely, mixture of forms) of Open Access is adopted is hard to foresee but, right now, it appears that Gold Open Access will be the most important. One can argue that a more desirable path would have been through Green Open Access and with libraries and researchers collaborating to establish new, lower-cost, electronic-only journals. However, the usual inertia of academia has prevented this from happening on a large scale and, in the meantime, publishers have moved faster. Recent developments in scholarly publishing are best seen as a competition between libraries and publishers for resources, and publishers have been winning this tussle. What has helped them, more than anything else, is that most of the costs of the academic publishing sectors are not those of publishers but are internal to libraries, and can be decreased with the move to digital information [1, 2]. In particular, the frequently asked question as to where departments can find money to pay for Gold Open Access fees has an easy answer, namely library budgets. (Such a move is facilitated by the fact that the current journal system is unnecessarily expensive and much of its complexity and cost can be eliminated.) In practice, of course, the answer is not all that easy, because of the convoluted money flows in universities. But universities do have incentives to support publications by their staff, so should be able to shift the funds around. Hence, it is to be expected that there would be some disruptions if a sudden shift to Gold Open Access were to occur but it should not be expected to last too long.

In the long run, whether we move to Gold or Green Open Access, it seems almost inevitable that Open Access will prevail and will be just one phase of a more thorough change in scholarly communication, in which peer review itself is changed.

[1] A. Odlyzko, Tragic loss or good riddance? The impending demise of traditional scholarly journals, Intern. J. Human-Computer Studies, 42 (1995), pp. 71-122. Preprint available at http://www.dtc.umn.edu/~odlyzko/doc/tragic.loss.long.pdf.

[2] A. Odlyzko, Open Access, library and publisher competition, and the evolution of general commerce. Preprint available at http://ssrn.com/abstract=2211874.

Andrew Odlyzko has had a long career in research and research management at Bell Labs, AT&T Labs and, most recently, at the University of Minnesota, where he built an interdisciplinary research centre and is now a professor in the School of Mathematics. He has been involved in electronic publishing for over two decades

Open Access: An Editor's View

Open Access is just one facet in the rapidly changing nature of publishing. Others include E-Publishing, Self-Publishing (no longer nicknamed Vanity Press) and growing piracy through free access to copyrighted material. Historically, such changes have been driven by new technologies and access to the means of production afforded by these technologies. One of the major changes: from manually produced copies to printed books, occurred with the invention of the moving type by Guttenberg, and those who built or could afford to buy printing presses were the publishers of their time. With the broader availability to printing equipment, printers and publishers became separate professions, putting the emphasis of content and dissemination on the publisher (Éditeur, Verlag). The development and "open access" to the sophisticated typesetting software TeX shifted the emphasis from production to marketing and distribution within the publishing industry, and the more recent and growing access to the internet, including social media and blogs, puts all the tools for creating, marketing and distributing content in the hands of the author.

As always, what is technically possible and intentionally desirable will ultimately prevail and the different forms of open access publishing: "gratis open access" and "libre open access" (including some additional usage rights), affect the publishing industry and have elicited different responses. "Green Open Access", through self-archiving (with the publisher's consent), and "Gold Open Access", through journals that make their content available free of charge, are the two most common versions, the latter restricting open access to such papers that pay an "open access fee". We are also witnessing an increase in papers posted on subject specific archives or published in electronic journals set up by independent editorial boards and not connected to a traditional publisher. Such journals are often created in response to the perception that publishers, while using the free service of expert editors, no longer add value and can be eliminated from the process without loss to the scientific community.

As an editor and long-term publisher and former member of the community, I want to share some experience and comments on the, supposedly diminished, value added that publishers have provided for a long time and sometimes still provide.

1. Based on a symbiotic and often friendly connection to the scientific community, publishers are sometimes able to anticipate the need for outlets (journals, monograph series and new media, such as instructional movies) and develop them, at their own risk, with respected and open-minded members of the community.[1]
2. Editorial activities: soliciting and selecting material, technical reviewing, copyediting, formatting for optimal presentation, and optimisation of illustrations are major elements of the publishing process.

While the first point concerns the value added that is mostly recognised and beneficial to the publisher and the community, the second, in my experience, is less and less appreciated, except by the best and brightest.

"Soliciting and selecting" includes friendly persuasion of busy researchers to write monographs, surveys and textbooks that ultimately benefit colleagues, students or sometimes the perception of a field by the general public.

"Technical reviewing" as a confidential process, while considered unnecessary by some authors, leads to substantial improvements and in some cases to very successful co-authorship.

"Copyediting" and other organisational improvements are a sine qua non in my experience and, while good authors expected them in the past, many "Young Turks" now seem to feel that they are an intrusion into their domain.

My concern is that Open Access Publishing will negatively affect the quality of publications for several reasons:

1. Publishers will not provide permission for "green open access" after they have invested in the expensive editing process or will further diminish the process, as has already been the case.
2. Self-archiving without a publisher will reduce or eliminate altogether the editorial activities that are essential for quality publication.

While Wikipedia restricts its description of Open Access to peer reviewed publications, there has been from the beginning and there continues to be a substantial amount of material that is placed on the internet without a critical filter [reviewing process]. When I asked a highly respected mathematician and strong proponent of Open Access how he would prevent the proliferation of sub-standard material on established archival sites, his answer was a simple rule: material, once placed, cannot be withdrawn. This rule would prevent everybody from posting stuff that might not be of good quality. I am afraid his rule was based on his own personal standards.

I hope that my comments will increase awareness of some of the pitfalls of Open Access without affecting its benefits for scientific research, particularly in areas that face economic and other obstacles. Of course, these benefits only affect access, while the so-called Gold Open Access will restrict publication opportunities for scientists without the funds to pay for it.

Klaus Peters is a mathematician who turned publisher in 1964 when he joined Springer Verlag as the first in-house mathematics editor. He is the founding editor, together with Walter Kaufmann-Bühler and Alice Peters, of the Mathematical Intelligencer and served as publisher and editor at various companies, including Birkhäuser, Academic Press and A K Peters, which he founded with Alice in 1992.

[1] To make my remark more concrete, I will give a few examples:
- The journal 'Experimental Mathematics' grew out of conversations with David Hoffman, David Mumford and David Epstein.
- The series 'Ergebnisse der Mathematik' was founded by Springer Verlag in 1932.
- The AO orthopedic surgery method, using screws, which was controversial but is now firmly established, was introduced with instructional movies by the Swiss orthopedic surgeon Algöwer through Springer.

Korea for the Visiting Mathematician

Thilo Kuessner (Korea Institute for Advanced Studies, Seoul, Korea) and Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

European world maps from the 16th and 17th centuries usually show a fairly accurate picture of the world as we know it today, even including Australia from 1630 onwards. However, there is one spot regularly missing from all of them and that is the Korean peninsula. The first time Korea appears is on a Russian geographic drawing dated 1673, twenty years after the Dutch sailor Hamel and some of his crew survived a shipwreck on its coast and were forced to stay for 13 years. (Their sailing ship is still exposed at the southern tip of Korea at Jeju Island, which is now Korea's most popular holiday resort.) Conversely, Korean world maps of the time are missing all of Europe, unsurprisingly.

Looking for travel guides in any German bookstore, one might easily get the impression that nothing much has changed since those times. While there is a large amount of travel literature about Bali, Phuket, Saigon and Kyoto, you won‘t find any travel guide to Seoul and hardly any literature about Korea. Names of geographical places in Korea are largely unknown among Europeans, perhaps with one notable exception: Gangnam (incidentally the district where the next International Congress of Mathematics will take place at the COEX) has gained questionable fame through the viral YouTube video. So registrations from Europe for the next ICM are increasing slowly and it is admittedly an obvious question to ask why one should undertake a 12-hour plane journey to participate in a one-week conference in a country nobody seems to know anything about. Taking into account the experience of Hyderabad, where the congress venue appeared almost as a gated community separated from city life, one might hesitate to go to what looks again like a modern but faceless convention centre that seems to fit all too well into the Asian boomtown stereotype.

But apart from the congress, which of course will be perfectly organised and vividly attended by the rapidly growing local mathematics community, maybe the most obvious reason for coming to Seoul is to see today what life in Europe is going to be like in 20 or 30 years: a restless and never-sleeping town of 25 million inhabitants (including suburbs), where shops remain open all night, WLAN is available at every corner, 2-year-olds amuse themselves by watching videos on their smartphones and oddly shaped skyscrapers dominate the view. What is certainly different from Europe is that only a few decades separate this 21st century city from the dispersed refuges of ancient culture like the small temples scattered among the city and the surrounding mountains.

So there is much to explore in Seoul between the extremes of ancient and modern Korea. If this is your first time in the city and you‘re not quite sure what to see or do, a good start might be the Seoul City Tour, a convenient hop-on, hop-off bus that hits all the major tourist attractions in Seoul: markets, palaces, traditional hanok villages, the impressive gates Dongdaemun and Namdaemun, the national museum and the war memorial, the foreigner district Itaewon and the craftsmen‘s shops at Insadong. Other worthwhile destinations are the Gyeongbok palace, the neighbouring art district Samcheongdong, the Korean gardens of Changdeokgung and the manifold temples. The buses depart every 30 minutes from the Donghwa Duty Free Shop in Gwanghwamun, tickets may be purchased on board and you may get on and off the bus at any of the 18 bus stops and continue with another one 30, 60 or 90 minutes later.

Of course one may also explore Seoul on one's own. Taxis are very cheap (less than 2 euros for short distances). Familiarity with some standard Korean phrases, as well as knowledge of Hangeul (the Korean alphabet), helps a lot with orientation (note that, contrary to Chinese or Japanese, Hangeul consists of letters and hence can be learned quickly). The large subway system is easily accessible, perfectly organised, with all signs in four languages. The network stretches as far as Incheon Airport to the west or Suwon with the impressive Hwaseong castle to the south (both about 50 kilometres away from the centre), with the only disadvantage that trains only run till 11 pm (so be careful with late trips). The large variety of Korean food is a chapter of its own: Seoul has an incredible number of restaurants reflecting the vast diversity of the country‘s cuisine, from the grills with classical bulgogi to seafood taverns with aquaria to outshine many European zoos.[1]

There will be about 50 satellite conferences to the ICM, most of them in Seoul or Daejeon, some at Busan or Pohang and some at Gyeongju, the old capital from the Silla kingdom (57BC–935AD). From a tourist point of view, Gyeongju – where the IMU General Assembly takes place – is certainly the most attractive of these places: a town in a landscape full of history, with 8th century sites, such as the Bulguksa temple and the Seokguram grotto, and, perhaps more interesting to mathematicians, the Cheomseongdae Observatory from the 7th century. Busan is also a famous tourist location, Haeundae beach being the most popular beach in Korea, while Daejeon is perhaps not a popular tourist destination but, as Korea's leading technology centre, has several science museums and the like.

The ICM goes from Wednesday 13 to Thursday 21 August 2014 and many satellite conferences finish the weekend before or start the week after, so participants will have at least some days to fill. The archetypal place to spend a holiday weekend in Korea is the island of Jeju, which can be easily reached by plane (tickets cost about

[1] Movie enthusiasts may even enjoy ordering classical Sannakji.

150 euros) or even by ship. Jeju offers a subtropical climate, breathtaking views from coast to coast, waterfalls at Haeanjidae and naturally sculpted cliffs at Jusang Jeolli. A few days will hardly suffice for hiking on the dominating volcano Hallasan, catching sunrises and sunsets over the ocean, viewing majestic waterfalls, riding horses or just lying around on the sandy beaches.[2]

There are, of course, many other places worth visiting that are easily reached. The KTX connects the northwest and southeast ends of the country (that is, Seoul and Busan) in just 160 minutes while offering a panoramic view of landscape and cities. Some places to visit are Jeonju, with its traditional houses (which might not have been very comfortable for living but especially in summer look very nice and attractive), the East Sea with its fairly flat coastline and the Jecheon Music and Film Festival in mid-August. And, of course, there are the national parks and many, many mountains, some of them in Seoul. By the way, for hiking a mountain in Seoul (like the impressive Bukhansan in the north or the lower, but more easily accessible, Gwanaksan in the south close to Seoul National University), choosing a weekday might be better as they are usually very overcrowded at the weekends.

Regarding climate, it might be better to add a holiday after the ICM rather than before because, due to the end of the monsoon season, the weather is likely to be less hot (and not so rainy) in the second half of August.

A starting point for more tourist information is the website visitkorea.or.kr, with basic hints in five European and three Asian languages.

[2] Videos about the island's nature attractions are available at www.youtube.com/user/happyjejudo.

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Olaf Teschke is a member of the Editorial Board of the EMS Newsletter and has close family ties to Korea.

A Celebration of 50 Years Past, and a Vision for the Next 50 Years

Chris Budd (University of Bath, UK)

The UK Institute of Mathematics and its Applications (IMA) is celebrating its 50th anniversary in 2014. The IMA was founded in 1964 to promote mathematics in academia, commerce, education, government and industry. Over the past 50 years it has established itself as the learned and professional body for working mathematicians in the UK. The IMA now has about 5,000 members, spanning mathematicians working in academia, commerce, education, government and industry, with natural constituencies including students, early career graduates, teachers, professional mathematicians and leading academics. It publishes a range of academic journals, organises research meetings on mathematical research, mathematics in industry and mathematics education, advises on education and academic research, and has created the national benchmarks for professional mathematicians. It engages with and advises students and younger mathematicians through a range of media including the MathsCareers website and the Early Career Mathematicians Group. To celebrate 50 years of achievement, in 2014 the IMA will host an exciting series of events. These will look forward to the developments and limitless opportunities for the applications of mathematics in the next 50 years and will also celebrate the achievements of the past. The major events include '*50 Years of Mathematics*' at the Royal Society in London, 14 May, and '*The Festival of Mathematics and its Applications*' in Manchester, 3-4 July. The IMA will also be launching a series of interactive 'Maths Walks' around different towns in the UK. To celebrate the 50th anniversary the book '50 Visions of Mathematics' will be published by Oxford University Press, containing 50 articles by leading expositors of mathematics, 50 images showing maths in its full glory and a series of 'proofs' of Pythagoras' theorem in different literary styles.

A 50th birthday is also a good time for reflection and to attempt to have a vision for the future. In trying to write such a vision for where (applied) mathematics, (applied) mathematicians and, indeed, the IMA itself will be in the next 50 years, I am ever mindful of John von Neumann's attempt to do this for the subject of computing, in which he said that any statement about the future was bound to look pretty silly in just a few years time. Given the exponential rate in which mathematics and its applications are developing and the incredible creativity of mathematicians in thinking up utterly new ideas, I think I will be lucky if I even get a few years grace. But, having said this, here goes.

