NEWSLETTER

OF THE EUROPEAN MATHEMATICAL SOCIETY

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European Mathematical Society
25 years

September 2015
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25th EMS Anniversary
The first three ECMs
The EMS from 1999 to 2006
EMIS

Interview
Abel Laureate
John F. Nash Jr.

Abel Science Lecture
Soap Bubbles and Mathematics

Feature
Diagonals of Rational Fractions

Discussion
Mathematics between Research, Application, and Communication

Journals published by the

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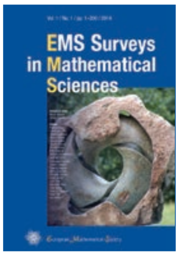

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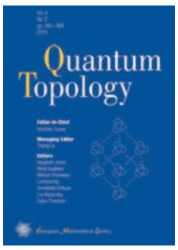

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Editor-in-Chief: Vladimir Turaev (Indiana University, Bloomington, USA)

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Aims and Scope

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EMS Executive Committee

EMS Agenda

2015

26–28 October
Meeting of the EMS Committee for Education
Siena, Italy
Dr Winter (Deutsche Telekom Stiftung), as an invited guest, will give a talk about DIGITAL LEARNING
Contact: Günter Törner, guenter.toerner@uni-due.de

5 November
Annual Meeting of the Applied Mathematics Committee of the EMS, Frankfurt Airport, Germany
http://www.euro-math-soc.eu/committee/applied-math

27–29 November
Executive Committee Meeting, Steklov Institute
Moscow, Russia

2016

18–20 March
Executive Committee Meeting, Institut Mittag-Leffler
Djursholm, Sweden

2–3 April
Presidents Meeting
Budapest, Hungary

9 April
Annual Meeting of the Committee for Developing Countries of the EMS, ICTP
Trieste, Italy
http://euro-math-soc.eu/EMS-CDC/

16–17 July
EMS Council, Humboldt University,
Berlin, Germany

EMS Scientific Events

2015

18–20 September
EMS-LMS Joint Mathematical Weekend
Birmingham, UK

22 October
25th Anniversary of the EMS, Institut Henri Poincaré,
Paris, France

2016

16–20 March
27th Nordic Congress of Mathematicians
Stockholm, Sweden
Bernoulli Society-EMS Joint Lecture: Sara van de Geer (ETH Zurich)

18–22 July
7th European Congress of Mathematics, Berlin, Germany
http://www.7ecm.de/

Editorial: Twenty Five, and What Next?

Pavel Exner (Czech Technical University, Prague, Czech Republic)

Twenty-five years is a period of barely one generation but a community with a source of inspiration and proper organisation can achieve a lot in that time. This is certainly the case with our society and the forthcoming anniversary provides a natural opportunity to look back and appreciate what has been done. There are historical surveys such as David Wallace's account of the first decade of the EMS and I am sure others will be written by those who have taken an active part in the work that has brought us to where we are now.

At the same time, the society is still young, even if it could be said to be a new building made of old bricks. On a human timescale, 25 is an athletic age, indicating someone full of strength still looking ahead. This is true in any field but mathematics reminds us that this juvenile vigour can be mental as well as physical; there is no need to recall the names of the greats who, even at that age, had come up with insights that fundamentally changed large parts of our discipline.

With an appreciation of what we have done, let us think, initially, of what we have to do and what we should do. If I try to condense our mission into a single phrase, our goal and duty is to create conditions, in the environment in which we historically and culturally belong, to make mathematical creativity flourish. We are doing that in a world that is far from peaceful, with numerous tensions arising all the time. It is vital, in this situation, to avoid splits in the community; a good historical reminder is the dispute about participation of the mathematicians of the "central powers" that followed the Great War. However, even if we manage to face such challenges with common sense, the outside world will remain a powerful source of disturbance to a mathematician's life. Often, triumph and tragedy are just inches apart, as we were painfully reminded in May this year.

If you think of how fast things are changing, it is good to recall the issue of the so-called Juncker plan. As we all know, by a concerted effort of numerous players in the European research scene, including many of our member societies, it has finally been achieved that the scientifically most valuable components of the Framework Programme, the European Research Council and the Marie Curie Sklodowska Action will be spared budget cuts. Without any doubt, this is a success and all who contributed to it deserve sincere thanks. At the same time, one cannot overlook how fast this fact was overshadowed on the European agenda by other and bigger problems. This is true as I am writing this editorial and who knows what the situation will be in two months when this issue of the newsletter appears.

Fortunately, there are important mathematical events that are approaching whatever the political weather might be. The dominant among them is our principal meeting, the congress to open next Summer in Berlin. Its organisers are working hard and we all believe it will be a successful meeting. I also use this opportunity to remind everyone that the call for prize nominations is open. It is in our common interest that no remarkable talent or achievement is left out of this prize competition.

With the seventh congress in view, we have to think about the eighth. At the moment, the Executive Committee has received two bids and we hope that the authors of the proposals will be able to offer attractive options to the delegates of next year's council meeting. And speaking of congresses, we cannot ignore the ICM. We understand that mathematics is global nowadays, and a series of congresses in emergent centres is important for developments in those parts of the world. However, we also think that it would be useful if the congress returns to Europe after a 16-year break and we will support endeavours in this direction.

Another issue to be addressed concerns the EMS Publishing House, the history of which started in the middle of the 25 year period (and it should be said that it started from zero). Looking at what we have now, it is a remarkable achievement and the society owes a lot to the many people who made it possible, and to Thomas Hintermann in particular. Of course, the growth brings new challenges that have to be appropriately addressed. These days, the Executive Committee appoints a Scientific Advisory Board, to be consulted both about specific projects as well as the strategy of Publishing House development. This is the first step in making this enterprise stronger and more useful to the European mathematical community, and other steps will follow. Having a solid and profitable publishing house will help us better fulfil the obligations we have towards young mathematicians and those in economically weaker countries.

This editorial is not meant, of course, as a society to-do list. There are many more questions to be solved and more proper occasions to discuss them. Mentioning some of the issues we are going to face here, I am trying to underline that the EMS is entering the second quarter century of its life as a strong and self-confident company, prepared to meet the challenges that await us.

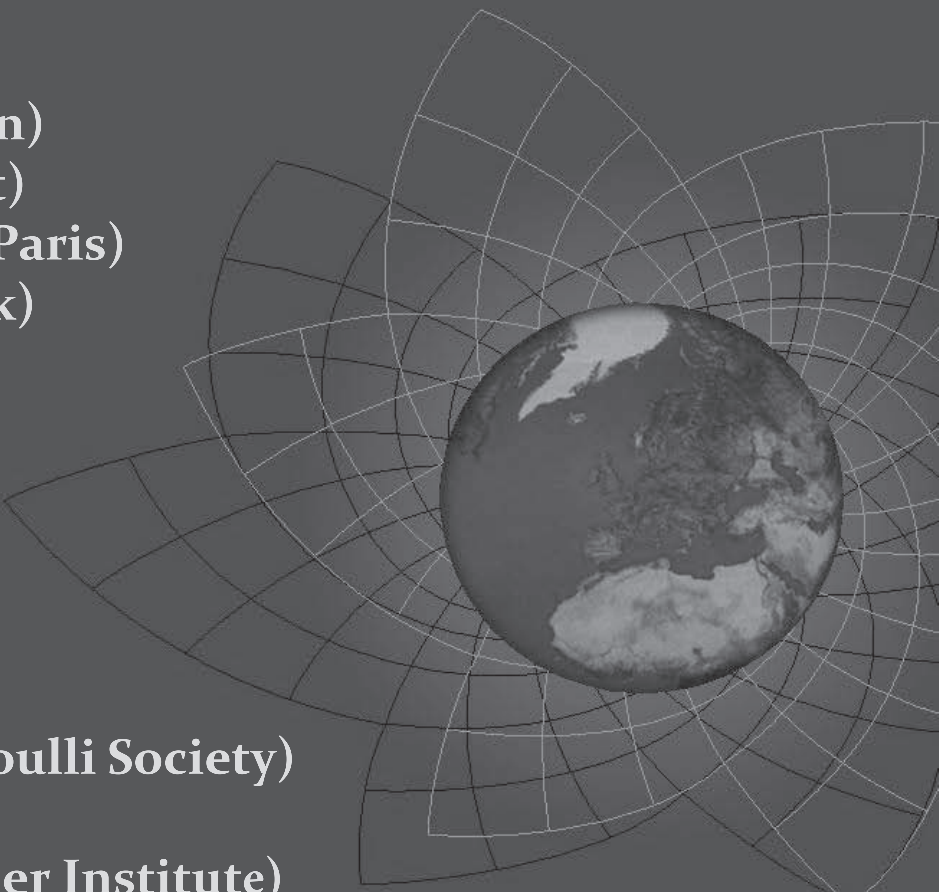
European Mathematical Society
25th ANNIVERSARY
"Challenges for the next 25 years"

Institut Henri Poincaré
Paris, France
22 October 2015

Institut
Henri
Poincaré

Speakers

Hendrik Lenstra (Leiden)
László Lovász(Budapest)
Laure Saint-Raymond (Paris)
Andrew Stuart (Warwick)



Panel

J.P. Bourguignon (ERC)
Peter Bühlmann (Bernoulli Society)
Maria Esteban (ICIAM)
Ari Laptev (Mittag-Leffler Institute)
Roberto Natalini (Istituto per le Applicazioni del Calcolo)

Société Mathématique de France
SMF

SFdS
SOCIETE FRANÇAISE DE STATISTIQUE

SMAI

Background Preparation for the First European Congress of Mathematics (1992): Creation of a New Medium

Mireille Chaleyat-Maurel (Emeritus Professor, University Paris Descartes, France) and Michèle Chouchan (Paris, France)

This joint presentation gives the background preparation for the first European Congress of Mathematics (ECM) in July 1992, which was supported by the European Mathematical Society (EMS). This paper is written through the eyes of the press officers and it will therefore not cover the mathematical programme or the richness of the ECM.

It was during the planning meeting in Poland, in October 1990, that the EMS decided to bring its support and help to this event.

When we began to think, almost 25 years ago, of how to build a different style of communication, it is important to remember the way things were around 1990. Imagination and originality were necessary…

How could we entice journalists (mainly scientific but also the popular press) to introduce mathematics into a cultural field where it was not generally accepted? At that time, the congress attracted more than 1,300 people from 58 countries but we still had to overcome the common "shock/horror" response when mathematics was mentioned, even if one of the main subjects was the relationship between mathematics, mathematicians and the general public.

The symposium "Mathématiques à Venir", held in December 1987, established a first step and was rather well covered by the generalist press. It was the only one. That's why Mireille Ch-M. (MCM) suggested to the Organisation Committee (OC) the creation of a new way of presenting the information, a sort of tool aimed at journalists so that they could easily use it without any need for further explanation. It must be correct as well as pleasant to read. Actors, places, descriptions: it could look like a piece of theatre with very serious subjects, leading to other mathematical areas and audiences.

Fulbert Mignot and François Murat, who had, respectively, been president and treasurer of the OC for more than a year, dedicated themselves to the establishment and development of the project. At that time, Michèle Ch. (MCh) was a journalist and producer for the national French radio (France Culture). She managed interviews, writing and editing for this new magazine. We called it "*Échanges et maths*". It was sent, in the first instance, mostly to the French media, as well as to some foreign scientific journalists, included in their press packs.

A few weeks before the congress in Paris, many people in the world had been touched by the Rio Declaration about the Environment (Earth Summit in June 1992), and the President of the International Mathematical Union (IMU) Jacques-Louis Lions and the General Director of UNESCO Federico Mayor had decided that the year 2000 would be dedicated to mathematics. The first European Congress of Mathematics in Paris, founded by Max Karoubi, can be thought of as the preparation (and reason) for its success.

Twenty-five years … it is such a short and yet, in many ways, such a long time. The EMS is much younger than the IMU. During this period, a lot of very profound changes, both scientific and geopolitical, have been apparent. The perception of mathematics in society is a little less bad.

"At this time, when a young student was asked about contemporary mathematicians, he answered with some caution the names of Einstein, Pythagoras…" remembers Nicole El Karoui, member of the OC, a specialist in financial mathematics. This field was not yet considered as it may be today in universities and meetings. Mathematics appeared (and maybe still appears) disconnected from daily life, more than philosophy, for instance.

About the representation of mathematics and mathematicians in the outside world, Philippe Boulanger, director of the magazine *Pour la science*, and the mathematician Jean-Michel Kantor were very severe:

> *"What is the image of mathematics in general public? It cannot be said that it is bad. Rather it is evanescent. The general public, outside of the professional mathematical milieu, does not have a clue of what contemporary mathematical research is about. (…) It is fashionable to be proud of being ignorant of it, as Bouvard or Pécuchet would have put it: such ignorance is almost a sign of sanity.*
> *The fetishistic example of Bourbakism also gave mathematics an image of a science consisting of abstract structures, in which general synthesis came before examples and formalism concealed meaning."* (in *Actes du Colloque, Round Table A: Mathematics and the General Public*, p. 3).

More and more professionals and partners succeeded in working together to improve the perception of mathematics. Last year, the Fields medals were announced on the front page of the French newspaper *Le Monde*. In 1992, it was much more difficult: MCM was badly received when she tried to join journalists from Antenne 2, the main public TV channel. "Maths is too off-putting," said the person she was talking to. In France Culture, it was also difficult to broadcast mathematical topics despite the international impact of two hours on the subject of Nicolas Bourbaki by MCh in 1988. As a matter of (small…) interest, the two authors of this article met at this time and became friends as they worked toward achieving a programme on pure and applied mathematics.

The EMS gathered together the European countries. Israel and Turkey were later also admitted. More than its geopolitical aspect, this First European Congress had to promote a better understanding of the connections between pure and applied mathematics. The Société Mathématique de France (SMF) and the Société de Mathématiques Appliquées et Industrielles (SMAI) contributed to achieving this aim.

During the meeting, mathematicians also had to show they wished, and were able, to share knowledge and research with other sciences. In the first *Échanges et maths*, Pr. Friedrich Hirzebruch (first President of the EMS, Max Planck Institut für Mathematik, Bonn) was quoted as saying: "The EMS is happy to be the patron of the congress during its first year of life. An original and attractive program has been prepared: a few chairmen were chosen and invited, which is usual for any congress of mathematics; on the other side, a list of round tables treating recent and important subjects can interest or could be interesting for every European mathematician."

During the inauguration of this mathematical week, Henri Cartan spoke about this new style of presentations, "which will give something different from all the other international meetings". The history of mathematical research has been influenced by these conferences: a lot of mathematicians today will recognise their own fields in those that were presented in July 1992.

Lectures, round tables (16) and exhibitions were received by the University Paris I Panthéon Sorbonne. Three days in the week were spent in the well known "Grand Amphithéâtre de la Sorbonne", particularly the round table "Mathematics and the General Public". It was at the same place that David Hilbert announced his 23 problems for the 20th century. The French Minister of Research and Space Hubert Curien supported the congress and was present at its inauguration.

"*Échanges et maths*"

We decided to establish a newspaper, published at first monthly, then more frequently (eight issues appeared between February and July 1992) and then fortnightly from the beginning of May 1992. In the last edition of *Échanges et maths*, MCM included our own question: "Will the fact that professional mathematicians show themselves in the 'arena' or explain their work, asking themselves and one another, be enough to give a taste and curiosity for a large field of human knowledge? We wish it of course."

After all this time since the beginning of the project, we think we can say, without boasting, that it is a very good production. It is possible to find many other evolutions but, for the most part, they don't seem to have changed a great deal, except for the fact that more women are considered able to bear the responsibilities of research.

We succeeded in treating all sorts of topics that could be interesting for scientific journalists as well as generalist ones. For example, we wrote articles about libraries and the choice of their books, or about relations with the general public. Pr. Marta Sanz, who was the President of the EMS from 2010 to 2014, borrowed our idea for the third Congress of Barcelona in 2000.

The magazine comprised four pages of A4 and contained recurring columns: an editorial signed by MCM, a description of a personality, two (or sometimes even three) themes to be developed during the round tables, and some practical information. The 8th edition contained photos so that the members of the OC could be quickly identified during the congress. Henri Cartan, whose portrait was in a good place in *Échanges et maths* number 2, gave us a picture signed by Harcourt Studio: he was pleased to do it but anxious we might lose it!


Échanges et Maths

La lettre du Congrès Européen de Mathématiques - Numéro 2 - Mars 1992

Éditorial

De la culture à la politique

Avec ce second numéro d'Échanges et maths nous ouvrons le dossier « culture européenne » du congrès de juillet prochain. Disons-le tout net : la fameuse tour d'ivoire est un cliché périmé même si le « grand public » français, souvent indifférent à ses mathématiciens, n'en a pas encore pris conscience. Ces quelques jours de congrès favoriseront une ouverture aux champs sociaux et culturels les plus larges.
Les mathématiques ne sont pas seulement ce langage universel et utilitaire auquel on a parfois voulu les limiter. Le congrès européen aura notamment pour tâche d'élargir et de modifier cette perception.
Nous poursuivons donc notre série de portraits en reprenant ici quelques propos de Henri Cartan, grande figure des mathématiques du vingtième siècle — il est l'un des fondateurs du groupe Bourbaki — soutenant l'idée d'un réseau mathématique européen.
A ce propos il était nécessaire d'expliciter, avec Daniel Gabay, le sens d'un projet destiné à lutter contre une pénurie critique de nouveaux chercheurs, et à intégrer les conditions politiques et économiques nouvelles dans les pays de l'Est.
Pour leur part, Catherine Goldstein et Jim Ritter nous offrent d'examiner les mythes et les réalités que recouvre la notion mal éclaircie de « mathématiques européennes ».
Il nous faut le répéter, ce congrès est aussi pour nous l'occasion d'une confrontation entre l'histoire et la prospective.

Mireille Chaleyat Maurel

Le portrait du mois : Henri CARTAN

Sommaire

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ECM
PARIS
6 - 10 juillet 1992

Even though none of us were active feminist militants, we wished to insist on the place of women in mathematics. Their image was often caricatured outside the community and their professional competence, or their numbers, not sufficiently appreciated. So we decided to describe two of them (Michèle Vergne and Mireille Martin-Deschamps) in the first and the last editions so that their personalities and their scientific work could be demystified.

The meeting was widely announced and described, even in some regional newspapers, thanks to a few "congress satellites", in Orléans for instance. Several mathematicians were interviewed: Jean-Pierre Bourguignon as President of the SMF in *Le Monde* (with an announcement on the front page), Jean-Pierre Kahane in *Révolution*, and Pierre-Louis Lions. Detailed articles were published in *Libération*, *Le Monde* and *La Lettre du Monde de l'Education*, and there were specific media and radio broadcasts (France Culture). British and American media also published articles. A very important relay was the Portuguese newspaper *Publico*: the journalist José Victor Malheiros sent his chronicle every day.

The round table "Mathematics and the General Public" was commentated on the most. This was actually the wish of the OC: for its members, it was a sort of challenge and marked the progress in such a meeting.

Over 25 years, the EMS has grown from a small group of societies to the present day, where they have around 60 member societies. It continues to organise congresses every four years and prestigious prizes have been given to very young researchers at each event since the first in 1992.

As Fulbert Mignot wrote in his summary (*La Gazette des mathématiciens*, n° 57, July 1993):

> *"What has come out of the whole congress is that mathematics has a complex role in human culture: first it has its own life; a result is justified and appreciated because of its generality, its interest, even the difficulty of its demonstration and of what can be expected after that. But at the same time it can be a support for other fields (physics, biology, human sciences…) or a language for a better communication between these sciences."*

In 1992, we wished to go forward and give better access to these facets of mathematics. What will be the situation and its social image in 25 years? Better yet, we hope.

Thanks to Claire Ropartz, who provided access to the SMF archives.

Mireille Chaleyat-Maurel [Mireille.Chaleyat@math-info.univ-paris5.fr], on the left, is an Emeritus professor at the University Paris Descartes and the editor of EMS e-news.

Michèle Chouchan (michele.chouchan@wanadoo.fr) was in charge of the scientific programme for the French radio "France Culture" for a number of years. She received the "Prix D'Alembert" of the SMF for her work on mathematics and mathematicians. Later, she was Director of Information for an industrial and scientific public institute. She has published books about the perception and management of nuclear, health and human risks and about the relationship between science and society.

Second European Congress of Mathematics Budapest, 22–26 July 1996

László Márki (Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary)

The Second European Congress of Mathematics was organised by the János Bolyai Mathematical Society and held in Budapest, 22–26 July 1996. The Scientific Committee, the Prize Committee and the Round Table Committee of the congress were nominated by the Executive Committee of the European Mathematical Society and the Organising Committee was appointed by the János Bolyai Mathematical Society.

The story of the congress started four years earlier. At the EMS Council meeting in 1992, two bids were presented to hold the second congress: by Barcelona and by Hungary (the Hungarian bid offered three site options and not only Budapest). After several interventions in favour of this or that bid, the delegate from Romania explained that having the congress in Hungary would show to young mathematicians in the poor post-communist countries that important mathematical meetings can take place in this region and that this could motivate them not to emigrate. This may have been the decisive argument for several delegates to give their vote for Hungary.

As time passed and the congress came closer, the Executive Committee of the EMS worried more and more: it was not clear how the amount of money necessary for holding the congress would come together. Raising funds in post-communist Hungary was far more difficult than doing so in France. The Hungarian organisers felt more comfortable in spite of the difficulties, based on experience from the past that, in the end, the necessary means would be available. Jean-Pierre Bourguignon, President of the EMS, personally negotiated at several places in France and in Brussels in order to raise funds. Finally, the necessary means were available when it was necessary, due to the efforts on both sides.

The congress had 724 registered participants from 58 countries; in addition, a large number of Hungarian students could attend the event without being registered. The scientific programme took place at two sites: the

Budapest Convention Centre (opening and closing ceremonies, plenary lectures and a film on Paul Erdős) and the Budapest University of Technology (parallel lectures, talks of prize winners, round tables, a poster session, scientific films and exhibitions).

The Scientific Committee, chaired by Jürgen Moser (ETH Zürich), chose the "Unity of Mathematics" as the theme of the congress and selected speakers with this in mind. There were 10 plenary lectures and 36 parallel lectures, each of 50 minutes. Among the plenary lectures (the list can be found in the proceedings of the congress), there was one whose title sounded quite exotic for most of the mathematicians present: the lecture by Jacques Laskar (CNRS, Observatoire de Paris) on the stability of the solar system.

Following the tradition set by the First European Congress of Mathematics in Paris, 1992, the European Mathematical Society awarded prizes to ten young European mathematicians to reward their talent, acknowledge the outstanding work they had accomplished and to encourage them in their current research. The Prize Committee, chaired by László Lovász (Yale University, New Haven), chose the following recipients for the prize:

Alexis Bonnet (France, applied analysis),
William T. Gowers (UK, geometry of Banach spaces),
Annette Huber (Germany, algebraic geometry and category theory),
Aise J. de Jong (Netherlands, arithmetic algebraic geometry),
Dmitri Kramkov (Russia, statistics and financial mathematics),
Jiří Matoušek (Czech Republic, algorithmic geometry and functional analysis),
Loic Merel (France, number theory),
Grigory Perelman (Russia, Riemannian geometry),
Ricardo Pérez-Marco (Spain, dynamical systems),
Leonid Polterovitch (Israel, symplectic geometry).

Nine prizes were presented at the opening ceremony of the congress by Gábor Demszky (Mayor of Budapest) and Jean-Pierre Bourguignon (President of the European Mathematical Society). Grigory Perelman declined the prize, referring to a gap in a proof in one of his papers, in spite of a statement of the Prize Committee that his achievements deserved the prize even without that paper. Six of the prize winners agreed to give 30-minute talks at the congress.

Also following the tradition set by the 1992 Paris congress, the scientific programme included round tables, aimed at discussing the relationship between mathematics and society. The Round Table Committee, chaired by Bernard Prum (Université Paris 5), approved the following topics:

Electronic literature in mathematics (Chair: Bernd Wegner),
Mathematical games (Chair: David Singmaster),
Demography of mathematicians (Chair: Jean-Pierre Bourguignon),
Women and mathematics (Chair: Kari Hag),
Public image of mathematics (Chair: Roland Bulirsch),
Mathematics and Eastern Europe (Chair: Doina Cioranescu),
Education (Chair: Vagn Lund Hansen).

Mathematics related films were also shown during the congress. The greatest success among them was the film "N is a number – a portrait of Paul Erdős", made by George Paul Csicsery, on screen at the Convention Centre on the last day of the congress. (Paul Erdős died two months after the congress.)

Those wishing to exhibit a poster had to submit it in advance and the authors of the best posters could get their local expenses covered by sponsors of the congress.

The proceedings of the congress were published by Birkhäuser Verlag in 1998 as Volumes 168 and 169 in the series *Progress in Mathematics*. The proceedings contained written versions of 40 lectures, 4 addresses read at the opening ceremony and accounts of 5 round tables.

From the side events, let us first note the banquet that was held in some of the central exhibition halls of the Hungarian National Gallery in the rebuilt former Royal Palace on Castle Hill of Buda. Several folk and classical music concerts were organised during the congress and, of these, the percussion group Amadinda performed to great acclaim.

Eighteen satellite conferences took place before or after the congress, in Hungary, Austria, Poland, the Czech Republic, Slovakia, Romania and Italy. With one exception, they were specialised conferences of the usual kind.

The exception was the Junior Mathematical Congress held in Miskolc, Hungary, 29 July – 2 August. The idea of such an event also came from the Paris Congress, where mathematicians lectured to teenagers in a mainly francophone setting. The Miskolc Junior Congress involved a more international group of participants and more active participation of the attendees. Announced as for the age group 12–18, it was attended by 350 children; the youngest of them was an 11-year old girl, already a first-year student of Moscow State University. Beside lectures by 'grown-up' mathematicians (among them Paul Erdős, Robert Tijdeman, Jean-Pierre Bourguignon, Peter Michor and Miklós Laczkovich), 35 young participants also gave talks, organised in language sections. These speakers were selected on the basis of abstracts submitted in advance and 17 of them received a prize for their talk, presented by the President of the EMS Jean-Pierre Bourguignon. The best participants had an opportunity for a personal discussion with Paul Erdős and the EMS decided to reimburse the costs of the three brightest participants.

At the congress, as well as at several satellite conferences, we had large numbers of participants from poor countries, made possible by funds obtained from many sponsors (including the IMU, the EMS, the Hungarian Government and public sources, and several private foundations), even enabling, in many cases, travel expenses to be covered.

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3ecm: Retrospective Musings

Sebastià Xambó-Descamps (Universitat Politècnica de Catalunya, Barcelona, Spain)

If I were asked to specify the most decisive factor in the organisation of the 3rd European Congress of Mathematics (3ecm) in July 2000, Barcelona, the answer would be the Catalan mathematical community. Rallying round the Catalan Mathematical Society (Societat Catalana de Matemàtiques, SCM), they enthusiastically embraced the mission entrusted to them by the EMS (of hosting what would, at that point, be the largest mathematical event celebrated in Spain) and provided strong, sustained and effective support throughout. With the hindsight of the 15 years that have elapsed since then, this factor above all others stands in my mind as contributing to the success that is attributed to that endeavour. Since then, the mathematical community that responded so aptly to that challenge has grown and evolved with ever increasing recognition from the worldwide mathematical community and all signs are that this trend will continue in the years to come. The recent creation of the Barcelona Graduate School of Mathematics (http://www.bgsmath.cat/), which has just been awarded a *María de Maeztu* distinction from the Spanish Government (Ministerio de Economía y Competitividad), with funding for four years, is one of the outstanding and tangible signs of the vitality of that trend.

It is of course a duty to recognise that many personalities, institutions and organisations played a fundamental role, even a decisive role, in enacting the 3ecm, in particular those that were present at the opening ceremony table: Andreu Mas-Colell, on behalf of the Catalan Government and the universities (as Minister for Universities, Research and Information Society); Ramón Marimón, on behalf of the Spanish Government (as Secretary of State for Scientific and Technological Policy); Joan Clos (Mayor of Barcelona); Manuel Castellet (President of the Institut d'Estudis Catalans, IEC, and Director of the Centre de Recerca Matemàtica, CRM); Rolf Jeltsch (President of the EMS); and the undersigned, as President of the SCM and the 3ecm Executive Committee. In the background at that stage, there was the assured reliability of our mathematical community. This critical support was embedded in the Executive Committee (6 members) and the Organising Committee (16 members), for their composition was a balanced representation of the universities, particularly those imparting degrees in mathematics, and of all the major specialities. Moreover, each member of the Organising Committee was assisted by several members of their institution.

Since many details about the structure, organisation and unfolding of the 3ecm can be found in the early sections of [1] and [2], including several speeches of the opening and closing ceremonies, and also in the reports in [3], it may be appropriate here to contribute with some comments on those events and their significance by focusing on a few additional snapshots that would perhaps find a fitting place if a more detailed history were to be written.

Among the activities predating the congress, some stand out in my memory as momentous turning points. Before the first ECM (Paris, 1992), the SCM made a bid for the organisation of the second ECM, but the winner was Budapest. The bid was presented once more for the next ECM. On this occasion, there were three more bids: Brighton, Copenhagen and Torino. Following the policy of the EMS, two members of the EMS Executive Committee visited Barcelona in the Spring of 1995: Mireille Chaleyat-Maurel and Aatos Lahtinen. Their mission was to assess the Barcelona offer. Of the very few pictures I have of that visit, one was taken while visiting the Faculty of Mathematics of the University of Barcelona (UB) and another while visiting the Faculty of Mathematics and Statistics (FME) of the Technical University of Catalonia (Universitat Politècnica de Catalunya, UPC). In the UB picture, Joaquín M. Ortega Aramburu, as Vice-president of the SCM, and I are flanking Aatos Lahtinen and Marta Sanz-Solé (who by then was Dean of the Mathematics Faculty). In the UPC picture, it is Joan Solà-Morales (then Dean of the FME) and I who flank Aatos Lahtinen. I much regret that in both cases Mireille had disappeared behind the camera taking the pictures.

The whole visit seemed to go quite smoothly but, as it turned out, just not enough: the EMS Executive Committee decided to recommend Brighton. This was a bit disappointing for the folks in Barcelona. The SCM was (and is) one of the scientific societies of the Institut d'Estudis Catalans (IEC) and this institution had assured, by a letter of Manuel Castellet to Jean-Pierre Bourguignon, then the President of the EMS, that it would provide full support to the SCM. The university rectors and political authorities had also expressed an unconditional commitment. What could be done? After some consultations

Barcelona, Spring 1995. From left to right: Sebastià Xambó-Descamps, Marta Sanz-Solé, Aatos Lahtinen and Joaquín M. Ortega Aramburu.

about whether the EMS Executive Committee decision was the final word, the reaction of the SCM was to prepare a defence of the bid to be pleaded in front of the EMS General Assembly to be held in Budapest on the occasion of the second ECM. Happily, the approach turned out to be successful. This was a high honour but also a huge responsibility, which would have been impossible to meet without the willingness and instinct of all to play well in a large orchestra with almost no possibility of any rehearsing.

The 3ecm was considered, thanks to the IMU, one of the main events of the World Mathematical Year (WMY) and this had a positive effect on the public perception of mathematics in general and of the congress in particular. The Barcelona Metropolitan Transportation company issued one million tickets with Pythagoras' theorem and another million with Archimedes' lever law. The visits of the Scientific Committee (14 members), chaired by Professor Sir Michael Atiyah, had a higher impact in the press than they would have had under other circumstances. The SCM increased its affiliation from around 400 in 1995 to over 1,000 five years later. The Catalan Parliament issued an official declaration in support of the WMY and later organised a one-day event with lectures and activities on various aspects of mathematics and its teaching. There were similar activities in Madrid organised by the Spanish Parliament. We had the impression that we were entering the 21st century (or the third millennium, as some prefer to say) sailing in a favourable wind.

Shaping the 21st Century was the slogan chosen for the 3ecm to convey the idea that the presence of mathematics in society would continue to increase in the future and that we mathematicians should care about it. One consequence of this axiom was the planning of mini-symposia on 10 topics, chosen by the Scientific Committee, that were expected to have great interest in the 21st century: Computer algebra; Curves over finite fields and codes; Free boundary problems; Mathematical finance: theory and practice; Mathematics in modern genetics; Quantum chaology; Quantum computing; String theory and M-theory; Symplectic and contact geometry and Hamiltonian dynamics; and Wavelet applications in signal processing. The state of each topic was to be charted by up to 10 specialists (see [1] or [3] for details) and together (more than 80 specialists were involved) they would provide a sort of collective overview, if not on the whole of mathematics as Hilbert did in 1900, at least on some promising avenues. The 3ecm motto also framed, to a large extent, the agenda of the nine round tables and, in fact, it was adopted as the title of the last one.

Opening of 3ecm. From left to right: S. Xambó-Descamps, R. Marimon, J. Clos, A. Mas-Colell, M. Castellet, R. Jeltsch.

Regarding the composition of the round tables, there were two unexpected events, the response to which illustrates the ductility with which the congress developed. The table "How to Increase Public Awareness of Mathematics" was to be chaired by Felipe Mellizo, a journalist working for the main public Spanish radio (popularly Radio Nacional) but who sadly died shortly before the congress. It was decided, nevertheless, to maintain the round table as an homage to Mellizo, and Vagn Lundsgaard Hansen, then holding the chair of the EMS Committee for Raising Public Awareness, agreed to act as a moderator. Moreover, Miguel de Guzmán, Chair of the Round Tables Committee, managed to give an introductory speech on the life and works of Felipe Mellizo. The other episode was in relation to the last round table: for some reason Mikhail Gromov (IHES), whose name was on the panel, did not come to Barcelona. However, Jean-Pierre Bourguignon was up to the task of rescuing the situation by replacing him at very short notice, a stance that was much appreciated by everybody.

3ecm Prize Winners. From left to right: Wendelin Werner, Paul Seidel, Emmanuel Grenier, Michael McQuillan, Vincent Lafforgue, Dominic Joyce, Stefan Nemirowski, Semyon Alesker, Raphaël Serf. Missing: Dennis Gaitsgory.

Prominent snapshots are related to the prizes awarded at the opening ceremony. Jacques-Louis Lions was Chair of the Prize Committee, which was composed of 16 members. There were 10 prizes (equivalent to 6,000 euros each, although the release into circulation of the new currency in Spain still had to wait some 18 months) to be awarded

Wendelin Werner receiving the Prize from S. Xambó-Descamps and M. Castellet.

for extraordinary work carried out whilst under the age of 32. Five of the awardees were French, two were English (UK) and the other three were from Israel, USA and Russia. Among them was Wendelin Werner, of whom I keep a cherished picture of the moment he received the prize (by chance it was my turn); six years later he was to win one of the Fields Medals bestowed at the Madrid ICM. The Felix Klein Prize was awarded to David C. Dobson, now at the Department of Mathematics of the University of Utah. The Ferran Sunyer i Balaguer Prize in the year 2000, won by Juan-Pablo Ortega and Tudor Ratiu for their memoir Momentum Maps and Hamiltonian Reduction, was also scheduled to be awarded as the last event in the prize ceremony. It was a truly magic moment.

Related to the prizes, I would like to mention one aspect of the organisation that perhaps went rather unnoticed but which illustrated the deep involvement of the SCM. I am thinking about the fundraising involving over 20 sponsors (see [1]) that resulted in over 300 grants for young researchers from many countries. It required a great deal of effort from the SCM. Today, with the seemingly never-ending economic crisis, it seems an amazing feat that we could go that far.

I should also say that the 3ecm owed a lot to another fund that is easily overlooked: the many volunteers (including over 100 students) who undertook all sorts of tasks, which explains why we could accomplish so many things with means that appear somewhat derisory when compared to those allocated to later events. Among all the many people to thank, the members of the 3ecm Executive Committee deserve a special mention: Vice-President Rosa M. Miró-Roig (in charge of programming and activities), Marta Sanz-Solé (organisational secretary), Carles Casacuberta (information and publications), Julià Cufí (finance) and Marta València (infrastructure). Its six members, together with Rolf Jeltsch, were on the closing ceremony table, one of the few occasions in which they appear together in pictures. Let me add that Marta Sanz-Solé was at that moment serving on the EMS Executive Committee and was to become President of the EMS for the four-year period 2011–2014.

3ecm Closing Ceremony. From left to right: J. Cufi, R.M. Miró-Roig, S. Xambó-Descamps, R. Jeltsch, M. Sanz-Solé, C. Casacuberta, M. Valencia.

One sad aspect in looking back at past events is reliving the painful moments in which colleagues and distinguished personalities depart us. Among the illustrious people that contributed to the 3ecm, four deaths were especially sorrowful. Jacques Louis Lions passed away in the Spring of 2001, less than a year after the 3ecm, just after his 73rd birthday. Miguel de Guzman died rather unexpectedly in the Spring of 2004 at the age of 68. Vladimir Arnold, who served as Vice-President of the Programme Committee (chaired by Michael Atiyah), died in 2010 just before reaching the age of 73. And Friedrich Hirzebruch, who had generously accepted chairing the round table "Building Networks of Cooperation in Mathematics", passed away in the Spring of 2012 aged 84.

IEC, from left to right: M. Castellet, M. Atiyah, V. Arnold, S. Xambó-Descamps.

