NEWSLETTER

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4 Presidents of the EMS.
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European Mathematical Society
25 years

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Cover photograph: P. Exner, President, and J.-P. Bourguignon, M. Sanz-Solé, A. Laptev, former presidents, at the celebration for the 25th anniversary of the EMS.

EMS Agenda

2016

3–6 March
Ethics Committee Meeting, Warsaw, Poland
Contact: matarne@math.aau.dk

18–20 March
Executive Committee Meeting, Institut Mittag-Leffler, Djursholm, Sweden

2–3 April
Presidents Meeting, Budapest, Hungary

9 April
Annual Meeting of the Committee for Developing Countries of the EMS, ICTP Trieste, Italy
http://euro-math-soc.eu/EMS-CDC/

15–16 April
ERCOM meeting, St. Petersburg, Russia

16–17 July
EMS Council, Humboldt University, Berlin, Germany

EMS Scientific Events

2016

15 March
Diderot Mathematical Forum 2016 “Biomedical Applications of Mathematics”
Universidad Complutense de Madrid, Spain; Université Paris Descartes, France; Politecnico di Milano, Italy

16–20 March
27th Nordic Congress of Mathematicians
Stockholm, Sweden
Bernoulli Society-EMS Joint Lecture: Sara van de Geer (ETH Zurich)

30 March–3 April
EUROMATH, Thessaloniki, Greece

11–15 July
EMS-IAMP Summer School in Mathematical Physics on “Universality, Scaling Limits and Effective Theories”
Roma, Italy
http://www.smp2016.cond-math.it/

18–22 July
7th European Congress of Mathematics, Berlin, Germany
http://www.7ecm.de/

25–26 August
Second Caucasian Mathematics Conference (CMC-II)
Lake Van, Turkey

2018

1–9 August
ICM 2018
Rio Centro Convention Center, Rio de Janeiro, Brazil

The EMS Jubilee: Challenges for the Next 25 Years

Richard Elwes (EMS Publicity Officer; University of Leeds, UK))

The roots of mathematics, like those of humanity itself, lie in Southern Africa. At the dawn of civilisation, thinkers in Mesopotamia and Egypt made early breakthroughs in notation and technique. Indian and Chinese mathematicians produced insights which remain with us today, while the Persian and Arabic traditions developed the subject over hundreds of years. Today, as in so many areas of life, the USA is a modern powerhouse. But, in our desire to give credit where it is due, and to honour the contributions of cultures which are too often overlooked, we should not get carried away. Our own continent of Europe has been home to a multitude of mathematical advancements since the time of Pythagoras. Just occasionally, it is worth reflecting and celebrating this glorious tradition.

Indeed, many European nations have their own illustrious mathematical histories. In the 19th and early 20th centuries, this led to the founding of plethora of national and regional mathematical societies, of which the oldest surviving is the Dutch *Koninklijk Wiskundig Genootschap*, founded in 1778. The European Mathematical Society is thus a latecomer, not born until 1990 in the Polish town of Mądralin. This year therefore, our society has reached 25 years of age, a youthful milestone which was celebrated in magnificent style at the Institut Henri Poincaré in Paris, on 22nd October. The day opened with an address from the Society's President, Pavel Exner, and comprised 4 plenary talks followed by a panel discussion on the state of European mathematics, as we look to the challenges of the next 25 years and beyond. Several of the themes from that conversation were well represented in the day as a whole, and perhaps it is worth drawing them out.

A major focus was the need for mathematics to be an outward-facing discipline, in several senses. Every aspect of today's society is influenced by life-transforming technologies whose design and operation relies on sophisticated mathematics. According to recent reports in UK, France, and Netherlands, the economic impact of Mathematics is enormous, to the tune of 9% of all jobs and 16% of Gross National Product. The market thus presents an unprecedented and growing demand for mathematical expertise, with data science in particular being an area of explosive growth.

In his plenary talk, Andrew Stuart spoke eloquently about one pressing challenge in this arena: the relationship between mathematical models and data. Taking the example of numerical weather forecasting, a technology as technically demanding as it is socially important, he discussed the difficulty of incorporating observational data into theoretical models, giving appropriate weight to both. He offered the opinion that mathematicians today stand in a similar position relative to Data, as they did to Analysis in the time of Fourier: we already have the basic language and techniques, but a revolution is surely imminent.

The guests for the EMS Jubilee at Institut Henri Poincaré. Photo courtesy of Elvira Hyvönen.

Why do mathematicians do mathematics? In truth, the answer is not usually because of its societal benefits or economic impact. At an individual level, we do it because we enjoy it. Depending on your perspective, we are either playful people who enjoy amusing ourselves with puzzles, or deep-thinkers who provide answers to some of the most profound questions our species can ask. (The paradox of our subject is that this distinction is, in fact, no distinction at all.)

The day saw two talks in this vein. The opening lecture by Hendrik Lenstra was an entertaining investigation of profinite number theory, meaning the structure of $\hat{\mathbb{Z}}$, the profinite completion of the integers. One delightful discussion involved the profinite extension of that staple of recreational mathematics, the Fibonacci numbers. Generations of school-students and lay-people have been amused by this recursive sequence, and as Lenstra showed, there is plenty of enjoyment to be had for professional mathematicians too. (See his essay in the Newsletter of the European Mathematical Society 61, September 2006, 15–19.)

In the afternoon session, we were treated to a talk from László Lovász on geometric representations of graphs. Here were beautiful problems and deep theorems, whose origins lie in puzzles accessible to school-children. He began with the theorem of Koebe that every planar graph has a circle representation: a set of non-overlapping discs in the plane, with two vertices joined by an edge if and only if their corresponding discs touch. Extending this is the Cage Theorem first proved by Andre'ev: that every 3-connected planar graph is isomorphic to the 1-skeleton of a convex 3-polytope, where every edge of the polytope touches a given sphere.

Another sense in which mathematics needs to outward facing is in the need to engage with our colleagues in other sciences. Mathematics has a long-standing relationship with physics, of course, but in the modern era biologists, chemists, medics, and others all have increasing needs of our expertise.

Laure Saint-Raymond delivered her plenary lecture on the subject of kinetic theory, a remarkable success story which other subjects would surely love to emulate. Here, mathematics has wonderfully illuminated an apparently intractable question: how to predict or analyse the (seemingly inherently unpredictable) behaviour of a gigantic system of interacting particles. Through the language of entropy and chaos theory, these terms have been picked apart and robust results obtained. In particular, she spoke of recent breakthrough work of her own with Isabelle Gallagher and Benjamin Texier, wherein the Boltzmann equation is rigorously derived as the limit of a system of hard spheres.

The formal part of the day was brought to an end with the panel discussion comprising Jean-Pierre Bourguignon, Maria Esteban, Roberto Natalini, Peter Bühlmann, and Ari Laptev, a fascinating and wide-ranging conversation touching on mathematical education, outreach, lobbying, publishing, ethics, and application. Afterwards, the assembled company retired to the splendid surroundings of the Mairie du 5ème Arrondissement for a welcome from the Mayor, Mme Florence Berthou, and to drink the health of the European Mathematical Society: Here's to another 25 years!

Pavel Exner (EMS President), Maria Esteban (ICIAM President elect), Florence Berthou (Maire du 5ème Arrondissement, Paris).

(Readers not able to attend the jubilee in Paris will find more coverage of the event in a special 100th edition of the Newsletter, next year.)

Announcement of the Next Meeting of the EMS Council Berlin, July 16 and 17, 2016

The EMS Council meets every second year. The next meeting will be held in Berlin, 16-17 July 2016, in the Senate Meeting Room in the Main Building of the Humboldt University at Unter den Linden 6. The council meeting starts at 14:00 on 16 July and ends at lunchtime on 17 July.

Delegates

Delegates to the council are elected for a period of four years. A delegate may be re-elected provided that consecutive service in the same capacity does not exceed eight years. Delegates will be elected by the following categories of members.

(a) Full Members

Full members are national mathematical societies, which elect 1, 2, 3 or 4 delegates according to their membership class. The membership class is decided by the council, and societies are invited to apply for the new class 4, which was introduced by the 2008 Council. However, the number of delegates for the 2016 Council is determined by the current membership class of the society.

Each society is responsible for the election of its delegates.

There is an online nomination form for delegates of full members. The nomination deadline for delegates of full members is 24 April 2016.

(b) Associate Members

Delegates representing associate members shall be elected by a ballot organised by the Executive Committee from a list of candidates who have been nominated and seconded by associate members, and who have agreed to serve. In October 2015, there were two associate members and, according to our statutes, these members may be represented by (up to) one delegate.

The delegate whose term includes 2016 is Mats Gyllenberg.

There is an online nomination form for delegates of associate members. The nomination deadline for delegates of associate members is 17 March 2016.

(c) Institutional Members
Delegates representing institutional members shall be elected by a ballot organised by the Executive Committee from a list of candidates who have been nominated and seconded by institutional members, and who have agreed to serve. In October 2015, there were 42 institutional members and, according to our statutes, these members may be represented by (up to) four delegates.

The delegates whose terms include 2016 are Joaquium Bruna and Alberto Pinto. The delegate who can be re-elected is Sverre Olaf Smalo.

There is an online nomination form for delegates of institutional members. The nomination deadline for delegates of institutional members is 17 March 2016.

(d) Individual Members
Delegates representing individual members shall be elected by a ballot organised by the Executive Committee from a list of candidates who have been nominated and seconded, and who have agreed to serve. These delegates must themselves be individual members of the European Mathematical Society.

In October 2015, there were 2650 individual members and, according to our statutes, these members may be represented by (up to) 27 delegates. However, this number may have increased by the time we call the election (if any) for individual members.