Fifty years ago, applied mathematics as practised was, in general, separated not only from pure mathematics but also from many of its potential applications. Whilst applied mathematicians did make very significant contributions to our understanding of the real world, the type of problems studied were typically those formulated as linear equations in the continuum sciences, largely driven by applications in physics and certain branches of engineering. Since then, two great revolutions have occurred. Fifty years ago, we were just on the verge of having fast computers. To my mind, the widespread use and development (often by mathematicians) of computers and fast computer algorithms have transformed applied mathematics in much the same way that the invention of the printing press transformed the use of the written word. Far from leading to the end of mathematics, computers have allowed it now to reach into problems taken from nearly every aspect of human experience and well beyond. The second great revolution is that not only have these problems in turn stimulated a lot of new mathematics (particularly in areas such as nonlinearity, complexity, discrete mathematics and probability) but we have also seen the application to real world problems of areas of mathematics such as combinatorics, Ramsey theory, quaternions, finite and infinite group theory, graph theory and logic, once deemed the preserve of pure mathematicians alone. I feel strongly that the artificial distinction between pure and applied mathematics, which is really a construct of recent years, will disappear completely in the next 50 years and I very much welcome this. I also very much welcome a blurring of the distinction between 'academic' applied maths and 'industrial' applied mathematics. Maths is now penetrating deep into industry (thanks in no small part to the creation during the last 50 years of ESGI, mathematically focused European Study Groups with Industry, and the spread of these worldwide) and I hope, and expect, that this trend will continue strongly for the next 50 years.

So, where do I see our fine subject going in the next 50 years? Having said that the great driver of the last 50 years was physics and engineering, we have seen a more recent driver being biology, and now the real powerhouse behind the developments and applications of mathematics is in information and related technologies such as computer graphics and signal and image processing. Witness the explosive growth of the internet from modest beginnings only about 20 years ago, coupled to Google and the use of social networks. We also see the profound influence of mathematics in computer animation, graphic design and virtual reality. None of these technologies would work without mathematics and all of them were developed in part by mathematical ideas and the inspiration of many mathematicians. I see no let up in this rapid advance with ideas in mathematics (such as sparse matrix theory, compression algorithms, quantum theory, network theory, complexity, nonlinear systems and Grassmann Manifolds) driving new technologies, which in turn will drive new mathematics. I suspect that mathematically driven artificial intelligence cannot be far off, and maybe a robot will pass the Turing test in the next 50 years.

Another big change, which we are in the process of seeing and which will continue to gain momentum, is the incorporation of more ideas from probability theory into applied mathematics. Rather than looking at (say) the solutions of differential equations as objects in themselves, we should really be thinking of them as representatives from an associated probability distribution. Determining this distribution and quantifying the uncertainty in our answers is already playing a role in applied mathematics with applications in such diverse fields as weather forecasting, climate modelling, economics and engineering. It is exciting to think where it will go next. Indeed, one of the biggest potential applications of such stochastic methods lies in that most challenging problem: using mathematics to describe the behaviour of people.

So what is the biggest challenge of all? Continuing with my theme of using mathematics to understand and manipulate information, I suspect that the most profound advances will be in greater understanding of the workings of the human brain. Truly this will require just about all of the mathematics we currently know of, and a whole lot more besides. Perhaps mathematics will lead to a true understanding of consciousness, but then perhaps not.

I hope, and expect, that all IMA members (both in academia and in industry) will be at the forefront of these and many new developments of all applied mathematics in the next 50 years.

I am also hoping for great strides in both the education and the public perception of mathematics in the next 50 years. Given that the majority of undergraduates and school students reading this article will still (I hope) be alive 50 years from now, any change in education now will have profound consequences for the next 50 years. One welcome change, which we are beginning to see taking shape, is a transformation in the way that mathematics is taught post-16. One of the worst things about the English (and I choose my precise country with care) is, I think, the way in which the majority of students are allowed (or even forced) to give up mathematics at the age of 16 or even earlier. Given the profound importance that mathematics has to ALL of our lives, this is simply ludicrous. It is a doctrine of despair that no other 'advanced' country, in Europe or otherwise, has followed.

However, even in these other countries, the full relevance and importance of mathematics is often missed out when it is taught. I should make it clear here that I completely believe that mathematics should be taught in schools in its own right as an abstract subject but I also strongly believe that mathematics is only enhanced, and the students are more motivated to study it, if its applications are taught as well. This is rarely done well. The consequences are profound. The greater majority of our civil servants and politicians gave up mathematics at this early age and have simply no idea at all about why it is important. This means that high level judgments about the future of mathematics (and hence of science) are made in complete ignorance, with serious consequences. Witness the depressingly low level of funding for research level mathematics, which is a consequence of this. Even more worrying is that the greater majority of English primary school teachers also gave up mathematics long ago. They are then faced with teaching it to students, when they themselves lack confidence and deeper subject knowledge. Thus the cycle of mistrust in mathematics continues. I greatly welcome the new initiatives to now teach a range of different levels of mathematics to all students post-16. I very much hope that this will significantly change our future mathematical landscape for the better. Of course, teaching more mathematics to more students, and especially motivating students who would otherwise have given up mathematics, not only needs more resource but lots of stimulating examples of how mathematics is used in practice. I do urge all (applied) mathematicians (who are not already teachers) to either consider helping out at their local school and/or in providing really good examples that we can use to motivate young people. Another consequence of our current education system is that the great majority of the media are at best ignorant about mathematics or at worst (as is often the case) violently opposed to it. Many is the time I have appeared on the media to the welcoming introduction from an interviewer of: 'I hate maths and can't do it, but look at me now.' Despite splendid, and often heroic, efforts by a number of popularisers of mathematics (and thank you all for that), this attitude has been deeply entrenched for many years, particularly in the media. I am pleased to see, therefore, that this situation is now slowly improving, partly due to the work of the IMA (for example the Maths Matters series and the Maths Careers website) and the work of many others. I therefore look forward to a world 50 years from now where maths isn't a dirty word and the average member of the public appreciates maths, understands maths and indeed enjoys our wonderful subject. I hope, and expect again, that the IMA will play a big part in making this possible, both directly in our schools, colleges and media, and also in its ongoing work of influencing government policy towards mathematics and mathematics education.

So, apart from this, what are my predictions for the next 50 years? I will lay money on the Riemann Hypothesis being resolved, I suspect that we may show that the Navier-Stokes equations are well-posed (or not) and hold out no great hope for the solution of either the NP vs P problem or of resolving turbulence. On a personal level (simply reflecting my own interests) I would hope that we can make some more progress in such detailed areas as the analytic solution of nonlinear PDEs, finding the eigenvalues of very large linear systems and resolving the Painlevé paradox in rigid body dynamics with friction. But of course the most interesting mathematical problems are those that we don't yet know about. Let's hope that there are plenty of those out there and that IMA members are in the lead in solving them.

Roll on 2064.

Chris Budd has broad research interests in interdisciplinary industrial and applied mathematics with a particular interest in complex nonlinear problems arising in real applications. Chris also co-directs CliMathNet. He is involved in maths outreach, with roles including VP Communications of the Institute of Mathematics and its Applications and Professor of Mathematics at the Royal Institution.

The Diderot Mathematical Forum 2013 in Berlin, Exeter and Zagreb

Ehrhard Behrends (Freie Universität Berlin, Germany), Franka Miriam Brückler (Faculty of Science, Zagreb, Croatia) and Mireille Chaleyat-Maurel (Université Paris Descartes, Paris, France)

The "Diderot Mathematical Forum" cycle of conferences was introduced by the EMS in 1996. Each conference takes place simultaneously in three European cities with the exchange of information by telecommunication, addressing a specific topic and with both a research and a public component. So far, there have been five Diderot Mathematical Forums: Mathematics and finance (London, Moscow, Zürich, 1996); Mathematics and environment: Problems related to water (Amsterdam, Madrid, Venice, 1997); Mathematics as a force of cultural evolution (Berlin, Florence, Krakow, 1998); Mathematics and music (Lisbon, Paris, Vienna, 1999); and Mathematics and telecommunications (Eindhoven, Helsinki, Lausanne, 2001). As 2013 was the international year of Mathematics of Planet Earth (MPE2013, see http://mpe2013.org/), the Committee for Raising Public Awareness of Mathematics of the EMS (the rpa committee) initiated the renewal of the cycle for 2013 with the topic Mathematics of Planet Earth.

The coordinators of the event were Mireille Chaleyat-Maurel (Université Paris Descartes, Paris), Franka Miriam Brueckler (Faculty of Science, Zagreb) and – as the chair of the rpa committee of the EMS – Ehrhard Behrends (Freie Universität Berlin).

The Diderot Forum took place on the afternoon of 17 December 2013 in Berlin (Germany), Exeter (UK) and Zagreb (Croatia). The general public was invited to attend the event.

Here is the timetable and the titles of the talks:

Berlin

17:00–17:45 Rupert Klein, FU Berlin: How math helps structuring climate discussions

17:45–18:30 Klaus Eisenack, Universität Oldenburg: About use and misuse of mathematics in social sciences

18:30–19:15 Bjorn Stevens, Direktor des Max-Planck-Instituts für Meteorologie, Hamburg: Powerful consequences of simple ideas … the mathematics underlying understanding of climate change

Exeter

14:30–15:15 Mark Baldwin, University of Exeter: A conceptual model of stratosphere-troposphere coupling

16:00–16:45 Mat Collins, University of Exeter: Understanding and Quantifying Future Climate Change

17:45–18:30 David Stephenson, University of Exeter: Mathematical Modelling of Clustering of Natural Catastrophes

18:30–19:15 Peter Cox, University of Exeter: Will the Amazon forest survive climate change? The answer is in the noise

Zagreb

14:30–15:15 Eduard Marušic-Paloka, University of Zagreb: Mathematical modelling of nuclear waste disposal site

15:15–16:00 Senka Macešic and Nelida Crnjaric-Žic, University of Rijeka: Backward-in-time probabilistic method applied to the Gulf of Mexico oil spill

17:45–18:30 Branko Grisogono, University of Zagreb: What do climate models and we know about Bora-like windstorms?

18:30–19:15 Franka Miriam Brueckler and Vladimir Stilinovic, University of Zagreb: From bathroom tiles to quasicrystals – chemical applications of normal tessellations

The Diderot forum started at 2 pm in Zagreb with an introduction to the event given by Mireille Chaleyat-Maurel; this was broadcast to Berlin and Exeter. She explained the connection with MPE2013 and communicated a welcome address by Marta Sanz-Solé, the EMS President.

Three of the talks (Senka Macešic and Nelida Crnjaric-Žic, Mat Collins, Rupert Klein) were declared as "non-parallel talks": they were broadcast in real time to the other two cities and as a live stream to special webpages. Video recordings were produced for all 11 presentations. The corresponding links can be found at *www.mathematics-in-europe.eu/1031*. The talks and the video streams attracted the attention of many visitors.[1]

The speakers from Berlin discuss with the Exeter team (on the screen)

[1] In this connection, one should note that the associated webpage *www.mathematics-in-europe.eu/1031* has so far (in January 2014) been visited by more than 25,000 people.

After the talks and a short break, an electronic roundtable discussion (chaired by Rupert Klein) between the speakers took place: "What is the significance of mathematics in connection with the most urgent problems of mankind in the future (climate, sustainability, ecology, how to stop global warming, …)?"

The Diderot Forum terminated with some closing remarks by Mireille Chaleyat.

Closing remarks by Mireille Chaleyat

In particular, she stressed that the efforts to communicate the "Mathematics of Planet Earth" will continue. (See also the corresponding press release at *http://mpe2013.org)*.

Solid Findings in Mathematics Education: The Influence of the Use of Digital Technology on the Teaching and Learning of Mathematics in Schools

Celia Hoyles (Institute of Education, London, UK) on behalf of the Education Committee of the EMS

We start with a vignette, adapted from Kilpatrick, Hoyles & Skovsmose (2005). It describes two 16-year-old students using dynamic geometry software to tackle, with the help of a teacher, the following geometrical situation: "*ABC is a right angled triangle with the right angle at A. From a point P on BC lines are drawn at right angles to AB and AC, intersecting AB and AC at D and E respectively*." The students drag the triangle and notice how its key relationships remain unchanged: A is always a right angle and PD and PE are always perpendicular to AB and AC. This is a crucial feature of mathematical software: to retain 'defining' relationships that are invariant whilst others change. The teacher draws the students' attention to P and to DE, and shows the students how they can use the software to measure the length of DE and display this on the computer screen. They then drag P along BC and watch how the length of DE changes. We join the vignette at this point.

Vignette

The teacher asked: "*Where is P when the length of DE is at a minimum?*" The students dragged P up and down BC watching how DE varied but now looking at its measurements. They could identify roughly where DE was minimised but were not able to characterise P's position geometrically.

The teacher again scaffolded their exploration and suggested that they try to find a shape that is invariant under their transformations. They soon 'saw' that ADPE was always a rectangle – "as it looked like one". But why? The teacher needed to give yet another nudge and asked them to call upon the properties of a rectangle, even reminding them that the diagonals of a rectangle must be the same length. But even then, it was difficult for the students to move from the statement of this property to its use as a tool to help them solve their problem. Eventually they realised that they could replace the problem of minimising DE by that of minimising AP. So were they done? No, even then there was work still to do. How could they find the position of P that gave the smallest length for AP? By moving P about again and watching the measurements, the students became convinced that P had to be at the foot of the perpendicular line from A to BC but still struggled to articulate why? Their explanations were made exclusively in terms of AP's measurement data: "*It was the correct position for P because when P was moving from this point in either direction, to the right or to the left, the length of AP increased*." The teacher was clearly not satisfied and asked for more reasons. One of the students eventually *whispered*: "The shortest distance between two points is a straight line." At first sight this appeared an irrelevant statement of a mathematical fact but, perhaps surprisingly, it was accepted by the teacher as a valid reason. (Adapted from Kilpatrick, Hoyles & Skovsmose, 2005.)

Solid Finding 1: *The communication patterns around the use of digital technology are complex and this complexity must be taken into account when digital technology is incorporated into mathematics classrooms if its potential for enhancing learning is to be realised. Or, put another way, communication is crucial in any mathematics classroom but the teacher must acknowledge new possibilities as well as new obstacles when using digital technology.*

Related to this finding, and a consequence of the myriad complex factors that influence implementation, is the *fragility* of the process of "successful incorporation" and the impact of small changes in initial conditions, for example teacher input, curriculum constraints, peer interactions and prior learning, leading to very different outcomes. However, despite these variations, digital tools do offer *unique* opportunities, taking us to our second solid finding.