To finish on a livelier note, let me quote a few sentences of David Brannan's report in [3]: "All the 3ecm activities took place in the Palau de Congressos of Barcelona, close to the Olympic Stadium. There was an Opening Reception in the Galeria of the Palau de Congressos and a wonderful Congress Dinner in the gardens of the Palau de Pedralbes. The weather was excellent throughout the congress, and the local organisation superb. It was an outstanding success, thoroughly enjoyed by everyone who participated. Lectures were given by those nine Prize-winners who were present in Barcelona. Overall the programme for the five days was very full and exciting, but with many opportunities for everyone to sit quietly to talk with colleagues as well as to enjoy a varied diet of excellent talks. All participants were agreed that the arrangements and the whole meeting had been excellent, and were really sad to leave Barcelona after 3ecm!"

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[1] C. Casacuberta, R.M. Miró-Roig, J. Verdera and S. Xambó-Descamps (eds.): *3rd European Congress of Mathematics*. Progress in Mathematics, Volumes 201 and 202. Birkhäuser, 2001. Downloadable from http://www.euro-math-soc.eu/ECM/.

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[3] *EMS Newsletter* No. 37, September 2000. Includes D. Brannan's report 3ecm in Barcelona, pp. 8–9, and a report on the Prizes awarded at 3ecm, pp. 12-13.

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EMS Presidency 1999–2002

Rolf Jeltsch (ETH Zürich, Switzerland)

Let me first note how I came to the EMS. I became aware that the EMS existed only in September 1991 when I received the first issue of the EMS Newsletter. Somehow, in the applied mathematics community, the EMS was not very prominent. At that time, only three applied mathematics societies were members: SMAI, GAMM and the IMA. Hence, I did not pay much attention to the EMS. This changed in 1994 when a delegate of GAMM to the council meeting, Professor Rainer Ansorge, called me up at ETH. Because of his vacation, he asked me to attend the council meeting at ETH, 12–13 August 1994, following the ICM 1994 in Zurich. The President of GAMM Reinhard Mennicken suggested me for the one opening on the Executive Committee (EC). I lost out to Alberto Conti. After the council, Peter Michor asked me to join the Electronic Publishing Committee. I was promised that this would not involve travelling and hence I agreed. This was my first involvement with the EMS. In 1995, the President of the EMS Jean-Pierre Bourguignon visited ETH and asked me whether I would be willing to be a candidate for the EC. In the Summer of 1996, the only week in which I could take a vacation with my family coincided with 2ecm and the council meeting in Budapest. Naturally, I did not attend these events but was elected anyway. The next problem was to become an individual member. It turned out that I was the first member of GAMM who wanted to become an individual member. Hence it was easier to join the EMS through the Swiss Mathematical Society.

The first EC meeting I attended was in Vienna in the Spring of 1997, the second on Capri in the Autumn. As indicated above, the EMS was not very visible in the community of applied mathematicians. For this reason, the EC had set up a search committee to find an applied mathematician for the next president. Unfortunately, this committee could not produce a valuable person in Capri. Hence, I was asked to be a candidate. I was elected in Berlin at the council meeting and took office in 1999. For an excellent description of the history of the EMS, I want to direct you to the "History of the European Mathematical Society 1990–1998" by David A. R. Wallace, which you can find on our website.

Before going into some of the actions and initiatives during my presidency, let me thank my home institution ETH Zurich for its generous support, not just financially

After the EC meeting, the Executive Committee were transported by an ETH bus to the home of the president for a drink. From left to right: Bernd Wegner, David Brannan, Anatoly Vershik, Marta Sanz-Solé and Tuulikki Mäkeläinen both at the back, Jean-Pierre Bourguignon, Olli Martio hidden behind Robin Wilson, and Andrzej Pelczar.

Lunch break at the Executive Committee meeting in Zürich, 9–10 October 1999. From left to right: David Brannan, Mireille Chaleyat-Maurel, Olli Martio, Tuulikki Mäkeläinen, Luc Lemaire and Anatoli Vershik.

but also for letting me take a sabbatical before my term as president. Thanks go also to my predecessor Jean-Pierre Bourguignon for his excellent work and for doing the housekeeping while I was still on sabbatical in Australia the first two months of my term.

This article is not to be understood as a meticulous and precise history of my term. During my term, I accumulated about 45 thick folders that rest in a steel cabinet directly under the roof of the ETH main building. But I have no time at the moment to plough through these documents since I accepted working on a project at the University of São Paulo in São Carlos in Brazil, starting 1 April.

The first question people might ask me is: 'Why did you, as an applied mathematician, accept being a candidate for the president of a society which has not been able to attract the applied math community?' The answer is simple. I did not and do not believe in the division into 'pure' and 'applied' mathematics and I am not afraid of 'pure' mathematicians. Both are needed and we have to work together. Another issue was the big change that electronic communication, the web and financial exploitation by commercial publishers was bringing to our profession.

Personally, I was fascinated by the fact that the EMS was a very young organisation and hence one could still have some impact. While I was still on sabbatical in Brisbane, Australia, I made a long 'to-do list' of problems to be attacked. Many of those were more of a practical or technical nature but some were fundamental.

A very important issue was whether the EMS should start publishing. Already, my predecessor Jean-Pierre Bourguignon had thought about this. He organised a meeting with societies during the Berlin council to discuss this issue. Clearly, the societies in the East were using their publishing of journals to be able to use exchange agreements with Western societies to have 'cheap' access to Western journals. Some of the Western societies, e.g. the London Mathematical Society, did their own publishing. It took me some time to get the courage to start the EMS Publishing House. Fortunately, Thomas Hintermann of Birkhäuser seemed to be willing to start a publishing house on his own but in collaboration with EMS. He was directed to me even before I was president and we discussed the matter. The most difficult problem appeared to be the initial financing and, to a lesser extent, the structure. My vision was that the publishing house should be owned by the mathematicians but run by professionals. At my first EC meeting as president, 17–18 April 1999, in Barcelona, I presented a working plan for its creation. This was approved and a working group was formed. Clearly, there was the danger that the publishing house would not be successful and could even go bankrupt. This could hurt the EMS. For this reason, the legal form of a foundation with a seat in Zurich was chosen. This had the additional advantage that a takeover by a big publisher was not possible. However, to start a foundation in Switzerland one needed 50,000 Swiss francs as untouchable funds. The EMS provided 10,000 euros and ETH allowed me to use, for the rest, some leftovers of third party funds from industry. In June 1999, Thomas Hintermann and I made a business plan. We assumed that with 600,000 Swiss francs 'à fonds perdu' we would be in the black by 2005. Unfortunately, I always received negative answers from possible supporters. In June 2000, I wrote to the Swiss Conseil Federal Madame Ruth Dreyfuss for support and was waiting for a reply during the 3ecm in Barcelona. Despite all these setbacks, the many discussions during the congress and the apparent need gave me the courage to announce, in my closing speech, the creation of the EMS Publishing House. Later, I approached the President of ETH, Olav Kübler, for support. He im-

Executive Committee meeting, 28-30 September 2002. Picture in the park in front of the monument of Sofia Kovalevskaya. From left to right: Ari Laptev (Host), Doina Cioranescu (EC), Luc Lemaire, (VP), Nina Uraltseva (Prize Committee Chair 4ecm), Victor Buchstaber (EC), Tuulikki Mäkeläinen (Helsinki office), Bodil Branner, Robin Wilson (Newsletter), Sir John Kingman (President-elect), Rolf Jeltsch, David Salinger (Secretary-elect), Helge Holden (Secretary-elect), Carles Casacuberta (Publication Officer) and David Brannan.

During 3ecm in Barcelona, the President of the American Mathematical Society and the EMS sign the first reciprocity agreement of the EMS. From left to right: Rolf Jeltsch, Felix Browder.

Hike on Saturday afternoon at the brainstorming weekend in Berlingen on the 'Digital Mathematical Library'.

mediately agreed to pay half of the needed funds. Unfortunately, it turned out that, according to Swiss law, one is not allowed to use tax money to support organisations that are in competition with the private sector. It needed more than a year to find ways that ETH could support the publishing house. You will find more details about this in the article by Thomas Hintermann in the EMS Newsletter of March 2012.

Nowadays, I am impressed by how many journals and books it produces. Clearly, this could not have been achieved without the courage of Thomas Hintermann (who committed himself to building a publishing company), the continued support by ETH and the Seminar for Applied Mathematics at ETH, and the equally committed Manfred Karbe, who joined the team at a later stage.

I was elected as president to make the EMS more attractive to applied mathematicians. To achieve this, I called for an informal brainstorming weekend in Berlingen at the lower end of Lake 'Bodensee'. It was impressive to see how enthusiastic the participants were about the topic. The result was the so-called Berlingen Declaration. In addition, it made sense to create conferences with topics which interested applied mathematicians. The first was the EMS-SIAM Conference in Berlin in 2001. In a sense, the AMAM conference in 2003, with the full title 'Applied Mathematics – Applications of Mathematics', in Nice, France, was a succession. It was jointly organised by the EMS-SMF-SMAI. Unfortunately, this 'series' of more applied mathematics oriented conferences was not really continued after my term. The creation of the EMS Felix Klein Prize for an individual or a group to solve an industrial problem had already increased the visibility to applied mathematics circles. How did this come about? I had attended the International Congress on Industrial and Applied Mathematics in Edinburgh, where Helmut Neunzert, the Director of the 'Fraunhofer Institute for Industrial Mathematics' in Kaiserslautern, received the Pioneer Prize (see iciam.org). He spontaneously sponsored the EMS Felix Klein Prize.

It was natural that the EMS would join the International Council for Industrial and Applied Mathematics (ICIAM) as a member. This happened in the year 2000. To open up the EMS to individuals who did not live in Europe, we made reciprocity agreements with the American Mathematical Society and the Australian Mathematical Society during my term.

One of the most important tasks mathematical societies have to do is interact with 'local' politics, funding agencies and education systems. For the EMS, this was the European Union and hence the EU Commission but also the European Science Foundation. I was happy that my predecessor Jean-Pierre Bourguignon had already worked very hard on this and that I could rely on an excellent Vice-President Luc Lemaire, who was already, before my time, the Relation Officer to the EU. What I learned was that these processes take a very long time but that one has to be continuously on top of it. I remember very well that Luc and I were at a meeting in Brussels. We complained in a coffee break to the Commissioner Philippe Busquin that the

European Research Advisory Board (EURAB), a high-level, independent, advisory committee created by the European Commission to provide advice on the design and implementation of EU research policy, did not have one mathematician on the board despite being made up of 45 top experts from EU countries and beyond. Busquin immediately said that the scientific members had been proposed by the European Science Foundation (ESF) and that we should complain to its Secretary General Eric Banda. Since Dr Banda was at the same coffee break, we immediately went to him and he said that the ESF Committee on Physical and Engineering Sciences (PESC) had made the suggestions. It turned out that despite the fact that PESC should care about mathematics too, hardly any mathematicians were members. The reason was that PESC members are delegated from countries and most countries felt that physics and engineering are more important than mathematics. Due to our intervention, ESF then changed its selection policy and, due to this, mathematics became much better represented. An important part of the work of the EMS in Brussels was to ensure that the framework programmes were developed such that they became more accessible for mathematicians. In fact, Luc Lemaire organised at least two special meetings with the Commissioner Philippe Busquin: one in 2000 with Jean-Pierre Bourguignon and one in 2002 with Philippe Tondeur, the Director of the Mathematical Science Program of the NSF of the USA. In those days, the framework programmes of

LIMES Meeting in Nantes, group photo, 2002.

the EU seemed to be more run by the ideas of politicians and lawyers rather than scientists. The intention of such interventions was that mathematics would receive more support. Another item was that in these framework programmes, large infrastructures were supported. It was difficult to explain that, in a sense, mathematics also has 'large infrastructures', e.g. Zentralblatt-MATH and the digital mathematical library (DML). The EMS was successful in getting the proposal LIMES approved in the Summer of 1999. It had a financial volume of 1,700,000 euros and the aims were to improve Zentralblatt-MATH with the help of FIZ Karlsruhe, the EMS and about half a dozen teams. To be better prepared to apply for these framework programme funds, I called for a second Berlingen meeting on 'Infrastructure for Mathematics in Europe' in the Spring of 2002. In particular, we discussed: the digitalisation of hard copies of mathematical articles, journals and books (DML); making Zentralblatt-MATH available to a large European infrastructure; the EMS Publishing House; and a possible new policy on scientific meetings by the EMS. In fact, one result was an application for the DML project with more than 40 partners. Unfortunately, this was not successful.

Our intervention with the ESF and PESC resulted in the EMS being invited as a guest to the PESC meetings. I participated for the first time in Reykjavik, Iceland, in the Spring of 2002. There was a lively discussion on the question of whether a European Research Council should be created. Later, in July, I received an urgent email from Jean-Pierre Bourguignon, who informed me that under the Danish presidency, there would be a conference on this topic on 7–8 October and the list of people who would be invited had no mathematicians. Unfortunately, I could not attend as I was scheduled during this time to receive an honorary degree (my first) from the North University of Baia Mare, Romania. The President-elect Sir John Kingman and Bodil Branner attended instead. As indicated above, processes are very slow. As President of GAMM, I made my speech at the opening of GAMM's annual conference in Luxembourg on 29 March 2005. In front of me sat the Minister for Education and Research of Luxembourg. As Luxembourg had the presidency of the EU, about two weeks later he hosted the ministers of research in the EU countries to decide about the creation of the ERC. In 2007, the ERC was founded and now we can see that it has been extremely successful. In 2014, my predecessor Jean-Pierre Bourguignon became its second president.

Looking back at my period in office, I was very lucky that the very interesting 'World Mathematical Year 2000' happened during my term. Of course, it was a strenuous time and after more than 70 flights I had contracted a back problem. However, it gave the opportunity to present the EMS and its activity on a worldwide level and throughout all of Europe.

Another great event in my term was the creation of the Abel Prize by the Norwegians. On 18 April 2001, I received an email by Ragni Piene asking for the support of the EMS to the idea for such a prize. She wrote that she needed such a letter within two days! Of course, Ragni Piene had also discussed the matter with Bodil Branner, who was a member of the EC. Hence, Bodil had asked John Hubberd to write a judgment of the importance of Abel's work to warrant the naming of this prestigious prize after him. Therefore, I had no problem meeting the short deadline. Bodil Branner had suggested that the EMS Council in 2002 should be held in conjunction with the Abel bi-centennial conference in Oslo. Hence, we could participate and were present when the Abel Prize was announced. I had the privilege of delivering the thank you speech to the Norwegians. Since the EMS and the IMU were allowed to make suggestions for members of the Abel Prize Committee, both organisations were consulted on how the committee should operate. This was an interesting discussion. For example, there was the question of whether the prize committee members should be publicly known or not. While the IMU favoured their solution used for the Fields Medal Committee, I was for the Norwegian point of view of making the names public. It turned out that by law the authors of a document signed by the Norwegian king had to be publicly known. Since members of the Abel Prize Committee have terms of either two or four years, they would be known after the first year. Hence, it made sense to make the members public. It was also great to be able to meet many of these winners of the Abel Prize at the ceremonies, which are always displays of Norwegian style in arts, music and their way of life.

The Editor-in-Chief of the Newsletter Lucia Di Vizio asked us a second question: 'What did you appreciate the most on a personal level, in the whole experience at the EMS?' My answer is that I found many good friends in Europe but also in the whole world. I met many idealistic, enthusiastic and hard working colleagues. I had the opportunity to learn much more on the diversity of mathematics but suffered of course from the lack of time to attend conferences in my research field.

At this point, I really have to add that the Executive Committee always worked as a great team. They helped me a lot in solving impossible looking tasks. Without them, not as much could have been achieved. And, of course, I must mention Tuulikki Mäkeläinen, the Secretary of the EMS office in Helsinki. Whenever I needed help, she replied instantaneously and precisely. I am happy to see how the EMS has developed further. Special moments will always stay in my mind: Carles Casacuberta and Bodil Bran-

Working on the ferry from Stockholm to Helsinki. From left to right: Doina Cioranescu (EC), Sir John Kingman (President-elect), Luc Lemaire, Robin Wilson (Newsletter), Mina Teicher, Olli Martio and Tuulikki Mäkeläinen (Helsinki office).

ner dancing on the streets in Barcelona, a complete pig being brought in for dinner in Bendlewo, the Norwegian King saying that 'a young nation needs heroes', being driven on the backseat of a motorcycle through heavy traffic in Granda and walking among army tanks in Kiew (which were exercising a parade for the 10th anniversary of independence from the Soviet Union), being at the grave of our famous Swiss mathematician Euler in St. Petersburg and also being in his office, the problem during the last EC meeting which was carried out on the ferry from Stockholm to Helsinki and the captain deciding when the one-hour time shift is made, the flags of Abel decorating Karls Johans gate in Oslo, ….

The most difficult question posed by the Editor-in-Chief was: 'How do you see the EMS in 25 years? What are the most important challenges that the EMS will have to face and overcome?' Looking at the exceptional development of the EMS and the dramatic increase in activities in its first 25 years, it is difficult to guess how it will be in the future. It is clear that publishing is currently going through a new development and soon everything will be electronic. I assume that the EMS Publishing House will adjust to these changes and will have simultaneously developed into a major player in this area. Already during my term, it has been hardly possible to fulfil the regular duties of a professor at my institution and I think this will become increasingly more difficult. I assume that at some point, we will need an Executive Director similar to the AMS and SIAM and a substantially larger office with more staff to support the President of EMS and the members of the governance. One could even think that EMS presidents would take a leave of absence from their home institution and be fully paid by the EMS. Currently, there are many countries where more than 90% of the mathematicians in academia are nationals from that country. However, to produce very high level mathematics, one needs international teams. It will be a big challenge for the EMS to support such a healthy development. I assume that the language barrier in the educational system will be solved, at least on the PhD and Master's levels. However, I am convinced that the cultural richness of Europe will still be present in 2040.

Rolf Jeltsch retired from the Swiss Federal Institute of Technology, ETH Zurich, in 2011. He has been the President of the International Council for Industrial and Applied Mathematics (ICIAM), the International Association for Applied Mathematics and Mechanics (GAMM) and the Swiss Mathematical Society. He works mainly in numerical analysis and scientific computing.

The European Mathematical Society, 2003–2006

Sir John Kingman (University of Bristol, UK)

When I took over as the fourth president, the society was well established and had developed sound ways of working (under the presidencies of Fritz Hirzebruch, Jean Pierre Bourguignon and Rolf Jeltsch). The next four years were to prove that the foundations were firm but a society like ours cannot stand still. It constantly needs to ask itself how it can better fulfil its purposes and how it can best respond to changing circumstances.

I was fortunate to inherit a group of officers and members of the Executive Committee who shared this view and who were willing to work hard to ensure that the EMS added real value to European mathematics. Helge Holden had just taken over from David Brannan as secretary, Olli Martio was an assiduous and canny treasurer, and Bodil Branner, Luc Lemaire and later Pavel Exner gave great service as vice-presidents. Above all, we were loyally and efficiently supported by Tuulikki Mäkeläinen in the permanent office in Helsinki.

A fundamental issue for the EMS is how best to complement the work of both the national mathematical societies and the International Mathematical Union. Some things are best done at a national level and others are global but there are problems and opportunities that are most effectively handled by the European mathematical community acting in concert. And, at the time, Europe was itself changing rapidly, including, among other things, the breakup of the Soviet Union and the enlargement and increasing ambition of the European Union.

One important function of the society is to raise the profile of mathematics, both with the general public and with decision-makers across Europe. It is easy for the contribution of mathematics to be taken for granted but modern life would be impossible without both traditional applied mathematics and the ability of mathematicians to deliver new mathematics to meet new challenges. This case must be put forward repeatedly and convincingly, at every level and at every opportunity. The fact that European mathematicians can speak with one voice does impress those who make decisions and who allocate resources in individual countries and in international organisations.

To take just one example, when the EU decided to set up a European Research Council (ERC), it seemed as though mathematics, as a comparatively cheap discipline, might not be represented. I was able, on behalf of the EMS, to approach Chris Patten, who was heading the panel to select council members and, as a result, Pavel Exner was a founding member of the ERC.

It is vitally important to know what is being discussed at different levels of the EU bureaucracy in Brussels and we were very fortunate that Luc Lemaire had close links and was able both to warn us of what might affect mathematics for good or ill and to exert a quiet influence when decisions were being taken. Sometimes it is useful to make a fuss in public but sometimes more can be achieved behind the scenes.

On several occasions, we had pleas for help from mathematicians in countries whose governments were taking decisions that would damage the subject. The EMS was able to support the national societies in their protests, and governments are sometimes impressed when their local scientists receive strong backing from colleagues in other countries. Success is not guaranteed but its probability can be increased.

The authority with which the society can speak (as well as its financial strength) is greater if it clearly represents the whole mathematical community in Europe. Our founding fathers cleverly set it up to include both mathematical societies as corporate members and individual mathematicians as members in their own right. We tried, with some success, to encourage more individual members, whose personal support was greatly appreciated, while also extending the range of societies associated with us. In doing so, we took a broad view of what constitutes mathematics and what constitutes Europe. The society covers many countries outside the European Union, some indeed outside a narrow geographical definition of Europe. It also welcomes the involvement of learned bodies in areas of traditional and more modern applications of mathematics; I was, for instance, delighted that some of the national statistical societies (though not, alas, my own) joined up.

The willingness of mathematicians to join, and become active in, the EMS depends on whether they see value to them and their work. The high level political issues are important but how does the society help particular mathematicians with their own scientific activities? One obvious answer is the organisation or support of international conferences and summer schools, both general and specialised. Of these, the most important are the European Congresses of Mathematics, held every four years so as to interleave with the International Congresses organised by the IMU. Thus, mathematicians have the opportunity to attend a large and high quality congress every two years. The four-year cycle means that every president has exactly one ECM in their term of office and it is one of the greatest highlights of their reign. I was lucky to have the Stockholm ECM in 2004, a most memorable event very ably organised by Ari Laptev.

Another aspect of great importance is that of scientific publication. A number of worrying trends were becoming clear. Many journals were produced by commercial publishers that tried to hold university libraries to ransom by charging exorbitant subscriptions. Librarians were then forced to discontinue what they saw as less important journals, including some published by learned societies reliant on publication income. There were similar problems with book publication and it could be difficult to find a publisher who would keep the price to what libraries could afford.

This had led Rolf Jeltsch, during his presidency, to set up the EMS Publishing House and to recruit Thomas Hintermann to run it. This was only possible because Rolf was backed by his university, ETH Zurich, which gave essential financial support in the early stages. It was vital that this commercial venture should not put the society at risk, so the publishing house belonged legally to a new European Mathematical Foundation, based in Zurich and operating under Swiss law. Although the EMS was not at risk, its officers were, since the president, secretary and treasurer sat *ex officio* on the board of the foundation. Had the publishing house failed, the personal consequences for the officers might have been severe but I was assured that the Zurich prisons were very civilised and comparatively comfortable.

Fortunately, I did not have to test this hypothesis because the publishing house slowly grew stronger and now publishes an impressive range of books and journals, including, of course, our own excellent JEMS. It does not try to compete with the publishing activities of national learned societies but it will, over time, offer an alternative to the greedy commercial publishers, who profit from the unpaid labour of authors, editors and referees.

By this time, of course, it had become clear that the future of mathematical publication lay in electronic rather than print media. Journal articles, in particular, are expected to be available on computer screens, and almost all new publication allows this. But mathematicians need to look back to past literature and it seemed to the executive committee that it was an achievable aim to digitise the whole of mathematical literature, right back to the distant past. (The advantage of exponential increase is that, if time is reversed, the result converges rapidly.) The project turned out to be more difficult than it seemed at first, largely because of copyright issues, but progress has been made and the aim is a noble one.

The enormous expansion of the mathematical literature increases the importance of the reviewing journals Zentralblatt-MATH and MathSciNet. It is good for the community to have two such operations, and the EMS was concerned for the health of the one based in Europe. We collaborated with the Heidelberg Academy and the publishers Springer to ensure that the scientific standard remained high and that the various sources of support remained in place.

When the countries of Eastern Europe that had been under the sway of the USSR regained their freedom, there was a fine flowering of mathematical activity and an enrichment of the historic links across the continent. The EMS was able to encourage this in a number of ways, for instance by giving financial help for travel to scientific meetings. The modest resources of the society limit the direct support that can be offered but we can apply leverage to get further help from bodies such as the EU, UNESCO and the European Science Foundation. It is vital to give young mathematicians the opportunity to interact with their peers in other countries and such interaction often leads to fruitful and lasting collaboration.

We worried too about the slow progress being made in encouraging more women to take up, and persevere with, mathematical research. Fortunately, we had an active committee looking for imaginative ways of improving the situation and there is no doubt that the mathematical world is moving in the right direction. We have now celebrated the first female Fields Medallist; when will a woman win the Abel Prize?

During my working life, the whole style of mathematical research has changed dramatically. In the old days, one worked away alone or with local colleagues, writing to other mathematicians to ask for reprints or to announce progress. Once or twice a year, one went to conferences to talk about one's results and to exchange ideas and problems, to return refreshed to one's desk to carry on the search for new theorems and calculations. Of course, individual mathematicians pursued burning ambitions to solve great problems but the overall flavour of the mathematical world was steady and sedate.

Today, the subject moves ever faster. To keep up, a typical mathematician has a network of electronic communication with others working on related problems all around the world. A new idea is shared at the speed of light and at once exploited by colleagues (or destroyed by a counterexample). Learned bodies like the EMS must come to terms with this new paradigm and must continually ask what they can best contribute to the overall advance of mathematics. There will always be a need for face-to-face meetings at which new collaborations can be set up. There will always be a need for quality control of different sorts of publication, by rigorous peer review. There will always be a need to argue the case for the resources that the subject needs to flourish; mathematics may be cheap but it is not free.

My four years as president convinced me that as well as thriving national societies and the global mission of the IMU, Europe needs its own continent-wide society so that European mathematics has an effective voice. I am sure that the European Mathematical Society will continue to be an essential feature of the mathematical scene and I wish it every success in the future.

Sir John Kingman, educated at Cambridge and Oxford, became a lecturer at Cambridge, moved to Sussex and then held a chair in Oxford. In 1985 he became Vice-Chancellor of the University of Bristol, and in2001, Director of the Isaac Newton Institute in Cambridge. He did "distinguished research on queueing theory, on regenerative phenomena and on mathematical genetics" for which he was awarded the Royal Medal of the Royal Society. He has been President of the London Mathematical Society and of the Royal Statistical Socety. Sir John Kingman was the first chairman of the Statistics Commission and chaired a government inquiry into the teaching of English.

EMIS – 20 Years of Cooperation of the EMS with FIZ Karlsruhe/zbMATH

Bernd Wegner (Technische Universität Berlin, Germany) and Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

Introduction

The European Mathematical Information Service (EMIS) was founded in 1994 as a cooperative venture between the EMS and FIZ Karlsruhe, hosted at the editorial office of zbMATH in Berlin. It provides a variety of electronic offerings in mathematics. Most prominent among them are the Electronic Library ELibM (a collection of databases) and access to projects dealing with electronic information and communication in mathematics. Initially, EMIS also served to distribute information on the activities of the EMS and on conferences and mathematical advertisements of general interest, like job opportunities. But this was discontinued after more than 10 years because, with the growing activities of the EMS and the more complex requirements of the EMS to establish an improved administration of their membership, a more professional and independent website had to be developed.[1]

An additional argument was that, from the beginning, EMIS was distributed worldwide through a growing

[1] See also M. Raussen, "EMS Website www.euro-math-soc.eu upgraded", *EMS Newsletter* 94, 6–7 (2014).

The EMIS portal

number of mirror sites, which is not suitable for the website of a society. The central server for EMIS (http://www.emis.de/) was installed in March 1995 at the editorial office of Zentralblatt MATH in Berlin and the mirror sites were set up very soon after. As a rule, their installation had to be approved by the EMS and the corresponding national mathematical society. They form a worldwide network, providing automatically updated copies of EMIS all over the world. The mirror sites improve the visibility and the accessibility of EMIS on the one hand; on the other hand, as they are run on a voluntary basis, requirements leading to additional effort at a mirror side have to be kept to a minimum. World Wide Web access to the contents of EMIS is free to all users, except for the full usage of some databases. In these restricted cases, a link leads directly to the corresponding system of database gateways and the user is subject to the conditions for accessing that database. The user will be able to do searches but if their institution does not subscribe to the service, only a restricted amount of information will be available as a result.

The concept of the Electronic Library (ELibM)

In the first 10 years, the library expanded from a few journals to a collection of more than 60 journals and several monographs and proceedings, and some interesting, innovative electronic offerings. The involvement of the EMS in EMIS was very helpful for the acquisition of journals and other freely accessible electronic publications in mathematics.[2] Even at that time, the collection was the biggest repository of freely accessible mathematics on the web, and the following figures show that the ELibM is still a top offering: it currently comprises 78,328 articles from 112 journals.

One important condition for posting journals through EMIS is that content once given to EMIS cannot be locked again if a journal decides to become accessible through paid subscriptions only. There are some cases of discontinued journals, where only a restricted period is available through EMIS.[3] In such a way, the ELibM serves as a distributed open archive. The archival function also corresponds well to the moving-wall policy employed by some journals, for which the articles remain open after a fixed period of time has elapsed.

Connecting electronic libraries and databases

Databases and project information form the other substantial part of the EMIS system. As already described by J.-P. Bourguignon in his article about the history of the EMS during the mid-1990s,[4] a key activity during this time was the development of advanced European information systems in mathematics, especially the online databases Zentralblatt MATH (zbMATH) and MathEduc, of which the EMS assumed responsibility as an editorial institution and played an important role in guiding their evolution. A number of projects pushed forward both the extent and the quality of the services. Though it is unfair to name only a few, the following have to be mentioned: the LIMES project,[5] which set up a distributed network of European editorial units for the service; the ERAM project, which

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2 See also B. Wegner, "EMIS – the involvement of EMS in publishing mathematics", *Proc. Int. Conf. Scholarly Communication and Academic Presses*, Florence, 22 March 2001, 41–45 (2002).

3 For more information, see, for example, M. Jost and B. Wegner, "EMIS 2001 – a world-wide cooperation for communicating mathematics online", *Lect. Notes Comput. Sci.* 2730, 87–94 (2003).

4 J.-P. Bourguignon, "Recollection of a very exciting time", *EMS Newsletter* 96, 15–17 (2015).

5 O. Ninnemann, "LIMES – An Infrastructure for the Benefit of Mathematicians in the Information Society", *Lect. Notes Comput. Sci.* 2730, 122–131 (2003).

achieved both the digitisation of the *Jahrbuch über die Fortschritte der Mathematik* and more than 700,000 full text pages of key journals, making the mathematics of almost a whole century freely available in electronic form;[6] the ViFaMath project, which resulted in a close interconnection of ELibM and zbMATH, allowing an integrated search of metadata and full text; and the EuDML project,[7] which succeeded in the formation of the largest open access library in mathematics, now connecting collections like the ELibM, NUMDAM, archives from the ERAM project and many European national digital libraries in a standardised format with enhanced functions. The 242,146 mathematical publications now assembled from five centuries form a considerable fraction of the world's mathematical literature, maintained by an initiative of several European institutions under the umbrella of the EMS.

Current developments

Presently, the growth and development of the ELibM is particularly influenced by the changes in the publication landscape. On the one hand, commercial digitisation efforts limit potential growth; on the other hand, it is influenced by several ramifications of Open Access Publishing, especially its "gold" variety. As has been discussed on several occasions in this newsletter,[8] there are a growing number of journals that give rise to doubts about their standards. Technical advancements have made it rather easy today to set up an electronic submission system and a journal website, which has led to the creation of many journals that seem to be primarily a vehicle for collecting author fees or serving to enlarge publication and citation counts of editorial board members. Obviously, the main challenge for such installations is to gain recognition and visibility; hence, there have been many recent requests to ELibM for inclusion by journals that do not fulfil the basic requirements (often they are not even related to mathematics). Though it is obviously not easy to have conclusive proof of a dysfunctional peer review, standards as defined by the IMU Best Current Practices for Journals[9] or the EMS Code of Practice[10] provide good guidelines. Such an evaluation requires an increasing amount of resourcing; here, the ELibM also benefits from its collaboration with zbMATH, though the scope is not identical (zbMATH will also index relevant articles from journals that are not predominantly mathematical whereas it is usually to be expected that the majority of articles from an ELibM journal would contain mathematics research). Journals that accept randomly generated articles or contributions that are almost exclusively from their own editorial boards can be quickly detected. In one extreme case, a journal adopted into ELibM had to be discontinued when its editor demanded the deletion of a published article (which claimed to provide an elementary proof of the Riemann hypothesis) from the ELibM system (as it had already been deleted on the publisher's site). It was not just the problematic content but also the violation of the principle of permanence of publications that left no alternative than to judge this as severe editorial misbehaviour; indeed, the design of the ELibM archival system would basically forbid such an action.

Due to such developments, the majority of inclusion requests to ELibM must now be turned down; on the other hand, the logic of quantitative measures that prevails makes it increasingly more difficult to achieve the inclusion of relevant quality journals. Nowadays, publishers often find it more attractive to maintain access to content on their site only, which allows the evaluation of access statistics and a detailed analysis of user behaviour. This has also been driven by the introduction of "alternative metrics" based on access figures, a number that seems (if possible) even less suitable to rank mathematical research than citation counts.[11] From the viewpoint of digital mathematics libraries, which should ideally evolve into an open, globally distributed ecosystem, this is an even less desirable development.

On the other hand, the archival option of the ELibM gains an ever increasing importance in this context, especially for the growing number of publications for which open access after a moving-wall period could be achieved through the efforts of the mathematical community and the editorial boards. A typical example is the *Journal of Algebraic Combinatorics*: since 2013, all content older than five years is available via ELibM and will remain so, independent of future changes of the publisher or its policy. Such efforts, also strongly supported within the EuDML framework, will hopefully drive convergence to a system that makes the large majority of relevant mathematical literature freely accessible.

Bernd Wegner [wegner@math.tu-berlin.de] has been a professor at TU Berlin since 1974 and is Editor-in-Chief of Zentralblatt MATH and MathEduc. He has participated in several electronic library and mathematics knowledge management projects. He has also been involved in the organisation of meetings on these subjects. He has been a member of the electronic publishing committee of the EMS for a longer period.

Olaf Teschke [teschke@zblmath.fiz-karlsruhe.de] is member of the Editorial Board of the EMS Newsletter, responsible for the Zentralblatt Column.

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6 B. Wegner, "EMANI, ERAM and Other European Activities Contributing to a Global Digital Library", in: *Mathematics. Digital Libraries: Advanced Methods and Technologies, Digital Collections. Conference Proceedings RCDL 2002*, Dubna, Volume 2, 71–83 (2002).

7 T. Bouche, "Introducing EuDML, the European Digital Mathematics Library", *EMS Newsletter* 76, 11–16 (2010).

8 For example, "Open access – four opinions", *EMS Newsletter* 91, 39–43 (2014) and "EMS Paper on Open Access", *EMS Newsletter* 95, 7–8 (2015).

9 http://www.mathunion.org/fileadmin/CEIC/bestpractice/bp-final.pdf.

10 http://www.euro-math-soc.eu/system/files/uploads/COP-approved.pdf.

11 See, for example, O. Teschke, "Negligible numbers", *EMS Newsletter* 82, 54–55 (2011) and "Guarding your searches: data protection at zbMATH" (with J. Holzkämper), *ibid*. 92, 54–55 (2014).

The 2015 Kamil Duszenko Prize Goes to Thomas Church

Tadeusz Januszkiewicz (Polish Academy of Sciences, Warsaw, Poland)

Kamil Duszenko was an extremely bright mathematics PhD student in Woclaw who worked on geometric group theory. He was also a great pianist, an avid mountain climber and a passionate bridge player. He participated in Polish and International Mathematical Olympiads, winning a Gold Medal at the International Mathematical Olympiad in 2004. Later, he worked for the Polish Mathematical Olympiad, coaching participants, proposing problems and grading competitions. In 2009, he was awarded the Marcinkiewicz Prize of the Polish Mathematical Society for the best student paper in the country.

In 2013, he submitted his PhD thesis and was diagnosed with acute lymphoblastic leukaemia. He fought the sickness for a year through several sessions of chemotherapy. Wroclaw and Warsaw mathematical communities rallied behind him, raising 75,000 euros for his treatment and searching for a potential bone marrow donor. A donor for Kamil was not found, but more than 100 students and faculty registered in the pan-European donor's database. Kamil died on 23 July 2014. He was 28.

In early 2015, Kamil's mother Iza Mironowicz established a prize in his memory. The prize is administered by the Wroclaw Mathematician's Foundation and supported by the Institute of Mathematics of the Polish Academy of Sciences, the Institute of Mathematics of Wroclaw University and the Polish Mathematical Olympiad. The prize is worldwide, carries a cash award of 3,000 euros and, for the term 2015–2019, will be awarded to a young (up to 5 years after PhD) researcher in geometric group theory. In 2020, there will be a review to decide which field of mathematics the prize will subsequently go to. Alongside the usual obligations associated with a prize (visiting Poland and interacting with students and faculty in Wroclaw and Warsaw), the laureate will meet and interact with Mathematical Olympiad participants. In 2015, the jury (Mladen Bestvina Jan Dymara, Tadeusz Januszkiewicz and Piotr Przytycki) invited over 40 eminent researchers on geometric group theory to nominate candidates.