Here is a list of the current delegates of individual members whose terms include 2016:

Arne Ball
Vasile Berinde
Maria Esteban
Vincenzo Ferone
Christian Kassel
Luis Narvaáez Macarro
Jiří Rákosník
Oriol Serra

Here is a list of the delegates of individual members who could be re-elected to the 2016 Council:

Peter Benner
Thierry Bouche
Mireille Chaleyat-Maurel
Krzysztof Ciesielski
Mirna Džamonja
Pavel Exner
Vincent Heuveline
Arne Jensen
Paul C. Kettler
Ari Laptev
José Francisco Rodrigues
Marie-Françoise Roy
Stepan Agop Tersian
Robin Wilson

There is an online nomination form for delegates of individual members. The nomination deadline for delegates of individual members is 17 March 2016.

Agenda
The Executive Committee is responsible for preparing the matters to be discussed at council meetings. Items for the agenda of this meeting of the council should be sent as soon as possible, and no later than 24 April 2016, to the EMS Secretariat in Helsinki.

Executive Committee
The council is responsible for electing the President, Vice-Presidents, Secretary, Treasurer and other Members of the Executive Committee. The present membership of the Executive Committee, together with their individual terms of office, is as follows.

President: Pavel Exner (2015–2018)

Vice-Presidents: Franco Brezzi (2013–2016)
Martin Raussen (2013–2016)

Secretary: Sjoerd Verduyn Lunel (2015–2018)

Treasurer: Mats Gyllenberg (2015–2018)

Members: Alice Fialowski (2013–2016)
Gert-Martin Greuel (2013–2016)
Laurence Halpern (2013–2016)
Volker Mehrmann (2011–2018)
Armen Sergeev (2013–2016)

Members of the Executive Committee are elected for a period of four years. The president can only serve one term. Committee members may be re-elected, provided that consecutive service does not exceed eight years.

The council may, at its meeting, add to the nominations received and set up a Nominations Committee, disjoint from the Executive Committee, to consider all candidates. After hearing the report by the Chair of the Nominations Committee (if one has been set up), the council will proceed to the election of Executive Committee posts.

All these arrangements are as required in the Statutes and By-Laws, which can be viewed on the webpage of the council:

http://www.euro-math-soc.eu/governance

The nomination forms for delegates can be found here

http://www.euro-math-soc.eu/nomination-forms-council-delegates

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LMS-EMS Joint Mathematical Week-end in Birmingham, UK

Richard Elwes (EMS Publicity Officer; University of Leeds, UK)

The week-end 18–20 September saw mathematicians from around the world congregate at the University of Birmingham, UK, for a conference in celebration of two birthdays: the 150th of the venerable London Mathematical Society (LMS), and the 25th of the relatively youthful European Mathematical Society.

Under the watch of the Joseph Chamberlain Memorial Clock-tower (or 'Old Joe', the world's tallest free-standing clock-tower), participants divided between parallel sessions on the themes of Algebra, Combinatorics, and Analysis, and reunited for plenary talks from some of mathematics' current leading lights.

After warm greetings from Terry Lyons and Pavel Exner, the two societies' respective Presidents, and from Andrew Schofield, Head of Birmingham University's College of Engineering and Physical Sciences, the meeting got underway with a plenary talk from Noga Alon (Tel Aviv and Princeton), on the subject of *Graphs, vectors and integers.* His focus was Cayley Sum Graphs of finite Abelian groups, and the role they play in subjects from Graph Theory to Information Theory. Aner Shalev (Jerusalem) later delivered the day's second plenary talk, on *Groups in Interaction*, discussing several instances of interplay between group theory and other subjects, including probability theory, algebraic geometry, and number theory.

Away from the lecture theatres, mathematicians were spotted enjoying Balti curry (a famous Birmingham creation, along with the postage stamp and the pneumatic tyre) and enjoying the outstanding collection of paintings at the Barber Institute of Fine Arts, next door to the School of Mathematics.

Stefanie Petermichl (Toulouse) delivered the first plenary session of Saturday 19th, on *Optimal control of second order Riesz transforms on multiply-connected Lie groups*, discussing progress on controlling the norms of certain classical operators on groups. She was followed by Béla Bollobás (Cambridge and Memphis) speaking on *Percolation and random cellular automata*. He paused during his talk to pay tribute to two friends who had recently passed away: Ian Cassels, Head of Mathematics during his PhD at Cambridge, and Bollobás's own graduate student Charles Read (Leeds). The day's final plenary session was from Timothy Gowers on the subject of *Interleaved products in highly non-Abelian groups*, an algebraic problem motivated by a question in cryptography.

The conference dinner took place on Saturday evening in Birmingham University Staff House, where delicious food was consumed, and many glasses were raised in cheerful celebration of the two societies' birthdays.

Rounding off the meeting on Sunday 20th was Keith Ball (Warwick), with an entertaining plenary talk exploring *The probabilistic character of high-dimensional objects*. Then with hearty thanks to the organisers, Chris Parker, Anton Evseev, Maria Carmen Reguera and Andrew Treglown, and with congratulations to EIisa Covato (Bristol) and Robert Hancock (Birmingham) winners of the graduate student poster competition, an excellent celebratory weekend drew to a close.

A Message from the Ethics Committee

Arne Jensen (Aalborg University, Denmark)

One of the tasks given to the Ethics Committee is to consider possible violations of the Code of Practice, in particular concerning plagiarism.
If you, as an author, believe that you are the victim of plagiarism and wish to bring a case to the Ethics Committee, you should undertake the following steps:

1. Gather detailed information on the suspected plagiarism.
2. Give this material to a colleague whose opinions you respect and ask this person to evaluate your case.

Assuming that this colleague agrees with you:

3. Contact the author(s) and the editor(s) of the journal involved, presenting your evidence of plagiarism.

This step may resolve the case. If no replies or negative replies are received, you may consider bringing the case to the Ethics Committee. You should start by contacting the Chair informally to ensure that your submission satisfies the requirements of the Code of Practice. Then make a formal submission.

4. The Committee will acknowledge receipt and will decide whether there is a prima facie case. If so, the committee will consult experts as specified in the Code of Practice.

Arne Jensen
Chair (until end of 2015)
Ethics Committee

The Code of Practice can be found at
http://www.euro-math-soc.eu/committee/ethics

IMU Committee for Women in Mathematics (CWM) Funding Call for 2016

Marie-Françoise Roy (Université de Rennes 1, France) and Caroline Series (University of Warwick, Coventry, UK)

The IMU's Committee for Women in Mathematics (http://www.mathunion.org/cwm/) invites proposals for funding of up to € 3000 for activities or initiatives taking place in 2016, and aimed at establishing or supporting networks for women in mathematics, preferably at the continental or regional level, and with priority given to networks and individuals in developing or emerging countries. CWM's help could include, for example, funding meetings, travel for individuals for consultation purposes, or advice and support in creating websites. Other ideas for researching and/or addressing problems encountered by women in mathematics may also be considered.

Proposers should write a short account (no more than two pages) explaining the nature of their activity and how it fulfills the above aims, as well as indications on how CWM money would be spent and other funding which may be available. There will be one call for applications regarding activities in 2016 with deadline of **15th January 2016**. It is anticipated that further calls will be made in subsequent years.

Applications should be sent to info-for-cwm@mathunion.org

Successful applications will be informed no later than February 29, 2016. Depending on demand, successful applications may not be funded in full. Successful applicants will be asked to send before the end of 2016 a short report of the activity with details of how the budget was spent.

IMU-CWM Committee
October 2015

5ECM in Amsterdam

André Ran (Vrije Universiteit Amsterdam, The Netherlands) and Herman te Riele (CWI, Amsterdam, The Netherlands)

In 2001, the two of us worked together for the first time on the organisation of the Dutch Mathematical Conference, which is an annual congress of the Dutch Royal Mathematical Society (KWG). Apparently, we did a nice job because, later in the year, we received phone calls from our respective bosses telling us that we were the chosen victims to try to get the fifth European Congress of Mathematics to Amsterdam. We said yes (reluctantly) and were joined by Jan Wiegerinck from the Korteweg-de Vries Institute of Mathematics of the University of Amsterdam, making a triumvirate. Thus, it became a joint effort of the three mathematics institutes in Amsterdam.

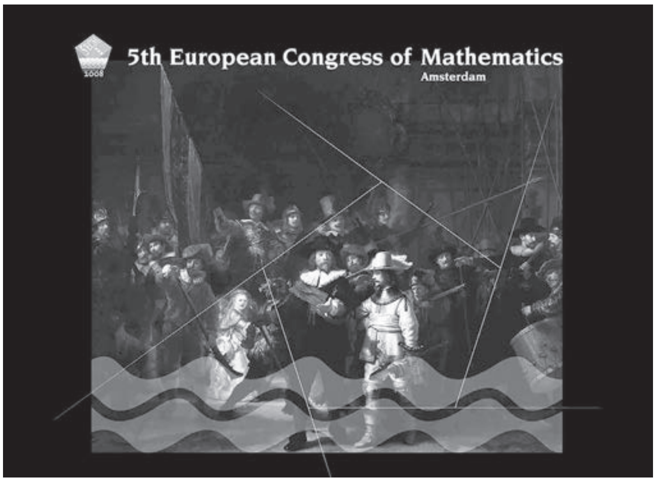
In our innocence, we thought we would get it to Amsterdam and then the big shots in the Dutch mathematical world would take over to actually organise it. So, we thought it would be a job spread over a few years, to write a bid-book and negotiate with the selection committee of the EMS. No big deal. Boy, how wrong we were.

Initially, everything seemed to go as planned. The bid-book was a piece of cake and we obtained promises of support from ministries, the municipality, scientific organisations, etc. The site visit in 2003 was a very pleasant experience and, in 2004, the decision was taken that 5ECM would be held in Amsterdam. To get an idea of what this would entail, the "triumvirate" participated in 4ECM in Stockholm and we got a lot of good ideas. By then, it was already abundantly clear to us that now we had secured 5ECM for the Netherlands, and for Amsterdam in particular, it was up to us to actually organise it.