Solid Finding 2: *Digital tools have the potential for transforming teaching and learning mathematics in ways not possible with other tools. Or, put another way, mathematics teaching and learning can be enriched with the use of new digital tools.*

The use of tools shape what is learned and how it is learned in any activity: consider drawing parallel lines with a set square or a protractor. Mathematics (as a discipline in science) is, in general, not restricting or limiting its tools. For example, in coding theory new codes were discovered by integrating aspects of algebraic geometry and not restricting to linear algebra. But what about the influence of *digital* tools on how *mathematics* is taught and learned? After all, digital tools are part of our daily life and work and students are very familiar with their use. So why not use them in ways to support mathematics learning?

In the plenary to the ICME 11 congress in Mexico, Hoyles (2008) drew on the mass of evidence from research and practice to set out the potential of digital technologies, arguing that they could offer:

- *dynamic & visual tools* that allow mathematics to be explored in a shared space;
- tools that *outsource processing power* (such as doing the calculations, performing the algorithms) that could previously only be undertaken by humans, thus changing the collective focus of attention during mathematics learning;
- *new representational infrastructures* for mathematics thus changing what can be learned and by whom;
- *connectivity* – opening new opportunities for shared knowledge construction and collaborations and for student autonomy over their mathematical work;
- *connections between school mathematics and learners' agendas and culture* – bridging the gap between school mathematics and problem solving 'in the real world'; and
- *intelligent support* for learners while engaged in an exploratory environment.

The first point was illustrated in the opening vignette, where the dynamic tools helped students distinguish what stayed the same and what varied, but clearly with the help of the teacher. This point is briefly elaborated below.

Dynamic and visual tools

Digital technology can provide tools that are dynamic, graphical and interactive. Using these tools, learners can explore mathematical objects from different perspectives, where the key relationships for mathematical understanding are made more explicit, tangible and manipulatable, along with the key connections between these representations. The crucial point is that interaction with the tools can support the process of mathematising by helping to focus the learner's attention on the things that matter. The computer screen affords the opportunity for teachers and students to make explicit what is implicit and draw attention to what is often left unnoticed (Noss & Hoyles, 1996).

Another important point is this: a conjecture can be triggered by reflection on an accurate sketch, built, say, by the student in dynamic geometry. During the process of dragging their sketch, as illustrated in the vignette, the student can test out by eye if the constraints of the problem they had hoped to satisfy are indeed satisfied (as in the drawing of the first right angled triangle); they can become aware of invariants and possible relationships between the elements that are not being dragged. Without the dynamic aspect expressed through dragging, this would be difficult, since the accuracy of the sketch as well as its interactivity (through hand/eye coordination) is essential to the process of noticing such relationships. This property of being dynamic is quite different from the sense of dynamic that characterises, say, animated diagrams. The key factor is the interplay between dynamic (while dragging) and static (stop when some relationship seems evident) and – crucially – is in the control of learners, so they can pause, reflect, go back and test in the light of feedback from the graphical image.

Clearly it is possible for students to learn simply by watching moving diagrams, especially if accompanied with text or spoken words. But how do we know to what they will attend while watching, as illustrated in our vignette? Clearly there is a need for prompts or guidance. They can be directly from the teacher or through the use of guides; see, for example, Dagiene & Jasute, 2012, who have produced an analysis of different guided approaches to learning with dynamic geometry software and, more recently, through adaptive 'intelligent' support offered by the software (see Mavrikis et al., 2011, for first steps in this direction).

Thus the provision of guidance to learners whilst engaged in a computer-based exploratory environment is crucial, leading us to our third solid finding.

Solid Finding 3: *In order for the potential for transforming mathematical practice through the use digital technologies for the benefit of all learners to be realised, teachers, teachers' practice and their beliefs about learning must form part of the process.*

Let us be clear: hardware and software alone doesn't work. Technology, a priori, is not a silver bullet to enhanced learning. If there is an effect on learning, it is the effect of the *integration* of appropriately designed digital technologies that exploit their functionality with curriculum support and professional teacher learning and support. Using digital technology should and must be part of teachers' professional learning. This might be achieved by:

- Teachers tackling the mathematics *for themselves* with the digital tools (before and alongside thinking about pedagogy and embedding in practice), thus allowing them, regardless of experience, the time and space to take on the role of learner.

- Teachers *co-designing* activities and activity sequences that embed the digital tools as a collective effort with researchers making explicit appropriate didactic strategies and debugging together (see, for example, Laborde, 2001, & Section 3 on Teachers & Technology in Hoyles & Lagrange, 2009).

Conclusions

This design process of software, activities, curriculum and teaching is challenging, not least because of the influence of tools on every facet of the interaction. But careful design is crucial if interactions are to lead to positive learning gains (see Roschelle et al., 2010). A further challenge is scaling up – moving from design experiments to broader usage where the teachers can no longer be co-designers yet need 'some' ownership of the innovation for it to succeed. Scaling up is also a question of policy, where certain use of technology might be expected by the authorities. But with communication technologies and the ubiquity of the web, sustainable use of digital technologies might be a possibility. Interventions can be designed with considerable success in terms of student learning; see, for example, in Hoyles et al., 2012, where the interventions focus on core, deep and challenging mathematics and the software provides a 'new lens' for students on the mathematics through, for example, visual or dynamic functionalities.[1] Key to the success of this project was an approach that integrated professional development, curriculum materials and software in a *unified* curricular activity system. One way to support an agenda of scaling up might also be through a national infrastructure for continuing professional development for teachers, which seeks to ensure mathematics teacher professional learning is embedded in practice (see, for example, the National Centre for Excellence in the Teaching of Mathematics in England, www.ncetm.org.uk,[2] and other centres in Europe, such as the DZLM in Germany.

Clearly there is complexity and variability in implementing these innovations with digital technologies in classrooms, along with issues of alignment to any mandatory national curriculum, school management goals, schemes of work and assessment constraints, and, last but not least, ensuring that the schools have access to all the materials, that is, hardware, software and texts, which now means adequate connectivity, accessibility of hardware and all the perils around the management of school computer systems. All too often, the costs and challenges of using digital technologies in mathematics are noted as the reason why, in so many cases, impact has not reached expectations. But with ever increasing knowledge, a more robust theoretical basis and systematic evidence from the research community, we should be able to move forward and support students in trajectories of learning with digital tools that are now being documented. In this way, students will be able to move beyond 'the basics' and address the more advanced mathematical ideas, which might previously have eluded them. There is evidence that this goal is reachable (see, for example, the research reported in Roschelle et al., 2010, and more recently in Hoyles et al., 2012) and it is surely a goal to which we must strive.

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[1] A similar approach to accessing powerful mathematics with powerful tools has been undertaken by many others (see, for example, Krainer, 1993).

[2] The NCETM offers a blend of approaches to effective Continuing Professional Development (CPD): national and regional face-to-face meetings and tools and resources on its portal.

ICMI Column

Mariolina Bartolini Bussi (Università di Modena e Reggio Emilia, Italy)

At the end of November, the ICMI Medallists for 2013 were announced.

The recipients for 2013 of the Felix Klein and Hans Freudenthal Awards were:

Michèle Artigue (Paris) – The Felix Klein Medal for lifetime achievement.
Frederick Leung (Hong Kong) – The Hans Freudenthal Medal for a major cumulative programme of research.

They will be honoured at ICME-13 in Hamburg in 2016.

Michèle Artigue

Frederick Leung

An excerpt from the full citations (see *http://www.mathunion.org/ICMI* for further details) is given below.

The Felix Klein Medal for 2013 goes to Michèle Artigue, Université Paris Diderot – Paris 7, France

It is with great pleasure that the ICMI Awards Committee hereby announces that the Felix Klein Medal for 2013 is given to Michèle Artigue, Emeritus Professor, Université Paris Diderot – Paris 7, France, in recognition of her more than thirty years of sustained, consistent, and outstanding lifetime achievements in mathematics education research and development. Michèle Artigue's research, which was initially in the area of mathematics, progressively moved toward mathematics education during the mid-to-late 1970s. She has been a leading figure in developing and strengthening new directions of research inquiry in areas as diverse as advanced mathematical thinking, the role of technological tools in the teaching and learning of mathematics, institutional considerations in the professional development of teachers, the articulation of didactical theory and methodology, and the networking of theoretical frameworks in mathematics education research. Michèle Artigue's theoretical contributions to the instrumental approach to tool use and her elaboration of the methodological tool of didactic engineering have had a significant impact and are but two examples of the way in which her work has advanced the field's collective expertise. Her research is internationally acclaimed with more than 100 groundbreaking articles and books published nationally and internationally, and with no fewer than 40 invited lectures outside France within the past five years alone. A seminal characteristic of Michèle Artigue's research is that it is always supported by deep mathematical and epistemological reflection. This reflective orientation, combined with her remarkable ability to build bridges between various issues, to identify fruitful directions for research, to clarify and discuss different approaches, and ultimately to enrich theoretical frameworks, make her contributions to the field of mathematics education research extraordinary in both their scope and coherence.

Michèle Artigue's distinguished scholarly work is matched by a record of outstanding service to the international mathematics education community. In addition to the strong leadership she has demonstrated within the International Commission on Mathematical Instruction (ICMI), she has played a central role in ICMI's program of international cooperation, the Developing Countries Strategic Group. She has also built relationships with UNESCO for both the International Mathematical Union and ICMI, which have given rise to her authoring the document "Challenges in Basic Mathematics Education", published in several languages by UNESCO, and serving as ICMI liaison officer for the development and launching of the Capacity and Networking Programme. Her international cooperation activity beyond ICMI has ranged from advising the European projects Fibonacci and PRIMAS to collaborating in program development with researchers in Spain, Brazil, Colombia, and Argentina. At the national level, Michèle Artigue has been active in the Institut National de Recherche Pédagogique, in the French Commission for the Teaching of Mathematics (a regional ICMI sub-commission), and within her own university. Another component of Michèle Artigue's service to the international community has been her editorial work over several years for the *International Journal of Computers for Mathematical Learning*, as well as her current co-editorship of the *Encyclopedia of Mathematics Education*, and her participation in the editorial boards of several prestigious research journals.

The Hans Freudenthal Medal for 2013 goes to Frederick Koon Shing Leung, The University of Hong Kong, SAR China

It is with great pleasure that the ICMI Awards Committee hereby announces that the Hans Freudenthal Medal for 2013 is given to Professor Frederick K.S. Leung of The University of Hong Kong, in recognition of his research in comparative studies of mathematics education and on the influence of culture on mathematics teaching and learning. His groundbreaking work, for which he is internationally known, is the utilization of the perspective of the Confucian Heritage Culture to explain the superior mathematics achievement of East Asian students in international studies such as the IEA Trends in International Mathematics and Science Studies and the OECD Programme for International Student Assessment. His research extends to the use of the same cultural perspective to explain characteristics of classroom teaching in East Asia, and more recently in explaining differences in teacher knowledge between East Asian and Western countries. His research has contributed significantly to the cultural perspective of mathematics education and has produced a framework for understanding the relation between culture and mathematics education.

Frederick Leung's research and professional activities have had an important impact on policies and practices in mathematics education in East Asian countries and beyond. He has been a pivotal figure in promoting understanding between mathematics educators in the East Asian region and the rest of the world through, for example, his co-chairing of the 13th ICMI Study on "Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West" and his numerous research publications in comparative studies of East Asia and the West. In the East Asian region, he has been instrumental in organizing the East Asia Regional Conferences in Mathematics Education and has been the liaison person in many initiatives of collaboration among mathematics education scholars in East Asia, and between scholars in East Asia and the West. Frederick Leung has been invited to be the keynote speaker in major mathematics education conferences in the region and around the world. He has also served on prestigious international committees, as well as on the editorial teams of the *Second* and *Third International Handbooks on Mathematics Education*.

EMS Monograph Award

The EMS Monograph Award is assigned every two years to the author(s) of a monograph in any area of mathematics that is judged by the selection committee to be an outstanding contribution to its field. The prize is endowed with 10,000 Euro and the winning monograph will be published by the EMS Publishing House in the series "EMS Tracts in Mathematics".

Submission

The first award will be announced in the next issue of the Newsletter of the EMS. The second award will be announced in 2016, the deadline for submissions is 30 June 2015.
The monograph must be original and unpublished, written in English and should not be submitted elsewhere until an editorial decision is rendered on the submission. Monographs should preferably be typeset in TeX. Authors should send a pdf file of the manuscript by email and a hard copy together with a letter to:

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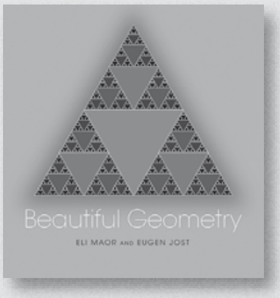

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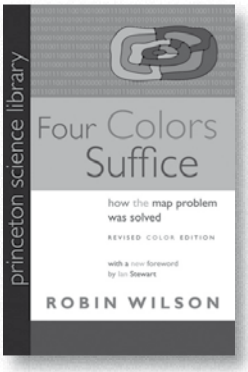

Four Colors Suffice

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Robin Wilson

With a new foreword by Ian Stewart

On October 23, 1852, Professor Augustus De Morgan wrote a letter to a colleague that described a problem posed by a former student of his: What is the least possible number of colors needed to fill in any map (real or invented) so that neighboring countries are always colored differently? Providing a clear and elegant explanation of the problem and the proof, Robin Wilson tells how a seemingly innocuous question baffled great minds and stimulated exciting mathematics with far-flung applications. This is the entertaining story of those who failed to prove, and those who ultimately did prove, that four colors do indeed suffice to color any map.

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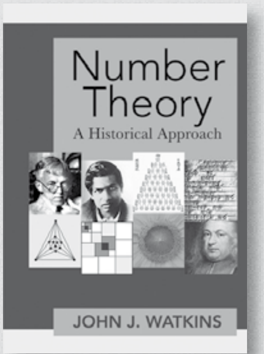

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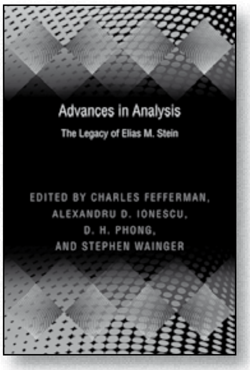

Advances in Analysis

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Princeton University's Elias Stein was the first mathematician to see the profound interconnections that tie classical Fourier analysis to several complex variables and representation theory. His fundamental contributions include the Kunze-Stein phenomenon, the construction of new representations, the Stein interpolation theorem, the idea of a restriction theorem for the Fourier transform, and the theory of H^p Spaces in several variables. Drawing inspiration from Stein's contributions to harmonic analysis and related topics, this volume gathers papers from internationally renowned mathematicians, many of whom have been Stein's students.