The winner of the Kamil Duszenko Prize in 2015 is Thomas Church from Stanford University (who received his PhD in 2011 from the University of Chicago). Tom's research up to now is in three main directions: low dimensional topology with an emphasis on the mapping class group, cohomology of arithmetic groups with an emphasis on high dimensional cohomology, and, perhaps most importantly, the new phenomenon he discovered and studied with collaborators, which is called representation stability.

Homological stability for, say, the braid groups B_n says that, for fixed k, the maps induced by standard inclusions $B_n \to B_{n+1}$ on (co)homology spaces: $H^k(B_{n+1}, \mathbb{Q}) \to H^k(B_{n+1}, \mathbb{Q})$ are isomorphisms for large n.

Thomas Church

A similar homological stability holds for many other important classes of groups. For arithmetic groups, it plays a crucial role in algebraic K-theory.

Representation stability deals with the cohomology of the pure braid groups P_n, which are kernels of the standard maps from braid groups B_n to S_n, the symmetric groups.

As a vector space, $H^k(P_n,\mathbb{Q})$ does not stabilise. However, if one considers the natural representation of S_n on $H^k(P_n,\mathbb{Q})$, a stability pattern emerges. For this, one has to compare representations of S_n and S_{n+1}. Concretely, it is done in terms of Young diagrams. Abstractly, one develops a formalism of "FI-modules", which are functors on the category of finite sets and embeddings.

As with homological stability, representation stability is a general phenomenon. It shows up among (families of) arithmetic groups, outer automorphism groups of the free groups, mapping class groups of surfaces, etc. It is clearly a beautifully simple, very general principle and it has already proven its usefulness in helping to solve several outstanding problems.

More on the Kamil Duszenko Prize can be found at http://kamil.math.uni.wroc.pl/en/ and http://fmw.uni.wroc.pl/o-fundacji/nagroda-duszenki/nagroda-im-kamila-duszenki.

More on Tom Church's work can be found at math.stanford.edu/~church/.

2015 Henri Poincaré Prizes Awarded

The International Association of Mathematical Physics (IAMP) has awarded the 2015 Henri Poincaré Prizes for Mathematical Physics to: *Thomas Spencer*, Institute for Advanced Study, Princeton; *Herbert Spohn*, Technische Universität München; and *Alexei Borodin*, Massachusetts Institute of Technology.

Spencer was honoured "for his seminal contributions to the theory of phase transitions, the theory of disordered systems, and constructive quantum field theory, including his proofs of the existences of broken symmetry phases and Anderson localization, and his use of novel supersymmetry methods". Spohn was honoured "for his seminal contributions to the theory of transitions from microscopic to macroscopic physics, including his derivation of kinetic and diffusive behaviour from classical and quantum systems, and his work on the fluctuation behaviour of surface growth models". Borodin was honoured "for his seminal contributions to the theory of big groups, to determinantal processes and most notably to the elucidation of Macdonald processes, which have important applications to the statistical physics of directed polymers, tiling models and random surfaces".

The Henri Poincaré Prize, which is sponsored by the Daniel Iagolnitzer Foundation, recognises outstanding contributions that lay the groundwork for novel developments in mathematical physics. It also recognises and supports young people of exceptional promise who have already made outstanding contributions to the field. The prize is awarded every three years at the International Congress on Mathematical Physics. This year's prizes were awarded on 27 July in Santiago de Chile. For previous winners, selection committee members and laudations, see http://www.iamp.org/page.php?page=page_prize_poincare.

Laure Saint-Raymond: the First Lady of the Simons Lectures at MIT

Elisabetta Strickland (University of Rome Tor Vergata, Italy), EMS Women in Mathematics Committee

The Department of Mathematics of the Massachusetts Institute of Technology, USA, annually presents the Simons Lecture Series to celebrate the most exciting mathematical work by the very best mathematicians of our time. The format of this lecture series has evolved since its inception in 1999 and now includes two weeks of lectures – one in pure mathematics and the other in applied mathematics – given each Spring.

The financial backing for these lectures is offered by Jim Simons, the American mathematician, hedge fund manager and philanthropist who, through his foundation, supports many projects in mathematics and in research in general.

The first Simons lecturer in 1999 was Laurent Lafforgue, the French mathematician, known for his outstanding contributions to the Langlands' programme in the fields of number theory and analysis, and as a Fields Medallist at the 2002 International Congress of Mathematicians in Beijing.

Through the years, the list of speakers has always been outstanding, including, in pure mathematics, Andrei Okounkov, John Conway, Nigel Hitching, Grigory Perelman, Wendelin Werner, Robert MacPherson, Manjul Bhargava, Raphaël Rouquier, Ben Green, Terry Tao, Alexander Lubotzky, Akshay Venkatesh and Étienne Ghys.

This year, for the first time, one of the two lecturers was a woman, the French mathematician Laure Saint-Raymond, of the École Normale Supérieure, who gave a series of three lectures, entitled "From particle systems to kinetic equations". The applied mathematics lectures were given by Leslie Greengard of New York University on "New mathematical approaches in acoustics and electromagnetics".

Laure Saint-Raymond, born in 1975, works in partial differential equations. She finished her PhD at Paris Diderot University in 2000, under the supervision of Françoise Golse. She was named full professor of mathematics in 2002, at the age of 27, at Pierre and Marie Curie University, Paris. In 2008, she received one of the ten European Mathematical Society prizes for her work on the hydrodynamic limit of the Boltzmann equation related to Hilbert's sixth problem. She gave an invited talk at the International Congress of Mathematicians in 2014 in Seoul, in the "Partial Differential Equations" section. She is currently Vice-Head of the Department of Mathematics at the École Normale Supérieure.

Laure is one of the best possible examples of a successful work-life balance, as she is also the mother of six children.

News from CIRM

A new look for CIRM's website

CIRM has a new website, which has been designed by its own dedicated team with the aim of promoting the research activities that are hosted there. One of the intentions of the website is to give visitors an overview of its conference-hosting facilities. Another aim is to provide organisers with an online toolkit to manage their event as efficiently as possible and make it visible to the international mathematics community.

The English version of the website is now available at www.cirm-math.com (the French version will be available shortly).

International research in mathematics available on film and online on YouTube

In October 2014, CIRM launched its own Audiovisual Mathematics Library. This is a dedicated platform, based on a corpus of talks and conferences given by mathematicians from around the world when visiting CIRM. The audiovisual library has all the functionalities of a high-level documentary research database, including enriched material and film catalogues.

To give the research talks wider coverage, they are now also available online via YouTube in HD but without chapter sections and indexing. They are classified by playlist: Algebraic and Complex Geometry, Number Theory, etc.

On CIRM's YouTube channel 'CIRMchannel' (https://www.youtube.com/user/CIRMchannel), you will also find interviews with mathematicians, talks aimed at the general public, etc.

A new call for proposals for the Jean-Morlet Chair

With candidates appointed for all semesters until July 2017, the Jean-Morlet Chair (www.chairejeanmorlet.com) is now looking for proposals for the second semester of 2017. Online applications can be presented until 30 September 2015. The Chair is currently held by François Lalonde (Montréal) and Andrei Teleman (Aix-Marseille) and focuses on 'Moduli Spaces in Symplectic Topology and Gauge Theory'.

The building project '2R–CIRM'

CIRM aims to double its hosting facilities by 2018 to respond to increased demand.

The centre hosts large conferences and workshops, as well as small groups and research in pairs, but its infrastructure is no longer sufficient to satisfy requests for future events. As a result, a new project entitled '2R-CIRM' plans to extend and renovate one of the current buildings, in partnership with national and local authorities (the *PACA* region). Not only will it allow CIRM to host two large events at the same time but it will also add to the onsite accommodation capacity. Work will start in July 2016 for an opening due in July 2017. From 2018, CIRM will be able to increase and diversify its offer as a centre for hosting events, particularly in relation to the LabEx initiatives that it is involved in (CARMIN and ARCHIMEDE), as well as the Jean-Morlet Chair.

Announcement of the New SCM journal "Reports@SCM"

Reports@SCM is a non-profit, electronic, open-access research journal on mathematics recently launched by the Societat Catalana de Matemàtiques (SCM) and originating from the desire to help students and young researchers with their first steps into the world of research publication.

Reports@SCM publishes short papers (a maximum of 10 pages) in all areas of mathematics. Articles must be written in English (with an abstract in Catalan), be mathematically correct and contain some original, interesting contribution. All submissions will follow a peer review process before being accepted for publication.

Research announcements containing preliminary results of a larger project are also welcome. In this case, authors are free to publish any future extended versions elsewhere, with the only condition being to make an appropriate citation to Reports@SCM.

We especially welcome contributions from researchers in the initial period of their academic careers, such as Master's or PhD students. Special care will be taken to maintain a reasonably short average time between the receipt of a paper and its acceptance, and between its acceptance and its publication.

More information and submissions at http://revistes.iec.cat/index.php/Reports.

Report on the EMS-SCM 2015 Joint Meeting

Àngel Calsina and Albert Ruiz (both Universitat Autònoma de Barcelona, Cerdanyola del Vallès, Spain)

The first Edinburgh Mathematical Society – Societat Catalana de Matemàtiques joint meeting took place in Barcelona, 28–30 May 2015.

The meeting focused on research areas of interest for the two scientific communities. The Scientific Committee was composed of Àngel Calsina, Jose Figueroa O'Farrill, Jim Howie and Marta Sanz-Solé, who together selected five plenary speakers and six thematic sessions. Each thematic session was coordinated by a member of each society. The members of the Organising Committee were Àngel Calsina, Jozsef Farkas, Xavier Jarque, Joan Mateu and Albert Ruiz.

In the opening session, Tony Carbery (President of the EMS) and Xavier Jarque (President of the SCM), welcomed all the participants and highlighted the opportunity to progress and strengthen collaboration between the two communities.

Beside plenary talks and parallel sessions, there was a poster session on Friday evening just before the conference dinner. The event had 102 participants from both societies and all the activities were held at the headquarters of the Institut d'Estudis Catalans, in the historical centre of the city of Barcelona.

Here are some details of the plenary talks:

Roberto Emparan's lecture was devoted to an original approach to the analysis of the equations of Einstein's theory of General Relativity in relation to the existence and shape of black holes, considering the number of spacetime dimensions as an adjustable parameter and perturbing the much easier equations that arise in the limit where the spacetime dimension tends to infinity. It was proven that, in this limit, the equations are often analytically tractable and that the shape of a black hole is determined by the same equations that describe minimal area surfaces.

The lecture given by *Istvan Gyongy* was a brief introduction to the theory of stochastic differential equations and applications to nonlinear filtering problems. New results on the innovation problem were given.

The plenary lecture by *Carles Simó* dealt with return maps to domains close to broken separatrices and their relation with dynamical properties of the system. In particular, the study of these return maps allows one to obtain realistic quantitative estimates of the boundaries of the confining region of confined motions. The methods can be applied to the restricted three-body problem and

Attendees at the joint meeting

compared to careful numerical computations in order to obtain deeper understanding of the dynamics. The comparisons are also the source of new problems.

Enric Ventura gave a talk on algorithmic group theory. His talk gave a solution of the conjugacy problem for free-by-cyclic groups and the developments on this subject when trying to generalise these results to free-by-free groups.

The lecture by *Jim Wright* was devoted to reviewing the definition of the classical Lebesgue constants of interpolation for continuous periodic functions and presenting recent results linking Lebesgue constants for functions with a sparse spectrum with extensions and generalisations of the work of Jean Bourgain on pointwise ergodic theorems along a sparse subset of integers.

Thematic sessions were distributed in pairs as parallel talks and covered the following topics:

Results on Dirichlet problems for parabolic equations with variable coefficients were presented in the session on *Analysis*. One lecture was devoted to a nonlinear mean value property associated to nonlinear differential operators such as the p-Laplacian. A characterisation of Hankel operators on Bergman spaces belonging to the Schatten class was also presented. The spectral properties of commutators between a pseudodifferential operator and a Hölder continuous function were discussed, with applications to noncommutative geometry and complex analysis of several variables.

Some topics covered in the *Geometric Group Theory* thematic session were properties of the conjugacy grown series of a non-elementary hyperbolic group, the Higman-Thompson group 2V, limit groups over partially commutative groups, the automorphism group of the McCullough-Miller space, palindromic automorphisms of right-angled Artin groups and metric estimates for a finitely generated group.

Talks in *Geometry and Mathematical Physics* were on zeta values and Feynman amplitudes, examples of quantum systems that are related to error correction codes and combinatorial design, Poisson manifolds of "symplectic" type and quantum data hiding and reading.

The topics treated in the parallel session on *Mathematical Biology* ranged from biological evolution, such as algebraic tools which add information to the phylogenetic reconstruction problem, population dynamics models for HIV within-host rapid evolution, birth-death branching processes arising in bacterial and cancerous cell populations, and dynamical systems tools to understand the fast-slow dynamics of stochastic gene expression, to epidemic outbreaks modelled on complex networks and viscoelastic models of plant cell walls.

Several numerical methods for stochastic differential equations were presented in the *Stochastics* session, in particular in order to compute stochastic travelling waves, which are often of interest in applications of models of neural tissue. Theoretical results about uniqueness of some non-Lipschitz stochastic differential equations were also introduced.

Applications to mathematical finance were considered. In particular, a variance estimator was proposed of an asset return with price driven by a diffusion process with jumps and assumed noise in market behaviour.

The strong law of large numbers and central limit theorem asymptotics for the maximal path length on a random directed graph on the integers were covered in another presentation.

Finally, the *Topology* session included talks on algebraic and low dimensional topology. More precisely, in talks on homotopy theory some results were presented on group cohomology, fusion systems, simplicial complexes and decomposition spaces. Another talk was on low dimensional topology, inluding some aspects of the geometry of discrete subgroups of isometries of symmetric spaces and the Heegaard Floer homology of orientable three-manifolds.

More information and details can be found on the meeting webpage: http://emsscm2015.espais.iec.cat/.

Àngel Calsina [acalsina@mat.uab.cat] is a professor of applied mathematics at the Universitat Autònoma de Barcelona. He is interested in partial differential equations and mathematical biology, mainly in structured population dynamics and mathematical modelling of biological evolution. He was a member of the Scientific Committee of the SCM from 2002 to 2014.

Albert Ruiz [albert@mat.uab.cat] is an associate professor in mathematics at the Universitat Autònoma de Barcelona and the Secretary of the Societat Catalana de Matemàtiques. He is interested in homotopy theory and its interactions with group theory.

Interview with Abel Laureate John F. Nash Jr.

Martin Raussen (Aalborg University, Denmark) and Christian Skau (Norwegian University of Science and Technology, Trondheim, Norway)

This interview took place in Oslo on 18 May 2015, the day before the prize ceremony and only five days before the tragic accident that led to the death of John Nash and his wife Alicia.
Nash's untimely death made it impossible to follow the usual procedure for Abel interviews where interviewees are asked to proof-read and to edit first drafts. All possible misunderstandings are thus the sole responsibility of the interviewers.

The prize

Professor Nash, we would like to congratulate you as the Abel laureate in mathematics for 2015, a prize you share with Louis Nirenberg. What was your reaction when you first learned that you had won the Abel Prize?
I did not learn about it like I did with the Nobel Prize. I got a telephone call late on the day before the announcement, which was confusing. However, I wasn't entirely surprised. I had been thinking about the Abel Prize. It is an interesting example of a newer category of prizes that are quite large and yet not entirely predictable. I was given sort of a pre-notification. I was told on the telephone that the Abel Prize would be announced on the morning the next day. Just so I was prepared.

But it came unexpected?
It was unexpected, yes. I didn't even know when the Abel Prize decisions were announced. I had been reading about them in the newspapers but not following closely. I could see that there were quite respectable persons being selected.

Youth and Education

When did you realise that you had an exceptional talent for mathematics? Were there people that encouraged you to pursue mathematics in your formative years?
Well, my mother had been a school teacher, but she taught English and Latin. My father was an electrical engineer. He was also a schoolteacher immediately before World War I.

While at the grade school I was attending, I would typically do arithmetic – addition and multiplication – with multi-digit numbers instead of what was given at the school, namely multiplying two-digit numbers. So I got to work with four- and five-digit numbers. I just got pleasure in trying those out and finding the correct procedure. But the fact that I could figure this out was a sign, of course, of mathematical talent.

John F. Nash jr. and his wife Alicia were received by His Majesty King Harald V. at the Royal Palace. (Photo: Håkon Mosvold Larsen/ NTB Scanpix.)

Then there were other signs also. I had the book by E.T. Bell, "Men of Mathematics", at an early age. I could read that. I guess Abel is mentioned in that book?

Yes, he is. In 1948, when you were 20 years of age, you were admitted as a graduate student in mathematics at Princeton University, an elite institution that handpicked their students. How did you like the atmosphere at Princeton? Was it very competitive?
It was stimulating. Of course it was competitive also – a quiet competition of graduate students. They were not competing directly with each other like tennis players. They were all chasing the possibility of some special appreciation. Nobody said anything about that but it was sort of implicitly understood.

Games and game theory

You were interested in game theory from an early stage. In fact, you invented an ingenious game of a topological nature that was widely played, by both faculty members and students, in the Common Room at Fine Hall, the mathematics building at Princeton. The game was called "Nash" at Princeton but today it is commonly known as "Hex". Actually, a Danish inventor and designer Piet Hein independently discovered this game. Why were you interested in games and game theory?

Well, I studied economics at my previous institution, the Carnegie Institute of Technology in Pittsburgh (today Carnegie Mellon University). I observed people who were studying the linkage between games and mathematical programming at Princeton. I had some ideas: some related to economics, some related to games like you play as speculators at the stock market – which is really a game. I can't pin it down exactly but it turned out that von Neumann[1] and Morgenstern[2] at Princeton had a proof of the solution to a two-person game that was a special case of a general theorem for the equilibrium of n-person games, which is what I found. I associated it with the natural idea of equilibrium and of the topological idea of the Brouwer fixed-point theorem, which is good material.

Exactly when and why I started, or when von Neumann and Morgenstern thought of that, that is something I am uncertain of. Later on, I found out about the Kakutani fixed-point theorem, a generalisation of Brouwer's theorem. I did not realise that von Neumann had inspired it and that he had influenced Kakutani.[3] Kakutani was a student at Princeton, so von Neumann wasn't surprised with the idea that a topological argument could yield equilibrium in general. I developed a theory to study a few other aspects of games at this time.

You are a little ahead of us now. A lot of people outside the mathematical community know that you won the Nobel Memorial Prize in Economic Sciences in 1994.
That was much later.

Yes. Due to the film "A Beautiful Mind", in which you were played by Russell Crowe, it became known to a very wide audience that you received the Nobel Prize in economics. But not everyone is aware that the Nobel Prize idea was contained in your PhD thesis, which was submitted at Princeton in 1950, when you were 21-years-old. The title of the thesis was "Non-cooperative games".

Did you have any idea how revolutionary this would turn out to be? That it was going to have impact, not only in economics but also in fields as diverse as political science and evolutionary biology?
It is hard to say. It is true that it can be used wherever there is some sort of equilibrium and there are competing or interacting parties. The idea of evolutionists is naturally parallel to some of this. I am getting off on a scientific track here.

But you realised that your thesis was good?
Yes. I had a longer version of it but it was reduced by my thesis advisor. I also had material for cooperative games but that was published separately.

Did you find the topic yourself when you wrote your thesis or did your thesis advisor help to find it?
Well, I had more or less found the topic myself and then the thesis advisor was selected by the nature of my topic.

Albert Tucker[4] was your thesis advisor, right?
Yes. He had been collaborating with von Neumann and Morgenstern.

Princeton

We would like to ask you about your study and work habits. You rarely attended lectures at Princeton. Why?
It is true. Princeton was quite liberal. They had introduced, not long before I arrived, the concept of an N-grade. So, for example, a professor giving a course would give a standard grade of N, which means "no grade". But this changed the style of working. I think that Harvard was not operating on that basis at that time. I don't know if they have operated like that since. Princeton has continued to work with the N-grade, so that the number of people actually taking the courses (formally taking courses where grades are given) is less in Princeton than might be the case at other schools.

Is it true that you took the attitude that learning too much second-hand would stifle creativity and originality?
Well, it seems to make sense. But what is second-hand?

Yes, what does second-hand mean?
Second-hand means, for example, that you do not learn from Abel but from someone who is a student of abelian integrals.

In fact, Abel wrote in his mathematical diary that one should study the masters and not their pupils.
Yes, that's somewhat the idea. Yes, that's very parallel.

While at Princeton you contacted Albert Einstein and von Neumann, on separate occasions. They were at the Institute for Advanced Study in Princeton, which is located close to the campus of Princeton University. It was very audacious for a young student to contact such famous people, was it not?
Well, it could be done. It fits into the idea of intellectual functions. Concerning von Neumann, I had achieved my proof of the equilibrium theorem for game theory using the Brouwer fixed-point theorem, while von Neumann and Morgenstern used other things in their book. But when I got to von Neumann, and I was at the blackboard, he asked: "Did you use the fixed-point theorem?"

"Yes," I said. "I used Brouwer's fixed-point theorem."

I had already, for some time, realised that there was a proof version using Kakutani's fixed-point theorem, which is convenient in applications in economics since the mapping is not required to be quite continuous. It has certain continuity properties, so-called generalised

[1] 1903–1957.
[2] 1902–1977.
[3] 1911–2004.
[4] 1905–1995.

continuity properties, and there is a fixed-point theorem in that case as well. I did not realise that Kakutani proved that after being inspired by von Neumann, who was using a fixed-point theorem approach to an economic problem with interacting parties in an economy (however, he was not using it in game theory).

What was von Neumann's reaction when you talked with him?
Well, as I told you, I was in his office and he just mentioned some general things. I can imagine now what he may have thought, since he knew the Kakutani fixed-point theorem and I did not mention that (which I could have done). He said some general things, like: "Of course, this works." He did not say too much about how wonderful it was.

When you met Einstein and talked with him, explaining some of your ideas in physics, how did Einstein react?
He had one of his student assistants there with him. I was not quite expecting that. I talked about my idea, which related to photons losing energy on long travels through the Universe and as a result getting a red-shift. Other people have had this idea. I saw much later that someone in Germany wrote a paper about it but I can't give you a direct reference. If this phenomenon existed then the popular opinion at the time of the expanding Universe would be undermined because what would appear to be an effect of the expansion of the Universe (sort of a Doppler red-shift) could not be validly interpreted in that way because there could be a red-shift of another origin. I developed a mathematical theory about this later on. I will present this here as a possible interpretation, in my Abel lecture tomorrow.

There is an interesting equation that could describe different types of space-times. There are some singularities that could be related to ideas about dark matter and dark energy. People who really promote it are promoting the idea that most of the mass in the Universe derives from dark energy. But maybe there is none. There could be alternative theories.

John Milnor, who was awarded the Abel Prize in 2011, entered Princeton as a freshman the same year as you became a graduate student. He made the observation that you were very much aware of unsolved problems, often cross-examining people about these.
Were you on the lookout for famous open problems while at Princeton?
Well, I was. I have been in general. Milnor may have noticed at that time that I was looking at some particular problems to study.

Milnor made various spectacular discoveries himself. For example, the non-standard differentiable structures on the seven-sphere. He also proved that any knot has a certain amount of curvature although this was not really a new theorem, since someone else[5] had – unknown to Milnor – proved that.

[5] István Fáry.

John F. Nash jr. at the Common Room, Institute of Advanced Study, Princeton. (Courtesy of the Institute for Advanced Study. Photo by Serge J.-F. Levy.)

A series of famous results

While you wrote your thesis on game theory at Princeton University, you were already working on problems of a very different nature, of a rather geometric flavour. And you continued this work while you were on the staff at MIT in Boston, where you worked from 1951 to 1959. You came up with a range of really stunning results. In fact, the results that you obtained in this period are the main motivation for awarding you the Abel Prize this year.

Before we get closer to your results from this period, we would like to give some perspective by quoting Mikhail Gromov, who received the Abel Prize in 2009. He told us, in the interview we had with him six years ago, that your methods showed "incredible originality". And moreover: "What Nash has done in geometry is from my point of view incomparably greater than what he has done in economics, by many orders of magnitude."

Do you agree with Gromov's assessment?
It's simply a question of taste, I say. It was quite a struggle. There was something I did in algebraic geometry, which is related to differential geometry with some subtleties in it. I made a breakthrough there. One could actually gain control of the geometric shape of an algebraic variety.

That will be the subject of our next question. You submitted a paper on real algebraic manifolds when you started at MIT, in October 1951. We would like to quote Michael Artin at MIT, who later made use of your result. He commented: "Just to conceive such a theorem was remarkable."

Could you tell us a little of what you dealt with and what you proved in that paper, and how you got started?
I was really influenced by space-time and Einstein, and the idea of distributions of stars, and I thought: 'Suppose some pattern of distributions of stars could be selected; could it be that there would be a manifold, something curving around and coming in on itself that would be in some equilibrium position with those distributions of stars?' This is the idea I was considering. Ultimately, I de-

veloped some mathematical ideas so that the distribution of points (interesting points) could be chosen, and then there would be some manifold that would go around in a desired geometrical and topological way. So I did that and developed some additional general theory for doing that at the same time, and that was published.

Later on, people began working on making the representation more precise because I think what I proved may have allowed some geometrically less beautiful things in the manifold that is represented, and it might come close to other things. It might not be strictly finite. There might be some part of it lying out at infinity.

Ultimately, someone else, A. H. Wallace,[6] appeared to have fixed it, but he hadn't – he had a flaw. But later it was fixed by a mathematician in Italy, in Trento, named Alberto Tognoli.[7]

We would like to ask you about another result, concerning the realisation of Riemannian manifolds. Riemannian manifolds are, loosely speaking, abstract smooth structures on which distances and angles are only locally defined in a quite abstract manner. You showed that these abstract entities can be realised very concretely as sub-manifolds in sufficiently high-dimensional Euclidean spaces.

Yes, if the metric was given, as you say, in an abstract manner but was considered as sufficient to define a metric structure then that could also be achieved by an embedding, the metric being induced by the embedding. There I got on a side-track. I first proved it for manifolds with a lower level of smoothness, the C^1-case. Some other people have followed up on that. I published a paper on that. Then there was a Dutch mathematician, Nicolaas Kuiper,[8] who managed to reduce the dimension of the embedding space by one.

Apart from the results you obtained, many people have told us that the methods you applied were ingenious. Let us, for example, quote Gromov and John Conway. Gromov said, when he first read about your result: "I thought it was nonsense, it couldn't be true. But it was true, it was incredible." And later on: "He completely changed the perspective on partial differential equations." And Conway said: "What he did was one of the most important pieces of mathematical analysis in the 20th century." Well, that is quite something!

Yes.

Is it true, as rumours have it, that you started to work on the embedding problem as a result of a bet?

There was something like a bet. There was a discussion in the Common Room, which is the meeting place for faculty at MIT. I discussed the idea of an embedding with one of the senior faculty members in geometry, Professor Warren Ambrose.[9] I got from him the idea of the realisation of the metric by an embedding. At the time, this was a completely open problem; there was nothing there beforehand.

I began to work on it. Then I got shifted onto the C^1-case. It turned out that one could do it in this case with very few excess dimensions of the embedding space compared with the manifold. I did it with two but then Kuiper did it with only one. But he did not do it smoothly, which seemed to be the right thing – since you are given something smooth, it should have a smooth answer.

But a few years later, I made the generalisation to smooth. I published it in a paper with four parts. There is an error, I can confess now. Some 40 years after the paper was published, the logician Robert M. Solovay from the University of California sent me a communication pointing out the error. I thought: "How could it be?" I started to look at it and finally I realised the error in that if you want to do a smooth embedding and you have an infinite manifold, you divide it up into portions and you have embeddings for a certain amount of metric on each portion. So you are dividing it up into a number of things: smaller, finite manifolds. But what I had done was a failure in logic. I had proved that – how can I express it? – that points local enough to any point where it was spread out and differentiated perfectly if you take points close enough to one point; but for two different points it could happen that they were mapped onto the same point. So the mapping, strictly speaking, wasn't properly embedded; there was a chance it had self-intersections.

But the proof was fixed? The mistake was fixed?

Well, it was many years from the publication that I learned about it. It may have been known without being officially noticed, or it may have been noticed but people may have kept the knowledge of it secret.[10]

May we interject the following to highlight how surprising your result was? One of your colleagues at MIT, Gian-Carlo Rota,[11] professor of mathematics and also philosophy at MIT, said: "One of the great experts on the subject told me that if one of his graduate students had proposed such an outlandish idea, he would throw him out of his office."

That's not a proper liberal, progressive attitude.

Partial differential equations

But nevertheless it seems that the result you proved was perceived as something that was out of the scope of the techniques that one had at the time.

Yes, the techniques led to new methods to study PDEs in general.

[6] 1926–2008.
[7] 1937–2008.
[8] 1920–1994.
[9] 1914–1995.

[10] The result in Nash's paper is correct; it has been reproved by several researchers (notably Mikhail Gromov) using the general strategy devised by Nash. Nash gave his own account on this error in the case of embeddings of non-compact manifolds in the book *The essential John Nash* (eds. Harold W. Kuhn and Sylvia Nasar), Princeton University Press, 2002.

[11] 1932–1999.

Let us continue with work of yours purely within the theory of PDEs. If we are not mistaken, this came about as a result of a conversation you had with Louis Nirenberg, with whom you are sharing this year's Abel Prize, at the Courant Institute in New York in 1956. He told you about a major unsolved problem within non-linear partial differential equations.

He told me about this problem, yes. There was some work that had been done previously by a professor in California, C.B. Morrey,[12] in two dimensions. The continuity property of the solution of a partial differential equation was found to be intrinsic in two dimensions by Morrey. The question was what happened beyond two dimensions. That was what I got to work on, and de Giorgi[13], an Italian mathematician, got to work on it also.

But you didn't know of each other's work at that time?

No, I didn't know of de Giorgi's work on this, but he did solve it first.

Only in the elliptic case though.

Yes, well, it was really the elliptic case originally but I sort of generalised it to include parabolic equations, which turned out to be very favourable. With parabolic equations, the method of getting an argument relating to an entropy concept came up.

I don't know; I am not trying to argue about precedents but a similar entropy method was used by Professor Hamilton in New York and then by Perelman. They use an entropy which they can control in order to control various improvements that they need.

And that was what finally led to the proof of the Poincaré Conjecture?

Their use of entropy is quite essential. Hamilton used it first and then Perelman took it up from there. Of course, it's hard to foresee success.

It's a funny thing that Perelman hasn't accepted any prizes. He rejected the Fields Prize and also the Clay Millennium Prize, which comes with a cash award of one million dollars.

Coming back to the time when you and de Giorgi worked more or less on the same problem. When you first found out that de Giorgi had solved the problem before you, were you very disappointed?

Of course I was disappointed but one tends to find some other way to think about it. Like water building up and the lake flowing over, and then the outflow stream backing up, so it comes out another way.

Some people have been speculating that you might have received the Fields Medal if there had not been the coincidence with the work of de Giorgi.

Yes, that seems likely; that seems a natural thing. De Giorgi did not get the Fields Medal either, though he did get some other recognition. But this is not mathematics, thinking about how some sort of selecting body may function. It is better to be thought about by people who are sure they are not in the category of possible targets of selection.

When you made your major and really stunning discoveries in the 1950s, did you have anybody that you could discuss with, who would act as some sort of sounding board for you?

For the proofs? Well, for the proof in game theory there is not so much to discuss. Von Neumann knew that there could be such a proof as soon as the issue was raised.

What about the geometric results and also your other results? Did you have anyone you could discuss the proofs with?

Well, there were people who were interested in geometry in general, like Professor Ambrose. But they were not so much help with the details of the proof.

What about Spencer[14] at Princeton? Did you discuss with him?

He was at Princeton and he was on my General Exam committee. He seemed to appreciate me. He worked in complex analysis.

Were there any particular mathematicians that you met either at Princeton or MIT that you really admired, that you held in high esteem?

Well, of course, there is Professor Levinson[15] at MIT. I admired him. I talked with Norman Steenrod[16] at Princeton and I knew Solomon Lefschetz,[17] who was Department Chairman at Princeton. He was a good mathematician. I did not have such a good rapport with the algebra professor at Princeton, Emil Artin.[18]

The Riemann Hypothesis

Let us move forward to a turning point in your life. You decided to attack arguably the most famous of all open problems in mathematics, the Riemann Hypothesis, which is still wide open. It is one of the Clay Millennium Prize problems that we talked about. Could you tell us how you experienced mental exhaustion as a result of your endeavour?

Well, I think it is sort of a rumour or a myth that I actually made a frontal attack on the hypothesis. I was cautious. I am a little cautious about my efforts when I try to attack some problem because the problem can attack back, so to say. Concerning the Riemann Hypothesis, I don't think of myself as an actual student but maybe some casual – whatever – where I could see some beautiful and interesting new aspect.

Professor Selberg,[19] a Norwegian mathematician who was at the Institute for Advanced Study, proved back in

[12] 1907–1984.
[13] 1928–1996.

[14] 1912–2001.
[15] 1912–1975.
[16] 1910–1971.
[17] 1884–1972.
[18] 1898–1962.
[19] 1917–2007.

John F. Nash jr. with last year's Abel laureate Yakov Sinai (right) and Michael Th. Rassias (left). (Photo: Danielle Alio, Princeton University, Office of Communications.)

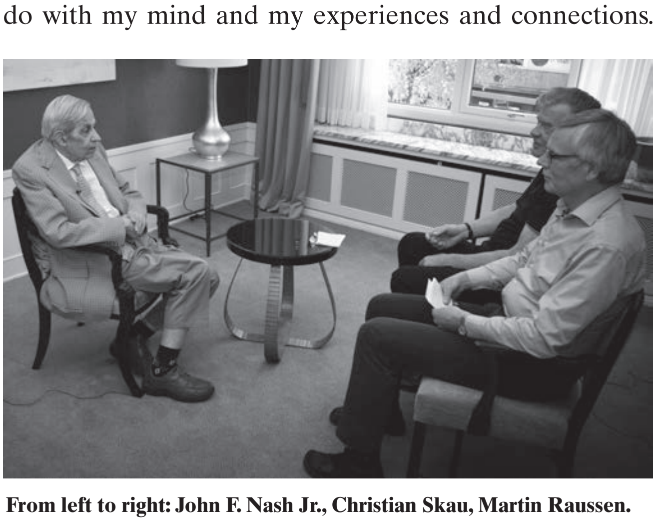
the time of World War II that there was at least some finite measure of these zeros that were actually on the critical line. They come as different types of zeros; it's like a double zero that appears as a single zero. Selberg proved that a very small fraction of zeros were on the critical line. That was some years before he came to the Institute. He did some good work at that time.

And then, later on, in 1974, Professor Levinson at MIT, where I had been, proved that a good fraction – around 1/3 – of the zeros were actually on the critical line. At that time he was suffering from brain cancer, which he died from. Such things can happen; your brain can be under attack and yet you can do some good reasoning for a while.

A very special mathematician?

Mathematicians who know you describe your attitude toward working on mathematical problems as very different from that of most other people. Can you tell us a little about your approach? What are your sources of inspiration?

Well, I can't argue that at the present time I am working in such and such a way, which is different from a more standard way. In other words, I try to think of what I can do with my mind and my experiences and connections. What might be favourable for me to try? So I don't think of trying anything of the latest popular nonsense.



From left to right: John F. Nash Jr., Christian Skau, Martin Raussen. (Photo: Eirik F. Baardsen, DNVA.)

You have said in an interview (you may correct us) something like: "I wouldn't have had good scientific ideas if I had thought more normally." You had a different way of looking at things.

Well, it's easy to think that. I think that is true for me just as a mathematician. It wouldn't be worth it to think like a good student doing a thesis. Most mathematical theses are pretty routine. It's a lot of work but sort of set up by the thesis advisor; you work until you have enough and then the thesis is recognised.

Interests and hobbies

Can we finally ask you a question that we have asked all the previous Abel Prize laureates? What are your main interests or hobbies outside of mathematics?

Well, there are various things. Of course, I do watch the financial markets. This is not entirely outside of the proper range of the economics Nobel Prize but there is a lot there you can do if you think about things. Concerning the great depression, the crisis that came soon after Obama was elected, you can make one decision or another decision which will have quite different consequences. The economy started on a recovery in 2009, I think.

It is known that when you were a student at Princeton you were biking around campus whistling Bach's "Little Fugue". Do you like classical music?

Yes, I do like Bach.

Other favourite composers than Bach?

Well, there are lot of classical composers that can be quite pleasing to listen to, for instance when you hear a good piece by Mozart. They are so much better than composers like Ketèlbey and others.

We would like to thank you very much for a very interesting interview. Apart from the two of us, this is on behalf of the Danish, Norwegian and European Mathematical Societies.

After the end of the interview proper, there was an informal chat about John Nash's main current interests. He mentioned again his reflections about cosmology. Concerning publications, Nash told us about a book entitled *Open Problems in Mathematics* that he was editing with the young Greek mathematician Michael Th. Rassias, who was conducting postdoctoral research at Princeton University during that academic year.

Martin Raussen is professor with special responsibilities (mathematics) at Aalborg University, Denmark. Christian Skau is professor of mathematics at the Norwegian University of Science and Technology at Trondheim. They have together taken interviews with all Abel laureates since 2003.