And then the hard work began. Obviously, many who had pledged support notified us that that was meant as support in spirit and not in actual euros. Finding funding was a constant worry for us but Jan took it upon himself to organise this and managed to come up with interesting sponsors. It became clear that support was a matter of asking the right persons and we finally managed to get a balanced budget, as well as a nice location. All kinds of other organisational matters were solved for us by the people from the organising agency ICS International Conference Services B.V. (now called MCI Amsterdam).

The matter of a suitable venue was actually solved during the site visit in 2003. We had several options but the RAI convention centre in the southern part of town turned out to be an excellent choice, with several hotels nearby and easy train connections to Schiphol Airport.

In the meantime, the scientific organising committee, headed by Lex Schrijver, and the prize committee, headed by Rob Tijdeman, had jointly put together a very interesting programme.



By the time the congress was about to start, we still had some small items on our list to do but, on the weekend before the congress, these were finally put to rest. On Monday 14 July 2008 at 8:45 in the morning, about 800 mathematicians from more than 60 European countries gathered at the RAI convention centre in Amsterdam for the opening ceremony.

The opening ceremony was a spectacular affair. A *tableau vivante* of Rembrandt van Rijn's most famous painting "The Nightwatch" was created on the stage by a group of re-enactors. This was accompanied by drum rolls from the drummers in the painting. In their midst was the painter himself, who was played by the first speaker of the opening session Robbert Dijkgraaf, currently Director of the Institute for Advanced Study in Princeton, at the time a professor at the University of Amsterdam and Chairman of the KNAW (Royal Dutch Academy of Sciences). An excellent opening address by Dijkgraaf was followed by a warm welcome by Ari Laptev, then president of the European Mathematical Society. After that came the ceremony announcing the prize winners: 10 young mathematicians were on stage for the EMS prizes, together with the winner of the Felix Klein prize. All in all, the conference was off to a good start.

There were several highlights from the conference: of course the 10 plenary lectures (including those of Richard Taylor and László Lovász) and 33 invited lectures, the lectures by the 10 EMS prize winners sponsored by the Dutch Foundation *Compositio Mathematica*: Arthur Avila, Alexei Borodin, Ben Green, Olga Holtz, Bo'az Klartag, Alexander Kuznetsov, Assaf Naor, Laure Saint-Raymond, Agata Smoktunowicz and Cédric Villani, and a lecture by the Felix Klein prize winner Josselin Garnier. Also, there were three science lectures: one on quantum information theory by Ignacio Cirac, one on climate change by Tim Palmer and one on mathematical biology by Jonathan Sherratt.

The 5ECM conference also incorporated the annual conference of the Royal Dutch Mathematical Society (Koninklijk Wiskundig Genootschap, KWG) and two highlights were prizes awarded at those meetings. On Monday evening, there was the ceremony around the Brouwer prize. This is a prize that is awarded by the KWG every three years, and 2008 happened to be one of those years. The recipient was Phillip Griffiths (Princeton), who gave his lecture on Monday evening in the auditorium of the Vrije Universiteit. The ceremony was preceded by organ music, played by mathematician Jozef Steenbrink, and followed by the presentation of the Brouwer Medal and a welcome reception. In the afternoon, Dirk van Dalen gave an historical lecture about L.E.J. Brouwer. Also during 5ECM, the Beeger prize was awarded to Dan Bernstein. The Beeger prize is given by CWI Amsterdam at every other meeting of the KWG and, again, 2008 happened to be a year for this.

During lunch breaks, movies were shown about Kurt Gödel and Wolfgang Döblin. Two round table meetings took place during the conference: one on mathematics and industry, and one on mathematics in developing countries. On Thursday afternoon, the Philips PhD prize lectures were presented and the Philips prize was awarded to Erik Jan van Leeuwen of CWI Amsterdam. All in all, the conference programme had many prize lectures, special lectures and other features, like the selling of Brouwer stamps and of Gaussian prime tablecloths by KWG. A special issue of Nieuw Archief voor Wiskunde called *Amsterdam Archive* (http://www.nieuwarchief.nl/serie5/pdf/naw5-2008-09-2-091.pdf) was included in the conference bag. It presents a Dutch view of the world of mathematics. The proceedings of 5ECM were published in 2010 by the EMS.

No conference goes without its problems and upheavals. For us, the most notable was the following. An extremely well known mathematician was stopped when entering the main lecture hall by a student: "Sir where is your badge?" "I don't have one, but my name is…," to which the student answered: "Well, sir, I do not care who you are; we are under strict orders not to let anyone in without a badge." We do apologise but we had to be strict; nobody got in without a badge, not even one of the most prominent mathematicians of the last half century. Notably, the problem was solved and the person in question very much enjoyed the president's dinner later that evening at the head office of ING, close to the conference venue.

On Wednesday, the congress dinner party was held in Hotel Arena. That turned out to be more than a bit misleading: the party was actually held in an old church and was, like the rest of the congress, a memorable event.

For us as organisers, the week went by in a blur. There was always something to be done but our team got us through the week in one piece. Nevertheless, we were quite relieved when the closing ceremony was over on Friday 18 July, around six in the afternoon. We survived the experience and we are happy that we were able to give so many of our colleagues from around the world a nice conference in what we hope they found a very enjoyable and hospitable city.

Herman te Riele joined CWI Amsterdam in 1970, until his retirement by the end of 2011. He carried out research on numerical mathematics and computational number theory, often with help of large-scale computers. Besides that he has been and still is active in organizational matters concerning mathematics (like member of the board of the Royal Dutch Mathematical Society (since 2003), the organization of the BeNeLux Mathematical Congress in Amsterdam (22–23 March, 2016), the organization of the fifth European Congress of Mathematics (Amsterdam, 2008), secretary of ERCOM (European research Centers on Mathematics (2006–2009) and secretary of the Review Committee for mathematical research at six Dutch Universities (2009–2010).

André Ran is the Desmond Tutu professor in mathematics at Vrije Universiteit Amsterdam, where he has been since 1985. He also holds an extra-ordinary professorship at North West University in South Africa. His research interests are linear algebra, operator theory and systems and control theory.

6ECM in Kraków – Organizer's Reminiscences

Stefan Jackowski (University of Warsaw, Poland)

6ECM, held in Kraków, 2–7 July 2012, was the most recent ECM, thus many readers may still remember it, if not as participants then as readers of the EMS Newsletter, where a detailed report "A Dozen Facts about the 6th European Congress of Mathematics" was published in September 2012. This article will therefore be mainly devoted to my personal reminiscences related to the congress and to what happened behind the scenes.

A long time ago – Mądralin 1990

I think the first time I heard about the idea of the European Congresses of Mathematics was in October 1990, at a meeting in a residence of the Polish Academy of Science in Mądralin near Warsaw, when the European Mathematical Society was founded. I must confess that, at the time, the idea of having another big congress, similar to the ICM, in which prizes similar to Fields Medals were awarded, did not appeal to me. I was hoping to hear some new ideas of European collaboration.

I can't remember why I attended the meeting of prominent representatives of the European mathematical community. At that time, I was less than 40 and I did not have any formal position. I think I was invited because my wife and I had had the honour of receiving Michael Atiyah at home when he came to Warsaw to discuss the organisation of the ICM 1983. Sir Michael was also the key person in the process of founding the EMS and the most distinguished participant of the Mądralin meeting.

His negotiating partner back in the 1980s and 1990 was Professor Bogdan Bojarski from the Polish Academy of Sciences. In some Soviet bloc countries, academies had a monopoly on representing the scientific community internationally, thus the Polish mathematical community couldn't be formally represented by the Polish Mathematical Society – a limited monopoly lasted long after the fall of the system. The second host in Mądralin was Professor Andrzej Pelczar, then President of the Polish Mathematical Society (PTM), who later became Vice-President of the EMS (1997–2000).

Utrecht 2008

It was Andrzej Pelczar's idea to invite the European Congress of Mathematics to Kraków, his beloved hometown, and to Jagiellonian University, his alma mater. Having this in mind, he promoted, in 2006, the upgrade of the Polish Mathematical Society membership of the EMS and then prepared a bid which was submitted to the Executive Committee of the EMS in 2007. I signed the bid as President of the PTM, assuming that all the work would be done in Kraków under Andrzej's supervision and my role would just be formal representation of the society. Organisation of the ECM was not on my list of priorities when I became President of the PTM in 2005. Then I got involved in the organisation of a joint meeting with the AMS in Warsaw in 2007 and, after its success, I planned similar meetings with Germany and Israel. I believed that bilateral meetings, where both partners collaborate on the programme, served the Polish community better then a big international meeting, where the local organisers are primarily responsible for financial support and logistics, with little influence on the scientific programme.

Three cities wanted to organise 6ECM and submitted their bids: Kraków, Prague and Vienna. Representatives of the EMS Executive Committee visited all three cities in late October 2007. After the visit they presented their evaluation, writing about Kraków as follows:

> *"The conference venue in Krakow was extremely beautiful and practical. Everything needed for the conference is available in one lovely building, close to the city centre. Also, this is owned by the University, so the price will be low. Andrzej Pelczar (the former Rector) has great local influence and the bid is strongly supported by other Universities. The team behind him was not so obvious to us, though. Another possible factor in Krakow is the football competition* [Euro 2012, organised jointly by Poland and Ukraine – SJ]*, which will almost certainly inhibit outreach activity."*

The EMS Executive Council met in Utrecht just before the 5ECM in Amsterdam and chose Kraków for 6ECM. I think the decisive role was played by the strong determination of Andrzej Pelczar (a figure well known to the EMS) to organise the congress, as well as the visiting team's very positive impression of the congress venue: the Auditorium Maximum of Jagiellonian University. The other two cities presented very attractive outreach programmes but couldn't offer as convenient facilities.