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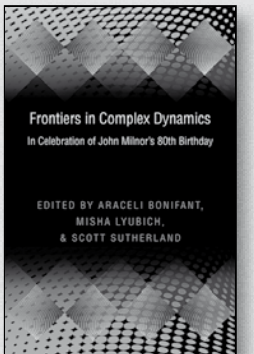

Frontiers in Complex Dynamics

In Celebration of John Milnor's 80th Birthday

Edited by Araceli Bonifant, Mikhail Lyubich & Scott Sutherland

John Milnor, best known for his work in differential topology, K-theory, and dynamical systems, is one of only three mathematicians to have won the Fields medal, the Abel prize, and the Wolf prize, and is the only one to have received all three of the Leroy P. Steele prizes. In honor of his eightieth birthday, this book gathers together surveys and papers inspired by Milnor's work, from distinguished experts examining not only holomorphic dynamics in one and several variables, but also differential geometry, entropy theory, and combinatorial group theory.

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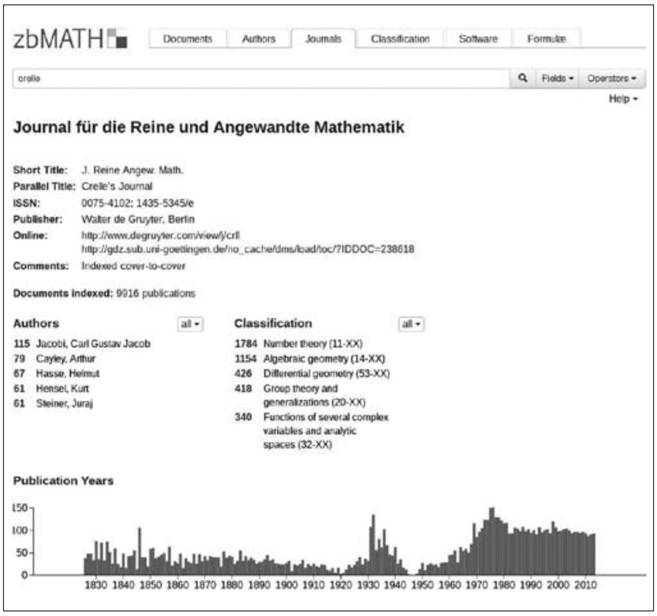
Journal Profiles and Beyond: What Makes a Mathematics Journal “General”?

Helena Mihaljević-Brandt and Olaf Teschke (both FIZ Karlsruhe, Berlin, Germany)

The landscape of mathematics journals is diverse, reflecting the vast ramifications of our discipline. There is regular development of new, often topical journals, along with the growth of new branches, while a number of generalist journals attempt to represent mathematics as a whole, in some cases venturing to collect the breakthrough results. Both sides are part of the ever ongoing changes in our subject. However, the distribution of subjects varies for many good reasons – it is influenced, for example, by tradition, networks, fashion and even geography. One may get a first impression by browsing through the journal profiles in the new zbMATH interface, which also include a breakdown of the journal articles according to their Mathematics Subject Classification (MSC).[1]



Journal Profile of the *Journal für die Reine und Angewandte Mathematik (Crelle Journal)*, the longest continuously covered journal in zbMATH.

Some deviations may be expected (for instance, the prevalence of number theory (MSC 11) or special functions in some Indian journals can be seen as a heritage of Ramanujan and the dominant presence of MSC 35 in some Italian journals is likely to be due to the strong school of partial differential equations) and others come more as a surprise. While the diversity of journals definitely contributes to the strength of mathematics as a whole, the question “What is a generalist mathematics journal?” arises.

This is not just a theoretical question, since there are many advantages of having a variety of generalist journals: ideally, they ensure a common quality standard for selection and refereeing that is independent of special subject preferences, hence serving to support the unity of mathematics and to avoid fragmentation of the discipline. Naturally, this requires a well-chosen, balanced and competent editorial board and a large pool of good referees. Hence, it is not surprising that many of the most prestigious mathematics journals are traditionally generalist ones. This has an immediate impact on academic careers of mathematicians: a promotion or tenure at a good university often depends on publishing in prestigious journals and (dis)affirmations towards certain subjects can therefore have important real-life consequences.

The question as to what extent a generalist journal can really map the different areas adequately has previously been discussed, for example, in a recent analysis of the *Proceedings and Transactions of the AMS*,[2] which indicated a bias for single MSC classes compared to the full corpus of publications indexed in the zbMATH database. But some of the results are not surprising: a negative bias pertains, for example, to areas from mathematical physics or economy, which is plausible since many papers, though mathematically relevant, will rather appear in physics or economics journals. Due to this effect, MSC classes > 65 will usually be less represented in generalist mathematics journals.

So, what might be a reasonable criterion based on the MSC for labelling a mathematics journal “general”? This is not so easy to answer: many classes of the MSC differ significantly in size and intersections with other classes, and, more importantly, the tree structure of the MSC is only a very coarse approximation to the structure of mathematics. So for our study, we formed 10 MSC clusters, taking into account significant overlaps among the classes, and formulated a criterion based on a (rather low) minimal occurrence for each cluster.

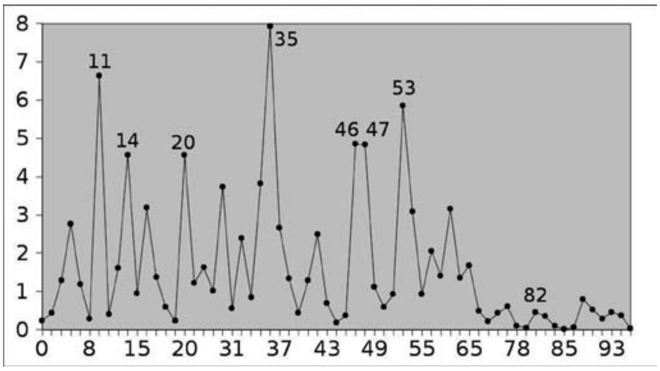
By applying this to 1768 current journals for the articles published in the years 2000–2014 (a period in which the main MSC classes have been stable), this criterion turned out to be quite selective: only 99 journals would be considered “generalist” in terms of our criterion. (For comparison: there are twice as many journals which satisfy the weaker criterion which allows a journal to stay below the required minimum in one but arbitrary MSC-cluster.) As expected, most of these journals are very well known and have a long tradition. But, for instance, even the *Crelle Journal* missed our criterion in two clusters: it

[1] The MSC is a classification scheme maintained by Mathematical Reviews and zbMATH and is used to categorise publications in mathematics. For more details see, for example, http://zbmath.org/classification/.

[2] J. Grcar, Topical bias in generalist mathematics journals, *Notices Am. Math. Soc.* 57, No. 11, 1421–1429 (2010).

contains just too few papers in differential equations and probability/statistics. So, while being sufficiently general in our definition is not a criterion of quality, it seems to be harder to fulfill than we initially expected.

For our set of general mathematics journals we wanted to find out the deviation in the MSC classes from the average in this set. Of course, this average is more strongly influenced by journals with many (but possibly shorter) articles but still differs significantly from the overall zbMATH distribution.



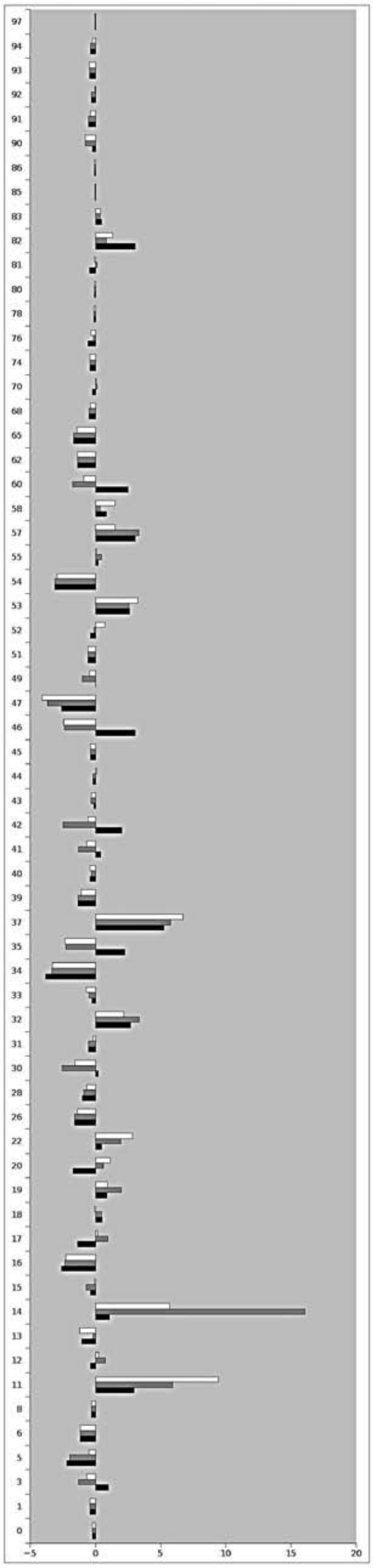
Percentage of articles in generalist journals with respect to their primary classification (in terms of the MSC). The heavyweight subjects are partial differential equations (35), number theory (11), differential geometry (53), functional analysis (46), operator theory (47), algebraic geometry (14) and group theory (20), while numerical mathematics (65), computer science (68), statistics (62), physics (70–86) and economics (91) are much less represented in this set of journals, in contrast to the overall distribution in zbMATH.

Among our generalist journals, the *Mathematical Notes/ Matematicheskie Zametki* is closest to the average, the *Proceedings and the Transactions of the AMS* follow at ranks 3 and 4, respectively. For these two journals Grcar showed in the aforementioned study that their MSC-distribution is rather untypical for the overall average of zbMATH, indicating that the 'subject bias' from the full zbMATH database is rather inherent in the generalist approach.

We evaluated the MSC distribution for some of the famous journals more closely. The comparison of the *Annals of Mathematics* (white), *Inventiones Mathematicae* (grey) and *Acta Mathematica* (black) in the figure below (see right column) is a good example of the overall picture: subjects with recent breakthroughs have the largest positive bias, for the *Annals* in number theory and dynamical systems and for *Inventiones* in algebraic geometry and number theory. Numerically, *Acta* is closest to the average distribution, though it is sometimes considered as a journal with a focus on analysis.

An extreme case is *Mathematika* which has the largest distance to the average among our generalist journals. A large contribution to its high distance value comes from its strength in convex geometry, which distinguishes it from the average profile. So, while being generalist is certainly a distinctive feature of a journal, the deviation provides a quite typical footprint.

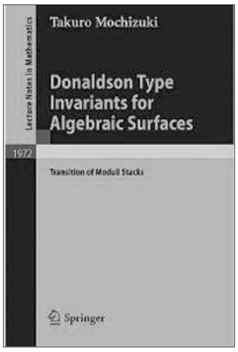
As a final observation, we would like to remark that only very few open access journals (even with a policy of general scope) can yet be characterised as generalist. The main factor seems to be the age: journals usually need some time to grow into a broad coverage of mathematics. However, those that qualify as generalist tend to be rather close to the average. Whether and how more detailed information could be read off from journal profiles would be the subject of a following (possibly larger) contribution.



Subject bias of *Acta* (black), *Annals* (white) and *Inventiones* (grey) from the average among the generalist mathematics journals.

For photos and short CVs of the authors please see EMS Newsletter issue 88.

Book Reviews



Takuro Mochizuki

Donaldson Invariants for Algebraic Surfaces

Springer-Verlag Berlin-Heidelberg
Year ?
ISBN 978-3-540-93912-2

Reviewer: Marina Logares

This monograph defines and studies an algebro-geometric version of Donaldson invariants, by using moduli stacks of semistable sheaves with arbitrary ranks on a polarised projective surface.

Recall that Donaldson invariants were successfully applied in the study of low dimensional differential topology until the appearance of Seiberg-Witten invariants.

Although Seiberg-Witten invariants seem to give the same information in almost all cases and they are easier to compute, the author chooses to generalise Donaldson invariants because he seeks an application to problems of universal relations among invariants which are generalisations of the "Kotschick-Morgan conjecture" and the "Witten conjecture". Both conjectures claim the existence of universal relations among invariants.

Kotschick and Morgan proved that the Donaldson invariant, $\Phi_d^{X,g}$, for X a compact, simply-connected, oriented, real, 4-dimensional C^∞-manifold with a Riemannian metric g and d half the dimension of the moduli space of antiself dual connections associated to X and g, depends only on a chamber defined by the metric g. They also conjecture that the difference $\Phi_d^{X,g_1} - \Phi_d^{X,g_2}$ is a polynomial with coefficients depending only on certain degree 2 cohomology of X and its homotopy type. In [2] Göttsche wrote a "wall crossing formula" for $\Phi_d^{X,g_1} - \Phi_d^{X,g_2}$ using modular forms. In [4] Witten conjectured that Donaldson and Seiberg-Witten invariants are related by an explicit formula (see also [1]).

The aim of the author with this monograph is to settle a basis for understanding the universal relations behind these conjectures.

The author solves the following problems. Let $\mathcal{M}^{ss}(y)$ be the moduli stack of semistable torsion free sheaves of type y, where *type* denotes the set of cohomology classes obtained as Chern character of some torsion free sheaves on X. The first problem consists of generalising the construction of Donaldson type invariants using $\mathcal{M}^{ss}(y)$ from rank 2 to higher ranks. But there are two main issues: we should take integrals over $\mathcal{M}^{ss}(y)$ but for higher ranks the moduli stack is not Deligne-Mumford and for Artin stacks there is no well established theory of 0-cycles. Also $\mathcal{M}^{ss}(y)$ may not be pure dimensional and hence it may not have fundamental classes. The author solves these problems by considering some related moduli stacks like the moduli stack of parabolic L-Bradlow pairs, constructing an obstruction theory for them and using the so-called virtual fundamental classes already used in Gromov-Witten theory.

After constructing the invariants, the author focuses on the universal relations among invariants and the geometry behind them. So, the next problem consists of clarifying the dependence on the polarisation of X and expressing the generalised Donaldson invariant as the sum of integrals over the products of moduli spaces of objects of rank one. The author uses enhanced master spaces to obtain the transition formulas.

Notice that Joyce in [3] has developed a theory of invariants that can be applied to the category of coherent sheaves on a smooth projective surface whose anti-canonical sheaf is nef. Joyce also studies wall-crossing phenomena of the invariants for a variation of semi stability conditions. But it seems that Joyce's invariant is different from the one given by Mochizuki since the first is "motivic".

