NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

Photograph taken by Håkon Mosvold Larsen/NTB scanpix

Interview
Yakov Sinai

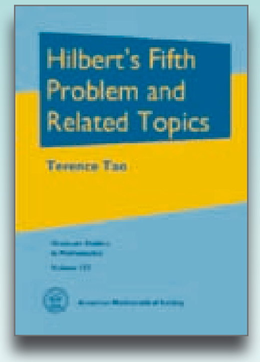
Features
Mathematical Billiards and Chaos
About ABC

Societies
The Catalan Mathematical Society

September 2014
Issue 93
ISSN 1027-488X

European Mathematical Society

American Mathematical Society



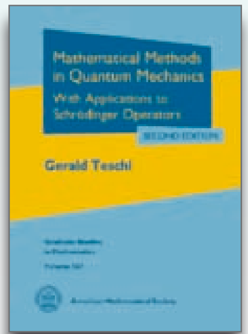
HILBERT'S FIFTH PROBLEM AND RELATED TOPICS

Terence Tao, *University of California*

In the fifth of his famous list of 23 problems, Hilbert asked if every topological group which was locally Euclidean was in fact a Lie group. Through the work of Gleason, Montgomery-Zippin, Yamabe, and others, this question was solved affirmatively. Subsequently, this structure theory was used to prove Gromov's theorem on groups of polynomial growth, and more recently in the work of Hrushovski, Breuillard, Green, and the author on the structure of approximate groups. In this graduate text, all of this material is presented in a unified manner.

Graduate Studies in Mathematics, Vol. 153

Aug 2014 338pp 9781470415648 Hardback €63.00



MATHEMATICAL METHODS IN QUANTUM MECHANICS

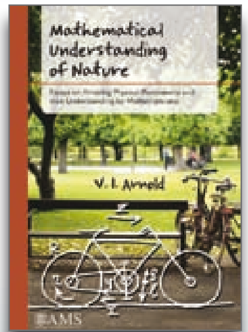
With Applications to Schrödinger Operators, Second Edition

Gerald Teschl, *University of Vienna*

Quantum mechanics and the theory of operators on Hilbert space have been deeply linked since their beginnings in the early twentieth century. States of a quantum system correspond to certain elements of the configuration space and observables correspond to certain operators on the space. This book is a brief, but self-contained, introduction to the mathematical methods of quantum mechanics, with a view towards applications to Schrödinger operators.

Graduate Studies in Mathematics, Vol. 157

Nov 2014 356pp 9781470417048 Hardback €61.00



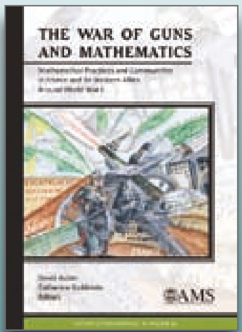
MATHEMATICAL UNDERSTANDING OF NATURE

Essays on Amazing Physical Phenomena and their Understanding by Mathematicians

V. I. Arnold

This collection of 39 short stories gives the reader a unique opportunity to take a look at the scientific philosophy of Vladimir Arnold, one of the most original contemporary researchers. Topics of the stories included range from astronomy, to mirages, to motion of glaciers, to geometry of mirrors and beyond. In each case Arnold's explanation is both deep and simple, which makes the book interesting and accessible to an extremely broad readership.

Oct 2014 167pp 9781470417017 Paperback €27.00



THE WAR OF GUNS AND MATHEMATICS

Mathematical Practices and Communities in France and Its Western Allies around World War I

Edited by David Aubin, *Sorbonne Universités, université Pierre et Marie Curie, Institut de mathématiques de Jussieu-Paris Rive Gauche* & Catherine Goldstein, *CNRS, Institut de mathématiques de Jussieu-Paris Rive Gauche*

For a long time, World War I has been shortchanged by the historiography of science. By focusing on a few key places (Paris, Cambridge, Rome, Chicago, Brno, and others), this book gathers studies representing a broad spectrum of positions adopted by mathematicians about the conflict and suggests a new vision of the long-term influence of World War I on mathematics and mathematicians.

History of Mathematics, Vol. 42

Nov 2014 424pp 9781470414696 Hardback €115.00

To order AMS titles visit **www.eurospanbookstore.com/ams**

CUSTOMER SERVICES:
Tel: +44 (0)1767 604972
Fax: +44 (0)1767 601640
Email: eurospan@turpin-distribution.com

FURTHER INFORMATION:
Tel: +44 (0)20 7240 0856
Fax: +44 (0)20 7379 0609
Email: info@eurospangroup.com

AMS
American Mathematical Society

distributed by **Eurospan** | group

European Mathematical Society

Newsletter No. 93, September 2014

Editorial: Opportunities and Challenges in Electronic Publication – *J. Rákosník & O. Teschke* ... 3
Meeting of Presidents in Istanbul – *S. Huggett* ... 6
EMS Council Meeting in Donostia/San Sebastián – *S. Huggett* ... 7
The Support Given by the ERC to Mathematical Research – *J.-P. Bourguignon* ... 9
The French Learned Societies and the EMS – *G. Allaire, A. Gégout-Petit & M. Peigné* ... 11
Interview with Yakov Sinai – Abel Laureate 2014 – *M. Raussen & C. Skau* ... 12
Speech in Honor of the Abel Prize Laureate – *J.-P. Bourguignon* .. 20
Mathematical Billiards and Chaos – *D. Szász* ... 22
Around ABC – *P. Mihăilescu* ... 29
An Exhibit of the Works of Siméon-Denis Poisson in Paris – *Y. Kosmann-Schwarzbach* ... 35
Stein Arild Strømme (1951–2014) – *S. Xambó* ... 37
The Gender Gap in Mathematics from the PISA Point of View – *E. Strickland* ... 45
The Catalan Mathematical Society and Mathematics in Catalonia – *J. Solà-Morales Rubió* ... 47
ICMI Column – *J.-L. Dorier* ... 49
The Klein Project – *B. Barton* ... 49
Solid Findings – *Education Committee of the EMS* ... 50
zbMATH Author Profiles: Open Up For User Participation – *H. Mihaljević-Brandt & N. Roy* ... 53
Book Reviews ... 56
Problem Corner – *T. M. Rassias* ... 59

The views expressed in this Newsletter are those of the authors and do not necessarily represent those of the EMS or the Editorial Team.

ISSN 1027-488X

Published by the
EMS Publishing House
ETH-Zentrum SEW A27
CH-8092 Zürich, Switzerland.
homepage: www.ems-ph.org

For advertisements and reprint permission requests contact: newsletter@ems-ph.org

EMS Executive Committee

President

Prof. Marta Sanz-Solé
(2011–2014)
University of Barcelona
Faculty of Mathematics
Gran Via de les Corts
Catalanes 585
E-08007 Barcelona, Spain
e-mail:ems-president@ub.edu

Vice-Presidents

Prof. Franco Brezzi
(2013–2016)
Istituto di Matematica Applicata
e Tecnologie Informatiche del
C.N.R.
via Ferrata 3
I-27100 Pavia, Italy
e-mail: brezzi@imati.cnr.it

Dr. Martin Raussen
(2013–2016)
Department of Mathematical
Sciences, Aalborg University
Fredrik Bajers Vej 7G
DK-9220 Aalborg Øst
Denmark
e-mail: raussen@math.aau.dk

Secretary

Dr. Stephen Huggett
(2007–2014)
School of Computing and
Mathematics
University of Plymouth
Plymouth PL4 8AA, UK
e-mail: s.huggett@plymouth.ac.uk

Treasurer

Prof. Jouko Väänänen
(2007–2014)
Department of Mathematics
and Statistics
Gustaf Hällströmin katu 2b
FIN-00014 University of Helsinki
Finland
e-mail: jouko.vaananen@helsinki.fi
and
Institute for Logic, Language
and Computation
University of Amsterdam
Plantage Muidergracht 24
NL-1018 TV Amsterdam
The Netherlands
e-mail: vaananen@science.uva.nl

Ordinary Members

Prof. Alice Fialowski
(2013–2016)
Eötvös Loránd University
Institute of Mathematics
Pázmány Péter sétány 1/C
H-1117 Budapest, Hungary
e-mail: fialowsk@cs.elte.hu

Prof. Gert-Martin Greuel
(2013–2016)
University of Kaiserslautern
Department of Mathematics
Erwin-Schroedinger Str.
D-67663 Kaiserslautern
Germany
e-mail: greuel@mathematik.uni-kl.de

Prof. Laurence Halpern
(2013–2016)
Laboratoire Analyse, Géométrie
& Applications
UMR 7539 CNRS
Université Paris 13
F-93430 Villetaneuse, France
e-mail: halpern@math.univ-paris13.fr

Prof. Volker Mehrmann
(2011–2014)
Institut für Mathematik
TU Berlin MA 4–5
Strasse des 17. Juni 136
D-10623 Berlin, Germany
e-mail: mehrmann@math.TU-Berlin.DE

Prof. Armen Sergeev
(2013–2016)
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8
119991 Moscow, Russia
e-mail: sergeev@mi.ras.ru

EMS Secretariat

Ms. Terhi Hautala
Department of Mathematics
and Statistics
P.O. Box 68
(Gustaf Hällströmin katu 2b)
FIN-00014 University of Helsinki
Finland
Tel: (+358)-9-191 51503
Fax: (+358)-9-191 51400
e-mail: ems-office@helsinki.fi
Web site: http://www.euro-math-soc.eu

EMS Publicity Officer

Dmitry Feichtner-Kozlov
FB3 Mathematik
University of Bremen
Postfach 330440
D-28334 Bremen, Germany
e-mail: dfk@math.uni-bremen.de

EMS Agenda

2014

24 October
Meeting of the Applied Mathematics Committee
London, UK
Contact: José Antonio Carrillo, carrillo@imperial.ac.uk,
Chair of the AMS Committee

21–23 November
Executive Committee of the EMS, Barcelona, Spain

EMS Scientific Events

2014

1–5 September
EMS Summer School in Applied Mathematics
Seventh European Summer School in Financial Mathematics
Oxford, UK
http://www2.maths.ox.ac.uk/euroschoolmathfi14/

1–5 September
EMS Summer School, Dyadic Harmonic Analysis and Related Topics, Santander, Spain
http://www.imus.us.es/SANTALO14/

5–6 September
Caucasian Mathematical Conference, Tbilisi, Georgia
Roland Duduchava: roldud@gmail.com (President of the Georgian Mathematical Society)
Armen Sergeev: sergeev@mi.ras.ru (Member of the Executive Committee of the EMS)

5–11 October
EMS Summer School in Applied Mathematics
Stochastic Analysis with Applications in Biology, Finance and Physics, Będlewo, Poland
http://bcc.impan.pl/14EMSschool/

10–15 November
Mathematical Models in Biomedicine
Instituto per le Applicazioni del Calcolo, CNR, Roma, Italy
EMS Lecturer 2014: Professor Miguel A. Herrero

2015

7–10 January
Aspects of Lie Theory, INdAM, Rome, Italy
http://www1.mat.uniroma1.it/~bravi/indam2015.html
EMS distinguished speaker: Vera Serganova (Berkeley, CA, USA)

10–13 June
AMS-EMS-PMS Congress, Porto, Portugal
http://aep-math2015.spm.pt/

18–20 September
EMS-LMS Joint Mathematical Weekend, Birmingham, UK

Date to be fixed
25th Anniversary of the EMS, Institut Henri Poincaré, Paris, France

2016

18–22 July
7th European Congress of Mathematics, Berlin, Germany
http://www.7ecm.de/

Editorial: Opportunities and Challenges in Electronic Publication

Jiří Rákosník (Academy of Sciences of the Czech Republic, Prague, Czech Republic) and Olaf Teschke (FIZ Karlsruhe, Germany)

Exactly a year ago, Bernard Teissier started his editorial in this newsletter claiming: "The advent of electronic publication has been changing our documentation practices for a quarter of a century." This is perfectly true. The changes have had a positive impact; they offer new possibilities but they also generate many new questions, difficulties, problems and challenges.

A simplistic opinion has spread that "electronic" publication has made it fast and easy. Undoubtedly electronic devices, the internet and TeX and other typesetting and imaging tools have greatly improved the technical quality of prints, facilitated the exchange and circulation of manuscripts, supported author collaboration, increased the speed of preparation of documents and contributed to the ever growing volume of scholarly literature produced in mathematics. To handle this burst requires sophisticated tools, thoughtful arrangements and new paradigms. It is the task of the Electronic Publishing Committee of the EMS[1] to follow closely the developments in this area and derive suggestions helping to build the appropriate electronic infrastructures.

TeX is wonderful if one knows how to use it. However, authors mostly learn TeX by trial and error and TeX allows a great deal of improper creativity. To get a satisfying appearance and printing quality often requires a lot of editorial work by the publisher, which cannot be safely avoided even if the publisher requires the use of a particular style. To reach an overall consensus on a publishing style is absolutely impossible but creating a reasonable platform which could be used by a variety of small publishers might be effective. A successful example of this kind is the French project CEDRAM. Recently, there have been other community attempts like the Episciences platform[2] and a project proposal in preparation by FIZ Karlsruhe to add Publishing Platform features to the ELibM.

The more complex the electronic publishing ecosystem, the more important the question of permanence of documents and their long-term preservation. The development is going in the natural and probably best direction that electronic resources are distributed in different places. However, this makes the problem more complex. Preservation of printed texts has already been proved. Of course, there are deterrent examples like the Library of Alexandria or the novel "The Name of the Rose" by Umberto Eco. And we cannot be sure that the documents currently printed with the new technologies on cheap (acid) paper will be legible after one or two centuries. In any case, to destroy a server or even to delete a file is much easier and faster than to put a library to fire and much faster than the degradation of paper and chemical ink. Archiving should be ensured primarily by the document producers – publishers, authors and their institutions. But the safety of archives must be considerably increased by proper technical tools and arrangements, backlogging, duplications, mirror sites, etc. There are commercial organisations like Portico and programmes like LOCKSS which publishers and libraries can use to archive their digital collections.

For ease of creating and handling electronic documents, we are paying a serious price: the fight against ever growing plagiarism and doubtful publishing practices.[3] It is an ethical problem which also requires efficient systemic and technical measures. Publishers should promote strict ethical rules and implement policies and efficient tools preventing and revealing plagiarism. While there are technical tools available to indicate suspicious papers, the vast majority of the revealed cases are due to community efforts, by the work of editors, referees, reviewers and even bloggers. The reason is both in the nature of mathematics – mathematics content is usually not suitable for solely text-based analysis – and in technical circumstances (many plagiarism cases are copied from older non-digital material, sometimes even from advanced text-books!). It is crucial to have independent platforms that document manifestations of plagiarism and doubtful publications sustainably. The reviewing services in mathematics have fulfilled this role for many years and evolved into modern databases which offer a powerful tool to detect and keep records of these issues.

The quickly developing environment of electronic publishing offers numerous opportunities, some of which can be achieved relatively soon and some rather in the realm of visions for farther future. Interlinking of documents, databases and other information resources is becoming standard. One click may lead the reader directly to another source of information either inside or outside the document and the reader would be rather surprised if a newborn electronic document did not provide such a possibility. Structured search through databases and digital libraries is a commonly required feature, which does not appear automatically. Its proper functioning depends mainly on quality metadata enhanced according to widely accepted standards. This is a nontrivial task requesting a lot of "hand-made" work, which is starting at publish-

1 http://mathsci.ucd.ie/~tpunger/EPC_EMS/.
2 J. Demailly: Episciences: A Publishing Platform for Open Archive Overlay Journals, *EMS Newsletter* 87, 31–32.
3 Some examples were outlined in the discussion "Open Access – Four Opinions", *EMS Newsletter* 91, 39–43.

ers and possibly ending at the creator of a digital library, and by no means all issues pertaining efficient and reliable identifier generation can be considered to be solved. While document identifiers exist in a stable framework, author identifiers are just about to evolve into an integrated network, and this issue for other relevant objects (like terms, definitions, theorems, etc.) is completely open. Until this is solved, search through full-texts in mathematics represents a tough task because they are rich in mathematical expressions, formulas, diagrams and other creatures difficult to interpret. There exist more-or-less efficient tools for searching mathematical formulas, e.g. the MIaS (Math Indexer and Searcher) developed at Masaryk University in Brno.

Linkage to social networking sites, from the most common Facebook and Twitter to the specialised Mendeley, CiteULike and BibSonomy, represents another interesting direction of development. Apparently, it has not yet found wide acceptance among mathematicians but the younger generation may change that soon. The same can be said for annotations – systems for blogging, providing comments to the displayed texts, managing discussion threads, enabling personalisation of the tools, etc.

A new feature emerging in electronic journals, in particular those devoted to applications of mathematics, is that papers have attachments in different formats, e.g. software, animations, videos and experimental datasets. This puts new requirements on electronic archives, digital libraries, metadata schemes and display methods. Questions often appear on how to handle so-called grey literature, informally published material such as reports, blogs and manuscripts that may be difficult to trace in conventional ways because they are neither published commercially nor widely accessible. This is a complex problem requiring special treatment which may not fit into the basic scheme of digital libraries.

Many other visionary ideas have been presented in the recent report "Developing a 21st Century Global Library for Mathematics Research", initiated by the IMU and the US National Research Committee and supported by the Alfred P. Sloan Foundation.[4]

The foreseen features mostly assume semantically enriched texts. To provide them requires not only technical equipment but also an efficient organisation and voluntary cooperation of stakeholders: authors, publishers and content providers.

There is a successful example of a facility where some of the ideas have already been turned into reality: the European Digital Mathematics Library (EuDML), created as a pilot project partly funded by the European Commission from February 2010 to January 2013.[5] The EuDML represents a technical infrastructure providing a unified access point for the digital mathematical content hosted by a number of different organisations across various countries and a cooperation model with a variety of stakeholders allowing the building of a reliable, enduring, global reference library to eventually become exhaustive. The EuDML has developed a detailed scheme for quality, standardised metadata and a central system for enhancing metadata of local content providers. One of the basic principles in the EuDML Policy requires that all items included in the EuDML must be open access after a finite embargo period. Once documents contributed to the library are made open access due to this policy, they cannot revert to closed access later on.

The EuDML provides interlinking between items and links to the databases zbMATH and MathSciNet. It offers users a linkage to various social networking sites. Registered users can enjoy features such as the annotations component and personalisation of the library. This is also the space where users could be dragged into active participation. The search through metadata and full-texts also includes search for formulas. The tools are still being developed and reliability of the outputs is limited by the technical quality of digital documents. Of course, we cannot expect too much of scans of old prints.

To ensure that these resources and services remain stable and there is a sustainable public service to the worldwide scientific community, it is crucial that they remain under public control. To this purpose, the international association without legal personality called the EuDML Initiative was established in 2014 by 12 partnering organisations including the EMS. The EMS plays an eminent role, which is supported by the fact that three members of the Electronic Publishing Committee became members of the association Executive Board and the EMS will nominate the Scientific Advisory Board. Thierry Bouche from the Cellule MathDoc in Grenoble is the Chair of the Initiative and Aleksander Nowiński from the ICM in Warsaw is in charge of the Technical Committee. The first tasks of the association include improving the technology built during the project, developing workflow to ease providing content into the EuDML, reviewing automatic metadata harvesting and ingesting, and enhancing the annotation service and formula search. In order to increase the amount of content, negotiations will continue with Euclid, Math-Net.Ru and in particular the EMS Publishing House about ingesting their collections. Archiving and long-term preservation is an important issue in the EuDML. According to its policy, the digital full-text of each item contributed to EuDML must be archived physically at one of the EuDML member institutions. The initiative will investigate creating a private LOCKSS network which might offer a prototype for further content providers, especially smaller publishers and member societies.

Naturally, all these activities generate costs. EuDML partners agreed that during the next few years they will use their own resources to cover expenses. There is an urgent task to create a long-term funding strategy. Research and development funding mostly aims at new developments and little attention is paid to the fact that the established facilities also have necessary running costs, not to mention that the project funding itself is not a reliable resource for long-term activities. The EuDML represents an important infrastructure for mathematics in

[4] Available at http://www.nap.edu/catalog.php?record_id=18619.
[5] See T. Bouche: Introducing EuDML, the European Digital Mathematics Library, *EMS Newsletter* 76, 11–16.

Europe and as such – perhaps together with zbMATH, the recently established EU-MATH-IN, etc. – it deserves sustainable support.

In a nutshell, another resource maintained by the EMS may illustrate how future semantic enrichment may work. The Encyclopedia of Mathematics (EoM)[6] has evolved into an open access, community-driven network of specialised mathematics information, which is now gradually linked both to the MSC as well as to relevant publications. In this way, the EoM is somewhat orthogonal to databases and digital libraries. The desirable interlinking between both kinds of resources is just beginning: at a document level, there already exist some (but not complete) links from the EoM to MathSciNet and zbMATH (and hence, through the latter, to EuDML) but not in the opposite direction. A complete interlinking from EuDML would be most useful if generated on the level of full-texts but this also represents the most demanding issue.

The progress in the domain of electronic publishing is moving ever faster. It is difficult to guess what will happen in a few decades and to formulate a firm universal and long-term strategy. We have to carefully follow the situation. Most of the issues mentioned above demand development of reasonable common standards and formulation of best practice principles, which should then be promoted by the EMS and its member societies.

It is natural that many of the features mentioned here have been in the scope of the Electronic Publishing Committee. The members of the current committee as well as their predecessors have been personally involved in the EuDML, EoM, zbMATH and other activities connected with electronic publishing, archiving and dissemination of digital mathematical literature.

[6] http://www.encyclopediaofmath.org.

Jiří Rákosník [rakosnik@math.cas.cz] the Chair of the Electronic Publishing Committee. He works as a researcher in the Department of Topology and Functional Analysis in the Institute of Mathematics ASCR in Prague. His research interest is the theory of function spaces, in particular spaces with variable exponents. He is also active in the field of digitisation and digital mathematical libraries and represents the EMS in the EuDML Initiative.

Olaf Teschke is member of the Electronic Publishing Committee and the Executive Board of EuDML initiative. In the Editorial Board of the EMS Newsletter, he is responsible for the zbMATH Column.

New Editor Appointed

Vladimir Kostic is an assistant professor at the Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad. After obtaining his PhD in numerical mathematics in Novi Sad, he did postdoctoral research at the Technical University of Berlin. His main research interests cover subjects in numerical and applied linear algebra, modelling and optimisation of dynamical systems, and stability and synchronisation of dynamical systems – especially networks of oscillators as well as biological and artificial neural networks. Among his other professional activities, he has been a member of the organising committees of five scientific meetings. His webpage can be accessed at http://sites.pmf.uns.ac.rs/vladimir.kostic.

A Note of Clarification from the EMS Executive Committee

This concerns the announcement of the new Editorial Board of JEMS published in the June 2014 issue of the EMS Newsletter.

A remark was received pointing out that the paragraph "The EMS is greatly indebted to the departing editorial board, which under the leadership of Professor Brézis has raised the journal to its current high rank", might lead to the interpretation that the rank of the journal was low under the first editorial board, with Professor Jürgen Jost as Editor in Chief.

With this note, we would like to make clear that this would not reflect the opinion of the Executive Committee. Acknowledgement to the founding editorial board is clearly expressed in the sentences "Today, JEMS is one of the strongest mathematical journals, thanks to the skill and dedication of the first two editorial boards".

EMS Executive Committee

Report on the Meeting of Presidents of Mathematical Societies, in Istanbul

Stephen Huggett (University of Plymouth, UK)

Introductions

Everybody present introduced themselves, saying which Society they represented.

Betul Tanbay welcomed us to the Boğaziçi University and gave a presentation of the history and recent activity of the Turkish Mathematical Society.

Reports from the European Mathematical Society

Marta Sanz-Solé gave her report as President of the EMS, concentrating on plans to increase the individual membership of the EMS, the work of the Committees (especially the Publications Committee and the Committee for European Solidarity), the new EMS Monograph Award, and the journals published by the EMS.

This led to a brief discussion of gold open access, which focussed on the discrimination between rich and poor academic communities and the important role of libraries and learned societies.

Marta Sanz-Solé announced the EMS Council in Donostia (San Sebastián), June 28–29, 2014, reminding Presidents about the nomination of delegates and asking for suggestions for the agenda.

Preparations for the 7ECM in Berlin in 2016 were presented by Jürg Kramer. Marta Sanz-Solé reported on the Scientific and Prize Committees for the 7ECM, thanked the *Foundation Compositio Mathematica* for contributing 50% of the prize money, and described the very interesting Hirzebruch Lecture initiative.

Marta Sanz-Solé gave a brief description of the EU-MATHS-IN project, and Gert-Martin Greuel did the same for EuDML.

Martin Raussen gave a presentation on the development of the EMS web site:

www.euro-math-soc.eu

There was a comment that what is currently regarded as "user-friendly" can in fact be very unfriendly. Presidents were also invited to comment on the EMS news sent by email.

Marta Sanz-Solé reported on Horizon 2020. She explained that she had been collecting data on the 7th Framework Programme which indicated that one would have expected mathematics to receive about 6% of the funding, whereas it had only received about 2%. We need to encourage more applications, or the funding will further decrease. Another problem was that there were open calls for "experts", which has led to them being of extremely variable quality. The EMS is trying to address this. In discussion it was suggested that perhaps our researchers need training in how to apply for these grants. It was also noted that the success rate for mathematics in the European Research Council system has been much higher.

Discussion on how national societies can influence at the national level suitable policies for mathematics

Günter Törner gave a presentation on the topic in the context of education. He emphasised the importance of collaboration in this work.

Terry Lyons gave a presentation on the same topic, but concentrating on activity in the UK. He emphasised the importance of trying to understand the other side's point of view, he talked a little about the Deloitte Report, and he noted that the people who are having such an enormous effect in the economy are those we sometimes regard as our "failures".

Marta Sanz-Solé drew attention to the document "Mathematical Sciences in the Netherlands", which is available on the EMS web site. In the ensuing discussion the following points were made.

- In France the declining standards at entry to University are posing a serious threat to the survival of many small departments.
- In Hungary there is an annual conference for mathematics teachers, with prizes awarded by the Hungarian Mathematical Society. This is a good form of recognition for teachers.
- In Finland there has been a very successful initiative for arousing interest in the natural sciences among school children. Even extremely small children are invited to visit Universities. It is crucially important that mathematics teachers have to do the same degree as mathematicians.
- Joint conferences between mathematicians and teachers have had a big effect in Romania: mathematics is now much more visible, and the number of students is increasing.
- There has been a dramatic drop in standards in Russia caused by the unified testing regime. Mathematicians are doing their best to modify the system in such a way as to limit the damage.
- In Turkey in 1980 there was a separation of the undergraduate degrees for teachers and mathematicians, and the bad effects are still being felt, one of which is that mathematicians now have much more difficulty in gaining access to politicians. Also, the exclusive focus on preparing students for the University entrance exam causes such damage that it takes the first two years at University before the student recovers.

Marta Sanz-Solé emphasised the importance of reports such as the Deloitte Report, and encouraged Mathematical Societies to try to produce similar reports in their own countries. The EMS will be keeping all such reports in a folder on the web.

Presentations

- Gert-Martin Greuel presented IMAGINARY.
- Nikolai Andreev described the work of the ETUDES laboratory at the Steklov Institute: http://www.etudes.ru/en/
- Armen Sergeev presented the first Caucasian Mathematical Conference: http://www.euro-math-soc.eu/cmc/
- Gregoire Allaire presented the new SMAI Journal on Computational Mathematics.

Closing

On behalf of all the participants, Marta Sanz-Solé thanked the local organisers of this meeting for their excellent preparation and hospitality and their warm welcome.

EMS Council Meeting in Donostia/ San Sebastián on the 28th and 29th June 2014

Stephen Huggett (University of Plymouth, UK)

Mrs Itziar Alkorta, Vice Minister for Universities and Research in the Basque Government, welcomed everybody to Donostia/San Sebastián and wished us a successful Council. The President thanked her, and gave her own welcome to all the delegates and guests.

The agenda was approved, and so were the minutes of the last Council meeting, in Kraków in 2012. The President asked for nominations from the floor of candidates for election to the Executive Committee. There were none. Maria Esteban, Arne Jensen, and Elizabetta Strickland were then elected Scrutineers.

[Note: many of the presentations referred to below are available to the delegates. Readers interested in the details are invited to contact their delegate.]

Reports, Finance, Membership, and By-Laws

The President presented her report, noting in addition the Joint Mathematical Weekend with the London Mathematical Society to be held in Birmingham in September 2015. She also took the opportunity to thank Cédric Villani for his offer to host the EMS 25th Anniversary event at the Institut Henri Poincaré in 2015.

The Secretary gave a short report on the meetings of the Executive Committee since the last Council. Then Vice-President Martin Raussen gave a report on the redevelopment of the EMS web site, describing the planned improvements to its security and usability.

The Treasurer gave a presentation of the Society's finances, focussing on two possible budgets, which differed in that one left the membership fees unchanged while the other included a small increase. He emphasised the proportion of the expenditure on scientific projects. The Treasurer then presented the financial statements and auditors' reports for 2012 and 2013. These were approved by Council, which then approved the Treasurer's proposal to increase the membership fees, and having done so it approved the budget, with the new fees. Finally, the Council appointed Nicola Bellomo and Stefan Jackowski as lay auditors, and PricewaterhouseCoopers as the professional auditors, for the years 2015 and 2016.

The Treasurer presented and explained the membership statistics, which led to a brief discussion on the variation from one country to another of the number of people who join the EMS through their national societies. It was also noted that many mathematicians from quite poor regions in Europe join the American Mathematical Society instead of the EMS. The President concluded the discussion by remarking that we need to show potential members what we do as a Society.

The Council approved by acclaim the change of class of the Italian Mathematical Union from 3 to 4, and the Finnish Mathematical Society from 1 to 2.

The Secretary introduced the proposed change to one of the By-Laws, which would now read:

> *RULE 12: Delegates representing individual members shall be elected by a ballot organized by the Executive Committee from a list of candidates who have been nominated and seconded, and have agreed to serve. These delegates must themselves be individual members of the European Mathematical Society.*

The Council approved this change.

Elections

Council elected the following people to the Executive Committee:

Pavel Exner, President
Sjoerd Verduyn Lunel, Secretary
Mats Gyllenberg, Treasurer

Also, Volker Mehrmann was re-elected as Member at large.

European Congresses of Mathematics

The President gave a report on the preparations for the 7th ECM in Berlin, focussing on the procedures adopted by the Executive Committee in choosing the Chairs and members of the Scientific Committee and the various Prize Committees. She was very pleased to announce that the Hirzebruch Lecture Committee consisted of Michael Atiyah, Jean-Pierre Bourguignon, Günter Ziegler, and Jürg Kramer. She thanked *Compositio Mathematica* for its significant financial support for the prizes.

Volker Mehrmann gave a presentation on behalf of the Organizing Committee of the 7th ECM.

The President announced that the call for bids for the 8th ECM would be published in the December issue of the Newsletter.

EMS Committees

Each of the eleven EMS Committees gave a presentation to Council, followed by an opportunity for questions to be asked.

Arne Jensen, Chair of the Ethics Committee, gave a presentation on the work of the Committee, focussing on the Code of Practice. Gert-Martin Greuel commented that plagiarism is encountered almost every week by Zentralblatt, and that if appropriate they can stop indexing the journal.

The President introduced the discussion on the draft paper on Open Access prepared by the Publications Committee. She emphasised that it was an early draft, and thanked the Committee for their work. Then the Treasurer gave a presentation of the paper, after which the President invited people to send comments to Bernard Teissier or to the EMS office. Arne Jensen drew attention to the lists of "good" journals, which will be made available on the Ethics Committee's web page.

Mathematics in Europe

The President gave a presentation on Horizon 2020.

Jean-Pierre Bourguignon, President of the European Research Council, gave a presentation on its work. The President thanked him for standing as President of the ERC, and asked Council for comments and questions. Grégoire Allaire asked whether it would be possible for the ERC to give smaller grants, but to more people. Jean-Pierre Bourguignon replied that he and the ERC Board are in favour of this, but that so far the Scientific Council of the ERC has voted against such a move. Nicola Bellomo remarked that it is a shame that the funding for research networks has gone, and he also explained the difficulty in forming the teams necessary for research in applied mathematics. Gert-Martin Greuel asked about interdisciplinary research, and Jean-Pierre Bourguignon replied that it is clear that such projects are not so well supported. This was a hard problem, but he would be trying to address it. Jean-Pierre Bourguignon finished by noting that some institutions are taking control of the budgeting part of the applications, which he regards as unacceptable.

Publication Activities

On behalf of the Editor-in-Chief, Jorge Buescu gave a presentation on the Newsletter. The President thanked all the Editors for their work, and noted the suggestion to think of a section aimed particularly at young mathematicians.

The President drew attention to the report (in the papers) from the Director of the EMS Publishing House, and described its relationship with the EMS, noting that the Board of the European Mathematical Foundation controls the policy of the publishing house. The EMS Publishing House now has a turnover of 1 million euros, arising from 18 journals (of which it owns 8) and 10 to 15 books per year. The new *Surveys in Mathematical Sciences* would be free to EMS members for one year, during which a subscription system would be decided on. She also noted the policy on journal pricing, and pointed out that the prices per page had been published in the Newsletter. Finally, the President noted that the Publishing House was at a critical stage in its development, and some fundamental decisions on its structure would have to be taken quite soon.

Gert-Martin Greuel gave a brief report on Zentralblatt, drawing attention to his written report (in the papers). He described the significant improvements in the system, and highlighted the fact that it is a not-for-profit and community-driven enterprise.

Projects

Thierry Bouche gave a demonstration of Googling a research article, getting the EuDML entry for it as the first hit, going to the EuDML site, and then finding the paper itself. He observed that in many cases this would not be possible without the EuDML database, and noted that the site also contains other useful metadata such as citations, the Zentralblatt review, and so on. The President thanked Thierry Bouche and Jiří Rákosník for all their work.

Maria Esteban gave a presentation on the EU-MATHS-IN project.

Future meetings

The next Meeting of Presidents would be hosted by the Austrian Mathematical Society and held in Innsbruck in 2015 (on a date to be fixed). The President proposed a vote of thanks to the Austrian Mathematical Society.

The next Council meeting would be held in Berlin just before the ECM in 2016, at the Humboldt University.

Closing

The President thanked the Spanish mathematical societies and the University of the Basque Country for hosting the Council, and in particular Luis Vega for the excellent local organisation. She also thanked the delegates for helping to make the Council so smooth and productive.

Martin Raussen thanked the President, Secretary, and Treasurer for all their work during their terms of office.

The Support Given by the European Research Council to Mathematical Research

Jean-Pierre Bourguignon (IHÉS, Bures-sur-Yvette, France), President of the ERC

The European Research Council (ERC) was created in 2007 as one of the components of the programme "Ideas" of the 7th Framework Programme of the European Commission. Its creation was achieved after a long and intense lobbying period by the scientific community, and the strong support of a number of politicians and commissioners, Philippe Busquin among them (commissioner in charge of research from 1999 to 2004). The ERC is original for several reasons, which concur to bring it in line with the scientific needs of researchers:

- An application is presented by a single Principal Investigator;
- Support is provided for up to five years to ambitious projects aimed at performing "frontier research";
- The ERC Scientific Council is responsible for determining how the budget of the programme is to be spent;
- The choice of evaluators is also left to the Scientific Council;
- The selection of scientists receiving grants is made only on the quality of the projects and of the applicants;
- There are less bureaucratic rules for the management of projects.

In just seven years, the programme has established itself as a reference. More than 4500 contracts have been put in place for a little more than 7 billion euros. The Scientific Council decided to give three types of grants: "Starters" for scientists 2 to 7 years after their PhD, "Consolidators" for those 7 to 12 years after their PhD, and Advanced Grants, with increasing levels of support from 1.5 to 3 million euros. It also decided to dedicate two thirds of the money available to younger researchers.

An Executive Agency has been set up, alongside the European Commission, with the duty of managing the programme. It has done an excellent job and has quickly won the confidence of the scientific community whilst having very low administrative costs (less than 3% of the overall budget).

The Scientific Council decided to deal with all fields of knowledge through 25 panels for each type of grant, the panels covering mathematics being the only ones that are disciplinary. Each panel consists of 12 to 16 members and identifying this number of high level scientists willing to dedicate a substantial amount of their time to the programme requires a sustained effort on the part of the Scientific Council, and in particular the vice-presidents in charge of the three domains: Physics-Engineering (including the panels dealing with mathematical projects), Life Sciences and Social Sciences and Humanities.

Altogether, among the 4500 ERC grants, 239 mathematical projects have been selected with 137 submitted by younger people and 102 being advanced grants. The average success rate in mathematics is in line with the overall success rate, i.e. about 11%, showing that the selection is very tough.

The success of the ERC has paved the way to continuing it in Horizon 2020, the present framework programme supporting research and innovation during the period 2014–2020. The ERC is now one of the essential components of the "Excellent Science" pillar of Horizon 2020. All its essential features have been kept. Its budget for the 7-year period has been set at 13.1 billion euros, a significant increase when compared to the overall ERC budget for FP7. This covers two periods: a flat budgetary period 2014–2015–2016, during which the amount available annually is approximately 1.7 billion euros, slightly below the level of the 2013 ERC budget; and a period of regular increase from 2017 to 2020, where the budget goes from 1.8 to 2.2 billion euros.

The 2014 Starting Grant and the 2014 Consolidator Grant calls are already in the process of being evaluated. The Advanced Grant call will be closed in October and evaluated in the early part of 2015. All 2015 calls, some to be open at the end of 2014, will be evaluated during 2015.