Back home

When Polish representatives returned home from Utrecht, difficult negotiations between the PTM and Jagiellonian University began. Since both institutions signed the bid and accepted responsibility for the congress, it seemed natural to specify the contributions expected of each of them. Public universities in Poland are subject to detailed and restrictive regulations concerning spending money, whereas a society, as a private entity, can be much more flexible. At the time, Andrzej Pelczar was very active locally and at the national and international level trying to get support and attract interest in the congress. It was all stopped by Andrzej's sudden death on 18 May 2010.

Some people couldn't imagine ECM without Andrzej. I even heard comments that it would be better to give up organisation of the congress. For my PTM collaborators and me, this was unthinkable. My second term as PTM president was ending on 31 Dec 2010 but, under these new circumstances, I decided to stay for a third term, 2011–13.

The unexpected absence of Professor Pelczar created many misunderstandings. We were not aware of various traditional and oral commitments that had been made, e.g. concerning funding. The misunderstandings became clear when I was invited to report on preparations of the congress at the Executive Committee meetings in Lausanne, Firenze and Ljubljana. In Ljubljana, the Executive Committee decided that for the next congress (2016), a document would be signed by the EMS President and the Chair of the Organising Committee, listing the commitments agreed to by the Organising Committee.

After Andrzej's death, the Dean of Mathematics at Jagiellonian University Roman Srzednicki (a former student of Andrzej) assumed his duties. In the Summer of 2010, we both gave presentations on the preparations of the 6ECM at the Executive Council Meeting in Sofia. Later, President of the EMS Marta Sanz-Solé and I made a presentation to the IMU General Assembly in Hyderabad. In January 2011, Marta visited Kraków to discuss progress in the organisation of the congress. Although several representatives of the university and city authorities received her and assured her of their support, she probably realised that organisation of the congress was at a crossroads.

The lobby of the 6ECM. Photo by Ada Pałka.

In the Spring of 2011, a new Executive Organising Committee (EOC) was founded, consisting of three representatives of the PTM: Krystyna Jaworska (Secretary and Treasurer of the PTM, Military University of Technology, Warszawa), Wacław Marzantowicz (Vice-President of the PTM, Adam Mickiewicz University, Poznań) as well as the author (Chair of the EOC) and three representatives of Jagiellonian University (UJ). Shortly after, Roman got sick and had to withdraw from organisational work. Representatives of UJ were changing; finally, from January 2012, they were Zbigniew Błocki (Director of the Mathematics Institute UJ, Vice-Chair of the EOC), Piotr Tworzewski (Vice-Rector of the UJ) and Robert Wolak (UJ). The EMS Executive Committee was seriously worried but, to support the smooth and more efficient organisation of the congress, the change was accepted.

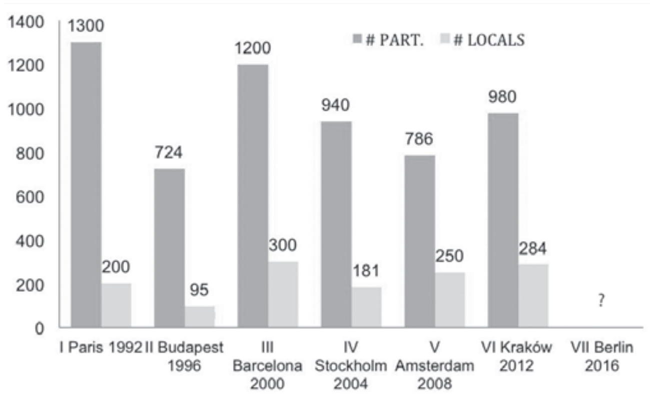
Last lap

Invitations to plenary and invited speakers were sent out in May 2011. The President of the EMS, and representatives of the PTM and UJ as co-organisers of the congress, signed the invitations. We offered to cover all the usual expenses. However, at that time, applications for support were still pending. The First Announcement was distributed in July 2011, a year before the congress. Pre-registration was opened on the 6ECM website.

At the end of June 2011, the 8th European Conference on Mathematical and Theoretical Biology was held at Auditorium Maximum, the future 6ECM venue. Members of the EOC went there to monitor how a conference of a size comparable to the ECM would fit in the building. Observations were carefully analysed and this helped us to arrange the space more comfortably. In particular, we noticed that there were not enough seats in the lobbies for informal discussions and registration booths took up too much space. Thus, just before the 6ECM, we rented a number of sofas and comfortable chairs; instead of registration booths, we just put tables at the side.

I distinctly remember the day at the beginning of July when, while on vacation, I worked on some 6ECM grant application – from that day, preparations of the 6ECM became my full-time job, which continued until the last day of the congress a year later. My closest collaborator was Krystyna Jaworska, who travelled with me many times from Warsaw to Kraków. She was in charge of finances of the 6ECM, starting from grant applications and ending in settling bills. She was involved in almost every aspect of preparation of the 6ECM. The EOC was also greatly helped by a large group of young Kraków mathematicians and students.

Registration of participants opened on the 6ECM website in early November 2011. Based on the experience of past ECMs, we were concerned that participation may turn out to be low [see diagram below]. At the conference on mathematical biology mentioned above, there were 950 participants, out of which over 600 presented their results. We heard comments that it was difficult to get support for coming to the 6ECM without the opportunity to speak, and complaints about an unbalanced scientific programme. Some mini-symposia organisers needed more time. As a response to these signals, the EOC proposed holding Satellite Thematic Sessions (STSs) during the congress. After announcing it in the Second Announcement, we received many applications from participants who wanted to organise STSs. They submitted proposals of subjects and speakers. Making a complete list of speakers, etc., was left to the STS organisers. Proposals were accepted by the EOC; in some cases, we suggested merging proposals into one session. Only registered 6ECM participants were eligible to speak at an STS. I believe it helped to increase participation in the congress. Some people told me that they came to Kraków mainly for STSs. There were 15 STSs held during the 6ECM with over 150 talks.



Participants of the past ECMs.

The congress

There are two gratifying moments in a conference organiser's life: a smooth opening and a successful closing.

The opening of the 6ECM was well-attended by state officials. The Minister of Science and Higher Education (Professor of Law Barbara Kudrycka) attended and gave a well-received speech, assuring the audience of the Government's high regard for mathematics: "Poland is very aware of the importance of science education, which of course includes mathematics." She went on to say: "The Government is aware of the special role of mathematical abilities in the labour market." She mentioned an increase of funds for fundamental research, in particular for mathematics. After the speeches, the President of the EMS, together with the chairs of the prize committees, presented the EMS prizes to the recipients. They gave lectures during the congress. In the booklet devoted to 2012

Students wearing 6ECM T-shirts at the receptionist desk. Photo by Ada Pałka.

Prize Winners, a list of the past-winners of the EMS prize for young mathematicians was included, with information on which of them had later received the Fields Medal (9 out of 50).

Adrian Constantin gave the first plenary lecture with the appealing title "Some mathematical aspects of water waves". The plenary speakers not only gave lectures but also answered two questions asked by the editors of the special volume of the PTM journal "Wiadomości matematyczne". The first concerned the motivation of interest in the subject of the lecture; the second concerned challenges in the field of the research. The answers took up 30 pages of the volume distributed to all participants. The volume also contained over 20 survey articles by foreign mathematicians on ideas and results of selected Polish mathematicians and their influence on mathematics. Articles were devoted to Stefan Banach, Karol Borsuk, Samuel Eilenberg, Józef Maria Hoene-Wroński, Andrzej Lasota, Stanisław Łojasiewicz, Wacław Sierpiński, Hugo Steinhaus, Witold Wolibner, Tadeusz Ważewski and Antoni Zygmund, and many other names were mentioned.

The next best attended event after the opening was certainly the congress banquet, which was held at the cloister of the functioning Franciscan Monastery in Kraków Old Town. Before the banquet, there was a guided tour to a nearby church of great historical and artistic value.

At the closing ceremony, prizes for the 10 best research posters were presented. The poster session turned out to be a great success. From over 300 submissions, the poster committee selected 186 posters, which were displayed in a suitably arranged basement (garage) in the auditorium. A jury, consisting of *ad hoc* invited congress participants, awarded the prizes. Publishers who exhibited their publications during the congress funded prizes – mostly books or electronic subscriptions.

Marta told me that there is a tradition of presenting a report on the congress at the closing ceremony. I wondered what kind of report might be interesting to the most faithful participants who had stayed to its very end. I asked my collaborators to prepare statistics concerning the structure of participation of the congress and presented them to the audience – most are included in my article report "A Dozen Facts about the 6th European Congress of Mathematics".

Finally, I invited all 6ECM organisers that were present up to the podium. The President of the EMS expressed thanks to the speakers and the participants, as well as the organisers, and declared 6ECM closed. [photo in Newsletter Sept 2012]

Christian Baer, the President of Deutsche Mathematiker-Vereinigung, invited 6ECM participants to the next 7ECM in Berlin.

After the congress – was it worth it?

After the 6ECM experience, I like congresses more than I did before and I'm more convinced that the effort to organise them is sensible. As Marta said at the opening speech, the congress is a "feast of mathematics" and it helps to maintain unity of mathematics – which is exceptional in contemporary science. It did serve this purpose! While the function of congresses and conferences as a means of transmitting scientific information is diminishing, their role in maintaining personal relations between mathematicians may be growing. You could see it in the lobby of the 6ECM. It is a great opportunity to meet people from different fields, whom you don't see at the specialised meetings. In today's world, it is particularly important to promote close contact between European mathematicians.