The book is structured in the following way. The first chapter provides the reader with a thorough introduction, where the main goals are explained. The second chapter reviews some results from geometric invariant theory, Deligne-Mumford stacks, etc., all of them with examples to be used in the rest of the book. The third chapter recalls definitions of some structures on torsion free sheaves, such as orientation, parabolic structure, L-section and reduced L-section. The relation between moduli stacks of oriented, reduced L-Bradlow pairs and unoriented, unreduced L-Bradlow pairs, and finally moduli stacks of semistable sheaves is also shown. Then, after recalling the concepts of Hilbert polynomials for torsion free sheaves with additional structure, stability, Harder-Narasimhan and Jordan-Höoder filtration, the author reviews the boundedness of δ-semistable L-Bradlow pairs when δ is varied, and finishes by recalling 1-stability and 2-stability conditions and moduli schemes of quotient sheaves with some additional structures. The fourth chapter recalls the construction of moduli stacks by using geometric invariant theory. There, the author constructs an enhanced master space and shows that it is Deligne-Mumford and proper, and describes the fixed point set with respect to a natural torus action. This fixed point set, when the oriented sheaves are considered, is isomorphic to the product of moduli stacks of objects with lower ranks. These results form a focus of the book together with the construction of obstruction theories for parabolic L-Bradlow pairs and related moduli stacks developed in chapter five. Obstruction theories for master spaces are also discussed. The obstruction theories of chapter five are shown to be perfect and have virtual fundamental classes in chapter six. This solves the problem about the integration process to obtain the invariant. In this chapter some relations are given of the virtual fundamental classes for some moduli stacks. Chapter seven is where the invariants are constructed and their transition functions studied. The ground field in this chapter is assumed to be the complex numbers for simplicity. Transition functions are given for simple cases and they are used to construct the invariants. Some related weak wall crossing formulas for rank 2 and 3 are also given.

From my point of view this is a foundational book for those who are interested on invariants for projective surfaces but also for those interested in the obstruction theory for moduli stacks of sheaves on X with some extra structure.

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[2] L. Göttsche, Modular forms and Donaldson invariants for 4-manifolds with $b_+ = 1$. *J. Amer. Math. Soc.* 9 (1996), 827–843.

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[4] E. Witten, Monopoles and 4-manifolds. *Math. Res. Lett.* 1 (1994), 769–796.

Marina Logares obtained a PhD at Universidad Autónoma de Madrid in 2006. She spent a year from 2006 to 2007 working as a postdoctoral researcher at Max Plank Institut für Mathematik Bonn, Germany, and then moved to the Centro de Matemática da Universidade do Porto, Portugal, with a postdoctoral contract from 2007 till the end of 2009. Early in 2010 she joined the "Instituto de Ciencias Matemáticas" (ICMAT) as a postdoctoral researcher and is an honorary professor at the mathematics department of the "Universidad Autonoma de Madrid".

R. Natalini and A. Piazzi, Editors

Comics & Science

CNR (Consiglio Nazionale delle Ricerche), 2013

ISBN 9788888485003

Reviewer: P. Vannucci

Comics & Science' is a booklet published by CNR (*Consiglio Nazionale delle Ricerche*, the Italian National Centre for Scientific Research), or more precisely by the *Istituto per le Applicazioni del Calcolo "M. Picone"* (editors R. Natalini and A. Piazzi).

The publication is intended to build a bridge between science and entertainment, and to make amazing science popularisation. The main content of this journal is *Misterius*, a comic strip by Leo Ortolani, the famous Italian cartoonist. *Misterius* is a parody of a typical TV programme of science popularisation and the whole strip is an exhilarating series of strange situations where the main character, a sort of TV scientific anchorman, tries to explain some scientific mysteries: do there exist some numbers that are more important than others? Yes – for instance, the phone number of Monica Bellucci. Meanwhile, Turing's machine is a car used by the celebrated mathematician to get around and so on.

Going on, the reader is sarcastically introduced to mathematics, discovering the life of an unlikely mathematician Jean-Pierre Bagolot (whose main discovery is *frazzo*, a number between 6 and 7) throughout the early 20th century of science, encountering Fermi, Gödel and so on.

The next mathematician to appear is not invented but real: the Fields Medallist Cédric Villani, teased by the author because of his peculiar and very personal look. The story goes on with a parody on encyclopedia and with a lot of humorous remarks, unfortunately understandable only to native Italians.

Besides the strip, the booklet also contains three articles. The first is an interview with the author Leo Ortolani. We discover that the strip is the result of a meeting between the author and some scientists; the idea was to propose something entertaining, funny and new for science popularisation; why not comics?

The result is a little bit ambiguous: the strip is really very amazing, at least for Italians, because it is permeated with a lot of references to Italian (sub-)culture. Nevertheless, to be precise, it is surely not science popularisation but probably, and more properly, science parody, and somewhat also science denigration.

Actually, the exhilarating interview with the author lets the reader understand that this strip is a sort of cold revenge of the author against the (I presume) scientific nightmares of his youth (he confesses to be a geologist). The result is somewhat grotesque, exaggerated and voluntarily sardonic. Clearly, the author does not make the effort to enter, in some way or other, into the science; he wants to remain a profane (in the etymological sense of the word) of science and prefers to delight himself, and the reader, with parody.

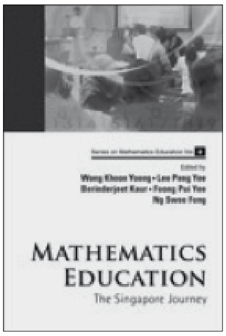
The second article, from Cédric Villani, previously appeared in the French *La gazette des mathématiciens*. It deals less with mathematics popularisation than with the way non-scientific papers and interviews should be done, to preserve the author from unpleasant surprises. The celebrated French mathematician seems to appreciate the nickname *Lady Gaga of mathematics* and assumes completely the role of herald of *mathemediatics*. His point of view is that the look of mathematics needs to be positive.

The main trouble of mathematics today is, according to Villani, the bad reputation that the discipline has with young people, along with the rapid loss of students and teachers of mathematics. He also gives a necessary (but not sufficient) condition to stopping the decline: to talk more and more publicly of science, in a word, to establish direct and stable contacts with the public and the media. The question of how to popularise science remains and is not really approached. The strip *Misterius* is probably a good way to laugh at science but some doubts remain on whether it is a good way to attract young people to science.

The last article is a semi-serious, short history of the Fields Medal, written by Stefano Pisani, the Editor in Chief of the website *Maddmaths!: MAtematica Divulgazione Didattica*.

This short booklet was first presented during the *Festival della Scienza* in Genova, on 26 October, and then during a public debate on comics and science, on 2 November, organised in the context of *Lucca Comics and Games*, the historical and international event on comics and games held annually in Lucca, Tuscany. Leo Ortolani contributed to the debate but he could not, on that occasion, meet Cédric Villani, who was invited to the same festival to talk on the same subject but on the day before.

Paolo Vannucci is a professor of mechanics at the University of Versailles and a member of the LMV (Laboratoire de Mathématiques de Versailles). His main research activities concern the use of tensor invariants in plane anisotropy, the optimal design of structures and methaeuristics for optimisation.



W. K. Yoong, L. P. Yee, B. Kaur, F. P. Yee and N. S. Fong (eds.)

Mathematics Education: The Singapore Journey

World Scientific, Singapore
2009
ISBN: 9789812833754

Reviewer: Mariolina Bartolini Bussi (Member of the Editorial Board)

Reviews from the series on Mathematics Education of the World Scientific Publishing Co. (Singapore) began in issue 89, offering the European readership a detailed introduction to mathematics education in some countries that are not so well known to Western researchers. The reviews included:

- Issue 89: two books about Chinese Mathematics Education (Fan & al., 2004, and Li & Huang, 2013).
- Issue 90: two books about Russian Mathematics Education (Karp & Vogeli, 2010–11).

Now it is the turn of the book edited by W. K. Yoong, L. P. Yee, B. Kaur, F. P. Yee and N. S. Fong in 2009 about *Mathematics Education: The Singapore Journey*. A further book on Singaporean mathematics education (on Mathematical Modelling) is forthcoming.

The subtitle is explained by the editorial team on page 525. From the start, they use a geographical metaphor about the Landscape of the Singapore Mathematics Education Journey (p. 1). The final chapter (Looking Forward and Beyond) begins with a quotation from Lao Zi: '*A journey of a thousand miles begins from beneath one's feet*', and another quotation from Withman hinting at the risks of every uncharted journey (*A Passage to India*). Hence the book aims to depict a partial history of Singaporean mathematics education. Singapore became well known internationally in the mathematics education community after high achievement in the Third International Mathematics and Science Study (TIMSS[1] in 1995, 1999 and 2003), concerning students from the 4th grade and the 8th grade. Singapore's textbooks were also introduced into the US in order to improve primary mathematics education (Beckmann, 2004). This introduction was facilitated by the fact that the original textbooks had been produced and published in English, as English is the school language. This is the first feature of the Singaporean education system that deserves some comment. Singapore is a small country of 700 square kilometres. The population in 2006 was 4.48 million. Singaporean citizens and permanent residents make up 80% of the population, the rest being foreign workers. The profile of the local population by descent is: Chinese (75.2%), Malays (13.6%), Indians (8.8%) and others (2.4%). The government has adopted a bilingual policy: English is the language of administration and of school, hence all the academic subjects are taught in English from primary to tertiary level, whilst the mother tongue is a compulsory subject. Most Singaporeans are now bilingual in English

1 http://timss.bc.edu and, for a short summary of data, http://en.wikipedia.org/wiki/Trends_in_International_Mathematics_and_Science_Study.

and their mother tongue. The difference between school and home language for mathematics proficiency might be addressed as a research question but it is not considered here (Kaur, personal communication).

The Singapore Mathematics Education Journey contains 23 chapters, an introduction and concluding remarks. The book is structured in three parts: the first part concerns Singapore's Education and Mathematics Teacher Education; the second part concerns some specific topics in the Teaching and Learning of Mathematics; and the third part concerns the role of Singaporean students in International Comparative Studies in Mathematics Education. The following will focus on some specific chapters which address issues that are supposed to be typical of the Singaporean education system.

First Part

In Chapter 1, a short overview of the Singaporean mathematics curriculum is presented. Singapore has a 6-2-2 structure with six years of primary school (starting when children are 6 years old), two years of lower secondary school and two years of upper secondary school, with pre-school education comprising nursery at age 3 and kindergartens one and two at ages 4 and 5. The focus of the Singaporean mathematics curriculum is summarised by the so-called "Pentagon framework", which has been used with minor changes since 1990 (p. 33).

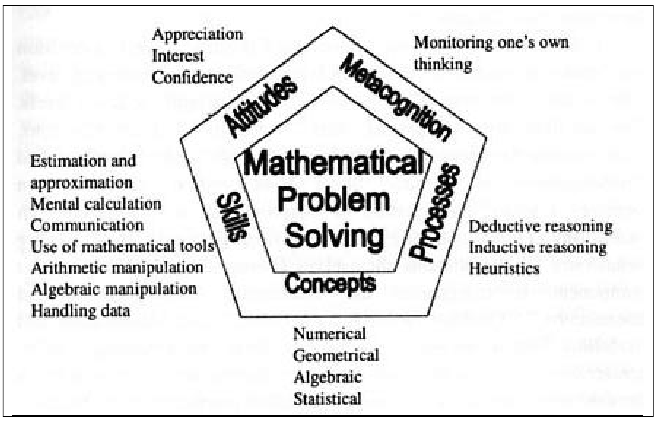


Figure 1: The Pentagon framework

Mathematical problem solving is at the core and is defined inclusively to "cover a wide range of situations from routine mathematical problems to open-ended investigations that make use of relevant mathematics" (p. 35). In particular, the so-called model method was introduced as a "Singapore creation" to help young children (from the 1st grade of primary school) to solve challenging problems without algebra. This model method will be covered in more depth below, as it appears in several chapters in order to show, on the one hand, that it is very useful in developing algebraic thinking from the very beginning of primary school and, on the other hand, that it may be an obstacle towards the learning of standards of algebraic symbolism in secondary school. This method, described below, is credited to Professor Kho, who introduced it in the 1980s, but, as shall be argued later, it can also be found in Western contemporary texts, although it is not certain whether they were known in Singapore or not.

The system of Mathematics Teacher Education and Development is described in five chapters which address different issues.

Chapter 2 concerns the general presentation of Pre-Service and In-Service Programmes. Some specific issues, which characterise Singaporean in-service education are then carefully illustrated: the creation of *Mathematics Centres of Excellence* around which school clusters and zones are organised, to promote mathematics learning among teachers (Chapter 3); the adoption of the methodology of Lesson Study, following the Japanese model (Chapter 4, see below); an experimental study on teacher change within a large research project which investigated the effects of using word problems that require students to engage in sense-making (Chapter 5); and the description of *Master Teachers*, who are role models of teaching excellence and are officially appointed, through an accreditation process, that draws on strong pedagogical knowledge they have demonstrated over the years.

All the chapters contribute to give the idea of a complex system where the implementation and the success of the mathematics curriculum are supported by strong programmes for teacher development. Due to space constraints, I shall only give some details on the methodology of *Lesson Study*, which is popular, with some differences dependent upon cultural issues, in many countries but, above all, in the Confucian heritage area in the Far East. In particular, the Asia-Pacific Economic Cooperation (APEC) Human Resource Working Group (HRDWG) adopted a five year Lesson Study project in mathematics from 2006 on the proposal of Thailand and Japan for stimulating mathematics and science education. I personally took part in one of the annual meetings of the APEC project in Tokyo (2010)[2] with the presentation to hundreds of interested teachers of some exemplary research lessons in either classrooms or a large auditorium. Hence I had a personal experience of this teacher-directed form of teacher development. In the introduction to Chapter 4, a short history of Lesson Study is presented from an international perspective (see also Isoda et al., 2007, and Imprasitha et al., forthcoming). The introduction reads (with several references omitted here):

> *Lesson Study is a cycle of activities in which teachers design, implement and improve one or more research lessons and make positive changes in instructional practice and student learning. From the beginning of this cycle, Lesson Study groups set long and short-term goals, plan and conduct lessons over a period that can last beyond a year. Planning is done collaboratively. One of the teachers carries the planned lesson out with team members observing the lesson and taking careful notes on how pupils respond to the lesson and how well they learn. Observers and the team of teachers then meet to review the evidence gathered during the lesson, discuss it, reflect upon ways to improve the lesson, revise it, and then teach it in another class. Obser-*

2 http://www.criced.tsukuba.ac.jp/math/apec/apec2009/index_en.php.

vations and findings are documented and shared with other teachers. For over five decades in Japan, teachers have used Lesson Study […] to improve their practices. […]. In recent years, Lesson Study has been introduced to many states in the U.S., implemented widely in Hong Kong in the form of Learning Study, in China in the form of Action Education, and in many APEC member countries (p. 105).