In view of the steady increase in the number of applications, the Scientific Council has decided to introduce some restrictions on repeated applications: a scientist whose project receives the lowest grade C when evaluated will not be allowed to present another project for two years; a scientist whose project receives a grade B when evaluated, i.e. is considered of good quality but not competitive enough to pass to the second stage of evaluation, will have to wait one year before submitting a new proposal. These rules were introduced with the 2013 calls so that the effects of these restrictions could be felt for the first two calls in 2014: the Starting Grant call had the same number of applicants as the 2013 call and the Consolidator Grant call witnessed a 30% decrease when compared to 2013, bringing the number of applicants in line with the equivalent 2012 call.

More information can be found on the ERC website erc.europa.eu or through the national contact points in the various countries, a list of which can be found at erc.europa.eu/national-contact-points. The ERC can also be followed on Facebook at EuropeanResearchCouncil or on Twitter at ERC_Research.

New International Women In Maths Website

Caroline Series (University of Warwick, Coventry, UK)

In March 2013 the Executive Committee of the International Mathematical Union (IMU) approved the establishment of an Advisory Group for Women in Mathematics, charged with creating and overseeing a section of the IMU website entitled Women in Mathematics (WiM). Opportunities for women vary widely from country to country and a main aim is to enhance the participation of women in all mathematical communities. The new WiM site was launched at the International Congress of Women Mathematicians on August 12th just prior to the International Congress of Mathematicians, at the address http://www.mathunion.org/wim/

The site includes information about organizations, people, events, resources and initiatives of interest to women mathematicians world-wide. In order to maximize the usefulness of this site, we welcome suggestions from the community. Indeed, advice concerning items for inclusion is important to us.

The Advisory Group may be contacted at info-for-wim@mathunion.org.

The WiM Advisory Group: Ingrid Daubechies (Chair) (USA), Petra Bonfert-Taylor (USA), Carla Cedarbaum (Germany), Nalini Joshi (Australia), Sunsook Noh (Korea), Marie-Françoise Ouedraogo (Burkina Faso), Dušanka Perišić (Serbia), Claudia Sagastizábal (Brazil), Caroline Series (UK), and Carol Wood (USA).

David Hilbert's Famous 1930 Radio Address

Janet Beery (University of Redlands, USA)

The Mathematical Association of America (MAA) invites you to visit Convergence, its free online journal on mathematics history and its use in teaching, to see and hear "Hilbert's Radio Address," an article that includes an audio recording, transcription, and translation into English of Hilbert's 4-minute radio version of his longer 1930 address with its famous finale, "Wir müssen wissen; wir werden wissen."
http://www.maa.org/publications/periodicals/convergence/david-hilberts-radio-address

The French Learned Societies and the EMS

Grégoire Allaire (École Polytechnique, Palaiseau, France), Anne Gégout-Petit (Université de Lorraine, Vandœuvre-lès-Nancy, France) and Marc Peigné (Université F. Rabelais, Tours, France)

The presidents of the French learned societies in mathematics Grégoire Allaire (SMAI), Anne Gégout-Petit (SFdS) and Marc Peigné (SMF) share their experiences of the academic year 2013-2014 with the EMS.

During the 2013–2014 academic year, the three French mathematical learned societies, the SMF (French Mathematical Society), the SMAI (French Society for Applied and Industrial Mathematics) and the SFdS (French Society of Statistics) had the opportunity to meet the EMS three times.

The first was in November 2013 when the EMS Executive Committee held a meeting at the Institut Henri Poincaré in Paris. The two days featured a meeting with the three French societies, which gave them a good opportunity to introduce themselves and describe their missions as well as the specific context for mathematics in France. The discussion that followed focused mainly on the future of publications, the economic models that are currently emerging and the ways to promote the network of academic mathematics libraries at a national level. It was underlined that one of the specificities of French research is the role played by the CNRS (National Centre for Research and Science) in managing the mathematics library network and in helping academic publications – not only those published by the learned societies but also those coming directly from universities or other institutes (Ecole Polytechnique, IHES, IHP, etc.). The French societies, reflecting the position of the French mathematical community, are very much convinced that academic publications must keep their share of the market and are a key alternative to the numerous commercial publishers. Therefore, they continue to support the development of well-established academic journals, while at the same time encouraging the creation of new, quality, open access electronic journals (a project is planned for the next academic year to that end). The medium-term goal is to reinforce this policy, which mirrors the one proposed by the EMS at a European level, the main difficulty being to secure long-term financing for this model of publication within the framework of European laws, especially regarding open access issues. It is obvious that the EMS and the French mathematical societies are all at a crucial development stage in their publishing activities and are facing quite similar challenges, a hot topic that was continued more informally during the convivial evening that followed.

A second opportunity to meet arose in April 2014 during the EMS Presidents Meeting in Istanbul. This was a great opportunity for presidents of national mathematics societies to interact and discuss issues of common interest. The three French societies were represented by G. Allaire, the President of the SMAI. Among the important topics discussed in Istanbul, such as scientific activities (conferences, summer schools, etc.), mathematical education and public awareness, participants were particularly concerned with two crucial issues for the future. One is the creation of EU-MATHS-IN, the European network which connects national networks of mathematicians collaborating with industry and society. In the context of the EU Research and Innovation programme HORIZON 2020, which never mentions mathematics (!), it is essential that we convince governments, funding agencies and, more broadly, the general public that mathematics is a key to research, innovation and technological developments for Europe's global competitiveness. The second remarkable issue is the impact of two Deloitte reports which recently appeared in the United Kingdom and in the Netherlands. These reports, which deliver very precise figures on the economic impact of mathematics at a national level, are instrumental in convincing parliament members and high ranking executive officers in administration and government that mathematics deserves to be highly supported. The French mathematical societies are considering recruiting a consulting company similar to Deloitte to report on the influence of mathematics in the French economy.

The last meeting took place during the EMS Council meeting in Donostia/San Sebastian at the end of June 2014. The main purpose of the meeting was to elect new officers, hear reports from the various committees and discuss broad issues concerning the EMS. But equally important were all the informal discussions between representatives from all over Europe. The three French learned societies really appreciated the open-minded discussions with the EMS and found it very helpful to have access to a wider overview of the mathematical context within Europe.

Grégoire Allaire is Professor at Ecole Polytechnique in Paris since 2000, specialist of numerical analysis in various aspects, he has been president of SMAI between july 2012 and june 2014.

Anne Gégout-Petit is Professor at Université de Lorraine since 2013, specialist of statistics, she is president of the SFdS since july 2013.

Marc Peigné is Professor at University F. Rabelais Tours since 2000, specialist of probability theory and ergodic geometry, he is president of the SMF since June 2013.

Interview with Yakov Sinai – Abel Laureate 2014

Martin Raussen (Aalborg, Denmark) and Christian Skau (Trondheim, Norway)

The Prize

Professor Sinai – first of all we would like to express our congratulations. You have been selected as the 12th winner of the Abel Prize and you will receive the prize tomorrow. We are curious; did you have any expectations beforehand? How did you receive the information?

In early March this year I got to know that the Abel Committee were interested in taking my photograph. A friend of mine told me this and I thought this must mean something because this had never happened before. And then there was a telephone call from the Norwegian Academy of Science and Letters informing me about the prize.

And this was on the same day that the prize was announced here in Oslo?

Yes. That happened on 26 March.

Youth

You were born in Moscow in 1935 into a family of scientists. Your parents were both biologists and your grandfather was well known in mathematics. We suppose this had important consequences for the development of your interests?

Definitely, yes. How could I say no to this question? Everything was about mathematics and mathematical events. But, at that time, I preferred to play volleyball.

The influence of mathematics was not as direct as you may think. I participated in many Olympiads in mathematics during my school years but never had any success and never won any awards. I say this to young people who have never won in Olympiads; there may be compensation in the future.

At this time, my grandfather was of a great age and he did not have the energy to push me into mathematics. And I also have a half-brother, G.I. Barrenblatt, who worked at Moscow State University and who was convinced that I should pursue a career in mathematics.

Do you remember when you found out that you had an exceptional talent for mathematics?

If at all, it happened very late. I was a graduate student when I brought my paper on entropy to my advisor A.N. Kolmogorov and he said: "At last you can compete with my other students." But I am not sure that he was right and that I have an exceptional talent for mathematics.

Yakov Sinai (Photo: John Jameson/Princeton University)

You must have entered school at about the same time that Nazi Germany invaded Russia. How did the war influence your first years at school?

I entered school in 1943 after my family returned from the evacuation of Moscow. At that time boys and girls studied separately; at the end of each year, we had about 10 exams. Before the evacuation, life was different. It was forbidden to leave windows open in the apartments in Moscow because it had to be dark. In 1943, windows were allowed to be open again. In Moscow there were no clear signs of war. But life was hard because of the time of Stalin. People had to behave in a special way.

And that also influenced life at school?

It was everywhere; you could be expelled from school or even sent to prison for being controversial.

Were there teachers with a lot of influence on you, in particular in mathematics?

We had a very good teacher in mathematics at our high school. His name was Vasily Alekseevich Efremov and he was a great old-style schoolteacher. He always brought us his problems in accurate handwriting on a piece of paper which he distributed among the students. Because of the well organised and inspiring work, mathematics was very popular among us. We discussed and tried to solve his problems. At this time I was not among the best in the class. There were definitely other students who were much better than me.

What was your age at this point?

This was still in high school just before I entered university. Thus, I was probably 16 or 17 years old.

Student at MSU-Mech-Mat

You entered the Faculty of Mechanics and Mathematics at Moscow State University in 1952 still a teenager. How was it to study at this famous institution as such a young student?
We had a number of very good professors there. For example, the lecture course in analysis was given by M.A. Lavrentiev, who was a very famous scientist at the time. He was also involved in administration but was a great teacher and his lectures were very interesting. We also had a very good lecturer in classical mechanics, Chetaev. I was his student in the second year. Moreover, we had lectures in geometry given by Bachvalov, who was famous in Russia but not so much known in the West. There is a story about him: when we entered the university on 1 September, he came into the room and said: "Let's continue." And that was the beginning of his lecture course.

In algebra, we had lectures by Dynkin, who was an excellent teacher for those who had started to study. These were lectures at a very high level. Dynkin used to hand out interesting problems for the enthusiastic students. Among such students in my year I could particularly mention I. Girsanov, who became a famous probabilist, and L. Seregin.

Was it Dynkin who inspired your first paper in mathematics?
Yes. I was a student of Dynkin during the second and third years and I wrote the first paper under his supervision. I solved a problem that he formulated for me; this became my first published paper when I was a student in the third year. I loved the work I did and still do.

Dynkin wanted me to work on problems on Markov processes in the style of Feller. The papers by Feller became very popular in Moscow at that time and Dynkin suggested that I should continue along this line. However, I was not very excited and interested in it.

To what extent were mathematics and mechanics integrated in the curriculum?
These were independent parts of the curriculum. Everybody could attend lectures within each branch. I was attending lectures in mathematics and mechanics but also, to a minor degree, some lectures in physics. But on the whole it was mainly in the mathematics department.

We imagine that besides Dynkin, Kolmogorov and Gelfand must have been very important figures for you?
Kolmogorov had many students and I became one of them. His students had complete freedom to work on any problem. Kolmogorov loved to discuss their results with them. There were several cases when Kolmogorov wrote their papers in order to teach them how to write mathematical texts.

Kolmogorov organised a seminar, which was initially a seminar on random processes and later became a seminar on dynamical systems and ergodic theory. I began to attend, together with other mathematicians like Arnold, Alekseev, Tikhomirov and others. Later I became a student of Kolmogorov. At that time he was also interested in problems of entropy in different compact and functional spaces. Questions of this type were very much discussed at that time.

But Gelfand tried to recruit you as a graduate student as well?
Yes. Gelfand organised his famous seminar, which was attended by many mathematicians of different generations. I took part in it for many years. It happened, if I remember it correctly, in 1955 when Gelfand was writing a famous volume in his series of books on distributions. Gelfand was interested in probability theory and he wanted me to become his student. We had some discussions about it and I told him that I wanted very much to work on problems related to probability theory but I had already written a paper under the supervision of Dynkin. He asked me: "Do you want to have probability theory as an appetiser or as a dessert?" I answered: "I want it as a main course." That was the end of the story…

This did not mean that our contact came to an end; we met many times, especially when he worked on problems in representation theory, which were connected with problems in ergodic theory, like the theory of horocycles and others. We discussed this many times. I attended Gelfand's seminars for many years because Gelfand had the remarkable ability to explain difficult mathematical topics in a clear and simple way.

Dynamical systems. Entropy and chaos

Could you explain what a dynamical system is?
We understand dynamical systems as objects that describe all types of evolution. The most interesting case is non-linear dynamical systems, when the equation for the dynamics of the evolution is non-linear. There can be many different phenomena, which require deep analysis.

And among these dynamical systems, what is an ergodic system?
I have a very good example for an ergodic system which I always explain to my students. Suppose you want to buy a pair of shoes and you live in a house that has a shoe store. There are two different strategies: one is that you go to the store in your house every day to check out the shoes and eventually you find the best pair; another is to take your car and to spend a whole day searching for footwear all over town to find a place where they have the best shoes and you buy them immediately. The system is ergodic if the result of these two strategies is the same. The entropy characterises the growth of the number of possibilities in dynamics. I heard the first explanation of this role of entropy from I.M. Gelfand.

Ergodic theory originally came from physics, in particular from the study of Hamiltonian equations. Can

you explain in general terms what chaos is and how one can measure it?

This is the subject of my lecture, which I will give the day after tomorrow, but I can summarise it briefly here. The main question concerns the difference between chaos theory and probability theory. In probability theory one deals with statistical experiments, say you toss a coin 100 times. One can have many different series resulting from this experiment and study the result.

If you consider the problem of chaos and for example want to measure the temperature at the same point you make the measurement during the year, you now have only one realisation of the temperature. You cannot have a hundred realisations of the temperature at a given place and at a given time. So the theory of chaos studies the series when the results of measurements have a limit as time tends to infinity and how to describe this limit. The existence of the limit actually follows from some hypothesis about the equations of motion. This gives the existence of the distribution, which determines the value of all kinds of averages (or, it is better to say, the existence of the averages and also finding their values).

Then, the question is: what are the equations of motion which determine the distribution and these averages? The basic statement in chaos theory is that the dynamics must be unstable. Instability means that small perturbations of the initial conditions lead to large perturbations in the dynamics after some time.

Then there is a mathematical theory that says that if the system is unstable the time averages exist and there is a possibility of calculating them. This is the general description of what is done in chaos. A more precise description requires more mathematics.

How do you measure chaos? Does entropy come into the picture here?

If we understand chaos as mentioned already, i.e. as the existence of time averages and also properties related to mixing, then there is a natural description of chaos in terms of some special distribution. Entropy is used in the theory of unstable systems and it characterises how many types of dynamics a given system can have. It is certainly a very useful notion because the positivity of the entropy determines other properties of the systems that can be studied.

Physicists always expressed their hope that entropy would allow them to understand turbulence (see, for example, the paper by B. Chirikov and the books by G. Zaslavski, R. Sagdeev and others). It is hard to say that this hypothesis is true. On the other hand there are many situations in physics where systems have small entropy.

Definition of entropy for dynamical systems

Kolmogorov had come up with the definition of entropy for Bernoulli shifts but then he changed it to a definition that was not invariant. Then you came with the correct definition. What is now called the Kolmogorov–Sinai theorem gives an efficient way to compute the entropy.

Kolmogorov started his seminar with von Neumann's theory of dynamical systems with pure point spectrum, which he explained in a purely probabilistic way. Later I found this approach in the book by Blanc-Lapierre and Fortet. Everything in Kolmogorov's seminars was very exciting. At that time we believed that the main problem in ergodic theory was to extend the theory of von Neumann to systems with continuous spectrum that can be constructed in terms of the second homology group of the spectrum with coefficients in the ring of bounded operators. It did not work but the idea remained.

At that time, Kolmogorov spent his time primarily on problems in information theory and the concept of dimension of linear spaces. I do not know how it happened but one day Kolmogorov came to his lecture and presented his definition of entropy. Using modern terminology, one can say that he gave the definition of entropy for Bernoulli shifts and thus proposed a new invariant for this class of dynamical systems. It was certainly a great result. Kolmogorov wrote his text. He submitted it for publication and left for Paris where he spent the whole semester. As is known, the text that was submitted for publication was different from what he explained in class. In his paper he introduced a new class of systems which he called quasi-regular. Later they were called K-systems (K for Kolmogorov). For this class of systems he introduced the notion of entropy. While Kolmogorov was away, I was thinking about a definition of entropy that could work for all dynamical systems. Later it appeared in my paper on entropy.

At that time, there was a clear feeling that for dynamical systems appearing in probability theory, the entropy is usually positive, while for dynamical systems generated by ODEs it should be zero. Thus, there seemed to be a possibility to distinguish dynamical systems in probability theory from dynamical systems in analysis.

How about your connection with Rokhlin?

The story about my connection with Rokhlin, who later became a close friend of mine, started when Kolmogorov's paper on entropy appeared in 1958. At that time, Rokhlin lived in a small provincial town Kolomna not far from Moscow. He had a very good graduate student, Leonid Abramov. There are several general theorems that were proven by Abramov, like the entropy of special flows, and other things like Abramov's formula, etc. When Rokhlin heard about the paper by Kolmogorov, he sent Abramov to Moscow to find out what had been really done, what was the situation and if possible to bring the text.

When Abramov came to Moscow, he found me, we talked a lot and I taught him everything I knew. Abramov then invited me to Kolomna to talk to Rokhlin and I accepted the invitation. I remember my first visit to Kolomna very well. Rokhlin had an apartment there, which was very orderly; everything was very accurate and he was dressed very well. We began to talk and he made a very strong impression on me.

Rokhlin had great experience in ergodic theory because he had already published several papers in this field. His doctoral thesis was also about this subject.

Rokhlin formulated a number of interesting problems in ergodic theory. Some of them were connected with Rokhlin's theory of measurable partitions. This theory became very useful in ergodic theory because through it one can understand conditional probabilities in probability theory much better.

One of the problems that I began to work on under the influence of Rokhlin was the calculation of entropy for group automorphisms of the two-dimensional torus. At that time it was not known that Kolmogorov's definition had to be modified; the analysis was rather difficult and I could not achieve anything. Following the ideology of that time, I tried to prove that the entropy was zero but all my attempts failed. Then I visited Kolmogorov and showed him my drawings. He said that it was clear in this case that the entropy must be positive. After that I proved the result.

At that time there was no question about publication of my paper because Kolmogorov's paper on entropy had been published and it was not clear why another definition of entropy was needed. However, after some time, Rokhlin pointed out his result about the deficiencies in the definition by Kolmogorov. It became clear that I had to publish my paper with the definition and the calculation of the entropy for the automorphism which I had already done at that time.

This was the beginning of my contact with Rokhlin. After that, he organised a seminar on ergodic theory in Moscow, which was attended by Arnold, Anosov, Alexeev and others. In parallel, he had a seminar in topology where Novikov was the central figure.

Later Rokhlin moved to Leningrad (Saint Petersburg) and I used to go there to give talks at his seminar on later results.

Billiard systems

You then came up with an extremely interesting example of an ergodic system: the so-called billiards. Can you explain what these are?

A billiard, as people know, is the motion of a ball on the billiard table. An interesting mathematical theory arises if you allow the table to have a more or less arbitrary form. A natural question, which was actually raised by the Russian physicist Krylov long before the theory of entropy appeared, was: Which billiard systems have the same instability as the dynamics of particles moving in a space of negative curvature? Particles moving in a space with negative curvature yield the best example of unstable systems. The theory of billiards says that if the boundary of the table is concave then the system is unstable in the sense we previously described. If we consider two initial conditions with different values of the velocities then the corresponding trajectories diverge exponentially. If you consider a typical unstable billiard, namely the usual square billiard with a circle removed, then the difference between this billiard and the usual billiard is that for the unstable billiard the particles come to the holes much faster than for the usual billiard.

This may become a little technical now. You proved a very important result about systems with positive entropy. Given a system with positive entropy you can find a Bernoulli shift – which is a so-called factor – with the same entropy. This implies that if you have two Bernoulli shifts with the same entropy, they are what is called weakly isomorphic. Ornstein proved later that entropy is a complete invariant for Bernoulli shifts. It follows then from the work of Ornstein that the billiard example is the most chaotic system and is actually a Bernoulli flow, right?

From Ornstein's theorem it follows that if we have two ergodic billiard systems with the same value of entropy then they are isomorphic. This is a remarkable and great result.

So coin tossing is, in a sense, similar to the deterministic billiard system – an amazing fact.

My result says that if you have a system with positive entropy there could be subsystems that move like Bernoulli shifts.

What about billiard systems in higher dimensions? Is anything known there?

A lot of things are known. We have, for example, the result from the Hungarian mathematician Nándor Simányi who is in Alabama now. He studied multi-dimensional dynamical systems that eventually become unstable and have positive entropy and are ergodic.

You introduced Markov partitions in your study of Anosov diffeomorphisms. This led to what later became known as the Sinai–Ruelle–Bowen measure, also referred to as the SRB-measure. Would you please explain?

First of all, there was my paper where I constructed this measure for the case of the so-called Anosov systems, or just hyperbolic systems. Then there was a paper by Bowen and Ruelle where they extended this construction to systems considered by Smale, that is, Axiom A systems with hyperbolic behaviour.

These measures are important if you study irreversible processes in these systems. Suppose you start with some non-equilibrium distribution and consider the evolution and you ask how a non-equilibrium distribution converges to the equilibrium one. The result of the theory says that the evolution is in a sense very non-uniform, along some directions the expansion is very small and all the time averages behave very well and converge to a limit. But along other directions this convergence is very erratic and hence it can only be studied using probability theory. So the measures, which are called SRB-measures, are the ones which are smooth along some unstable directions and are very irregular along other directions. This is a class of measures that appears in the theory of evolution of distributions in the case of chaotic systems.

Are the SRB-measures related to Gibbs measures?

Yes. These measures are examples of Gibbs measures. But the Gibbs measures are a much more general object.

Mathematics and Physics

Let's go back to more general questions, starting with the interplay between mathematics and physics. May we begin with the physicist Eugene Wigner, who in 1960 published the paper "The unreasonable effectiveness of mathematics in the natural sciences", in which he gave many examples showing how mathematical formalism advanced physical theory to an extent that was truly amazing. Do you have a similar experience?

My impression is that this effectiveness of mathematics is no longer a surprise for people. There are so many cases, for instance the fact that string theory is practically a mathematical theory for physics. Some time ago Joel Lebowitz organised a discussion about this phrase of Wigner – in particular how it can be that mathematics is so effective. The conclusion was that this is just a well-established fact.

In my generation, there was a group of young mathematicians who decided to study physics seriously. However, there were different points of view of how to do mathematical physics. F.A. Berezin always stressed that mathematicians should prove only results that are interesting for physicists. R.L. Dobrushin and I always tried to find in physical results some possibilities for mathematical research.

On the other hand, there seems to be influence going in the opposite direction. Physicists have had a noteworthy impact on questions in quantum geometry and sometimes even in number theory. They have come up with formalisms that were not really developed in mathematics but nevertheless led to correct predictions which could be verified only after lengthy mathematical development.

So mathematics is effective but you can say that it is not effective enough.

You published in 2006 an article with the title "Mathematicians and Physicists = Cats and Dogs?" What is the main message of that paper?

I wanted to show examples where mathematicians and physicists look at the same problems differently. One example for this is the following story: my student Pirogov and I worked on problems in the theory of phase transitions in statistical physics. We proved several theorems and I went to meet the famous Russian physicist Ilya Lifshitz to show him our theory; Lifshitz replaced Lev Landau when Landau had his car accident, severely incapacitating him. When I presented the theory he stopped me and said: "It's very simple what you are talking about." He started to write formulas which eventually gave our results. I left him very much embarrassed and I started to think why this had happened. I realised that the final result of our theory was an obvious statement for him. He certainly did not know how to prove it but he did not need the proof. He just used it as an obvious fact.

There is a famous quotation of the great Gauss: "Now I have the result. The only thing remaining is the proof." So intuition does play an important role in mathematics…

I can also tell the following story, again connected to Gelfand. I explained to him a theorem, which we obtained together with Robert Minlos. And Gelfand said: "This is obvious. All physicists know this." So we asked him if it was so obvious, should we write a text of 200 pages with complete proofs? He looked at us and said: "Certainly, yes!"

A Jewish mathematician in the Soviet Union

May we continue with a political question? You mentioned that being at school in the time of Stalin was not easy; life was still difficult for you when you entered university and started your career. You came from a Jewish family; in the Soviet Union, at least sometimes, a latent anti-Semitism prevailed…

I can mention two cases in my career when I encountered anti-Semitism. The first one was the entrance examination, which I failed. The influence of my grandfather, who was Head of the Chair of Differential Geometry, and the help from the President of Moscow University, I.G. Pertrovsky, were needed in order to give me the possibility of being admitted to the University. This was a clear sign that things were not simple.

The other case arose with my entrance examination to graduate school. This exam was about the history of the Communist Party; I was very bad in this topic and failed the exam (I don't want to discuss the details). But P.S. Alexandrov – who was Head of the Mathematical Department at Moscow State University – together with Kolmogorov, visited the Head of the Chair of the History of the Party and asked her to allow me to have another attempt. She gave permission and I got a B on the second attempt, which was enough to enter the graduate school. The result was not clear a-priori and it could have gone either way.

In spite of these obstructions, it is quite obvious that many famous Russian mathematicians were and are of Jewish origin. This is quite amazing – can you offer any explanation?

First something trivial: Jews had more traditions in learning than other nations. They study the Bible, the Talmud and other religious books and spend a lot of time doing this, which is conducive to learning. At that time, following the Jewish religion was strictly forbidden. People still did, however, but under very high pressure. If you do something under pressure you work more. There is some kind of conservation law. This is my opinion of why Jews could succeed.

You had to be much better in order to get the same opportunities?

I think it would be wrong to say that we had this feeling. We certainly tried to prepare for all exams and competitions; the result was not clear a-priori but there was always a hope that something could come out of it.

Perhaps another reason is (especially under Stalin but also later) that a lot of very intelligent people were attracted to the natural sciences because there were fewer restrictions than in, say, history or political science…
That is certainly true. I can give you one example: at the time, Mech-Mat, the Faculty of Mechanics and Mathematics, had many graduate students that came from other countries. The rule was that they could only have advisors who were members of the Party. But there were students who wanted to work with Arnold, with me or perhaps with some other people. The way out of this situation was the following: there were a number of people in the Party who became the students' official advisors but the students actually worked with professors and mathematicians who were not members of the Party.

East and West

You told us that you were not allowed to travel for many years, and this happened to a lot of Russian mathematicians at the time. Did these obstructions hamper or delay progress in science? Did it have the effect that Russian mathematics did not get recognition in the West that it deserved?
It is very difficult to answer your question because you are asking what would have happened if something didn't happen. It is just impossible to say. It certainly caused harm but it is not clear how big it was.

Arnold was rather adamant about the lack of recognition. As a consequence of bad communication between East and West, results by Russian mathematicians during the isolation period were sometimes later rediscovered in the West. Therefore Russian mathematicians did not get the credit they deserved.
I have perhaps a special point of view concerning this. The question is whether some results can be stolen or not. My point of view, to which many people probably won't agree, is that if a result can be stolen, it is not a very good result.

Tell us about the Landau Institute for Theoretical Physics in the Russian Academy of Sciences – your workplace for many years.
For many years the Landau Institute was the best institution in Russia. It was organised after Lev Landau's untimely death as a result of a car accident. Its director I.M. Khalatnikov had a remarkable talent to find gifted people all over Russia and to invite them to the institute. After several years, the Landau Institute had a very strong group of physicists like Abrikosov, Gorkov, Dzyaloshinski, A.B. Migdal, Larkin, Zakharov, Polyakov, A.A. Migdal and many others. The group of mathematical physicists was headed by S.P. Novikov and was much smaller.

It turned out that there was a big area of theoretical physics in which mathematicians and physicists could understand each other very well. They could even work on similar problems. Among these mathematicians I can name Novikov, Krichever, Khanin, Shabat and Bogoyavlenskiy. Sometimes we invited physicists to explain to us their results in our seminars. The tradition of discussing problems of mutual interest still prevails.

You moved in 1993 from the Landau Institute to Princeton University while still maintaining your position in Moscow. Why was it so attractive for you to go to the USA?
That is an easy question. First of all I had many friends at Princeton. When we met we always had many points for discussion and common interests. Another reason was that many people had escaped from Russia so the situation there was no longer what it was before. In previous times everybody was in Moscow and St. Petersburg and you could call everyone to ask questions or to have discussions. Now that became impossible. The working conditions were better in the West and in particular at Princeton.

You have now been in the USA for more than 20 years and you must know the American system almost as well as the Russian one. Could you tell us about how they compare from your perspective?
Concerning academic lives, it seems to me that they are more or less similar. However, I must stress that I was never a member of any scientific committee at Mech-Mat at Moscow State University and I was never invited to participate in any organisational meetings. Now I am Chairman of the Scientific Council at the Institute of Transmission of Information.

Teaching and collaboration

You have been teaching courses and seminars for almost all your career. Do you have a particular technique or philosophy?
First of all, I like to teach undergraduate courses rather than graduate courses for the following reason: when you teach undergraduate courses you can easily see how your students become cleverer and more educated as they absorb new notions and connections and so on. When you teach graduate courses, the subject matter is usually a narrow piece of work and students are mostly interested in some special issues that are needed for their theses. For me, that is less attractive.

My basic principle is as follows: if people do not understand my explanations then this is my fault. I always ask students to ask questions as much as possible. Students who have asked me many questions during the lecture course have better chances for a good mark.

You have an impressive list of students that have done well after graduation under your supervision. Grigory Margulis, just to mention one name, won the Fields medal in 1978 and he will give one of the Abel lectures related to your work later this week…
I think the reason for this is not because of me but because of the types of problems we worked on. We did

very interesting mathematics and formulated interesting problems that students were attracted to. This is my explanation. Many students preferred to work independently and I was never against this.

You are a very good example of the fact that mathematicians can flourish in late age as well. We came across a paper on number theory that you published this year together with two of your students. You have also published other papers related to number theory so you must have kept an interest in that aspect of ergodic theory?
Yes, definitely. In the field we are working in there are many problems that are more natural for ergodic theory than for number theory. I don't want to be specific but we had a paper that was more natural for an ergodic theorist than for a number theorist, so we could get the results more easily.

Many joint papers appear on your list of publications. Apparently you like to have a lot of collaborators.
Well, I would say that they like it! And I'm not against it. It has never happened that I have asked someone to be my co-author. I can only talk about some problems and explain why they are interesting.

But you are right, I have had many co-authors. I very much liked collaborating with Dong Li, who is now a professor at the University of British Columbia. When we work on the same problem we call each other many times a day. There are many others of my students with whom I liked to work. It's different to work with different people. Certainly I can work with Russian mathematicians as well as with mathematicians from other countries. Sometimes I like to work alone but with age, I need co-authors.

You have only published one paper with Kolmogorov but you have mentioned that you would have liked to publish more papers with him.
At a certain time, Kolmogorov decided that the Soviet Union did not have enough applied statistics. He worked on theoretical statistics and found many beautiful and deep results but he was not satisfied with the fact that the theorems in applied statistics were not used for practical purposes. He found a problem related to the motion of the rotational axis of the earth that could be studied with the help of mathematical statistics. French observatories published data about the axis of rotation every two weeks and Kolmogorov wanted to construct statistical criteria that could predict this motion. He wanted us to work on this problem and invited a very good geophysicist Yevgeny Fyodorov, who was one of the main experts in this field. We were sitting there – Kolmogorov and Fyodorov were present. Kolmogorov said: "Look at these people; they prefer to write a paper for *Doklady* instead of doing something useful." (*Doklady* was the leading Russian journal.) In our joint paper (by M. Arató, A. Kolmogorov and myself) written on this occasion, practically everything was done and written by Kolmogorov. Later, M. Arató wrote a big monograph on that subject.

In other cases, I often tried to explain my latest results to Kolmogorov. Sometimes his reaction was unexpected: "Why did you work on that problem? You are already a grown-up?" But usually his reaction was very friendly. I regret very much that we never worked together; perhaps the reason is a difference in style.

Wasn't it Kolmogorov who said that he spent a maximum of two weeks on a problem?
Kolmogorov used to stress that he did not have papers on which he worked for a long time. He mostly prepared his papers, including the proof and the text, in just two weeks and this was a major difference in our approaches. Kolmogorov was a person with a strong temperament and he could not do anything slowly. I worked on some of my papers for years.

He was a towering figure, not only in Russian mathematics but worldwide in the 20th century.
Yes, definitely. Can I tell you one more story about him? It was when Kolmogorov was close to 80. I asked him how it happened that he was a pure mathematician, even though he worked on concrete physical problems like turbulence. He answered that he was studying the results of concrete experiments. He had a lot of papers with results from experiments lying on the floor. He was studying them and in this way he came up with his hypotheses on turbulence.

So his intuition was motivated by physical considerations?
Yes. He subscribed to physical journals and one could say he was into physics in a big way.

Is that also true for you? Do you think mainly in terms of algebraic or analytic formulas? Or is it geometric intuition or even a mixture of all of that?
It depends on the problem. I can come to the conclusion that some problem must have a specific answer. I just told a journalist the story about a problem in which I knew there should be a definite answer. I worked on this problem for two years and at the end of that time I discovered that the answer was one-half!

In general, I probably prefer to develop theories, sometimes to find the right concepts rather than solving specific problems.

Have you had what we sometimes call a Poincaré moment, where all of sudden you see the proof?
Ideas often come unexpectedly, sometimes like revelations. But it happens only after a long period, maybe years, of difficult work. It did not happen while trying to find a taxi or something similar. It was very hard work for a long time but then suddenly there was a moment where it became clear how the problem could be solved.

If you yourself made a list of the results that you are most proud of, what would it look like?
I like all of them.

Mountaineering

You mentioned Arnold who died four years ago, an absolutely brilliant Russian mathematician. Arnold is, among many other things, known for his contributions to the so-called KAM theory. You both followed Kolmogorov's course and seminars in 1958. You told us that there was a close friendship already between your grandfathers.
Both you and Arnold loved the outdoors and hiking. You once went to the Caucasus Mountains together and you have to tell us the story about what happened when you stayed in the tents with the shepherds.
That is a very funny story. The weather was very bad; there was a lot of rain. We came to the shepherds' tent and they let us in and we could dry our clothes. We had lost our tent in the mountains so we decided to go back to try to find it. We started to walk back but these shepherds had some very big dogs – Caucasus dogs, a really big race. The shepherds weren't there any longer and when the dogs found out that we were leaving, they surrounded us and started to bark ferociously. Arnold began to yell back with all the obscenities he knew and the dogs did not touch him. But they attacked me. They didn't touch my skin but they ripped my trousers apart. Finally, the shepherds came back and we were saved.

We would like to ask one final question that has nothing to do with mathematics: you have certainly focused on mathematics during your life but surely you have developed other interests also?
I was interested, especially in former years, in many different sports. I was a volleyball player and I liked to ski, both downhill and cross-country. I also liked mountaineering but I cannot say I was a professional. I climbed often with a close friend of mine, Zakharov, who worked on integrable systems. We were climbing in the mountains together and once we were on a very difficult 300 metre long slope, which took us four hours to get down from! We had to use ropes and all sorts of gear. Nowadays, my possibilities are more limited.

From left to right: Yakov Sinai, Martin Raussen, Christian Skau (Photo: Eirik F. Baardsen)

Thank you very much for this most interesting conversation. We would like to thank you on behalf of the Norwegian, the Danish and the European Mathematical Societies.
Thank you very much.

Martin Raussen is associate professor of mathematics at Aalborg Univesity, Denmark. Christian Skau is professor of at the Norwegian University of Science and Technology at Trondheim. They have together taken interviews with all Abel laureates since 2003.

Speech in Honour of the Abel Prize Laureate

Opening the Banquet at Akershus Castle

Jean-Pierre Bourguignon (IHÉS, Bures-sur-Yvette, France), President of the ERC

Dear Minister Røe Isaksen,
Dear Abel Prize laureate Professor Yakov Sinai,
Your Excellencies, Distinguished guests, Friends, ladies and gentlemen.

It is a great privilege to address you tonight to honour and congratulate the recipient of the 2014 Abel Prize.

This prestigious prize celebrates the memory of the towering figure of Niels Henrik Abel in the history of mathematics. The international mathematical community is very grateful to the Norwegian Government because this prize is now a new reference in the very selective, yet expanding, circle of awards accessible to mathematicians. Further, it gives due recognition to mathematics as one of the major components of the scientific endeavour, itself a very significant step in the efforts of Mankind to understand and master the world we live in.

To tell you the whole truth, it was late in December when I was approached on behalf of Minister Røe Isaksen to deliver this speech. I was honoured to give my consent well before the name of the laureate was known, confident that the choice of the Abel Prize winner 2014 by the Abel Committee would be impeccable, as it has been for the past eleven years. When the name of Professor Yakov Sinai was released, my expectation was confirmed. However, the pleasure of speaking of him here this evening is even greater, as I have the highest respect for him, not only as a mathematician, but also as a person.