During preparations for the congress, I profited a lot from the experience of the organisers of the past ECMs and ICMs. But these were informal conversations and passing on of various materials, etc. I think it would be useful to create an ECM advisory committee which would preserve know-how, experience and knowledge about good practices. There is no reason to re-invent the wheel every time; it is enough to improve it.

A great reward for the organisers was receiving many kind letters after the congress. Let me quote one – from the founder of the first ECM in Paris in 1992:

> *"I take this opportunity to congratulate you again for the organization of the 6th ecm. It was perfect. (…). On the top, the statistics you wrote at the closing ceremony were quite interesting. I am happy that my idea – called "foolish" by influential French mathematicians in the 90's – has finally taken a good shape thanks to European mathematicians like you. (…). Max Karoubi (organiser of the 1ECM in Paris)."*

Stefan Jackowski lectures at the University of Warsaw (UW). His research interests are in algebraic topology. He has been: Dean of the Faculty of Mathematics, Informatics and Mechanics UW (1990–96, 1999–2005); President of the Polish Mathematical Society (2006–2013); Member of the EMS Executive Council (2008–14); EMS lay auditor for 2015–16; and Chair of the Executive Organising Committee of 6ECM (2011–12). Since 2012, he has been a member of the Committee for Evaluation of Research Units at the Polish Ministry of Science and Higher Education.

Problems for Children 5 to 15 Years Old

Vladimir Arnold

I wrote these problems in Paris in the spring of 2004. Some Russian residents of Paris had asked me to help cultivate a culture of thought in their young children. This tradition in Russia far surpasses similar traditions in the West.

I am deeply convinced that this culture is developed best through early and independent reflection on simple, but not easy, questions, such as are given below. (I particularly recommend Problems 1, 3, and 13.)

My long experience has shown that C-level students, lagging in school, can solve these problems better than outstanding students, because the survival in their intellectual "Kamchatka" at the back of the classroom "demanded more abilities than are requisite to govern Empires", as Figaro said of himself in the Beaumarchais play. A-level students, on the other hand, cannot figure out "what to multiply by what" in these problems. I have even noticed that five year olds can solve problems like this better than can school-age children, who have been ruined by coaching, but who, in turn, find them easier than college students who are busy cramming at their universities. (And Nobel prize or Fields Medal winners are the worst at all in solving such problems.)

1. Masha was seven kopecks short of the price of an alphabet book, and Misha was one kopeck short. They combined their money to buy one book to share, but even then they did not have enough. How much did the book cost?

2. A bottle with a cork costs $1.10, while the bottle alone costs 10 cents more than the cork. How much does the cork cost?

3. A brick weighs one pound plus half a brick. How many pounds does the brick weigh?

4. A spoonful of wine from a barrel of wine is put into a glass of tea (which is not full). After that, an equal spoonful of the (non-homogeneous) mixture from the glass is put back into the barrel. Now there is a certain volume of "foreign" liquid in each vessel (wine in the glass and tea in the barrel). Is the volume of foreign liquid greater in the glass or in the barrel?

5. Two elderly women left at dawn, one traveling from A to B and the other from B to A. They were heading towards one another (along the same road). They met at noon, but did not stop, and each of them kept walking at the same speed as before. The first woman arrived at B at 4 PM, and the second arrived at A at 9 PM. At what time was dawn on that day?

6. The hypotenuse of a right-angled triangle (on an American standardized test) is 10 inches, and the altitude dropped to it is 6 inches. Find the area of the triangle.

American high school students had been successfully solving this problem for over a decade. But then some Russian students arrived from Moscow, and none of them was able to solve it as their American peers had (by giving 30 square inches as the answer). Why not?

7. Victor has 2 more sisters than he has brothers. How many more daughters than sons do Victor's parents have?

8. There is a round lake in South America. Every year, on June 1, a Victoria Regia flower appears at its center. (Its stem rises from the bottom, and its petals lie on the water like those of a water lily). Every day the area of the flower doubles, and on July 1, it finally covers the entire lake, drops its petals, and its seeds sink to the bottom. On what date is the area of the flower half that of the lake?

9. A peasant must take a wolf, a goat and a cabbage across a river in his boat. However the boat is so small that he is able to take only one of the three on board with him. How can he transport all three across the river? (The wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage.)

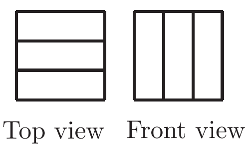
10. During the daytime a snail climbs 3 cm up a post. During the night it falls asleep and slips down 2 cm. The post is 10 m high, and a delicious sweet is waiting for the snail on its top. In how many days will the snail get the sweet?

11. A hunter walked from his tent 10 km. south, then turned east, walked straight eastward 10 more km, shot a bear, turned north and after another 10 km found himself by his tent. What color was the bear and where did all this happen?

12. High tide occurred today at 12 noon. What time will it occur (at the same place) tomorrow?

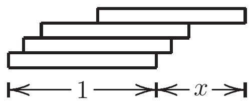
13. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2 cm thick, and the front and back covers are each 2 mm thick. A bookworm has gnawed through (perpendicular to the pages) from the first page of volume 1 to the last page of volume 2. How long is the bookworm's track? [This topological problem with an incredible answer – 4 mm – is totally impossible for academicians, but some preschoolers handle it with ease.]

14. Viewed from above and from the front, a certain object (a polyhedron) gives the shapes shown. Draw its shape as viewed from the side. (Hidden edges of the polyhedron are to be shown as dotted lines.)

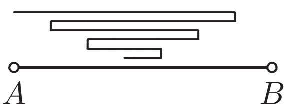


15. How many ways are there to break the number 64 up into the sum of ten natural numbers, none of which is greater than 12? Sums which differ only in the order of the addends are not counted as different.

16. We have a number of identical bars (say, dominoes). We want to stack them so that the highest hangs out over the lowest by a length equal to *x* bar-lengths. What is the largest possible value of *x*?



17. The distance between towns *A* and *B* is 40km. Two cyclists leave from *A* and *B* simultaneously traveling towards one another, one at a speed of 10km/h and the other at a speed of 15 km/h. A fly leaves *A* together with the first cyclist, and flies towards the second at a speed of 100 km/h. The fly reaches the second cyclist, touches his forehead, then flies back to the first, touches his forehead, returns to the second, and so on until the cyclists collide with their foreheads and squash the fly. How many kilometers has the fly flown altogether?

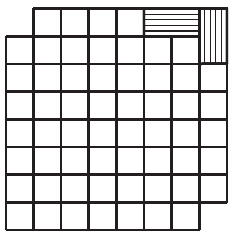


18. Vanya solved a problem about two pre-school age children. He had to find their ages (which are integers), given the product of their ages.

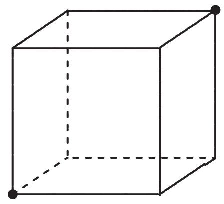
Vanya said that this problem could not be solved. The teacher praised him for a correct answer, but added to the problem the condition that the name of the older child was Petya. Then Vanya could solve the problem right away. Now you solve it.

19. Is the number 140 359 156 002 848 divisible by 4 206 377 084?

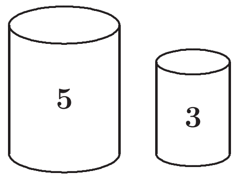
20. One domino covers two squares of a chessboard. Cover all the squares except for its two opposite corners (on the same diagonal) with 31 dominoes. (A chessboard consists of $8 \times 8 = 64$ squares.)



21. A caterpillar wants to slither from the front left corner of the floor of a cubical room to the opposite corner (the right rear corner of the ceiling). Find the shortest route for such a journey along the walls of the room.

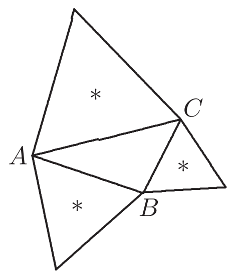


22. You have two vessels of volumes 5 liters and 3 liters. Measure out one liter, leaving the liquid in one of the vessels.

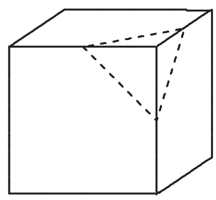


23. There are five heads and fourteen legs in a family. How many people and how many dogs are in the family?

24. Equilateral triangles are constructed externally on sides *AB*, *BC*, and *CA* of a triangle *ABC*. Prove that their centers (marked by asterisks on the diagram) form an equilateral triangle.

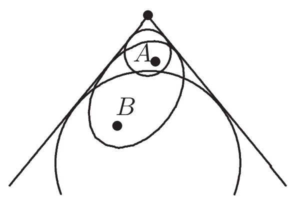


25. What polygons may be obtained as sections of a cube cut off by a plane? Can we get a pentagon? A heptagon? A regular hexagon?

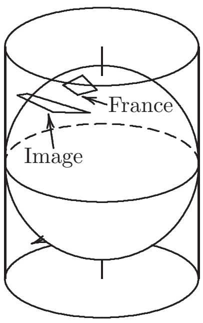


26. Draw a straight line through the center of a cube so that the sum of the squares of the distances to it from the eight vertices of the cube is (a) maximal, (b) minimal (as compared with other such lines).

27. A right circular cone is cut by a plane along a closed curve. Two spheres inscribed in the cone are tangent to the plane, one at point A and the other at point B. Find a point C on the cross-section such that the sum of the distances *CA* + *CB* is (a) maximal, (b) minimal.



28. The Earth's surface is projected onto a cylinder formed by the lines tangent to the meridians at the points where they intersect the equator. The projection is made along rays parallel to the plane of the equator and passing through the axis of the earth that connects its north and south poles. Will the area of the projection of France be greater or less than the area of France itself?