In Europe a form of Lesson Study (called Learning Study as it is more focused on students' learning) has been implemented in Sweden, following Marton's research (Kullberg, 2010). From an international perspective, it is evident that all these realisations are biased by cultural context, including the features of the intended curriculum in a given country and the system of beliefs of teachers, students, parents and the wider society. This influence is highlighted in the book in the few lines below:

[the implementations in those countries] undergo a contextualizing and adapting process to fit the local contexts and to address inherent problems of education beyond borders. This adaptation process provides an important site of inquiry for education researchers to test the vitality of Lesson Study and understand what conditions and factors lead to successful implementation in different educational systems (p. 105).

An important difference that is expected to inhibit an easy application of Lesson Study in the West is related to the teachers' belief that their own classroom is a "private" zone, where the freedom of lesson design, implementation and analysis is given to teachers and where privacy is a major value. This resistance is reinforced, in some cases, by institutional constraints. In Italy, for instance, it is not so easy to visit a classroom, especially if the number of visitors is more than one. Safety reasons (e.g. the numbers of people allowed to stay in a class according to the size, the number of doors, etc.) are used by principals to forbid entrance and the admission of student teachers during practical work is also regulated by specific and complex agreements. No photos or video are allowed and copies of students' protocols are limited. The difference in privacy protection of both teachers and students was made evident in a recent International Conference on Mathematics Education (ICME 12 in Seoul, 2012). During a plenary panel on *Math education in East Asia (Korea-China-Japan)*, chaired by Frederick Leung (see issue 85 of this newsletter), two video clips of mathematics classrooms were shown on the same slide, one from a country from the Far East and another from the US. The former was "normal" whilst in the latter the students' faces were hidden by pixellation. The side-by-side comparison between the video clips was more efficient for the audience than a long speech.[3]

[3] See the report on *The Teacher Development Continuum in the United States and China* (2010) at http://www.nap.edu/catalog.php?record_id=12874.

The Singaporean realisation of Lesson Study, described in Chapter 4, draws on Vygotsky, Leont'ev and Engestroem's cultural-historical activity theory and on Wenger's notion of community of practice. The former gives a general framework, usually represented by the triangle composite model of the human activity system (p. 107).

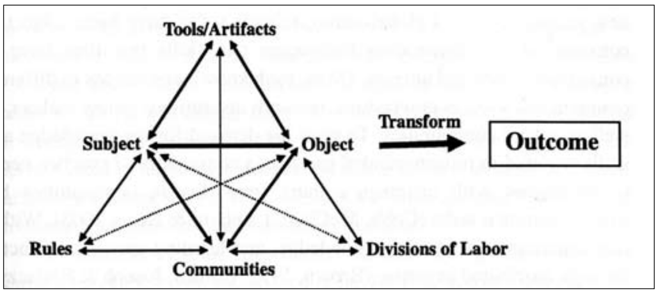


Figure 2: The human activity system

In the activity system, the human subject's (can be an individual or a group) actions and activities are mediated by both tools and artifacts (represented in the upper triangle in the above figure) and communities of practices (represented in the lower triangle) at the same time. Therefore these actions and activities are also mediated by rules and norms governing the communities and the participation of the members is mediated by division of labour. In the process of participation, subjects are transformed and such transformation represents learning from social practice or praxis (p. 107).

This framework is used by the authors to describe Singapore's journey through Lesson Study and is applied to the discussion of three case studies concerning *long division with remainder* (involving six teachers from grade 1 to grade 3), the *areas of rectangles* (involving five teachers from grade 4) and *equivalent fractions* (involving 18 teachers from different grades up to grade 5). For each experiment, several data are briefly reported and discussed concerning planning sessions, research lessons and post-lesson discussions. In some cases there are also further cycles where the research lesson is repeated after post-lesson discussions.

This short summary may give the impression that only primary school is addressed in Lesson Study. This might be true in some countries, as the effort to change and improve instruction usually takes place from the beginning of school entrance. In Japan, however, where the tradition of Lesson Studies is more ancient, several experiments are also carried out in secondary school.

The usefulness of Lesson Study in Singapore is emphasised by the authors, who discuss the problems of the increasing number of teachers retiring from the service and the younger age and lesser experience of the teaching force. Lesson Study has become a promising pathway for effective induction of new teachers into the teaching profession. The authors also comment on the length of the process of Lesson Study (with cycles lasting at least one year and often more): "the slow pace can often discourage both practitioners from action and policy-makers from funding such work (p. 127)". Yet drawing on the case studies, they insist on the advantages of a slow pace when huge challenges are facing educators. This is an important lesson from a country that has reached the highest rates in international assessment and, nevertheless, is acting in order to improve the education system more and more.

Second Part

The second part of the volume concerns some particular topics of the Singaporean mathematics curriculum. The model method (already mentioned above) is the focus of several chapters (Chapter 7 concerns the model method as a visual tool to support algebraic word problem solving at the primary level; Chapter 8 concerns the roles of working memory and the model method; Chapter 10 concerns the model method for multiplicative word problems on speed; and Chapter 11 offers a review of research on mathematical problem solving in Singapore, including a part on the model method); hence four chapters out of twelve focus on the model method. This interest is consistent with the great number of papers on the model method, published in the most important research journal on mathematics education in Singapore: *The Mathematics Educator*[4] (published since 1996). In short, the model method is introduced from the beginning of primary school to represent data and unknowns in word problems, to solve in an effective way early arithmetic and algebraic problems.

> *A very simplistic description of the model method is that a series of rectangles is used as external and hence visual representation of the information presented in word problems. The quantity and qualitative relationship embodied by the rectangles are then represented by a series of arithmetic equations. Hence the model method requires students to work with three different modes of representation. (i) text, (ii) pictorial, and (iii) numerical expressions. Translations are between these modes, text to pictorial to numerical, although the order may not be in this sequence (p. 172).*

A mathematics teacher from Singapore (Mr Koh) is credited to have introduced the model method in the early 1980s.

> *When they use the model method, young students' impulsion to compute is effectively curtailed as they are required to process the text based information and think of how to represent pictorially the quantitative and qualitative information (p. 173).*

Mr Koh introduced this method in Singapore but the model method was known in other countries too (Schmittau & Morris, 2004, and see also the review of Russian Mathematics Education prepared by Bartolini Bussi in issue 90 of this newsletter, p. 53 ff.). Moreover, the idea of using geometric methods to solve algebraic problems is very ancient and dates back to Euclid, who, in the second book of *Elements*, solved problems, which today could be named algebraic, with recourse to figures. This tradition through Arabic mathematics was influent also in the West before the introduction of algebraic symbolism. See, for instance, Figure 3 for the solution of a quadratic equation by al-Khwārizmī.[5]

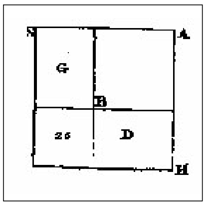


Figure 3: al-Khwārizmī's drawing

In some cases (e.g. China and Russia) the rectangles quickly evolve to become lines.

> *The model method is a visual and concrete representation of the text-based information. The structure of the model drawing also shows the link between the arithmetic expressions and the text-based information. The very physical nature of the model drawing means that specific text-based information can be pointed and hence manipulated (p. 179).*

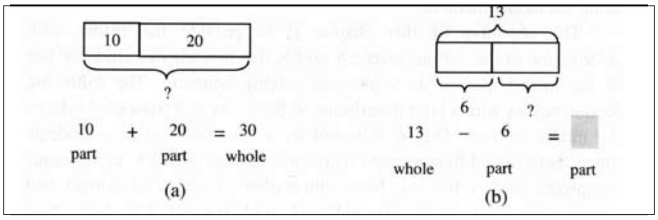


Figure 4: Additive problems (part-whole relationships, p. 174)

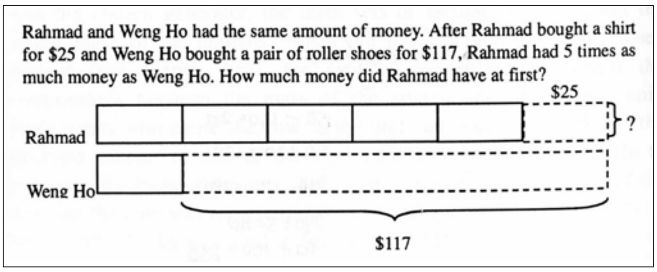


Figure 5: Multiplicative problems (p. 175)

The possibility of manipulating both data and unknowns is more abstract and difficult with algebraic symbolism. In Figure 5, it is evident that a new "unit" (for the amount of money) is introduced, i.e. the total amount of money held by Weng Ho. This attitude of manipulating units in a flexible way is a good way to be introduced to algebraic thinking. It is worthwhile to quote the app for tablet *Thinking Blocks*,[6] developed in the US but following the Singaporean curriculum, for introducing primary school students to several cases of the model method for solving word problems (see Figure 6).

The model method is really effective at primary level but may become an obstacle for the transition to meth-

[4] http://math.nie.edu.sg/ame/matheduc/.

[5] http://en.wikipedia.org/wiki/Muḥammad_ibn_Mūsā_al-Khwārizmī.

[6] http://www.mathplayground.com/thinkingblocks.html.

ods based on algebraic manipulation, hence for the transition to modern algebra.

Given an algebraic word problem, it is common practice among many secondary students to construct a model drawing and then its equivalent algebraic equation and then revert to arithmetic methods to evaluate the unknown unit, hence dispensing with the need to transform equations. Students asked whether it was necessary to use letter-symbolic algebra to solve word problems when the model method was just as effective (p. 198).

This issue is briefly discussed in Chapter 4 where the *Algebra Study* is introduced to look at "how to support students make the transition between model methods and letter-symbolic methods" (p. 198). A software tool has been designed to help students make the transition. Unfortunately, this tool is hosted on an online website with terms of use allowing access for Singaporean teachers and students only (Yen Yeen Peng, Singapore Ministry of Education, personal communication).

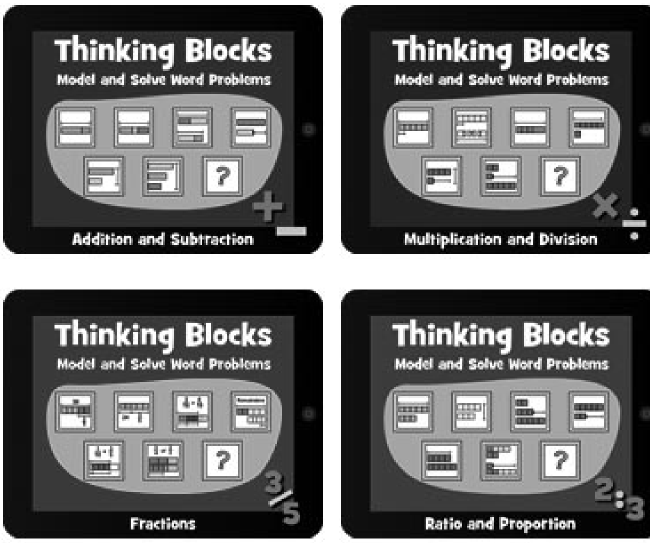


Figure 6: Thinking blocks

The other chapters of the second part address statistics and graphs (Chapter 9), ICT (Chapter 12), mathematics anxiety (Chapter 13), cooperative learning (Chapter 14), mathematics curriculum for the gifted (Chapter 15) and for low achievers (Chapter 16), kindergarten numeracy (Chapter 17) and the need for a new paradigm for mathematics assessment (Chapter 18). All these chapters offer interesting discussions about the issues at stake and witness the great efforts that a small state devote to the improvement of mathematics teaching at all levels, in spite of the already high performances in international assessment.

Third Part

The third part of the volume addresses *Comparative Studies in Mathematics Education*. Besides the well known TIMSS project (Chapters 19–20), other international projects are reported: the Kassel Project, concerning a longitudinal study of samples of pupils in 18 countries, including Singapore (Chapters 21–22); and a comparative study of perceptions *about my "best" mathematics teacher* of elementary school students from Singapore and Brunei Darussalam (Chapter 23).

No mention appears of the extraordinary results gained by Singapore in OECD-PISA[7] since 2009 (i.e. the year of entrance of Singapore in the assessment), as this book was printed in 2009, whilst the results of PISA 2009 were released at the beginning of 2013. It is worthwhile noting that in both the 2009 and 2012 PISAs, Singapore was second after Shanghai, showing that the very good performances in TIMSS have been confirmed in PISA.

At the end of this journey through Singaporean mathematics education, it is evident that Western mathematics educators have a lot to learn from this country. The solution, however, is not simply to translate or to use a textbook from Singaporean schools. As already emphasised in the review of Chinese Mathematics Education prepared by Bartolini Bussi (issue 89 of this newsletter, p. 60 ff.) it is important to exploit others' traditions to challenge our assumptions and to understand the principles of a working innovation. In the case of Singapore, I believe that the adoption of Lesson Study as a methodology for teacher development and the continuous search for flaws and possible improvement of innovation, in spite of the already excellent performances, is something to be learnt.

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[7] http://www.oecd.org/pisa/ and, for a summary, http://en.wikipedia.org/wiki/Programme_for_International_Student_Assessment.

Solved and Unsolved Problems

Themistocles M. Rassias (Athens, Greece)

In Mathematics the art of proposing a question must be held of higher value than solving it.
Georg Cantor (1845–1918)

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

123. Compute the limit

$$\lim_{n\to\infty} \frac{(n+1)^\alpha \ln^\beta(n+1) - n^\alpha \ln^\beta n}{n \ln n},$$

where $\alpha, \beta \in \mathbb{R}$.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania and Columbus State University, Georgia, USA*)

124. For a positive integer k, where $k \geq 2$, define the sequence

$$a_n^{(k)} = \sum_{j=0}^{k} (-1)^k \binom{k}{j} \sqrt{n+k-j}, \quad n = 1, 2, \ldots$$

Compute the limit $\lim_{n\to\infty} n^\alpha \cdot a_n^{(k)}$, where $\alpha \in \mathbb{R}$.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania and Columbus State University, Georgia, USA*)

125. (Torricelli-Steiner point in the metric space $C[0,1]$.)
Consider a triangle $f_1 f_2 f_3$ in the space of continuous functions $C[0,1]$ with vertices $f_1(x) = x$, $f_2(x) = x + 1$, $f_3(x) = \sin 2x$. Find a point $f \in C[0,1]$ for which the sum of distances to the vertices of the triangle is minimal.
(Let us recall that the distance between points f and g in the space $C[0,1]$ is defined by $\|f - g\| = \max_{x\in[0,1]} |f(x) - g(x)|$.)