Professor Sinai's many outstanding contributions to mathematics touch several fields, most of which are connected to real world problems. Professor Sinai has greatly expanded our understanding of the evolution of all kinds of systems, from the motion of the solar system, which is somewhat chaotic if one looks at it over long periods of time, to the behaviour of molecules in a gas, that is the fundamental basis for thermodynamics, the key physical theory for all sorts of engines, including the Earth.

His work exemplifies the way mathematicians impact other disciplines by bringing about new concepts and the development of techniques that enable us to gain insight into very complicated and intricate situations. Some of his work has also dealt with strictly mathematical questions.

As a matter of fact, it is in such a context that I was first confronted with one of his achievements. In 1986, he gave a dramatic simplification of a very elaborate proof, more than 200 pages long, of a hard theorem first proved by Michel Herman, with whom I was very close. The new approach, inspired by physicists, gives a direct way of obtaining the result, making it accessible even to young students. The great ingenuity of the method in itself was an impressive achievement.

When asked, Professor Sinai says that his domain is "ergodic theory". In one of his articles, he quotes the physicist I.M. Lifshitz on the following definition: "*Ergodic theory is a theory which explains that every shoelace sooner or later becomes disentangled.*" This is not the place to go into much more detail about what ergodic theory really is but it is enough to say that it contributes to predicting the unpredictable evolution of a large class of chaotic systems.

But how and when did the consideration of chaotic behaviours capture the interest of mathematicians? This piece of history gives me the opportunity to mention that interest in scientific achievements is not recent here in Scandinavia. In 1885 the still young journal *Acta Mathematica* launched a prize competition under the high patronage of King Oskar II of Sweden and Norway. The topic proposed for the Prize was the N-body problem in celestial mechanics. This referred to the fact that, whilst Isaac Newton, thanks to mathematical concepts and tools he had created, showed that the 2-body problem had an explicit beautiful solution, the 3-body problem already escaped the understanding of the most distinguished mathematicians. And, as you all know, the solar system we live in has many more bodies, even if one neglects the huge number of small asteroids.

Henri Poincaré, pressed to consider the question, finally submitted a memoir in 1888 resulting in his receiving the Prize. We now know that the manuscript he submitted contained a major error, that he could identify thanks to the pressing questions of Lars Edvard Phragmen, the young mathematician in charge of publishing it. In the final corrected version, he gave a description of the chaotic behaviour of the planets around the sun when there is more than one of them and even if one of the bodies considered has a small mass. This led him to write his famous *Leçons nouvelles sur la mécanique céleste* in three volumes. These are regarded as the founding building blocks of the qualitative approach to the theory of dynamical systems.

Professor Sinai's contributions, extending vastly those of his master Andrey Kolmogorov, provide a deep linkage of these questions to probability theory. He explored systematically the stochastic aspects of these chaotic behaviours to gain understanding of the phenomena met. Professor Sinai's work is often classified as mathematical physics, as it is truly a monument of mathematical theory. His work is highly relevant to physicists in their understanding of chaotic evolutions. However, he considers himself a mathematician. Let me share with you the opinion of a physicist on him: "*I consider Yakov Sinai as one of the truly great scientists of our time. He is both very imaginative and very broad, since his work covers a number of areas of mathematics and physics.*"

Professor Sinai is therefore very knowledgeable about the relations between mathematicians and physicists. This is the question he addresses in an article entitled "Mathematicians and Physicists: cats and dogs?" You may have a strong opinion on who may be the cats and who may be the dogs. I read the article very carefully to try and figure out on which side Professor Sinai was putting himself, as I care about this matter. I must admit I could not find the slightest hint. Very recently though, when having a look at the book by Vladimir Arnold and André Avez from which, as a student, I learned of ergodic theory, I may have finally found something: one of the most significant examples of dynamics proposed in the book is illustrated by considering how a cat's face is transformed when one applies a specific (ergodic) transformation once, then twice, then three times, and so on.

The awfully distorted image of the face of the poor cat submitted to such a treatment prompted the authors to say that they looked for the permission of the Animal Protection Society to reproduce it.[1] And who is thanked by the authors for reading the manuscript and improving it? Yakov Sinai. I am very happy therefore to conclude that I am a dog!

Alas, the search may not be over: indeed, in the recent interview Professor Sinai gave to a journalist of Nature, he says: "*Mathematics and physics must go together as horse and carriage*," This was after he received the Prize. You see to what extent prizes transform people.

Let me close in renewing my most sincere congratulations to the Norwegian people on the 200th anniversary of their very progressive Constitution, to the Norwegian government for its wonderful support of mathematics through the Abel Prize and several other very significant actions to help young people develop interest in Mathematics, and of course to Professor Yakov Sinai for the example provided as a mathematician and as a man.

[1] Yakov Sinai announced in the Abel lecture he gave the morning following the banquet that Vladimir Arnold never had a cat, but did have a dog.

Mathematical Billiards and Chaos

Domokos Szász (Budapest University of Technology and Economics, Hungary)

1 Introduction

On 20 May 2014, Yakov G. Sinai of Princeton University and Landau Institute for Theoretical Physics, Russian Academy of Sciences, was awarded the Abel Prize at a ceremony at University Aula in Oslo [1]. In conjunction with Sinai's award, there were four mathematical lectures, including the Prize Lecture by Sinai himself ("How everything has been started? The origin of deterministic chaos") and distinguished mathematical lectures by Gregory Margulis of Yale University ("Kolmogorov-Sinai entropy and homogeneous dynamics") and Konstantin Khanin of the University of Toronto ("Between mathematics and physics"). I gave the Science Lecture of the same title as this article, which was based on my talk in Oslo. The videos of all four lectures can be seen at http://www.abelprize.no/artikkel/vis.html?tid=61307.

My goal was to recite Sinai's most far-reaching results and their most significant after-effects on billiards, which were described in the citation as follows:

> Sinai has been at the forefront of ergodic theory. He proved the first ergodicity theorems for scattering billiards in the style of Boltzmann, work he continued with Bunimovich and Chernov. He constructed Markov partitions for systems defined by iterations of Anosov diffeomorphisms, which led to a series of outstanding works showing the power of symbolic dynamics to describe various classes of mixing systems.

By its definition: *The Science Lecture is intended for the broader scientific community and aims to highlight connections between the work of the Abel Laureate and other sciences*. Moreover, the request of the Prize Committee emphasised that the Science Lecture is also meant for students of mathematics. To accomplish this task, I will use many illustrations and metaphors.

Sinai's results on billiards have exceptional significance in science. For an explanation I will go back to the birth of statistical physics.

2 Atomic theory and the birth of statistical physics

Atomic theory assuming or claiming that matter is composed of discrete units called atoms or molecules goes back to thinkers from ancient Greece and India. Extensive scientific theories relying on this assumption only formed in the 19th century, whereas – however surprising it may sound – its scientific confirmation only occurred in the 20th century. Sinai's works mentioned in the citation originate from two fundamental questions raised by 19th century statistical physics.

The first puzzle is related to Ludwig Boltzmann's ergodic hypothesis. Basic laws of statistical physics were formulated by classics of the theory (Maxwell, Gibbs and Boltzmann) on a macroscopic level by assuming that on the microscopic level atoms move obeying laws of Newtonian mechanics. In Boltzmann's picture and in today's language, macroscopic concepts of statistical physics: temperature, pressure etc. appear as the result of the law of large numbers valid in the microscopic system. His ergodic hypothesis claims that the equilibrium expected value of a measurement can also be obtained as the limit of time averages, under Newtonian dynamics, of the same measurement when both time and the size of the system tend to infinity (see [3, 4]).

The second problem also goes back to the 19th century. By the progress of microscopes, inquisitive scholars could observe an increasingly wide circle of phenomena. For instance, in 1827, Robert Brown, a British botanist, inspected pollen of the plant Clarkia pulchella suspended in water. To his great surprise, pieces of pollen showed an erratic, random motion. He was even more astonished when probing minerals including liquids isolated for millions of years: tiny particles suspended in them displayed the same erratic motion. Is it imaginable that a living substance can survive in the inclusion for such a long time? Or can this chaotic motion be explained by atomic theory? Is it the result of collisions of the observed 'Brownian' particle with atoms of the liquid?

This latter idea was quantified in 1905 in the derivation of the diffusion equation by Einstein, a believer in atomic theory. The diffusion equation is, indeed, the appropriate model for the chaotic motion of the Brownian particle and it is worth noting that the same equation also depicts the flow of heat in many substances. It is less known that Louis Bachelier – motivated by the fluctuations of stock prices – as early as 1900 introduced the stochastic process solving the diffusion equation and also describing Brownian motion (Einstein does not seem to have known about this work). The precise mathematical definition of this process was provided in the early 1920s by Norbert Wiener and most often it is called a Wiener process (or a Brownian motion process).

3 Ergodic hypotheses: From Boltzmann to Sinai

In today's language, the basic object of ergodic theory is a group ($S^{t+s} = S^tS^s,\ \ t, s \in \mathbb{R}$) of measure preserving transformations $\{S^t|t \in \mathbb{R}\}$ on some probability space $(M, \mathcal{F}, \mu)$. Here the non-empty set M is the phase space and μ is a probability measure on it. Measure preserving means that the dynamics $\{S^t\}$ leaves the probability μ invariant, i.e. $\forall A \subset M$ and t $\in \mathbb{R}$ one has $\mu(S^tA) = \mu(A)$. (Later we will forget about the $\sigma-$algebra $\mathcal{F}$.)

For Boltzmann's formulation of the ergodic hypothesis, consider N particles in a vessel that – for simplicity – can be the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ (or the cube $[0, 1]^3$). Thus the phase space of the system is $M_N = \{(q_i, v_i)|1 \le i \le N,\ q_i \in \mathbb{T}^3,\ v_i \in \mathbb{R}^3, \sum v_i^2 = 1\}$. Of course, the dynamics $S^t_N x_N$ ($t \in \mathbb{R}$ – time, $x_N \in M_N$), the invariant probability μ_N and the measurement $f_N : M_N \to M_N$ all depend on N.

Conjecture 1 (Boltzmann's ergodic hypothesis)**.** For a typical initial phase point $x \in M$,

$$1/T \int_0^T f_N(S^t_N x)\,\mathrm{d}t \to \int_{M_N} f_N(y)\,\mathrm{d}\mu_N(y)$$

as $T, N \to \infty$ (or, in other words, time averages converge to the equilibrium average).

This is an hypothesis of an ingenious theoretical physicist. It is far from having a precise mathematical meaning (for instance, neither the sense of convergence of time averages nor the way that T and N tend to ∞ is specified) though it is known that Boltzmann had, indeed, made a lot of tricky calculations to convince himself at least that the conjecture holds. (According to Boltzmann's picture, a physical system in equilibrium – in its time evolution – (1) not only goes through all possible phase points of the system, but it does that in such a way that (2) the relative sojourn times in subsets are close to the equilibrium law of the system. Claim (2) is just the content of his ergodic hypothesis whereas claim (1) in general fails.) Despite this deficit of exactness, the conjecture and likewise the derivations of his famous equation or his H-theorem and other statistical arguments were known to several top mathematicians of the era, so much so that David Hilbert – in his celebrated lecture on ICM1900 about 23 problems of mathematics for the 20th century – included one, namely the 6th with the title *Mathematical Treatment of Axioms of Physics*. Its issue was to give a precise mathematical background to related arguments of theoretical physicists and Hilbert himself said:

> … I refer to the writings of Mach, Hertz, Boltzmann and Volkmann …

The response of mathematics was not immediate. Inspired by many sources, modern measure theory was first founded in the first decade of the 20th century; later in the 1920s – pressed by the necessity to provide mathematical framework for quantum physics then in statu nascendi – functional analysis and the theory of operators were created. In the academic year 1926/27 John von Neumann visited Hilbert in Göttingen. Until this visit, his main interest was pure mathematics and the reason for the visit was his shared interest with Hilbert in the foundations of set theory. Nevertheless, during this year Neumann got deeply involved in operator theory, which certainly helped him understand the mechanism behind the convergence of time averages and allowed him to prove the first ever ergodic theorem. Consider a fixed dynamical system (M, S^t, μ) as formulated at the beginning of this section.

Definition 1. (M, S^t, μ) is *ergodic* if $\forall x \in M$, $\forall t \in \mathbb{R}$ $f(S^t x) = f(x)$ implies $f = \text{const}$ for μ – a.e. $x \in M$. (In other words, every invariant function is constant almost everywhere.)

For an illustration, let us try to decide whether the simplest 'physical' dynamics, i.e. uniform motion on the Euclidean 2-torus is ergodic.

Example 1: Geodesic motion on the (Euclidean) 2-torus $\mathbb{T}^2$. $M = \mathbb{T}^2 \times \mathcal{S}$, $(\mathcal{S} = \mathbb{T}^1)$. For $x = (q, v) \in M$, $S^t x = (q + tv, v)$, μ = area. We note that the restriction $\|v\| = 1$ reflects the conservation of energy. This 'boring' dynamics is, of course, nonergodic since $f(q, v) = v$ is a non-constant invariant function.

Theorem 1 (Neumann's Ergodic Theorem, 1931)**.** Assume (M, S^t, μ) is ergodic and $f : M \to \mathbb{R}$ is a nice function. Then, as $T \to \infty$,

$$1/T \int_0^T f(S^t x)\,\mathrm{d}t \overset{L_2}{\Longrightarrow} \int_M f(y) d\mu(y) \qquad (*)$$

For instance, if – in an ergodic system – one takes the indicator function of a set A:

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

then according to the ergodic theorem the average time spent by the system in the set A converges to the probability $\mu(A)$ of the set as $T \to \infty$. (Later, George Birkhoff (1931) and Khinchin (1933) proved that $(*)$ also holds for μ–a. e. $x \in M$.)

A survey, written in 1932 by Birkhoff and Koopmans about the history of the first ergodic theorems wrapped up with:

> It may be stated in conclusion that *the outstanding unsolved problem in the ergodic theory* is the question of the truth or falsity of *ergodicity for general Newtonian systems.*

Indeed, since the search for ergodic theorems was motivated by physics, it is inevitably a major question of ergodic theory to clarify the mechanism lying behind the ergodic behaviour of physical motions. Once uniform motion in a Euclidean domain was not ergodic it was a natural idea to investigate uniform – or in other words geodesic – motion in a hyperbolic domain.

Example 2: Geodesic motion on the hyperbolic octagon. Let us consider the Poincaré disc model of hyperbolic (i.e. Bolyai-Lobatchevsky) geometry: $D = \{z \mid |z| < 1\}$. At a point $z \in D$ the hyperbolic metric $(ds)_{\text{hyp}}$ can be expressed through the Euclidean metric $(ds)_{\text{Euc}}$ as follows: $((ds)_{\text{hyp}})^2 = \frac{((ds)_{\text{Euc}})^2}{(1-r^2)^2}$, where $r = |z|$. As is well known, D supplied with this metric is a surface of constant negative curvature. In this case the phase space is $M_\infty = \{x = (q, v) | q \in D, v \in \mathbb{R}^2, |v| = 1\} = D \times \mathcal{S}$. The orbits of uniform, i.e. geodesic, motion are circular arcs orthogonal to the circle ∂D of infinities (see Figure 1). Completely analogously to the Euclidian case, here one should also factorise with respect to some discrete subgroup (there it was $\mathbb{Z}^2$). Here too, there is an abundance of possibilities and we select one of the simplest cases: we obtain a hyperbolic octagon O and it is compact and the invariant measure is finite (see Figure 2).

Theorem 2 (Hedlund, Hopf (1939))**.** The geodesic motion in the hyperbolic octagon (i.e. on $M = O \times S$) is ergodic.

The value of this result is not only that it represented the first system of physical flavour whose ergodicity was ever established but it also revealed the mechanism that can lead to ergodicity: it was profoundly connected to the hyperbolic feature of the geometry. The hyperbolicity of the dynamics can be easily illustrated by a favourite paradigm of ergodic theory courses: baker's map $T : [0, 1]^2 \to [0, 1]^2]$ (see Figure 3). T is defined as follows:

$$T(x, y) = \begin{cases} (2x, y/2) & \text{if } 0 \le x \le 1/2, \\ (2x - 1, \frac{y+1}{2}) & \text{if } 1/2 < x < 1. \end{cases}$$

In the figure, one can easily see that – at each step – in the horizontal direction the map expands by a factor of 2 and in

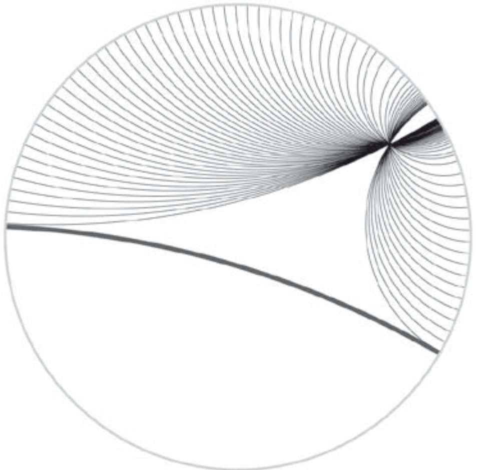


Figure 1. Geodesics in the hyperbolic plane

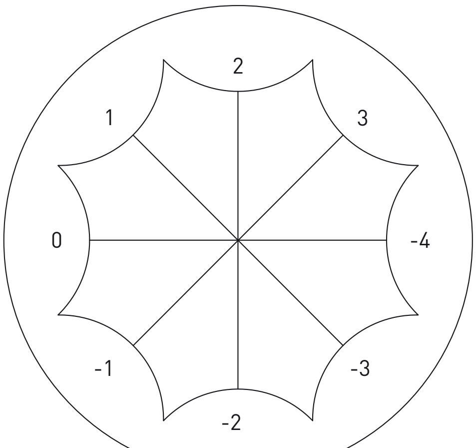


Figure 2. A hyperbolic octagonal torus (edges with i and $-i$ sides is meant to be glued together)

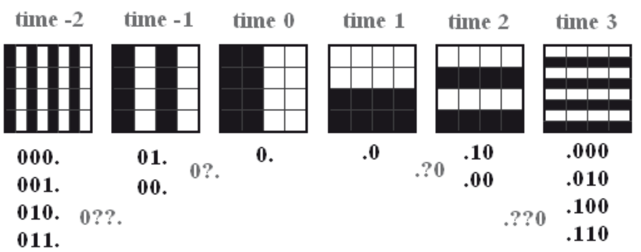


Figure 3. Iterates $-2, -1, 0, 1, 2$ of the baker map

the vertical one it contracts by the same factor of 2 and thus the area remains invariant. The composed effect of expansion and contraction is just hyperbolicity.

The proof also relied on constructions of Hadamard and introduced the method of Hopf chains that is still essentially the only general method for establishing ergodicity of hyperbolic systems.

Not only many top mathematicians but often top theoretical physicists have also been aware of the interesting goings on in each other's science. Thus, in 1942, an excellent Russian statistical physicist N. S. Krylov was playing around with the ideas of Hedlund and Hopf. His calculations led him to a remarkable observation: at collisions of elastic hard ball particles some hyperbolicity seems to appear.

In the 1960s, there was remarkable progress in the theory of smooth hyperbolic dynamical systems. Here, the main names to be mentioned are Anosov, Sinai and Smale. These systems provide far-reaching generalisations of geodesic flows on compact manifolds of negative curvature. On ICM1962 Sinai, combining Krylov's observation and the momentum of this progress, came up with the sensational conjecture:

Conjecture 2 (Sinai's form of Boltzmann's ergodic hypothesis, 1962)**.** A system of $N(\geq 2)$ identical elastic hard balls on $\mathbb{T}^\nu$ ($\nu \geq 2$) is ergodic (modulo the natural invariants of motion: energy, momentum and centre of mass, which are, of course, invariant functions).

This conjecture is fully in line with the programme formulated by Birkhoff and Koopman. Nevertheless, at that time, it deeply surprised experts who had been thinking that – in accordance with Boltzmann – for ergodicity the number of particles should tend to infinity. According to Sinai, two hard balls should already produce ergodicity?! Now we know that this is the peculiarity of the hard core interaction of balls. (For typical interactions the system of a fixed number of particles will not be ergodic since KAM islands will appear. What one can hope for then is that the larger the number of particles, the closer to 1 the measure of the largest ergodic component.) The reader interested in the early history of the Boltzmann-Sinai ergodic hypothesis up to 1996 can find more details in [16].

4 Sinai Billiards. Verification of the Boltzmann-Sinai Hypothesis

Sinai, 1970, and Case $N = \nu = 2$

The difficulty of verification of Sinai's Conjecture, today called the Boltzmann-Sinai ergodic hypothesis (or BSEH for short), is shown by the fact that Sinai needed eight years to complete the proof for the 'simplest' case of two planar – not too small – discs (see [12]). His approach used the introduction of dispersing billiards, also called Sinai billiards.

Definition 2. A billiard is a dynamical system $(M, \{S^t | t \in \mathbb{R}\}, \mu)$, where $M = Q \times S_{d-1}$, Q is a domain in $\mathbb{R}^d$ or $\mathbb{T}^d$ with piecewise smooth boundary, μ is the uniform probability measure on M and $\{S^t | t \in \mathbb{R}\}$ is the billiard dynamics: uniform motion in Q with velocity $\in S_{d-1}$ and elastic reflection at ∂Q. Smooth pieces of ∂Q are called scatterers. (Figure 4 illustrates a Sinai billiard on the 2-torus.)

Billiards had already been used before Sinai, e.g. planar billiards in an oval by Birkhoff. The reason for Sinai's introduction of dispersing billiards was based on the following simple fact.

Fact. *The system of $N \geq 2$ elastically colliding identical hard balls on $\mathbb{T}^\nu$ is isomorphic to a billiard whose scatterers are (subsets of) spheres when $N = 2$ and (subsets of) cylinders when $N > 2$.*

This isomorphic billiard can be obtained by first putting together all the centres of the individual particles into an $N\nu$-dimensional vector. Because of the hard core condition no

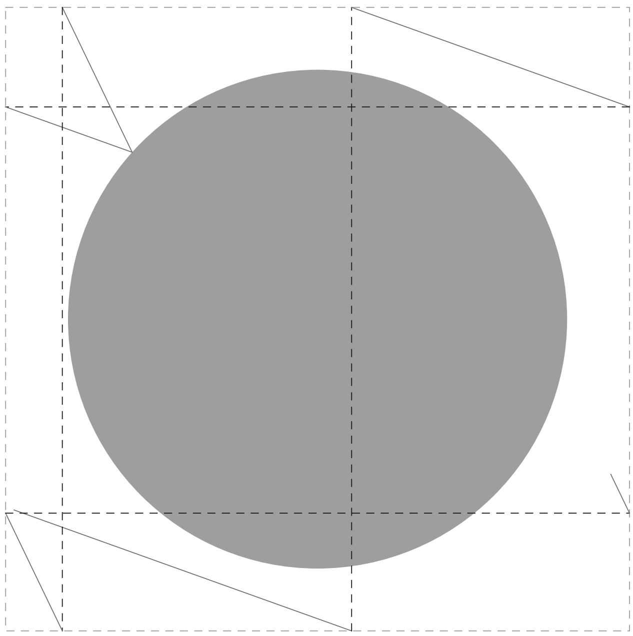


Figure 4. Orbit of a Sinai billiard (with one circular scatterer) on the 2-torus

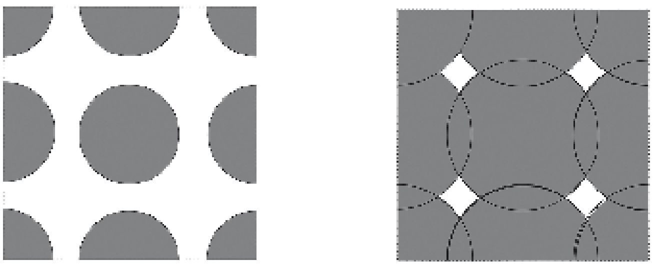


Figure 5. Sinai billiard isomorphic to dynamics of two elastic disks

pair of centres can be closer than $2R$, where R denotes the radii of the balls. These excluded sets are cylinders with spherical bases. Now, by taking into account conservation laws, the dimension of the 'billiard table' gets reduced by ν (total momentum and 'centre of mass' are assumed to be zero). Thus the dimension of the isomorphic billiard is $d = (N-1)\nu$.

Definition 3. A billiard is semi-dispersing (dispersing) if the scatterers are convex (strictly convex) as seen from the billiard table Q. Dispersing billiards are also called Sinai billiards.

Thus the system of two balls is isomorphic to a dispersing billiard and that of $N \geq 3$ balls to a semi-dispersing billiard. For instance, in Figure 5, one can see the isomorphic billiards corresponding to the case $N = \nu = 2$ (cases $R < 1/4$ and $R > 1/4$).

Definition 4. A billiard is said to have finite horizon if there is no infinite orbit without any collision. Otherwise it has infinite horizon. (For instance, in Figure 5, the first billiard has infinite horizon whereas the second one has finite horizon.)

Sinai's celebrated 1970 result is:

Theorem 3 ([12])**.** Every planar dispersing billiard with finite horizon is ergodic. (In particular, the system of two elastic discs is ergodic if $R > 1/4$.)

Later we will illustrate why dispersing billiards are hyperbolic but first we reveal a fundamental difficulty arising in billiards: the singularity of the dynamics. For both purposes it is useful to consider the evolution of fronts: smooth one-codimensional submanifolds Σ in the configuration space Q lifted to the phase space by adjoining to each point $q \in \Sigma$ an orthogonal unit normal vector $\in S_{d-1}$ as a velocity vector (this procedure has two continuous versions; later we will be interested in convex fronts, where the curvature operator of Σ is non-negative definite and then we will assume that the attached normal vectors open up). Indeed, as is shown in Figure 6, in the case of a tangential collision of a front, its image is not smooth anymore; in fact, after the collision its image is still continuous but is not differentiable anymore. Moreover, the readers can convince themselves that if the orbit hits an intersection point of different scatterers then, afterwards, the dynamics is not uniquely defined and the trajectory will have at least two smooth branches. In view of the singularity of the dynamics, the main achievement of Sinai's 1970 work was to show that *the theory of* smooth *hyperbolic dynamical systems can be extended to that of hyperbolic systems with* singularities – *at least in 2D*. This extension may seem to be a harmless generalisation but, in fact, it is a wonder that it is possible at all. From Hadamard, Hedlund and Hopf, and later from Anosov, Sinai and Smale, it was known that the main tools of hyperbolic theory were the so-called unstable (and stable) invariant manifolds possessing two important properties: 1) they were smooth and 2) the time evolution of the unstable ones had an exponentially expanding effect (in the case of stable manifolds, their time reversed evolution had it; see baker's map in Figure 3 considered earlier). Sinai showed that, in the case of singular systems, smooth pieces of invariant manifolds still exist for almost every point; they can, however, be arbitrarily small. This fact made the application of classical techniques, though possible, still a rather hard task. Sinai coped with this and his style was also optimal: though the subject was new he avoided overexplaining ideas; he provided as much information as was needed.

His forthcoming paper with Bunimovich could drop the condition on the finiteness of the horizon and with this the verification of the BSEH for two discs is completed.

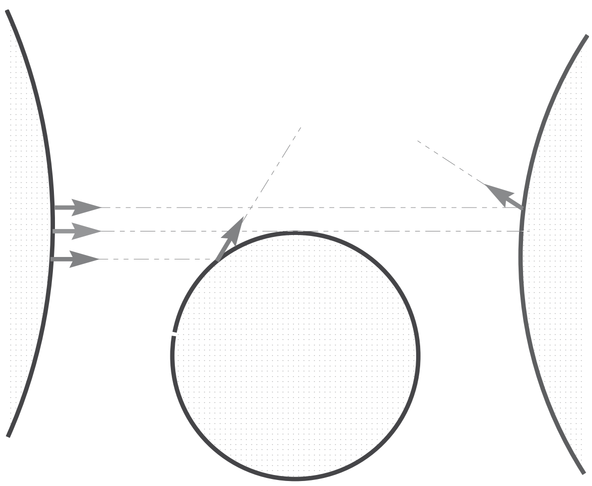


Figure 6. Tangential singularity of a billiard

Theorem 4 ([7])**.** Every planar dispersing billiard with infinite horizon is ergodic. In particular, two elastic discs on the 2-torus is ergodic.

Chernov-Sinai, 1987, and Case $N = 2, \nu \geq 2$

To proceed with the proof of the BSEH, there were two tasks: to also treat 1) the multidimensional case; and 2) the semi-dispersing case. Returning to the question of why these dispersing type billiards are hyperbolic, this is based on the following, intuitively transparent facts:

Fact 1. For a dispersing billiard – after one collision – any convex front becomes strictly convex.

Fact 2. For a semi-dispersing billiard – convexity of a front can only improve.

These facts are illustrated in Figures 7 and 8. Indeed, Figure 7 should convince the reader that after a collision with a strictly convex scatterer even a hyperplanar front, the extreme case of a convex one, will become strictly convex. On the other hand, Figure 8 shows that after a collision with a cylindrical scatterer, the direction(s) of a hyperplanar front, parallel with the generator of the cylinder, will remain linear, implying that the image of the hyperplanar front will not be strictly convex.

If we accept that for hyperbolicity, strict convexity of the image of any incoming convex front is necessary then what we can hope for is that this at least happens after collisions with several different scatterers. This explains the following fundamental definitions.

Definition 5. A phase point $x = (q, v) \in M$ is a hyperbolic point if the hyperplanar front through q and orthogonal to v becomes strictly convex – maybe only after several collisions (see Figure 9).

Definition 6. A subset $U \subset M$ belongs to one ergodic component if $\forall x \in U$, $\forall t \in \mathbb{R}$, $f(S^t x) = f(x)$ implies $f =$ const for μ – a. e. $x \in U$. (Compare this with Definition 1.)

Now we are in a position to formulate the main result of a beautiful work of Chernov and Sinai, born 17 years after Sinai's classical paper.

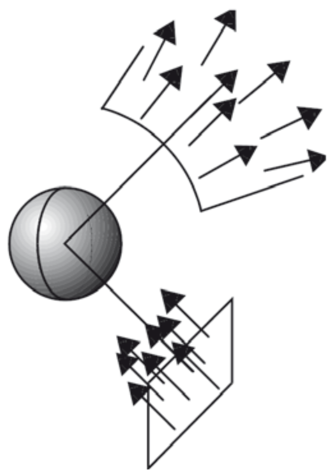


Figure 7. Collision in a dispersing billiard

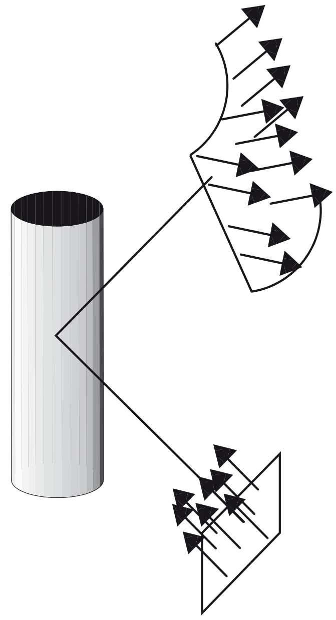


Figure 8. Collision in a semi-dispersing billiard

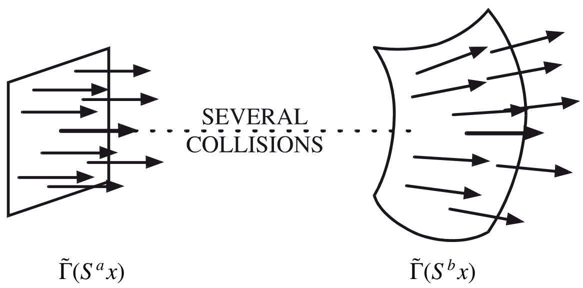


Figure 9. Hyperbolic orbit

Theorem 5 ([11])**.** For a semi-dispersing billiard ($d \geq 2$!), under some additional conditions every hyperbolic point has an open neighbourhood which belongs to one ergodic component.

A simple but important corollary of this theorem is:

Corollary 1 (simple)**.**
1. Every dispersing billiard is ergodic ($d \geq 2$).
2. The system of TWO hard balls on any $\mathbb{T}^\nu$ is ergodic.

Case $N \geq 2, \nu \geq 2$

Here we are satisfied to just list some main steps leading to the complete verification of the BSEH by Simányi in 2013.

- Going from the case $N = 2$ to the case $N > 2$ required the solution of additional topological, ergodic-theoretical and algebraic problems. In [10] this was done for the case $N = 3, \nu \geq 2$ of the BSEH.
- [6] found a model of hard balls with $N \geq 2, \nu \geq 2$ whose ergodicity could be shown. The technical ease of the model is that the balls are localised in it and as a result only balls in neighbouring cells can collide.
- By introducing algebraic geometric methods into the problem, [15] could prove that, for any $N \geq 2, \nu \geq 2$, typical systems of hard balls are hyperbolic.
- Though hyperbolicity is actually the antechamber of ergodicity, there still remained a lot of work to be done until the

complete verification of the BSEH. Simányi pursued this goal in a series of delicate works, until he could finish the arguments in [14].

5 Dynamical theory of Brownian motion

Section 2 already referred to Brown's discovery of the chaotic motion of particles suspended in a fluid. Einstein's derivation for it from microscopic assumptions was heuristic and used conditions which were not satisfied. In fact, one of his main aims with this work was to derive an estimate for Avogadro's number on the basis of atomic theory. This was so successful that it inspired the French experimental physicist Jean Baptiste Perrin to experimentally test Einstein's calculation, which he did in 1908. His observations were in full agreement with Einstein's calculations and this finally led to a complete acknowledgment of atomic theory (for this work Perrin deservedly received the Nobel Prize for Physics in 1926).

Since Einstein's explanation of Brownian motion was heuristic, the question was still left completely open to provide a microscopic and mathematically rigorous derivation for it. An intermediate step came first from probability theory. Random walks have been a classical and favourite model of stochastics. The simple symmetric random walk (SSRW) is a discrete time Markov process $\{S_n, n \in \mathbb{Z}_+\}$ on $\mathbb{Z}^d$; $d \geq 1$, such that $\forall n \in \mathbb{Z}_+, \forall k \in \mathbb{Z}^d$, $\mathbb{P}(S_{n+1} = k \pm e_j | S_n = k) = \frac{1}{2d}$, where e_j is any of the d unit coordinate vectors in $\mathbb{R}^d$. To make the ideas more transparent, several figures are presented: Figure 10 shows several trajectories of 1D SSRWs; Figure 11 shows one orbit of a 1D Brownian motion; Figures 12-14 present orbits of 2D SSRWs for 2500, 25000 and 250000 steps, respectively; and finally Figure 15 shows an orbit of a 2D Brownian motion. These figures suggest that – independently of the dimension d – if one looks at an orbit of a SSRW from a distance, then an orbit of a Brownian motion arises.

The mathematical formulation of this claim is the following: take a large parameter $A \in \mathbb{R}_+$ and rescale time with A and space with $\sqrt{A}$ (this is in accordance with the fact that the variance of S_n is $\sqrt{n}$). In other words, consider the rescaled trajectory $X_A(t) = \frac{S_{At}}{\sqrt{A}}, t \in \mathbb{R}_+$ of the SSRW (so far S_{At} has only been defined for integer values of At; we

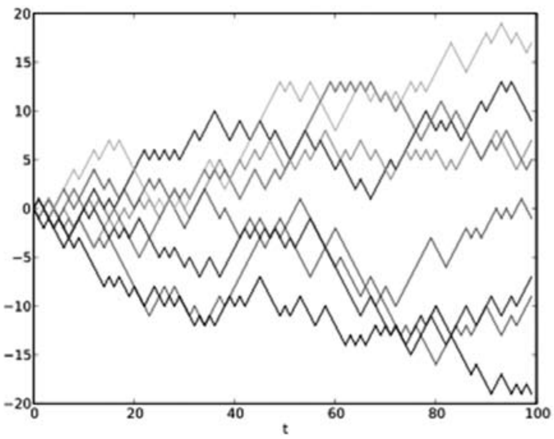


Figure 10. Trajectories of a 1D RW

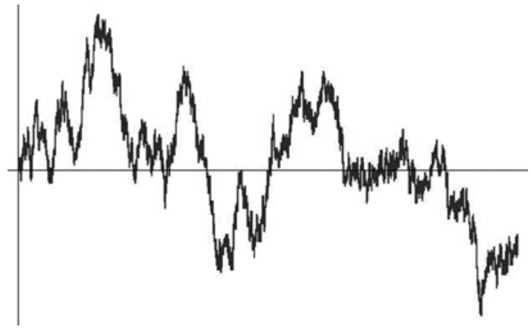


Figure 11. Linear Brownian Motion

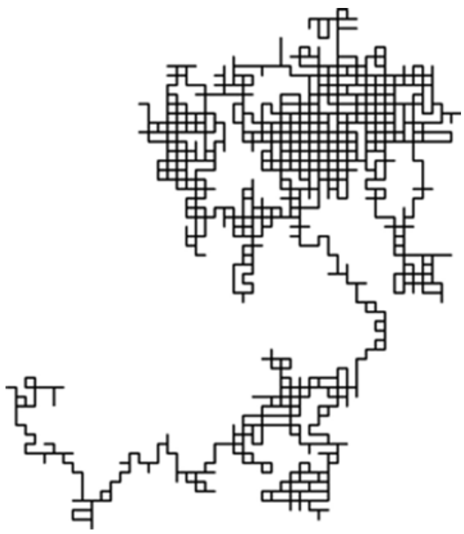


Figure 12. Orbit of a 2D RW (2500 steps)

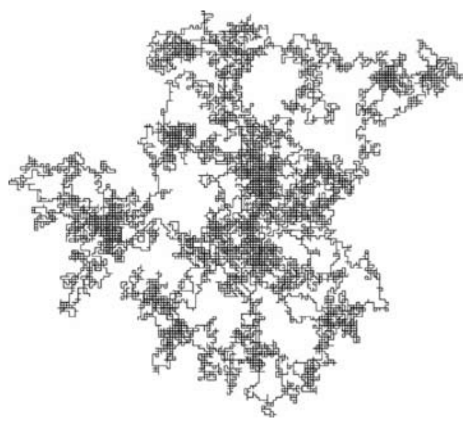


Figure 13. Orbit of a 2D RW (25000 steps)

overcome this deficiency by taking the piecewise linear extension). Then a classical beautiful result of probability theory says that $X_A(t) \Rightarrow W(t) \quad \text{as} \quad A \to \infty$, where $W(t)$ is the Wiener process. (The convergence is understood in the weak sense of probability measures.) The meaning of this claim is the following: one starts with an SSRW, a stochastic model of the motion of an individual particle, and then one obtains the Wiener process, the model of Brownian motion in an appropriate scaling. The true result would be an analogous one for a deterministic model. In probability theory and in statistical physics there had been a lot of effort into finding such models.