Soap Bubbles and Mathematics[1]

Frank Morgan (Williams College, Williamstown, USA)

Soap bubbles are a serious topic in mathematics and one with lots of applications. At big international conferences, I meet scientists and engineers designing foams to extinguish fires, to force oil from underground and to model biological cells; Figure 1 shows some human cells that look and act a lot like soap bubble clusters. Colleagues at Trinity College, Dublin, study solid foams of cellular materials, like the bees' honeycomb, that are light and strong and useful for designing lighter parts for airplanes and cars. Modern foams provide comfortable running shoes, mattresses, pillows and seat cushions. Bakers seek the perfect size bubbles or holes in their bread. Coffee artisans seek the perfect foam for their cappuccino. The Beijing Olympics Swimming Cube, where Michael Phelps earned his gold, used a soap bubble cluster model.

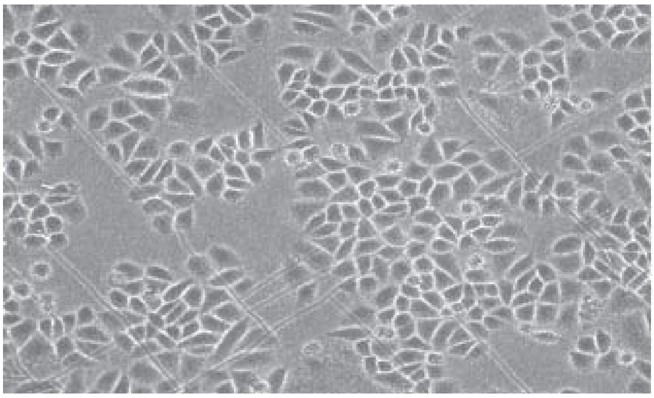


Figure 1. Human cells look and act like soap bubbles. (UC San Diego, all rights reserved.)

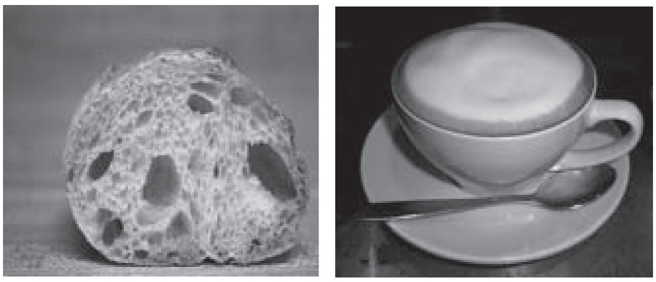


Figure 2: Bakers seek the perfect size bubbles or holes in their bread (applepiepatispate.com/bread/peter-reinharts-french-bread/). Coffee artisans seek the perfect foam for their cappuccino (wikipedia.org)

Figure 3a: The Beijing Olympics Swimming Cube, where Michael Phelps earned his gold, used a soap bubble cluster model.

[1] Write-up of Abel Science Lecture, 20 May 2015, Oslo.

Figure 3b. Inside the Bejing Olympics Swimming Cube.

Such clusters of millions of soap bubbles can get quite complicated, so mathematicians start with the simplest example: a single soap bubble. It has one salient feature: its shape – it is round.

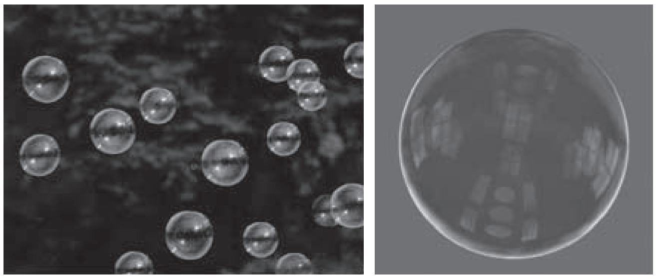


Figure 4: Soap bubbles are round. (Left: 4freephotos.com; right: John M. Sullivan, used by permission, all rights reserved.)

Why are soap bubbles so beautifully round? Zenodorus essentially proved in 200 BC that the round circle minimises perimeter for a given area. Two thousand years later, H. Schwarz proved that the round sphere minimises perimeter (surface area) for a given volume. So soap bubbles minimise surface area or energy. Similarly, soap bubble clusters, as in Figure 5, seek the least-perimeter way to enclose and separate several regions of prescribed volume.

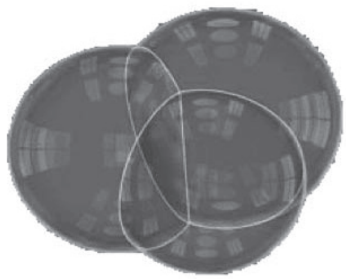


Figure 5. A cluster of *n* bubbles seeks the least-perimeter way to enclose and separate n regions of prescribed volume. (John M. Sullivan, used by permission, all rights reserved.)

Such soap bubble clusters are a difficult mathematical topic because their beautiful smoothness and simplicity break down at the singular places they come together. For this reason, much serious study had to wait for the advent of the very technical "Geometric Measure Theory", pioneered by, for example, L.C. Young at Wisconsin, Ennio De Giorgi in Italy, E.R. Reifenberg in England, and Herbert Federer and Wendell Fleming at Brown. The isometric embedding theorem of Abelist John Nash was used to generalise results from Euclidean space to Riemannian manifolds until more recent intrinsic methods. In 1976, Jean Taylor climactically proved mathematically the classification of soap film singularities, which Plateau had observed 100 years earlier. Here's a little guessing contest to test your intuition.

Question 1. Can one soap film cross straight through another?
Answer. No – not that it breaks, just that it merges and deflects.

Question 2. To obtain the shortest road system connecting the three corners of an acute triangle, where should the junction go?
A. Where the angle bisectors meet.
B. Where the perpendicular bisectors of the sides meet.
C. Where the roads meet at equal angles.
Answer. Where the roads meet at equal angles. This can be demonstrated with soap films, as in Figure 6.

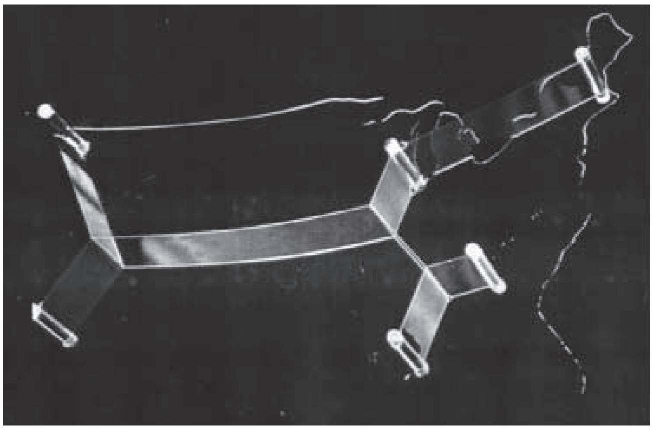


Figure 6: When soap films, striving to minimise length, reach a happy equilibrium, they meet at equal 120° angles. (Photo by Gordan Graham, all rights reserved.)

That's one way that soap films meet: in threes at 120°. And we've seen one thing that does not happen: one soap film will not cross straight through another.

Question 3. In how many different ways can pieces of soap film come together?
A. 1.
B. 2.
C. 3.
D. 4–10.
E. 11–100.
F. More than 100.
G. Infinitely many.

Answer. The answer to most such mathematics questions is either 1 or infinity but, in this case, the answer is 2. We've already seen that three surfaces can meet along a line or curve at 120°. In addition, four such curves (and six surfaces) can meet at equal angles of about 109°, as in the triple bubble of Figure 5. And that's it. That is Jean Taylor's theorem. Incidentally, it was Abelist Louis Nirenberg, with collaborators David Kindelehrer and Joel Spruck, who proved that the curve where three soap films meet must be real-analytic.

Take a closer look at the lower place where four curves meet in the triple bubble. Under a microscope, the four curves will look straight, like the lines from the centre of a regular tetrahedron to its four vertices. And if you dip a tetrahedron in soapy water, that's just what you get:

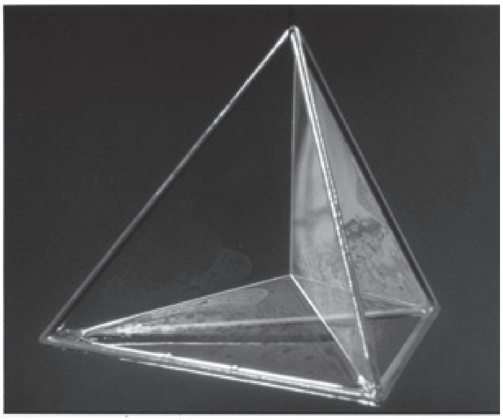
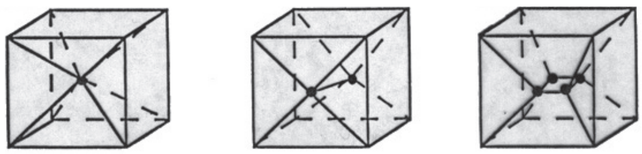


Figure 7: In the soap film on a tetrahedron, soap films meet in threes along four lines that meet in the centre at equal angles of about 109°. (Photo by F. Goro, all rights reserved.)

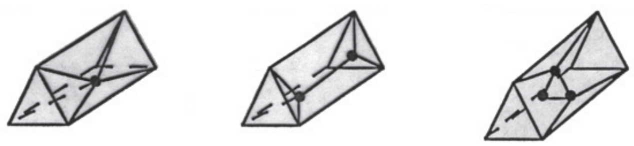
Question 4. The soap film on a cubical frame meets in the centre of the frame:



A. In a point. B. In a line. C. In a square.

Answer. In a square. This is the only way all of the lines can meet in fours.

Question 5. The soap film on a long triangular prism meets in the centre:



A. In a point. B. In a line. C. In a triangle.

Answer. In a line. Not in a point because lines cannot meet in sixes. Not in a triangle because it's a long triangular prism. (In a shorter prism, they could meet in a triangle.)

Jean Taylor, with her advisor Fred Almgren, wrote a beautiful *Scientific American* article that explained why surfaces meeting in threes along curves and curves meeting in fours at points are the only kinds of singularities that can occur, even in the most complicated clusters of millions of bubbles. The proof, published in *Annals of Mathematics*, considers the way the soap films meet a tiny sphere about a singularity: in arcs of great circles meeting in threes at 120°. There are just 10 such nets on the sphere and most of them give rise to complicated singularities, like the cone over the cube with everything meeting at a point, that resolves into four simpler singularities, as Jean discovered using physical soap films. The hard part is to show that the resulting linear approximation is a good one.

Many fundamental open questions remain, such as the existence of a least-area soap film on a given smooth boundary curve. There have been some famous partial results. In 1936, Jesse Douglas and Tibor Rado independently proved the existence of a least-area disc. Such a disc may pass through itself as soap films will not, in which case there is a soap film of less area. For this work, Douglas won one of the first Fields Medals in 1936. Work in geometric measure theory (1960–1979) proved the existence of a minimiser in some special classes of soap films with limitations on the kinds of singularities. In 1960, E.R. Reifenberg proved a least-area soap film among those with nontrivial homology, but some soap films have trivial homology; some can be deformation retracts onto their boundaries!

Another open question is whether the standard triple bubble of Figure 5 is the least-perimeter way to enclose and separate three given volumes of air. What about the standard double bubble?

Figure 8: The Double Bubble Conjecture states that this standard double bubble is the least-perimeter way to enclose and separate two given volumes of air. (John M. Sullivan, used by permission, all rights reserved.)

We mathematicians thought we knew how to prove it optimal, so I asked one of my students, Joel Foisy, to write down the proof as part of his undergraduate thesis. But when we talked to the mathematicians who knew how to prove it, it turned out that they didn't. Of course, we all believed it. Two separate bubbles are wasteful; it is more efficient to share a common wall. A bubble inside a

Figure 9: I have blown a bubble inside a bubble but it is unstable: it pops out as soon as it comes into contact with the outer bubble. (Photo by Jeff Bauer of Citco, used by permission.)

bubble is even worse, since it causes the outer bubble to be bigger. After some instruction from the professional Bubble Guy Tom Noddy, I have blown a bubble inside a bubble but it is unstable: it pops out as soon as it comes into contact with the outer bubble.

Question 6. Consider a double bubble consisting of a large spherical cap, a small spherical cap and a surface between them. That separating surface:
A. Is flat.
B. Bows into the small bubble.
C. Bows into the big bubble.
Answer. The separating surface bows into the big bubble because the smaller bubble has more pressure, as you know from when you blow up a birthday party balloon (how the pressure is greater when it's small and tightly curved and how it gets easier as it grows larger).

Are there any other possibilities? Yes, but none that we've ever seen. To describe them, we cannot rely on photographs. The computer simulation of Figure 10 shows an exotic double bubble: one bubble on the inside with a second bubble wrapped around it in a toroidal innertube. Now, this double bubble is unstable and has much more area than the standard double bubble so it doesn't contradict the conjecture. But it does make you realise that there may be many other possibilities which neither we nor the bubbles have thought of yet.

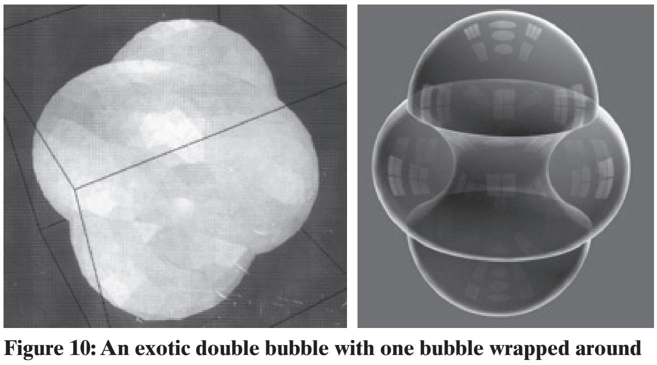


Figure 10: An exotic double bubble with one bubble wrapped around another. (John M. Sullivan, used by permission, all rights reserved.)

There are more possibilities, as in Figure 11, where the first blue inner bubble has another component, a thinner innertube wrapping around the fatter red innertube, connected to the inner bubble by a thread of zero area if you like. Or maybe there could be layers of innertubes on innertubes. Or maybe the bubbles could be knotted as in Figure 12.

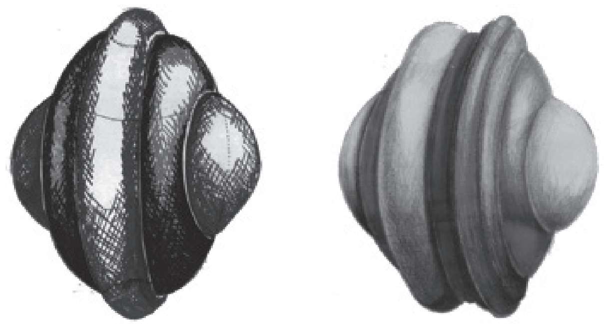


Figure 11: Layers of innertubes. (Drawings by Yvonne Lai, former undergraduate research student, all rights reserved.)

Figure 12: Bubbles knotted about each other.

Or maybe, as in Figure 13, the double bubble could be totally fragmented into millions of pieces, maybe with empty space trapped inside.

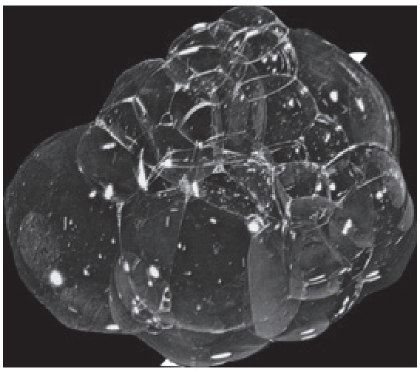


Figure 13: A fragmented double bubble. (Photo by F. Goro, all rights reserved.)

Alas, there are innumerable possibilities to rule out in order to prove that the conjectured standard double bubble is best. Yet in this gallery of possibilities, there shines a ray of hope: they all look unstable and are very area expensive. On this basis, the work to narrow down the possibilities went forward. First of all, a proof outlined by Stanford mathematician Brian White showed that the minimiser has to have lots of symmetry and has to be a surface of revolution. Starting from this proof, Michael Hutchings, a former undergraduate research student, now a professor of mathematics at the University of California at Berkeley, showed that the total number of components is at most three, as in Figure 11a, although they could, in principle, be quite lopsided. The final argument, developed with my collaborators from Granada and Spain, Manuel Ritoré and Antonio Ros, proved the cases of one or two innertubes around a central bubble to be unstable and therefore not minimising.

The proof that a perimeter-minimising double bubble must be a surface of revolution starts with the lemma that some vertical plane splits both volumes in half. Of course it's easy, starting at any angle, to find a vertical plane that splits the first volume in half but it is unlikely to split the second volume too. The idea is to do this with all starting angles. If at first the big half of the second bubble lies before you, 180° later it will lie behind you and, by the Intermediate Value Theorem, there is some starting angle for which the second volume as well as the first is split in half. Now turn that plane horizontal and repeat the argument to get a second plane, perpendicular to the first, which splits both volumes in half. We may assume that these two planes are the xz- and yz-planes, meeting along the z axis. It is the z axis that turns out to be the axis of rotational symmetry.

The final instability argument, which we'll describe in the case of one innertube, is suggested in the working illustration of Figure 14. The bubble on the left has a yellow innertube about it from top to bottom. The way to reduce area and thus prove instability is to rotate the left half to the left and the right half to the right. The top gets fatter, the bottom gets thinner, but the net volume of each bubble remains the same. At the joints at top and bottom, cusps form, which can be smoothed to reduce area slightly.

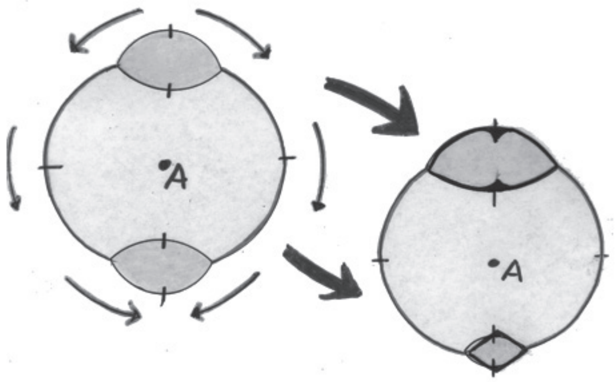


Figure 14: This exotic bubble can be shown to be unstable by rotating the left half to the left and the right half to the right.

So now we have a good answer to the question: 'Why are double bubbles the shape they are?'

Double Bubble Theorem. *The standard double bubble shape minimises surface area.*

Of course to really explain why these are the only double bubbles we see, we'd like to prove a stronger result:
Conjecture. *The standard double bubble is the only* stable *(connected) double bubble.*

I love this conjecture because we mathematicians have no idea how to prove it but a kindergarten student could settle it tomorrow by blowing a different kind of double bubble.

Amazingly enough, on the heels of our proof of the Double Bubble Theorem, a group of undergraduates generalised the result to hyperbubbles in four-dimensional space R^4. Where we had six difficult cases, they had 200. In another five years, the leader of the group Ben Reichardt generalised the result to all dimensions.

In curved universes the best single bubble remains mysterious. Yes – it is a round sphere in all dimensions of Euclidean space, the sphere or hyperbolic space. But even in a space as symmetric as CP^2 (complex projective space), it remains conjectural that geodesic balls minimise perimeter.

For unit-volume convex bodies in R^n, it is conjectured that the least perimeter to fence off a given fraction of the volume is greatest for the round ball, as in Figure 15. The result is known only in the plane for exactly half the area (Esposito, Ferone, Kawohl, Nitsch and Trombetti, 2012).

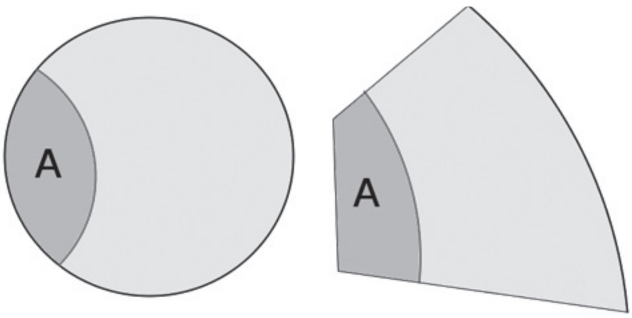


Figure 15: It is conjectured that among convex bodies of unit area, the round disc is the most expensive place to fence off a given fraction of the area. (Bryan Brown)

Inside a cube, it is conjectured that the least-area way to partition off a given fraction of the volume is a spherical cap about a corner, a quarter cylinder about an edge, or a plane, as in Figure 16, although the Lawson surface of Figure 17 comes within 0.03% of beating the cylinder.

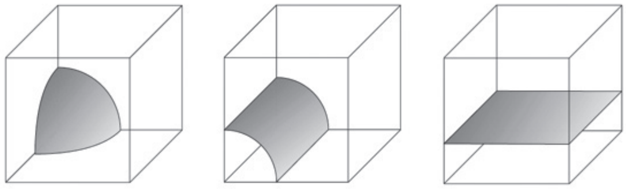


Figure 16: The conjectured least-perimeter partitions of the cube. (Antonio Ros, used by permission, all rights reserved.)

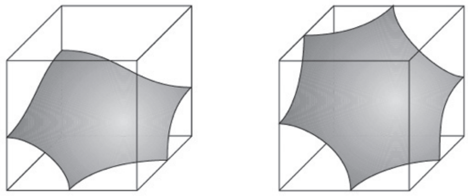
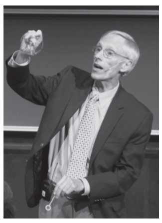


Figure 17: The Lawson and Schwarz surfaces come close to beating the conjecture. (Antonio Ros, used by permission, all rights reserved.)

In the subRiemannian Heisenberg group, Pierre Pansu conjectured in 1982 that the optimal single bubble is a rotationally symmetric union of spiralling geodesics. The latest claim of a proof was announced on 29 January 2015, but subsequently withdrawn.

In certain "asymptotically Schwarzschild" black-hole cosmologies, optimal big bubbles have been proven to be nearly round spheres (Brendle, Eichmair and Metzger), with much gradual progress, most recently on asymptotically conical manifolds (Choddosh, Eichmair and Volkmann).

In any case, one way to understand the universe is to start with soap bubbles.



Frank Morgan [Frank.Morgan@williams.edu] studies optimal shapes and minimal surfaces. He has published over 200 articles and six books, including "Calculus" and "The Math Chat Book", based on his live call-in TV show and column. Founder of the NSF "SMALL" Undergraduate Research Project, inaugural winner of the Haimo national teaching award, past Vice-President of the MAA and of the AMS, he is Atwell Professor of Mathematics and Chair at Williams College and incoming Editor of Notices of the American Mathematical Society.

For background, references and more information, see Morgan's *Geometric Measure Theory* book.

Diagonals of Rational Fractions

Gilles Christol (Université Pierre et Marie Curie, Paris, France)

First introduced to study properties of Hadamard's product [6], the diagonal of a power series could appear to be a somewhat artificial notion. However Furstenberg proved [16] that, over a finite field, the diagonals of rational fractions are exactly the algebraic functions. Our aim is to explain why, over $\mathbb{Q}$, they make a very interesting class of functions, sharing most of the properties of algebraic ones.

1 Introduction

Recently, both physicists and combinatorialists have encountered a lot of powers series with integer coefficients that are D-finite, i.e. solutions of linear differential equations with polynomial coefficients. They all appear to be *diagonals of rational fractions* (DRFs) [5]. For instance, [1] gives a (complete in some sense) list of 403 differential equations of "Calabi-Yau type", and the (unique) analytic solution near 0 of each one can be shown (more or less easily) to be a DRF.[1]

The main aim of this paper is to explain why this could be a general fact. Although they are seemingly elementary objects, DRFs will actually lead us into algebraic geometry, somewhere between Weil's conjectures and motives theory.

We will limit to working over the field $\mathbb{Q}$. Actually, a large number of the results remain true over any field of characteristic zero but we are mainly interested in the arithmetic aspect of the topic and more precisely in a "for almost all p" theory ("*almost all*" meaning "all but a finite number"). This means that we are concerned with properties that can be expressed through reductions modulo p^h, for almost all primes p and all $h \geq 1$. For instance, over fields of finite characteristic, DRFs are exactly the algebraic functions. This implies that DRFs have algebraic reductions modulo p for almost all p.[2] In addition, there are actually non-algebraic DRFs!

In some respects, it would be more convenient to replace $\mathbb{Q}$ with a number field but this would introduce technical difficulties that would obscure things without introducing new ideas.

2 Diagonals of rational fractions

2.1 Definition

For $F(x_1, \ldots, x_n) = \sum_{(s_1,\ldots,s_n)\in\mathbb{N}^n} a_{s_1\cdots s_n}\, x_1^{s_1} \cdots x_n^{s_n}$ in $\mathbb{Q}[[x_1, \ldots, x_n]]$, let's set:

$$\mathrm{Diag}(F) \stackrel{\mathrm{def}}{=} \sum_{s=0}^{\infty} a_{s\cdots s}\, x^s \quad \in \mathbb{Q}[[x]] .$$

A power series f in $\mathbb{Q}[[x]]$ is said to be a *diagonal of a rational fraction (DRF) of n variables* if there is a rational fraction $F = \frac{P}{Q} \in \mathbb{Q}(x_1, \ldots, x_n)$ with Q *invertible* in $\mathbb{Q}[[x_1, \ldots, x_n]]$ (i.e. with $Q(0, \ldots, 0) \neq 0$), such that $f = \mathrm{Diag}(F)$.

We will denote by $\mathcal{D}_n$ the set of DRFs of n-variables and by $\mathcal{D} = \bigcup_{n\geq1} \mathcal{D}_n$ the set of all DRFs. It is almost obvious that $\mathcal{D}_n \subset \mathcal{D}_m$ for $n < m$.

Remark 1. For a given DRF f, the power series F, such that $f = \mathrm{Diag}(F)$, is far from unique. For instance, for any $F \in x_1\mathbb{Q}[[x_1, x_2]]$, setting $G(x_1, x_2) = F(x_1x_2^2, x_2)$, we get $\mathrm{Diag}(G) = 0$.

Considering algebraic functions instead of rational ones does not increase the set of diagonals, as shown by the following proposition.

Proposition 2 ([15])**.** Let $F \in \mathbb{Q}[[x_1, \ldots, x_n]]$ be algebraic over $\mathbb{Q}(x_1, \ldots, x_n)$. Then, its diagonal belongs to $\mathcal{D}_{2n}$.

When $n = 1$, this is a consequence of an explicit formula due to Furstenberg [16]: let $f \in x\mathbb{Q}[[x]]$ such that $P(x, f(x)) = 0$ with $P'_y(0, 0) \neq 0$. Then,

$$f^m(x) = \mathrm{Diag}\left(\frac{y^m\, P(xy, y)}{P'_y(xy, y)}\right), \quad (m \geq 0). \tag{1}$$

General algebraic functions in a single variable can be reduced to that case by separating roots. However, even if theoretically easy, this is algorithmically a rather complex process.

When $n = 1$, the converse of Proposition 2 is also true: any $f \in \mathcal{D}_2$ is algebraic over $\mathbb{Q}(x)$. This is easily seen because residues of a rational fraction are algebraic over the field of its coefficients (see Equation (3) below).

However, in general, elements of $\mathcal{D}_n$ are not algebraic for $n > 2$: Furstenberg [16] pointed out the function

$$\sum_{s=0}^{\infty} \binom{2s}{s}^2 x^s = (1 - 4x)^{-1/2} \star (1 - 4x)^{-1/2},$$

which is both in $\mathcal{D}_3$ and not algebraic over $\mathbb{Q}(x)$.

2.2 Properties

DRFs can be characterised by their coefficients.

Proposition 3 ([24] Theorem 15.1)**.** $\sum u_n x^n \in \mathbb{Q}[[x]]$ is a DRF if and only if the sequence u_n is a binomial sum, i.e. it can be obtained from *binomial coefficients* and *geometric sequences* by means of *affine changes of index* and *finite sums* (see [24] page 111 for a precise definition).

A representative example is given by Apery's numbers $a(s) = \sum_{k=0}^{s} \binom{s}{k}^2 \binom{s+k}{k}^2$, for which one has

$$\sum_{s=0}^{\infty} a_s x^s = \mathrm{Diag}\, \frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4}.$$

---

1 Some of them (#242 , #244, ...) involve harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ in their coefficients but this is only an artefact, as shown by Maillard's formula

$$\sum_{k=1}^{n} k\binom{2n-2k}{n-k}\binom{2k}{k}(H_{n-k} - H_k) = \binom{2n}{n}(2n + 1) - 4^n(n + 1) .$$

2 Actually, DRFs have algebraic reductions modulo p^h for almost all p and all $h \geq 1$, "bad" primes being the (finitely many) ones involved in the denominator of some coefficients

However, the most striking fact is the stability of the set $\mathcal{D}$ under many operations.

Proposition 4. $\mathcal{D}$ is stable under:
1. Sum, and (Cauchy) product ($\mathcal{D}$ is a sub-algebra of the $\mathbb{Q}$-algebra $\mathbb{Q}[[x)]]$).
2. Derivative $f \mapsto \frac{d}{dx} f$.
3. Hadamard $\sum a_s x^s \star \sum b_s x^s = \sum a_s b_s x^s$ and Hurwitz $\sum a_s x^s \mathsf{H} \sum b_s x^s = \sum_{s,t} \binom{s+t}{s} a_s b_t x^{s+t}$ products.
4. "Decimations" $\psi_{r,d} : \sum a_s x^s \mapsto \sum a_{r+ds} x^s$.
5. Algebraic changes of variable: $f \mapsto f \circ g$ for $g \in x\mathbb{Q}[[x]]$ algebraic over $\mathbb{Q}(x)$.

Points 1 to 4 are rather easily checked. Point 5 is more subtle and is based on Equation (1). Complete proofs can be found in [10].

The following statements can also be made.

Proposition 5. The invertible elements of $\mathcal{D}$:
1. For the Cauchy product are the algebraic functions.
2. For the Hadamard product are rational functions.

The proof of 1 is based on two rather deep results: D-finiteness of both f and $1/f$ implying algebraicity of f [17] and Grothendieck's conjecture being true for first order linear differential equations over curves [13].

A proof of 2 for $\mathcal{D}_2$ is given in [4] and is easy to generalise. We now state the properties we will mainly be concerned with. Recall that "for almost all" (notation faa) means for all but a finite number. A property that is true for almost all p will be said to be *global*.

Proposition 6. Any DRF $f \in \mathbb{Q}[[x]]$ is:
1. *Globally bounded:* f has a non-zero radius of convergence (in $\mathbb{C}$) and $\exists\, c,\, d \in \mathbb{N}^*,\ d\, f(c\, x) \in \mathbb{Z}[[x]]$.
2. *D-finite:* $\exists\, L \in \mathbb{Z}[x][\frac{d}{dx}],\ L \neq 0$, such that $L(f) = 0$.

The proof of 1 is straightforward but the proof of 2 is much deeper. It was first proved as a corollary of Theorem 23. Then, more elementary and direct proofs were given [25]. The last step was to find algorithms to compute the differential equation L. An overview on these topics can be found in [24] where the most efficient known algorithms are given.

Remark 7. The (non-equivalent) absolute values on the field $\mathbb{Q}$ are known to be the classical one and the p-adic ones (p prime). For each, $\mathbb{Q}$ has a completion, namely $\mathbb{R}$ and $\mathbb{Q}_p$. The completion $\mathbb{Z}_p$ of $\mathbb{Z}$ coincides with the unit disc of $\mathbb{Q}_p$. Property 6.1 means firstly that the function f has a non-zero radius of convergence in all these completions and secondly that, for almost all prime p (i.e. for p not dividing d or c), it belongs to $\mathbb{Z}_p[[x]]$ and hence is a bounded (by 1) function on $\mathbb{Z}_p$.

Let A be any (finite) ring. A power series $\sum a_s x^s \in A[[x]]$ is said to be *p-automatic* if there is a deterministic finite p-automaton[3] that returns a_n when inputting the digit sequence of n in base p. For people not interested in automata, let's just say that a power series $f \in \mathbb{Z}/p^h\mathbb{Z}[[x]]$ is p-automatic if and only if it is, in some sense, algebraic over $\mathbb{Z}/p^h\mathbb{Z}(x)$ (see Theorems 40 and 41 for precise statements).

Proposition 8. Any DRF f is *globally automatic*: for almost all p and for all h, $f \mod p^h$ is p-automatic.

For f in $\mathcal{D}_2$, this is proved in [7] and is easily generalised. It is connected with Property *4* of Proposition **4**.

2.3 Conjectures

The properties of Proposition 6 could be characteristic of DRFs; we will give a definition and set a working hypothesis.

Definition 9. Any power series in $\mathbb{Q}[[x]]$ satisfying Properties 6-1 and 6-2 will be said to be a *pseudo-diagonal*.

Conjecture 10. Every pseudo-diagonal is a DRF.

In Section 4, this conjecture will be proved to be true under a large number of circumstances, including examples introduced by physicists and combinatorialists. However, at the end of the section, we will see that it remains widely open for some (hypergeometric) functions.

Remark 11. It is natural to imagine seemingly more general conjectures, for instance, replacing the field $\mathbb{Q}$ by any number field or replacing the field $\mathbb{Q}(x)$ by any finite extension, i.e. working on a curve instead of the projective line. In [10], both extensions are shown to be consequences of Conjecture 10.

Proposition 8 suggests a weak form of the conjecture.

Conjecture 12 (weak)**.** Every pseudo-diagonal is globally automatic.

Assuming the seemingly out of reach geometric Conjecture 19, we will give an almost-proof of Conjecture 12. To have an actual proof, a better understanding of the p-adic cohomology for families of varieties over a finite field is still needed.

Conjecture 10 is reminiscent of a more classical one.

Conjecture 13 (Grothendieck)**.** If, for almost all p, $L \in \mathbb{Z}[x][\frac{d}{dx}]$ has a complete set of solutions in $\mathbb{F}_p[x]$ then all its solutions are algebraic over $\mathbb{Q}(x)$.

A first difference between Conjectures 10 and 13 is that Conjecture 13 needs to know *all* solutions of the differential equation L and hence, when proved for a particular set S of differential equations (for instance Picard–Fuchs ones as Katz did) then it is not proved for all the sub-equations (i.e. right factors) of the $L \in S$.

On the other side, Conjecture 10 requires a solution in $\mathbb{Z}_p[[x]]$ and not only in $\mathbb{F}_p[x]$, but it can be seen that both requirements are almost equivalent.

There is another deep but less visible difference between the two conjectures. Actually, in Conjecture 10, the underlying differential equation has singularities involving logarithms (with non-finite local monodromy); they cannot be cancelled by any algebraic pullback as is done for Conjecture 13.

3 A p-automaton $\mathcal{M} = \{M, m_0, e_i, s\}$ is built from a finite set M, an input element $m_0 \in M$, maps $\{e_i : M \to M\}_{0 \le i < p}$ satisfying $e_0(m_0) = m_0$ and an output map $s : M \to A$.
When inputting $n = n_h p^h + \cdots + n_1 p + n_0$ $(0 \le n_i < p)$, it outputs $s \circ e_{n_0} \circ e_{n_1} \circ \cdots \circ e_{n_h}(m_0) \in A$.

3 Integral representations

3.1 G-functions

Definition 14. A G-function is a $f \in \mathbb{Q}[[x]]$ such that:
1. f is D-finite;
2. f has a non-zero radius of convergence in $\mathbb{C}$; and
3. its radii of convergence $\mathtt{Ray}_p(f)$ in $\mathbb{Q}_p$ satisfy

$$\prod_{p \text{ prime}} \mathtt{Ray}_p^{\leq 1}(f) > 0,$$

where $\mathtt{Ray}_p^{\leq 1}(f) = \min(1, \mathtt{Ray}_p(f))$.

Remark 15. Any pseudo-diagonal f is a G-function because, by globally boundedness, it satisfies the condition

$$(\forall p)\ \mathtt{Ray}_p(f) > 0\,, \quad (\text{faa}\, p)\ \mathtt{Ray}_p^{\leq 1}(f) = 1. \tag{2}$$

Definition 16 (Galočkin condition)**.** For $L \in \mathbb{Z}[x][\frac{d}{dx}]$, one can define, for each prime p, a *generic radius of convergence* $\mathtt{Ray}_p(L)$. Roughly speaking, it is the minimum of the radii of convergence of solutions of L near any point of the unit disc (of any extension of $\mathbb{Q}_p$). Galočkin's condition requires that $\prod_p \mathtt{Ray}_p(L) > 0$.

Now, we have a marvellous theorem, which was first proved by D. and G. Chudnowsky [12] (actually, the proof contained a mistake that was later corrected by Y. André). It is yet more remarkable when it is completed, as here, with a previous result of N. Katz [21]. As far as we know, it is the only theorem allowing one to go from properties of a single solution to properties of the (minimal) differential equation.

Theorem 17 (Chudnovsky)**.** Let f be a G-function. The *minimal* $L \in \mathbb{Z}[x][\frac{d}{dx}]$ such that $L(f) = 0$ fulfils Galočkin's condition. In particular, L only has regular singular points with rational exponents.

3.2 Geometric conjecture

The following conjecture is classical even if it is not precise.

Conjecture 18. G-functions come from geometry.

A weak form of this conjecture says that any G-function satisfies Condition (2) of remark 15. Such results seem entirely out of reach at the moment.