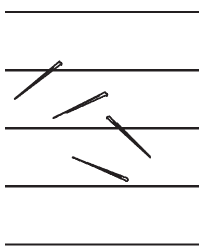


29. Prove that the remainder upon division of the number 2^{p-1} by an odd prime p is 1 (for example: $2^2 = 3a + 1$, $2^4 = 5b + 1$, $2^6 = 7c + 1$, $2^{10} - 1 = 1023 = 11 \cdot 93$).

30. A needle, 10 cm long, is thrown randomly onto ruled paper. The distance between neighboring lines on the paper is also 10 cm. This is repeated N (say, a million) times. How many times (approximately, up to a few per cent error) will the needle fall so that it intersects a line on the paper?

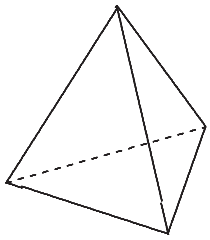
One can perform this experiment with $N = 100$ instead of a million throws. (I did this when I was 10 years old.)

[The answer to this problem is surprising: $\frac{2}{\pi}N$. Moreover, even for a curved needle of length $a \cdot 10$ cm the number of intersections observed over N throws will be approximately $\frac{2a}{\pi}N$. The number $\pi \approx \frac{355}{113} \approx \frac{22}{7}$.]

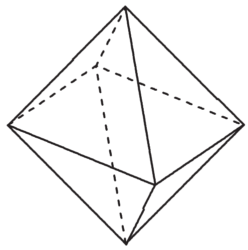


31. Some polyhedra have only triangular faces. Some examples are the Platonic solids: the (regular) tetrahedron (4 faces), the octahedron (8 faces), and the icosahedron (20 faces). The faces of the icosahedron are all identical, it has 12 vertices, and it has 30 edges.

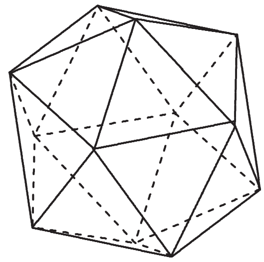
Is it true that for any such solid (a bounded convex polyhedron with triangular faces) the number of faces is equal to twice the number of vertices minus four?



tetrahedron
(tetra = 4)



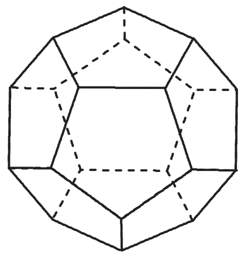
octahedron
(octa = 8)



icosahedron
(icosa = 20)

32. There is one more Platonic solid (there are 5 of them altogether): a dodecahedron. It is a convex polyhedron with twelve (regular) pentagonal faces, twenty vertices and thirty edges (its vertices are the centers of the faces of an icosahedron).

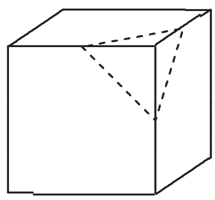
Inscribe five cubes in a dodecahedron, whose vertices are also vertices of the dodecahedron, and whose edges are diagonals of faces of the dodecahedron. (A cube has 12 edges, one for each face of the dodecahedron). [This construction was invented by Kepler to describe his model of the planets.]



33. Two regular tetrahedra can be inscribed in a cube, so that their vertices are also vertices of the cube, and their edges are diagonals of the cube's faces. Describe the intersection of these tetrahedra.

What fraction of the cube's volume is the volume of this intersection?

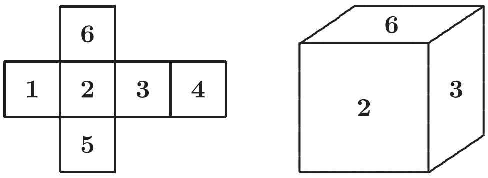
33^{bis}. Construct the section cut of a cube cut off by the plane passing through three given points on its edges. [Draw the polygon along which the plane intersects the faces of the cube.]



34. How many symmetries does a tetrahedron have? A cube? An octahedron? An icosahedron? A dodecahedron? A symmetry of a figure is a transformation of this figure preserving lengths.

How many of these symmetries are rotations, and how many are reflections in planes (in each of the five cases listed)?

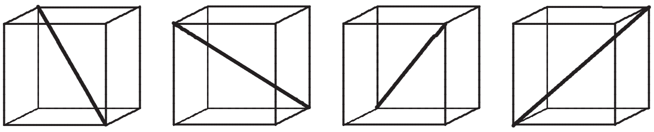
35. How many ways are there to paint the six faces of similar cubes with six colors (1,...,6) [one color per face] so that no two of the colored cubes obtained are the same (that is, no two can be transformed into each other by a rotation)?



36. How many different ways are there to permute n objects?

For $n = 3$ there are six ways: $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$. What if the number of objects is $n = 4$? $n = 5$? $n = 6$? $n = 10$?

37. A cube has 4 major diagonals (that connect its opposite vertices). How many different permutations of these four objects are obtained by rotations of a cube?



38. The sum of the cubes of several integers is subtracted from the cube of the sum of these numbers. Is this difference always divisible by 3?

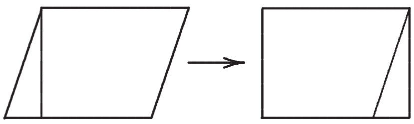
39. Answer the same question for the fifth powers and divisibility by 5, and for the seventh powers and divisibility by 7.

40. Calculate the sum

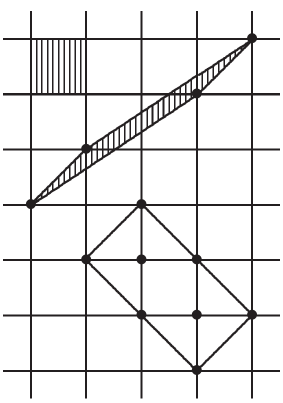
$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}$$

(with an error of not more than 1% of the correct answer).

41. If two polygons have equal areas, then they can be cut into a finite number of polygonal parts which may then be rearranged to obtain both the first and second polygons. Prove this. [For spatial solids this is not the case: the cube and the tetrahedron of equal volumes cannot be cut this way!]



42. Four lattice points on a piece of graph paper are the vertices of a parallelogram. It turns out that there are no other lattice points either on the sides of the parallelogram or inside it. Prove that the area of such a parallelogram is equal to that of one of the squares of the graph paper.



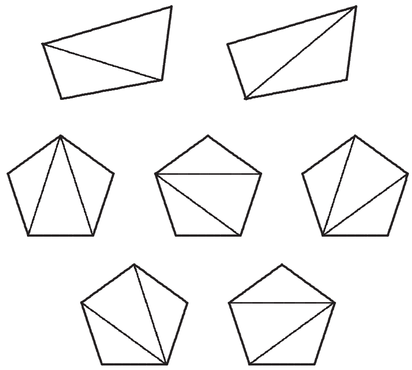
$a = 2, b = 2$

43. Suppose, in Problem 42, there turn out to be a lattice points inside the parallelogram, and b lattice points on its sides. Find its area.

44. Is the statement analogous to the result of problem 43 true for parallelepipeds in 3-space?

45. The Fibonacci ("rabbit") numbers are the sequence $(a_1 = 1), 1, 2, 3, 5, 8, 13, 21, 34, \ldots$, for which $a_{n+2} = a_{n+1} + a_n$ for any $n = 1, 2, \ldots$. Find the greatest common divisor of the numbers a_{100} and a_{99}.

46. Find the number of ways to cut a convex n-gon into triangles by cutting along non-intersecting diagonals. (These are the *Catalan* numbers, $c(n)$). For example, $c(4) = 2$, $c(5) = 5$, $c(6) = 14$. How can one find $c(10)$?



47. There are n teams participating in a tournament. After each game, the losing team is knocked out of the tournament, and after $n - 1$ games the team left is the winner of the tournament.

A schedule for the tournament may be written symbolically as (for example) $((a, (b, c)), d)$. This notation means that there are four teams participating. First b plays c, then the winner plays a, then the winner of this second game plays d.

How many possible schedules are there if there are 10 teams in the tournament?

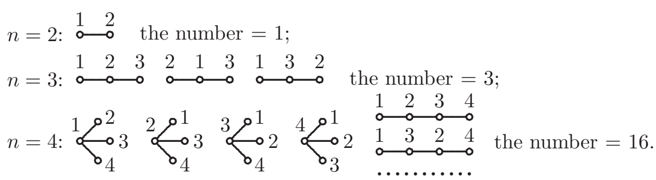
For 2 teams, we have only (a, b), and there is only one schedule.

For 3 teams, the only possible schedules are $((a, b), c)$, or $((a, c), b)$, or $((b, c), a)$, and are 3 possible schedules.

For 4 teams we have 15 possible schedules:

$(((a, b), c), d)$; $(((a, c), b), d)$; $(((a, d), b), c)$; $(((b, c), a), d)$;
$(((b, d), a), c)$; $(((c, d), a), b)$; $(((a, b), d), c)$; $(((a, c), d), b)$;
$(((a, d), c), b)$; $(((b, c), d), a)$; $(((b, d), c), a)$; $(((c, d), b), a)$;
$((a, b), (c, d))$; $((a, c), (b, d))$; $((a, d), (b, c))$.

48. We connect n points $1, 2, \ldots, n$ with $n - 1$ segments to form a tree. How many different trees can we get? (Even the case $n = 5$ is interesting!)



49. A permutation $(x_1, x_2, \ldots, x_n)$ of the numbers $\{1, 2, \ldots, n\}$ is called a *snake* (of length n) if $x_1 < x_2 > x_3 < x_4 > \cdots$.

Examples.

$$n=2,\ \text{only}\ 1<2, \qquad \text{the number} = 1;$$

$$n=3, \quad \left.\begin{array}{l} 1<3>2 \\ 2<3>1 \end{array}\right\}, \qquad \text{the number} = 2;$$

$$n=4, \quad \left.\begin{array}{l} 1<3>2<4 \\ 1<4>2<3 \\ 2<3>1<4 \\ 2<4>1<3 \\ 3<4>1<2 \end{array}\right\}, \ \text{the number} = 5;$$

Find the number of snakes of length 10.