Sinai, with Bunimovich, later also including Chernov [8, 5], considered a deterministic model of Brownian mo-

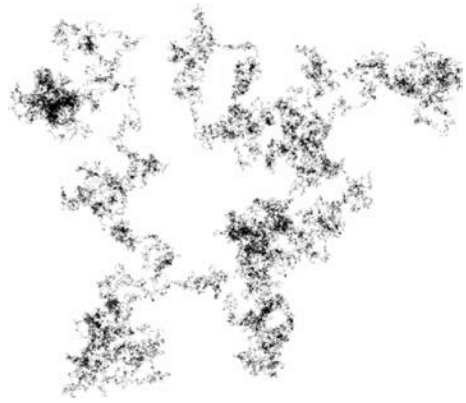


Figure 14. Orbit of a 2D RW (250000 steps)

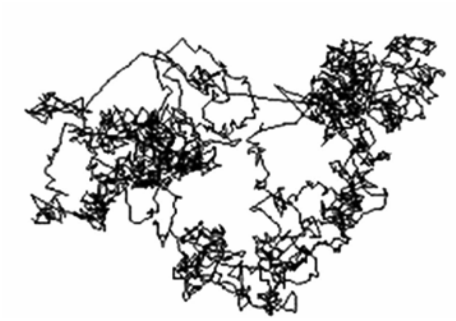


Figure 15. Orbit of a 2D Brownian motion

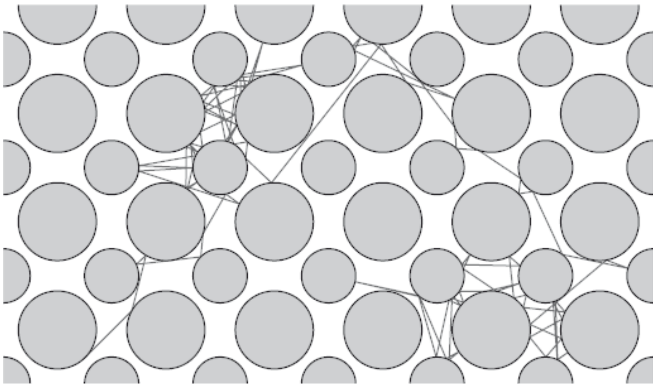


Figure 16. Orbit of a 2D Lorentz process (finite horizon)

tion introduced in 1905 by the Dutch physicist Hendrik Antoon Lorentz. He designed the so-called Lorentz process to describe the motion of a classical electron in a crystal. (The Nobel Prize winner Lorentz is widely known for the Lorentz transformation of relativity theory.) Mathematically speaking, the periodic Lorentz process is a $\mathbb{Z}^d$-extension of a Sinai billiard and Figure 16 illustrates its finite horizon version for $d = 2$. We emphasise that it is a deterministic motion $L(t)$. In fact, it is a billiard trajectory on the plane with only the initial point $L(0)$ selected at random. By using the same scaling which we used for the SSRW, let us define $L_A(t) = \frac{L(At)}{\sqrt{A}}$.

Theorem 6 ([8, 5]). Let $L(t)$ be a planar finite horizon Lorentz process with periodic scatterers. Then

$$L_A(t) \Rightarrow W(t) \quad \text{as} \quad A \to \infty.$$

It was mentioned that the periodic Lorentz process is an extension of the Sinai billiard. The proof of ergodicity of the planar Sinai billiard was quite a breakthrough but – even after its proof – the proof of the above theorem required at least as much creative effort as the proof of ergodicity. One more remark: some of Sinai's achievements, like the proof of ergodicity and the theorem above, are solutions of outstanding problems of the utmost importance. However, the methods for the above theorem were prepared by Sinai's theory-building activity also emphasised in the part of the citation at the start of this paper. In fact, starting from 1968, Sinai was deeply involved in the construction and applications of Markov partitions for smooth hyperbolic systems. Here, we only circumscribe the essential philosophy of this fundamental activity. Once one can construct a Markov partition, satisfying some good properties for a dynamical system, then instead of the original dynamical system one can investigate the isomorphic symbolic system. This is not a Markov chain but often it can be efficiently approximated by higher order Markov chains of longer and longer memory. These are, however, objects of probability theory permitting the applications of its artillery for the understanding of the dynamical behaviour of the system. This philosophy culminated in Sinai's work [13]. The works [8, 5] were based on constructing Markov partitions (or weaker and more flexible objects, called Markov sieves) for planar Sinai billiards.

As happened with ergodicity, the construction of effective Markov approximations for planar billiards raised the demand and opened the way for multidimensional extensions. This is still beyond reach, yet there exist more flexible variants of Markov approximations. We mention two important and promising ones:

1. L. S. Young [18] introduced the method of Markov towers, also providing a new proof for the previous theorem of Sinai and co-authors. The method is also applicable for treating stochastic properties of planar singular hyperbolic systems, in general. Thus, for instance, it also works for dissipative maps, like the Hénon-map, or unimodal maps of the interval. Another advantage of this method was that it permitted optimal, exponential bounds for the speed of correlation decay, i.e. for relaxation to equilibrium. An important generalisation of the tower method was given by Bálint and Tóth [2], where Markov towers were constructed for multidimensional finite horizon dispersing billiards under an additional 'complexity condition'. Further, by the application of Young's tower method, Szász and Varjú [17] could extend the aforementioned theorem of Sinai and co-authors to the 2D infinite horizon case by showing that, in this case,

$$\frac{L(At)}{\sqrt{A \log A}} \Rightarrow W(t) \quad \text{as} \quad A \to \infty.$$

2. In [9], Chernov and Dolgopyat made the method of Markov approximations even more flexible by introducing the method of standard pairs. This also permitted them to extend the circle of mechanical models where variants of Brownian motion can be derived.

Section 1 explained the significant classical problems of statistical physics that billiards – and the Lorentz process – help to understand and hopefully answer. In fact, billiard models seem to be the most appropriate ones for this purpose. As a conclusion, it is noted that the derivation of laws of statistical physics from microscopic principles is still a fundamental goal of mathematicians and physicists. I have only mentioned here the ergodic hypothesis and the dynamical theory of Brownian motion. This approach has had a lot of successes and by the progress of mathematical techniques, which owe their birth to Sinai, it is able to cover wider and wider circles of phenomena (recently much effort has been focused on understanding heat transport).

Bibliography

[1] http://www.abelprize.no.
[2] P. Bálint and I. P. Tóth, *Exponential Decay of Correlations in Multi-dimensional Dispersing Billiards*, Annales Henri Poincaré 9(7) 1309–1369 (2008).
[3] L. Boltzmann, *Einige allgemenine Sätze über das Wärmegleichgewicht*, Wien. Ber. **63**, 679–711 (1871).
[4] L. Boltzmann, *Über die Eigenschaften monozyklischer und amderer damit vervandter Systeme*, Creeles Journal **98**, 68–94 (1884).
[5] L. A. Bunimovich, N. I. Chernov and Ya. G. Sinai, *Statistical properties of two dimensional dispersing billiards,* Russian Math. Surveys **46** 47–106, 1991.
[6] L. Bunimovich, C. Liverani, A. Pellegrinotti and Y. Suhov, *Ergodic systems of n balls in a billiard table*, Comm. Math. Phys. **146**, 357-396, (1992), Ergodic systems of n balls in a billiard table.
[7] A. Bunimovich and Ya. G. Sinai, *Fundamental theorem in the theory of dispersing billiards*, Mat. Sbornik, **90** 1973, 415–431.
[8] L. A. Bunimovich and Ya. G. Sinai, *Statistical properties of Lorentz gas with periodic configuration of scatterers,* Comm. Math. Phys. **78** 479–497, 1981.
[9] N. Chernov and D. Dolgopyat, *Brownian Motion–1,* Memoirs AMS, vol. 198, no 927 (2009).
[10] A. Krámli, N. Simányi and D. Szász, *The K–Property of Three Billiard Balls*, Annals of Mathematics, **133**, 37–72 (1991).
[11] Ya. G. Sinai and N. I. Chernov, *Ergodic properties of some systems of 2-D discs and 3-D spheres*, Usp. Mat. Nauk, **42**, 1987, 15.
[12] Ya. G. Sinai, *D*ynamical systems with elastic reflections: Ergodic properties of dispersing billiards, Russ. Math. Surv. **25** (1970), 137–189.
[13] Ya. G. Sinai, *Gibbs measures in ergodic theory*, Russ. Math. Surv. **27** 21–69, 1972.
[14] N. Simányi, *Singularities and non-hyperbolic manifolds do not coincide*, Nonlinearity **26** 1703–1717, 2013.
[15] N. Simányi and D. Szász, *Hard ball systems are completely hyperbolic*, Annals of Mathematics, **149**, 1999, 35–96.
[16] D. Szász, *Boltzmann's Ergodic Hypothesis, a Conjecture for Centuries?* Appendix in *Hard Ball Systems and the Lorentz Gas*, Springer Verlag, Encyclopaedia of Mathematical Sciences, vol. 101, 421–446, 2000.
[17] D. Szász and T. Varjú, *Limit Laws and Recurrence for the Planar Lorentz Process with Infinite Horizon*, J. Statist. Phys. **129**, 59–80, 2007.
[18] L.–S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. Math. **147** (1998) 585–650.

Domokos Szász is a professor emeritus of mathematics at the Budapest University of Technology. He is the founder of the Budapest schools of mathematical statistical physics and of (hyperbolic) dynamical systems. His current interest is the rigorous derivation of laws of statistical physics (diffusion, heat conduction) from microscopic assumptions. He is a Vice-President of the Hungarian Academy of Sciences (and was Director of its Mathematical Institute from 1993 until 1996).

Around ABC

Preda Mihăilescu (Universität Göttingen, Germany)

1 Let x, y, z be *integers* and x', y', z' be their derivative ... (Diophantine equations in polynomial rings)

Diophantine equations have often been an invitation to expand the realm of known mathematics. Who does not know Fermat's equation $x^n + y^n = z^n$ and the impact it had upon developments in number theory, up until Wiles' proof of the (semisimple case of the) Tanyiama-Shimura Conjecture? In a similar vein, one may consider the *rational Catalan-equation* $x^n - y^m = 1,\ x, y \in \mathbb{Q}$, which essentially amounts[1] to solving the following equation in pairwise coprime integers:

$$x^p + y^q = z^{pq}, \quad \text{with } 2 < p \neq q \text{ primes and } x, y, z \in \mathbb{N} \text{ being coprime.} \quad (1)$$

Differentiating, we find the following linear system:

$$x^p + y^q = z^{pq},$$
$$px^{p-1}x' + qy^{q-1}y' = pqz^{pq-1}z',$$

which yields, after elimination of y:

$$x^{p-1}(pyx' - qy'x) = qz^{pq-1}(pyz' - y'z).$$

We assumed that x, y, z are pairwise coprime, so x^{p-1} should divide $pyz' - y'z$. The reader will have noticed until now that we switched from (1) to the analogous equation[2] in the poly-

1 We do not claim that the two equations are equivalent but the readers may convince themselves that solving the first reduces with no substantial effort to solving the second.

2 We use here a variant of the exposition trick used by Granville and Tucker in their nice introductory paper [7] on the ABC Conjecture.

nomial ring $\mathbb{Z}[X]$. We note that $pyz' = y'z$ implies, for non-constant polynomials, that $\log(z^p) = \log(Cy)$ and thus $z^p = Cy$, which is in conflict with the assumption that $(y, z) = 1$. We conclude that

$$(p-1)\deg(x) \leq \deg(pyz' - y'z) \leq \deg(y) + \deg(z) - 1,$$

since the degree of the derivative is one less than the degree of the initial polynomial. Adding $\deg(x)$ on both sides we obtain a symmetric expression on the right side. The same can be achieved by eliminating x and z, the resulting identities being:

$$\begin{aligned} p\deg(x) &\leq \deg(x) + \deg(y) + \deg(z) - 1, \\ q\deg(y) &\leq \deg(x) + \deg(y) + \deg(z) - 1, \\ pq\deg(z) &\leq \deg(x) + \deg(y) + \deg(z) - 1. \end{aligned}$$

By adding up, we obtain

$$(3-p)\deg(x) + (3-q)\deg(y) + (3-pq)\deg(z) \geq 3 > 0.$$

Since $q > p \geq 3$ and the degrees of all involved polynomials are positive, the left side must be negative – so we have obtained a proof for the fact that the equation $x^n - y^m = 1$ has no non-trivial solutions $(x, y) \in \mathbb{Z}[X]^2$.

The Diophantine equation (1) is assumed to have no integer solutions; unlike the case of Fermat's equation prior to Wiles' epochal proof, hardly any partial results are known[3] for (1). But we see that a relatively simple proof can be provided for the case of rational functions. The analogous proof for the case of Fermat's equation was apparently already known to Liouville [7], whose more intricate proof involved integration. The reason why these proofs are so simple in the polynomial domain is related to taking derivatives – a property which also has important consequences in computational number theory and cryptography, for instance in factoring algorithms.

But how far can we go with solving Diophantine equations in the polynomial domain? The answer is *very far*, in a sense that was made precise by Richard C. Mason in 1983 – in fact, Mason rediscovered a result which had been previously published by W. Wilson Stothers [19]. In order to state this result, it will be useful to introduce a definition. Let A be a commutative ring with unique factorisation, i.e. for each $a \in A$ there is a factorisation as a product of finitely many prime elements $a = \varepsilon \prod_{i=1}^{s} p_i^{e_i}$ which is unique up to order of the factors and possible invertible elements in $\varepsilon \in A^*$. Assuming that the prime elements are made unique in A by some choice (for instance, if $A \in \mathbb{Z}$, one may ask that the primes are positive, which makes them unique; otherwise, $-2, 2$ are primes of $\mathbb{Z}$ which differ by the invertible element -1), we define the *radical* of a as the square-free product of the primes dividing a, namely:

$$a = \varepsilon \prod_{i=1}^{s} p_i^{e_i} \quad \Rightarrow \quad \mathrm{rad}(a) = \prod_{i=1}^{s} p_i.$$

If K is any field of characteristic[4] 0 – for instance, the field of the complex numbers $\mathbb{C}$ or the rationals $\mathbb{Q}$ – the ring $A = K[X]$ has unique factorisation and its prime elements are the irreducible polynomials, while the invertible elements are exactly the non-vanishing constant polynomials, namely $K[X]^* = K^*$. While investigating Diophantine equations in function fields, Mason noticed that in $K[X]$, one may go beyond the classical equations already discussed above and pose the very general question: *'Under what conditions does the equation $a + b = c$ have non-trivial solutions in polynomials $a, b, c \in K[X]$ which have no common root in an adequate extension of K?'* The answer is given by the following result, [9], [19]:

Theorem 1 (Mason – Stothers)**.** Let K be a field of characteristic 0. If A, B, C are non-constant polynomials in $K[X]$ with $A + B + C = 0$ and $\gcd(A, B, C) = 1$, then

$$\max\{\deg(A), \deg(B), \deg(C)\} \leq \deg(\mathrm{rad}(A \cdot B \cdot C)) - 1.$$

Proof. The simple proof that we provide here is due to Noah Snyder [17], who was a high school student at the time he presented the proof. We start by noting that for arbitrary $f \in K[X]$, the quotient $g := f/\mathrm{rad}(f) \in K[X]$. This follows directly from the factorisation of f into irreducible polynomials. Next we remark that $g \mid f'$. Indeed, if $f = \prod_i p_i^{e_i}$ then $g = \prod_i p_i^{e_i - 1}$ and $p_i \mid g$ iff $e_i > 1$. The derivative is

$$f' = f \cdot \left(\sum_i e_i p_i' / p_i \right) = g \cdot \sum_i e_i p_i',$$

hence the claim. We also note that $\gcd(A, B) = \gcd(A, C) = \gcd(A, C) = 1$, since any common divisor of two of the three polynomials will automatically divide the third, due to the relation $A + B = C$. This relation also implies

$$C'B - CB' = AB' - A'B, \tag{2}$$

since $C'B - CB' = (-A' - B')B + (AB' + BB') = AB' - BA'$. We let $P = \{A, B, C\}$ be the set consisting of the three given polynomials A, B, C. We use P also as an index-set, letting $g_U = U/\mathrm{rad}(U)$ for $U \in P$ and note that $g_U \mid (C'B - CB')$ for all $U \in P$. This fact follows from the definition if $U \neq A$, and from (2) if $U = A$. Since the polynomials $U \in P$ are coprime, so are, a fortiori, the polynomials $g_U,\ U \in P$; therefore g_U are coprime factors of $C'B - CB'$. Noting that coprimality also implies $\mathrm{rad}(A)\mathrm{rad}(B)\mathrm{rad}(C) = \mathrm{rad}(ABC)$, we finally obtain

$$\frac{A}{\mathrm{rad}(A)} \cdot \frac{B}{\mathrm{rad}(B)} \cdot \frac{C}{\mathrm{rad}(C)} \Big| \, (C'B - CB'), \qquad \text{hence}$$

$$\frac{ABC}{\mathrm{rad}(ABC)} \Big| \, (C'B - CB').$$

Assume now that $\deg(A) \geq \deg(\mathrm{rad}(ABC))$; then

$$\begin{aligned} \deg\left(\frac{ABC}{\mathrm{rad}(ABC)}\right) &= \deg(ABC) - \deg(\mathrm{rad}(ABC)) \\ &\geq \deg(ABC) - \deg(A) \\ &= \deg(BC) > \deg(C'B - CB'). \end{aligned}$$

But we have seen that $\frac{ABC}{\mathrm{rad}(ABC)}$ divides $C'B - CB'$, so it cannot have larger degree unless $C'B - CB' = 0$. But in this case, if B, C are non-constant, and thus non-vanishing, it follows upon division by BC that $\frac{d}{dX}(\log(C/B)) = 0$ and thus B, C differ by a constant, which is impossible for non-constant and coprime polynomials. We have obtained a contradiction showing that $\deg(A) < \deg(\mathrm{rad}(ABC))$. In the same way, we conclude that $\deg(U) < \deg(\mathrm{rad}(ABC))$ for all $U \in P$ and, due to coprimality, $\max_{U \in P}(\deg(U)) < \deg(\mathrm{rad}(ABC))$, which completes the proof. □

3 But see [10].

4 i.e. K is not a finite or infinite extension of a finite field.

2 The *ABC* Conjecture

The elegant Theorem 1 on polynomials generalises some Diophantine equations, for which one expects there to be no solutions in the integers – see, for example, the books of P. Ribenboim [15] and M. Th. Rassias [14] for some exciting elementary introductions to Diophantine equations. One is thus tempted to assume that some appropriate generalisation *might* hold in the integers. However, in lack of derivation, both the statement and the proof of an appropriate generalisation are not evident. After discussing these questions, and also confronting minimal numerical data, Masser and Oesterlé decided in favour of stating the following:

Conjecture 1 (ABC Conjecture – Masser and Oesterlé)**.** For arbitrary $\varepsilon > 0$ there exists a constant $K(\varepsilon) > 0$ such that if $A, B, C \in \mathbb{Z}$ are three mutually coprime integers verifying the equation $A + B = C$, then

$$\max(|A|, |B|, |C|) < K(\varepsilon) \cdot (\mathrm{rad}(ABC))^{1+\varepsilon}.$$

Stated in this form, the ABC Conjecture is ineffective with respect to the value of $K(\varepsilon)$; despite the power of this conjecture, its application to Diophantine equations will thus result in statements of the type: *Equation X has no integer solutions, except possibly for finitely many, which are bounded in terms of $K(\varepsilon)$ for some adequate $\varepsilon > 0$.* Numerical investigations suggest that an explicit conjecture of the following type might hold:

Conjecture 2 (Effective ABC)**.** Let $A, B, C > 1$ be mutually coprime integers verifying $A + B = C$. Then

$$\max(A, B, C) \le \mathrm{rad}(ABC)^2. \tag{3}$$

Diophantine applications

One easily verifies that Fermat's Last Theorem follows from this conjecture: it implies that $|XYZ|^{n-6} < 1$ and thus the exponent $n < 6$, reducing the problem to the investigation of the cases $n = 3, 4, 5$, which have been known from the 19th century [15].

For the generalisation $aX^n + bY^n + cZ^n = 0$ with fixed $a, b, c \in \mathbb{Z}$ and $(X, Y, Z) = 1$ we obtain, by letting $A = aX^n, B = bY^n, C = cZ^n$:

$$\max(|aX^n|, |bY^n|, |cZ^n|) \le \mathrm{rad}(abcXYZ)^2 \le (abc)^2 \cdot (XYZ)^2,$$

$$\text{hence} \qquad (|XYZ|)^{n-2} \le |abc|.$$

This shows that the solutions are bounded by $|abc|^{1/(n-1)}$, so there are, in particular, at most finitely many solutions. In order to exclude small solutions in an orderly way, some additional techniques are required.

For the Catalan equation $X^m - Y^n = 1$ one directly obtains $|X^mY^n| < (XY)^4$ and thus $|X^{m-4}Y^{n-4}| < 1$, an inequality which has no integer solutions for $m, n \ge 4$. The remaining cases had been solved for several decades, using simple, but sometimes unexpected, elementary methods – like in the case of Ko Chao's solution for $X^2 - Y^q = 1$, which he found in 1961. More generally, one may consider the *binary* equation $aX^m + bY^n = c$ for which Conjecture 3 below implies that the integer solutions must verify $|X^{m-4}Y^{n-4}| < |abc|$. For $m, n > 4$ we obtain bounded solutions, while for the remaining pairs of exponents, a case by case investigation is required.

Let us now consider the more general *Generalized Fermat* equation:[5]

$$X^p + Y^q + Z^r = 0, \qquad (X, Y, Z) = 1. \tag{4}$$

We assume here that the exponents are coprime integers, thus $(p, q, r) = 1$, and let $\delta = 1 - (1/p + 1/q + 1/r)$. For $\delta < 0$, the equation (4) has infinitely many solutions.[6] However, if $\delta > 0$, Henri Darmon and Andrew Granville have proved that the more general "super-Fermat" equation

$$A_0X^p + B_0Y^q + C_0Z^r = 0, \qquad (X, Y, Z) = 1, \tag{5}$$

for fixed parameters $A_0, B_0, C_0 \in \mathbb{Z}$ and coprime exponents with $\delta(p, q, r) > 0$, has at most finitely many solutions for every fixed triple of exponents. The proof uses Faltings' proof of the Mordell Conjecture and is, in particular, ineffective, i.e. it cannot provide explicit bounds for the finitely many possible solutions. For $A_0 = B_0 = C_0$ and $\delta > 0$, the following 10 solutions are known:

$$1^r + 2^3 = 3^2,$$
$$13^2 + 7^3 = 2^9,$$
$$2^5 + 7^2 = 3^4,$$
$$3^5 + 11^4 = 122^2,$$
$$33^8 + 1549034^2 = 15613^3,$$
$$43^8 + 96222^3 = 30042907^2,$$
$$2^7 + 17^3 = 71^2$$
$$17^7, +76271^3 = 21063928^2,$$
$$1414^3 + 2213459^2 = 65^7,$$
$$9262^3 + 15312283^2 = 113^7.$$

A conjecture of Tijdeman and Zagier claims that these are all non-trivial solutions for (4) in the case $\delta(p, q, r) > 0$. Note that no particular solutions are known for (4) in which all the exponents $p, q, r \ge 3$; the banker Andrew Beal has offered \$1,000,000 for a proof or disproof of this statement, which is also called the *Beal Conjecture*.

But what answers in relation to equation (5) does the ABC Conjecture provide? We may assume that the exponents satisfy $2 \le p \le q \le r$ and that they are not all equal – otherwise, we are in the case of Fermat's Last Theorem. Letting $A = X^p, B = Y^q$ and $C = -Z^r$, we deduce from Conjecture 3 below that $|X^2| \le |X^p| \le \max(|A|, |B|, |C|) < \mathrm{rad}(ABC)^2 \le (XYZ)^2$, while $|X^pY^qZ^r| < \mathrm{rad}(ABC)^6 \le (XYZ)^6$. We obtain explicit contradictions if $p > 6$; for small exponents, tedious case distinctions are required.

If explicit constants $K(\varepsilon)$ are known in Conjecture 1 then one may choose $\varepsilon = 1/6$, deducing that $|X^pY^qZ^r| \le K(1/6)|XYZ|^{3.5}$. This yields upper bounds for $|X|, |Y|, |Z|$ if $p > 3$. Otherwise, some case distinction is required. Consider the case where $p = q = 3, r = 3 + d$ and $d > 0$, let $\varepsilon > 0$ be small and $k = K(\varepsilon)$ and let $r = |A_0B_0C_0|$. Let $A = A_0x^p, B = B_0y^q, C = C_0z^r$ and $R = A_0B_0C_0$, while $N = \max(|A|, |B|, |C|)$. Then Conjecture 1 implies

$$N < k\mathrm{rad}(|R|xyz)^{1+\varepsilon} < k|Rxyz|^{1+\varepsilon},$$

5 We adopt the terminology used by Don Zagier. Other authors use the term *Fermat-Catalan* for the same equation. We reserve this term for the smallest common generalisation, which is the equation $X^n + Y^n = Z^m$.

6 See, for instance, [5] for more details.

and, a fortiori,

$$|Rx^p y^q z^r| = |R| \cdot |(xyz)^3| \cdot |z|^d < k|R|^{3(1+\varepsilon)}|xyz|^{3(1+\varepsilon)},$$

hence $|z|^d < k'|xyz|^{3\varepsilon}$ for $k' = k \cdot R^{2+3\varepsilon}$. We may assume without loss of generality that $N = |A_0x^3|$, since the other possibilities can only occur when x, y, z are within fixed bounds depending on k and R. We have $|C_0z^r| = |A_0x^3 \pm B_0y^3| \geq |A_0x^3| - |B_0y^3| \geq k''N^2$ for some $k'' > 0$, hence

$$N^{\frac{2d}{3+d}} k''' < |z|^d < k'N^{9\varepsilon}.$$

We may choose $\varepsilon < \frac{1}{18} \cdot \frac{2d}{3+d}$, thus obtaining a bound $N^{\frac{d}{3+d}} < (k'/k''')^2$ for the largest solution. The case when $q > 3$ is easier and can be treated similarly.

As a general pattern, we see that the ABC Conjecture allows one to show that trinomial exponential Diophantine equations have bounded solutions, provided the exponents occurring are sufficiently large. Otherwise, some additional techniques may be required. For the Fermat-Catalan equation, one may for instance use cyclotomic methods for proving that the solutions X, Y, Z are themselves divisible by some large powers, which may be sufficient for completing a general proof.

Further consequences and variants

Consequences of the ABC Conjecture have been actively investigated over the 1990s. N. Elkies found in 1991 that it implies effective versions of the Mordell Conjecture – which had been proved (ineffectively) by G. Faltings. M. van Frankenhuijsen additionally proved in his PhD thesis [22] that the conjecture also implies the theorem of K. Roth in Diophantine Approximation. Having such a powerful number theoretic tool at hand, a natural question that arises is whether it can be of any use in proving the major open standing question of analytic number theory: the Riemann Conjecture. While Granville and Stark proved in [6] that the L-series of certain quadratic characters have no Siegel zeroes, it is generally accepted that the ABC Conjecture is essentially *independent* of the Riemann Conjecture.

Concerning the ABC Conjecture itself, there have been various intense numerical investigations attempting to find triples $a, b, c > 1$ with $a + b = c$ which maximise the quotient $Q(a, b, c) := \frac{\log(\max(a,b,c))}{\log(\mathrm{rad}(abc))}$, also called the *quality* of the triple. The triple $2 + 3^{10} \cdot 109 = 23^5$ was discovered by Éric Reyssat; with $Q = 1.629912$, it has so far the largest quality known. One may find up-to-date tables on Bart de Smit's webpage [4] of special triples and more related literature. Several authors (C. Stewart and R. Tijdeman [18], Stewart and Yau [20], Robert, Stewart and Tenenbaum [16] and M. van Frankenhuijsen [21]) have investigated lower bounds dependent on various assumptions on ABC-triples.

Improving upon estimates of van Frankenhuijsen, Cameron Stewart formulated conjectures which also provide *lower bounds* for the constant $K(\varepsilon)$ in Conjecture 1. The conjecture of Stewart is:

Conjecture 3 (Stewart)**.** Let $\varepsilon > 0$. There exists $K(\varepsilon) > 0$ such that for any abc-triple with $R = \mathrm{rad}(abc) > 8$ and $c = \max(a, b, c)$, we have

$$c < K(\varepsilon)R \cdot \exp\left((4\sqrt{3} + \varepsilon) \cdot \left(\frac{\log R}{\log\log R}\right)^{1/2}\right).$$

Furthermore, there exist infinitely many abc-triples for which

$$c > R \cdot \exp\left((4\sqrt{3} - \varepsilon) \cdot \left(\frac{\log R}{\log\log R}\right)^{1/2}\right).$$

Also starting from investigations of ABC triples with high quality, Alan Baker introduced the observation that highly composite integers tend to provide triples of large quality; if $\omega(n)$ is, as usual, the multiplicative function that counts the number of distinct prime factors of the integer n then Baker proposed the following alternative inequality for the ABC Conjecture:

$$\max(|A|, |B|, |C|) < Kr \cdot \frac{r^\omega}{\omega!}, \quad r := \mathrm{rad}(ABC), \qquad \omega = \omega(|ABC|). \tag{6}$$

Note that there is no small positive constant ε in this inequality, and the only constant $K > 0$ is absolute. The function $\Theta(n) = \sharp\{x \leq n \ : \ x | n^n\}$ gives the number of integers less than n which decompose into primes dividing n; Andrew Granville noticed that the right side in (6) approaches the value of this function and proposed the alternative inequality:

$$\max(|A|, |B|, |C|) < K' r\Theta(r), \qquad r := \mathrm{rad}(ABC). \tag{7}$$

So far, we have focused primarily on consequences of the ABC Conjecture for Diophantine equations. There are, however, further consequences in number theory that have been proved. Most importantly, ABC is known to be equivalent to a slightly older conjecture of Lucien Szpiró, relating conductors and discriminants of elliptic curves. More surprisingly, in order to derive a sought after proof of a famous conjecture of Greenberg on the vanishing of Iwasawa's constants λ and μ in the cyclotomic $\mathbb{Z}_p$-extensions of totally real fields, Ichimura [8] formulates and then assumes the truth of a variant of the ABC Conjecture for number fields. In his slides [23] Michel Waldschmidt yields a nice overview of known consequences of the conjecture in various areas of number theory.

Carl Pomerance's contribution [13] on computational number theory to the Princeton Mathematical Companion contains valuable interpretations of the heuristics on the relation between the constants $\varepsilon, K(\varepsilon)$ in Conjecture 1. In this context he also mentions Andrew Granville's interesting remark that the four term inequality

$$\mathrm{rad}(abcd)^{1+\varepsilon} \leq \max\{|a|, |b|, |c|, |d|\}$$

with $(a, b, c, d) = 1$ has more solutions than one would have expected under the assumption of statistical independence of such solutions. The fact is related to the existence of polynomial identities of the type $(x + 1)^5 = (x - 1)^5 + 10(x^2 + 1)^2 - 8$, giving rise to infinitely many solutions.

Mochizuki's Inter-Universal Teichmüller Theory

In the autumn of 2012 the Japanese number theorist Shinichi Mochizuki, former PhD student of Gerd Faltings, announced the development of a new theory which includes the ABC Conjecture among its consequences. Mochizuki was known for his previous work involved with topics related to anabelian geometry, concerning aspects like p-adic Teichmüller and Frobenioids.

In his announcement, Mochizuki explained having developed a new theory, which he denotes IUTT – *Inter-Universal Teichmüller Theory*; one of the important "philosophical" motivations that he mentions is the fact that schemes over $\mathbb{Z}$, which are objects of mathematics one would try to relate to questions such as the ABC Conjecture, are too rigid and fail from being sensible about particularities of the integers. Shortly after the announcement, a group of mathematicians including Meyjhong Kim and Vesselin Dombrovski pointed out that there had to be infinitely many counterexamples to the quantitative inequality derived by Mochizuki from his theory and which implied the ABC Conjecture.

The challenge was accepted by Mochizuki, who understood the reason for his error and promised a correction of the statement and proof of his result; these were indeed published on Mochizuki's homepage [11] several months later. While very few mathematicians appear to have tackled the details of Mochizuki's construction, multiple blog discussions and even publications in large public journals of mathematics read like a fascinating ongoing drama. Those interested speculate mostly on whether Mochizuki is being sufficiently cooperative with possible attempts by specialists to understand his work. Opinions range from the expectation that an author making a valuable new contribution should present it from the start in such a way that any specialist from the field can verify the steps of the proof without difficulty, to a more realistic attitude, accepting that before verifying the proof in detail, experts must spend some time familiarising themselves with the new ideas and concepts that Mochizuki has developed. Quite independently of these rather speculative discussions, based more on second-hand information and the fact that it was purported that Mochizuki did not accept any international invitation to present his talk, the verification process started in a small way. Presently, Go Yamashita, a Japanese research fellow, and Mohamed Saïdi, a well-established Algerian expert of anabelian geometry and presently a professor in Exeter, have been exposed independently to intensive seminars by Mochizuki concerning his work, over a period of roughly half a year. After that, Yamashita started his own seminar, in which he is presenting his understanding of the theory to three further Japanese experts, intending to write his own report afterwards. The general impression is that the time required for *getting acquainted* with the IUTT is roughly half a year; as Mochizuki adds in his *verification progress report* [12], this means *not one month, but not several years*. Saïdi expresses having understood the motivation of the theory and having gained confidence that it may be true; he states being positively impressed by the fact that a deep extension of anabelian algebra may be at hand and that algebra and in particular Galois theory may provide the tools for proving deep analytic results.

In addition, Mochizuki's report, together with the very detailed motivational papers that can be found on his homepage, definitely contradict the perception that he might be "socially non-cooperative". The reason for turning down invitations to deliver talks on his result is explained by the impossibility of reducing the weight of the new theory to something that can be explained in one day or even in one week. A more tangible criticism that is brought stems from the fact that the theory, exposed over several hundreds of pages, is supported by precious few theorems. There are no independent (non-trivial) applications where one can see it put to work in a verifiable way, before offering a proof to a conjecture of the importance of ABC. This is certainly a problem, and it is equally clear that the acceptance of the theory would increase dramatically if the author or some of the experts acquainted with his work could succeed in identifying some particular case, problem or aspect which has either a known solution or is easier to understand and to which IUTT can be successfully applied. While such a special case has not been provided, it remains true that the situation about Mochizuki's proof is still totally undecided and the more experts in anabelian geometry or other fields of number theory that get acquainted with the details of the theory, the more chances there are for understanding the proof and possibly discovering some intermediate questions and steps that may help to gather community understanding and acceptance for the new results. According to the opinions of those who have already been exposed to the details, the least one can say is that some important ideas have been developed that have the potential to enlarge the fields of algebra and number theory. It is equally certain that we shall have to equip ourselves with a good deal of patience until such time as a competent and well-verified opinion arises about Mochizuki's theory *and* its application for proving some variant of the ABC Conjecture.

Given the fact that the verification of a proof of such dimensions requires a very important amount of work, which is expected to go without adequate acknowledgement, one may ask the question, if it would not be conceivable to make a small amount of the Beal price available for the work of providing a final, community accepted form of Mochizuki's proof.

3 Diophantine variations

Although matters of taste play an important role in mathematics, and when talking in private many mathematicians will be willing to recognise the value they give to aesthetics of proofs, the professional discussion is restricted to statements and proofs.

Tianxin Cai, a Chinese number theorist who is most reputed internationally for his volumes of poetry that have been translated into various European tongues, has recently started to develop his own variant of classical Diophantine questions. The common pattern consists of *reversing additive and multiplicative symmetries*, in a sense which we will allow the reader to understand in their personal way, starting from several examples that are taken from work of Cai and his students.