We will use the following avatar of Conjecture 18.[4]

Conjecture 19. Any pseudo-diagonal power series has an integral representation

$$f(x) = \int_\gamma F(x,\, x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_n,$$

where
- F is an algebraic function of $x,\, x_1,\, \ldots,\, x_n$; and
- γ is an n-cycle on the variety V of F ("not depending" on x).

In other words $f(x) = \int_\gamma \omega(x)$, where
- ω is an n-differential form on a smooth quasi-projective variety $V \to S$ of relative dimension n, defined over an open set $S \in \mathbb{P}_1$ that contains a punctured disc centred on 0; and
- γ is an n-cycle on V (actually on V_x and not depending on x up to homotopy).

3.3 Integral representations for DRFs

Insisting on giving a special role to one of the variables, we give it the index 0, hence considering DFRs in $n+1$ variables.

For $F = \frac{P}{Q} \in \mathbb{Q}[[x_0, \ldots, x_n]]$, a repeated application of the residue theorem easily shows that:

$$\operatorname{Diag}\big(F\big)(x) = \frac{1}{(2i\pi)^n} \int_\gamma \omega(x), \tag{3}$$

- $\gamma = \prod_{i=1}^n \gamma_i$, $\gamma_i = \{|x_i| = \varepsilon\}^{\circlearrowleft}$ (*evanescent cycle*); and
- $\omega = F\Big(\frac{x}{x_1 \cdots x_n},\, x_1, \ldots, x_n\Big)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$.

In other words, ω is an n-differential form on $V \to S$ with

$$\begin{aligned} V \quad &: \{(x_0, \ldots, x_n)\,;\ x_0 \cdots x_n\, Q(x_0, \ldots, x_n) \neq 0\} \subset \mathbb{A}_S^{n+1}, \\ V \to S \ &: (x_0, \ldots, x_n) \mapsto x = x_0 \cdots x_n \qquad \text{(onto map)}. \end{aligned}$$

Remark 20. As Q is invertible in $\mathbb{Q}[[x_0, \ldots, x_n]]$, one has $Q(0, \ldots, 0) \neq 0$ and, in particular, $\exists \eta$ such that $Q(x_0, \ldots, x_n) \neq 0$ for $|x_i| < \eta$. Now, for $\varepsilon < \eta$ and $0 < |x| \leq \varepsilon^{n+1}$, $\gamma \subset V$ because, on γ,

$$0 < |x_0| = \left|\frac{x}{x_1 \cdots x_n}\right| = |x|\varepsilon^{-n} \leq \varepsilon < \eta\,.$$

3.4 De Rham cohomology

Sections 3.4 and 3.5 briefly summarise a rather abstract and multiform theory. The reader ready to accept finiteness properties of spaces $H^n_{\mathrm{DR}}(V)$ and $\mathcal{H}^n_{\mathrm{DR}}(V/S)$ can skip these sections. He will find the down-to-earth objects we are concerned with in Remarks 24 and 26. Actually, these remarks generally contain all we need.

Let V be an n-dimensional smooth algebraic complex variety, let V_{an} be the n-dimensional underlying analytic complex variety and let $V_\mathbb{R}$ be the $2n$-dimensional underlying real differential variety.

Each of them is endowed with the sheaves (of $\mathbb{C}$-vector spaces) of m-differential forms ($\Omega^m = \bigwedge^m \Omega^1$):

$$\Omega^m(V) \overset{\text{def}}{=} \{\textit{algebraic } m\text{-differential forms on } V\},$$

$$\Omega^m(V_{an}) \overset{\text{def}}{=} \{\textit{holomorphic } m\text{-differential forms on } V_{an}\},$$

$$\Omega^m(V_\mathbb{R}) \overset{\text{def}}{=} \{C^\infty\ m\text{-differential forms on } V_\mathbb{R}\}.$$

Definition 21. The *de Rham cohomology of* $V_\mathbb{R}$ is the cohomology $H^i_{\mathrm{DR}}(V_\mathbb{R})$ of the global section complex:

$$0 \longrightarrow \Gamma\Omega^0(V_\mathbb{R}) \overset{d}{\longrightarrow} \Gamma\Omega^1(V_\mathbb{R}) \longrightarrow \cdots \longrightarrow \Gamma\Omega^{2n}(V_\mathbb{R}) \longrightarrow 0.$$

By Poincaré's lemma, this complex is a resolution of the constant sheaf $\mathbb{C}$. Hence, $H^i_{\mathrm{DR}}(V_\mathbb{R}) = H^i(V(\mathbb{C}), \mathbb{C})$ and dimensional finiteness of $H^i_{\mathrm{DR}}(V_\mathbb{R})$ follows.

To avoid considering global sections, it is possible to replace the simple complex of vector spaces by a complex of

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4 *Why limit ourself to the "central dimension" n?*
Actually, introducing more general cohomology spaces would not be an actual generalisation in view of the following statements.
Theorem (Lefschetz). *Let V be an n-dimensional complex projective algebraic variety.*
- *For W a hyperplane section such that $V - W$ is smooth,*

$$H^i_{\mathrm{DR}}(V) \to H^i_{\mathrm{DR}}(W) \text{ is } \begin{cases} \textit{an isomorphism for} & i < n-1, \\ \textit{injective for} & i = n-1. \end{cases}$$

- *For* $r \geq 1$, *one has* $H^{n-r}_{\mathrm{DR}}(V) \overset{\sim}{\to} H^{n+r}_{\mathrm{DR}}(V)$.

sheaves but at the cost of looking at its *hypercohomology*. Sheaves of C^∞-differential forms being fine, we have

$$H^i_{\mathrm{DR}}(V_{\mathbb{R}}) = \mathbb{H}^i(\Omega^0(V_{\mathbb{R}}) \to \cdots \to \Omega^{2n}(V_{\mathbb{R}}))\ .$$

This leads us to give the following definitions.

Definition 22. The *analytic and algebraic de Rham cohomologies* are given by

$$H^i_{\mathrm{DR}}(V_{an}) = \mathbb{H}^i(\Omega^0(V_{an}) \to \cdots \to \Omega^n(V_{an}))\ ,$$
$$H^i_{\mathrm{DR}}(V) = \mathbb{H}^i(\Omega^0(V) \to \cdots \to \Omega^n(V))\ .$$

There is an analytic Poincaré lemma such that:

$$H^i_{\mathrm{DR}}(V_{an}) = H^i(V(\mathbb{C}), \mathbb{C}) = H^i_{\mathrm{DR}}(V_{\mathbb{R}})$$

and $H^i_{\mathrm{DR}}(V_{an})$ is finite dimensional. But there is no algebraic Poincaré lemma. To obtain dimensional finiteness of $H^i_{\mathrm{DR}}(V)$, we have to use a much deeper statement.

Theorem 23 (Grothendieck comparison theorem)**.** The natural map $H^i_{\mathrm{DR}}(V) \to H^i_{\mathrm{DR}}(V_{an})$ is an isomorphism.

Remark 24. In particular, when V is an *affine variety*,

$$H^n(V(\mathbb{C}), \mathbb{C}) = H^n_{\mathrm{DR}}(V) = \Omega^n(V)/d(\Omega^{n-1}(V))\ .$$

3.5 Relative de Rham cohomology

Actually, we are not looking at one variety but at a family of varieties. However, definitions can be made in the same way by just replacing cohomology with *higher direct images* ([18] III-8). More precisely, let $f : V \to S$ be a smooth morphism of smooth algebraic varieties over $\mathbb{Q}$ and let $\Omega^m(V/S)$ be the algebraic S-differential forms on V ([18] II-8) enjoying the following characteristic property:

$$f^*\big(\Omega^1(S)\big) \longrightarrow \Omega^1(V) \longrightarrow \Omega^1(V/S) \longrightarrow 0\ .$$

Then, the relative de Rham cohomology is defined by

$$\mathcal{H}^i_{\mathrm{DR}}(V/S) \overset{\mathrm{def}}{=} \mathbf{R}^i f_*(\Omega^\bullet(V/S))\ .$$

Proposition 25 ([22])**.** *$\mathcal{H}^i_{\mathrm{DR}}(V/S)$ is a sheaf on S endowed with an integrable connection called the Gauss-Manin connection.*

Remark 26. Roughly speaking, it is a differentiation under the integral sign. To compute $\frac{d}{dx}\int\omega$ with $\omega = f(x, x_1, \ldots, x_\ell)$ $dx_1 \wedge \cdots \wedge dx_n$, where variables $x_{n+1}, \ldots, x_\ell$ are linked by $\ell - n$ relations $g_j(x, x_1, \ldots, x_\ell) = 0$, one assumes $\frac{d}{dx}(x_i) = \frac{d}{dx}(dx_i) = 0$ for $1 \le i \le n$ and computes the other derivatives $\frac{d}{dx}(x_i)$ $(i > n)$ by means of the relations $g_j = 0$. The point is that the value so computed for $\frac{d}{dx}(\omega)$ does not depend on the choice of the "independent" variables $x_1, \ldots, x_n$ up to an *exact differential form* (i.e. it lies in $d(\Omega^{n-1}(V))$ and so disappears when integrating on a cycle γ).

Now, $\mathcal{H}^n_{\mathrm{DR}}(V/S) \otimes \mathbb{Q}(x)$ is a $\mathbb{Q}(x)$ finite dimensional vector space endowed with an action of the derivative $\frac{d}{dx}$. The *solutions* of this *module with connection* are given by $\int_\gamma$ for the various cycles γ in V.

For a given differential form $\omega \in \Omega^n(V)$, there is $L_\omega \in \mathbb{Z}[x][\frac{d}{dx}]$ such that $L_\omega(\omega) \in d(\Omega^{n-1}(V))$. The *periods* $\int_\gamma \omega(x)$ are solutions of L_ω (over S).

The *minimal equation* of $\int_\gamma \omega(x)$ is the minimal monic $L_{\omega,\gamma} \in \mathbb{Q}[x][\frac{d}{dx}]$ such that $L_{\omega,\gamma}(\int_\gamma \omega(x)) = 0$.

4 Toward a proof of Conjecture 10

4.1 To be or not to be evanescent

Let f be pseudo-diagonal. By Conjecture 19, it has an integral representation $f(x) = \int_\gamma \omega(x)$ for $\omega \in \Omega^n(V/S)$. The function f being defined on a punctured disc centred on 0, S contains this punctured disc. By *Hironaka's theorem on the resolution of singularities* [19], we can extend S to 0 in such a way that V_0 is a normal crossings divisor. By the *semi-stable reduction theorem* [23], up to ramifying x and replacing $\mathbb{Q}$ by some finite extension, one can even suppose V_0 to be reduced.

Practically, this means we can choose, for each point P in V_0, *local coordinates* $x_0, x_1 \ldots, x_n$ such that, near P, the "equation" of V is $x_0\, x_1 \cdots x_r = x \quad (0 \le r \le n)$, i.e. $x_i = 0$ $(0 \le i \le r)$ are the (local) equations of divisors contained in V_0.

Now, the integral representation of f can be written

$$f(x) = \int_\gamma F(x_0, \ldots, x_n)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{x_r} \wedge dx_{r+1} \wedge \cdots \wedge dx_n,$$

where F is an algebraic function.

Remark 27. Coefficients of F are algebraic numbers. A Galois argument shows that they can be taken in $\mathbb{Q}$.

Remark 28. The differential ω could have multiple poles along some of the divisors. An easy but rather tedious computation shows that these multiple poles can be eliminated by adding an exact differential form to ω.

Finally, when γ is (homotopic to) the evanescent cycle (see 3.3) around P, one finds

$$f(x) = \mathrm{Diag}\,(x_{r+1} \cdots x_n\, F(x\, x_{r+1} \cdots x_n, x_1, \ldots, x_n))\ .$$

It remains to decide wether this is the case. For that purpose, we have to introduce filtrations.

4.2 Filtrations

The space $\mathcal{H}^n_{\mathrm{DR}}(V/S)$ is filtered by the order of poles along V_0. ω has weight at least $n + k$ if, near any $P \in V_0$, it can be written:

$$\omega = F(x_0, \ldots, x_n)\, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_k}{x_k} \wedge dx_{k+1} \wedge \cdots \wedge dx_n$$

for some algebraic F with $F(0, \ldots, 0) \neq 0$.

The space $\mathcal{S}$ of solutions (near 0) of the Picard–Fuchs equation L_ω is endowed with the so-called *monodromy filtration* that we now define. Let T be the monodromy operator (turning once anticlockwise around 0) and let $N \overset{\mathrm{def}}{=} T - 1$. Then, $N(\log x) = 2i\pi$ and $N(f) = 0$ for any analytic function f. Then, N, acting on $\mathcal{S}$, is a nilpotent operator. The monodromy filtration is entirely determined by asking the operator N to decrease the weight by 2. More explicitly, when $f \in \mathcal{S}$ is associated to exactly k solutions involving logarithms, i.e.

$$f\ ,\quad \{f\ \log(x) + \cdots\}\ , \ldots ,\quad \{f\ \log^k(x) + \cdots\},$$

then these solutions have the following weights

$$-k\ ,\quad -k+2 \quad ,\ldots, \quad +k.$$

The key point is the following statement:

Theorem 29 ([27])**.** *The two filtrations on $\mathcal{H}^n_{\mathrm{DR}}(V/S)$ and $\mathcal{S}$ just defined are set in duality through $(\gamma, \omega) \to \int_\gamma \omega(x)$.*

$$M_k\,(\mathcal{H}^n_{\mathrm{DR}}(V/S)) = \mathtt{Ann}\ M_{n-k+1}\ (\mathcal{S}).$$

Corollary 30 ([11]). Solutions of the Picard–Fuchs equation L_ω with minimum weight (i.e. $-n$) are DRFs.

In particular, minimal weighted solutions of any Picard–Fuchs equation are globally bounded. In some respect, this looks like a p-adic analogue to the Deligne weight-monodromy conjecture [14].

4.3 The case of hypergeometric functions

Definition 31. The Pochhammer symbol is defined by

$$(a)_0 = 1\,, \quad (a)_s = a(a+1)(a+2)\dots(a+s-1), \quad (s \geq 1)$$

and the (generalised) hypergeometric functions are

$${}_{n+1}F_n(\mathbf{a};\mathbf{b};x) = \sum_{s=0}^{\infty} \frac{(a_1)_s \dots (a_{n+1})_s}{(b_1)_s \dots (b_n)_s} \frac{x^s}{s!},$$

where $\mathbf{a} = (a_1, \dots, a_{n+1})$ and $\mathbf{b} = (b_1, \dots, b_n)$ are in $\mathbb{Q}^n$.

They are D-finite, satisfying $L_{\mathbf{a};\mathbf{b}}(f) = 0$ with

$$L_{\mathbf{a};\mathbf{b}} = x \prod_{i=1}^{n+1} \left(x\frac{d}{dx} + a_i \right) - x\frac{d}{dx} \prod_{i=1}^{n} \left(x\frac{d}{dx} + b_i - 1 \right).$$

An integral representation can be given recursively:[5]

$${}_1F_0(a_1;;x) = (1-x)^{a_1}\,, \; {}_{n+1}F_n\,(\mathbf{a}, a_{n+1}; \mathbf{b}, b_n; x) =$$

$$\text{cste} \int_0^1 t^{a_{n+1}-1}(1-t)^{b_n - a_{n+1}-1} \;{}_nF_{n-1}\,(\mathbf{a};\mathbf{b};t\,x)\,dt$$

(the integration path seems not to be a cycle but it is classical to close it by coming back from 1 to 0 on the other side of the cut).

Then, it is a pseudo-diagonal if and only if it is globally bounded.

Remark 32. For hypergeometric functions ${}_{n+1}F_n$, the relative dimension is n and the differential equation is of order $n+1$. A solution f of the equation is of minimum weight, namely $-n$, if and only if there is a solution $f \log^n x + \cdots$. In that case, f is the unique analytic solution near 0 and the monodromy has maximum nilpotent order, namely $N^n \neq 0$. This is exactly the MUM (Maximum Unipotent Monodromy) condition required to be a Calabi-Yau differential equation. It seems likely that the convenient condition should be a "solution with minimum weight" instead of MUM. Unfortunately, it is much harder to check the minimum weight condition than the MUM condition because one almost always knows the order of the differential equation but the relative dimension of the integral representation is rarely known.

Now, it is easy to check that the monodromy weight ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is minus the number of integers amongst the b_i. Hence, Corollary 30 says that ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is a DRF when $\mathbf{b} = (1,\dots,1)$. But this can be checked straightforwardly:

$${}_{n+1}F_n(\mathbf{a}; 1, \dots, 1; x) = (1-x)^{a_1} \star \cdots \star (1-x)^{a_{n+1}}\,.$$

The following theorem proves Conjecture 10 for functions of weight 0.

Theorem 33 ([3]). Let a_i and b_i be in $\frac{1}{N}\mathbb{Z}$ and $b_i \notin \mathbb{Z}$. Then, the following conditions are equivalent:

1. ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is globally bounded and of weight 0.
2. $(\forall k,\ (k,N) = 1)$, $\exp(2i\pi\,k\,a_i)$ and $\exp(2i\pi\,k\,b_i)$ are intertwined on the trigonometric circle.
3. The monodromy group of $L_{\mathbf{a};\mathbf{b}}$ is finite.
4. The function ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is algebraic over $\mathbb{Q}(x)$.

In the general case (weight $k \in]-n, 0[$), it is still easy to decide whether ${}_{n+1}F_n(\mathbf{a};\mathbf{b};x)$ is globally bounded ([9]). We know many examples. The older one is ${}_3F_2(\frac{1}{9},\frac{4}{9},\frac{5}{9};\frac{1}{3},1;x)$ but there are a lot of others: $(\frac{1}{7},\frac{2}{7},\frac{4}{7};\frac{1}{2})$, $(\frac{1}{11},\frac{2}{11},\frac{6}{11};\frac{1}{2})$, etc. Moreover, one knows several globally bounded ${}_4F_3$ with weight -2 or -1.[6]

Now, the open question is to decide whether or not these pseudo-diagonals are actually DRFs.

5 A quasi-proof of Conjecture 12

5.1 Proof overview

In this section, p will be a fixed prime.

The p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ can be extended to $\mathbb{Q}_p(x)$ by setting

$$\left| \frac{\sum a_i x^i}{\sum b_i x^i} \right|_p \overset{\text{def}}{=} \frac{\max |a_i|}{\max |b_i|}\,.$$

This absolute value is usually called the "*Gauss norm*". Let E_p be the completion of $\mathbb{Q}_p(x)$ for $|\cdot|_p$.

Definition 34. An *E_p-differential module $\mathcal{H}$* is a finite dimensional E_p-vector space endowed with a $\mathbb{Q}_p$-linear action of $\frac{d}{dx}$ satisfying, for $a \in E_p$ and $m \in \mathcal{H}$,

$$\frac{d}{dx}(am) = \frac{d}{dx}(a)m + a\frac{d}{dx}(m)\,.$$

A morphism $\theta : M \to N$ of E_p-differential modules is a morphism of E_p-modules such that $\theta(\frac{d}{dx}m) = \frac{d}{dx}\theta(m)$.

For instance, the E_p-differential module $E_p[\frac{d}{dx}]/E_p[\frac{d}{dx}].L$ is associated to each $L \in E_p[\frac{d}{dx}]$. The cyclic base theorem says that this is an equivalence of categories.

Definition 35. Let $\mathcal{H}$ be an E_p-differential module. Its *Frobenius* $\mathcal{H}^\varphi$ is its inverse image by the change of variable $x \to x^p$. When the E_p-differential modules $\mathcal{H}$ and $\mathcal{H}^\varphi$ are isomorphic, then $\mathcal{H}$ will be said to *have a Frobenius*. If G is the

5 *Another integral representation for* ${}_3F_2$: let $N \geq 2$, let V_x be defined by equations

$$x_1^N + y_1^N + 1 = 0\,,\; x_2^N + y_2^N + 1 = 0\,,\; x_3^N + y_3^N + 1 = 0\,,$$
$$x_1x_2x_3 = x$$

(hence of relative dimension 2) and let

$$\omega(x) = x_1^p y_1^q x_2^r y_2^s y_3^t \frac{dx_1}{x_1} \frac{dx_2}{x_2}\,.$$

Then, for a suitable cycle γ, one has

$$\int_\gamma \omega(x) = \text{cste}\;{}_3F_2(\frac{-t}{N}, \frac{-p-q}{N}, \frac{-r-s}{N}; \frac{N-r}{N}, \frac{N-p}{N}; x^N)\,.$$

In this setting, the fiber V_0 is the union of three families of N divisors with almost normal crossing. For instance, the divisors of the first family have the following equations:

$$x_1 = 0\,,\; y_1^N + 1 = 0\,,\; x_2^N + y_2^N + 1 = 0\,,\; x_3^N + y_3^N + 1 = 0\,.$$

6 Given an example, one can get others by "permutation": $a_i \to k\,a_i$, $b_i \to k\,b_i$ $(k,N) = 1$. The number of distinct examples so obtained is difficult to foresee:
$(\frac{2}{9},\frac{4}{9},\frac{5}{9};\frac{1}{3},1) \to (\frac{2}{9},\frac{5}{9},\frac{7}{9};\frac{2}{3},1) \to (\frac{1}{9},\frac{7}{9},\frac{8}{9};\frac{1}{3},1) \to (\frac{4}{9},\frac{5}{9},\frac{8}{9};\frac{2}{3},1)$
but $(\frac{1}{9},\frac{4}{9},\frac{7}{9};\frac{1}{3},1) \;\to\; (\frac{2}{9},\frac{5}{9},\frac{8}{9};\frac{2}{3},1)$. Let's notice that when listing such examples, one can limit oneself to the *primitive* case, excluding Hadamard products of simpler ones.

matrix of $\frac{d}{dx}$ in some base of $\mathcal{H}$, this happens if and only if there exists an invertible matrix $H \in \mathrm{Gl}(E_p)$ such that $px^{p-1}G(x^p)\,H = \frac{d}{dx}(H) + H\,G$.

Roughly speaking, when $L \in E_p[\frac{d}{dx}]$, its "Frobenius" is the $L^\varphi \in E_p[\frac{d}{dx}]$ such that $L^\varphi(f(x^p)) = 0$ as soon as $L(f(x)) = 0$.

Remark 36. Most of the E_p-differential modules $\mathcal{H}$ we will encounter "come from" $\mathbb{Q}(x)$-differential modules $\mathcal{H}_\prime$, namely $\mathcal{H} = E_p \otimes_{\mathbb{Q}(x)} \mathcal{H}_0$. Then, the Frobenius is still defined over $\mathbb{Q}(x) : \mathcal{H}^\varphi = E_p \otimes_{\mathbb{Q}(x)} \mathcal{H}_0^\varphi$ but the isomorphism between $\mathcal{H}$ and $\mathcal{H}^\varphi$ comes very rarely from a $\mathbb{Q}(x)$-morphism.

Now, let $V, S, \omega \in \mathcal{H}^n_{\mathrm{DR}}(V/S)$, γ and $f(x) = \int_\gamma \omega(x)$ be defined as in Conjecture 19. We will use the following notation:

- $V \to V_p \stackrel{\text{def}}{=}$ modulo p reduction of V (it does exist faa p),
- $\mathcal{H}^n_p(V_p/S_p) \stackrel{\text{def}}{=}$ "*p-adic relative cohomology*" of V_p (to be defined).

Then, to deduce Conjecture 12 from Conjecture 19, it would be enough to prove the following points:

1. $\mathcal{H}^n_p(V_p/S_p)$ is an E_p-module with connection and there is an injection $\mathcal{H}^n_p(V_p/S_p) \to \mathcal{H}^n_{\mathrm{DR}}(V/S) \otimes E_p$. By functoriality, the Frobenius $V_p \to V_p$ endows the E_p-differential module $\mathcal{H}^n_p(V_p/S_p)$ with a Frobenius.
2. Let $f \in \mathbb{Z}_p[[x]]$ be a solution of some differential equation $E_p[\frac{d}{dx}]$ with a Frobenius (i.e. the corresponding differential module has a Frobenius). Then, for all $h > 1$, it is p-automatic modulo p^h.

We will explain how to prove Point 1 in Section 5.2 and we will prove Point 2 in Section 5.3.

5.2 Cohomology of varieties in characteristic p

Point 1 just gives a precise sense to the imprecise sentence "any Picard Fuchs equation is endowed with a Frobenius for almost all p". The prototype result of this type is the following.

Theorem 37. Dwork-Katz theory [20]. Let $V \subset \mathbb{P}^{n+1}_{\mathbb{Q}(x)}$ be a *non-singular and in general position* hypersurface with (homogenous) equation

$$F(x, X_0, ..., X_{n+1}) = 0\,, \quad F \in \mathbb{Q}[x][X_0, ..., X_{n+1}]$$

and $R(x) := \texttt{resultant}\{X_i \frac{\partial F}{\partial X_i}\}_{0\le i\le n+1} \neq 0$ (i.e. the $X_i \frac{\partial F}{\partial X_i}$ have no common zero). Then, $\mathcal{H}^n_p(V_p/S_p)$ (and $\mathcal{H}^n_p((\mathbb{P}^{n+1} - V)_p/S_p)$) can be defined, with "good" p being those verifying $|R|_p = 1$, as

- $\mathcal{H}^n_p(V^\emptyset_p/S_p) \sim \mathcal{L}/(D_x\mathcal{L} + \sum_i D_{X_i}\mathcal{L})$,
- $\mathcal{H}^n_p((\mathbb{P}^{n+1} - V)^\emptyset_p/S_p) \sim \mathcal{L}/(\sum_i D_{X_i}\mathcal{L})$,
  where $V^\emptyset = V \bigcap \{X_0 \cdots X_{n+1} \neq 0\}$ and D is the derivative $\frac{d}{dx}$ "twisted" by $e^{-\pi x F}$.

Remark 38. As we know V/S only through Conjecture 18, it is hazardous to presume its properties. Nevertheless, by removing some points if necessary, we can assume V and S to be smooth affine varieties and, using Bertini's theorem, we can also assume the morphism $V \to S$ to be smooth. So, the problem is not really the non-singular property but the hypersurface assumption. Dwork proposed generalisations of Theorem 37 concerning some particular complete intersections but, to this day, we do not have a satisfactorily general statement.

To totally agree with our approach in Section 5.1, a *functorial p*-adic cohomology theory for varieties in characteristic p would be needed. For smooth affine varieties, it does exist, namely the Monsky-Washnitzer theory [26]. The case of general varieties is also solved in [2] using the clever notion of "special modules". Unfortunately, there is not yet any relative theory. Among the numerous difficulties to be overcome, there is the non-validity of Bertini's theorem in characteristic p.

5.3 Frobenius and p-automaticity

Firstly, there is an elementary statement.

Proposition 39. For any solution f of a differential equation with Frobenius, there exist d and $a_i \in E_p$ such that f is a solution of a difference equation in the Frobenius:

$$\sum_{i=0}^{d} a_i(x)\, f(x^{p^i}) = 0\,, \qquad (a_0\, a_d \neq 0). \qquad (C_d)$$

Then, $f(x^p)$ being not far p-adically from f^p, one gets:

Theorem 40 ([8] 6.5)**.** Any $f \in \mathbb{Z}_p[[x]]$ satisfies a relation (C_d) if and only if it is a limit (for the Gauss norm) of functions in $\mathbb{Z}_p[[x]]$ *algebraic* over E_p.

Finally, there is the p-adic generalisation of a classical statement over finite fields.

Theorem 41 ([8] 8.1)**.** When $f \in \mathbb{Z}_p[[x]]$ is *algebraic* over E_p, then, for all $h \ge 1$, it is *p-automatic* modulo p^h.

Summarising the three statements, we get that any $f \in \mathbb{Z}_p[[x]]$ solution of a differential equation with Frobenius (as pseudo-diagonals should be for almost all p) is p-automatic modulo p^h for all h.

Remark 42. The proof does not make any distinction between globally bounded functions and locally bounded functions (i.e. whose denominators of coefficients can contain infinitely many primes, each one with bounded powers). For instance, the function ${}_3F_2\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; \frac{5}{3}, 1; x\right)$ is so proved to be globally automatic.

6 Self-adjoint differential equations

Definition 43. Let $L^* = \sum(-1)^i \frac{d^i}{dx^i} f_i(x)$ be the adjoint of $L = \sum f_i(x)\frac{d^i}{dx^i}$. Then, L is *self-adjoint* when the modules associated to L and L^* are *isomorphic*.

For irreducible L, self-adjointness means existence of M and Q such that $\deg M < \deg L$ and $L\,M = Q\,L^*$.[7]

To be self-adjoint is a rather hard constraint but when computing irreducible factors of differential equations satisfied by DRFs, they (almost) all appear to be self-adjoint! As the (irreducible) minimal differential equation of ${}_3F_2(\frac{1}{9}, \frac{4}{9}, \frac{5}{9}; \frac{1}{3}, 1; x)$ is not self-adjoint, it is natural to ask if that is an indication it is not a DRF.

7 As $\hom(L, L^*) = L \otimes L = \mathrm{ext}^2(L) \oplus \mathrm{sym}^2(L)$, L is self-adjoint if and only if $\mathrm{ext}^2(L)$ or $\mathrm{sym}^2(L)$ has a rational non-zero solution. It can be seen in its Galois differential group.

The first answer is that the DRF ${}_3F_2(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; 1, 1; x)$ also has a non-self-adjoint irreducible differential equation.

To go further, we call on the Deligne-Steenbrink-Zucker theorem, saying that Gauss-Manin connections are "variations of *polarised* mixed Hodge structures". In particular, the associated graded modules are Gauss-Manin connections of smooth and projective varieties. They are self-adjoint by Poincaré duality. Turning back to the Picard–Fuchs differential equation we started with, this means that it is a product of not necessarily irreducible self-adjoint factors. Now, for rather small examples, it is no more surprising that these self-adjoint factors are very often irreducible.

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Gilles Christol was a professor at the University Pierre et Marie Curie in Paris and is now retired. His main research topic was p-adic differential equation theory. He worked on this topic, in particular, with B. Dwork and then with Z. Mebkhout.

Mathematics between Research, Application, and Communication

Gert-Martin Greuel (University of Kaiserslautern, Germany)

Introduction

Possibly more than any other science, mathematics of today finds itself between the conflicting demands of research, application, and communication. A great part of modern mathematics regards itself as searching for inner mathematical structures just for their own sake, only committed to its own axioms and logical conclusions. To do so, neither assumptions nor experience nor applications are needed or desired.

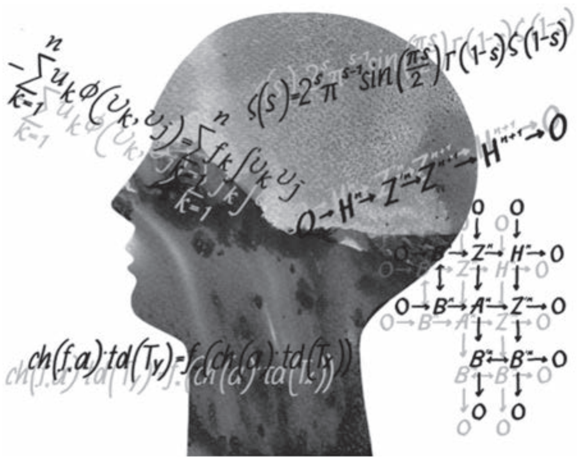
On the other hand, mathematics has become one of the driving forces in scientific progress and moreover, has even become a cornerstone for industrial and economic innovation. However, public opinion stands in strange contrast to this, often displaying a large amount of mathematical ignorance.

Wise Words

Let me introduce my conception regarding research, application, and communication by first quoting some celebrated personalities, and developing my own point of view afterwards.

Research

In this context, I mean by research pure scientific work carried out at universities and research institutes. Trying to explain concrete mathematical research to a non-mathematician is one of the hardest tasks, if at all possible. But it is possible to explain the motivation of a mathematician to do research.



Painting by Boy Müller.

Therefore, my first quote refers to this motivation and from Albert Einstein (physicist 1879–1955) from 1932:

> *"The scientist finds his reward in what Henri Poincaré calls the joy of comprehension, and not in the possibilities of application to which any discovery of his may lead."*

Indeed, the "joy of comprehension" is both motivation and reward at the same time and from my own experience I know that most scientists would fully agree with this statement.

Application

One could hold numerous lectures on the application of mathematics, probably forever. The involvement of mathematics in other sciences, economics, and society is so dynamic that after having demonstrated one application one could immediately continue to lecture on the resulting new applications.

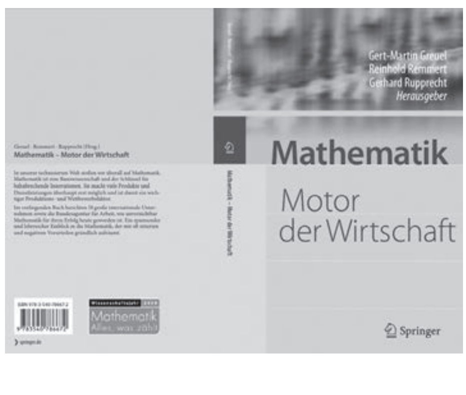
The statement concerning the application of mathematics consists of three quotes, in chronological order:

> *"Mathematics is the language in which the universe is written."* Galileo Galilei (mathematician, physicist, astronomer; 1564–1642).

> *"Mathematical studies are the soul of all industrial progress."* Alexander von Humboldt (natural scientist, explorer; 1769–1859).

> *"Without mathematics one is left in the dark."* Werner von Siemens (inventor, industrialist; 1816–1892)

In 2008, the German Year of Mathematics, the book *Mathematik – Motor der Wirtschaft* (*Mathematics – Motor of the Economy*) was published, giving 19 international enterprises and the German Federal Employment Agency a platform to describe how essential mathematics has become for their success. The main point of this publication was not to demonstrate new mathematics, but to show that, in contrast to a great proportion of the general public, the representatives of economy and industry are well aware of the important role of mathematics.



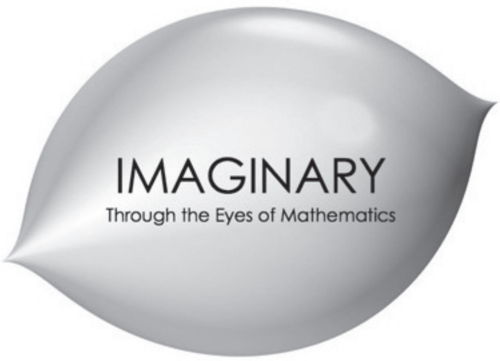
The above statements seem to indicate that there is a direct relation between mathematical research and applications. In fact, recent research directions in geometry are motivated by new developments in theoretical physics, while research in numerical analysis and stochastic is often directed by challenges from various fields of application. On the other hand, the development of an axiomatic foundation of mathematics is guided by trying to formalize mathematical structures in a coherent way and not by the motivation to understand nature or to be useful in the sense of applications. Partly due to this development, it appears that the relationship between research-orientated or pure mathematics on the one hand and application-orientated or applied mathematics on the other hand is not without its strains. Some provocative statements in this article will illustrate this.

Communication

Each of us, whether a mathematician or not, is aware how difficult it is to communicate mathematics. Hans Magnus Enzensberger (German poet and essayist, born 1929) discussed the problem of communicating mathematics on a literary basis in 1999. He writes:

> *"Surely it is an audacious undertaking to attempt to interpret mathematics to a culture distinguished by such profound mathematical ignorance."*

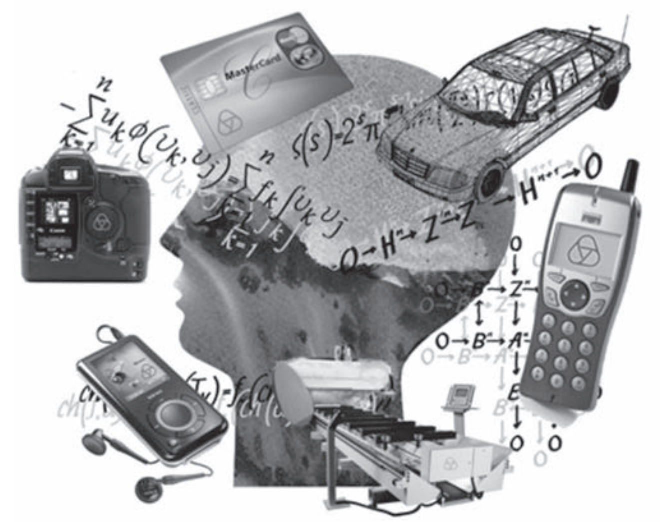
The exhibition *IMAGINARY – Through the Eyes of Mathematics* is one attempt to interpret and communicate mathematics to a broad audience and there exist many other examples of successful communication. Nevertheless, the problem remains and will be discussed later when I shall give some reasons why it is so difficult.



Research and Application – Theory and Practice

Everyday Applications of Mathematics

When we talk about the application of mathematics, we face the fact that mathematics is essential for new and innovative developments in other sciences as well as in the economy and for industry. I do not claim that only mathematics can provide innovation, but it is no exaggeration to claim that mathematics has become a key technology behind almost all common and everyday applications, which includes the design of a car, its electronic components and all security issues, safe data transfer, error correction codes in digital music players and mobile phones, the optimization of logistics in an enterprise and even the design and construction of large production lines. We may say that *Mathematics is the technology of technologies*.