(*Vladimir Protasov, Department of Mechanics and Mathematics, Moscow State University, Russia*)

126. Let $k \geq 1$ be an integer. Prove that

$$\sum_{i_1, i_2, \ldots, i_k = 1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k (i_1 + i_2 + \cdots + i_k)^2} = (-1)^k k! \sum_{n=k}^{\infty} (-1)^n \frac{s(n,k)}{n! \cdot n^2},$$

where $s(n,k)$ denotes the Stirling numbers of the first kind.
Deduce that

$$\sum_{i_1, i_2 = 1}^{\infty} \frac{1}{i_1 i_2 (i_1 + i_2)^2} = \frac{1}{2}\zeta(4).$$

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Romania*)

127. Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. Then for any $t, x \in [a,b]$ prove that

$$|tf(x) - xf(t)| \leq \begin{cases} \|f - \ell f'\|_\infty |x - t| & \text{if } f - \ell f' \in L_\infty[a,b], \\ \frac{1}{2q-1} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a,b],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, & \end{cases} \quad (1)$$

where $\ell(t) = t$, $t \in [a,b]$.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

128. Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b) with $b > a > 0$. Then for any $x \in [a,b]$, prove the inequality

$$\left| \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b \frac{f(t)}{t} dt \right| \leq \frac{2}{b-a} \|f - \ell f'\|_\infty \left(\ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x} \right), \quad (2)$$

where $\ell(t) = t$, $t \in [a,b]$. The constant 2 is the best possible.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

II Two new open problems

129*. For $\alpha > 0$; $m, k = 1, 2, 3, \ldots$, examine whether there exists a way to express

$$(\alpha k)_m = (\text{term independent of k})_k,$$

where the rising factorial is defined by

$$(n)_m = n(n+1)\ldots(n+m-1).$$

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

130*. Define

$$T_{n,m}(x) = (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_0^1 p_{n,k}(t) t^m dt, x \in [0,1],$$

where

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!} \binom{n}{k} (nx)_k (n - nx)_{n-k}, \quad p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k},$$

with

$$(x)_n = x(x+1)(x+2)\ldots(x+n-1).$$

Is it possible to have a recurrence relation between $T_{n,m+1}(x)$ and $T_{n,m}(x)$?

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

III Solutions

115. Prove that for every $x, y > 0$, the following inequality holds

$$x2^y + y2^{-x} > x + y.$$

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. If $x = y$, it follows that $2^x + 2^{-x} > 2$, which is true because $x > 0$. Assume that $x \neq y$, and let $y = x_1 - x_2, x = x_2 - x_3$ for some $x_1 > x_2 > x_3 > 0$. The inequality is equivalent to

$$\frac{2^{x_1-x_2} - 1}{1 - 2^{x_3-x_2}} > \frac{x_1 - x_2}{x_2 - x_3},$$

that is

$$\frac{2^{x_1} - 2^{x_2}}{x_1 - x_2} > \frac{2^{x_2} - 2^{x_3}}{x_2 - x_3}. \quad (3)$$

By the Mean Value theorem, there are $c_1 \in (x_2, x_1), c_2 \in (x_3, x_2)$ such that

$$\frac{2^{x_1} - 2^{x_2}}{x_1 - x_2} = 2^{c_1} \ln 2$$

and

$$\frac{2^{x_2} - 2^{x_3}}{x_2 - x_3} = 2^{c_2} \ln 2.$$

Because $c_1 > c_2$, we have $2^{c_1} > 2^{c_2}$, and (3) is proved. □

Also solved by Mihály Bencze (Brasov, Romania), Tim Cross (King Edward's School, Birmingham, UK), G. C. Greubel (Department of Physics, Old Dominion University, Newport News, VA, USA), Soon-Mo Jung (Chochiwon, Korea), P. T. Krasopoulos (Athens, Greece), Angel Praza (University of Las Palmas de Gran Canaria, Spain), Daniel Vacaru (Pitesti, Romania), Con Amore Problem Group (Department of Mathematics, Institute of Curriculum Research, Copenhagen, Denmark)

116. Prove that for every positive integer $n \geq 3$, the following inequality holds

$$n\sqrt[n]{n} + \frac{n+1}{\sqrt[n+1]{n+1}} > 2n + 1.$$

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. Consider the function $f : (0, \infty) \to \mathbb{R}$, defined by

$$f(t) = \sqrt[n]{t} - \sqrt[n+1]{t}.$$

We have

$$f'(t) = \frac{1}{n} t^{\frac{1}{n+1}-1} \left(t^{\frac{1}{n(n+1)}} - \frac{n}{n+1} \right).$$

Thus, if $t \geq (\frac{n}{n+1})^{n(n+1)}$ then $f'(t) \geq 0$. Therefore f is strictly increasing on the interval $[(\frac{n}{n+1})^{n(n+1)}, \infty)$ and hence for

$$x > y \geq \left(\frac{n}{n+1} \right)^{n(n+1)},$$

we have $f(x) > f(y)$. That is,

$$\sqrt[n]{x} + \sqrt[n+1]{y} > \sqrt[n]{y} + \sqrt[n+1]{x}. \quad (4)$$

Setting $x = n^{n+1}$ and $y = (n+1)^n$, we have $x > y$, since this inequality is equivalent to

$$\left(1 + \frac{1}{n}\right)^n \leq n,$$

which is clearly true because for every $n \geq 3$ we have

$$\left(1 + \frac{1}{n}\right)^n < e < n.$$

From relation (4) the desired inequality follows. □

Also solved by Mihály Bencze (Brasov, Romania), Soon-Mo Jung (Chochiwon, Korea), P. Krasopoulos (Athens, Greece), Angel Praza (University of Las Palmas de Gran Canaria, Spain), Daniel Vacaru (Pitesti, Romania)

117. Let f be a real-valued function defined on an open interval I of the real line. Prove or disprove the following statements:

(a) If for every $t \in I$ we have

$$\lim_{h \to 0} (f(t+h) - f(t-h)) = 0$$

then f is continuous on I.

(b) If for every $t \in I$ we have

$$\lim_{h \to 0} \frac{f(t+h) - f(t-h)}{h} = 0$$

then f is constant on I.

(c) If f is continuous on I and for every $t \in I$ we have

$$\lim_{h \to 0} \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} = 0$$

then f is a linear function.

(*Richard A. Zalik, Auburn University, USA*)

Solution by the proposer. Let c be an arbitrary but fixed point in I. The function $f(t)$, defined to be equal to 1 if $t \neq c$ and 0 if $t = c$, disproves both (a) and (b). On the other hand, (c) is correct. The left side of the displayed formula on (b) is called the first Schwarz derivative. The left side of the displayed formula on (c) is called the second Schwarz derivative: we shall denote it by $f^{(2)}$. Clearly every differentiable function has a first Schwarz derivative. However, the function $g(t)$ that equals $t \sin(1/t)$ if $t \neq 0$ and 0 if $t = 0$ is not differentiable at 0 but has a Schwarz derivative there.
Here is a proof of (c). It suffices to prove the assertion for any closed subinterval $[a, b]$ of I. Let

$$g(t) := f(t) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(t - a) \right]$$

and

$$p_\varepsilon(t) := g(t) + \varepsilon(t - a)(t - b),$$

where ε is arbitrary but fixed. By construction p_ε is continuous on $[a, b]$, vanishes at a and b, and its second Schwarz derivative equals 2ε on $[a, b]$. If p_ε were to have its maximum value at an interior point t_0 of $[a, b]$, this would readily imply that $p_\varepsilon^{(2)}(t_0) \leq 0$, which is a contradiction. It follows that $p_\varepsilon(t) \leq 0$ on $[a, b]$. Applying the same reasoning to the function

$$q_\varepsilon(t) := -g(t) + \varepsilon(t - a)(t - b)$$

yields $q_\varepsilon(t) \leq 0$ on $[a, b]$. Combining the inequalities we obtained for $p_\varepsilon(t)$ and $q_\varepsilon(t)$ we deduce that

$$|g(t)| \leq \varepsilon |(t - a)(t - b)|.$$

Since ε is arbitrary we conclude that $g(t) = 0$, and the assertion follows. □

Also solved by Mihály Bencze (Brasov, Romania), Soon-Mo Jung (Chochiwon, Korea)

118. Let $[a,b]$ be a closed bounded interval of the real line. Assume that f is a continuous function of bounded variation and that g is a strictly increasing continuous function, both defined on $[a,b]$. For $a \le \alpha < \beta \le b$, let $V(f,\alpha,\beta)$ denote the total variation of f on $[\alpha,\beta]$. Let $c \in [a,b]$ be arbitrary but fixed, and define $v(f,t)$ to equal $V(f,c,t)$ on $[c,b]$ and $-V(f,t,c)$ on $[a,c]$. Finally, let $q(t) := g(t) + v(f,t)$ and $h(t) := f[q^{-1}(t)]$. Prove that $h(t)$ is absolutely continuous on $[q(a),q(b)]$.

(*Richard A. Zalik, Auburn University, USA*)

Solution by the proposer. The hypotheses imply that $q(t)$ is strictly increasing and continuous on $[a,b]$; thus q^{-1} exists and is strictly increasing on $[q(a),q(b)]$. If $a \le s_1 < s_2 \le b$ then

$$|f(s_2) - f(s_1)| \le V(f,s_1,s_2)$$
$$= v(f,s_2) - v(f,s_1) \le v(f,s_2) - v(f,s_1) + g(s_2) - g(s_1) = q(s_2) - q(s_1).$$

Thus, if $(\alpha_1,\beta_1),(\alpha_2,\beta_2),\dots,(\alpha_n,\beta_n)$ are disjoint subintervals of $[q(a),q(b)]$, we have:

$$\begin{aligned}\sum_{i=1}^{n} |h(\beta_i) - h(\alpha_i)| &= \sum_{i=1}^{n} |f(q^{-1}(\beta_i)) - f(q^{-1}(\alpha_i)| \\ &\le \sum_{i=1}^{n} |q(q^{-1}(\beta_i)) - q(q^{-1}(\alpha_i)| \\ &= \sum_{i=1}^{n} (\beta_i - \alpha_i).\end{aligned}$$

□

Also solved by Mihály Bencze (Brasov, Romania), Soon-Mo Jung (Chochiwon, Korea)

119. Let $h : [0,\infty) \to \mathbb{R}$ be a continuous function and let $f : [0,\infty) \to \mathbb{R}$ be a twice differentiable function which satisfies the inequality

$$f''(x) - 5f'(x) + 6f(x) \ge h(x) \quad \text{for} \quad x \ge 0,$$

with initial conditions $f(0) = f'(0) = 0$.
Prove that

$$f(x) \ge \int_0^x \left(e^{3(x-t)} - e^{2(x-t)}\right) h(t)dt, \quad x \ge 0.$$

(*Ovidiu Furdui and Dorian Popa, Technical University of Cluj-Napoca, Romania*)

Solution by the proposer. Let

$$g(x) = f'(x) - 3f(x).$$

The differential inequality implies that $g'(x) - 2g(x) \ge h(x)$, which in turn implies that

$$](g(x)e^{-2x})' \ge \left(\int_0^x e^{-2t}h(t)dt\right)'.$$

It follows that the function

$$u(x) = g(x)e^{-2x} - \int_0^x e^{-2t}h(t)dt$$

is an increasing function on $[0,\infty)$. Thus

$$u(x) \ge u(0) = 0 \text{ for } x \ge 0.$$

This implies that

$$g(x)e^{-2x} - \int_0^x e^{-2t}h(t)dt \ge 0,$$

which yields that

$$f'(x) - 3f(x) \ge e^{2x}\int_0^x e^{-2t}h(t)dt.$$

Equivalently,

$$(f(x)e^{-3x})' \ge e^{-x}\int_0^x e^{-2t}h(t)dt$$

or

$$(f(x)e^{-3x})' - \left(\int_0^x e^{-t}\left(\int_0^t e^{-2y}h(y)dy\right)dt\right)' \ge 0.$$

Thus, the function

$$v(x) = f(x)e^{-3x} - \int_0^x e^{-t}\left(\int_0^t e^{-2y}h(y)dy\right)dt$$

is increasing on $[0,\infty)$. It follows that $v(x) \ge v(0)$ for $x \ge 0$, which implies that

$$f(x)e^{-3x} - \int_0^x e^{-t}\left(\int_0^t e^{-2y}h(y)dy\right)dt \ge 0.$$

This means that

$$f(x) \ge e^{3x}\int_0^x e^{-t}\left(\int_0^t e^{-2y}h(y)dy\right)dt. \tag{5}$$

We calculate the preceding integral by parts, with

$$u(t) = \int_0^t e^{-2y}h(y)dy, \quad u'(t) = e^{-2t}h(t),$$

$v'(t) = e^{-t}$ and $v(t) = -e^{-t}$, and we get that

$$\int_0^x e^{-t}\left(\int_0^t e^{-2y}h(y)dy\right)dt = -e^{-x}\int_0^x e^{-2y}h(y)dy + \int_0^x e^{-3t}h(t)dt. \tag{6}$$

Combining (5) and (6) we obtain that

$$f(x) \ge \int_0^x \left(e^{3(x-t)} - e^{2(x-t)}\right)h(t)dt, \ x \ge 0,$$

and the problem is solved. □

Also solved by G. C. Greubel (Department of Physics, Old Dominion University, Newport News, VA, USA), Soon-Mo Jung (Chochiwon, Korea)

120. Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Show that

$$\begin{aligned}&\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\ &\le \int_a^x \left(\bigvee_t^x (f)\right)dt + \int_x^b \left(\bigvee_x^t (f)\right)dt \\ &\le (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f) \\ &\le \begin{cases}\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a),\end{cases}\end{aligned} \tag{7}$$

for any $x \in [a,b]$, where $\bigvee_c^d (f)$ denotes the total variation of f on the interval $[c,d]$.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia, and University of the Witwatersrand, Johannesburg, South Africa*)

Solution by the proposer. We start with the following equality that can be easily proved integrating by parts:

$$f(x)(b-a)-\int_a^b f(t)\,dt=\int_a^x (t-a)\,df(t)+\int_x^b (t-b)\,df(t), \tag{8}$$

which holds for any $x \in [a,b]$ and $f : [a,b] \to \mathbb{C}$, a function of bounded variation on $[a,b]$.