The Hilbert–Warring problem aims to establish, for integers $k > 1$, the value $g(k)$ which is the minimal number of kth power terms that are required in order to ascertain that *every* integer $n = \sum_{i=1}^{g(k)} n_i^k$, i.e. it can be represented in at least one way as a sum of $g(k)$ kth powers. The problem was first raised in a special case by Warring in 1770 and has received much attention ever since. Cai has *dualised* the problem by asking about the minimal integer $\overline{g}(k)$ such that every integer n can be written as a sum of $\overline{g}(k)$ integers, the product of which is a

kth power. Thus:

$$n = \sum_{i=1}^{\overline{g}(k)} n_i, \qquad \text{such that} \qquad \prod_{i=1}^{\overline{g}(k)} n_i = N^k.$$

His variant of the Fermat problem is then [2]:

$$a + b = c, \quad \text{such that} \quad abc = N^n. \tag{8}$$

By now the reader may well have understood the process by which Cai derives *dual* Diophantine questions from classical ones. We thus propose that they derive their own variants[7] of the generalised Fermat equation $x^p + y^q + z^r = 0$ with $1/p + 1/q + 1/r < 1$ and of the Pilai equation $x^p - y^q = m$.

We now analyse the dual Fermat equation; obviously, any possible solution of the classical Fermat equation would be a solution of the dual equation (8), too. If $(a, b, c) = 1$ we recover the Fermat equation – the additional difficulty arises from the possibility of having a non trivial common divisor. Let $d = (a, b, c)$ and $a' = a/d; b' = b/d; c' = c/d$. Then $a' + b' + c' = z^n/d^3$ and if $z = z_1 \cdot z_2$, with $\mathrm{rad}(z_2) = \mathrm{rad}(d)$ and $(z_1, z_2) = 1$, then there is a decomposition $z_1 = uvt; z_2^n/d^3 = u'v't'$ in mutually coprime factors, such that

$$\begin{aligned} a = u^n \cdot u' \cdot d; \quad b = v^n \cdot v' \cdot d; \quad c = t^n \cdot t' \cdot d, \\ u^n u' + v^n v' = t^n t'. \end{aligned} \tag{9}$$

The identity in the second line is the new variant of the Fermat equation. Most interestingly, there does not seem to exist any constructive application of the ABC Conjecture to (8) that would work in general for the case when $d \neq 1$. Note, however, that $z_1 = 1$ would imply that (9) becomes $u' + v' = t'$ subject to the restriction $u'v't' = z^n/d^3$, in which case the exponential character of the equation (9) is shaded away by the common divisor. If $n \equiv 0 \bmod 3$, say $n = 3m$, we obtain $u' + v' = t'$ and $u'v't' = (z^m/d)$. Since $(u', v', t') = 1$, we recover the Fermat equation for exponent 3, which is known to have no solution. The cases $n \not\equiv 0 \bmod 3$ may, though, have solutions that cannot be excluded even if the ABC Conjecture were proved.

As we see, number theory is continuously advancing. While a possible proof of the ABC Conjecture may simultaneously solve many important Diophantine equations, a variation of the same brings them back in a flavour which may be resistant to attempts at solutions using the conjecture.

Bibliography

[1] Tianxin Cai and Deyi Chen: *A new variant of the Hilbert-Waring problem*. Math. Comp. **82** (2013), no. 284, pp. 2333–2341.

[2] Tianxin Cai, Deyi Chen and Yong Zhang: *A new generalization of Fermat's Last Theorem*. Arxiv: http://arxiv.org/abs/1310.0897.

[3] Henri Darmon and Andrew Granville: *On the equations $x^p + y^q = z^r$ and $z^m = f(x, y)$*, Bull. London Math. Soc. **129** (27 part 6, Nov.1995), pp. 513–544.

[4] Bart de Smit: *ABC triples*, http://www.math.leidenuniv.nl/~desmit/abc/.

[5] Noam Elkies: *The ABC's of Number Theory*, Harvard College Mathematical Review, **1** (2007), pp. 57–76.

[6] Andrew Granville and Harold M. Stark: *ABC implies no 'Siegel zero' for L-functions of characters with negative discriminant*, Invent. Math. **139** No. 3, (2000), 509–523.

[7] Andrew Granville and Tom Tucker: *It's as easy as ABC*, Notices of the AMS, Vol **49**, Nr. 10 (2002), pp. 1224–1231.

[8] Humio Ichimura: *A note on Greenberg's conjecture and the ABC Conjecture*, Proc. Am. Math. Soc. **126** (1998), pp 1315–1320.

[9] Richard. C. Mason: *Diophantine Equations over Function Fields*, London Math. Soc. Lecture Notes Series **96**, Cambridge University Press (1984).

[10] Preda Mihăilescu: *A cyclotomic investigation of the Fermat–Catalan equation*, Arxiv.

[11] Shinichi Mochizuki's homepage: http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html.

[12] Shinichi Mochizuki's verification report: http://www.kurims.kyoto-u.ac.jp/~motizuki/research-english.html.

[13] Carl Pomerance: *Computational Number Theory*, in T. Gowels (ed.), Princeton Companion to Mathematics, pp. 348–363.

[14] Michael Th. Rassias: *Problem-Solving and Selected Topics in Number Theory. In the Spirit of the Mathematical Olympiads*, Springer (2011).

[15] Paulo Ribenboim: 13 *Lectures on Fermat's Last Theorem*, Springer (1979).

[16] O. Robert, Cameron L. Stewart and Gérald Tenenbaum: *A refinement of the abc conjecture* (2013), http://www.iecl.univ-lorraine.fr/~Gerald.Tenenbaum/PUBLIC/Prepublications_et_publications/.

[17] Noah Snyder: *An alternative proof of Mason's Theorem*, Elem. Math. **55** (2000), pp. 93–94.

[18] Cameron L. Stewart and Robert Tijdeman: *On the Oesterlé-Masser conjecture*, Monatsh. Math. 102 (1986), 251–257.

[19] W. Wilson Stothers: *Polynomial identities and hauptmoduln*, Quarterly J. Math. Oxford, (1981), **32**, pp. 349–370.

[20] Cameron Stewart and Kunrui Yu: *On the ABC Conjecture*, Math. Ann. **291** (1991), pp. 225–230.

[21] Machiel van Frankenhuijsen: *A lower bound in the ABC Conjecture*, J. Number Th. **82**, (2000), pp. 91–95.

[22] Machiel van Frankenhuijsen: *Hyperbolic spaces and the ABC Conjecture*, PhD Thesis Katholieke Universiteit Nijmegen (1995).

[23] Michel Waldschmidt: *On the abc-Conjecture and some of its consequences*, Lahore (2013), http://www.math.jussieu.fr/~miw/articles/pdf/abcLahore2013VI.pdf.

Preda Mihăilescu [preda@uni-math.gwdg.de] was born in Bucharest, 1955. He studied mathematics and computer science in Zürich, receiving his PhD from ETH-Zürich. He was active during 15 years in the industry, as a numerical analyst and cryptography specialist. In 2002, Mihăilescu proved Catalan's Conjecture. This number theoretical conjecture, formulated by the French mathematician E. C. Catalan in 1844, had stood unresolved for over a century. The result is known as Mihăilescu's Theorem. He is currently a professor at the Institute of Mathematics of the University of Göttingen.

7 In Chinese tradition, complementary aspects in dual pairs are frequently endowed with yin and yang characters. In mathematics, too, even numbers are yin and odd ones are yang. For Cai, equations also have yin and yang aspects – the traditional, additive ones being probably yin while the variations proposed by him are yang.

You missed it!

An exhibit of the works of Siméon-Denis Poisson in Paris

Yvette Kosmann-Schwarzbach (Paris, France)

Introduction

Everyone has heard of the Poisson distribution, of Poisson brackets and of many other "Poisson something"s. But who was Poisson and what did he do to acquire such fame? At the initiative of its director, Brigitte Laude, the research library, "Mathématiques Informatique Recherche" (MIR), of the Université Pierre et Marie Curie in Paris, on the Jussieu campus, organised an exhibit "Siméon-Denis Poisson. Les mathématiques au service de la science", which was on display from 19 March to 19 June 2014 aiming to present some of Poisson's works and their modern continuation.

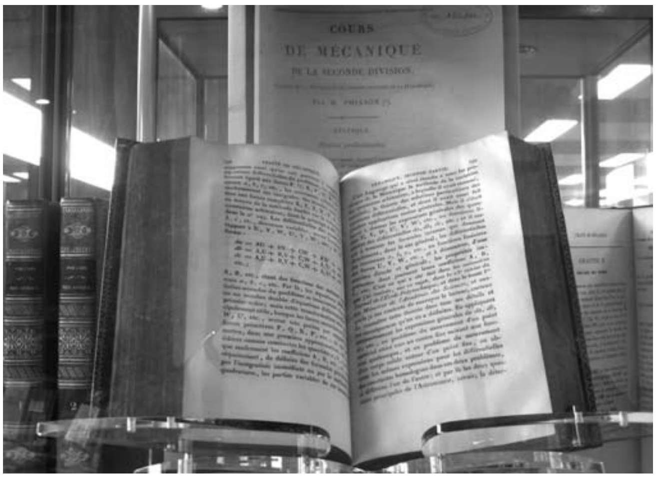
From Poisson's baptism certificate, one day after he was born in Pithiviers (a small town south of Paris), to the list of students admitted to the École Polytechnique in Year 7 of the Republic, where he is no. 1 at the top of a finely written page, to the postcard representing his statue erected in 1851 in his native city, 11 years after he died, we follow Poisson in his brilliant career as a professor at the École Polytechnique (he was appointed as the successor to Fourier at the age of 21), later at the Faculté des Sciences (since its creation in 1809) and, after 1812, as an influential member of the Académie des Sciences. Poisson never travelled but his works did: several of his books and articles were translated into German and even into English, influencing George Green among others.

All Poisson's books

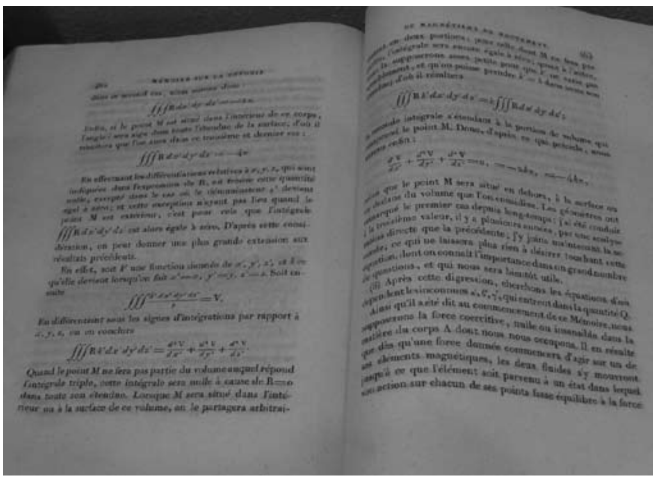
Between 1799, one year after his admission to the École Polytechnique at the age of 17, and his death in 1840, Poisson wrote approximately 200 memoirs, articles, notes and reports, and ten books. Not long before he died, he

The exhibit at the MIR, UPMC, Jussieu, Paris

Some 19th and 20th century books on the history of science



The Mechanics of Poisson (*Cours de mécanique*, 1809; *Traité de Mécanique*, 1811, Second edition 1833, opened at page 399)



Poisson's equation on page 463 of his *Mémoire sur la théorie du magnétisme en mouvement* (1823)

wrote a careful catalogue of his published works. The idea of the exhibit at Jussieu was primarily to present the original editions of all his books and that was almost achieved. Two books, the first version of the mechanics course he taught at the École Polytechnique and a slim, technical volume on the effect of firing a cannon, had to be represented by scanned copies of the title pages. But the two volumes of his 1811 *Traité de Mécanique*, with their fine leather binding and spines decorated in red and gold, on loan from an anonymous private collection, were displayed and, next to them, his "considerably enlarged" second edition, opened at page 399 where he cites Lagrange "who extended the method of variation of arbitrary constants to all the problems of Mechanics" and recalls that he had found and published [in 1809] a direct way to determine the inverses of the expressions found by Lagrange … that is, he had discovered the "Poisson brackets"! His famous *Recherches sur la probabilité des jugements*" (1837) was opened at page 206, where one reads

$$P = \left(1 + \omega + \frac{\omega^2}{1.2} + \frac{\omega^3}{1.2.3} + \dots + \frac{\omega^n}{1.2.3\dots n}\right) e^{-\omega}.$$

Other books and memoirs dealt with his contributions to, among other branches of mathematics and physics, the analysis of integrals and series, differential equations and integration in the complex domain, as well as to the theories of electricity and magnetism, heat, capillarity, elasticity and optics, all to be included in a "Traité de physique mathématique" that he would not complete before his death at the age of 58.

The rest of the exhibit contained contemporary and later evaluations, laudatory or critical, of Poisson's role in the development of mathematics and physics in the first half of the 19th century, and a selection of modern books where Poisson's work is extended: texts and research monographs on potential theory, probability theory and Poisson geometry in mathematics and physics.

Another chance

If you missed this exhibit, you have another chance, in fact two chances, to see it! A near duplicate display will be re-created in the mathematics library of the University of Illinois at Urbana-Champaign, on the occasion of the "International Conference on Poisson Geometry in Mathematics and Physics" (Poisson 2014) at the end of July 2014 (see http://www.math.uiuc.edu/Poisson2014/) and then, through the initiative of Alan Weinstein, at the University of California in Berkeley, on the occasion of the annual "Gone Fishing" conference in early November 2014 (see http://math.berkeley.edu/~libland/Gone_Fishing_2014.html).

If you do not plan to cross the Atlantic this year to see the Urbana or Berkeley versions of this exhibit, you can download the Paris catalogue from http://www.math-info-paris.cnrs.fr/bibli/ (archives des actualités, June 2014) where you will find some comments on the books pictured in the photographs illustrating this *ex post facto* announcement.

Articles published in the *Journal de l'École Polytechnique* and in the *Mémoires de l'Académie des Sciences*, 1811–1830

Yvette Kosmann-Schwarzbach (yks@math.polytechnique.fr) has published on Poisson geometry, written a book on "The Noether Theorems" (2006, English translation 2011) and, more recently, edited a book of essays on Poisson's mathematics and physics and current developments (2013). A few hours before the opening, she finished writing the notices for the catalogue of the exhibit at Jussieu. With Brigitte Laude, she wishes to thank the librarians and archivists who lent books and documents for this exhibit.

Stein Arild Strømme (1951–2014) *in memoriam*

Sebastià Xambó (Universitat Politècnica de Catalunya, Barcelona, Spain)

Figure 1. Stein A. Strømme as Director of the Mathematics Department, University of Bergen

Professor Stein Arild Strømme*, of the* University of Bergen *and associated with the* Oslo Centre of Mathematics and Applications*, passed away on 31 January this year at the age of 62. A well known specialist in algebraic geometry, in particular in the areas of moduli spaces, intersection theory and enumerative geometry, he was also Editor of* Acta Mathematica (1997–2000) *and a committed leader on the international stage with endeavours such as the* Europroj *project, the* Sophus Lie Center *at Nordfjordeid and the* Mittag-Leffler Institute *at Djursholm (Stockholm).*

The Maple package SCHUBERT*, developed jointly with* Sheldon Katz*, accredited him as a computational wizard. The aim of this article is to present a more detailed overview of these accomplishments, the milieu in which they took form and the visible influence on current undertakings.*

A masterpiece (and a big moment for mirror symmetry)

This high praise for *The number of twisted cubic curves on the general quintic threefold* [21] by one reviewer (Bruce Hunt) is aptly asserted in the opening sentence of his excellent review.[1] A reviewer (Susan Colley) of [23], which is the final version of [21], justly refers to it as "a *tour de force* of enumerative geometry".[2] She also notes that in *Bott's formula and enumerative geometry* [24] (which is clearly another *chef d'œuvre*) "the authors have obtained the number 317 206 375 of twisted cubics on a general quintic hypersurface in $\mathbf{P}^4$ in another way, namely by using a residue formula of Bott. This technique avoids some of the intricate and involved intersection-theoretic calculations used in the paper under review." The significance of these works, which goes far beyond their mathematical and computational virtuosity, is rooted in the context in which they were produced. In 1990, a principle of 'mirror symmetry' for Calabi-Yau manifolds was proposed by string theorists Brian Greene and Ronen Plesser.[3] This principle, together with other insights of the late 1980s relating Calabi-Yau manifolds and conformal field theories, was used by Philip Candelas and his collaborators (Xenia de la Ossa, Paul Green and Linda Parkes) to predict the number N_d of rational curves of any degree d contained in a generic quintic 3-fold in $\mathbf{P}^4$.[4] What Strømme and Ellingsrud did was an independent computation of N_3 according to the (algebraic geometry based) standards of enumerative geometry at that time.[5] At first (May 1991), their result disagreed with the physicists' prediction but after correcting a bug in their program they got the same answer and graciously recognised the error by sending the famous "Physics wins!" message (June 1991). If it was true that the numbers N_d, for $d \geq 4$, seemed beyond the reach of current enumerative geometry methods, the retrospect perspective on the positive significance of the computation of N_3 has been emphasised by leading researchers, as for example by Shing-Tung Yau in [68] (p. 170, my emphasis):

> This proved to be *a big moment for mirror symmetry*. The announcement of Ellingsrud and Strømme not only advanced the science of mirror symmetry, but also helped change attitudes toward the subject [… mathematicians] now came to realise there was something to be learned from the physicists after all.

In [31], Brian Greene had already presented these new connections between mathematics and physics with the words (p. 262, my emphasis):

> For quite some time, physicists have 'mined' mathematical archives in search of tools for constructing and analyzing models of the physical world.

1 MR1191425 (94d:14050)

2 MR1345086 (96g:14045)

3 B. R. Greene and M. R. Plesser. *Duality in Calabi-Yau Moduli Space*. Nuclear Physics B 338 (1990), 15-37. In the case of a Calabi-Yau 3-fold, Poincaré duality and Hodge duality imply that the Hodge numbers $h^{1,1}(X)$ and $h^{1,2}(X)$ determine all the other $h^{p,q}(X)$. The 'mirror symmetry' conjecture, made on the grounds of a 'physical duality', was that there ought to exist a Calabi-Yau 3-fold $\bar{X}$ such that $h^{1,1}(\bar{X}) = h^{1,2}(X)$ and $h^{1,2}(\bar{X}) = h^{1,1}(X)$.

4 P. Candelas, X. de la Ossa, P. Green, L. Parkes. *A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory*. Nuclear Physics B 359 (1991), 21–74.

5 $N_1 = 2875$ was determined by H. H. Schubert, [52]; $N_2 = 609250$, by S. Katz, [36].

> Now, through the discovery of string theory, physics is beginning to repay the debt and *to provide mathematicians with powerful new approaches to their unsolved problems*. String theory not only provides a unifying framework for physics, but it may well forge an *equally deep union with mathematics as well*.

Among the more striking cases of 'mining', let me just mention the 'eightfold way' approach (Murray Gell-Mann and Yuval Ne'eman) to the problem of classifying elementary particles. This happened about 30 years before the discovery of mirror symmetry and it involved a few of the representations of SU(3) that were described by Hermann Weyl over 30 years before that.[6] The eightfold way schemes somehow belong to the prehistory of string theory, for they were the first step on the way to building the so-called standard model (early 1970s), which in turn spawned a new fertile era in the relations between mathematics and physics, with leading figures such as Sir Michael Atiyah, Simon Donaldson and Edward Witten. We still have a bit more to say about this in a later section.

Small world[7]

A quick glance at the Proceedings of the Enumerative Geometry Conference Sitges 1987 [65] reveals that while there was one Scandinavian mathematician for every eight participants, they actually contributed nearly one third of the lectures and one third of the published papers. This is rather remarkable when contrasted with the participation in the previous, more general, conference Sitges'83 Week of Algebraic Geometry [4] in which there were just four Scandinavians[8] among a total of 71 participants, only two of them (Ulf Persson and Ragni Piene) as speakers.

The explanation of these figures is actually quite simple. The 1983 conference was on algebraic geometry in a general sense. It was the first conference organised in Barcelona on that topic and the invited speakers belonged to a number of specialties. The only enumerative geometer, for instance, was Ragni Piene, a student of Steven L. Kleiman (she got her PhD in 1976). By the way, the title of her contribution was *On the problem of enumerating twisted cubics* ([4], pages 329–337), a theme that is closely related to problems that were to interest Stein A. Strømme (as we will see later). But in the four years up to 1987, a small and enthusiastic enumerative geometry group was formed in Barcelona and it was decided that the second algebraic geometry conference would be focused on that area (including intersection theory). The high participation of Scandinavians on that second occasion simply reflected the fact that in those countries there were more people interested, or expert, in the conference topic than in any other country.

In the Sitges'87 conference, Strømme showed some of his cards: he lectured on the Chow ring of a geometric quotient, the substantive fruit of another collaboration with Ellingsrud which was later published in the Annals of Mathematics [20].

But it was actually two years earlier that I met Strømme for the first time. The setting was the conference on space curves held in Rocca di Papa, organised by Franco Ghione, Christian Peskine and Edoardo Sernesi (Springer LNiM 1266, 1987), and it happened through the good offices of Kleiman. I had proposed to him to go over some ideas about checking Schubert's 'big number' (number of twisted cubics that are tangent to 12 quadrics in $\mathbf{P}^3$) and he immediately arranged a few discussions of the three of us, which led to the paper [40].

The scientific participation of Strømme, however, was far larger than his part in [40], for he published two more papers in the same proceedings: [55] and [15]. The latter, in collaboration with Ellingsrud and Ragni Piene, develops a nice geometric technique for compactifying the space of twisted cubics, an insight that six years later was a keystone for the first computation of the number N_3 explained in the previous section.

Let me also say that Kleiman, who might be counted as Scandinavian,[9] submitted a wonderful paper on multiple-point formulas to Sitges'87: *Multiple-point formulas II: the Hilbert scheme* (38 pages). This was the long awaited sequel to the paper *Multiple-point formulas I: Iteration* (37 pages) that was published in Acta Mathematica in 1981.

The SCHUBERT package

The SCHUBERT Maple package [39], written by Stein A. Strømme and Sheldon Katz, was released in 1992 and was maintained, according to its website,[10] until 2011, with revisions and updates "to reflect new versions of Maple" by Jan-Magnus Økland. The package supports computations in intersection theory and enumerative geometry. It was frozen at release 0.999 but it may still be instructive to reflect on how its authors could achieve so much power with a remarkably compact code.[11] The somewhat technical digression that follows may be helpful for those interested in ideas about how such a system is built or extended, or even ported to other platforms.

To highlight a few characteristic features of the package in a suitably generic setting, I will glean a few ideas from [66], which will be referred to as SCHUBERT-WIT. It is not required to know anything about the interface WIT, for here we only rely on its rather self-explanatory pseudo-code-like expressiveness. This may also help those interested in getting acquainted with the Sage package CHOW (about which you will find a bit more at the end of this section).

One of the main ideas in SCHUBERT-WIT is that its objects are implemented as instances of `Class` definitions (or types). A `Class` definition can be thought of as a list of pairs `k : T`, where `k` denotes an identifier (a key) and `T` a type. It follows that an object can be construed as a list of pairs `k : t`, where

6 H. Weyl, *Gruppentheorie und Quantemechanik* (1928). See also the book *Essays in the history of Lie groups and algebraic groups*, by Armand Borel, especially Chapter III (History of Mathematics, Vol. 21, AMS, 2001).

7 Stolen from the title of David Lodge's 1984 novel (published by Penguin Books in 1995).

8 Audun Holme, Trygve Johnsen, Ulf Persson and Ragni Piene.

9 On account of being married to Beverly, who is Danish, but also because of the many shared undertakings with mathematicians from those countries.

10 http://stromme.uib.no/schubert/.

11 The length of the package file is 142 KB. It has about four thousand lines, of which more than half are devoted to help texts and examples.

t is an object of type T and where k runs through some subset of the Class definition keys. Objects are dynamic, in the sense that the number of keyed entries can be enlarged during a computation. The types T can be types of the underlying language (like Identifier, Integer, Polynomial, List or Vector) or other SCHUBERT-WIT types, such as Sheaf, Variety or Morphism.

Let us look at some simple examples (we refer to [56] or [64] for a quick introduction to basic concepts about intersection theory and Chern classes). The type Sheaf, for instance, is defined with the following code:

```
let Sheaf = Class
    rk : Element(Ring)
    ch : Vector
```

This means that the representation of a sheaf (an instance of Sheaf) is like a table with two entries, one with key rk (for the rank, possibly in symbolic form) and another with key ch (for the Chern character).

In order to work with sheaves, we first need some constructors, that is, functions that deliver a sheaf out of some suitable data. The simplest situation is when we know the Chern character x = [x.1,...,x.n] or, more generally, that the Chern character has the form pad(x,d), where d is an integer (a cut-off dimension) and pad(x,d) is [x.1,...,x.d] if d ≤ n or [x.1,...,x.n, 0,...,0] otherwise, with $d - n$ zeros. If these conditions can be assumed then we can define the constructor SH as follows (note the three different call forms):

```
SH(r:Element(Ring), x:Vector, d:Nat) :=
   instance Sheaf
       rk = r
       ch = if x==[] then [0] else pad(x,d)
SH(r:Element(Ring), x:Vector) :=
    SH(r,x,length(x))
SH(r:Element(Ring), x:Vector, X:Variety) :=
    SH(r,x,dim\X)
```

The expression dim\X yields the dimension of X. Indeed, as we will see below, one of the keys of the type Variety is dim, and the dimension of a variety X can be extracted either as X(dim) or dim\X (the operator \ swaps its operands and applies the second to the first).

In general, however, the data that we will have at our disposal for the construction of a sheaf is a (symbolic) vector of Chern classes c = [c.1,...,c.n] (Chern vector) and a cut-off dimension d for the Chern character. In that case we can use the following constructors:

```
sheaf(r:Element(Ring), c:Vector, d:Nat) :=
   SH(r,c2p(c,d))
sheaf(r:Element(Ring), c:Vector, X:Variety) :=
   SH(r,c2p(c,dim\X))
sheaf(r:Element(Ring),c:Vector) :=
   SH(r,c2p(c))
```

Here c2p(c,d) takes care of converting, using basic formulas from intersection theory, the vector of Chern classes to the Chern character vector, padded to the cut-off dimension d.

To construct vector bundles (locally free sheaves), we can use calls to sheaf that take advantage of additional properties of their Chern classes:

```
bundle(n:Element(Ring), c:Name, d:Nat) :=
   if is?(n,Integer) & n>=0
   then sheaf(n,vector(c,min(n,d)),d)
   else sheaf(n,vector(c,d))
bundle(n:Element(Ring), c:Name, X:Variety) :=
   bundle(n,c,dim\X)
bundle(n:Element(Ring), c:Name) :=
   bundle(n,c,DIM)
```

It is important to note that an expression such as vector(u,3), where u is a symbol, yields the symbolic vector [u1,u2,u3]. In the case that the rank data is a non-negative integer, the span of the Chern vector is at most min(n,d) and we use this fact in the first call. In the third call, DIM refers to the (current) default dimension.

In a similar way, varieties are objects of type VAR = Variety = Manifold, which is defined as follows:

```
let VAR = Class
    dim : Integer
    kind : Identifier
    gcs : Vector # of generating (Chow) classes
    degs : Vector # codimensions of the gcs
    monomials : List
    monomial_values : Table
    pt : Polynomial
    relations : Vector
    basis : List
    dual_basis : List
    tan_bundle : Sheaf
    todd_class : Vector
```

Among the several constructors of varieties, a simple illustration is given by the case of projective spaces (note the dynamic construction of the final object):

```
projective_space(n:Integer, h:Name) :=
   P=instance VAR
      dim =n
   P(kind) = if n==1 then _projective_line_
             elif n==2 then _projective_plane_
             else _projective_space_ end
   P(gcs)=[h]
   P(degs)=[1]
   P(relations)=[h^(n+1)]
   P(monomials)={{h^j} with j in 1..n}
   P(monomial_values)={h^n : 1}
   P(pt)=h^n
   P(tan_bundle)=sheaf(n,
      [binomial(n+1,j)*h^j with j in 1..n],P)
   P(todd_class)=todd_vector(P(tan_bundle),P)
   return P
```

The symbol h stands for the rational class of a hyperplane of the projective space of dimension n. Hence gcs is the vector [h] (a single component) and degs is [1]. There is a single relation ($h^{n+1} = 0$), and $1 = h^0, h^1, \ldots, h^n$ are all the nonzero monomials. The judgments pt = h^n and h^n : 1 express the fact that the intersection of n general hyperplanes is a point and that a point has degree 1.

A type such as VAR can be extended in order to define more specialised types. A nice example is the type GRASS = Grassmannian:

```
let GRASS = Class from VAR
    tautological_bundle : Sheaf
    tautological_quotient : Sheaf
```

This means that a grassmannian is a variety with two additional keys, one for the tautological bundle and another for the tautological quotient. As to constructors of grassmannians, they can be defined in a rather straightforward manner by using suitable formulas for the values of the different keys.

As an example of all this, here is a lovely script that computes the number of lines (27) on a generic cubic surface in 3-dimensional projective space:

```
G=grassmannian(1,3,c);
S=tautological_bundle\G;
F=symm(3,dual(S));
c4=chern(rk\F,F)
integral(G,c4)
```

Finally, let us consider the type MOR = Morphism:

```
let MOR = Class
    source : VAR
    target : VAR
    dim : Integer
    upperstardata : Table
    kind : Identifier
```

The key upperstardata is assigned a Table. For a given morphism f (an instance of MOR) the entries of the table upperstardata\f have the form x : u, where x runs trough gcs\target\f and u is the expression of f*x in terms of gcs\source\f.

The types MOR and VAR can be used to define the type BLOWUP:

```
let BLOWUP = Class from VAR
   exceptional_class : Element(Ring)
   blowup_map : MOR
   blowup_locus : VAR
   locus_inclusion : MOR
```

This class includes the ingredients that are involved when we blow up a variety X (this is captured in the first line) along a subvariety Y (blowup_locus). The locus_inclusion stands for the inclusion $Y \xrightarrow{i} X$ and blowup_map for the projection map $\widetilde{X} \xrightarrow{f} X$. The exceptional_class stands for the rational class of the exceptional divisor. Sure enough, we have a constructor blowup(i:MOR, e:Name, f:Name) that binds i to the locus inclusion, f to the blowup map and e to the exceptional class.

As an illustration, let us look at a script that computes the number (3264) of conics in a plane that are tangent to five smooth conics in general position:

```
P5=projective_space(5,H);
S =projective_space(2,h);
i=morphism(S,P5,[2*h]);
X=blowup(i,e,f);
integral(X,(6*H-2*e)^5)
```

Here P5 is the projective space of conics in a plane (for background, see, for example, [5]). The set of conics tangent to a given conic is a sextic hypersurface and hence its rational class is $6H$. These hypersurfaces all contain the surface S of double lines but their proper transforms on the blowing up of $\mathbf{P}^5$ along S do not intersect on the exceptional locus. As it turns out that the class of any of these proper transforms is $6H - 2e$, the number we are seeking is the degree of the intersection $(6H - 2e)^6$.

After the lecture [66], I learned that D. Grayson, M. Stillman, D. Eisenbud and S. A. Strømme were working on SCHUBERT2, a port of SCHUBERT to Macaulay2. But this project seems to have been displaced later on by CHOW, a Sage package written in Singular by M. Lehn and C. Sorger.[12] It seems clear that Strømme's illness diminished the time and energy he could devote to the project. One of the entries in his diary reads:

> Sunday, 24 Nov 2009: Getting stronger with each passing day. This morning I was able to go to church for the first time after surgery. I could even play the violin with the worship team like before :-) Looking forward to getting started with radiation therapy on Tuesday.

And Sheldon Katz, in a message he sent me on 2 February 2014, wrote:

> Stein Arild was very courageous and inspiring to many people while he was fighting his illness these past few years. He was diagnosed in the middle of the planning of a mini-conference at MSRI a few years ago in which he was to be a key participant, but he was still able to participate some by Skype. Against expectations, he recovered enough to do active and impressive research again. He wasn't trying to hide his health challenges but I'm not surprised that it wasn't well known since he was being productive again.

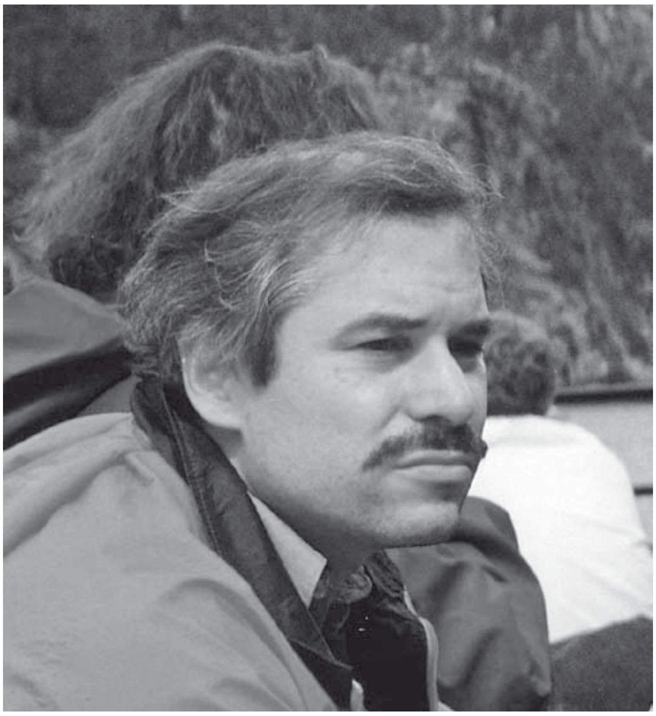


Figure 2. Sheldon Katz, coauthor of the SCHUBERT Maple package. Dyrkolbotn Conference on Mirror Symmetry, 1993

New directions

The deep 'repaying' impact of quantum field theory on mathematics, and in particular in enumerative geometry, has been extraordinary. In the words of S. Katz and D. Cox in [6] (first paragraph of the preface):

> The field of mirror symmetry has exploded onto the mathematical scene in recent years. This is part of an increasing connection between quantum field theory and many branches of mathematics.

Around and after the episode concerning the computation of N_3 (see the first section), there was a stream of important contributions in the direction of mathematical proofs of the pre-

12 See http://www.math.sciences.univ-nantes.fr/~sorger/chow/html/ for details. The user manual can be downloaded from this site.

Figure 3. Duco van Straten and Stein A. Strømme during a boat excursion at the 1993 Dyrkolbotn Conference on Mirror Symmetry

dictions of mirror symmetry but the most innovative developments actually came from applications of physical ideas to develop new mathematics. The rest of this section is an attempt to signpost a number of key highlights, with due regard to Strømme's contributions.

1992

M. Kontsevich proves a conjecture of Witten in his PhD thesis *Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function*. It was published as the widely acclaimed paper [42]. Kontsevich is also an invited speaker at the 1st ECM (Paris). The lecture on *Feynman diagrams and low dimensional topology* was published in the conference proceedings (First European Congress of Mathematics, 1992, Paris, Volume II, Progress in Mathematics 120, Birkhauser 1994, 97-121).

S.-S. Yau assembles [67], "the first book of papers published after the phenomenon of mirror symmetry was discovered". The collection includes 20 papers (see AMS for the table of contents), in particular the two landmark contributions *Mirror manifolds and topological field theory* (E. Witten, [62]) and *Topological mirrors and quantum rings* (C. Vafa, [60]). It also includes *Rational curves on Calabi-Yau threefolds* (S. Katz, [37]). The paper [21] was included in the first edition, seemingly without the permission of the authors, but was excluded in the updated version (1998). In a private communication, S. Katz says (19 July, 2014):

> The *Essays on Mirror Manifolds* volume included papers from the Workshop on Mirror Symmetry at MSRI in May 1991. This was the first workshop that brought such a large number of algebraic geometers and string theorists together, and was organised quickly following the work of Stein Arild Strømme/Geir Ellingsrud and Candelas et al. [. . .] Not many mathematicians have inspired short-notice international conferences at MSRI.

1993

G. Ellingsrud and S. A. Strømme publish [22].

The author of this article made a small contribution [63] to explain the new connections to 'physical mathematics' to the Spanish Differential Geometry community. This was shortly after he attended the 1993 Dyrkolbotn Conference (S. A. Strømme was one of the organisers and the picture with Duco van Straten was taken on that occasion). The papers [21, 22] were cited.

1994

M. Kontsevich and A. Givental plenary lectures *Homological algebra of mirror symmetry* and *Homological geometry and mirror symmetry* at the Zürich ICM were published in the congress proceedings as [44] and [29], respectively. M. Kontsevich also takes part in the conference *Moduli Space of Curves* (Texel Island) and his lecture on *Enumeration of rational curves via torus actions* was included in the proceedings as [43]. In this paper we read:

> The main body of computations is contained in section 3. Our strategy here is quite standard: we reduce our problems to questions concerning Chern classes on a space of rational curves lying in projective spaces (A. Altman - S. Kleiman, S. Katz), and then use Bott's residue formula for the action of the group of diagonal matrices (G. Ellingsrud and S. A. Strømme). As a result we get in all our examples certain sums over trees.

Concerning the work on Bott's formula and enumerative geometry [24] "Kontsevich realised that the method could be applied to the moduli stack of stable maps more easily than it could be applied to the Hilbert scheme of rational curves. This led to the rapid development of Gromov-Witten theory" (Sheldon Katz, private communication, 19 July 2014).

M. Kontsevich and Y. I. Manin publish the landmark paper [45]. V. Batyrev publishes [2]. G. Ellingsrud, J. Le Potier and S. A. Strømme present the work *Some Donaldson invariants of* $\mathbf{CP}^2$ to the Sanda/Kyoto Conference on Moduli of Vector Bundles and it is published in the proceedings as [13].