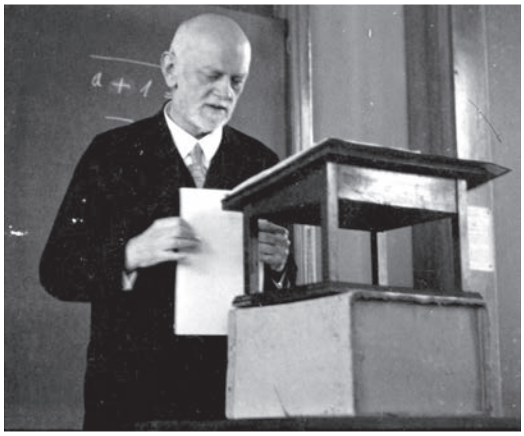


However, since the mathematical kernel of an innovation is in most cases not visible, the relevance of mathematics is either not acknowledged by the general public or simply attributed to the advances of computers.

Hilbert's Vision

It cannot be denied and is a simple and easily verifiable fact that mathematics is applied in our everyday life and that the application of mathematics in industry and the economy is a part of our utilization of nature. However, according to David Hilbert (mathematician; 1862–1943), mathematics, and only mathematics, is the foundation of nature and our culture in a fundamental sense:

> *"The tool implementing the mediation between theory and practice, between thought and observation is mathematics. Mathematics builds the connecting bridges and is constantly enhancing their capabilities. Therefore it happens that our entire contemporary culture, in so far as it rests on intellectual penetration and utilization of nature, finds its foundation in mathematics."*



David Hilbert, Lecture in Göttingen, 1932. Oberwolfach photo collection.

Based on his belief, Hilbert tried to lay the foundation of mathematics on pure axiomatic grounds, and he was convinced that it was possible to prove that they were without contradictions. The inscription on his gravestone in Göttingen expresses this vision with the words: "*We must know – we will know*". Today it is no longer possible to fully adhere to Hilbert's optimism, due to the work of Gödel on mathematical logic showing that the truth of some mathematical theories is not decidable within mathematics. But Hilbert's statement about the *utilization* of nature is truer than ever.

Gravestone of Hilbert: "We must know – we will know". (Photo: HilbertGrab" by Kassandro – Own work. Licensed under CC-SA 3.0 via Wikimedia Commons.)

On the other hand, this is no reason to glorify mathematics or to consider it superior to other sciences. First of all, the utilization of nature is not possible with mathematics alone. Many other sciences contribute, though differently, in the same substantial way. Secondly, the utilization of nature cannot be considered as an absolute value, as we know today. We are all a part of nature and we know that utilization, as necessary as it is, can destroy nature and therefore part of our life.

Pure Versus Applied Mathematics

I use the terms "pure" and "applied" mathematics although it might be better to say "science-driven" and "application-driven" mathematics. In any case, here is a very provocative and certainly arrogant quotation of Godfrey Harold Hardy (mathematician; 1877–1947) from his much quoted essay *A Mathematician's Apology*:

> *"It is undeniable that a good deal of elementary mathematics […] has considerable practical utility. These parts of mathematics are, on the whole, rather dull; they are just the parts which have least aesthetic value. The 'real' mathematicians, the mathematics of Fermat and Euler and Gauss and Abel and Riemann, are almost wholly 'useless'."*

Hardy distinguishes between "elementary" and "real" (in the sense of interesting and deep) mathematics. The essence of his statement has two aspects: elementary mathematics, which can be applied, is unaesthetic and dull, while "real" mathematics is useless.

Godfrey Harold Hardy. (Photo: Ghhardy@72" by Unknown – A mathematician's apology. Licensed under Public domain via Wikimedia Commons

I think Hardy is wrong in both aspects. Of course there exist interesting and dull mathematics. Mathematics is interesting when new ideas and methods prove to be fruitful in either solving difficult problems or in creating new structures for a deeper understanding. Routine development of known methods almost always turns out to be rather dull, and it is true that many applications of mathematics to, say, engineering problems are routine. However, this is not the whole story. Before applying mathematics, one has to find a good mathematical model for a real world problem, and this is often not at all elementary or trivial but a very creative process. This point is completely missing in Hardy's essay. Maybe, because he did not consider this as mathematics at all.

His other claim must also be refuted. Very deep and interesting results of "real" mathematics have become applicable, as we now know. That is, the border between interesting and dull mathematics is not between pure and applicable mathematics, but goes through any subdiscipline of mathematics, independent of whether it is applied or pure.

Applications Cannot Be Predicted – The Lost Innocence

Nowadays we know better than in Hardy's time that his statement about the uselessness of pure mathematics is wrong. The following quote by Hardy concerns his own research field, number theory.

Shortly after Hardy's death the methods and results from number theory became the most important elements for public-key cryptography, which today is used millions of times daily for electronic data transfer in mobile phones and electronic banking. For his claim that deep and interesting mathematics is useless, Hardy calls Gauß and Riemann and also Einstein his witnesses:

> *"The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present at any rate, almost as 'useless' as the theory of numbers. It is the dull and elementary parts of applied mathematics, as it is the dull and elementary parts of pure mathematics, that work for good or ill. Time may change all this. No one foresaw the applications of matrices and groups and other purely mathematical theories to modern physics, and it may be that some of the 'highbrow' applied mathematics will become 'useful' in as unexpected a way."*

The last sentence indicates that Hardy himself was skeptical about his own statements, although he did not really believe in the possibility of "real" mathematics becoming useful. However, the development of GPS, relying on the deep work of Gauss and Riemann on curved spaces and on Einstein's work on relativity, proves the applicability of their "useless" work.

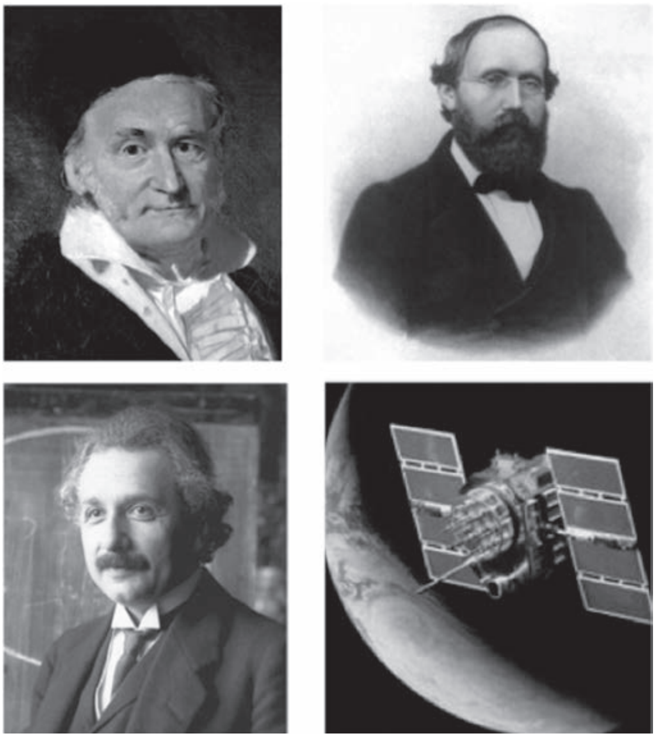
This is not about blaming Hardy because he did not foresee GPS or the use of number theory in cryptography. Nevertheless, his strong statements are somewhat surprising as he was of course aware that, for example, Kepler used the theory of conic sections, a development of Greek mathematics without intended purpose, in order to describe the planetary orbits. So, what was the reason that Hardy insisted on the uselessness of "real" mathematics?

In my opinion we can understand Hardy's strong statements, made in 1940 at the beginning of World War II, only if we know that he was a passionate pacifist. It would have been unbearable for him to see that his own mathematics could be useful for the purpose of war. He was bitterly mistaken.

We all know that nowadays the most sophisticated mathematics, pure and applied, is a decisive factor in the development of modern weapon systems. Without GPS, and hence without the mathematics of Gauss and Riemann, this would have been impossible. Even before that, the atomic bomb marked the first big disillusionment for many scientists regarding the innocence of their work. If there was ever a paradise of innocence with no possibility for mathematics to do 'good or ill' to mankind, it was lost then.

Applicability Versus Quality of Research

History shows, and the statements of Hardy prove this, that it is impossible to predict which theoretical developments in mathematics will become "useful" and will have an impact on important applications in the future. The distinction between pure and applied mathematics is more a distinction between fields than between applicability. Quite often we notice that ideas from pure research, only aiming to explore the structure of mathematical objects and their relations, become the basis for innovative ideas creating whole new branches of economic and industrial applications. Besides number theory for cryptography, I would like to mention logic for formal verification in chip design, algebraic geometry for coding theory, computer algebra for robotics, and combinatorics for optimization applied to logistics, to name just a few.



From Gauss through Riemann and Einstein to GPS. (Photos, clockwise from top left): "Carl Friedrich Gauss" by Gottlieb Biermann – Gauß-Gesellschaft Göttingen e.V. (Foto: A. Wittmann). Licensed under Public domain via Wikimedia Commons, "Georg Friedrich Bernhard Riemann" by unknown. Licensed under Public domain via Wikimedia Commons, "GPS Block IIIA" by USAF. Licensed under Public domain via Wikimedia Commons.)

Although the list of applications of pure mathematics could be easily enlarged, it is also clear that some parts of mathematics are closer to applications than others. These are politically preferred and we can see that more and more national and international programs support only research with a strong focus on applications or even on collaboration with industry.

I think the above examples show that it would be a big mistake, if applicability were to become the main or even the only criterion for judging and supporting mathematics. In this connection I like to formulate the following:

> Thesis. *The value of a fundamental science like mathematics cannot be measured by its applicability but only by its quality.*

History has shown that in the long run, quality is the only criterion that matters and that only high-quality research survives. It is worthwhile to emphasize again that any kind of mathematics, either science driven or application driven, can be of high or low quality.

In view of the above and many more examples, one could argue that we would miss unexpected but important applications by restricting mathematical research to a priori applicable mathematics. This is certainly true, it is, however, not the main reason why I consider it a mistake to judge mathematics by its applicability. My main reason is that it would reduce the mathematical sciences to a useful tool, without a right to understand and to further develop the many thousands of years of cultural achievements of the utmost importance. This leads us to reflect on freedom of research.

Freedom of Research and Responsibility

Freedom of research has many facets. One aspect is that of unconditional research, implying in particular that the scientist himself defines the direction of research.

In mathematics there is an even more fundamental aspect. Today's mathematics is often searching for inner mathematical structures, only committed to its own axioms and logical conclusions and thus keeping it free from any external restriction. This was clearly intended by the creators of modern axiomatic mathematics. Georg Cantor (mathematician; 1845–1918), the originator of set theory, proclaims:

> *"The nature of mathematics is its freedom"*

and David Hilbert considers this freedom as a paradise:

> *"Nobody shall expel us from the paradise created for us by Cantor".*

This kind of freedom was emphasized also by Bourbaki who clearly believed that the formal axiomatic method is a better preparation for new interpretations of nature, at least in physics, than any method that tries to derive mathematics from experimental truths. In this connection I like to present a very interesting but little known document from the early days of Oberwolfach after World War II.

As early as 1946 the first small meetings were held in Oberwolfach at the old hunting lodge "Lorenzenhof". Among the participants were mathematicians like the Frenchman Henri Cartan, whose home country had been an "arch-enemy" for centuries. His family had suffered tremendously under the regime of the National Socialists so that his participation was not at all a matter of course. The first famous guests visiting the Lorenzenhof were Heinz Hopf (a world-famous topologist from Zurich, a German of Jewish descent who had moved from Germany to Switzerland in 1931) and Henri Cartan (the "grand maître" of complex analysis from Paris). It was said that "*Without Hopf and Cartan Oberwolfach would have remained a summer resort for mathematicians, where in a leisurely atmosphere dignified gentlemen would polish classic theories*".

In August 1949 a group of young "wild" Frenchmen met in Oberwolfach who had taken up the cause of totally rewriting mathematics as a whole, based on the axiomatic method and aiming at a new unification. It was a truly bold venture that only young people would dare to take up. Some of their names have become famous, including Henri Cartan, Jean Dieudonné, Jean Pierre Serre, Georges Reeb, and René Thom. A photo from that time was only discovered a few years ago. It shows part of the group in the autumn of 1949. Cartan himself could not come, due to the consequences of a car accident.

From left to right: René Thom, Jean Arbault, Jean-Pierre Serre and his wife Josiane, Jean Braconnier and Georges Reeb. (Photo: Oberwolfach photo collection.)

The Gospel According to St Nicolas and the Freedom of Research

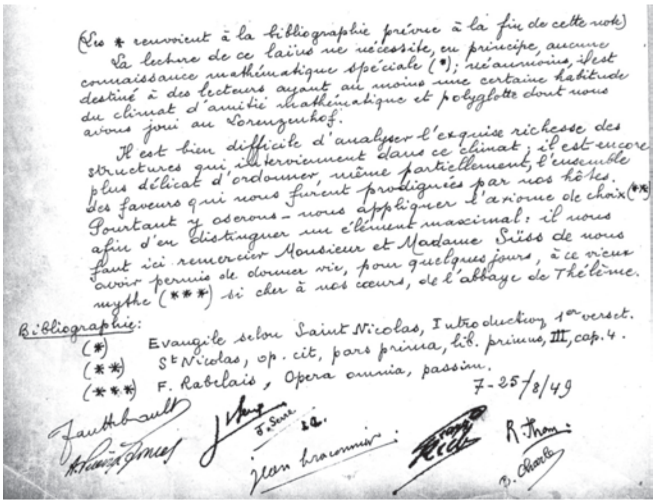
In the first guest book they wrote down the *Evangile selon Saint Nicolas* (see next page), a humorous homage on the Lorenzenhof and its famous Oberwolfach atmosphere, endorsed with mathematical hints. The name *Evangile selon Saint Nicolas* is an allusion to the works of Nicolas Bourbaki, an alias for that group of French mathematicians who wanted to rewrite mathematics entirely from scratch. During that time, almost no one in Germany had heard of Bourbaki. During the Nazi period, the so-called "Deutsche Mathematik" simply missed some important developments in mathematics, for instance in topology, the theory of distributions, and in complex and algebraic geometry. It is one of the most extraordinary achievements of the small Oberwolfach workshops that those mathematicians who stayed in Germany and were not expelled by the Nazis were able to join the world's elite again.

The myth of the Abbaye de Thélème mentioned by the authors of the *Evangile selon Saint Nicolas* refers to the motto of the Abbey of Thélème, a utopic and idealized "anti-monastery" from Rabelais' *Gargantua*: the motto was "Fay ce que voudras" (Do as you please). Even today the Oberwolfach Institute is sometimes compared to an isolated monastery where mathematicians live and work together, only devoted to their science.

Bourbaki has become a synonym for the modern development of mathematics being interested only in the development of its internal structures based on a few basic axioms. This restriction of the scientific objective implies a great freedom from external forces and independence from political and social influences.

On the other hand, there are reasons to question this freedom as an absolute value, because it does also imply, implicitly, freedom from responsibility for the consequences of its research. In this connection we must emphasize that this does not excuse the individual scientist as a human being from his responsibilities. The physicist Max Born wrote in 1963:

> *"Although I never participated in the application of scientific knowledge to any destructive purpose, like the construction of the A-bomb or H-bomb, I feel responsible."*



(Les * renvoient à la bibliographie prévue à la fin de cette note)
La lecture de ce laïus ne nécessite, en principe, aucune connaissance mathématique spéciale (*); néanmoins, il est destiné à des lecteurs ayant au moins une certaine habitude du climat d'amitié mathématique et polyglotte dont nous avons joui au Lorenzenhof.
Il est bien difficile d'analyser l'exquise richesse des structures qui interviennent dans ce climat; il est encore plus délicat d'ordonner, même partiellement, l'ensemble des faveurs qui nous furent prodiguées par nos hôtes. Pourtant y oserons-nous appliquer l'axiome de choix (**) afin d'en distinguer un élément maximal: il nous faut ici remercier Monsieur et Madame Süss de nous avoir permis de donner vie, pour quelques jours, à ce vieux mythe (***) si cher à nos cœurs, de l'abbaye de Thélème.
Bibliographie:
(*) Evangile selon Saint Nicolas, Introduction, 1er verset.
(**) St Nicolas, op. cit, pars prima, lib. primus, III, cap. 4.
(***) F. Rabelais, Opera omnia, passim.
7-25/8/49

Evangile selon Saint Nicolas from the first guest book of Oberwolfach. Translation:
Evangile selon Saint Nicolas: Reading these lines requires, in principle, no special (*) mathematical knowledge, yet they are intended for readers who have at least developed some feeling for the mathematical and multilingual friendly atmosphere, which we have enjoyed on the Lorenzenhof.
It is extremely difficult to analyze the exquisite diversity of structures that brings this atmosphere into existence; it is even much more difficult to classify the favors that were given to us by our hosts, in their entirety or only in part. Nevertheless, we dare here to apply the axiom of choice (**) in order to reward a maximal element: our thanks to Mr. and Mrs. Suss who allowed us for a few days to give life to this old myth (***) of the Abbaye de Thélème, which is close to our hearts so much.
Literature:
(*) Saint Nicholas gospel, introduction, first verse
(**) St. Nicholas, op cit. pars prima, lib. primus, III, Chap. 4
(***) F. Rabelais, Opera omnia, passim

It may be argued that the self-referential character of mathematics, as it appears in the visions of Hilbert and Bourbaki, is, at least partially, responsible for the lack of responsibility. This is emphasized by Egbert Brieskorn (mathematician, 1937–2013) who not only deplores this character but even goes a step further in claiming that this attribute implies the possibility of assuming and misusing power:

> *"The restriction on pure perception of nature by combining experiment and theoretical description by means of mathematical structures is the subjective condition to evolve this science as power. The development of mathematics as self-referential science enforces the possibility to seize power for science as a whole. […] It belongs to the nature of the human being to prepare and to take possession of the reality. We should not feel sorrow about that, however, we should be concerned that the temptations of power is threatening to destroy our humanity."*

The self-referential character appears clearly in Hilbert's and Bourbaki's concept of mathematical structures based on the axiomatic method. This concept was of great influence in the development of mathematics in the twentieth century. It was, however, never without objections and nowadays it is certainly not the driving force anymore. In theoretical mathematics the most influential new ideas arise from a deep interaction with physics, in particular with quantum field theory. Michael Atiyah (mathematician; born 1929) even calls this the "era of quantum mathematics".

Applied mathematics such as numerical analysis or statistics, on the other hand, has always been too heterogeneous to be adequately covered by Bourbaki's approach. It is often driven by challenging problems from real world applications. But I do not see that this fact makes it less vulnerable to the temptations of power, maybe even to the contrary.

Not denying the threat of misuse for any kind of mathematics, I like to point out that freedom of research is a precious gift, related to freedom of thought in an even broader sense.

Mathematicians for example are educated to use their own brains, to doubt any unsubstantiated claim, and not to believe in authority. A mathematical theorem is true not because any person of high standing or of noble birth claims it, but because we can prove it ourselves. In this sense I like to claim:

> Thesis. *Mathematical education can contribute to freedom of thought in a broad sense.*

On the other hand, being aware of the "lost innocence" and the fact that mathematics can be "for good or ill" to mankind, freedom must be accompanied by responsibility. The responsibility for the impact of their work, though not a part of science itself and not easy to recognize, remains the task for each individual mathematician.

> Thesis. *Freedom of research must be guaranteed in mathematics and in other sciences. It has to be defended by scientists, but it must be accompanied by responsibility.*

Mathematical Research – Popularization Versus Communication

Having described some of the process of mathematical research, let me now consider the challenge of communicating mathematics and its research results.

The Popularization of Mathematics is Impossible

I would like to start with a provocative quotation by Reinhold Remmert (mathematician; born 1930) from 2007:

> *We all know that it is not possible to popularize mathematics. To this day, mathematics does not have the status in the public life of our country it deserves, in view of the significance of mathematical science. Lectures exposing its audience to a Babylonian confusion and that are crowded with formulas making its audience deaf and blind, do in no way serve to the promotion of mathematics. Much less do well-intentioned speeches degrading mathematics to enumeration or even pop art. In Gauss' words mathematics is "regina et ancilla", queen and maidservant in one. The "usefulness of use-*

less thinking" might be propagated with good publicity. Insights into real mathematical research can, in my opinion, not be given.

Remmert's statement about the popularization of mathematics has a point. However, we have to distinguish between popularization and communication. While his statement may apply to popularization, it does, in my opinion, not apply to the communication of mathematics. Before I explain why, let me start with the difficulties that we face when trying to communicate mathematics.

Structural Difficulties

First of all we may ask why it is not be possible to communicate mathematics. What is different in mathematics compared to other sciences? One could argue that for any research, regardless of whether it's in physics, chemistry, or biology, very specialized knowledge is required so that the popularization of research on the one hand and profoundness and correctness on the other do not go well together. Nevertheless, due to your own experience you will all have the feeling that mathematics might fall into a special category. In my opinion there are two significant structural reasons why it is so difficult to communicate or even popularize mathematics.

The first reason is that objects in modern mathematics are abstract creations of human thought. I do not wish to enter into a discussion of whether we only discover mathematical objects, which exist independently of our thoughts, or whether these objects are abstractions of human experience. Except for very simple ideas, like natural numbers or elementary geometrical figures, mathematical objects are not perceived, even if one can argue that they are not independent of perception. Objects like e.g. groups, vector spaces, or curved spaces in arbitrary dimension cannot be experienced with our five senses. They need a formal definition, which does not rely on our senses. Gaining an understanding of mathematical objects and relations is only possible after a long time of serious theoretical consideration.

Another reason is that mathematics has developed its own language, more than any other science. This is necessarily a result from the previous point that mathematics cannot be experienced directly. Therefore, each mathematical term needs a precise formal definition. This definition includes further terms that must be defined, and so on, so that finally a cascade of terms and definitions is set up that make a simple explanation impossible. But even in ancient times the abstraction from objects of our perception has always been a decisive part of mathematics, which made it difficult to comprehend. In Euclid's words: "*There is no royal road to geometry*".

The language of mathematics requires an extremely compact presentation, a symbolism that allows replacing pages of written text by a single symbol. The peak of mathematical precision and compact information is a mathematical formula. But mathematical formulas frighten and deter. Stephen Hawking (physicist; born 1942) wrote in 1988: "…*each equation I included in the book would halve the sales*" .

The importance of "closed" mathematical formulas or equations might change in the future, being at least partly replaced by computer programs. However, this will not make communication easier.

These structural reasons support the thesis of Remmert that the nature of mathematical research cannot be popularized. And all mathematicians engaged in research would agree, that it is nearly impossible to feasibly illustrate to a mathematically untrained person the project one is currently working on. Actually, this experience applies not only to mathematically untrained people but even to mathematicians working in a different field.

The Need to Communicate Mathematics

Nonetheless, the statement that insights into the nature of mathematical research are not possible for a non-mathematician is for me hard to accept. Because this also implies quite a lot of resignation. As much as this statement might be true when limited to genuine mathematical research, it is not true when you take into consideration the fact that mathematical research has become a cultural asset of mankind during its development over 6,000 years.

Furthermore, it is my impression that everyone has a feeling for mathematics even if it is developed to different degrees. Each of you who has been around small children would know that already from an early age they take great pleasure in counting and natural numbers and have a quantitative grasp of their surroundings. They often love to solve little calculations. Regrettably, this interest often gets lost during schooling. I would even go so far as to introduce the following:

> Thesis. *In an overall sense, mathematical thinking is, after speech, the most important human faculty. It was this skill especially that helped the human species in the struggle for survival and improved the competitive abilities of societies. I believe that mathematical thinking has a special place in evolution.*

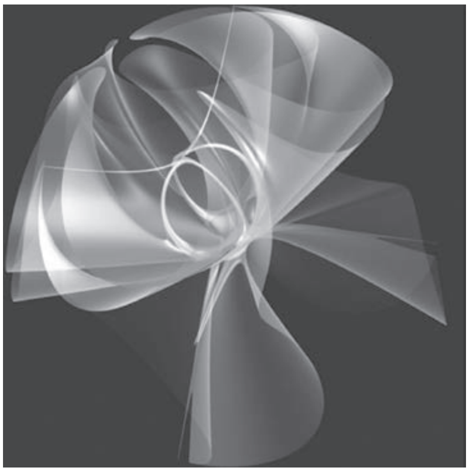
By mathematical thinking I mean analytic and logical thinking in a very broad sense, which is certainly not independent of the ability to speak. Of course, the development of mathematics as a science is a cultural achievement but, in contrast to languages, it developed in a similar way in different societies. We can face the fact that the importance of mathematics for mankind has grown continuously over the centuries, regardless of the cultural and social systems. No modern science is possible without mathematics and societies with highly developed sciences are in general more competitive than others. Attaching this value to mathematics, one must conclude the following:

> Thesis. *Society has the fundamental right to demand an appropriate explanation of mathematics. And it is the duty of mathematicians to face this responsibility.*

However, if mathematicians want to make their science easier to understand it will be at the expense of correctness. And that's a problem for mathematicians. All their professional training is necessarily based on being exact and complete. Mathematicians simply abhor to be inexact or vague. But in order to be understood by society, they will have to be just that. I admit that this remains a continual conflict for every mathematician.

How Can We Raise Public Awareness in Mathematics?

In my experience there are two approaches for raising public interest in mathematics and demonstrating its significance: First, by examples that show the applicability of mathematics, and second, by examples that demonstrate the beauty and elegance of mathematics.



Featherlight, an algebraic surface by Angelika Schwengers (using IMAGINARY's Surfer).

The first approach is certainly the favored one and it is often the only one accepted by politicians. However, we should not underestimate the second approach: it is often much more appealing and even crucial if we wish to get children interested in mathematics.

The elegance of a mathematical proof can really be intellectually fulfilling, e.g. the proof that the square root of 2 is an irrational number, or that there are infinitely many prime numbers. Both proofs can be given in advanced school classes. More accessible and therefore even more suitable for a larger audience is the beauty of geometrical objects. An example of this kind is the mathematical exhibition and communication platform IMAGINARY with its beautiful pictures, see www.imaginary.org.

Text sources

The text is based on the article (containing full references to the quotations): Gert-Martin Greuel: Mathematics between Research, Application and Communication. In: E. Behrends et al. (eds.), *Raising Public Awareness of Mathematics*, Springer-Verlag Berlin Heidelberg (2012), 359–386.

This is the first part of the book *Mathematics Communication for the Future – Mission and Implementation* by Gert-Martin Greuel.

It was first presented at the IMAGINARY panel of the ICM 2014 conference in Seoul 2014. The book is available under the Creative Commons License CC-BY-SA-NC-3.0 and can be downloaded at www.imaginary.org/background-material.

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Pierre Liardet (1943–2014) In Memoriam

Guy Barat (Université d'Aix-Marseille, France), Peter J. Grabner (Graz University of Technology, Austria) and Peter Hellekalek (University of Salzburg, Austria)

1 Introduction

Words and numbers cannot describe a man. The three authors of this paper had a special relationship with Pierre Liardet: as a former PhD student, as collaborators and friends. As such, we will try to give the reader an idea not only of the numerous contributions that Pierre made to number theory and related fields but also of the fine man behind these results.

Pierre Liardet was born in Gardanne, close to Aix-en-Provence, on 20 March 1943. After receiving his baccalauréat en mathématiques et technologie at the Lycée Saint-Éloi in Aix-en-Provence in 1961, he studied mathematics at the university in Marseille, finishing with his doctorate in 1970. Afterwards, he obtained a position as a "Maître Assistant" at the Université de Provence and submitted his Thèse d'État in 1975. Soon afterwards, he became a professor and later "Professeur de première classe" at this university. From 1990 to 2010, he directed the team "Dynamique, Stochastique et Algorithmique" (DSA).

He was an editor of the "Journal de Théorie des Nombres de Bordeaux" and of "Uniform Distribution Theory". In 2005, he organised the biannual international conference on number theory "Journées Arithmétiques". Pierre liked to travel and visited many countries all over the world. This is reflected by the fact that he had coauthors from many countries: Austria, Canada, China, France, Hungary, Japan, the Netherlands, Poland, Romania, USA. He was professor emeritus from 2012 and was still active in research until his last days. He passed away on 29 August 2014.

Even if his interests quickly turned to analysis, Pierre thought as an algebraist, as he was more interested in the structures governing mathematical objects and in the operations on them than in the objects themselves. In this sense, he was an inheritor of Bourbaki's tradition.

2 Algebraic number theory and algebraic geometry

Pierre Liardet wrote his PhD thesis [L2] under the advice of Gérard Rauzy on a subject from algebraic number theory. The main focus of this research was rational transformations that leave certain sets of algebraic numbers stable (see [L3]). One of his results was published in [L1] and concerned a stability property of the set Θ_n of algebraic integers with exactly n conjugates of modulus greater than 1, and all the other conjugates of modulus smaller than 1 (the case $n = 1$ gives the Pisot–Vijayaraghavan numbers).

Theorem 1. Let f be a non-constant rational function defined over an algebraic extension of $\mathbb{Q}$, and E be the complement in

Pierre Liardet 2012

Θ_n of a bounded set. Then

$$\forall \theta \in E : f(\theta) \in \Theta_n \Rightarrow \exists m \in \mathbb{N} : f(x) = \pm x^m.$$

After finishing his doctorate, he continued working in algebraic number theory and algebraic geometry until his Thèse d'État [L6] in 1975, devoted to the stability of algebraic properties of sets of polynomials and rational functions (see [L9, L10]). Two outstanding results from this work were published, namely the disproof of a conjecture of W. Narkiewicz [L4, L8] and the proof of a conjecture of S. Lang. The latter is formulated in terms of intersections of abelian varieties with algebraic curves in [L5]; we give the more elementary formulation from [L7].

Theorem 2. Let Γ_0 be a finitely generated subgroup of $\mathbb{C}^*$ and let $\Gamma = \{x \in \mathbb{C}^* \mid \exists n \in \mathbb{N}, n \neq 0 : x^n \in \Gamma_0\}$. If $P(X, Y) \in \mathbb{C}[X, Y]$ is such that $P(\alpha, \beta) = 0$ for infinitely many $\alpha, \beta \in \Gamma$ then there exist non-zero integers u, v and $a, b \in \Gamma$ such that $P(ax^u, bx^v)$ is identically zero.

After this fruitful period, he turned his interest mostly to ergodic theory, dynamical systems and uniform distribution of sequences.

3 Ergodic theory

Pierre started by reading the most important authors and assimilated the works of Conze, Furstenberg, Kakutani, Keane, Klaus Schmidt and others. He published papers in "pure" ergodic theory between 1978 [L11] and 2000 [L32]. The first paper where Pierre grappled with ergodic theory is [L11]. It is noteworthy that this paper already deals with the main themes of his later research articles.

Let X be a compact metrisable space, G a compact metrisable group and $\varphi\colon X \to G$ a continuous map. For T a transformation on X, one can define the skew product $(X\square_\varphi G, T_\varphi)$ by

$$T_\varphi\colon X\times G \to X\times G, \qquad T_\varphi(x,g) = (Tx, g\cdot\varphi(x))$$

and ask about relations between properties of (X,T) and properties of $(X\square_\varphi G, T_\varphi)$. Skew products furnish an inexhaustible source of examples, allow one to describe by isomorphism several flows associated to sequences (there are numerous such examples in Pierre's work, for instance in [L38]) and illustrate the general idea that a good way to understand mathematical objects is to describe how to act on them. For those reasons, Pierre had a continuous and deep interest in this type of product systems. In this first paper, beginning with considerations from uniform distribution theory, Pierre gave characterisations of ergodicity and weak mixing of large families of skew products, extending results of Furstenberg and Conze; he was also able to mould a theorem of Veech in a more precise form.

The use of ergodicity of certain types of skew products was one of Pierre's central techniques in proving results for certain sequences like *irregularities of distribution* (see Section 4). Usually, these sequences were shown to be the orbits of points in some other space under an *odometer* (adding machine transformation, see Section 5), which translates into the study of a group rotation. Further investigations on skew products can be found in [L13] and [L32]. K. Schmidt (probably) introduced him to $\mathbb{Z}^d$-actions, on which he published [L24] and [L26]. His deep interest on spectral approaches originates from here.

Paper [L27] deals with the speed of convergence in Birkhoff's ergodic theorem. Let T be a homeomorphism of the compact metric space X and $C_0(X)$ be the space of all real-valued continuous functions f having zero integral with respect to a fixed T-invariant aperiodic measure λ. Then there exists a G_δ set in $C_0(X)$ such that distributions of the random variables $\frac{1}{c_n}\sum_{0\le j<n} f\circ T^j$ are dense in the set of all probability measures on the real line, where $c_n \uparrow \infty$ and $c_n/n \to 0$. The second part of the paper is devoted to irrational rotations R_α on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Functions defined on $\mathbb{T}$ are often called *cocycles* in this context. Moreover, a cocycle F is a *coboundary* if there exists G such that $F = G - G\circ R_\alpha$. (Hence, coboundaries F make the sums $\sum F\circ T^n$ telescopic.) The last theorem in [L27] shows that if α has bounded partial quotients then there exists a dense G_δ subset of the set of absolutely continuous cocycles with zero integral that are not coboundaries. Similar results are shown when the sequence of partial quotients is not bounded, and convergence rates of $(S_n)_n$ are given for several subspaces of cocycles in the latter case.

Studies in the same flavour have been done for Anzai's skew product extensions of the two-dimensional torus, which Pierre investigated in [L20] and [L23]. The monograph [L20] contains an elaborate study of cocycles that are either absolutely continuous or step functions, and a general discussion on group extensions, which yield families of interesting examples. Pierre supervised four PhD theses on subjects related to his interests in ergodic theory: J. Mouline (1990), Y. Lacroix (1992), E. Olivier (1997) and C. Guille-Biel (1997).

Pierre with Oto Strauch at a conference in Smolence in 2012

4 Uniform distribution

Pierre's contributions were mainly focused upon two topics: sequences generated by certain digit expansions of real numbers and the application of notions of ergodic theory. Two keywords in this context are *digital sequences* and *irregularities of distribution*.

The dynamical systems approach that Pierre had chosen to study statistical properties of a sequence $\omega = (x_n)_n$ in a compact metrisable space X, with $X = [0,1)^s$, $s \ge 1$, the s-dimensional torus as the most important case, proceeds in two steps: first, one has to find the dynamical system behind the given sequence ω, and, second, one has to employ properties of this system to derive distribution properties of the sequence.

To illustrate this approach, let $\omega = (\mathbf{x}_n)_{n\ge 0}$ be a sequence on the torus $[0,1)^s$. For a (Lebesgue-) measurable subset A of $[0,1)^s$, let $\lambda_s(A)$ denote its Lebesgue measure. The quantity

$$R_N(A,\omega) = \sharp\{n, 0\le n<N : \mathbf{x}_n \in A\} - N\lambda_s(A)$$

is called *the local discrepancy* of the first N points of ω for the set A or, for short, *the remainder* of A.

For a given sequence ω, is there any hope that the remainders $R_N(A,\omega)$ stay bounded in N? In an impressive series of papers, Wolfgang Schmidt investigated this question of *bounded remainder sets* by techniques from metric number theory. Here, two aspects are central:

(i) for sets A with bounded remainder, determine the set of possible volumes $\lambda_s(A)$ (the so-called *admissible volumes*); and
(ii) identify all bounded remainder sets, at least in the case when they are s-dimensional intervals.

(i) can be related to topological dynamics and ergodic theory (see [3, 8]). From this approach, one obtains the result that the set of admissible volumes stems from eigenvalues of an isometric operator. The main contributions of Pierre in this context concern general versions of the *coboundary theorems* that are involved here. This allowed Pierre to give at least partial answers to the bounded remainder sets problem in the case of certain sequences (see [L16]). This area is still active, which is shown by the recent breakthrough in [4].

(ii) is much more difficult to attack and results are known only in very special cases for ω and for A, for example the case for Kronecker sequences $(n\alpha \mod 1)_{n\ge 0}$ in $[0,1)^s$

(see [5] for $s = 1$ and [L16] for $s > 1$). An important result of Pierre deals with polynomial sequences $(p(n) \mod 1)_{n\geq 0}$ (see [L16]). In the same paper, as an example to his general approach, he showed the following. Let α be irrational and let $s_g(n)$ be the sum of digits of n to the base $g \geq 2$. Then, the only intervals I of the 1-torus $[0, 1)$ that are bounded remainder sets for the sequence $(\alpha s_g(n))_{n\geq 0}$ in $[0, 1)$ are the trivial ones, that is to say, $|I| = 0$ or 1.

In the early years of Pierre's examination of uniform distribution and discrepancy theory, he gave an ingenious and transparent new proof of W. Schmidt's lower bound for the discrepancy (i.e. the supremum $D_N(\omega)$ of the local discrepancies for all intervals $A \subset [0, 1)$) of a sequence $\omega = (x_n)_{n\geq 0}$ in $[0, 1)$,

$$D_N(\omega) = \sup_A \left|\frac{R_N(A, \omega)}{N}\right| \geq C\frac{\log N}{N}, \text{ for infinitely many } N,$$

refining Schmidt's constant $C = \frac{1}{66 \log 4}$ to $\frac{3}{40 \log 5}$. Unfortunately, this result was only published in an institute report [L12]; the proof can be found in [2, pp. 41–44]. Pierre mentioned on several occasions that it was his dream to find the best possible constant C, and that its value should be an eigenvalue of an operator related to the problem.

A second series of Pierre's contributions to the theory of uniform distribution of sequences concerned the interplay between sequences related to numeration systems and the qualitative behaviour of such sequences. Historically, these investigations started in close cooperation with Coquet [L15] on notions of (statistical) independence of sequences and then proceeded to measure-theoretic aspects of numeration systems in cooperation with Grabner and Tichy [L30], and with Baláž and Strauch [L46].