Taking the modulus in (8) and using the property of the Riemann-Stieltjes integral of bounded variation integrators we have

$$\begin{aligned}&\left|f(x)(b-a)-\int_a^b f(t)\,dt\right| \qquad (9)\\ &\le \left|\int_a^x (t-a)\,df(t)\right|+\left|\int_x^b (t-b)\,df(t)\right|\\ &\le \int_a^x (t-a)\,d\left(\bigvee_a^t (f)\right)+\int_x^b (b-t)\,d\left(\bigvee_a^t (f)\right),\end{aligned}$$

for any $x \in [a,b]$.

Applying the integration by parts formula for the Riemann-Stieltjes integral, we also obtain

$$\begin{aligned}\int_a^x (t-a)\,d\left(\bigvee_a^t (f)\right) &= (t-a)\bigvee_a^t (f)\Big|_a^x-\int_a^x\left(\bigvee_a^t (f)\right)dt \qquad (10)\\ &=(x-a)\bigvee_a^x (f)-\int_a^x\left(\bigvee_a^t (f)\right)dt\\ &=\int_a^x\left(\bigvee_a^x (f)-\bigvee_a^t (f)\right)dt\\ &=\int_a^x\left(\bigvee_t^x (f)\right)dt\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)\,d\left(\bigvee_a^t (f)\right) &= (b-t)\bigvee_a^t (f)\Big|_x^b+\int_x^b\left(\bigvee_a^t (f)\right)dt \qquad (11)\\ &=\int_x^b\left(\bigvee_a^t (f)\right)dt-(b-x)\bigvee_a^x (f)\\ &=\int_x^b\left(\bigvee_a^t (f)-\bigvee_a^x (f)\right)dt\\ &=\int_x^b\left(\bigvee_x^t (f)\right)dt,\end{aligned}$$

for any $x \in [a,b]$.

Using (9)–(11) we deduce the first inequality in (7).

Since

$$\bigvee_t^x (f) \le \bigvee_a^x (f) \quad \text{for } t \in [a,x]$$

and

$$\bigvee_x^t (f) \le \bigvee_x^b (f) \quad \text{for } t \in [x,b],$$

it follows that

$$\begin{aligned}&\int_a^x\left(\bigvee_t^x (f)\right)dt+\int_x^b\left(\bigvee_x^t (f)\right)dt\\ &\le (x-a)\bigvee_a^x (f)+(b-x)\bigvee_x^b (f),\end{aligned}$$

for any $x \in [a,b]$, which proves the second inequality in (7).

The last part is obvious by the max properties and the fact that for $c,d \in \mathbb{R}$ we have

$$\max\{c,d\}=\frac{c+d+|c-d|}{2}.$$

The details are omitted. □

Also solved by Mihály Bencze (Brasov, Romania), Soon-Mo Jung (Chochiwon, Korea)

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR 15780, Athens, Greece, and by email to trassias@math.ntua.gr.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Geometry*.

Letter to the Editor

Ciro Ciliberto (Università di Roma "Tor Vergara", Rome, Italy)

Dear Editor,

I would like to draw the attention of the readers of the Newsletter of the EMS to what I think is malpractice which is more and more developing in Europe.

A few days ago a young (not Italian) mathematician, who applied for a Marie Curie fellowship to visit an Italian university, let me know the evaluation he had just received. This was quite high but I do not know whether it will suffice to get the fellowship. However, this is not the point. This young and brilliant colleague ended his message by forwarding to me, as an amusement, the following comments sent to him together with the evaluation:

> *"Weaknesses of the proposal: The inter and multidisciplinary connections are not adequately cited in the proposal. The proposal does not supply convincing arguments for timeliness. There are not adequate technological or socio-economic pieces of relevant information provided in the proposal."*

The above comments clearly refer to some of the points mentioned on page 52 of the guide for applicants in the call for applications of the Marie Curie Actions for 2013. It is stated there (the upper case is mine):

> *"IF RELEVANT, provide information on interdisciplinary / multidisciplinary and/or inter-sectoral aspects of the proposal."*

and

> *"Describe the SCIENTIFIC, technological, socio-economic OR other reasons for carrying out further research in the field covered by the project."*

The research project, which is in my opinion excellent, concerns various aspects of birational and enumerative geometry, which have always been, and still are nowadays, at the centre of the interest of algebraic geometers (like, for instance, the Riemann hypothesis is for number theorists: there are topics which never age). Therefore "timeliness", well elucidated in the project, should not have been under discussion in a competent evaluation.

The "inter and multidisciplinary connections" WITHIN mathematics were also well explained. These connections AWAY from mathematics were certainly not central, and I would say they were not RELEVANT in this case.

What really left me wordless was the final comment about the lack of "adequate technological or socio-economic pieces of relevant information provided in the proposal". Note that in the above statement I copied out, because of the presence of the "or", the "technological" and "socio-economic … reasons for carrying out further research in the field covered by the project" do not seem to be mandatory, given the presence of the SCIENTIFIC reasons well described and clearly prevalent in this case.

In any case, if weaknesses of a project in PURE MATHEMATICS are to be found either in a lack of "technological or socio-economic pieces of information", or in its SEEMING distance from applications and from immediate social usefulness, then it is clear that the future of ALL (not just pure) mathematics will be very dark and the suspicion of this rising trend is that it will eventually destroy the great tradition of basic, curiosity driven research, which lies at the foundation of the development of European science and speculation.

I think mathematicians, regardless of their specific fields of interest, should definitely fight against this trend, which is going to irreversibly jeopardise our discipline. Comments like those above should simply not be there!

Best regards,

Ciro Ciliberto
Professor of Higher Geometry at the
University of Roma Tor Vergara
President of the Italian Mathematical Union

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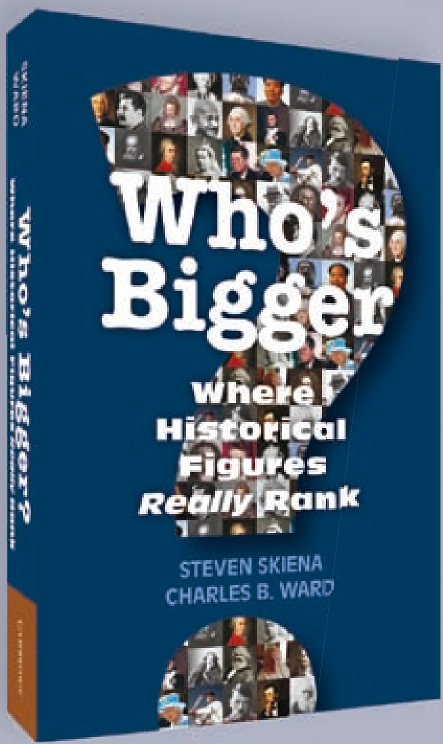
Who's Bigger?

Where Historical Figures Really Rank

Steven Skiena, State University of New York, Stony Brook
Charles B. Ward, Google, Inc., Mountain View, California

Hardback | 9781107041370 | December 2013 | £18.99



Is Hitler bigger than Napoleon? Washington bigger than Lincoln? Picasso bigger than Einstein? Quantitative analysts are rapidly finding homes in social and cultural domains, from finance to politics. What about history?

In this fascinating book, Steve Skiena and Charles Ward bring quantitative analysis to bear on ranking and comparing historical reputations. They evaluate each person by aggregating the traces of millions of opinions, just as Google ranks webpages. The book includes a technical discussion for readers interested in the details of the methods, but no mathematical or computational background is necessary to understand the rankings or conclusions. Along the way, the authors present the rankings of more than one thousand of history's most significant people in science, politics, entertainment, and all areas of human endeavor. Anyone interested in history or biography can see where their favorite figures place in the grand scheme of things.

Contents

Part I. Quantitative History: 1. History's most significant people; 2. Ranking historical figures; 3. Who belongs in Bonnie's textbook?; 4. Reading through the past; 5. Great Americans and the process of canonization; 6. The baseball hall of fame; 7. Historical time scales; Part II. Historical Rankings: 8. American political figures; 9. Modern world leaders; 10. Science and technology; 11. Religion and philosophy; 12. Sports; 13. The arts; 14. The performing arts; 15. Devils and angels; Part III. Appendices: A. Ranking methodology; B. Resources; C. Biographical dictionary.

'This is all fun: reputational face-offs are great entertainment. And, shrewdly, Skiena and Ward have an app. More seriously, historians will put quantitative analysis to good use - and their model may help historiographers grapple with Wikipedia.'
New Scientist

'I confess to simply liking the book. I still do not care about the great order of things; nonetheless, I very much appreciate a huge amount of fascinating detail that the book makes available at one's fingertips, and the orderly manner in which it does that.'
Alex Bogomolny, MAA Reviews

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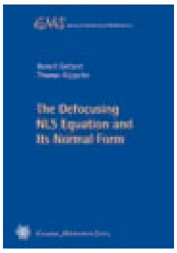
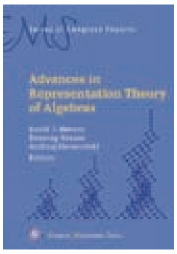


Benoît Grébert (Université de Nantes, France) and Thomas Kappeler (Universität Zürich, Switzerland)
The Defocusing NLS Equation and Its Normal Form (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-131-6. March 2014. 176 pages. Softcover. 17 x 24 cm. 32.00 Euro

The theme of this monograph is the nonlinear Schrödinger equation. This equation models slowly varying wave envelopes in dispersive media and arises in various physical systems such as water waves, plasma physics, solid state physics and nonlinear optics. More specifically, this book treats the defocusing nonlinear Schrödinger (dNLS) equation on the circle with a dynamical systems viewpoint. By developing the normal form theory it is shown that this equation is an integrable partial differential equation in the strongest possible sense. In particular, all solutions of the dNLS equation on the circle are periodic, quasi-periodic or almost-periodic in time and Hamiltonian perturbations of this equation can be studied near solutions far away from the equilibrium.
The book is not only intended for specialists working at the intersection of integrable PDEs and dynamical systems, but also for researchers farther away from these fields as well as for graduate students. It is written in a modular fashion, each of its chapters and appendices can be read independently of each other.



Advances in Representation Theory of Algebras (EMS Series of Congress Reports)
David J. Benson (University of Aberdeen, UK), Henning Krause (University of Bielefeld, Germany) and Andrzej Skowroński (Nicolaus Copernicus University, Toruń, Poland), Editors

ISBN 978-3-03719-125-5. 2014. 378 pages. Hardcover. 17 x 24 cm. 78.00 Euro

This volume presents a collection of articles devoted to representations of algebras and related topics. Dististinguished experts in this field presented their work at the International Conference on Representations of Algebras which took place 2012 in Bielefeld. Many of the expository surveys are included here. Researchers of representation theory will find in this volume interesting and stimulating contributions to the development of the subject.

François Labourie (Université Paris Sud, Orsay, France)
Lectures on Representations of Surface Groups (Zurich Lectures in Advanced Mathematics)
978-3-03719-127-9. 2013. 146 pages. Softcover. 17 x 24 cm. 32.00 Euro

The subject of these notes is the character variety of representations of a surface group in a Lie group. We emphasize the various points of view (combinatorial, differential, algebraic) and are interested in the description of its smooth points, symplectic structure, volume and connected components. We also show how a three manifold bounded by the surface leaves a trace in this character variety.
These notes were originally designed for students with only elementary knowledge of differential geometry and topology. In the first chapters, we do not insist in the details of the differential geometric constructions and refer to classical textbooks, while in the more advanced chapters proofs occasionally are provided only for special cases where they convey the flavor of the general arguments. These notes could also be used by researchers entering this fast expanding field as motivation for further studies proposed in a concluding paragraph of every chapter.

European Congress of Mathematics, Kraków, 2 – 7 July, 2012
Rafał Latała, Andrzej Ruciński, Paweł Strzelecki, Jacek Świątkowski, Dariusz Wrzosek and Piotr Zakrzewski, Editors

ISBN 978-3-03719-120-0. 2013. 824 pages. Hardcover. 16.5 x 23.5 cm. 108.00 Euro

The European Congress of Mathematics, held every four years, has become a well-established major international mathematical event. Following those in Paris (1992), Budapest (1996), Barcelona (2000), Stockholm (2004) and Amsterdam (2008), the Sixth European Congress of Mathematics (6ECM) took place in Kraków, Poland, July 2–7, 2012, with about 1000 participants from all over the world.
Ten plenary, thirty-three invited lectures and three special lectures formed the core of the program. As at all the previous EMS congresses, ten outstanding young mathematicians received the EMS prizes in recognition of their research achievements. In addition, two more prizes were awarded: the Felix Klein Prize for a remarkable solution of an industrial problem, and – for the first time – the Otto Neugebauer Prize for a highly original and influential piece of work in the history of mathematics. The program was complemented by twenty-four minisymposia with nearly 100 talks, spread over all areas of mathematics. Six panel discussions were organized, covering a variety of issues ranging from the financing of mathematical research to gender imbalance in mathematics.

Robert J. Marsh (University of Leeds, UK)
Lecture Notes on Cluster Algebras (Zurich Lectures in Advanced Mathematics)
978-3-03719-130-9. 2013. 132 pages. Softcover. 17 x 24 cm. 28.00 Euro

The aim of these notes is to give an introduction to cluster algebras which is accessible to graduate students or researchers interested in learning more about the field, while giving a taste of the wide connections between cluster algebras and other areas of mathematics.
The approach taken emphasizes combinatorial and geometric aspects of cluster algebras. Cluster algebras of finite type are classified by the Dynkin diagrams, so a short introduction to reflection groups is given in order to describe this and the corresponding generalized associahedra. A discussion of cluster algebra periodicity, which has a close relationship with discrete integrable systems, is included. The book ends with a description of the cluster algebras of finite mutation type and the cluster structure of the homogeneous coordinate ring of the Grassmannian, both of which have a beautiful description in terms of combinatorial geometry.

Isabelle Gallagher (Université Paris-Diderot, France), Laure Saint-Raymond (Université Pierre et Marie Curie, Paris) and Benjamin Texier (Université Paris-Diderot, France)
From Newton to Boltzmann: Hard Spheres and Short-range Potentials (Zurich Lectures in Advanced Mathematics)
978-3-03719-129-3. 2013. 150 pages. Softcover. 17 x 24 cm. 32.00 Euro

The question addressed in this monograph is the relationship between the time-reversible Newton dynamics for a system of particles interacting via elastic collisions, and the irreversible Boltzmann dynamics which gives a statistical description of the collision mechanism. Two types of elastic collisions are considered: hard spheres, and compactly supported potentials.
Following the steps suggested by Lanford in 1974, we describe the transition from Newton to Boltzmann by proving a rigorous convergence result in short time, as the number of particles tends to infinity and their size simultaneously goes to zero, in the Boltzmann-Grad scaling.
This book is intended for mathematicians working in the fields of partial differential equations and mathematical physics, and is accessible to graduate students with a background in analysis.

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