1995

G. Ellingsrud and S. A. Strømme publish [23] (see the first section of this note). V. V. Batyrev and D. van Straten publish [3]. W. Fulton and R. Pandharipande take part in the Santa Cruz *Algebraic Geometry* Conference and present *Notes on stable maps and quantum cohomology* (included in the proceedings as [28]). D. Morrison and M. Plesser publish [48].

1996
G. Ellingsrud and S. A. Strømme publish [24] (*Bott's formula and enumerative geometry*), for whose influence we refer to the comments under **1994**. A. Givental publishes [30] (*Equivariant Gromov-Witten invariants*). D. Morrison publishes the lecture notes *Mathematical aspects of Mirror Symmetry* (`alg-geom/9609021`). C. Voisin publishes [61], an English translation by R. Cooke of her French notes of the previous year. Publication of [32] (*Mirror Symmetry, II*; see AMS for the table of contents).

1998
M. Kontsevich is awarded the Fields Medal at the ICM-98 (Berlin). Among other contributions, his work on enumerative geometry (and its connection to Witten's conjecture) is cited. G. Ellingsrud and S. A. Strømme publish [25] *An intersection number for the punctual Hilbert scheme of a surface*.

1999
D. Cox and S. Katz publish the comprehensive text [6]. They cite the papers [23, 24] of G. Ellingsrud and S. A. Strømme. Publication of [51] (*Mirror Symmetry, III*; see AMS for the table of contents). Y. I. Manin publishes *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces* (for more information, see AMS).

2000
Two plenary lectures at the 3rd ECM (Barcelona 2000) are closely related to the topics of this timeline: *The Mathematics of M-Theory* (R. Dijkgraaf) and *Moduli, motives, mirrors* (Y. I. Manin). They were published in the proceedings as [8] and [46]. The latter surveyed the "mathematical developments that took place since M. Kontsevich's report at the Zürich ICM".

2002
Publication of [7] (*Mirror Symmetry, IV*; see AMS for the table of contents).

2003
Publication of the eight-author treatise [35] on *Mirror Symmetry* (*xx*+929 pp. – see AMS for the table of contents). According to Y. I. Manin, it is "a valuable contribution to the continuing intensive collaboration of physicists and mathematicians. It will be of great value to young and mature researchers in both communities interested in this fascinating modern grand unification project."

2005
R. Penrose publishes [50]. Although some critics have issued negative views of different aspects of this purported "complete guide to the laws of nature", it provides many insights to the ways mathematics and physics interact. The section 31.14, for example, is devoted to "The magical Calabi–Yau spaces and *M*-theory".

R. P. Thomas publishes [59].

2006
S. Katz publishes [38], a text based on "fifteen advanced undergraduate lectures I gave at the Park City Mathematics Institute" during the Summer of 2001. He cites the paper [23]. This book, together with the more advanced [6] (for a later stage), is a good starting point for readers wishing to know more about developments in algebraic geometry (and in enumerative geometry in particular) related to string theory. For physical and historical background, references such as [31] and [68] may also be good reading.

2007
J. Kock and I. Vainsencher publish [41]. This lovely text is based on notes in Portuguese prepared "to support a five-lecture mini-course given at the 22° Colóquio Brasileiro de Matemática and published by IMPA in 1999". Two papers of Strømme are cited [55, 57]. The latter paper is also cited in Hartshorne's *Deformation Theory* [34].

2010
M. Atiyah, R. Dijkgraaf and N. Hitchin publish the review *Geometry and Physics* [1] of the "remarkably fruitful interactions between mathematics and quantum physics in the last decades". The authors "point out some general trends" of these interactions and advance ideas about the kind of problems that will have to be solved in order to have a comprehensive understanding in mathematical terms of physical theories involved in a unified view of nature. This paper is included in the seventh volume of Atiyah's Collected Works (Oxford University Press, 2014).

A colleague, leader and fellow human being

On 1 February, Antonio Campillo, President of the Royal Spanish Mathematical Society, suggested that I write an article about Stein Arild Strømme, who had died the day before. After some preliminary contact with colleagues of the small world (Geir Ellingsrud, Sheldon Katz, Ragni Piene, Audun Holme and Steve Kleiman), I wrote to Professor Gunnar Fløystad (a colleague of Sein Arild at the Mathematics Department of the University of Bergen) to find out whether somebody else closer to Strømme had a similar intention. After a few messages, I concluded that I would try to go ahead and so I corresponded with the editors of the EMS Newsletter who very kindly indicated a few guidelines about how to frame such a work. My thanks to all the people named here for their invaluable help.

Professor Gunnar Fløystad, in particular, sent me the texts read at the funeral and the short obituary published in the local newspapers. Since they amount to an institutional appreciation, it is fitting to provide an English translation of the parts that are more relevant here[13] (a few footnotes have been inserted in order to provide information that readers cannot be assumed to know):

> [...] Stein Arild Strømme was born in Oslo in 1951[14] and studied mathematics at the University of Oslo. After his doctorate he came to Bergen in 1984 and soon gained permanent employment at the Department of Mathematics. His specialty was algebraic geometry. He soon gave a strong impression, both by his ability to grasp and solve problems and by his deep understanding and mastery of the field.[15] He was appointed as a professor in 1993.

13 I am grateful to Gunnar Fløystad for helping me polish my rather rough initial translation.

14 His mother Inger Johanne Strømme was a school teacher and his father Sigmund Strømme was a renowned and influential publisher.

15 This can be illustrated with the papers [19] and [17]. They appeared in

In the 1980s and 1990s, he produced significant mathematical articles in collaboration with Geir Ellingsrud, his brother-in-law. In 1988, jointly with Ellingsrud and Professor Christian Peskine of the University of Oslo, he received the NAVF's award for excellence in research. In 1999, he became head of the Mathematics Institute, a position that he held for 11 years. During this period, major changes took place at the institute: there were many new appointments to replace people who had retired and this happened under important changes in the funding and teaching at universities. Stein Arild created the balanced and strong research institute that we have today.

A book has been written about being a head of department. It says you must expect to make one enemy for each year in office. Stein Arild managed the feat of renewing the department while still remaining friends with all his colleagues. He was a member of the science faculty council consisting of deans, the director and other science department heads and he was highly valued and respected in this group.

Stein Arild was a man of quiet manner. He sensed rather than pushed his way through. He kept an online blog and we became worried if too much time passed since the last post. [...] He showed concern both for students and for us, his colleagues. When he stepped down as head of the department and the disease seized him, we especially noticed how much he cherished his colleagues. We nevertheless understood when he moved to Oslo and Lambertseter during his last year, to be close to his family.

Our thoughts go out to his wife Leikny and his two sons and their families.[16] Stein Arild has passed away but he has left us with many dear memories that will stay with us. Colleagues and friends at the department and science faculty thank you for what you were to us as a colleague, leader and fellow human being.

Let the distances in space and time be no barriers for the deep expressions of similar feelings of condolence by all his colleagues and friends from so many places around the world.

Remark

In the references below we have included some general treatises that are not cited in the text and which may have been agreeable to Stein Arild: [14], [49], [9], [27], [58], [33], [10], [26] and [11] (forthcoming).

Bibliography

[1] M. Atiyah, R. Dijkgraaf and N. Hitchin. Review: Geometry and Physics. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 368(1914):913–926, 2010.

[2] V. Batyrev. Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. *Journal of Algebraic Geometry*, 3:493–535, 1994. `arXiv:alg-geom/9310003`.

[3] V. V. Batyrev and D. van Straten. Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties. *Communications in Mathematical Physics*, 168:493–533, 1995 (see `arXiv:alg-geom/9310001`).

Inventiones and together they provide an explicit determination of the Chow ring of the Hilbert space of points in the plane. He also worked on vector bundles and moduli spaces: [16, 18, 47, 53, 54, 12].

16 Stein Arild had two sons: Kjetil Strømme, a publisher, with his first wife, and Torstein Strømme, an engineer, with his second wife (Leikny). Kjetil has a daughter, Vanja.

[4] E. Casas, G. Welters and S. Xambó-Descamps, editors. *Algebraic Geometry – Sitges (Barcelona) 1983*, volume 1124 of *Lecture Notes in Mathematics*. Springer-Verlag, 1986.

[5] E. Casas-Alvero and S. Xambó-Descamps. *The enumerative theory of conics after Halphen*. Number 1196 in Lecture Notes in Mathematics. Springer, 1986.

[6] D. A Cox and S. Katz. *Mirror symmetry and algebraic geometry*. Number 68 in Mathematical Surveys and Monographs. American Mathematical Society, 1999.

[7] E. D'Hoker, D. Phong and S.-S. Yau, editors. *Mirror symmetry, IV*. AMS & International Press, 2002.

[8] R. Dijkgraaf. The Mathematics of M-Theory. In C. Casacuberta, R. M. Miró, J. Verdera and S. Xambó-Descamps, editors, *3rd European Congress of Mathematics*, pages 1–19. Birkhäuser, 2001.

[9] D. Eisenbud. *Commutative Algebra: with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer, 1995.

[10] D. Eisenbud. *The geometry of syzygies*. Number 229 in Graduate Texts in Mathematics. Springer, 2005.

[11] D. Eisenbud and J. Harris. *3264 & All That: Intersection Theory in Algebraic Geometry*. Harvard University, 2013. pdf.

[12] G. Ellingsrud, L. Gruson, C. Peskine and S. A. Strømme. On the normal bundle of curves on smooth projective surfaces. *Inventiones Mathematicae*, 80 (1): 181–184, 1985.

[13] G. Ellingsrud, J. Le Potier and S. A. Strømme. Some Donaldson invariants of $\mathbf{CP}^2$. In *Moduli of vector bundles* (Sanda, 1994; Kyoto, 1994), number 179 in Lecture Notes in Pure and Applied Mathematics, pages 33–38. Dekker, 1996.

[14] G. Ellingsrud, C. Peskine, G. Sacchiero and S. A. Strømme, editors. *Complex projective geometry*, volume 179 of *Lecture Note Series*. London Mathematical Society, 1992.

[15] G. Ellingsrud, R. Piene and S. A. Strømme. On the variety of nets of quadrics defining twisted cubics. In *Space curves* (Rocca di Papa, 1985), number 1266 in Lecture Notes in Mathematics, pages 84–96. Springer, 1987.

[16] G. Ellingsrud and S. A. Strømme. Stable rank-2 vector bundles on $\mathbf{P}^3$ with $c_1 = 0$ and $c_2 = 3$. *Mathematischen Annalen*, 255 (1): 123–135, 1981.

[17] G. Ellingsrud and S. A. Strømme. On the homology of the Hilbert scheme of points in the plane. *Inventiones Mathematicae*, 87 (2): 343–352, 1987.

[18] G. Ellingsrud and S. A. Strømme. On the rationality of the moduli space for stable rank-2 vector bundles on $\mathbf{P}^2$. In *Singularities, representation of algebras, and vector bundles* (Lambrecht, 1985), number 1273 in Lecture Notes in Mathematics, pages 363–371. Springer, 1987.

[19] G. Ellingsrud and S. A. Strømme. On a cell decomposition of the Hilbert scheme of points in the plane. *Inventiones Mathematicae*, 91 (2): 365–370, 1988.

[20] G. Ellingsrud and S. A. Strømme. On the Chow ring of a geometric quotient. *Annals of Mathematics*, 130 (1): 159–187, 1989.

[21] G. Ellingsrud and S. A. Strømme. The number of twisted cubic curves on the general quintic threefold (preliminary version). In Yau [67], pages 181–222.

[22] G. Ellingsrud and S. A. Strømme. Towards the Chow ring of the Hilbert scheme of $\mathbf{P}^2$. *Journal für die reine und angewandte Mathematik*, 441: 33–44, 1993.

[23] G. Ellingsrud and S. A. Strømme. The number of twisted cubic curves on the general quintic threefold. *Mathematica Scandinavica*, 76 (1): 5–34, 1995.

[24] G. Ellingsrud and S. A. Strømme. Bott's formula and enumerative geometry. *Journal of the American Mathematical Society*, 9 (1): 175–193, 1996.

[25] G. Ellingsrud and S. A. Strømme. An intersection number for the punctual Hilbert scheme of a surface. *Transactions of the American Mathematical Society*, 350 (6): 2547–2552, 1998.

[26] B. Fantechi, L. Götsche, L. Illusie, S. L. Kleiman, N. Nitsure and A. Vistoli. *Fundamental algebraic geometry: Grothendieck's FGA explained*. Number 123 in Mathematical Surveys and Monographs. American Mathematical Society, 2005.

[27] W. Fulton. *Intersection theory* (2nd edition), volume 2 of *Ergebnisse der Mathematik un ihrer Grenzgebiete (*3. Folge) / *A series of modern surveys in mathematics*. Springer, 1998. Corrections and some additional references with respect to the 1st edition, 1984.

[28] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In J. Kollár, R. Lazarsfeld and D. Morrison, editors, *Algebraic Geometry (Santa Cruz, 1995)*, volume 62 of *Proceedings of Symposia in Pure Mathematics*, pages 45–96. American Mathematical Society, 1997.

[29] A. Givental. Homological geometry and mirror symmetry. In *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, volume 1, pages 472–480, 1994.

[30] A. Givental. Equivariant Gromov-Witten invariants. *International Mathematics Research Notices*, pages 613–663, 1996. alg-geom/9603021v2.

[31] B. Greene. *The elegant universe*. W. W. Norton, 2003.

[32] B. Greene and S.-S. Yau, editors. *Mirror symmetry, II*. AMS & International Press, 1996. Zentralblatt: "The papers of this volume will undoubtedly allow the reader to gain much insight into both the physics and the mathematics of the remarkable structure of mirror symmetry."

[33] G.-M. Greuel and G. Pfister. *A Singular introduction to commutative algebra* with contributions by O. Bachmann, C. Lossen and H. Schönemann, Springer, 2002.

[34] R. Hartshorne. *Deformation theory*, volume 257 of *Graduate Texts in Mathematics*. Springer, 2010.

[35] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslov. *Mirror symmetry*. Clay Mathematics Monographs. American Mathematical Society, 2003.

[36] S. Katz. On the finiteness of rational curves on quintic threefolds. *Composition Mathematica*, 60(2):151–162, 1986.

[37] S. Katz. Rational curves on Calabi-Yau threefolds. 1992. Available at `arXiv:hep-th/9202017` and included in [67].

[38] S. Katz. *Enumerative geometry and string theory*, volume 32 of *Student mathematical library*. American Mathematical Society, 2006.

[39] S. Katz and S. A. Strømme. Schubert: *A Maple package for intersection theory and enumerative geometry*. Available at http://stromme.uib.no/schubert/, 1992—.

[40] S. Kleiman, S. A. Strømme and S. Xambó. Sketch of a verification of Schubert's number 5819539783680 of twisted cubics. In *Space curves* (Rocca di Papa, 1985), number 1266 in Lecture Notes in Mathematics, pages 156–180. Springer, 1987.

[41] J. Kock and I. Vainsencher. *An invitation to quantum cohomology: Kontsevich's formula for rational plane curves*, volume 249 of *Progress in Mathematics*. Birkhäuser, 2007.

[42] M. Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Communications in Mathematical Physics*, 147:1–23, 1992.

[43] M. Kontsevich. Enumeration of rational curves via torus actions. In R. Dijkgraaf, C. Faber and G. van der Geer, editors, *The Moduli Space of Curves*, volume 129 of *Progress in Mathematics*, pages 335–368. Birkhäuser, 1995. `arXiv:hep-th/9405035`.

[44] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the ICM-1994 (Zürich), vol. I*, pages 120-139. Birkhäuser, 1995. Available at `arXiv:alg-geom/9411018`.

[45] M. Kontsevich and Y. I. Manin. Gromov-witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164(3):525–562, 1994. `arXiv:hep-th/9402147`.

[46] Y. I. Manin. Moduli, motives, mirrors. In C. Casacuberta, R. M. Miró, J. Verdera and S. Xambó-Descamps, editors, *3rd European Congress of Mathematics*, pages 53–73. Birkhäuser, 2001.

[47] J. Meseguer, I. Sols and S. A. Strømme. Compactification of a family of vector bundles on $\mathbf{P}^3$. In *Proceedings 18th Scandinavian Congress of Mathematicians* (Aarhus, 1980), number 11 in Progress in Mathematics, pages 474–494. Birkhäuser, 1981.

[48] D. Morrison and M. Plesser. Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties. *Nuclear Physics B*, 440(1):279–354, 1995.

[49] D. Mumford, J. Fogarty and F. C. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete* (2. Folge). Springer, 1994 (3rd edition).

[50] R. Penrose. *The road to reality*. Alfred A. Knopf, 2005.

[51] D. Phong, L. Vinet and S.-S. Yau, editors. *Mirror symmetry, III*. AMS & International Press, 1999.

[52] H. H. Schubert. Die *n*-dimensionale Verallgemeinerung der Anzahlen für die vielpunktig berührenden Tangenten einer punktallgemeinen Fläche *m*-ten Grades. *Mathematischen Annalen*, 26:52–73, 1886 (see page 72 for two ways of computing the number 2875 of lines contained in a general quintic 3-fold).

[53] S. A. Strømme. Deforming vector bundles on the projective plane. *Mathematischen Annalen*, 263 (3): 385–397, 1983.

[54] S. A. Strømme. Ample divisors on fine moduli spaces on the projective plane. *Mathematische Zeitschrift*, 187 (3): 405–423, 1984.

[55] S. A. Strømme. On parametrized rational curves in Grassmann varieties. In *Space curves* (Rocca di Papa, 1985), number 1266 in Lecture Notes in Mathematics, pages 251–272. Springer, 1987.

[56] S. A. Strømme. Quick and dirty introduction to intersection theory and Chern classes. Notes for the Regional Geometry Institute, Amherst, 1992. See http://stromme.uib.no/home/pub.html.

[57] S. A. Strømme. Elementary introduction to representable functors and Hilbert schemes. In *Parameter spaces* (Warsaw, 1994), number 36 in Banach Center Publications, pages 179–198. Polish Academy of Sciences, 1996.

[58] R. P. Thomas. Derived categories for the working mathematician. 2001. Available at `arXiv:math/0001045v2`.

[59] R. P. Thomas. The geometry of mirror symmetry. 2005. Available at `arXiv:math/0512412`.

[60] C. Vafa. Topological mirrors and quantum rings. In Yau [67], pages 96–119.

[61] C. Voisin. *Mirror Symmetry*, volume 2 of *SMF/AMS Texts and Monographs*. AMS/SMF, 1996.

[62] E. Witten. Mirror manifolds and topological field theory. In S.-T. Yau, editor, *Essays on Mirror Manifolds*. International Press, 1992.

[63] S. Xambó-Descamps. On Calabi–Yau manifolds: bridging enumerative geometry and string theory. In X. Gràcia, M. C. Muñoz and N. Román, editors, *Proceedings of the Fall Workshop on Differential Geometry and its Applications*. Barcelona '93 (http://www-ma4.upc.edu/dgdsa/barcelona93.html), pages 63–73, 1994.

[64] S. Xambó-Descamps. *Using intersection theory*. Number 7 in Aportaciones Matemáticas. Sociedad Matemática Mexicana, 1996.

[65] S. Xambó-Descamps, editor. *Enumerative Geometry – Sitges 1987*, volume 1436 of *Lecture Notes in Mathematics*. Springer-Verlag, 1990.

[66] S. Xambó-Descamps. WIT: A structured and comprehensive package for computations in intersection theory. Presented in January 2009 at the AMS/MAA Joint Mathematics Meeting held in Washington DC, in the AMS Special Session on

Computational Algebraic and Analytic Geometry for Low-dimensional Varieties (organised by M. Seppälä, T. Shaska and E. Volcheck), these slides summarise work done in the preceding years that aimed to better understand the functional structure of the SCHUBERT package. It was also an attempt to speed it up for enumerative geometry computations on cuspidal and nodal cubics carried out with Josep M. Miret.
[67] S.-T. Yau, editor. *Essays on mirror manifolds*. International Press (& AMS, for an updated version published in 1998 with the title *Mirror Symmetry, I*), 1992.
[68] S.-T. Yau and S. Nadis. *The shape of inner space. String theory and the geometry of the Universes' hidden dimensions*. Basic Books, 2010. Illustrations by Xianfeng (David) Gu and Xiaotian (Tim) Yin.

Sebastià Xambó [sebastia.xambo@upc.edu, http://www-ma2.upc.edu/sxd/] is a full professor of applied mathematics (information and coding theory) at the Universitat Politècnica de Catalunya.

The Gender Gap in Mathematics from the PISA Point of View

Elisabetta Strickland (University of Rome "Tor Vergata", Italy) on behalf of the WIM–EMS Committee

The Programme for International Student Assessment (PISA) is a triennial international survey (the last one started in 2012) that aims to evaluate education systems worldwide by testing skills and knowledge of 15-year-old students. To date, students representing more than 65 economies have participated in the assessment. The programme is an initiative of the Organisation for Economic Co-operation and Development (OECD), active since 1961, in which the member countries from across the world regularly turn to one another to identify problems, discuss and analyse them, and promote policies to solve them. PISA is unique because it develops tests which are not directly linked to the school curriculum. In their last survey, PISA asked students about their attitudes towards mathematics. The main facts that were outlined by the answers didn't actually state anything new but, as people are often sceptical about these problems, it's interesting to find out that even with a study quite different from the usual ones, the results are more or less the same. The main problems that have to be faced to fight the gender gap in mathematics are once again pointed out for public attention. These are:

1) In most countries and economies, girls underperform boys in mathematics. Amongst the highest achieving students, the gender gap in favour of boys is even wider.
2) The gender gap in mathematics performance mirrors the gender gap in students' drive, motivation and self belief.
3) Boys and girls tend to benefit equally when they persevere, are motivated to learn and have confidence in their abilities to learn mathematics. Consequently, the performance of both boys and girls suffer at the same rate when they lack motivation to learn and confidence in their own abilities.

Numbers in the survey state that many girls choose not to pursue careers in science, technology, engineering and mathematics because they do not have confidence in their ability to excel in mathematics, despite having the capacity and skills to do so.

Across the 65 countries involved in the survey, 57% of students intend to take additional mathematics courses after school finishes: 63% of boys but only 51% of girls.

Boys and girls are not equally likely to plan a career that involves a lot of mathematics, compared to careers that involve more science. On average, only 38% of girls but 53% of boys plan to pursue a career that involves a lot of mathematics rather than one that involves a lot of science. In addition, evidence from previous PISA cycles – when students were asked about the kind of career they expect to pursue as young adults – suggests that even those girls who envision pursuing scientific careers expect to work in fields that are different from those the boys expect to pursue.

Girls are over-represented among students who expect to work in the health and social fields, while boys are over-represented among 15-years-olds who expect to work as engineers or computer scientists.

Some facts about the survey are really quite interesting: for instance, the average girl in Shanghai-China scores 610 points in mathematics, well above the average performance of boys in every other country and school system that participated in PISA. Meanwhile, the average boy in Shanghai-China scores 557 points in reading, higher than the average performance of girls in every other participating country and school system, except for Hong Kong-China, Japan and Singapore.

Of course, gender differences in mathematics performance are much wider in some countries and economies than in others: the gender gap in mathematics is larger in Austria, Chile, Colombia, Costa Rica, Liechtenstein and

Luxembourg; no gender gap is observed in 23 countries and economies; and in Iceland, Jordan, Malaysia, Qatar and Thailand, girls outperform boys in mathematics.

Interestingly, in mathematics, the gender gap in favour of boys is largest among the best performing students. Among the poorest performing students, performance differences related to gender are small or non-existent.

Since it focused on mathematics performance, PISA 2012 collected detailed information about students' strengths and weaknesses in solving various types of mathematical problems.

For example, the gender gap in favour of boys is wider when looking at students' abilities to formulate concepts mathematically compared to when looking at students' abilities to employ or interpret mathematical concepts. Many students, particularly girls, feel anxious about mathematics and have low levels of confidence in their own abilities, even if they perform well in mathematics. What is particularly worrisome is that, even when girls and boys perform equally well, girls are more likely to feel anxious toward mathematics and have less confidence in their own mathematical skills and their ability to solve mathematics problems.

Gender gaps in drive, motivation and self belief are particularly troubling because these factors are essential if students are to achieve at the highest levels. And PISA results show that the relationship between drive, motivation and mathematics-related self belief on the one hand and mathematics performance on the other is particularly strong amongst the best performing students. Unless girls believe that they can achieve at the highest levels, they will not be able to do so.

Indeed, a substantial proportion of the difference in mathematics performance related to gender can be explained by the difference in boys' and girls' self belief and motivation to learn mathematics.

The gender gap in mathematics performance has remained stable in most countries since 2003, as has the gender gap in mathematics self belief. In the short term, changing these mindsets may require making mathematics more interesting to girls, identifying and eliminating gender stereotypes in textbooks, promoting female role models and using learning materials that appeal to girls. Over the long term, shrinking the gender gap in mathematics performance will require the concerted effort of parents, teachers and society as a whole to change the stereotyped notions of what boys and girls excel at, what they enjoy doing and what they believe they can achieve.

It's natural to agree with the PISA conclusion after the survey, which states that the gender gaps in mathematics require a concerted effort by parents and educators to challenge and eliminate gender stereotypes and bolster girls' belief in themselves.

Reference

Are boys and girls equally prepared for life?, OCSE 2014. www.oecd.org/pisa/pisaproducts/PIF-2014-gender-international-version.pdf.

Elisabetta Strickland is a full professor of algebra at the University of Rome "Tor Vergata" and is Deputy President of INdAM, the Italian National Institute for Advanced Mathematics. Since 2008, she has been a delegate for individual members on the EMS Council and, since January 2014, she has been a member of the Women in Mathematics Committee of the EMS. In 2009, she co-founded the first Gender Inter-university Observatory based in Rome, Italy.

The Catalan Mathematical Society and Mathematics in Catalonia

Joan de Solà-Morales (Polytechnic University of Catalonia, Barcelona, Spain), President of the Societat Catalana de Matemàtiques

I thank very much the editors of the EMS Newsletter, and especially Jorge Buescu, for the opportunity to make this presentation of the Catalan Mathematical Society (the *Societat Catalana de Matemàtiques* in our language – Catalan – and so with acronym SCM) and also to present a short overview of mathematical activity in Catalonia.

To begin with, let me recall that Catalonia is a country of around 7.5 million inhabitants (Spain has a total of 45 million) and a GDP of around 200 billion euros, therefore constituting something between 15% and 20% of Spain. Nowadays, its legal status, according to Spanish Constitution, is that of an Autonomous Community but Catalonia has a very strong feeling of identity, of which every informed observer is aware. These feelings of identity are rooted in our language (which is also spoken outside Catalonia with natural variants) but also in a long-standing tradition of cultural and artistic life, industrial and business activity, civil institutions, knowledge and research centres, sports, and political institutions and associations. The region of Barcelona is perhaps the most important metropolitan centre in Catalonia but Catalan life is not at all restricted to this area.

I present these facts to start explaining the reasons for the existence of our society. Of course, mathematics is an international discipline and we are happy with this characteristic but we, as mathematicians, are also part of our local community and are interested in keeping strong relations with our cultural, scientific and political institutions, and making mathematics as present as it can be in the life of our country.

As is natural, the SCM has very good relations with other mathematical societies operating in the whole of Spain, like the RSME (Royal Spanish Mathematical Society), the SEIO (Spanish Society of Statistics and Operations Research), the SEMA (Spanish Society of Applied Mathematics) and others. These good relations are based on many common interests but also on a natural, friendly, scientific fellowship and a deep respect for each other. Many of our adherents share membership with some of these other societies and also with the European Mathematical Society (EMS). The SCM also feels happy to be a full member of the EMS, something that gives us the possibility of having an independent international presence in Europe. As an EMS member, the SCM was the organiser of 3ECM in Barcelona, July 2000.

The life of the SCM is tied to the life of an old and important Catalan institution: the *Institut d'Estudis Catalans* (IEC, the Institute of Catalan Studies), the national academy of sciences and humanities. As with other academies, the IEC has its internal scientific life, which, in this case, includes the important task of taking care of the Catalan language from a philological point of view, but also (and in this point it is different from other academies) it has a good collection of affiliated societies, in many different subject areas. The SCM is one of these affiliated societies and we share with the IEC the headquarter facilities in Barcelona (administration support and part of our budget). Affiliated societies have very open membership rules, in contrast with the more restrictive policy of the main IEC members.

Under the name of Section of Mathematics, inside a formerly existing Catalan Society of Sciences, our society started its life in 1931. Nowadays, the SCM has around 850 members, mostly university professors, researchers and high school teachers. We see it as a positive point that university and secondary school mathematicians come together in the same society.

For a glimpse of the mathematical activity in Catalonia, we can say that there are seven public and three private universities, employing around 600 mathematicians in teaching and research, and also the *Centre de Recerca Matemàtica* (CRM), a prestigious centre of mathematical research whose activities started 30 years ago. The community of mathematical teachers in secondary schools is obviously much larger and the number of mathematicians working in companies has been increasing appreciably over the last few decades.

In these Catalan universities, three degrees in mathematics are taught and there are two degrees in statistics and several Master's and PhD programmes in mathematics, statistics and mathematical teaching. In the period 2000–2009 (the last period reported) around 250 PhD theses in mathematics were defended in Catalan universities. It is worth mentioning the recently created Barcelona Graduate School of Mathematics (BGSMath), a joint initiative of the Catalan universities together with the CRM to bring together the scientific activities for graduate students offered by these institutions.

There are four international mathematical journals published in Catalonia: *Collectanea Mathematica*, *Publicacions Matemàtiques*, *Qualitative Theory of Dynamical Systems* and SORT (*Statistics and Operations Research Transactions*); all of them are published in English.

Among the institutions that could be called *companions* of our SCM inside Catalonia are the FEEMCAT, a federation of associations of teachers of mathematics, and the *Societat Catalana de Estadística* (Statistics Catalan Society), which was created recently.

Let me now explain the main activities of the SCM, which can be summarised into meetings, publications and prizes. Concerning the meetings, we organise two annual meetings, one dedicated to research: the *Trobada Matemàtica*, and one dedicated to teaching: the *Jornada d'Ensenyament de les Matemàtiques*. The first of these meetings has been replaced this year (2014) by an international conference: the Barcelona Mathematical Days 2014, which will take place in November with the aim of repeating it every two years. Because of that, the *Trobada* this year has not been dedicated to research but to another subject of great interest for the SCM, namely Divulgation, Diffusion and Communication (DDC) in mathematics.

Apart from these regular meetings, we also organise the *Young Researchers Meeting* (though not every year) devoted to young PhD students and recent PhD holders. We also organise seminars and open talks devoted to very different aspects of mathematical activity, such as, for example, the distinguished lecture we organise on the occasion of the award of the Abel Prize. We also have a tradition of organising joint scientific conferences with other societies, such as the so-called CSASC conference (of the Czech, Slovenian, Austrian, Slovak and Catalan Societies) and the one we are planning for May 2015 with the Edinburgh Mathematical Society, which will take place in Barcelona.

Meeting of young researchers.

Publication is also an important activity of the SCM. We publish the *SCM-Notícies* (in Catalan), a printed bulletin of news of our society and the Catalan mathematical community, and the *Butlletí* (also in Catalan), a journal that publishes mathematical papers of an expositive or review nature. Both appear twice per year. The *Butlletí* is nowadays perhaps the highest, and almost the unique, representative of a scientific publication of Mathematics in Catalan language.

Cover of SCM-Notícies.

Another journal that is important for us is *Noubiaix*. It is a journal on mathematical teaching that we publish together with the FEEMCAT mentioned above. The publication deals with mathematical teaching at the primary, secondary and university levels. It is also written in Catalan and it is a good representation of the issues, concerns and achievements of the mathematical teaching community in Catalonia.

Finally, we have a new journal, the first issue of which is about to appear: *Reports@scm*. It is an electronic journal, in English, for research papers of a maximum of 10 pages, and its idea is to be a way for young researchers, like Master's or PhD students, to make their first steps in the world of mathematical publishing. But the journal is also open to other researchers and it is not, by any means, restricted to the local mathematical community. The papers are fully refereed.

The other important fields of activities of the SCM are the prizes and competitions. We have been quite successful in organising contests for secondary school students and sessions and short courses to prepare students to compete in them. We organise the *Cangorou Contest* in Catalonia and the SCM is a full member of the *Cangorou sans Frontières* organisation. This year, more than 20,000 students aged 14 to 17 voluntarily participated in this contest (only in Catalonia). We also organise the Spanish Mathematical Olympiad in collaboration with the RSME.

We also give other research-oriented prizes: the A. Dou Prize, for example, awarded for an article showing the importance of mathematics in our world; the E. Galois Prize, awarded for research work by a student; and, finally, the new Barcelona Dynamical Systems Prize, an international prize to be awarded every two years for a research paper in the area of dynamical systems.

Joan de Solà-Morales [Jc.sola-morales@upc.edu] is a professor of applied mathematics at the Polytechnic University of Catalonia (Barcelona). He is interested mainly in partial differential equations, infinite dimensional dynamical systems and industrial and applied mathematics.

ICMI Column

Jean-Luc Dorier (Université de Genève, Switzerland)

CIEAEM 66, Commission internationale pour l'enseignement et l'amélioration de l'enseignement des mathématiques
21–25 July 2014, IFÉ, ENS de Lyon
http://colloques.ens-lyon.fr/public/detailsEvent.html

The International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM) investigates the actual conditions and the possibilities for development of mathematics education, in order to improve the quality of mathematics teaching. The annual CIEAEM conferences are essential to realising this goal. The conferences are characterised by exchange and discussion of research work and its realisation in practice and by the dialogue between researchers and educators in all domains of practice.

The 66th Conference took place 21–25 July at Institut Français de l'Education – École Normale Supérieure de Lyon.

International Programme Committee: Gilles Aldon (F) Peter Appelbaum (USA), Françoise Cerquetti-Aberkane (F), Javier Diez-Palomar (ES), Benedetto Di Paola, Gail Fitzsimmons (AU), Uwe Gellert (D), Fernando Hitt (Ca), Corinne Hahn (F), François Kalavasis (Gr), Corneille Kazadi (Ca), Michaela Kaslova (CZ), Réjane Monod-Ansaldi (F), Michèle Prieur (F), Cristina Sabena (I), Sophie Soury-Lavergne (F)

Plenary speakers:
Michèle Artigue, Université Paris Diderot (F)
Paul Drijvers, Freudenthal Institute et Cito (NL)
Alejandro Gonzalez-Martin, Université de Montréal (Ca)
Franišek Kuřina, University of Hradec Kralove (CZ)
Tamsin Meaney, Malmö University (S)

The theme of CIEAEM 66 is "Mathematics and realities". It covers different aspects linked to the relationships of mathematics and its teaching with other disciplines but also to the questions of logic when performing mathematics. 'Realities' also means technologies that allow mathematics to be done in other ways, as well as allowing different ways to communicate, teach and learn. In a globalised world, 'realities' is also about bringing together cultures and practices, taking into account the teaching realities in multicultural classrooms.

The conference will be divided into four sub-themes:
- Mathematics and its teaching in relation to other disciplines.
- Logic when doing (or performing) mathematics.
- Realities, technologies and mathematical experiences.
- Multicultural mathematics in reality.

Jean-Luc Dorier [Jean-Luc.Dorier@unige.ch] is a professor of mathematics didactics at Geneva University. His first interest was in teaching and learning in linear algebra but has since been employed in research at all school levels from primary school up to university. Since January 2013, he has been an elected member of the ICMI Executive.

The Klein Project

Bill Barton (The University of Auckland, New Zealand)

Felix Klein's book Elementary Mathematics from an Advanced Standpoint *comprises his notes for university graduates becoming secondary school teachers and became the inspiration for the Klein Project. Felix Klein, as well as being an eminent mathematician, was, in 1908, the first president of the ICMI (International Commission on Mathematical Instruction).*

The Klein Project is a joint project of the IMU and the ICMI. It aims to produce material on contemporary mathematics for a secondary school teacher readership. It takes its inspiration from Felix Klein's "Elementary mathematics from an advanced standpoint", first published over 100 years ago.

At present, the materials, in the form of short "Vignettes", appear on a Blog *http://blog.kleinproject.org*. Most are translated into several languages, including Arabic, English, French, German, Italian, Mandarin, Portuguese and Spanish. A Vignette is a particular form of writing developed for the Klein Project. It encompasses new mathematics at an appropriate level for secondary school teachers, using a style to capture a teacher's attention and of a length that teachers may read as part of their busy professional lives. A Vignette needs a "hook" that will keep a teacher reading and a "point" that illustrates an important aspect of the discipline of mathematics. Vignettes are not a popularisation of mathematics – they contain substantial mathematical content at a level suitable for the target audience. Most Vignettes are produced by collaborations between mathematicians and teachers. Both groups have contributions to make to the content and style.

The Klein Project is also producing a book to be a modern version of Klein's volumes. Chapter authors are currently being sought.

An unanticipated extension of the Klein Project has been the formation of national groups meeting under its umbrella. These groups are self-organised and self-funded, and involve mathematicians, mathematics educators and secondary school mathematics teachers coming together to talk about contemporary mathematical topics and, sometimes, to work on Vignettes. The theme of the project seems to provide a space in which these groups can work together in a collaborative way without the divisions that appear when curriculum or assessment are discussed.

Anyone wishing to be more involved can contact Bill Barton, the convenor of the Design Team (b.barton@auckland.ac.nz) or Michèle Artigue, a member of the Design Team from France (michele.artigue@univ-paris-diderot.fr).