5 Systems of numeration and digital expansions

A significant number of Pierre's achievements concern numeration, sequences related to diverse representations of real, complex or natural numbers, and the underlying dynamical systems. Those themes traversed Pierre's mathematical life for 25 years and he was able to fruitfully exploit their interplay. One can emphasise three seminal papers in this area.

The first [L18] was very important for him, scientifically as well as personally. During the 1970s and the 1980s, there was intense interest in statistical properties of sequences, originating with the early works of Wiener. Let W be the set of complex sequences $u = (u_n)_n$ such that the correlations

$$\gamma_u(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} u_{m+n}\overline{u_n}$$

exist for all $m \in \mathbb{N}$. Extended to negative integers by $\gamma_u(-m) = \overline{\gamma_u(m)}$, the correlations $(\gamma_u(m))_m$ form a positive definite sequence, hence are the Fourier coefficients of some Borel measure on the one-dimensional torus $\mathbb{T}$ by the Bochner-Herglotz theorem. This measure is called the *spectral measure* of u. Investigations on this measure yielded to several concepts and results on pseudo-random sequences. Bésineau, Coquet, Kamae, Mendès-France, M. Queffélec, Rauzy, Rhin and others made several contributions to the subject. Pierre was both interested and engaged in that framework and published [L14], where the authors used their abilities in ergodic theory to show several results on spectral disjointness. Pierre published a continuation [L15] of this work, which contains, in a more general setting, the following result:

Theorem 3. Let θ be a q-normal real number (the sequence $(q^n\theta(mod 1))_n$ being uniformly distributed) and P a real polynomial. Then, the two sequences $(q^n\theta \pmod 1)_n$ and $(P(n) \pmod 1)_n$ are spectrally disjoint.

Meanwhile, Coquet passed away and this affected Pierre greatly. Coquet's wife communicated to Pierre the private drafts of her husband and Pierre wrote a synthesis of them in [L18]. This included the redaction of some proofs that had only been sketched by Coquet. In this paper, one encounters, for the first time in Pierre's work, the notion of *systems of numeration*, which is an increasing sequence of integers $(G_n)_n$ with $G_0 = 1$. Any integer $m \in \mathbb{N}$ can be expressed uniquely as

$$m = \sum_{k\geq 0} \varepsilon_k(m)G_k, \text{ provided that } \forall n\colon \sum_{k<n} \varepsilon_k(m)G_k < G_n.$$

Pierre developed the theory of systems of numeration in [L25], which is his most cited paper and the second that we will emphasise. Here, the closure of the set of representations is considered, namely the set

$$\mathcal{K}_G = \{(\varepsilon_k)_{k\geq 0};\ \forall n\colon\ \varepsilon_0(m)G_0 + \cdots \varepsilon_{n-1}(m)G_{n-1} < G_n\}.$$

An extension of the addition $\tau\colon m \mapsto m + 1$ is constructed on $\mathcal{K}_G$, which yields a dynamical system $(\mathcal{K}_G, \tau)$, called the *odometer*. Surjectivity, continuity and minimality are discussed. Special and important examples are Ostrowski systems, based on continued fraction expansions, where $G_n = q_n$, for $\alpha = [0; a_0, a_1, \ldots] \in [1/2, 1)$ and $\frac{p_n}{q_n} = [0; a_0, a_1, \ldots, a_n]$, and sequences $(G_n)_n$ satisfying linear recursions of particular types. The study of odometers has been pursued in several papers, especially [L31] and [L36]. Pierre, together with several coauthors, also published an overview on numeration in [L40], where they introduced the notion of *fibred numeration system* after Schweiger [7].

Arithmetic functions related to systems of numeration have been extensively studied by the Austrian school and Pierre collaborated on several papers on the subject. We cite [L29], [L35], which he liked very much, and [L38, L41, L42, L43]. Some of these papers are devoted both to arithmetic functions and odometers. Pierre liked to construct dynamical systems and to investigate them, deciphering their structures and comparing them through conjugacy and other accurate notions of isomorphism.

The third paper we want to emphasise is [L19]. Let us consider a trigonometric polynomial

$$P(e^{i\theta}) = \sum_{n=1}^{N} \epsilon_n e^{in\theta}, \tag{1}$$

with $\epsilon_n \in \{-1, 1\}$. Parseval's Theorem shows that $\|P\|_\infty \geq \sqrt{N}$ and it follows from results of Salem and Zygmund that the expected order of magnitude for $\|P\|_\infty$ is $(N \log N)^{1/2}$ for almost all such sequences $\epsilon_n \in \{-1, 1\}$. In 1951, Shapiro, in his thesis, gave an inductive construction of a sequence $(\epsilon_n)_n$ such that $|P(e^{i\theta})| \ll N^{1/2}$. Rudin rediscovered the sequence in 1959 and it has since been called the *Rudin-Shapiro sequence*. It turns out that ϵ_n has a simple digital interpretation: it counts the parity of the number of appearances of the string "11" in the binary expansion of n.

In [L19], Allouche and Liardet extended this result in a way that was typical of Pierre's thinking. First, counting occurrences of subwords of the type $a_0a_1 \cdots a_d$ with $a_0a_d \neq 00$ (*generalised Rudin-Shapiro sequences*), they focus on the properties of such sequences and introduce a general abstract machinery including those particular cases. The notion of a *chained map* on an alphabet $\mathcal{A}$ taking its values in a compact metrisable group G is introduced and investigated, especially through an appropriate matrix formalism. Those maps f satisfy the relation

$$f(\alpha\beta\gamma) = f(\alpha\beta)f(\beta)^{-1}f(\beta\gamma)$$

for non-empty words α, β and γ and are proven to be q-automatic if $\mathcal{A} = \{0, 1, \dots, q-1\}$ and if G is finite. Norm bounds for functions of the form (1) are obtained, where the exponential function is replaced by an irreducible representation of the group G. In a further section, the generalised Rudin-Shapiro sequences are studied from a dynamic point of view and the authors prove the following:

Theorem 4. Let $u\colon \mathbb{N} \to \mathbb{U}$ be a generalised multiplicative Rudin-Shapiro sequence. Then, u has a correlation function and its spectral measure is the Lebesgue measure.

The paper ends with the following announcement:

> *In a forthcoming paper we shall study the flows associated to chained sequences, proving that they are (except for degenerate cases) strictly ergodic and can be obtained as group extensions of an a-adic rotation; we shall also give the spectral study of these flows.*

Eighteen years later, this programme was completed in [L45].

Other number representations, such as continued fractions and Engel series, also caught Pierre's attention. Methods from ergodic theory, especially skew products, are used in [L17] to prove uniform distribution of the numerators p_n and denominators q_n of the convergents in residue classes modulo m. In [L21], the set $E(\alpha)$ consisting of all real numbers β such that $(\beta q_n(\alpha))_n$ tends to 0 modulo 1 is studied depending on diophantine properties of α.

In [L28], (infinite) automata are constructed that compute the partial quotients of $f(x)$ from the partial quotients of x for certain rational functions f. For instance, the partial quotients of $\sqrt[3]{2}$ can be computed by such an automaton. This work was continued in [L33], where (infinite) automata are given that compute the Engel series expansion from the continued fraction and *vice versa*.

Pierre's work in that context was the basis for seven doctoral theses: D. Barbolosi (1988), C. Faivre (1990), P. Stambul (1994), G. Barat (1995), N. Loraud (1996), M. Doudékova-Puysdebois (1999) and I. Abou (2008). Several of the results mentioned above were generalised and extended in their work.

6 Applications

It was very typical of Pierre, and his point of view of mathematics and science in general, that he was trying to make his knowledge, especially of ergodic theory, accessible to people working in other areas of mathematics, and also to more applied scientists, especially computer scientists.

At the Conference on Uniform Distribution Theory in Marseille in January 2008, Pierre brought up the idea of making methods from ergodic theory accessible for, and more popular amongst, people working on low-discrepancy sequences. This led to many informal discussions, an educational workshop "Dynamical Aspects of Low-Discrepancy-Sequences" held in Linz in September 2009 and, finally, a survey article [L49]. Pierre's part of this survey was a very general but still accessible and pedagogical description of the cutting-stacking construction for interval exchange maps, which originated in Rokhlin's work and was developed further by Kakutani.

Over the years, he cooperated intensively on cryptographic and algorithmic applications, where he contributed his mastery of dynamical systems. He always intended to make the descriptions of these notions and techniques understandable for his readers but he was also interested in the applications themselves. It is noteworthy that his son Pierre-Yvan is working in computer science, and Pierre was regularly discussing scientific issues with him, even if they did not formally collaborate. Pierre-Yvan likes to tell how his father helped him for a crucial counting that he needed for [6]. Father and son regularly climbed the Sainte-Victoire – Cézanne's mountain – and it was the setting for many mathematical discussions. Pierre-Yvan introduced his father to his cryptographic questions and Pierre used his theoretic knowledge to tackle the problems.

Obvious applications of dynamical systems are random number generators, which are used in different branches of computer science. Evolutionary algorithms are special stochastic search algorithms, which allowed Pierre to introduce his view of dynamical systems to optimise the implementation. The results of these efforts are the papers [L37, L39, L44]. The two papers [L47, L48] propose algorithms for the generation of random selections of k elements out of n. These contributions give a considerable improvement in the rate of convergence towards uniformity. Similarly, a randomisation in the construction of the implementation of finite fields by tower fields is proposed in [L50], in order to make the AES cryptosystem less vulnerable to side-channel attacks.

A further branch of Pierre's research with applications to computer science is the prediction of binary sequences with automata. This started with Annie Broglio's PhD thesis in 1991, which led to a paper [L22] and continued with [L34].

Pierre supervised three PhD theses on subjects related to applications in computer science, cryptography and automata: A. Broglio (1991), B. Peirani (1994) and S. Rochet (1998).

7 Concluding remarks

Pierre Liardet left a lasting influence in all research areas he worked in. His results on the intersection of algebraic curves with abelian varieties, solving a conjecture of Lang, were amongst the first results in this direction (see [1]).

After he left the area of algebraic geometry and started to work in ergodic theory, he was influenced by the work of Coquet and Mendès France. He contributed several important works on the dynamics of the orbit closure of sequences defined by digital functions. This had been of interest from work of Wiener and Mahler since 1927 and still provides interesting examples of dynamical systems. Besides his studies on the interplay of number theory and ergodic theory, he contributed to the development of ergodic theory itself by deep investigations about skew products. The propagation of the study of odometers and other digital representations as dynamical systems was his personal concern. By the dissemination of this idea, he influenced – directly and indirectly – many young mathematicians, especially in France and Austria. The influence of his driving idea of finding dynamical systems behind many mathematical questions even led to his cooperation with computer scientists. Reading the papers discussed in Section 6, one can immediately pin down his contribution, namely making apparent the dynamical system behind the seemingly unrelated problem originating from computer science.

Pierre Liardet had great influence on the current development of the theory of uniform distribution of sequences modulo one. He was one of the innovators to apply concepts and results of ergodic theory to open problems concerning irregularities of distribution of sequences, which has allowed this area to advance after about two decades at a stand-still (after the ground-breaking results of K. F. Roth and W. Schmidt in this field). Further, his insistence on what he called "the dynamic point of view" of sequences has changed the attitude with which various types of sequences are studied nowadays and has helped to develop a broader view of the phenomena that appear. Pierre was also one of the leading researchers to bring the theory of automata, ergodic theory and the theory of uniform distribution together to study the fine structure of various types of sequences.

Pierre liked to meet colleagues and organised several meetings and conferences, among others the Journées Arithmétiques in Marseille in 2005 and several meetings at the CIRM on number theory, ergodic theory, uniform distribution and their interplay. He also created more specialised events, like the "Journées de numération", which has become a regular meeting of researchers working in various aspects of digital expansions, ranging from automata theory to topology and diophantine approximation. The last conference in this series took place in Nancy in June 2015. Indeed, over the last 30 years, numeration has become a dynamical field and Pierre was one of the leading figures of the group.

We have given an overview of Pierre's scientific work, his achievements and his influence in several fields. Yet, besides his constant interest in science (especially mathematics but also medicine and computer science), he was a loving husband to Josy, a father to Pierre-Yvan, Frédéric (who survived his father by only four months) and Christine, and a caring grandfather to his grandchildren. Pierre was a good friend and colleague to many of us. He could be a missionary for his mathematical point of view but he was always open for discussion and dialogue. His mathematical thoughts and his approach to doing mathematics have influenced many of us (especially the three authors of this article). We will miss his constant input and encouragement.

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After having studied in the École Normale Supérieure de Cachan, Guy Barat went to Marseille to write his PhD under the supervision of Pierre Liardet on numeration systems and associated arithmetical functions. From that point onward, his principal duty has been teaching mathematics in Classes préparatoires. As a sideline, he still pursues research in number theory at the Institut de Mathématiques de Marseille (France) and at the Graz University of Technology (Austria).

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Why Maths?

Régis Goiffon (Université Claude Bernard Lyon 1, Villeurbanne, France)

As a consequence of the World Mathematical Year, UNESCO addressed the scientific community with a call for projects for the dissemination of mathematical knowledge that would follow on from the multiple actions developed for that event. One of the projects that was retained was the interactive exhibition "Pourquoi les Mathématiques?",[1] which, as it consisted of several modules, could be periodically enhanced with new exhibits/workshops. The exhibition is especially aimed at a young audience; for countries throughout the world, it is important to inspire children into mathematics and arouse their interest in occupations related to it. The future researchers are at school now!

The exhibition covers a dozen major topics, each represented in three parts illustrated by "manipulations". The descriptions are given in two languages: a local one and French or English. The idea is to do mathematics with both head and hands, naturally favouring the interactive and amusing aspects. For the visitors, each problem seems to be a game or a puzzle. But they quickly understand that to find a solution they need to cultivate a mathematical type of action and reasoning.

The addressed topics vary from mathematical tools adapted for genetics to applications of graph theory, to management of a telecommunication network and the equations used to study financial markets. Every exhibit/workshop is independent of the others. One can therefore follow the direction of imagination or interest: starting from mathematics of nature or the rotary engine or sphere packing or the four colour theorem. The difficulty level is also adjustable. The goal is not to learn methods and techniques (this is done at school) but to give a glimpse of what mathematical questions can be and what answers (often partial) one can give to them. In short, the aim is to show what it is to carry out research in mathematics. With this approach, the visitors to the exhibition

discover a different way to do mathematics, where curiosity is the driving force to the wish to understand.

Before the official premiere, "Why Maths?" was presented at the International Congress on Mathematical Education[2] 2004 in Copenhagen. Since then, the exhibition has been circulating all over the world; it has been to about 100 countries in Europe, Asia and Latin America, always with great success. It is interesting to note that whatever the country, the language or the culture of the audience, the reaction of the visitors is always the same. In Moscow, for example, the exhibition took place in the Lomonosov University over February–March 2012.[3] Over these two months, university and school students and participants of the Science Festival and the Mathematical Olympiad, as well as their teachers and professors, benefited from the exhibition. In Lyon, its presentation at the museum was such a success that mathematicians from the ENS and the ICJ[4] developed a "light" version of the exhibition to be able to visit educational institutions. Subsequently, every month, for a couple of days, four mathematicians meet students of all levels and ages. These activities are enriching for both sides; they show, through the large number of questions, the interest that young people have for mathematics and related professions.

[1] The original French name of "Why Maths?" (www.MathEx.org).

[2] The ICME is held every four years, alternating with the ICM.

[3] www.mathexpo.ru.

[4] ENS – École Normale Supérieure de Lyon; ICJ – Institut Camille Jordan de l'Université Claude Bernard Lyon 1.

Mathematics is intimately related to the core of our society and to the core of our daily life, not only because it is addressed, to some extent, by various other disciplines but also because it is unavoidable, maybe in a less visible manner, in the realisation of lots of objects that are milestones of modern life: from mobile phones to space probes, and also in the domains of weather forecasting, stock exchange fluctuations, numerical imaging, communication, health protection and numerous other crucial sectors of human activity. This may explain the strong demand from the public to discover the foundations of our developing world. But this is only a part of the explanation, as one should not forget another face of mathematics: the one that offers to the human spirit a "free" investigation field, the pleasure of purely "intellectual" challenge, revoking any "material" concern (at least for a time). This aspect, which is often present, for example, in various mathematical games and competitions, is able to seduce a great number of souls, whatever their age. The interest of "Why Maths?" is to combine both aspects and show that mathematics is beautiful, surprising and accessible to all.

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Photos: Régis Goiffon

The author thanks Dr. Vladimir Salnikov for translating the original French article.

To Infinity and Beyond or How We Can Try Everyday to Increase the Discreet Charm of Mathematics

Roberto Natalini (Consiglio Nazionale delle Ricerche, Rome, Italy)

During the Spring meeting in March in Prague, the EMS Executive Committee appointed Roberto Natalini as Chair of the Committee for Raising Public Awareness of Mathematics for the term 2015–2018, thus succeeding Ehrhard Behrends. The full list of the members of the committee can be found at http://www.euro-math-soc.eu/committee/raising-public-awareness.

In this article, Roberto Natalini illustrates some of the main ideas about public awareness strategies that are going to inform his actions over the coming years.

Promoting mathematics in Europe is a crucial issue, not only to sustain our discipline, for instance by asking our governments and the EU Commission for more support and funds, but more importantly for our society and culture. A basic mathematical knowledge is necessary to understand and to face social and political challenges, and the appreciation of mathematical culture is required to form modern and complete European citizens. Mathematics is the key enabling skill for technological innovation and more sustainable development. Mathematics is the key for a deeper understanding of reality. However, even nowadays, this centrality of mathematics to modern life is not well recognised by laymen or our governments and it is a hard but fundamental task to try to change this common misperception.

Clearly, education is one of the main fields where we have to fight to make mathematics appealing, interesting and useful for students, who are actually our future citizens at all levels. We have to understand and resolve the difficulties of teaching that we observe almost everywhere in Europe and, for these purposes, it is necessary to improve the image of mathematics in everyday life, building up an appealing and appropriate language to put the younger generations in contact with our scientific experiences.

Also, it is necessary to improve the internal degree of awareness in the mathematical community. These days, most of our colleagues are simply not aware of the great advances achieved by our discipline as soon as these advances occur just outside of their specific branch of interest; sometimes, communication is lacking even inside the same sub-domain of mathematics. Modern mathematics is more and more based on cross-interaction and pollination between different fields and disciplines. Improving communication could be of great importance for the future of our research.

However, the main concern of a global policy of promotion of public mathematical awareness has to be, in my opinion, about the strategies and the ideas to implement to have a real and deeper impact on the whole of society. We need to reach a larger audience to display our present and past achievements. We need to be more appealing in our presentations and more entertaining, while still keeping an adequate level of authoritativeness by trying to avoid the usual exaggerations and clichés. Clearly, the role of a European Committee is not to replace or duplicate national policies or activities: it is not useful and too expensive for the limited resources of the EMS. Instead, it is worth sharing different local experiences, coordinating a few international initiatives and events and, especially, thinking of a viable European communication strategy for mathematics in the near future.

To cut a long story short, I prefer to sketch some of the activities I proposed to the RPA EMS Committee for my time as chair. It is not an exhaustive list but just a collection of some starting ideas to baseline our actions.

Firstly, we have to improve the presence of our public awareness action on the web. The starting point is the site maintained by the EMS RPA Committee (http://www.mathematics-in-europe.eu/). It is now a useful collection of resources about mathematics but it is not yet satisfactory as an appealing aggregation point for a general audience. We aim to open the site to stories about people working in mathematics, illustrating new achievements and old problems with simple language, and showing amazing applications and connections with everyday life, art and culture. We want to try to be fun most of the time but also rigorous in presenting serious topics, proposing videos, podcasts, posters and web comics. The site should collect and promote in a more interactive way some of the more interesting popular resources about mathematics now dispersed on national sites.

A second key ingredient in our presence on the web is the increase of the use of social networks. We need to create a strong social community by establishing a continuous flow of interesting posts, mixing highbrow and lowbrow tones and vocabulary, with strong references to pop culture, literature, movies and comics. It would be useful for these purposes to organise a light network of young researchers dispersed in all participant countries, who can be quite devoted to this specific task. Connected to this Social Community, we aim to create a Facebook Group (which, unlike a Facebook page, is more similar to a Forum) to share various experiences and to increase participation around Europe. Some experiences at the national level in various countries are promising and the common feeling is that, after a short preliminary period, participation increases quite rapidly.

But there is not only the Web. In the coming months, we are trying to organise some great awareness activities. For instance, following the example of national events like Science Week and also looking at the experience of the American "Mathematical Awareness Month" (http://www.mathaware.org/index.html), which is promoted by all the main mathematical societies in the USA, we could find a common period (a week or a month, to be decided) to celebrate mathematics everywhere in Europe. Usually, these events, with a certain regularity and limitation in time, encompassing many kinds of activities (official posters, videos, comics, activities and science fairs in schools), each year devoted to a different theme, are able to aggregate different groups across countries by producing shared experiences and common practices. Also, they have more chance of having a serious impact on the media.

Also, we have to promote public awareness mathematical sessions as part of large audience events. Following the successful experience of Kraków at ECM 2012, with the activity "Mathematics in the streets", we are organising some new activities for the next congress of the EMS in Berlin in 2016. But this is not enough. We have to try to be present at global and local events with large audiences, not restricted to mathematical relevance: literature and movie festivals, comics conventions, science fairs, researchers' nights. They are all very interesting aggregation points, where we can create some unexpected mathematical moments with a well calibrated blend of science and entertainment, which is finally the core of our communication strategy.

This is a serious job and we need a lot of people to share our mission. For this reason, we invite all interested people across Europe: young bloggers, serious academicians (not too serious, though), science journalists, and PhD and postdoc students, to contact our committee for possible collaborations, directly to my personal email address: roberto.natalini@cnr.it. And remember: 'We need you!'

Roberto Natalini received his PhD in mathematics from the University of Bordeaux (France) in 1986. Since 2014, he has been Director of the Istituto per le Applicazioni del Calcolo "Mauro Picone" of the Italian National Research Council. Previously, he was Director of Research at the same institute (1999-2014), an associate professor at the University of Rome "La Sapienza" (1998-1999) and a researcher at the same institute (1988-1998). He has written more than 90 papers in international journals. His research themes include fluid dynamics, road traffic, semiconductors, chemical damage of monuments and biomathematics. He is on the Board of the Italian Society of Industrial and Applied Mathematics and he is Chair of the Committee for Raising Public Awareness of the European Mathematical Society. He is the coordinator of the Italian site MaddMaths! (http://maddmaths.simai.eu/), which supports the dissemination of mathematics.

European Mathematical Society

EMS Monograph Award by the EMS Publishing House

The EMS Monograph Award is assigned every two years to the author(s) of a monograph in any area of mathematics that is judged by the selection committee to be an outstanding contribution to its field. The prize is endowed with 10,000 Euro and the winning monograph will be published by the EMS Publishing House in the series "EMS Tracts in Mathematics".

The winners of the first award were announced in the June 2014 issue of the Newsletter of the EMS (see below). The second award will be announced in 2016. The submission deadline for the 2018 Award is 30 June 2017.

Submission of manuscripts

The monograph must be original and unpublished, written in English and should not be submitted elsewhere until an editorial decision is rendered on the submission. Monographs should preferably be typeset in TeX. Authors should send a pdf file of the manuscript by email to:

E-mail: award@ems-ph.org

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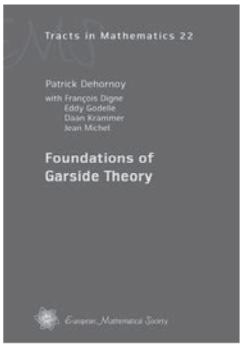
This series includes advanced texts and monographs covering all fields in pure and applied mathematics. Tracts will give a reliable introduction and reference to special fields of current research. The books in the series will in most cases be authored monographs, although edited volumes may be published if appropriate. They are addressed to graduate students seeking access to research topics as well as to the experts in the field working at the frontier of research.

Most recent volume: Winner of the EMS Monograph Award 2014



Vol. 22
Patrick Dehornoy with François Digne, Eddy Godelle, Daan Krammer and Jean Michel
Foundations of Garside Theory
978-3-03719-139-2. 2015. 710 pages. 108.00 Euro

This text is a monograph in algebra, with connections toward geometry and low-dimensional topology. It mainly involves groups, monoids, and categories, and aims at providing a unified treatment for those situations in which one can find distinguished decompositions by iteratively extracting a maximal fragment lying in a prescribed family. Initiated in 1969 by F. A. Garside in the case of Artin's braid groups, this approach turned out to lead to interesting results in a number of cases, the central notion being what the authors call a Garside family. At the moment, the study is far from complete, and the purpose of this book is both to present the current state of the theory and to be an invitation for further research.

There are two parts: the bases of a general theory, including many easy examples, are developed in Part A, whereas various more sophisticated examples are specifically addressed in Part B.

In order to make the content accessible to a wide audience of nonspecialists, exposition is essentially self-contained and very few prerequisites are needed. In particular, it should be easy to use the current text as a textbook both for Garside theory and for the more specialized topics investigated in Part B: Artin–Tits groups, Deligne-Lusztig varieties, groups of algebraic laws, ordered groups, structure groups of set-theoretic solutions of the Yang–Baxter equation. The first part of the book can be used as the basis for a graduate or advanced undergraduate course.

To appear: Co-Winner of the EMS Monograph Award 2014

Vol. 23
Augusto C. Ponce
Elliptic PDEs, Measures and Capacities. From the Poisson Equation to Nonlinear Thomas–Fermi Problems
978-3-03719-140-8. 2016. Approx. 350 pages. 58.00 Euro

European Mathematical Society Publishing House
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ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

13th International Congress on Mathematics Education (ICME-13)

Preparations for ICME-13 continue. The conference will take place in Hamburg, Germany, 24-31 July 2016. The second announcement (http://icme13.org/files/ 2nd_announcement.pdf) includes:

- A call for papers, posters, discussion groups and workshops (see pages 31–33 for guidelines and deadlines) and for support to cover some of the expenses.
- An invitation to mathematics educators from developing and economically disadvantaged countries to submit applications (depending on need) to attend the conference.

ICME-14

According to the ongoing tradition, about a year before an ICME, the Executive Committee of the ICMI selects the site of the following congress. Three countries submitted bids to host ICME-14 in 2020: Australia (Sydney), USA (Honolulu) and China (Shanghai).

The Executive Committee had to make a choice between excellent and outstandingly excellent proposals – after long deliberation, Shanghai was selected to host ICME-14.

The Executive Committee is very grateful to the mathematics education communities in the three countries for their enormous efforts and enthusiasm in preparing their bids and in hosting the delegation during the site visits.

Forthcoming ICMI study volumes

The ICMI is pleased to announce that two ICMI study volumes are already in print and will be available soon:

1. ICMI Study 21 on *Mathematics Education and Language Diversity*.
2. ICMI Study 22 on *Task Design*.

ICMI Study 23: Primary mathematics study on whole numbers – the study conference

Mariolina Bartolini Bussi and Xuhua Sun – Co-Chairs of ICMI Study 23

The discussion document of this study was presented in Issue 92 of this newsletter together with the introduction of the IPC.[1] This is a short report on the study conference (Macau-China, 3–7 June 2015).

A volume of about 650 pages covering the proceedings can be found on the website (http://www.umac.mo/fed/ICMI23/).

Five themes (each corresponding to a working group at the conference) were pursued:

1 *The why and what of whole number arithmetic.*
2 *Whole number thinking, learning and development.*
3 *Aspects that affect whole number learning.*
4 *How to teach and assess whole number arithmetic.*
5 *Whole numbers and connections with other parts of mathematics.*

Three plenary speeches were given:
Liping Ma: *The theoretical core of whole number arithmetic.*
Brian Butterworth: *Low numeracy – from brain to education.*
Hyman Bass: *Quantities, numbers, number names and the real number line.*

Three plenary panels took place:
Traditions in whole number arithmetic (chaired by Ferdinando Arzarello).
Special needs in research and instruction in whole number arithmetic (chaired by Lieven Verschaffel).
Whole number arithmetic and teacher education (chaired by Jarmila Novotná).

Many European mathematics educators took part in the conference, from Denmark, France, Germany, Israel, Italy, the Netherland and Sweden. In total, there were about 90 participants from the five continents. Five observers, invited by the University of Macau and the ICMI, came from the CANP (Capacity & Networking Project, Mathematical Sciences and Education in the Developing World), which is the major development focus of the international bodies of mathematicians and mathematics educators. Other observers came from the Great Mekong Area and mainland China.

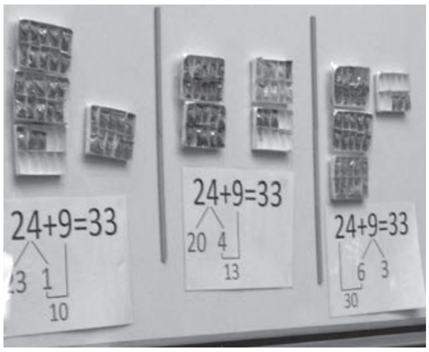
Beside the intense scientific programme, the social programme was designed around two main events:

A visit to primary school classrooms of the Hou Kong School for lessons on addition and subtraction.

A visit to the Macau Ricci Institute for a presentation on the activity of Matteo Ricci, the Italian Jesuit who was in China between the 16th and 17th centuries.

The visit to primary school classrooms allowed the participants to have direct experience of the so-called open-class (公開課), which is the main model of teacher education and development in Chinese mathematics education. Although the visit was arranged for the participants, it was included as part of standard school activity: in the school, a mathematics research group is constituted, with the aim of planning/designing/teaching/reflecting. Together with the participants of the conference, many teachers of the school (both mathematics teachers and English teachers for the purpose of translation) also took part in the les-

[1] International Programme Committee:
Co-Chairs: Maria G. (Mariolina) Bartolini Bussi (Italy), Xuhua Sun (China).
Members: Sybilla Beckmann (USA), Sarah González de Lora (Dominican Republic), Berinderjeet Kaur (Singapore), Maitree Inprasitha (Thailand), Joanne Mulligan (Australia), Jarmila Novotná (Czech Republic), Hamsa Venkatakrishnan (South Africa), Lieven Verschaffel (Belgium); and Abraham Arcavi (Israel – ICMI Secretary General) ex-officio.
ICMI Executive Advisors: Ferdinando Arzarello (Italy – ICMI President), Roger E. Howe (USA – ICMI Liason).



son, which had been planned in advance. Each participant had a copy of the teaching plan and was invited later to discuss with the teacher about the observed lesson. It was a very lively experience that allowed the participants to observe a research lesson for the introduction of a basic topic by means of carefully chosen artefacts and very detailed timing (40 minutes).

The visit to the Macau Ricci Institute, with a welcome by the Director of the Institute Artur K. Wardega, SJ, allowed the participants to listen to a general presentation of the activity of Matteo Ricci, SJ, who spent many years (from 1582 to 1610) in China organising the Chinese mission. The Jesuit missionaries were equipped with a solid humanistic and scientific culture.

Matteo Ricci, in particular, had studied mathematics in Rome with Christophorus Clavius and is known as the first translator of some important Western mathematical works. With the help of a Chinese student (Xu Guangqi), he translated into Chinese the first six books of Euclid's *Elements* (Clavius' version) and a European book on pen arithmetic. Professor Siu Man Keung (from Hong Kong University), a well known historian of mathematics, presented some features of the translation into Chinese of Euclid's *Elements* that are not so well known to Western mathematicians. Rather than a "literal" translation, it was a project to blend Euclid's text with Chinese mathematics traditions. These translations gave Chinese people their first access to real images of Western mathematics.

The ICMI study conference is the basis for the production of the study volume. The character of the volume is rather unique to ICMI studies and is different from proceedings, edited books and handbooks. Although the volume exploits the contributions appearing in the proceedings, the collective production was started during the conference, drawing on the discussions and cooperative works of participants. The preparation of the volume is in progress. The aim is to present the volume in July 2016 at the ICME13 in Hamburg.

As participants in other ICMI studies, we believe that this study has some peculiar features that we wish to emphasise:

- The preparation of a context form, to be filled in by each participant, to give the background information of the study and/or its theoretical statements.
- The invitation to submit video-clips with papers, to exploit the effectiveness of visual data in this age of web communication.
- The participation of IPC members as authors and not only as organisers and co-leaders of working groups.
- The scientific support offered to authors in the revision of their papers.
- The economic support offered to authors from the University of Macau.
- The supported participation of CANP observers.
- The involvement of both the IMU President (Professor Shigefumi Mori) and the ICMI President (Professor Ferdinando Arzarello) in the preparation of the conference.

This collective international effort has led us to the Macau Conference, a product of the fruitful cooperation between mathematicians and mathematics educators, when, for the first time in the history of ICMI, the issue of whole number arithmetic in primary schools has been addressed.

Mariolina and Xuhua in Macau, China, together with ICMI President Professor Ferdinando Arzarello.

Maria G. (Mariolina) Bartolini Bussi is a full professor in mathematics education at the University of Modena and Reggio Emilia (Italy). She is the Director of the University Programme for Pre-primary and Primary Teacher Education. She has been a member of the ICMI Executive Committee (2007/2012) and a member of the Editorial Board of the European Mathematical Society Newsletter (2006/2014). She is now the Co-Chair of ICMI Study 23.

Xuhua Sun is an assistant professor in education at the University of Macau, China, specialising in mathematics education from early childhood to secondary level. She conducts a range of research projects focused on children's mathematical development, curriculum and teacher professional development, with a special interest in Chinese history, culture and tradition in mathematics education. She is now the Co-Chair of ICMI Study 23.

INDRUM – International Network for Didactical Research on University Mathematics

Carl Winsløw (University of Copenhagen, Denmark)

Knowledge about *mathematics teaching* is, crucially, knowledge about *mathematics* itself. It has special characteristics for a given topic such as proportional reasoning or group theory. It involves knowledge about multiple modes of presenting and otherwise initiating the study of the topic, which could lead to different results among the students. There is no doubt that didactics of mathematics, the field which occupies itself with generating such knowledge, is a field in which mathematics itself is everywhere dense.

As regards the specific area of didactics of mathematics that focuses on the teaching and learning of mathematics in universities, a picture of the current "state of the art" can be found in the proceedings of the last three conferences of ERME, the European Society for Research in Mathematics Education (CERME-7, CERME-8 and CERME-9 – see [1]), specifically papers presented in the working group on "University Mathematics Education", together with a recent special issue of *Research in Mathematics Education* (Vol. 16, Issue 2, 2014). We should also mention here the new *International Journal of Research in Undergraduate Mathematics Education*, published by Springer from this year.

Emerging from the working groups at the CERMEs, and formally founded at CERME-9 in February 2015, a new *International Network for Didactical Research on University Mathematics* (INDRUM) now exists to "contribute to the development of research in didactics of mathematics at all levels of tertiary education, with a particular focus on building research capacity in the field and on strengthening the dialogue with the mathematics community" (citing the founding document). While the origin of the network is thus distinctly European, its scope and aims are clearly international.

As a first step, the network will organise an international conference (INDRUM 2016) in Montpellier, France, to be held 31 March – 2 April 2016. The conference will be the second in a series of "European Topic Conferences" organised by ERME to supplement and deepen the work of the CERMEs. Besides a plenary by Michèle Artigue (former President of the ICMI), it will feature work with contributed research papers in two thematic groups. More details on INDRUM 2016, including the call for papers, can be found at the conference website [2], where proposals for papers and posters may also be uploaded by registered participants.

On behalf of the scientific board of INDRUM, I heartily invite all readers to consider involving themselves in INDRUM, for instance by attending the conference mentioned above and perhaps even submitting a paper to one of the thematic working groups in the call. Mathematicians seriously interested in didactical research are especially welcome. Indeed, we are convinced that to advance our knowledge on university mathematics as an educational field, and thereby the success and development of this field, there is no way forward without mathematicians who invest themselves in systematic and internationally oriented didactical research on mathematics and, indeed, realise the inherent potentials of the mathematician as a didactician: "all mathematicians are practitioners and consequently *connoisseurs* of didactics as applied to mathematics" [3, p. 44].

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[1] http://www.mathematik.uni-dortmund.de/~erme/index.php?slab=conferences.

[2] http://indrum2016.sciencesconf.org.

[3] Brousseau, G. (1999). Research in mathematics education: Observation and … mathematics. In I. Schwank (Ed.) *European research in mathematics education* (Vol. 1, pp. 35–49). Osnabrück: Forschungsinstitut für Mathematikdidaktik.

Carl Winsløw is a full professor at the Faculty of Science, University of Copenhagen, Denmark. Since his PhD at the University of Tokyo (1994), he has pursued two research interests: von Neumann algebras and their subfactors, and the didactics of mathematics with a special focus on the teaching of analysis at university level.

Book Reviews

Eugenia Cheng

Cakes, Custard and Category Theory

Profile Books, 2015
ISBN 9781781252871
304 pages

Reviewer: Barbara Fantechi

Books about mathematics for the general public often focus on pretty pictures, numerical tricks, statistics/probability theory, applications and/or the history of the subject, leaving completely untouched many basic concepts such as manifolds or vector spaces. It is not a criticism about any specific book; however, the collection as a whole paints, for the interested but non-expert public, a picture of mathematics that I find misses several important features. I've complained about it to various people in the know (my institution organises a Master's in Science Communication, so there's ample choice) and have been repeatedly told that there is no way to explain such abstract ideas. So I was very excited at the idea of a book aiming to popularise category theory, surely one of the most abstract (and beautiful, and useful) parts of pure mathematics.