50. Let s_n denote the number of snakes of length n, so that

$$s_1 = 1,\ s_2 = 1,\ s_3 = 2,\ s_4 = 5,\ s_5 = 16,\ s_6 = 61.$$

Prove that the Taylor series for the tangent function is

$$\tan x = 1\frac{x^1}{1!} + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + \cdots = \sum_{k=1}^{\infty} s_{2k-1}\frac{x^{2k-1}}{(2k-1)!}.$$

51. Find the sum of the series

$$1 + 1\frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} s_{2k}\frac{x^{2k}}{(2k)!}.$$

52. For $s > 1$, prove the identity

$$\prod_{p=2}^{\infty} \frac{1}{1-\frac{1}{p^s}} = \sum_{n+1}^{\infty} \frac{1}{n^s}.$$

(The product is over all prime numbers p, and the summation over all natural numbers n.)

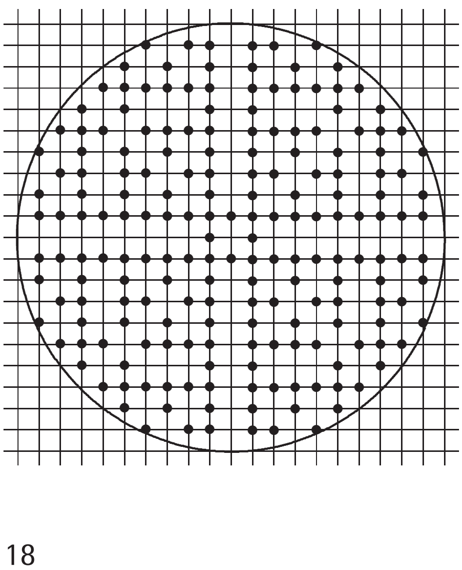
53. Find the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(Prove that it is equal to $\frac{\pi^2}{6}$, or approximately $\frac{3}{2}$.)

54. Find the probability that the fraction $\frac{p}{q}$ is in lowest terms.

This probability is defined as follows: in the disk $p^2 + q^2 \le R^2$, we count the number $N(R)$ of points with integer coordinates p and q not having a common divisor greater than 1. Then we take the limit of the ratio $N(R)/M(R)$, where $M(R)$ is the total number of integer points in the disk ($M \sim \pi R^2$).



$N(10) = 192$
$M(10) = 316$
$N/M = 192/316$
≈ 0.6076

55. The sequence of Fibonacci numbers was defined in Problem 45. Find the limit of the ratio a_{n+1}/a_n as n approaches infinity:

$$\frac{a_{n+1}}{a_n} = 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots.$$

Answer: "The golden ratio", $\frac{\sqrt{5}+1}{2} \approx 1.618$.

This is the ratio of the sides of a postcard which stays similar to itself if we snip off a square whose side is the smaller side of the postcard.

How is the golden ratio related to a regular pentagon and a five-pointed star?

56. Calculate the value of the infinite continued fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_{2k} = 1$, $a_{2k+1} = 2$.

That is, find the limit as n approaches infinity of

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

57. Find the polynomials $y = \cos 3(\arccos x)$, $y = \cos 4(\arccos x)$, $y = \cos n(\arccos x)$, where $|x| \le 1$.

58. Calculate the sum of the k^{th} powers of the n complex n^{th} roots of unity.

59. On the (x, y)–plane, draw the curves defined parametrically:

$$\{x = \cos 2t,\ y = \sin 3t\},\ \{x = t^3 - 3t,\ y = t^4 - 2t^2\}.$$

60. Calculate (with an error of not more than 10 % of the answer)

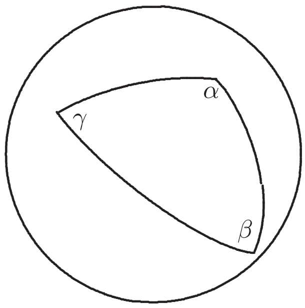
$$\int_0^{2\pi} \sin^{100} x\, dx.$$

61. Calculate (with an error of not more than 10 % of the answer)

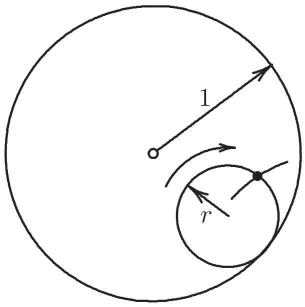
$$\int_1^{10} x^x\, dx.$$

62. Find the area of a spherical triangle with angles (α, β, γ) on a sphere of radius 1. (The sides of such a triangle are great circles; that is, cross-sections of the sphere formed by planes passing through its center).

Answer: $S = \alpha + \beta + \gamma - \pi$. (For example, for a triangle with three right angles, $S = \pi/2$, that is, one-eighth of the total area of the sphere).



63. A circle of radius r rolls (without slipping) inside a circle of radius 1. Draw the whole trajectory of a point on the rolling circle (this trajectory is called a hypocycloid) for $r = 1/3$, $r = 1/4$ for $r = 1/n$, for $r = p/q$, and for $r = 1/2$.

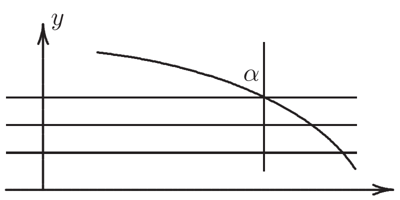


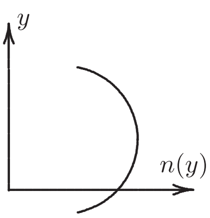
64. In a class of n students, estimate the probability that two students have the same birthday. Is this a high probability? Or a low one?

Answer: (Very) high if the number of the pupils is (well) above some number n_0, (very) low if it is (well) below n_0, and what this n_0 actually is (when the probability $p \approx 1/2$) is what the problem is asking.

65. Snell's law states that the angle α made by a ray of light with the normal to layers of a stratified medium satisfies the equation $n(y)\sin\alpha = \text{const}$, where $n(y)$ is the index of refraction of the layer at height y. (The quantity n is inversely proportional to the speed of light in the medium if we take its speed in a vacuum to be 1. In water $n = 4/3$).

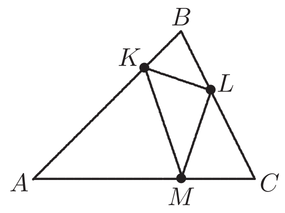
Draw the rays forming the light's trajectories in the medium "air above a desert", where the index $n(y)$ has a maximum at a certain height. (See the diagram on the right.)





(A solution to this problem explains the phenomenon of mirages to those who understand how trajectories of rays emanating from objects are related to their images).

66. In an acute angled triangle ABC inscribe a triangle KLM of minimal perimeter (with its vertex K on AB, L on BC, M on CA).



Hint: The answer for non-acute angled triangles is not nearly as beautiful as the answer for acute angled triangles.

67. Calculate the average value of the function $1/r$ (where $r^2 = x^2 + y^2 + z^2$ is the distance to the origin from the point with coordinates (x, y, z)) on the sphere of radius R centred at the point (X, Y, Z).

Hint: The problem is related to Newton's law of gravitation and Coulomb's law in electricity. In the two-dimensional version of the problem, the given function should be replaced by $\ln r$, and the sphere by a circle.

68. The fact that $2^{10} = 1024 \approx 10^3$ implies that $\log_{10} 2 \approx 0.3$. Estimate by how much they differ, and calculate $\log_{10} 2$ to three decimal places.

69. Find $\log_{10} 4$, $\log_{10} 8$, $\log_{10} 5$, $\log_{10} 50$, $\log_{10} 32$, $\log_{10} 128$, $\log_{10} 125$, and $\log_{10} 64$ with the same precision.

70. Using the fact that $7^2 \approx 50$, find an approximate value for $\log_{10} 7$.

71. Knowing the values of $\log_{10} 64$ and $\log_{10} 7$, find $\log_{10} 9$, $\log_{10} 3$, $\log_{10} 6$, $\log_{10} 27$, and $\log_{10} 12$.

72. Using the fact that $\ln(1 + x) \approx x$ (where ln means $\log_e$), find $\log_{10} e$ and $\ln 10$ from the relation[1]

$$\log_{10} a = \frac{\ln a}{\ln 10}$$

and from the values of $\log_{10} a$ computed earlier (for example, for $a = 128/125$, $a = 1024/1000$ and so on).

Solutions to Problems 67–71 will give us, after a half hour of computation, a table of four-digit logarithms of any numbers using products of numbers whose logarithms have been already found as points of support and the formula

$$\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for corrections. (This is how Newton compiled a table of 40-digit logarithms!).

73. Consider the sequence of powers of two: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, ... Among the first twelve numbers, four have decimal numerals starting with 1, and none have decimal numerals starting with 7.

Prove that in the limit as $n \to \infty$ each digit will be met with as the first digit of the numbers 2^m, $0 \le m \le n$, with a certain average frequency: $p_1 \approx 30\,\%$, $p_2 \approx 18\,\%, \ldots, p_9 \approx 4\,\%$.

74. Verify the behavior of the first digits of powers of three: 1, 3, 9, 2, 8, 2, 7, ... Prove that, in the limit, here we also get

1 Euler's constant $e = 2.71828\cdots$ is defined as the limit of the sequence $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. It is equal to the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$. It can also be defined by the given formula for $\ln(1 + x)$: $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.

certain frequencies and that the frequencies are same as for the powers of two. Find an exact formula for $p_1, \dots, p_9$.

Hint: The first digit of a number x is determined by the fractional part of the number $\log_{10} x$. Therefore one has to consider the sequence of fractional parts of the numbers ma, where $a = \log_{10} 2$.