Bill Barton [b.barton@auckland.ac.nz] was a secondary school teacher in New Zealand for many years, including teaching mathematics in the Maori language in a bilingual school. He joined the Department of Mathematics at the University of Auckland in 1993, carrying out research in ethnomathematics, mathematics and language, and undergraduate teaching. From 2010 to 2012 he was President of the ICMI.

Solid Findings: Concept Images in Students' Mathematical Reasoning

Tommy Dreyfus (Tel Aviv University, Israel) on behalf of the Education Committee of the EMS

Edwards and Ward (2004) reported on typical reasoning patterns they observed in undergraduate students. For example, Stephanie, a student in an introductory real analysis course, was able to explain the following definition of an infinite decimal:

> Let $c_1, c_2, \ldots, c_n, \ldots$, be an infinite sequence of integers with $0 \le c_i \le 9$. The number sup $\{0.c_1\, c_2 \ldots c_n \mid n = 1,2,3,\ldots\}$ is denoted by $0.c_1\, c_2 \ldots c_n \ldots$ and is called an infinite decimal.

In particular, Stephanie gave a detailed explanation of sup as a least upper bound. She also equated 0.3333… to 1/3. However, she was adamant that 0.9999… was smaller than 1. She justified this by explaining that 0.3333… equals 1/3 since one could divide 1 by 3 and get 0.3333…, whereas dividing 1 by 1 would never yield 0.9999…. For Stephanie, 0.9999 (with some finite number of 9s) was the least upper bound although one could always make a larger one by "adding another 9".

Similarly, Jesse gave an acceptable definition of continuity but nevertheless claimed that the absolute value function $y = |x|$ was not continuous at $x = 0$. He was aware that, according to the definition, the function should have been continuous but insisted that it was not.

Again, similarly, students in an introductory algebra course did not make use of the definition when asked to multiply cosets but rather made use of inappropriate notions they had acquired earlier such as the union of sets.

Many authors have reported, over the past 35 years, similar patterns in students' reasoning in other areas of mathematics. Vinner and Hershkowitz (1980) were the first ones to point out that students' geometrical thinking is frequently based on prototypes rather than on definitions. They have shown, for example, that junior high school students tend to think that the altitude in a triangle is the altitude on the base and a base must be drawn horizontally, i.e. parallel to the edge of the sheet or blackboard. Moreover, students tend to think that the altitude has to reach the base (rather than its extension). Hence they draw the altitude inside the triangle even in a triangle with an obtuse base angle. Students' prototype altitude is one that is inside the triangle and vertical, perpendicular to a horizontal base. This is based on the students' experience: they have seen many triangles in which the altitude is inside the triangle and vertical and few in which it is not. They might consider these few cases as exceptions (Lakatos might say monsters). This is so even if the students know and can recite the (general) definition of altitude in a triangle.

A similar situation occurs with sequences. A prototype sequence is one like $1 - 1/n$, whose terms monotonically approach the limit. Most sequences students meet are of this kind or, "at worst", oscillatory and convergent with each term closer to the limit than the previous one.

Students act as if all sequences are monotonically increasing or monotonically decreasing even if they know and can recite the definition of a converging sequence. They still tend to act according to the examples they have in their minds rather than according to the formal definition. These examples are far more readily accessible for them than the definition.

Elementary school students tend to think that multiplication makes bigger. Many high school students will reject functions such as the Dirichlet function because they do not fit the image of a typical function they are familiar with. This is so even for talented high school and college students who have no trouble reciting and explaining the meaning of the definition of a function as a mapping from one set to another (e.g. Vinner & Dreyfus, 1989). Similarly, high school students tend to believe that inflection points have horizontal tangents and that definite integrals must be positive since they represent area.

Mathematicians often look for common features or structure across different situations; this is how notions such as groups or topology emerged and allowed a unified view on many apparently disconnected situations. Mathematics educators look at commonalities in student reasoning in order to see a "bigger picture" in apparently disconnected student reasoning patterns. In the cases described above, the commonality is that students seem not to base their reasoning on the definition of the concepts under consideration (even though they are often aware of these definitions and can recite and explain them) but rather on something else. Vinner, Hershkowitz and Tall have proposed the term *concept image* for that "something else". They have described it, for geometry, as "the set of all pictures that have ever been associated with the concept in the student's mind" (Vinner & Hershkowitz, 1980, p. 177) and then generalised it beyond geometry as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 152). They noted that this structure may not be globally coherent and may have aspects which are at variance with the formal concept definition. Hence, in mathematics education, *the cognitive processes by which concepts are conceived must be distinguished from the mathematical concepts as formally defined.*

The notion of concept image accounts for the complex role played by images in students' mathematical reasoning. This complexity may be expressed from two different points of view. On the one hand, it is not possible to introduce a concept without giving examples; in geometry, that means drawing figures or showing models. On the other hand, particular instances of the concept never suffice to fully determine the concept. As a consequence, specific elements of the examples, even if not pertinent to the mathematical definition of the concept, become for the student key elements characterising the concept, e.g. the height of a triangle must be vertical in the sense explained above. It follows that images may deeply influence concept formation.

Hence, a concept image is personal. A specific student's concept image of a particular concept is influenced by all of this student's experiences associated with this concept. These include examples, problems the student has solved, prototypes the student may have met substantially more often than non-prototype examples, and different representations of the concept including visual, algebraic and numerical ones. In particular, experiences with computer software may cause a student's concept image of function, limit or tangent to have dynamic aspects. While potentially richer than static aspects, these dynamic aspects, just like static ones, may fit the formal concept definition fully or only partially. Investigating the potential of using interactive computer software such as dynamic geometry in order to support students' concept formation is a focus of research in mathematics education.

It may be of particular interest to the readership of this newsletter that students also typically have a concept image of proof. Moore (1994), for example, found that university students majoring in mathematics or mathematics education "had concept images of proof. The concept image of some students was that of proof as explanation, whereas for others proof was a procedure, a sequence of steps that one performs. It was not clear to what extent the students viewed proof as a piece of mathematical knowledge, an object. … they had some limited notions of the purpose of proof, but they probably were not ready to use proof and deductive reasoning as tools for solving mathematical problems and developing mathematical knowledge." (p. 264)

A solid finding of mathematics education research, supported by dozens of studies with students from elementary school to university, is that *students' mathematical reasoning is frequently based on their concept images rather than on a mathematical concept definition*. This makes sense to the students since definition is rarely an explicit object of instruction and students do not experience the need for an exact definition.

Research based on the notion of concept image is actively continuing and developing. This research has recently taken additional directions, beyond identifying students' only partially adequate concept images of specific mathematical notions. One such direction includes the design and experimentation of activities that help students improve their concept images and understand, at the micro-level, the knowledge constructing processes occurring during such learning activities; in an example of such research, Kidron (2008) has investigated the changes in a student's concept image of horizontal asymptote. Another recent development is represented by a study showing that undergraduate students' concept images in calculus are not necessarily only personal and cognitive but show variations according to a student's departmental affiliation. Specifically, Bingolbali and Monaghan (2009) found that mechanical engineering students' concept images of derivative developed in the direction of rate of change aspects whilst mathematics students' concept images of derivative developed in the direction of tangent aspects. We may thus expect that research based on the idea of concept image will continue to contribute to the improvement of mathematics education at all levels, including undergraduate studies.

Authorship

Even though certain authors have taken the lead in each article of this series, all publications in the series are published by the Education Committee of the European Mathematical Society. The committee members are Tommy Dreyfus, Qendrim Gashi, Ghislaine Gueudet, Bernard Hodgson, Celia Hoyles, Boris Koichu, Konrad Krainer, Maria Alessandra Mariotti, Mogens Niss, Juha Oikonnen, Núria Planas, Despina Potari, Alexei Sossinsky, Ewa Swoboda, Günter Törner and Lieven Verschaffel.

References

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, 68, 19–35.

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: student (mis)use of mathematical definitions. *The American Mathematical Monthly*, 111, 411–424.

Kidron, I. (2010). Constructing knowledge about the notion of limit in the definition of the horizontal asymptote. *International Journal of Science and Mathematics Education*, 9, 695–717.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27, 249–266.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the 4th Annual Meeting for the Psychology of Mathematics Education* (pp. 177–184). Berkeley, CA: PME.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the notion of function. *Journal for Research in Mathematics Education*, 20, 356–366.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limit and continuity. *Educational Studies in Mathematics*, 12, 151–169.

zbMATH Author Profiles: Open Up For User Participation

Helena Mihaljević-Brandt (FIZ Karlsruhe, Germany) and Nicolas Roy (FIZ Karlsruhe, Germany)

The zbMATH author database has been the subject of several contributions in this column, from its beginnings[1] to the introduction of profiles[2] and applications to career records[3]. Parallel to the steady improvement of the underlying data, several new features have been added since it began, partly driven by another formative trend of the current time – the growing importance of social media and user-generated information. Nowadays, various online providers offer tools for researchers with the goal of increasing the visibility of their scientific activities.

Many search engines have recognised the potential of such services, Google Scholar being perhaps the most prominent example.

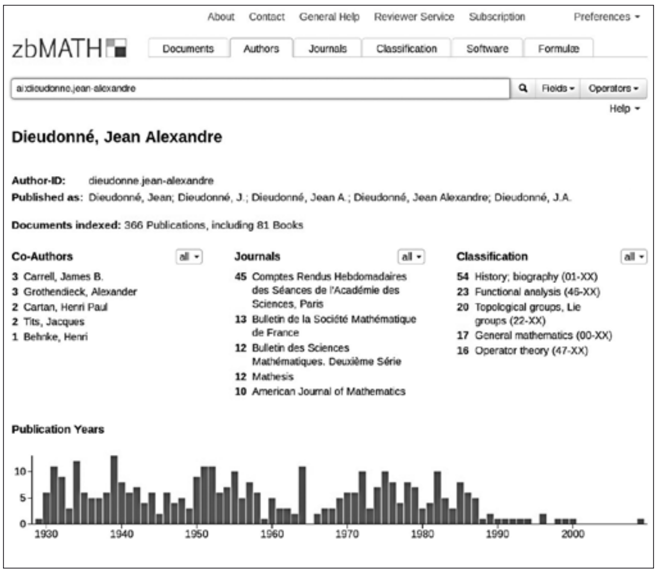
The social media hype of making a profile of oneself publicly available has also been brought into the scientific world by other commercial enterprises, e.g. ResearchGate, which attempts to form a kind of "Facebook for researchers". One could become intimidated by this growing jungle of services and associated author identifiers. The need for a global and sustainable authorship administration has been recognised by the initiative ORCID,[4] which has the potential of becoming a standard author identifier in the future.

On the other hand, in the field of mathematics, traditional indexing and reviewing services like zbMATH have several advantages due to their completeness, a high degree of data reliability, the amount of historical information and the detailed content information generated through classification, keywords and reviews. Reviewer communities are probably among the longest living – though initially paper-based – scientific networks, and have proven to be a durable form of scientific communication over many decades.

Hence, it is natural to also explore the benefits of integrating community features into the author facets of the databases.

Author profiles at zbMATH

By now, a typical author profile at zbMATH (as illustrated in the example of Jean Alexandre Dieudonné in the figure below) displays the author's identifier, together with name variations occurring in their publications. The assigned publications are broken down according to co-authors, journals and mathematical subjects. Furthermore, the author's contributions are also displayed in chronological order as a clickable diagram for easy visualisation of their scientific output. The author profile provides direct links to additional information on related objects in the zbMATH database such as co-authors, joint papers and documents published in certain journals. For instance, the number 3 in front of Dieudonné's co-author *Grothendieck, Alexander* is linked to the list of their joint publications.



Publications are filtered according to co-author, source, mathematical subject and publication year, respectively.

We recently started to connect our profiles to those of other services, e.g. ORCID, the Mathematics Genealogy Project and the Biographies of Women Mathematicians of Agnes Scott College. Along with further information added in the future, these links will extend the interoperability of the profiles. But what is the legitimacy and benefit of such a distributed approach, which connects several (possibly differing) profiles? For zbMATH, a striking argument for a mathematician is certainly the fact that the database aims at complete coverage of all mathematical publications at a research level.

Currently, about 3.4 million publications are indexed, from all areas of mathematics and its applications, going back to 1755 (and complete since 1868). Moreover, our database contains records that are rather difficult to retrieve using standard search engines, such as journal

[1] O. Teschke, "On authors and entities", Eur. Math. Soc. Newsl. 71, 43–44 (2009)

[2] O. Teschke and B. Wegner, "Author profiles at Zentralblatt MATH", Eur. Math. Soc. Newsl. 79, 43–44 (2011)

[3] H. Mihaljević-Brandt and L. Santamaría, "Show me your data! What publication records in zbMATH can say about women's careers in mathematics." Eur. Math. Soc. Newsl. 90, 51–52 (2013)

[4] The Open Researcher and Contributor ID (ORCID) is a recent non-profit initiative to assign unique identifiers to authors of research publications (http://orcid.org).

compilations from universities with rather low budgets or those published in languages other than English.

From the perspective of data reliability, it is important to mention that zbMATH indexes only peer-reviewed literature. All items undergo an internal editorial process and many of them, in particular in the core fields of mathematics, are additionally reviewed by an independent expert in the corresponding field. This editorial procedure ensures a high integrity level of mathematical content, also helping to detect malicious behaviour such as "plagiarism" or "non-sense" papers.

This quality control by the reviewing community enjoys a high level of confidence with mathematicians, despite the numerous alternatives mentioned above.

Last but not least, our service indexes the works of *all* mathematicians, including those who were active long before the era of the internet and hence have neither a homepage nor a social media profile and whose publications are far from being completely available in electronic form.

Author disambiguation – a multifaceted challenge

Currently, zbMATH has approximately 860,000 author identities. The trend of maintaining profile pages and registering on various platforms is quite recent and still not intensively picked up by mathematicians. For instance, less than 3,000 authors (i.e. less than 0.4%) from the zbMATH database have an ORCID ID, most of them from mathematical physics or computer science. Although the identifiers or links to these services are obviously a great help for the identification of authorship, they are almost never available in publications at the moment. Information like affiliation and email is a rather standard part of a publication's metadata but these tend to change, still leaving the name as the starting point for author identification.

The task of attributing an exact set of publications to a certain author based only on a person's name is far from trivial, due to several reasons:

- *Incompleteness*: The available data may be incomplete; some parts of the name may be missing or abbreviated.
- *Synonyms*: For a single author, one can face a great variability in names, in particular due to different spellings and translations. For instance the family name of the famous Russian mathematician Пафнутий Львович Чебышёв has been translated in more than 10 different ways (Čebišev, Tschebyschew, Chebyshev, Tchebichef, etc.)
- *Homonyms*: The same name may refer to multiple individuals. This is particularly acute in the case of certain East Asian names; for example, in China the surnames *Wang*, *Li* and *Zhang* account for more than 20% of the population. A search in zbMATH of all authors with name matching *Y. Wang* would give more than 800 results, suggesting the high level of ambiguity that a paper published under *Y. Wang* could have.

This explains why author name disambiguation is a longstanding research topic, with high relevance for bibliometric studies and publication retrieval, and why the field of entity recognition has gained a lot of interest recently.

A new community interface for zbMATH author profiles

The authors of a new publication record in zbMATH are either identified by means of existing profiles or else new profiles are created. The first step of this procedure is solved algorithmically by analysing the name string and additional information such as co-authors.

After such an algorithmic step, a post-processing procedure involves manual correction, which is often initialised through user requests. The zbMATH author identification team receives such requests on a daily basis, showing an increasing interest of the mathematical community in the availability of reliable author profiles.

As outlined above, the problem of author disambiguation is too difficult to be tackled completely algorithmically. On the other hand, there are too many publications incorporated every year into zbMATH (approximately 120,000 new items) and other information services to solve this task manually. We thus decided to open up this process at least partially to community input.

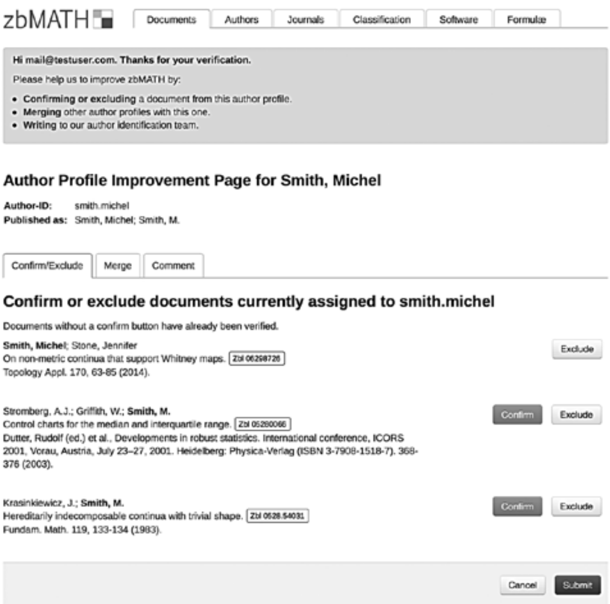
In the past, there has already been feedback to zbMATH to clean up author profiles or correct mistakes in attribution of publications, usually contributed via email. This approach is evidently error-prone since, for example, it requires hand typing of author or document identifiers, a process that is highly sensitive to typographical errors. Also, it is not very efficient or scalable because of the amount of time required for both writing and processing such an email request.

Therefore, there is now a publicly accessible interface[5] that allows users to improve the quality of publication data in author profiles.

Through this interface, the user has the opportunity to confirm correctly assigned publications or exclude incorrectly assigned publications and resolve, in this way, ambiguities in algorithmic assignment. They can also merge different profiles that use different names (or name variants) for the same person. The target users of this interface are of course the authors themselves but also their associates (colleagues, students, co-workers, etc.), with the obvious aim of keeping their own author profiles up-to-date. But the zbMATH author interface is a service for science itself and the community as a whole. Those people generally interested in quality of information content (e.g. librarians or historians) also have the opportunity to participate in this enhancement of information.

There is the natural question of possible abuse and the amount of authentication. Though one could also think of restricting the interface to a small community (like EMS members, who could employ their member accounts in this way) or limited functions (e.g. people would only be allowed to correct their own profiles), the chosen approach is as democratic as possible and raises

[5] https://zbmath.org/author-profile/edit/



Community interface for editing of author profiles.

no large barriers – users can simply utilise email authentication and possible errors or even abusive assignments are tested via internal heuristics before final confirmation.

Future developments

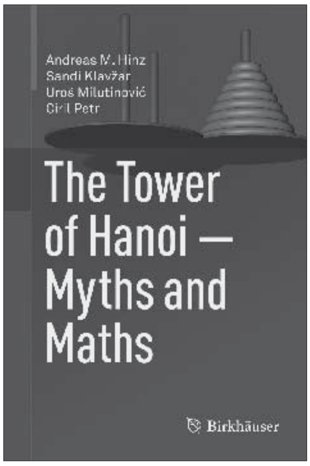
For a couple of months we have been testing a prototype of the author disambiguation interface among different user groups (zbMATH editors, mathematics students, librarians). Based on very positive feedback we are confident that the public version of the interface will enjoy high usage in the mathematical community.

Given an intensive usage of the interface, it could be enhanced by additional features, which would require secure authentication. Among the possible extensions could be the possibility of adding comments or tagging the content of publications, or uploading and matching additional metadata and sources such as references or photos, and much more.

Helena Mihaljević-Brandt was born in 1982 in Sarajevo and is currently working on the editorial board of zbMATH. She studied mathematics in Göttingen and obtained her PhD from the University of Liverpool in the field of complex dynamics.

Nicolas Roy received his PhD in mathematics in Grenoble in 2003 and then worked for seven years as a researcher and assistant lecturer at the Humboldt University in Berlin. Between 2011 and 2012, he acted as a developer and specialist for mathematics education for the e-learning platform Bettermarks.com. Since 2013, he has been member of the editorial board and responsible for author identification at zbMATH.org.

Book Reviews



Andreas M. Hinz, Sandi Klavžar, Uroš Milutović and Ciril Petr

The Tower of Hanoi – Myths and Maths

Springer Basel, Basel 2013
xiv, 335 p.
ISBN print
978-3-0348-0236-9
ISBN e-book
978-3-0348-0237-6

Reviewer: Jean-Paul Allouche

Three pegs, N discs, a tower, the Divine Rule, an anagram. The reader will have recognised the Hanoi Tower puzzle. Invented in 1883 by the French mathematician Lucas, under the anagram N. Claus, the Tower of Hanoi (also called the Tower of Brahma) consists of three pegs and N discs of sizes $1, 2, \ldots, N$. At the beginning of the game, the discs are on the same peg, in decreasing order with the largest disc below. At each step, the disc on the top of a peg can be moved to another peg, provided that the "divine rule" is respected: no disc can be put on a smaller one. At first view this puzzle seems to be one of those "mathematical" puzzles without real difficulty and/or interest. But this is absolutely not the case! In this nice book, the authors show a large number of mathematical results coming from or associated with the Tower of Hanoi.

If one thinks of computer science, the Tower of Hanoi is "the" classical example of the possibility of transforming recursive algorithms into iterative algorithms.

The reader might try to find "simple" (or "easy to write") algorithms to solve the Hanoi puzzle and then try to concoct an optimal solution (i.e., with a minimal number of moves). Another computer science flavoured result is the occurrence of a 2-automatic sequence describing the moves for N discs (for N integer or infinity). This result is a bridge towards number theory (recall that Lucas was a famous number theorist); binary expansions, integer sequences (e.g. the *ruler* sequence, also known as the Gros sequence), square-free sequences, fixed points of morphisms of the free monoid on two letters, the Pascal triangle, the Stern sequence, etc. show up in the detailed study of the Hanoi Tower. Number theory is not the only field touched by this puzzle. One can find links with graph theory, topology, fractals (e.g., the Sierpiński curve), etc. More unexpectedly, by looking at generalised Hanoi puzzles, one can find chemistry (the number of Kekulé structures of a specific class of benzenoid hydrocarbons with the molecular formula $C_{12n+2}H_{6n+4}$) and … psychology (with Shallice's Tower of London and Ward-Allport's Tower of Oxford).

Each time I open the book I discover a renewed interest in the Tower of Hanoi. I am sure that this will be the case for all readers, who will share my enthusiasm and enjoy all of the chapters, not to mention the numerous exercices, the bibliography with 352 references and the 21 conjectures or open problems listed at the end of the book.

Alex Oliver, and Timothy Smiley

Plural Logic

Oxford University Press, Oxford 2013
xiv, 336 p.
ISBN 978-0-19-957042-3

Reviewer: Louis F. Goble

The Newsletter thanks Zentralblatt MATH and Louis F. Goble for the permission to republish this review, originally appeared as Zbl 1273.03002.

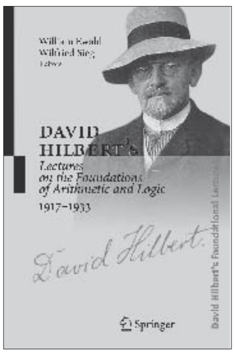
A short summary review like this cannot convey how interesting and important this book is. If it had been written a hundred years ago, the course of philosophy of logic, philosophy of mathematics, philosophy of language, not to mention formal logic itself, would have been quite different, and far more wholesome. As the title declares, the authors here present a logic of plurals, expressions that typically denote more than one individual, though they might denote merely single things, or nothing at all. These include proper names, e.g., 'British Isles'; definite descriptions, 'the authors of *Principia Mathematica*'; lists, 'Whitehead and Russell'; functional value terms, 'the wives of Henry VIII', '$\sqrt{4}$', 'the square roots of 4'; demonstratives, 'these', 'those'; etc. Such terms are ubiquitous in ordinary discourse, and essential too to mathematics and other formal disciplines. They become especially crucial to contexts of collective, as opposed to distributive, predication, e.g., 'Whitehead and Russell wrote *Principia Mathematica*', 'the premises entail the conclusion'. Yet plurals have been excluded from modern classical logic since its inception. This book provides a comprehensive corrective.

The book develops in three stages. The first, Chapters 2–4, presents the need for a rigorous and robust plural logic, arguing the inadequacy of alternative treatments of plural terms, such as analyses that 'change the subject' to regard such terms as actually denoting single, though perhaps exotic, things, like sets, or combinations, or wholes with parts, etc., or predicative analyses that would reduce plural terms to singular predicates standing for concepts or properties. The second stage, Chapters 5–10, presents the fundamentals of the authors' plural logic, focusing on the nature of both singular and plural terms, and the need to allow for empty terms, which denote nothing at all, and the denotation relation for them. Here are extended accounts of plural descriptions, multivalued functions, and lists. This part also discusses distributive and collective predicates, and introduces the key elements that enable the plural logic, plural quantification and the logical relation of inclusion 'is/are among', from which (plural) identity, 'is/are', may be defined.

The third stage, Chapters 11–14, presents the full formal plural logic itself. This takes three steps. Chapter 11 gives the base logic, a singular, first-order logic, which provides both a contrast to the subsequent plural logic and introduces many key components of that logic. This base logic is notably different from standard classical first-order logic in that it accommodates definite descriptions and functional terms as genuine terms, and it allows all kinds of terms to be empty; it also allows that nothing exists or that so many things exist that no single set-like object can stand as a domain. Chapter 12 develops an intermediate first-order 'mid-plural' logic; this extends the base logic with plural variables, a predicate for inclusion, and an operator for 'exhaustive plural description', while the non-logical vocabulary may now contain plural constants, predicates, and multivalued functions. There is, however, no quantification over plural variables, which only occur free. An axiomatization for the resulting system is proved sound and complete. This chapter also presents an algebra of plurals, analogous to Boolean algebra. Chapter 13 extends that logic to the full plural logic that does include quantification over (first-order) plural variables. The gain in expressive power is considerable, but at the cost of axiomatizability. Plural quantification can often mimic familiar second-order quantification, and a categorical plural arithmetic can be defined. Chapter 14 illustrates full plural logic at work to develop a Cantorian set theory, with the function sign '{ }' as the only non-logical primitive, membership being defined in terms of the inclusion relation. Through all of this formal development it is noteworthy that the metalanguage describing the plural logic is itself a plural language, e.g., the denotation predicate must be construed pluralistically. This is, of course, inevitable, and not different from metalinguistic characterizations of singular logic. The final chapter of the book, Chapter 15, is a postscript describing some topics yet to be dealt with, including pseudo-singular terms, which appear singular but are semantically plural; higher-level plural logic; higher-order plural logic; functions and empty terms; modality in plural logic; etc.

Throughout the book the exposition is clear; the arguments cogent; the formalism as transparent as can be. Proofs are relegated to appendices. This is a rewarding book. It deserves study in any course in philosophical or mathematical logic, and a place in every logician's library.

Lou Goble [lgoble@willamette.edu] received his Ph. D. in Philosophy from the University of Pittsburgh in 1967. He is now Professor of Philosophy (Emeritus) at Willamette University. His research interests include many aspects of philosophical logic, especially deontic logic and other forms of modal logic, relevant and substructural logics, paraconsistent logic, etc. He edited The Blackwell Guide to Philosophical Logic *(Blackwell, 2002) and, with J.-J. Ch. Meyer,* Deontic Logic and Artificial Normative Systems *(Springer, 2006).*



David Hilbert's Lectures on the Foundations of Arithmetic and Logic, 1917–1933

William Ewald and Wilfried Sieg (Editors)

Springer, Berlin 2013
xxv, 1062 p.
ISBN print 978-3-540-20578-4
ISBN e-book 978-3-540-69444-1

Reviewer: Victor V. Pambuccian

The Newsletter thanks Zentralblatt MATH and Victor V. Pambuccian for the permission to republish this review, originally appeared as Zbl 1275.03002.

Here we encounter mathematical logic and proof theory in *statu nascendi* in the lectures Hilbert held between 1917 and 1933, containing insights into the history of the subject that cannot be read from any published source during the first half of the 20th century. Far from having abandoned in 1904 the axiomatic point of view and the metamathematical investigations started in the work leading to the *Grundlagen der Geometrie* of 1899, only to return to it in 1921, as the published record and the record of Hilbert's talks would indicate, following the trail of the lectures held in Göttingen reveals that Hilbert never stopped thinking about metamathematical matters after 1904, and held lectures on foundational matters regularly. This third volume of Hilbert's foundational lectures sets off in 1917, the date that marks a break, the beginning of a new phase in Hilbert's thinking, that can be said to begin with his lecture on September 11, 1917 to the Swiss Mathematical Society in Zürich, published as the famous *Axiomatisches Denken* in 1918. It was then and there that he invited Paul Bernays to return to Göttingen as his assistant. Bernays's presence in Göttingen is an essential part of the story of mathematical logic and proof theory.

The lectures are grouped into four chapters: (1) On the principles of mathematics (from 1917/18), together with Bernays's unpublished *Habilitationsschrift* of 1918, *Beiträge zur axiomatischen Behandlung des Logik-Kalküls*; (2) On logic (from 1920); (3) On proof theory (1921/22–1923/24); (4) On the infinite (1924/25, 1931, 1933). Appendices contain reprints of the texts most relevant to those of the lectures. They are: D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*. Berlin: J. Springer (1928; JFM 54.0055.01), as well as the text of several lectures held by D. Hilbert (in one case together with a discussion by Weyl and Bernays, as well as a letter from the latter to the former), and published in *Abh. Math. Semin. Univ. Hamb*. 6, 65–85 (1928; JFM 54.0055.02); *Math. Ann*. 104, 485–494 (1931; JFM 57.0054.04); *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* 1931, 120–125 (1931; JFM 57.0055.01); *Math. Ann*. 102, 1–9 (1929; JFM 55.0031.01). There is a complete list of the titles of Hilbert's lecture courses between 1886 and 1934.

The 1920 lectures on the principles of mathematics are characterized by a very detailed motivation for the introduction of any new notion, for the extension of any logical calculus to a higher level, of the kind one would find today only in a comprehensive textbook of philosophical logic. The metamathematical concerns are, much like in the *GdG*, consistency and (in)completeness. The third concern of the *GdG*, the independence of axioms, is the central concern of Bernays's *Habilitationsschrift* in the case of propositional logic. It also contains, proved without any fanfare, what one would see today as a more important result, the completeness of the propositional calculus, in both the semantical sense and as Post-completeness (avant Post, a matter that was first signaled by R. Zach in *Bull. Symb. Log*. 5, No. 3, 331–366 (1999; Zbl 0942.03003)).

The lectures in (2) and (3) constitute a graduate introduction to proof theory, leading to first attempts at proving the consistency of weak systems of arithmetic, with the aim of tackling the consistency of arithmetic with induction by similar methods. The weak arithmetic is based on addition, the symbol 1, the = symbol (without any significance assumed, i.e. as a non-specified binary relation), two rules of inference (modus ponens and substitution for numerals (variable-free terms)), and the axioms $1=1$, $a=b \rightarrow a+1=\mathrm{b}+1$, $a+1=b+1 \rightarrow a=b$, $a=b \rightarrow (a=c \rightarrow b=c)$, $a+1 \neq 1$. From this, Hilbert shows why $a+1=1$ cannot be derived, and presents a similar proof that a weak form of induction is consistent for an axiom system in a language extended by a unary predicate Z (meant to designate that a variable is a numeral), to which the axioms $Z(1)$ and $Z(a) \rightarrow Z(a+1)$ have been added (and in subsequent lectures there are additional Z-axioms). The weak form of induction is here an inference schema, allowing one to deduce $Z(a) \rightarrow F(a)$ from $F(1)$ and $F(a) \rightarrow F(a+1)$. They also contain an in-depth critique of the logicist programme, and the indispensability of the axiomatic method, which finds an eloquent characterization in "Das Wesen dieser Methode besteht darin, dass man sich über die Voraussetzungen und die Methoden des Schliessens klar wird, die man in einer Wissenschaft gebraucht. Axiomatisch zu verfahren ist also nichts anderes, als mit Bewusstsein zu denken." (p. 363)

The lectures in (4) are addressed to a general audience (and were intended to be *allgemeinverständlich*), and are devoted to showing the central position of the infinite in mathematics, as well as in sciences depending on mathematics. They are very much worth reading today, as they explain with great clarity, in beautiful prose, using a wide variety of examples from physics and biology, the necessity of the road traveled in the history of mathematics, and in particular the necessity of going beyond the finite. Here Hilbert points out that the world around us shows no sign of infiniteness, neither in the sense of a homogeneous and indefinitely divisible continuum, nor in the sense of the infinitely large (although there cosmology had not yet provided a clear answer, only one of likelihood), but that we are nevertheless forced to make this un-natural assumption in the process of modeling nature. In his own words: "ein homogenes Kontinuum, das die fortgesetzte Teilbarkeit zuliesse und somit das Unendlich-Kleine realiseren würde, [wird] in der Wirklichkeit nirgends angetroffen. Die unendliche Teilbarkeit eines Kontinuums ist nur eine in Gedanken vorhandene Operation, nur eine Idee, die durch unsere Beobachtungen der Natur und die Erfahrungen der Physik und Chemie widerlegt wird." (p. 705) It is here that we also find the lecture "Ueber die Grundlagen des Denkens" of c. 1931, in which he introduces his version of an ω-rule: "The statement $(\forall x)A(x)$ is correct if $A(z)$ is correct whenever z is a numeral." (p. 766)

The totality of the lectures also offers a solid basis for judging aspects of the intuitionism-formalism debate, and to determine whether and how much of Brouwer's contention that formalism learned from intuitionism is accurate. What transpires from these lectures is that Hilbert clearly considered formalism to be only a tool devised to prove consistency, and not a philosophy of mathematics, and that he was guided in all his choices by actual mathematical practice. One can find passages that almost sound as if he were an intuitionist, such as in (c. 1931): "Erfahrung und reines Denken sind die Quellen unserer Erkenntnis. Um diese auszuschöpfen, bedienen wir uns einer gewissen Fähigkeit unseres Geistes, durch die wir schon im voraus a priori in der Vorstellung konkrete Objekte unmittelbar erleben, sodass dieselben für uns vollkommen in allen Teilen überblickbar sind und ihre Aufweisung, ihre Unterscheidung, ihre Aufeinanderfolge oder ihr Nebeneinandergereihtsein im Endlichen anschaulich da ist als etwas, das sich weder auf etwas anderes reduzieren lässt noch einer solchen Reduktion bedarf." (p. 765)

Victor Pambuccian [victor.pambuccian@asu.edu] was born in Romania, got his Ph. D. from the University of Michigan in 1993, and has been at Arizona State University since 1994, where he is currently a professor of mathematics. His main field of research is the axiomatic foundation of geometry, in which he has published a hundred papers. His translations of 20th century poetry written in Romanian, German, or French into English, and of Armenian poetry into German have appeared in various periodicals.

Solved and Unsolved Problems

Themistocles M. Rassias (National Technical University, Athens, Greece)

Where there is matter, there is geometry.
Johannes Kepler (1571–1630)

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

131. Let $ABCD$ be an isosceles trapezium with $AB = \frac{3}{2}$, angle $CAD = 90°$ and angle $CAB = 15°$. On the same part of the line CD with A and B we consider a point M such that $MC = \sqrt{2}$ and $MD = 1$. Find $MA + MB$.

(*Cristinel Mortici, Valahia University of Târgovişte, Romania*)

132. Consider the regular polygon $A_1A_2 \ldots A_n$. Find the minimum and the maximum of the lengths of the segments with ends on the sides of the polygon and passing through its centre.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania*)

133. In the convex quadrilateral $ABCD$, consider P the intersection point of its diagonals and E, F the projections of P on the sides AB and CD. Let M, N be the midpoints of the sides BC and AD. Prove that if $MN \perp EF$ then the quadrilateral $ABCD$ is cyclic or a trapezoid.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania*)

Einstein addition. Until recently, general Einstein addition of relativistically admissible velocities that need not be parallel has rested in undeserved obscurity. The following problems uncover the rich mathematical life that Einstein addition possesses.

Let $c > 0$ be any positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n = 1, 2, 3, \ldots$, equipped with common vector addition "+" and inner product "·". The home of all n-dimensional Einsteinian velocities is the c-ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}. \tag{1}$$

The c-ball $\mathbb{R}^n_c$ is the open ball of radius c, centred at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition is a binary operation "⊕" in the c-ball $\mathbb{R}^n_c$ given by the equation

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\} \tag{2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, where $\gamma_\mathbf{v}$ is the gamma factor

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \geq 1 \tag{3}$$

and where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. Einstein subtraction is denoted by $\ominus\mathbf{v} = -\mathbf{v}$, so that $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$. The pair $(\mathbb{R}^n_c, \oplus)$ forms the n-dimensional Einstein gyrogroup.

134.
(1) Prove that Einstein addition "⊕" and the gamma factor "$\gamma_\mathbf{v}$" are related by the *gamma identity*

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_\mathbf{u}\gamma_\mathbf{v}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \tag{4}$$

(2) Prove that Einstein addition is closed in the ball $\mathbb{R}^n_c$, that is, prove that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ then $\mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^n_c$.

(*Abraham A. Ungar, Department of Mathematics, North Dakota State University, USA*)

135. Prove that Einstein addition "⊕" obeys the *Einstein triangle inequality*

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{5}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$.

(*Abraham A. Ungar, Department of Mathematics, North Dakota State University, USA*)

A subset C of a metric space X is called a *Chebyshev set* if for each point of X there exists a unique nearest point in C. It is well known that every closed convex set in a finite dimensional Euclidean vector space is a Chebyshev set. The converse is also true and it was shown independently by L. N. H. Bunt (1934) and T. S. Motzkin (1935). The problem that we are going to state leads to an alternative proof of this fact for the case of a 2-dimensional Euclidean vector space.