Cakes, Custard and Category Theory is different from other books from the very beginning; instead of inspiring quotes, the prologue (and each of the book's chapters) starts with a recipe, which is the starting point for a complex path leading up to a nugget of content, with plenty of real-life analogies and humorous anecdotes along the way. The prologue starts by explaining that mathematics can apply to things other than numbers, by an analogy with a recipe for making clotted cream in a rice cooker, and goes on to negate several myths about mathematics and then introduce the aim of the book: presenting research work in a number-free part of mathematics, namely category theory.

The book is divided into two parts of approximately equal length, simply titled "Mathematics" and "Category Theory". The first part would make a very good book in its own right. Its core idea is to explain how mathematics works by putting the accent not on any specific topic (although a few are informally introduced, such as knots, groups and modular arithmetic) but on the way mathematicians think, introducing axioms, formal proofs and the chain of generalisation that is at the core of a lot of contemporary mathematics. This is done by mixing a small amount of more mathematical content (evidenced by a change of font) with a lot of analogies from everyday life.

The author isn't afraid of finding analogies everywhere, throughout experiences shared by people no matter their education or background; besides the recipes, there are comparisons with tidying up a desk, competitive sports versus fitness exercise, grocery shopping and even party conversations. The fact that a lot of mathematics isn't characterised by its object of study but by the methods applied is compared to someone having bought a new kitchen gadget and trying to find recipes to use it. I found particularly attractive the time and effort the author devotes to the difference between guessing a theorem, convincing oneself it is indeed true and writing a detailed proof; this is compared to the structure of Saint Paul's Cathedral in London, which has an outer dome that is a mark on the city's skyline, an inner dome that is meant to harmonise with the walls if a visitor looks up, and an intermediate, invisible structure that supports the other two.

On the other hand, sometimes the experiences referred to in analogy are personal ones. For instance, the difficulty of passing from a level of understanding to a more abstract one (say, from a specific second-degree equation to a general one or from a simple puzzle to a general linear system) is compared to the author's "block" when, in school, she was supposed to try the high-jump. This is important in several ways. For one thing, it acknowledges that each of us has things we can or can't do and, at the same time, it is a reminder that you can appreciate something difficult even if you can't do it yourself. Finally, it helps us get to know the author beyond her work interests. By the end of the book, one has gained a limited but satisfactory knowledge of the author as a very likeable person with many interests beyond mathematics, as well as glimpses of the occasional annoyances that go along with being a mathematician in everyday life. (Some parts are a bit exotic for non-UK residents, from the clotted cream the book opens with to the strange pricing system of train travel.)

The second half of the book builds on the examples and ideas previously introduced to present the language of categories as a kind of "metamathematics", a mathematical structure whose purpose it is to model and study mathematical structures. In case you wonder how it is even possible to explain to the layperson what many professional mathematicians find too abstract, the key to Cheng's approach is the same as in the first half of the book: putting the accent on the key steps of the process rather than on specific objects or results, and including a lot of real-life analogies. I was very impressed, for instance, with her ability to explain the notion of universal properties and objects and am considering recommending it as reading material for my students.

In case you're worried that this part of the book is all fluff and no content, there is indeed plenty of mathematics inside and, in fact, more than I would have expected.

For instance, there is an informal but understandable proof of the fact that the empty set is the initial object of the category of sets, and the set with one element is its final object. Another aspect I found very satisfying are the illustrations. There are no glossy colour pictures or computer-generated surfaces but rather simple photos and the kind of diagrams we would often draw during a lecture; the author repeatedly remarks that such diagrams are important tools for the understanding of mathematics both by students and researchers and as a way to use our real-world, geometric intuition to guide us along the paths of abstraction. As usual, appreciation for such images will vary; my favourite is a very pretty picture of a 2-commutative diagram but topologists might prefer the photo of two bagels that were placed too near each other in the oven and ended up as a perfect example of a genus 2 surface.

To whom would I recommend "Cakes, Custard and Category Theory"? In my opinion, the people who would most benefit from it are high-school students because it will give them a much better idea of what a university education in mathematics is about than most of what they learn at school. This is especially important in school systems where pupils are allowed to drop subjects like mathematics at an early stage, but I think it is a key part of education everywhere. This book is a very good counterpart to the average schoolbooks, which are often dry and uninspiring (disclaimer: this is based on the children's books I experienced in Italy – maybe other countries have books that are much better!). On a related note, it is a great book for mathematics teachers, especially those whose own mathematical education was focused on the computational side.

I remember (from a higher category conference) the definition of a person's category number, defined as "the smallest Nc such that for every N > Nc thinking of N-categories gives you a headache". Hence, Nc = –1 means you can't even stand thinking about sets, while Nc = ∞ means you can deal with ∞-categories without trouble. The second half of this book is strongly recommended for mathematicians whose category number is less than or equal to one, especially if they find that their own research would benefit by raising said number. The whole book would also be very appropriate for scientists in nearby disciplines: physics, computer science and more. I think they will find it understandable and enjoyable (although some of the more theoretical computer science people may find it too easy).

Do I think the book is successful in its aim of explaining category theory to a general public in an entertaining way? The answer, in my opinion, is a limited yes. We've known for thousands of years that there isn't an easy way into mathematics, and the book is demanding on the reader, who may or may not manage to keep going with it to the end. Luckily, it is structured in such a way that even a local lack of understanding won't impede progress and enjoyment; in fact, it is a book that almost demands a second reading to be more properly appreciated.

On a personal note, I do hope that the book eventually gets an Italian translation. One of the things that bothers me most about being a mathematician is that so many people I'm close to, from my family of origin to my pre-college and outside of academia friends, cannot even begin to grasp what it is I do all day, and what this mysterious "research" is that I get paid for. I think "Cakes, Custard and Category Theory" would be a great first step to fixing, or at least diminishing, this problem. In the meantime, I'll encourage my own (English-speaking teenage) children to read it, perhaps bribing them with one of the recipes.

Barbara Fantechi studied in Pisa under the supervision of Fabrizio Catanese; she has held positions at the Universities of Trento and Udine and has been a professor of geometry at Sissa in Trieste, Italy, since 2002. Her research interests are in algebraic geometry, in particular deformation theory, moduli problems, algebraic stacks and enumerative geometry.

Solved and Unsolved Problems

Themistocles M. Rassias (National Technical University, Athens, Greece)

It is through science that we prove,
but through intuition that we discover.
Henri Poincaré (1854–1912)

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

147. Prove or disprove the following. If $f : \mathbb{R} \to \mathbb{R}$ has both a left limit and right limit at every point then f is continuous, except perhaps on a countable set.

(*W. S. Cheung, The University of Hong Kong, Pokfulam, Hong Kong*)

148. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive real numbers. If

1. $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are both unbounded; and
2. $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$,

prove that the set $M = \left\{ \frac{a_n}{b_m} : m, n \geq 1 \right\}$ is everywhere dense in the interval $[0, \infty)$.

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

149. (a) Prove that

$$\lim_{n\to\infty}\left(2\zeta(3) + 3\zeta(4) + \cdots + n\zeta(n+1) - \frac{n(n+1)}{2}\right) = 0.$$

(b) *An Apéry's constant series.* Calculate

$$\sum_{n=2}^{\infty}\left(\frac{n(n+1)}{2} - 2\zeta(3) - 3\zeta(4) - \cdots - n\zeta(n+1)\right),$$

where ζ denotes the Riemann zeta function.

(*Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*)

150. We say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t) f(x) + t f(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0, 1]$.
Let $f, h : [a, b] \subset (0, \infty) \to \mathbb{R}$ be such that $h(t) = t f(t)$ for $t \in [a, b]$. Show that f is HA-convex on the interval $[a, b]$ if and only if h is convex on $[a, b]$.

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

151. Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a, b]$. Show that we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(t)}{a+b-t}dt \leq \frac{af(a) + bf(b)}{a+b}. \tag{2}$$

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

152. Let G be an arbitrary group written multiplicatively. Let $\sigma : G \to G$ be an anti-homomorphism (i.e., $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$) satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Let $\mathbb{C}$ be the field of complex numbers.

(i) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(xy) + f(\sigma(y)x) = 2 f(x) \tag{3}$$
for all $x, y \in G$.

(ii) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(xy) - f(x\sigma(y)) = 2 f(y) \tag{4}$$
for all $x, y \in G$.

(iii) Find all functions $f : G \to \mathbb{C}$ that satisfy the functional equation
$$f(x\sigma(y)) = f(x) f(y) \tag{5}$$
for all $x, y \in G$.

(*Prasanna K. Sahoo, University of Louisville, Louisville, USA*)

II Two new open problems

153*. For $0 \leq \beta < 1$, $n \in \mathbb{N}$, $x \geq 0$ and

$$L_{n,k}^{(\beta)}(x) = \frac{nx(nx + k\beta)^{k-1}}{k!}e^{-(nx+k\beta)},$$

we define

$$V_{n,m}^{\beta}(x) = \sum_{k=0}^{\infty}\frac{< L_{n,k}^{(\beta)}(t), t^m >}{< L_{n,k}^{(\beta)}(t), 1 >}L_{n,k}^{(\beta)}(x),$$

where $< f, g >= \int_0^\infty f(t)g(t)dt$. Is it possible to have a recurrence relation between $V_{n,m+1}^{\beta}(x)$ and $V_{n,m}^{\beta}(x)$ or between their derivatives?

(*Vijay Gupta, Netaji Subhas Institute of Technology, New Delhi, India*)

154*. For $a \geq 0$, $n \in \mathbb{N}$, $r \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$ and for $0 \leq k \leq n$, examine whether the integral

$$\int_0^1 e^{ax/(1-x)}x^{k+r}(1-x)^{n-k}dx$$

exists. If that is the case, compute its value.

(*Vijay Gupta, Netaji Subhas Institute of Technology, New Delhi, India*)

III Solutions

139. If we define $T^{\rho}_{n,r}(x)$ with $n \in \mathbb{N}, r \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$ and $\rho > 0$ as follows:

$$T^{\rho}_{n,r}(x) = \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} b^{\rho}_{n,k}(t) t^r dt,$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

$$b^{\rho}_{n,k}(t) = \frac{1}{B(k\rho, n\rho+1)} \frac{t^{k\rho-1}}{(1+t)^{k\rho+n\rho+1}},$$

prove that for $n\rho > r$, the following recurrence relation holds:

$$\left(n - \frac{r}{\rho}\right) T^{\rho}_{n,r+1}(x) = x(1+x)[T^{\rho}_{n,r}(x)]' + \left(\frac{r}{\rho} + nx\right) T^{\rho}_{n,r}(x).$$

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

Solution by the proposer.

By simple computation, we have

$$x(1+x)[v_{n,k}(x)]' = (k - nx)v_{n,k}(x)$$

and

$$[t(1+t)b^{\rho}_{n,k}(t)]' = \rho(k - nt)b^{\rho}_{n,k}(t).$$

Using the above identities, we can write

$$x(1+x)[T^{\rho}_{n,r}(x)]' = \sum_{k=1}^{\infty} x(1+x)[v_{n,k}(x)]' \int_0^{\infty} b_{n,k}\rho(t)t^r dt$$

$$= \sum_{k=1}^{\infty} (k - nx)v_{n,k}(x) \int_0^{\infty} b_{n,k}\rho(t)t^r dt$$

$$= \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} (k - nt + nt - nx) b_{n,k}\rho(t)t^r dt$$

$$= \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} \frac{1}{\rho}[t(1+t)b_{n,k}\rho(t)]' t^r dt + nT^{\rho}_{n,r+1}(x) - nxT^{\rho}_{n,r}(x).$$

Now, integrating by parts the last integral, we have

$$x(1+x)[T^{\rho}_{n,r}(x)]' = -\frac{r}{\rho}[T^{\rho}_{n,r}(x) + T^{\rho}_{n,r+1}(x)] + nT^{\rho}_{n,r+1}(x) - nxT^{\rho}_{n,r}(x).$$

□

Also solved by Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Cristinel Mortici (Valahia University, Targoviste, Romania), Socratis Varelogiannis (National Technical University of Athens, Greece)

140. If we define $T_{n,r}(x)$ with $n \in \mathbb{N}$ and $r \in \mathbb{N}^0 := \mathbb{N} \cup \{0\}$ as follows:

$$T_{n,r}(x) = \sum_{k=1}^{\infty} v_{n,k}(x) \int_0^{\infty} b_{n,k}(t) t^r dt,$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},$$

$$b_{n,k}(t) = \frac{1}{B(k, n+1)} \frac{t^{k-1}}{(1+t)^{k+n+1}},$$

prove that

$$T_{n,r}(x) = \frac{\Gamma(n-r+1)\Gamma(r+1)x}{\Gamma(n)} \;.{}_2F_1\,(n+1, 1-r; 2; -x)$$

(*Vijay Gupta, Department of Mathematics, Netaji Subhas Institute of Technology, New Delhi, India*)

Solution by the proposer. By definition, for $r \in \mathbb{N}$, we have

$$T_{n,r}(x)$$

$$= \sum_{k=1}^{\infty} \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{1}{B(k, n+1)} \int_0^{\infty} \frac{t^{k+r-1}}{(1+t)^{k+n+1}} dt$$

$$= \sum_{k=1}^{\infty} \frac{(n)_k}{k!} \frac{x^k}{(1+x)^{n+k}} \frac{1}{B(k, n+1)} B(k+r, n-r+1)$$

$$= \frac{\Gamma(n-r+1)}{\Gamma(n+1)} \frac{1}{(1+x)^n} \sum_{k=1}^{\infty} \frac{(n)_k}{k!} \left(\frac{x}{1+x}\right)^k \frac{\Gamma(k+r)}{\Gamma(k)}$$

$$= \frac{\Gamma(n-r+1)\Gamma(r)}{\Gamma(n+1)} \frac{1}{(1+x)^n} \sum_{k=1}^{\infty} \frac{(n)_k (r)_k}{k!(k-1)!} \left(\frac{x}{1+x}\right)^k$$

$$= \frac{\Gamma(n-r+1)\Gamma(r)}{\Gamma(n+1)} \frac{x}{(1+x)^{n+1}} \sum_{k=0}^{\infty} \frac{(n)_{k+1} (r)_{k+1}}{(2)_k . k!} \left(\frac{x}{1+x}\right)^k$$

$$= \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n)} \frac{x}{(1+x)^{n+1}} \sum_{k=0}^{\infty} \frac{(n+1)_k (r+1)_k}{(2)_k . k!} \left(\frac{x}{1+x}\right)^k$$

$$= \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n)} \frac{x}{(1+x)^{n+1}} \;.{}_2F_1\left(n+1, r+1; 2; \frac{x}{1+x}\right)$$

Using the transformation

$${}_2F_1\,(a, b; c; x) = (1-x)^{-a} .{}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right),$$

the result follows immediately. □

Also solved by Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Sotirios E. Louridas (Athens, Greece), Cristinel Mortici (Valahia University, Targoviste, Romania), Socratis Varelogiannis (National Technical University of Athens, Greece)

141. For a real number $a > 0$, define the sequence $(x_n)_{n \geq 1}$, $x_n = \sum_{k=1}^{n} a^{\frac{k^2}{n^3}} - n$.

(1) Prove that $\lim_{n\to\infty} x_n = \frac{1}{3} \ln a$.

(2) Evaluate $\lim_{n\to\infty} n(x_n - \frac{1}{3} \ln a)$.

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. Recall that

$$\lim_{t\to 0} \frac{a^t - 1}{t} = \ln a.$$

For $\varepsilon > 0$, there is $\delta > 0$ such that, for every real number t with $|t| < \delta$, we have

$$\ln a - \varepsilon < \frac{a^t - 1}{t} < \ln a + \varepsilon. \qquad (1)$$

There is an integer n_0 such that

$$\frac{k^2}{n^3} < \delta \text{ for all } n \geq n_0 \text{ and } k = 0, 1, \dots, n.$$

Inequality (1) gives, for $n \geq n_0$,

$$\ln a - \varepsilon < \frac{a^{\frac{k^2}{n^3}} - 1}{\frac{k^2}{n^3}} < \ln a + \varepsilon, \text{ where } k = 0, 1, \dots, n$$

and thus

$$\ln a - \varepsilon < \frac{\sum_{k=1}^n \left(a^{\frac{k^2}{n^3}} - 1\right)}{\sum_{k=1}^n \frac{k^2}{n^3}} < \ln a + \varepsilon.$$

It follows that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n \left(a^{\frac{k^2}{n^3}} - 1\right)}{\sum_{k=1}^n \frac{k^2}{n^3}} = \lim_{n\to\infty} \frac{x_n}{\sum_{k=1}^n \frac{k^2}{n^3}} = \ln a.$$

On the other hand, using the well known formula

$$\sum_{k=1}^n \frac{k^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3},$$

we have

$$\lim_{n\to\infty} \sum_{k=1}^n \frac{k^2}{n^3} = \frac{1}{3}$$

and the desired relation is proved.

2. Now, we will use the relation

$$\lim_{t\to 0} \frac{a^t - 1 - t\ln a}{t^2} = \frac{1}{2}\ln^2 a,$$

which can easily be derived using l'Hopital's rule. For $\varepsilon > 0$, there is $\delta > 0$ such that, for every real number t with $|t| < \delta$, we have

$$\frac{1}{2}\ln^2 a - \varepsilon < \frac{a^t - 1 - t\ln a}{t^2} < \frac{1}{2}\ln^2 a + \varepsilon. \qquad (2)$$

Consider n_0 such that

$$\frac{k^2}{n^3} < \delta \text{ for all } n \geq n_0 \text{ and } k = 0, 1, \dots, n.$$

For $n \geq n_0$, we have

$$\frac{1}{2}\ln^2 a - \varepsilon < \frac{a^{\frac{k^2}{n^3}} - 1 - \frac{k^2}{n^3}\ln a}{\left(\frac{k^2}{n^3}\right)^2} < \frac{1}{2}\ln^2 a + \varepsilon,$$
$$\text{where } k = 0, 1, \dots, n$$

and hence

$$\frac{1}{2}\ln^2 a - \varepsilon < \frac{x_n - \frac{n(n+1)(2n+1)}{6n^3}\ln a}{\sum_{k=1}^n \left(\frac{k^2}{n^3}\right)^2} < \frac{1}{2}\ln^2 a + \varepsilon.$$

The last relation is equivalent to

$$\frac{1}{2}\ln^2 a - \varepsilon < \frac{n\left(x_n - \frac{n(n+1)(2n+1)}{6n^3}\ln a\right)}{\frac{1}{n^5}\sum_{k=1}^n k^4} < \frac{1}{2}\ln^2 a + \varepsilon. \qquad (3)$$

Taking into account the well known limit

$$\lim_{n\to\infty} \frac{1}{n^5}\sum_{k=1}^n k^4 = \frac{1}{5},$$

from (3) it follows that

$$\lim_{n\to\infty} n\left(x_n - \frac{n(n+1)(2n+1)}{6n^3}\ln a\right) = \frac{1}{10}\ln^2 a. \qquad (4)$$

Relation (4) can be written as

$$\begin{aligned}
&\frac{1}{10}\ln^2 a \\
&= \lim_{n\to\infty} n\left[\left(x_n - \frac{1}{3}\ln a\right) - \left(\frac{n(n+1)(2n+1)}{6n^3}\ln a - \frac{1}{3}\ln a\right)\right] \\
&= \lim_{n\to\infty} n\left(x_n - \frac{1}{3}\ln a\right) - \lim_{n\to\infty} n\left(\frac{n(n+1)(2n+1)}{6n^3}\ln a - \frac{1}{3}\ln a\right) \\
&= \lim_{n\to\infty} n\left(x_n - \frac{1}{3}\ln a\right) - \frac{1}{3}\ln a \lim_{n\to\infty} n\left(\frac{n(n+1)(2n+1)}{2n^3} - 1\right) \\
&= \lim_{n\to\infty} n\left(x_n - \frac{1}{3}\ln a\right) - \frac{1}{3}\ln a \lim_{n\to\infty} n\left(\frac{n(n+1)(2n+1) - 2n^3}{2n^3}\right) \\
&= \lim_{n\to\infty} n\left(x_n - \frac{1}{3}\ln a\right) - \frac{1}{3}\ln a \lim_{n\to\infty} \frac{n(n+1)(2n+1) - 2n^3}{2n^2} \\
&= \lim_{n\to\infty} n\left(x_n - \frac{1}{3}\ln a\right) - \frac{1}{2}\ln a
\end{aligned}$$

and we find

$$\lim_{n\to\infty} n\left(x_n - \frac{1}{3}\ln a\right) = \frac{1}{10}\ln^2 a + \frac{1}{2}\ln a.$$

□

Also solved by Mihaly Bencze (Brasov, Romania), Soon-Mo Jung (Hongik University, Chochiwon, Korea), Sotirios E. Louridas (Athens, Greece), Cristinel Mortici (Valahia University, Targoviste, Romania), Socratis Varelogiannis (National Technical University of Athens, Greece)

142. Let $a < b$ be positive real numbers. Prove that the system

$$\begin{cases} (2a+b)^{x+y} = (3a)^x(a+2b)^y \\ (a+2b)^{y+z} = (2a+b)^y(3b)^z \end{cases}$$

has a solution (x, y, z) such that $a < x < y < z < b$.

(*Dorin Andrica, Babeş-Bolyai University of Cluj-Napoca, Romania*)

Solution by the proposer. The system is equivalent to

$$\left(\frac{2a+b}{3a}\right)^x = \left(\frac{a+2b}{2a+b}\right)^y = \left(\frac{3b}{a+2b}\right)^z.$$

We have

$$\frac{2a+b}{3a} > \frac{a+2b}{2a+b} > \frac{3b}{a+2b}.$$

Therefore, every solution (x, y, z) to the system satisfies $x < y < z$. Now, we have to show that we can find a solution (x, y, z) with the components in the interval (a, b).

Applying the Mean Value Theorem for the function $f(t) = \ln t$ on the interval $\left[a, a + \frac{b-a}{3}\right]$ yields

$$\frac{\ln(a + \frac{b-a}{3}) - \ln a}{\frac{b-a}{3}} = \frac{1}{x}, \text{ where } x \in \left(a, a + \frac{b-a}{3}\right).$$

Hence

$$\left(\frac{2a+b}{3a}\right)^x = e^{\frac{b-a}{3}}. \tag{1}$$

Using the same argument for the interval $\left[a + \frac{b-a}{3}, a + \frac{2(b-a)}{3}\right]$, we obtain

$$\frac{\ln\left(a + \frac{2(b-a)}{3}\right) - \ln\left(a + \frac{b-a}{3}\right)}{\frac{b-a}{3}} = \frac{1}{y},$$

where

$$y \in \left(a + \frac{b-a}{3}, a + \frac{2(b-a)}{3}\right).$$

Hence

$$\left(\frac{a+2b}{2a+b}\right)^y = e^{\frac{b-a}{3}}. \tag{2}$$

Applying the same argument for the interval $\left[a + \frac{2(b-a)}{3}, b\right]$, we get

$$\frac{\ln b - \ln\left(a + \frac{2(b-a)}{3}\right)}{\frac{b-a}{3}} = \frac{1}{z}, \text{ where } z \in \left(a + \frac{2(b-a)}{3}, b\right).$$

Therefore

$$\left(\frac{3b}{a+2b}\right)^z = e^{\frac{b-a}{3}}. \tag{3}$$

From equalities (1), (2) and (3), we get

$$\left(\frac{2a+b}{3a}\right)^x = \left(\frac{a+2b}{2a+b}\right)^y = \left(\frac{3b}{a+2b}\right)^z = e^{\frac{b-a}{3}},$$

with $a < x < a + \frac{b-a}{3} < y < a + \frac{2(b-a)}{3} < z < b$, and the conclusion follows. □

Also solved by Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), Cristinel Mortici (Valahia University, Targoviste, Romania), Socratis Varelogiannis (National Technical University of Athens, Greece)

143. A function $f : I \subset \mathbb{R} \to (0, \infty)$ is called *AH-convex (concave)* on the interval I if the following inequality holds

$$f((1-\lambda)x + \lambda y) \le (\ge) \frac{1}{(1-\lambda)\frac{1}{f(x)} + \lambda\frac{1}{f(y)}} \tag{AH}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$.
Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. Show that if $a, b \in I$ with $a < b$ then we have the inequality

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) \left[\frac{b-s}{b-a} f(b) + \frac{s-a}{b-a} f(a)\right] f(s) \tag{6}$$

for any $s \in [a, b]$.
In particular, we have

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2} \tag{7}$$

and

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \le (\ge) f(a) f(b). \tag{8}$$

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

Solution by the proposer. If the function $f : I \to (0, \infty)$ is AH-convex (concave) on I then the function $\frac{1}{f}$ is concave (convex) on I. Therefore the function f is differentiable almost everywhere on I and we have the gradient inequality for $\frac{1}{f}$:

$$\frac{1}{f(s)} - \frac{1}{f(t)} \le (\ge) \frac{f'(t)}{f^2(t)}(t-s),$$

which is equivalent to

$$\frac{f(t)}{f(s)} - 1 \le (\ge) \frac{f'(t)}{f(t)}(t-s) \tag{9}$$

for every $s \in [a, b]$ and almost every $t \in [a, b]$. This is an inequality of interest in itself.

Multiplying (9) by $f(t) > 0$ and integrating over $t \in [a, b]$, we have

$$\frac{1}{f(s)}\int_a^b f^2(t)\,dt - \int_a^b f(t)\,dt \le (\ge) \int_a^b f'(t)(t-s)\,dt. \tag{10}$$

Integrating by parts, we have

$$\int_a^b f'(t)(t-s)\,dt = f(b)(b-s) + f(a)(s-a) - \int_a^b f(t)\,dt$$

and by (10) we get the desired result (6).

We observe that (7) follows by (6) for $s = \frac{a+b}{2}$ while (8) follows by (6) for either $s = a$ or $s = b$. □

Also solved by Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), Cristinel Mortici (Valahia University, Targoviste, Romania)

144. Let $f : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function on $[a, b]$. Show that, if $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$ then we have the inequality

$$\begin{aligned} &\Phi\left(\frac{(s-a) f(a) + (b-s) f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\quad \le (\ge) \frac{s-a}{(b-a)^2}\int_a^b \Phi[f(a) - f(t)]\,dt \\ &\qquad + \frac{b-s}{(b-a)^2}\int_a^b \Phi[f(b) - f(t)]\,dt \end{aligned} \tag{11}$$

for any $s \in [a, b]$.
In particular, we have

$$\begin{aligned} &\Phi\left(\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\quad \le (\ge) \frac{1}{b-a}\int_a^b \frac{\Phi[f(a) - f(t)] + \Phi[f(b) - f(t)]}{2}\,dt. \end{aligned} \tag{12}$$

(*Sever S. Dragomir, Victoria University, Melbourne, Australia*)

Solution by the proposer. We have

$$\begin{aligned} &\frac{(s-a) f(a) + (b-s) f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \qquad (13) \\ &= \frac{1}{b-a}\int_a^b \left[\frac{(s-a) f(a) + (b-s) f(b)}{b-a} - f(t)\right] dt \\ &= \frac{1}{b-a}\int_a^b \frac{(s-a)[f(a) - f(t)] + (b-s)[f(b) - f(t)]}{b-a}\,dt \end{aligned}$$

for any $s \in [a,b]$.

Using Jensen's inequality for the convex (concave) function $\Phi : \mathbb{R} \to \mathbb{R}$, we have

$$\Phi\left(\frac{(s-a)\,f(a)+(b-s)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \tag{14}$$
$$=\Phi\left(\frac{1}{b-a}\int_a^b \frac{(s-a)\,[f(a)-f(t)]+(b-s)\,[f(b)-f(t)]}{b-a}dt\right)$$
$$\le (\ge)\,\frac{1}{b-a}\int_a^b \Phi\left(\frac{(s-a)\,[f(a)-f(t)]+(b-s)\,[f(b)-f(t)]}{b-a}\right)dt$$

for any $s \in [a,b]$.

By the convexity (concavity) of Φ, we also have

$$\Phi\left(\frac{(s-a)\,[f(a)-f(t)]+(b-s)\,[f(b)-f(t)]}{b-a}\right) \tag{15}$$
$$\le (\ge)\,\frac{s-a}{b-a}\Phi\,[f(a)-f(t)]+\frac{b-s}{b-a}\Phi\,[f(b)-f(t)]$$

for any $s, t \in [a,b]$.

Integrating (15) over $t \in [a,b]$, we get

$$\frac{1}{b-a}\int_a^b \Phi\left(\frac{(s-a)\,[f(a)-f(t)]+(b-s)\,[f(b)-f(t)]}{b-a}\right)dt \tag{16}$$
$$\le (\ge)\,\frac{s-a}{(b-a)^2}\int_a^b \Phi\,[f(a)-f(t)]\,dt+\frac{b-s}{(b-a)^2}\int_a^b \Phi\,[f(b)-f(t)]\,dt$$

for any $s \in [a,b]$.

Utilising (14) and (16), we deduce the desired result (11). □

Also solved by Soon-Mo Jung (Hongik University, Chochiwon, Korea), Panagiotis T. Krasopoulos (Athens, Greece), Cristinel Mortici (Valahia University, Targoviste, Romania)

Note. Xiaopeng Zhao (East China Normal University) also solved problem 79.

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions both by ordinary mail to Themistocles M. Rassias, Department of Mathematics, National Technical University of Athens, Zografou Campus, GR-15780, Athens, Greece, and by email to trassias@math.ntua.gr.
We also solicit your new problems with their solutions for the next "Solved and Unsolved Problems" column, which will be devoted to *Mathematical Analysis*.

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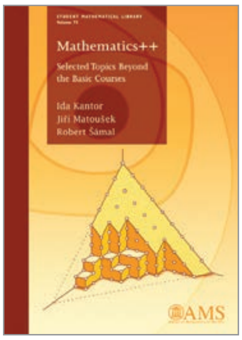

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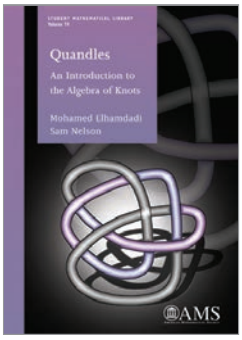

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Mohamed Elhamdadi, University of South Florida & Sam Nelson, Claremont McKenna College

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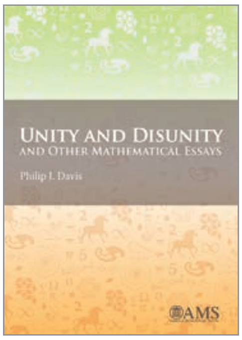

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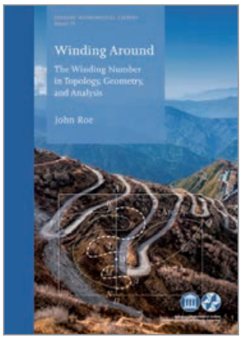

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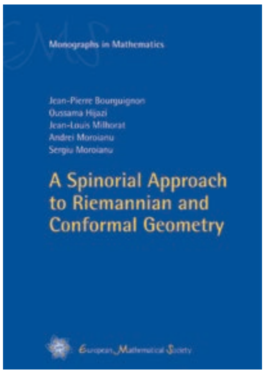
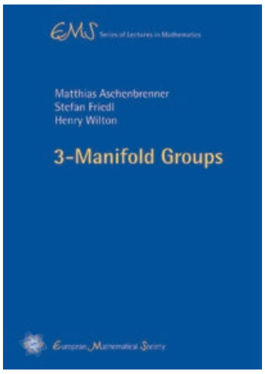


Jean-Pierre Bourguignon (IHÉS, Bures-sur-Yvette, France), Oussama Hijazi (Université de Lorraine, Nancy, France), Jean-Louis Milhorat (Université de Nantes, France), Andrei Moroianu (Université de Versailles-St Quentin, France) and Sergiu Moroianu (Institutul de Matematică al Academiei Române, Bucureşti, Romania)
A Spinorial Approach to Riemannian and Conformal Geometry (EMS Monographs in Mathematics)

ISBN 978-3-03719-136-1. 2015. 462 pages. Hardcover. 16.5 x 23.5 cm. 78.00 Euro

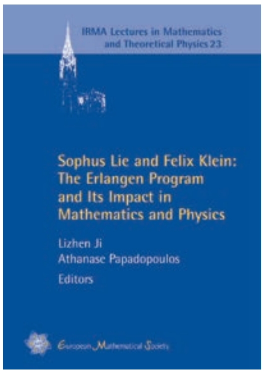
The book gives an elementary and comprehensive introduction to Spin Geometry, with particular emphasis on the Dirac operator which plays a fundamental role in differential geometry and mathematical physics.
After a self-contained presentation of the basic ingredients, a systematic study of the spectral properties of the Dirac operator on compact spin manifolds is carried out. The classical estimates on eigenvalues and their limiting cases are discussed and several applications of these ideas are presented, including spinorial proofs of the Positive Mass Theorem or the classification of positive Kähler–Einstein contact manifolds. Representation theory is used to explicitly compute the Dirac spectrum of compact symmetric spaces.
The special features of the book include a unified treatment of Spin^c and conformal spin geometry, an overview with proofs of the theory of elliptic differential operators on compact manifolds based on pseudodifferential calculus, a spinorial characterization of special geometries, and a self-contained presentation of the representation-theoretical tools needed in order to apprehend spinors. This book will help advanced graduate students and researchers to get more familiar with this domain of mathematics.



Matthias Aschenbrenner (University of California Los Angeles, USA), Stefan Friedl (University of Regensburg, Germany) and Henry Wilton (University of Cambridge, UK)
3-Manifold Groups (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-154-5. 2015. 230 pages. Softcover. 17 x 24 cm. 34.00 Euro

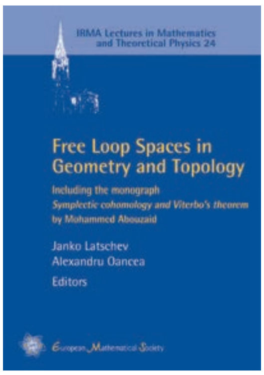
The field of 3-manifold topology has made great strides forward since 1982, when Thurston articulated his influential list of questions. Primary among these is Perelman's proof of the Geometrization Conjecture, but other highlights include the Tameness Theorem of Agol and Calegari–Gabai, the Surface Subgroup Theorem of Kahn–Markovic, the work of Wise and others on special cube complexes, and finally Agol's proof of the Virtual Haken Conjecture. This book summarizes all these developments and provides an exhaustive account of the current state of the art of 3-manifold topology, especially focussing on the consequences for fundamental groups of 3-manifolds.
As the first book on 3-manifold topology that incorporates the exciting progress of the last two decades, it will be an invaluable resource for researchers in the field who need a reference for these developments. It also gives a fast-paced introduction to this material – although some familiarity with the fundamental group is recommended, little other previous knowledge is assumed, and the book is accessible to graduate students.
The book closes with an extensive list of open questions of interest to graduate students and established researchers alike.



Sophus Lie and Felix Klein: The Erlangen Program and Its Impact in Mathematics and Physics
(IRMA Lectures in Mathematics and Theoretical Physics, Vol. 23)
Lizhen Ji (University of Michigan, Ann Arbor, USA) and Athanase Papadopoulos (Université de Strasbourg, France), Editors

ISBN 978-3-03719-148-4. 2015. 348 pages. Hardcover. 17 x 24 cm. 48.00 Euro

The Erlangen program expresses a fundamental point of view on the use of groups and transformation groups in mathematics and physics. The present volume is the first modern comprehensive book on that program and its impact in contemporary mathematics and physics. Klein spelled out the program, and Lie, who contributed to its formulation, is the first mathematician who made it effective in his work. The theories that these two authors developed are also linked to their personal history and to their relations with each other and with other mathematicians, incuding Hermann Weyl, Élie Cartan, Henri Poincaré, and many others. All these facets of the Erlangen program appear in the present volume.
The book is written by well-known experts in geometry, physics and history of mathematics and physics. It is addressed to mathematicians, to graduate students, and to all those interested in the development of mathematical ideas.



Free Loop Spaces in Geometry and Topology. Including the monograph *Symplectic cohomology and Viterbo's theorem* by Mohammed Abouzaid (IRMA Lectures in Mathematics and Theoretical Physics, Vol. 24)
Janko Latschev (University of Hamburg, Germany) and Alexandru Oancea (Université Paris 6, France), Editors

ISBN 978-3-03719-153-8. 2015. 496 pages. Hardcover. 17 x 24 cm. 78.00 Euro

One of the main purposes of this book is to facilitate communication between topologists and symplectic geometers thinking about free loop spaces. It was written by active researchers coming to the topic from both perspectives and provides a concise overview of many of the classical results, while also beginning to explore the new directions of research that have emerged recently. As one highlight, it contains a research monograph by M. Abouzaid which proves a strengthened version of Viterbo's isomorphism between the homology of the free loop space of a manifold and the symplectic cohomology of its cotangent bundle, following a new strategy.
The book grew out of a learning seminar on free loop spaces held at Strasbourg University and should be accessible to a graduate student with a general interest in the topic. It focuses on introducing and explaining the most important aspects rather than offering encyclopedic coverage, while providing the interested reader with a broad basis for further studies and research.

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