The following notation will be used. By E_2 we denote the Euclidean plane. For $x, y \in E_2$ we write $[x, y] = \{(1-\lambda)x + \lambda y : 0 \leq \lambda \leq 1\}$ for the closed segment and $(x, y) = \{(1-\lambda)x + \lambda y : 0 < \lambda < 1\}$ for the open segment with endpoints x, y. The set $D = D(x_0, r) = \{y \in E_2 : \|y - x_0\| \leq r\}$ is the closed disc with centre x_0 and radius $r > 0$. For a subset C of E_2 we denote by $conv(C)$ the convex hull of C. If X is a topological space and $Y \subseteq X$, we denote by $Int(Y)$ the *interior* of Y and by $Bd(Y)$ the *boundary* of Y. We will say that Y is a *retract* of X if there exists a continuous map $R : X \to Y$ such that $R(y) = y$ for every $y \in Y$.

136. Let C be a Chebyshev set in E_2.
(a) Show that for every closed disc $D = D(x_0, r)$ with $x_0 \in C$, the intersection $C \cap D$ is a Chebyshev set.
(b) Assume that C is a bounded set and let $K = \text{conv}(C)$. If the boundary of K is contained in C, show that the sets K and C coincide.
(Hint: Use Brouwer's theorem.)
(c) Let x_1, x_2 be two distinct points of C. Also let $x_0 \in (x_1, x_2)$ and let ζ be the perpendicular line to $[x_1, x_2]$ that passes through the point x_0. If D is a closed disc containing x_1, x_2, show that $D \cap \zeta \cap C \neq \emptyset$.
(Hint: If $D = D(z_0, r)$ does not satisfy the conclusion, show that $\{x_1, x_2\}$ is a retract of the set $[x_1, z_0] \cup [z_0, x_2]$.)
(d) Prove that C is convex.
(Hint: Combine the above.)

(*Vassilis Kanellopoulos, National Technical University of Athens, Department of Mathematics, Greece*)

II Two new open problems

137*.

(a) Calculate, if possible, in terms of known constants, the logarithmic integral

$$\int_0^1 \ln(1-x^2)\ln(1+x^2)dx.$$

(b) More generally, let $n \geq 2$ be an integer. Calculate

$$\int_0^1 \ln(1-x^n)\ln(1+x^n)dx.$$

Remark. The problem is motivated by the well known logarithmic integral [1, p. 244]:

$$\int_0^1 \ln(1-x)\ln(1+x)dx = \ln^2 2 + 2 - \zeta(2) - 2\ln 2.$$

References

[1] G. Boros, V. H. Moll, Irresistible Integrals. Symbolics, Analysis and Experiments in the Evaluation of Integrals, Cambridge University Press, Cambridge, 2004.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Romania*)

138*. Let ABC be a hyperbolic triangle in the Beltrami-Klein disc model or in the Poincaré disc model of the hyperbolic plane. Let M_a, M_b, M_c be the hyperbolic midpoints of the triangle sides BC, AC, AB, respectively, and let X, Y, Z be the points of tangency of the hyperbolic incircle of the hyperbolic medial triangle $M_aM_bM_c$ with sides M_bM_c, M_aM_c and M_aM_b, respectively.
Prove that the hyperbolic lines AX, BY, CZ are concurrent.

Remark. The Euclidean counterpart of this problem appears in the book by S. E. Louridas and M. Th. Rassias, *Problem-Solving and Selected Topics in Euclidean Geometry in the Spirit of the Mathematical Olympiads*, p. 84, Springer, New York, 2013, where one can find many interesting problems and solutions in Euclidean plane geometry.

(*Abraham A. Ungar, Department of Mathematics, North Dakota State University, USA*)

Solutions

123. Compute the limit

$$\lim_{n\to\infty} \frac{(n+1)^\alpha \ln^\beta(n+1) - n^\alpha \ln^\beta n}{n \ln n},$$

where $\alpha, \beta \in \mathbb{R}$.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania and Columbus State University, Georgia, USA*)

Solution by the proposer. We will use the following auxiliary result:

Lemma.

$$\lim_{n\to\infty} \frac{(n+1)^2 \ln(n+1) - n^2 \ln n}{n \ln n} = 2.$$

Proof of the Lemma. Applying l'Hopital's rule, we have

$$\begin{aligned}
&\lim_{x\to\infty} \frac{(x+1)^2 \ln(x+1) - x^2 \ln x}{x \ln x} \\
&= \lim_{x\to\infty} \frac{2(x+1)\ln(x+1) + x + 1 - 2x\ln x - x}{\ln x + 1} \\
&= 2 \lim_{x\to\infty} \frac{\ln(x+1) - \ln x}{\frac{1}{x}} \\
&= 2 \lim_{x\to\infty} \ln\left(1 + \frac{1}{x}\right)^x \\
&= 2.
\end{aligned}$$

Denote by L the limit in the problem. Using the result in the previous lemma, we can write

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{(n+1)^2 \ln(n+1) - n^2 \ln n}{n \ln n} \\
&\quad \cdot \lim_{n\to\infty} \frac{(n+1)^\alpha \ln^\beta(n+1) - n^\alpha \ln^\beta n}{(n+1)^2 \ln(n+1) - n^2 \ln n} \\
&= 2 \lim_{n\to\infty} \frac{(n+1)^\alpha \ln^\beta(n+1) - n^\alpha \ln^\beta n}{(n+1)^2 \ln(n+1) - n^2 \ln n}.
\end{aligned}$$

In order to evaluate the last limit we use the functions $f, g : (0,\infty) \to \mathbb{R}$, defined by $f(t) = t^\alpha \ln^\beta t$, $g(t) = t^2 \ln t$. Applying Cauchy's Theorem for the functions f and g on the interval $[n, n+1]$, we get

$$\frac{(n+1)^\alpha \ln^\beta(n+1) - n^\alpha \ln^\beta n}{(n+1)^2 \ln(n+1) - n^2 \ln n} = c^{\alpha-2} \ln^{\beta-1} c \cdot \frac{\alpha \ln c + \beta}{2\ln c + 1},$$

where $c = c_n \in (n, n+1)$.

Clearly, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(n+1)^\alpha \ln^\beta(n+1) - n^\alpha \ln^\beta n}{(n+1)^2 \ln(n+1) - n^2 \ln n} \\
&= \lim_{c\to\infty} c^{\alpha-2} \ln^{\beta-1} c \cdot \frac{\alpha \ln c + \beta}{2\ln c + 1} \\
&= \lim_{c\to\infty} (c^{\alpha-2} \ln^{\beta-1} c) \cdot \lim_{c\to\infty} \frac{\alpha \ln c + \beta}{2\ln c + 1} \\
&= \frac{\alpha}{2} \lim_{c\to\infty} (c^{\alpha-2} \ln^{\beta-1} c).
\end{aligned}$$

On the other hand, for the last limit we have

$$\lim_{c\to\infty} (c^{\alpha-2} \ln^{\beta-1} c) = \begin{cases} 1 \text{ if } \alpha = 2, \beta = 1 \\ 0 \text{ if } \alpha < 2, \beta \in \mathbb{R} \\ +\infty \text{ if } \alpha > 2, \beta \in \mathbb{R}. \end{cases} \tag{6}$$

Finally, we obtain

$$L = \begin{cases} 2 \text{ if } \alpha = 2, \beta = 1 \\ 0 \text{ if } \alpha < 2, \beta \in \mathbb{R} \\ +\infty \text{ if } \alpha > 2, \beta \in \mathbb{R}. \end{cases} \tag{7}$$

□

Also solved by Daniel Vacaru (Pitesti, Romania), Soon-Mo Jung (Chochiwon, Korea), M. Bencze (Brasov, Romania).

124. For a positive integer k, where $k \geq 2$, define the sequence

$$a_n^{(k)} = \sum_{j=0}^{k} (-1)^k \binom{k}{j} \sqrt{n+k-j}, \quad n = 1, 2, \ldots$$

Compute the limit $\lim_{n\to\infty} n^\alpha \cdot a_n^{(k)}$, where $\alpha \in \mathbb{R}$.

(*Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania and Columbus State University, Georgia, USA*)

Solution by the proposer. Write $a_n^{(k)}$ as

$$a_n^{(k)} = \sqrt{n}\sum_{j=0}^{k}(-1)^k\binom{k}{j}\sqrt{1+(k-j)\frac{1}{n}}. \qquad (8)$$

Now, using the well known result (see, for instance, Problem 2.3.39 in the book by W. J. Kaczor and M. F. Nowak, *Problems in Mathematical Analysis II* (Continuity and Differentiation), AMS, 2001)

$$\lim_{t\to 0}\frac{1}{t^k}\sum_{j=0}^{k}(-1)^k\binom{k}{j}f(1+(k-j)t) = f^{(k)}(1),$$

where f is a k^{th}-differentiable function on an interval containing 1, for the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\sqrt{t}$, we obtain

$$\lim_{t\to 0}\frac{1}{t^k}\sum_{j=0}^{k}(-1)^k\binom{k}{j}\sqrt{1+(k-j)t} = \frac{(-1)^{k-1}(2k-3)!!}{2^k},$$

since

$$f^{(k)}(t) = \frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)t^{\frac{1}{2}-k} = \frac{(-1)^{k-1}(2k-3)!!}{2^k}t^{\frac{1}{2}-k},$$

where $(2k-3)!! = 1\cdot 3\cdots(2k-3)$.

From (1) and from the above remark, it follows that

$$\lim_{n\to\infty} n^\alpha\cdot a_n^{(k)} = \lim_{n\to\infty} n^{\alpha-k+\frac{1}{2}}\cdot n^k\sum_{j=0}^{k}(-1)^k\binom{k}{j}\sqrt{1+(k-j)\frac{1}{n}}$$
$$= \frac{(-1)^{k-1}(2k-3)!!}{2^k}\lim_{n\to\infty} n^{\alpha-k+\frac{1}{2}}.$$

Finally, we obtain

$$\lim_{n\to\infty} n^\alpha\cdot a_n^{(k)} = \begin{cases} \frac{(-1)^{k-1}(2k-3)!!}{2^k} \text{ if } \alpha = k-\frac{1}{2} \\ 0 \text{ if } \alpha < k-\frac{1}{2} \\ +\infty \text{ if } \alpha > k-\frac{1}{2} \text{ and } k \text{ is odd} \\ -\infty \text{ if } \alpha > k-\frac{1}{2} \text{ and } k \text{ is even.}\end{cases} \qquad (9)$$

□

Also solved by Ulrich Abel (University of Applied Sciences, Friedberg, Germany), Soon-Mo Jung (Chochiwon, Korea), M. Bencze (Brasov, Romania).

125. (Torricelli-Steiner point in the metric space $C[0,1]$.)
Consider a triangle $f_1f_2f_3$ in the space of continuous functions $C[0,1]$ with vertices $f_1(x)=x$, $f_2(x)=x+1$, $f_3(x)=\sin 2x$. Find a point $f\in C[0,1]$ for which the sum of distances to the vertices of the triangle is minimal.
(Let us recall that the distance between points f and g in the space $C[0,1]$ is defined by $\|f-g\| = \max_{x\in[0,1]}|f(x)-g(x)|$.)

(*Vladimir Protasov, Department of Mechanics and Mathematics, Moscow State University, Russia*)

Solution by the proposer. After a shift by the vector $g(x)=-x$ we obtain a triangle with vertices

$$\varphi_1(x)=0,\ \varphi_2(x)=1,\ \varphi_3(x)=\sin 2x - x.$$

It suffices to solve the problem for the triangle $\varphi_1\varphi_2\varphi_3$. For an arbitrary function $\varphi\in C[0,1]$, let x_1 and x_2, respectively, denote its points of maximum and minimum (on the segment $[0,1]$) and let a be the difference between the maximal and minimal values. Then $\|\varphi-\varphi_1\|+\|\varphi-\varphi_2\| \geq$

$$|\varphi(x_1)-\varphi_1(x_1)|+|\varphi(x_2)-\varphi_2(x_2)| = |\varphi(x_1)|+|\varphi(x_2)-1|$$
$$= |\varphi(x_1)|+|\varphi(x_1)-a-1| \geq 1+a.$$

The maximum of the function φ_3 on the segment $[0,1]$ is equal to $\frac{\sqrt{3}}{2}-\frac{\pi}{6}$ and it is achieved at the point $\frac{\pi}{6}$. The minimum is attained at the point 1 and it is equal to $\sin 2 - 1$. The difference between the maximal and minimal values is $b=\frac{\sqrt{3}}{2}-\frac{\pi}{6}-\sin 2+1$ and $\|\varphi-\varphi_3\|\geq\frac{b-a}{2}$. Consequently,

$$\sum_{k=1}^{3}\|\varphi-\varphi_k\| \geq 1+a+\frac{b-a}{2} \geq 1+\frac{b}{2}.$$

This inequality becomes equality if and only if φ is the arithmetic mean of the maximum and the minimum of the function φ_3. Thus, $\varphi(x)=\frac{\sqrt{3}}{4}-\frac{\pi}{12}+\frac{\sin 2-1}{2}$ is a unique point of minimum and $f(x)=x+\varphi(x)$. □

Also solved by Frank Plastria (Universiteit Brussel, Belgium), Soon-Mo Jung (Chochiwon, Korea), Sotirios E. Louridas (Athens, Greece).

126. Let $k\geq 1$ be an integer. Prove that

$$\sum_{i_1,i_2,\ldots,i_k=1}^{\infty}\frac{1}{i_1i_2\cdots i_k(i_1+i_2+\cdots+i_k)^2} = (-1)^k k!\sum_{n=k}^{\infty}(-1)^n\frac{s(n,k)}{n!\cdot n^2},$$

where $s(n,k)$ denotes the Stirling numbers of the first kind.
Deduce that

$$\sum_{i_1,i_2=1}^{\infty}\frac{1}{i_1i_2(i_1+i_2)^2} = \frac{1}{2}\zeta(4).$$

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Romania*)

Solution by the proposer. We will be using in our analysis the following formula involving the Stirling numbers of the first kind (see [2, Entry 2, p. 76])

$$\log^k(1+z) = k!\sum_{n=k}^{\infty}s(n,k)\frac{z^n}{n!},\quad |z|<1. \qquad (10)$$

Also, if $a>0$ is a real number, one has that $1/a^2=\int_0^\infty e^{-at}tdt$ and this implies that

$$\sum_{i_1,i_2,\ldots,i_k=1}^{\infty}\frac{1}{i_1i_2\cdots i_k(i_1+i_2+\cdots+i_k)^2}$$
$$= \sum_{i_1,i_2,\ldots,i_k=1}^{\infty}\frac{1}{i_1i_2\cdots i_k}\int_0^\infty e^{-(i_1+i_2+\cdots+i_k)t}tdt$$
$$= \int_0^\infty t\left(\sum_{i=1}^{\infty}\frac{e^{-it}}{i}\right)^k dt$$
$$= (-1)^k\int_0^\infty t\ln^k(1-e^{-t})dt$$
$$= (-1)^{k+1}\int_0^1\frac{\ln y}{y}\cdot\ln^k(1-y)dy$$
$$\overset{(10)}{=} (-1)^{k+1}\int_0^1\frac{\ln y}{y}\left(k!\sum_{n=k}^{\infty}s(n,k)\frac{(-1)^n}{n!}y^n\right)dy$$
$$= (-1)^{k+1}k!\sum_{n=k}^{\infty}s(n,k)\frac{(-1)^n}{n!}\int_0^1 y^{n-1}\ln y dy$$
$$= (-1)^k k!\sum_{n=k}^{\infty}(-1)^n\frac{s(n,k)}{n!\cdot n^2}.$$

When $k = 2$, we have, since $s(n,2) = (-1)^n (n-1)! H_{n-1}$ (see [2, Entry 10, p. 77]), that

$$\begin{aligned}\sum_{i_1,i_2=1}^{\infty} \frac{1}{i_1 i_2 (i_1+i_2)^2} &= (-1)^2 \cdot 2! \sum_{n=2}^{\infty} (-1)^n \frac{s(n,2)}{n! \cdot n^2} \\ &= 2\sum_{n=2}^{\infty} \frac{H_{n-1}}{n^3} \\ &= 2\sum_{n=2}^{\infty}\left(\frac{H_n}{n^3} - \frac{1}{n^4}\right) \\ &= 2\sum_{n=1}^{\infty} \frac{H_n}{n^3} - 2\zeta(4) \\ &= 2\cdot\frac{5}{4}\zeta(4) - 2\zeta(4) \\ &= \frac{1}{2}\zeta(4).\end{aligned}$$

We have used in our calculations the well known formula (see [1, Problem 3.58, p. 148 and pp. 207–208])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4) = \frac{\pi^4}{72}.$$

□

References

[1] O. Furdui, Limits, Series and Fractional Part Integrals, Problems in Mathematical Analysis, Springer, London, 2013.
[2] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Insights, 2012.

Also solved by G. C. Greubel (Department of Physics, Old Dominion University, VA, USA), Soon-Mo Jung (Chochiwon, Korea).

127. Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. Then for any $t, x \in [a,b]$ prove that

$$|tf(x) - xf(t)| \tag{11}$$
$$\leq \begin{cases} \|f - \ell f'\|_{\infty} |x-t| & \text{if } f - \ell f' \in L_{\infty}[a,b], \\ \frac{1}{2q-1} \|f - \ell f'\|_p \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & \text{if } f - \ell f' \in L_p[a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases}$$

where $\ell(t) = t$, $t \in [a,b]$.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

Solution by the proposer. If f is absolutely continuous on the interval $[a,b]$ then f/ℓ is absolutely continuous on the interval $[a,b]$ that does not contain 0 and

$$\int_t^x \left(\frac{f(s)}{s}\right)' ds = \frac{f(x)}{x} - \frac{f(t)}{t}$$

for any $t, x \in [a,b]$, with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{s}\right)' ds = \int_t^x \frac{f'(s)s - f(s)}{s^2} ds,$$

we get the identity

$$tf(x) - xf(t) = xt\int_t^x \frac{f'(s)s - f(s)}{s^2} ds \tag{12}$$

for any $t, x \in [a,b]$.

Taking the modulus in (12) we have

$$\begin{aligned}|tf(x) - xf(t)| &= \left| xt\int_t^x \frac{f'(s)s - f(s)}{s^2} ds\right| \\ &\leq xt\left|\int_t^x \left|\frac{f'(s)s - f(s)}{s^2}\right| ds\right| := I\end{aligned} \tag{13}$$

and utilising Hölder's integral inequality we deduce

$$I \leq xt \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s)s - f(s)| \left|\int_t^x \frac{1}{s^2} ds\right| \\ \left|\int_t^x |f'(s)s - f(s)|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{s^{2q}} ds\right|^{1/q} & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \left|\int_t^x |f'(s)s - f(s)| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{s^2}\right\} \end{cases} \tag{14}$$

$$\leq \begin{cases} \|f - \ell f'\|_{\infty} |x-t| \\ \frac{1}{2q-1} \|f - \ell f'\|_p \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \\ \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}} \end{cases}$$

and the inequality (11) is proved. □

Also solved by Soon-Mo Jung (Chochiwon, Korea), M. Bencze (Brasov, Romania).

128. Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b) with $b > a > 0$. Then for any $x \in [a,b]$, prove the inequality

$$\begin{aligned}&\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \\ &\quad \leq \frac{2}{b-a} \|f - \ell f'\|_{\infty} \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x}\right),\end{aligned} \tag{15}$$

where $\ell(t) = t$, $t \in [a,b]$. The constant 2 is the best possible.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

Solution by the proposer. Applying Pompeiu's mean value theorem [1] (see also [2, p. 83]), for any $x, t \in [a,b]$ there is a ξ between x and t such that

$$tf(x) - xf(t) = [f(\xi) - \xi f'(\xi)](t - x),$$

giving

$$|tf(x) - xf(t)| \leq \sup_{\xi\in[a,b]} |f(\xi) - \xi f'(\xi)| |x-t| = \|f - \ell f'\|_{\infty} |x-t|$$

for any $t, x \in [a,b]$ or, by dividing by $x, t > 0$,

$$\left|\frac{f(x)}{x} - \frac{f(t)}{t}\right| \leq \|f - \ell f'\|_{\infty} \left|\frac{1}{x} - \frac{1}{t}\right| \tag{16}$$

for any $t, x \in [a,b]$.

Integrating over $t \in [a,b]$, we get

$$\begin{aligned}\left|\frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t} dt\right| &\leq \int_a^b \left|\frac{f(x)}{x} - \frac{f(t)}{t}\right| dt \\ &\leq \|f - \ell f'\|_{\infty} \int_a^b \left|\frac{1}{x} - \frac{1}{t}\right| dt\end{aligned} \tag{17}$$

and since

$$\int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| dt = \int_a^x \left(\frac{1}{t}-\frac{1}{x}\right) dt + \int_x^b \left(\frac{1}{x}-\frac{1}{t}\right) dt$$
$$= \ln\frac{x}{a} - \frac{x-a}{x} + \frac{b-x}{x} - \ln\frac{b}{x}$$
$$= \ln\frac{x^2}{ab} + \frac{a+b-2x}{x}$$
$$= 2\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x \in [a, b]$, we deduce from (16) the desired result (14).

Now, assume that (14) holds with a constant $k > 0$, i.e.

$$\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \le \frac{k}{b-a}\|f-\ell f'\|_\infty \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right) \quad (18)$$

for any $x \in [a, b]$.

Consider $f : [a, b] \to \mathbb{R}$, $f(t) = 1$. Then

$$\|f-\ell f'\|_\infty = 1, \quad \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt = \frac{1}{b-a}\ln\frac{b}{a}$$

and by (17) we deduce

$$\left|\frac{1}{x} - \frac{1}{b-a}\ln\frac{b}{a}\right| \le \frac{k}{b-a}\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x \in [a, b]$.

If we take in this inequality $x = a$, we get

$$\left|\frac{1}{a} - \frac{1}{b-a}\ln\frac{b}{a}\right| \le \frac{k}{b-a}\left(\ln\frac{a}{\sqrt{ab}} + \frac{b-a}{2a}\right) \quad (19)$$
$$= \frac{k}{2(b-a)}\left(\ln\frac{a^2}{ab} + \frac{b-a}{a}\right)$$
$$= \frac{k}{2(b-a)}\left(\ln\frac{a}{b} + \frac{b-a}{a}\right).$$

If we multiply (18) by $2(b-a)$, we get

$$2\left|\frac{b-a}{a} - \ln\frac{b}{a}\right| \le k\left(\frac{b-a}{a} - \ln\frac{b}{a}\right),$$

which implies that $k \ge 2$. □

References

[1] D. Pompeiu, Sur une proposition analogue au théorème des accroissements finis, *Mathematica* (Cluj, Romania), **22**(1946), 143–146.

[2] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, 2000.

Also solved by Soon-Mo Jung (Chochiwon, Korea), M. Bencze (Brasov, Romania).

Remark. Problem 115 was also solved by Pierluigi Vellucci (Rome, Italy)

130*. Define

$$T_{n,m}(x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t)\,t^m dt, x \in [0,1],$$

where

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k}, \quad p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k},$$

with

$$(x)_n = x(x+1)(x+2)\dots(x+n-1).$$

Is it possible to have a recurrence relation between $T_{n,m+1}(x)$ and $T_{n,m}(x)$?

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

Solution by G. C. Greubel (Department of Physics, Old Dominion University, VA, USA). The problem asks for a type of recurrence relationship for

$$T_{n,m}(x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n,k}(t)t^m dt,$$

where

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k}$$

and

$$p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}.$$

First consider the integral in $T_{n,m}(x)$. The integral can be evaluated in terms of the Beta function and is readily seen to be

$$\int_0^1 p_{n,k}(t)t^m dt = \binom{n}{k}B(k+m+1, n-k+1).$$

Using this result, $T_{n,m}(x)$ becomes

$$T_{n,m}(x) = \frac{2(n+1)!}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}(nx)_k(n-nx)_{n-k}\cdot\frac{\Gamma(n+1)\Gamma(k+m+1)}{k!\Gamma(n+m+2)}$$
$$= \frac{2(n+1)}{\binom{2n}{n}}\frac{\Gamma(m+1)\Gamma(2n-nx)}{\Gamma(n+m+2)\Gamma(n-nx)}\sum_{k=0}^{n}\frac{(-n)_k(nx)_k(m+1)_k}{k!(1)_k(nx-2n+1)_k}$$
$$= \frac{2(n+1)}{\binom{2n}{n}}\frac{\Gamma(m+1)\Gamma(2n-nx)}{\Gamma(n+m+2)\Gamma(n-nx)} \cdot {}_3F_2(-n, nx, m+1; 1, nx-2n+1; 1),$$

where ${}_nF_m$ is the generalised hypergeometric function. There are known recurrence relations between the parameters of a ${}_3F_2$ series. For example (this particular contiguous relationship is obtained from the Digital Library of Mathematical Functions, namely, http://dlmf.nist.gov/16.3)

$$(a-d)(a-e){}_3F_2(a-1,b,c;d,e;z)$$
$$= a(a+1)(1-z){}_3F_2(a+2,b,c;d,e;z)$$
$$+ a[d+e-3a-2+z(2a-b-c+1)]{}_3F_2(a+1,b,c;d,e;z)$$
$$+ [(2a-d)(2a-e) - a(a-1) - z(a-b)(a-c)]{}_3F_2(a,b,c;d,e;z).$$

When $z = 1$, this is reduced to

$$(a-d)(a-e){}_3F_2(a-1,b,c;d,e;1)$$
$$= a[d+e-a-b-c-1]{}_3F_2(a+1,b,c;d,e;1)$$
$$+ [a(2a+b+c-2d-2e+1) + de - bc]{}_3F_2(a,b,c;d,e;1).$$

Now, for $a = m + 1$, $b = -n$, $c = nx$, $d = 1$, $e = nx - 2n + 1$, the result obtained is

$$m(m + 2n - nx)_3F_2(-n, nx, m; 1, nx - 2n + 1; 1)$$
$$= -(m + 1)(m + n)_3F_2(-n, nx, m + 2; 1, nx - 2n + 1; 1)$$
$$+ \phi\, {}_3F_2(-n, nx, m + 1; 1, nx - 2n + 1; 1),$$

where $\phi = (m + 1)(2m - nx + 3n - 1) + n(nx + x - 2) + 1$. Since

$$\frac{\Gamma(n + m + 2)}{\Gamma(m + 1)} T_{n,m}(x) = \frac{2(n + 1)(n - nx)_n}{\binom{2n}{n}} \cdot {}_3F_2(-n, nx, m + 1; 1, nx - 2n + 1; 1),$$

it can be seen that

$$\frac{m(m + 2n - nx)\Gamma(n + m + 1)}{\Gamma(m)} T_{n,m-1}(x) = -\frac{(m + 1)(m + n)\Gamma(n + m + 3)}{\Gamma(m + 2)} T_{n,m+1}(x) + \phi \frac{\Gamma(n + m + 2)}{\Gamma(m + 1)} T_{n,m}(x),$$

which leads to the desired recurrence relation

$$\phi\, T_{n,m}(x) = (m + n)(m + n + 2)T_{n,m+1}(x) + \frac{m^2(m + 2n - nx)}{n + m + 1} T_{n,m-1}(x),$$

where $\phi = n(n - m)x + (n + m)(2m + 1) + nm$.

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.

We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Mathematical Analysis*.

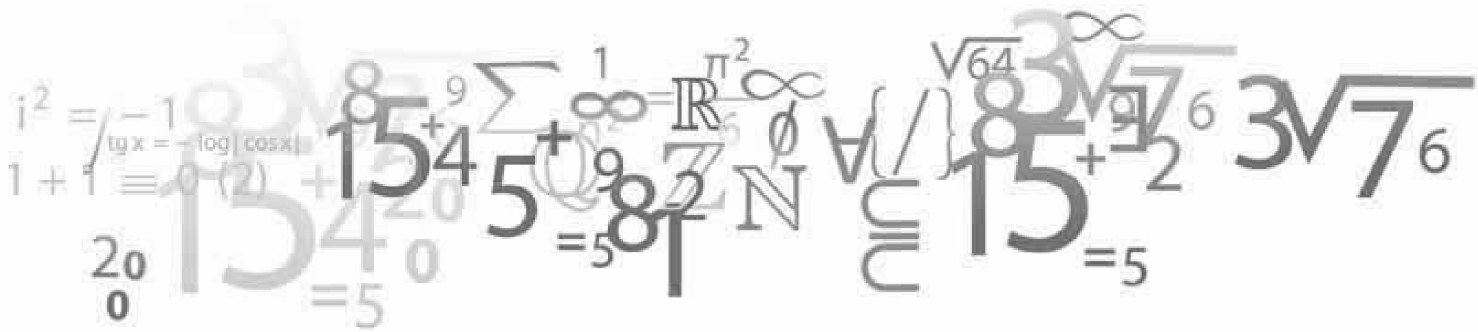
CALL FOR APPLICATIONS

FSMP
Fondation Sciences Mathématiques de Paris

ACADEMIC YEAR 2015-2016

The Foundation Sciences Mathématiques de Paris is pleased to offer the following opportunities to outstanding scientists and excellent students:

- ⇨ **Research chairs of excellence**
- ⇨ **Junior chairs**
- ⇨ **Postdoctoral positions**
- ⇨ **Master course scholarships**



How to apply

Procedures of applications available at

www.sciencesmaths-paris.fr

The call for applications begins in October 2014

FOUNDATION SCIENCES MATHÉMATIQUES DE PARIS - WWW.SCIENCESMATHS-PARIS.FR
IHP, 11 RUE PIERRE ET MARIE CURIE F75231 PARIS CEDEX 05
TEL : +33 1 44 27 68 03 - FAX : +33 1 44 27 68 04

FOUNDERS

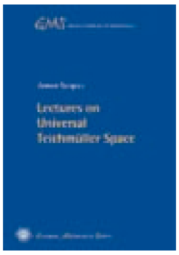
SCIENTIFIC PARTNERS

PARTNERS

New books published by the

European Mathematical Society

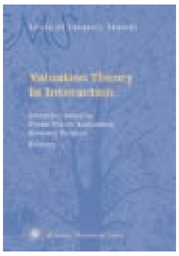
Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.



Armen Sergeev (Steklov Mathematical Institute, Moscow, Russia)
Lectures on Universal Teichmüller Space (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-141-5. 2014. 112 pages. Softcover. 17 x 24 cm. 24.00 Euro

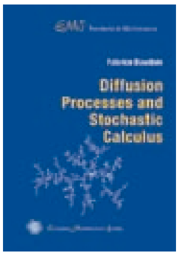
This book is designed for a one semester course for undergraduate students, familiar with basic differential geometry, complex and functional analysis.
The universal Teichmüller space $\mathcal{T}$ is the quotient of the space of quasisymmetric homeomorphisms of the unit circle modulo Möbius transformations. The first part of the book is devoted to the study of geometric and analytic properties of $\mathcal{T}$. It is an infinitedimensional Kähler manifold which contains all classical Teichmüller spaces of compact Riemann surfaces as complex submanifolds which explains the name "universal Teichmüller space".
Apart from classical Teichmüller spaces, $\mathcal{T}$ contains the space $\mathcal{S}$ of diffeomorphisms of the circle modulo Möbius transformations. The latter space plays an important role in the quantization of the theory of smooth strings. The quantization of $\mathcal{T}$ is presented in the second part of the book.



Valuation Theory in Interaction (EMS Series of Congress Reports)
Antonio Campillo (Universidad de Valladolid, Spain), Franz-Viktor Kuhlmann (University of Saskatchewan, Saskatoon, Canada) and Bernard Teissier (Université de Paris Rive Gauche, France), Editors

ISBN 978-3-03719-149-1. 2014. 670 pages. Hardcover. 17 x 24 cm. 98.00 Euro

Having its classical roots in algebraic number theory, algebraic geometry and the theory of ordered fields and groups, valuation theory has seen an amazing expansion into many other areas in recent decades. Moreover, having been dormant for a while in algebraic geometry, it has now been reintroduced as a tool to attack the open problem of resolution of singularities in positive characteristic and to analyse the structure of singularities. Driven by this topic, and by its many new applications in other areas, also the research in valuation theory itself has been intensified, with a particular emphasis on the deep open problems in positive characteristic. The book presents high quality research and survey papers and is of interest to researchers and graduate students who work in valuation theory, as well as a general mathematical audience interested in the expansion and usefulness of the approach.



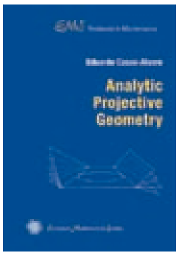
Fabrice Baudoin (Purdue University, West Lafayette, USA)
Diffusion Processes and Stochastic Calculus (EMS Textbooks in Mathematics)
978-3-03719-133-0. 2014. 287 pages. Hardcover. 16.5 x 23.5 cm. 48.00 Euro

The main purpose of the book is to present at a graduate level and in a self-contained way the most important aspects of the theory of continuous stochastic processes in continuous time and to introduce to some of its ramifications like the theory of semigroups, the Malliavin calculus and the Lyons' rough paths. It is intended for students, or even researchers, who wish to learn the basics in a concise but complete and rigorous manner. Several exercises are distributed throughout the text to test the understanding of the reader and each chapter ends up with bibliographic comments aimed to those interested in exploring further the materials.
The first part of the text is devoted the general theory of stochastic processes, we focus on existence and regularity results for processes and on the theory of martingales. This allows to quickly introduce the Brownian motion and to study its most fundamental properties. The second part deals with the study of Markov processes, in particular diffusions. Our goal is to stress the connections between these processes and the theory of evolution semigroups. The third part deals with stochastic integrals, stochastic differential equations and Malliavin calculus. Finally, in the fourth and final part we present an introduction to the very new theory of rough paths by Terry Lyons.

Emmanuel Hebey (Université de Cergy-Pontoise, France)
Compactness and Stability for Nonlinear Elliptic Equations (Zurich Lectures in Advanced Mathematics)

ISBN 978-3-03719-134-7. 2014. 304 pages. Softcover. 17 x 24 cm. 42.00 Euro

The book offers an expanded version of lectures given at ETH Zürich in the framework of a Nachdiplomvorlesung. Compactness and stability for nonlinear elliptic equations in the inhomogeneous context of closed Riemannian manifolds are investigated, a field presently undergoing great development. The author describes blow-up phenomena and presents the progress made over the past years on the subject, giving an up-to-date description of the new ideas, concepts, methods, and theories in the field. Special attention is devoted to the nonlinear stationary Schrödinger equation and to its critical formulation.
Intended to be as self-contained as possible, the book is accessible to a broad audience of readers, including graduate students and researchers.



Eduardo Casas-Alvero (Universitat de Barcelona, Spain)
Analytic Projective Geometry (EMS Textbooks in Mathematics)

ISBN 978-3-03719-138-5. 2014. 636 pages. Hardcover. 16.5 x 23.5 cm. 58.00 Euro

This book contains a comprehensive presentation of projective geometry, over the real and complex number fields, and its applications to affine and Euclidean geometries. It covers central topics such as linear varieties, cross ratio, duality, projective transformations, quadrics and their classifications – projective, affine and metric –, as well as the more advanced and less usual spaces of quadrics, rational normal curves, line complexes and the classifications of collineations, pencils of quadrics and correlations. Two appendices are devoted to the projective foundations of perspective and to the projective models of plane non-Euclidean geometries. The presentation uses modern language, is based on linear algebra and provides complete proofs. Exercises are proposed at the end of each chapter; many of them are beautiful classical results.
The material in this book is suitable for courses on projective geometry for undergraduate students, with a working knowledge of a standard first course on linear algebra. The text is a valuable guide to graduate students and researchers working in areas using or related to projective geometry.

Matheon – Mathematics for Key Technologies (EMS Series in Industrial and Applied Mathematics, Vol. 1)
Peter Deuflhard, Martin Grötschel, Frank Schmidt (all Konrad-Zuse-Zentrum, Berlin, Germany), Dietmar Hömberg, Volker Mehrmann, Martin Skutella (all Technische Universität Berlin, Germany), Ulrich Horst, Jürg Kramer (both Humboldt-Universität zu Berlin, Germany), Konrad Polthier, Christof Schütte (both Freie Universität Berlin, Germany) and Jürgen Sprekels (Weierstraß Institut für Angewandte Analysis und Stochastik, Berlin, Germany), Editors

978-3-03719-137-8. 2014. 466 pages. Hardcover. 17 x 24 cm. 48.00 Euro

Mathematics: intellectual endeavor, production factor, key technology, key to key technologies? Mathematics is all of these! The last three of its facets have been the focus of the research and development in the Berlin-based DFG Research Center Matheon in the last twelve years. Through these activities Matheon has become an international trademark for carrying out creative, application-driven research in mathematics and for cooperating with industrial partners in the solution of complex problems in key technologies.
Modern key technologies have become highly sophisticated, integrating aspects of engineering, computer, business and other sciences. Flexible mathematical models, as well as fast and accurate methods for numerical simulation and optimization open new possibilities to handle the indicated complexities, to react quickly, and to explore new options. Researchers in mathematical fields such as Optimization, Discrete Mathematics, Numerical Analysis, Scientific Computing, Applied Analysis and Stochastic Analysis have to work hand in hand with scientists and engineers to fully exploit this potential and to strengthen the transversal role of mathematics in the solution of the challenging problems in key technologies. This book presents in seven chapters the highlights of the research work carried out in the Matheon application areas: Life Sciences, Networks, Production, Electronic and Photonic Devices, Finance, Visualization, and Education.