Prove that these fractional parts are uniformly distributed over the interval from 0 to 1: of the n fractional parts of the numbers ma, $0 \le m < n$, a subinterval A will contain the quantity $k_n(A)$ such that as $n \to \infty$, $\lim(k_n(A)/n)$ = the length of the subinterval A.

75. Let $g : M \to M$ be a smooth map of a bounded domain M onto itself which is one-to-one and preserves areas (volumes in the multi-dimensional case) of domains.

Prove that in any neighborhood U of any point of M and for any N there exists a point x such that $g^T x$ is also in U for a certain integer $T > N$ (the "Recurrence Theorem").

76. Let M be the surface of a torus (with coordinates $\alpha \pmod{2\pi}, \beta \pmod{2\pi}$), and let $g(\alpha, \beta) = (\alpha + 1, \beta + \sqrt{2})$. Prove that for every point x of M the sequence of points $\{g^T(x)\}, T = 1, 2, \dots$ is everywhere dense on the torus.

77. In the notation of problem 76, let

$$g(\alpha, \beta) = (2\alpha + \beta, \alpha + \beta) \pmod{2\pi}.$$

Prove that there is an everywhere dense subset of the torus consisting of periodic points x (that is, such that $g^{T(x)} = x$ for some integer $T(x) > 0$).

78. In the notation of Problem 77 prove that, for almost all points x of the torus, the sequence of points $\{g^T(x)\}$, $T = 1, 2, \dots$ is everywhere dense on the torus (that is, the points x without this property form a set of measure zero).

79. In Problems 76 and 78, prove that the sequence $\{g^T(x)\}$, $T = 1, 2, \dots$ is distributed over the torus uniformly: if a domain A contains $k_n(A)$ points out of the n points with $T = 1, 2, \dots, n$, then

$$\lim_{n\to\infty} \frac{k_n(A)}{n} = \frac{\text{mes}\, A}{\text{mes}\, M}$$

(for example, for a Jordan measurable domain A of measure mes A).

Note to Problem 13. In posing this problem, I have tried to illustrate the difference in approaches to research by mathematicians and physicists in my invited paper in the journal *Advances in Physical Sciences* for the 2000 Centennial issue. My success far surpassed the goal I had in mind: the editors, unlike the preschool students on the experience with whom I based my plans, could not solve the problem. So they changed it to fit my answer of 4mm. in the following way: instead of "from the first page of the first volume to the last page of the second", they wrote "from the *last* page of the first volume to the *first* page of the second".

This true story is so implausible that I am including it here: the proof is the editors' version published by the journal.

This excerpt was taken from the book V. I. Arnold, Lectures and Problems: A Gift to Young Mathematicians *(AMS, 2015; ISBN-13: 978-1-4704-2259-2). To learn more about this book and to order a copy, visit www.ams.org/bookstore-getitem/item=mcl-17 . The Newsletter thanks the AMS and the MSRI for authorizing the reprint.*

Additive Eigenvalue Problem

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1 Introduction

The classical Hermitian eigenvalue problem addresses the following question. What are the possible eigenvalues of the sum $A + B$ of two Hermitian matrices A and B, provided we fix the eigenvalues of A and B. A systematic study of this problem was initiated by H. Weyl (1912). By virtue of contributions from a long list of mathematicians, notably Weyl (1912), Horn (1962), Klyachko (1998) and Knutson-Tao (1999), the problem was finally settled. The solution asserts that the eigenvalues of $A + B$ are given in terms of a certain system of linear inequalities in the eigenvalues of A and B. These inequalities are given explicitly in terms of certain triples of Schubert classes in the singular cohomology of Grassmannians and the standard cup product. Belkale (2001) gave an optimal set of inequalities for the problem in this case. The Hermitian eigenvalue problem has been extended by Berenstein-Sjamaar (2000) and Kapovich-Leeb-Millson (2009) for any semisimple complex algebraic group G. Their solution is again in terms of a system of linear inequalities obtained from certain triples of Schubert classes in the singular cohomology of the partial flag varieties G/P (P being a maximal parabolic subgroup) and the standard cup product. However, their solution is far from being optimal. In a joint piece of work with P. Belkale, we defined a deformation of the cup product in the cohomology of G/P and used this new product to generate a system of inequalities which solves the problem for any G optimally (as shown by Ressayre). This article is a brief survey of this additive eigenvalue problem. The eigenvalue problem is equivalent to the *saturated tensor product problem.*

This note was written during my visit to the University of Sydney and their hospitality is gratefully acknowledged.

Partial support from NSF grant number DMS-1501094 is also gratefully acknowledged.

2 Main Results

We now explain the classical Hermitian eigenvalue problem and its generalisation to an arbitrary connected semisimple group more precisely.

For any $n \times n$ Hermitian matrix A, let $\lambda_A = (\lambda_1 \geq \cdots \geq \lambda_n)$ be its set of eigenvalues written in descending order. Recall the following classical problem, known as the *Hermitian eigenvalue problem*. Given two n-tuples of nonincreasing real numbers: $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_n)$, determine all possible $\nu = (\nu_1 \geq \cdots \geq \nu_n)$ such that there exist Hermitian matrices A, B, C with $\lambda_A = \lambda, \lambda_B = \mu, \lambda_C = \nu$ and $C = A + B$. This problem has a long history, starting with the work of Weyl (1912), followed by works of Fan (1949), Lidskii (1950) and Wielandt (1955), and culminating in the following conjecture given by Horn (1962). (See also Thompson-Freede (1971).)

For any positive integer $r < n$, inductively define the set S^r_n as the set of triples (I, J, K) of subsets of $[n] := \{1, \ldots, n\}$ of cardinality r such that

$$\sum_{i \in I} i + \sum_{j \in J} j = r(r+1)/2 + \sum_{k \in K} k \tag{1}$$

and, for all $0 < p < r$ and $(F, G, H) \in S^p_r$, the following inequality holds:

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq p(p+1)/2 + \sum_{h \in H} k_h. \tag{2}$$

Conjecture 1. A triple λ, μ, ν occurs as eigenvalues of Hermitian $n \times n$ matrices A, B, C respectively such that $C = A + B$ if and only if

$$\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \mu_i$$

and, for all $1 \leq r < n$ and all triples $(I, J, K) \in S^r_n$, we have

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j.$$

Horn's conjecture above was settled in the affirmative (see Corollary 11) by combining the work of Klyachko [Kly] (1998) with the work of Knutson-Tao [KT] (1999) on the 'saturation' problem.

The above system of inequalities is overdetermined. Belkale (2001) proved that a certain subset of the set of inequalities suffices. Subsequently, Knutson-Tao-Woodward (2004) proved that the subsystem of inequalities given by Belkale forms an irredundant system of inequalities.

Now, we discuss a generalisation of the above Hermitian eigenvalue problem (which can be rephrased in terms of the special unitary group $SU(n)$ and its complexified Lie algebra $\mathfrak{sl}(n)$) to an arbitrary complex semisimple group. Let G be a connected, semisimple complex algebraic group. We fix a Borel subgroup B, a maximal torus $H \subset B$ and a maximal compact subgroup K. We denote their Lie algebras by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{k}$ respectively. We choose K such that $\sqrt{-1}\,\mathfrak{k} \supset \mathfrak{h}$. Let R^+ be the set of positive roots (i.e. the set of roots of $\mathfrak{b}$) and let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R^+$ be the set of simple roots. There is a natural homeomorphism $\delta : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation and $\mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_i(h) \geq 0 \,\forall\, i\}$ is the positive Weyl chamber in $\mathfrak{h}$. The inverse map δ^{-1} takes any $h \in \mathfrak{h}_+$ to the K-conjugacy class of $\sqrt{-1}h$.

For any positive integer s, define the *eigencone*

$$\begin{aligned}\bar{\Gamma}_s(\mathfrak{g}) :=& \{(h_1, \ldots, h_s) \in \mathfrak{h}_+^s \mid \exists (k_1, \ldots, k_s) \in \mathfrak{k}^s : \sum_{j=1}^{s} k_j \\ &= 0 \text{ and } \delta(k_j) = h_j \forall j\}.\end{aligned}$$

By virtue of the general convexity result in symplectic geometry, the subset $\bar{\Gamma}_s(\mathfrak{g}) \subset \mathfrak{h}_+^s$ is a convex rational polyhedral cone (defined by certain inequalities with rational coefficients). The aim of the *general additive eigenvalue problem* is to find the inequalities describing $\bar{\Gamma}_s(\mathfrak{g})$ explicitly. (The case $\mathfrak{g} = \mathfrak{sl}(n)$ and $s = 3$ gives the Hermitian eigenvalue problem if we replace C by $-C$.)

Let $\Lambda = \Lambda(H)$ denote the character group of H and let $\Lambda_+ := \{\lambda \in \Lambda : \lambda(\alpha_i^\vee) \geq 0 \,\forall$ simple coroots $\alpha_i^\vee\}$ denote the set of all the dominant characters. Then, the set of isomorphism classes of irreducible (finite dimensional) representations of G is parametrised by Λ_+ via the highest weights of irreducible representations. For $\lambda \in \Lambda_+$, we denote by $[\lambda]$ the corresponding irreducible representation (of highest weight λ).

Similar to the eigencone $\bar{\Gamma}_s(\mathfrak{g})$, one defines the *saturated tensor semigroup*:

$$\begin{aligned}\Gamma_s(G) =& \{(\lambda_1, \ldots, \lambda_s) \in \Lambda_+^s : ([N\lambda_1] \otimes \cdots \otimes [N\lambda_s])^G \\ &\neq 0, \text{ for some } N \geq 1\}.\end{aligned}$$

Then, under the identification $\varphi : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ (via the Killing form),

$$\varphi(\bar{\Gamma}_s(\mathfrak{g})) \cap \Lambda_+^s = \Gamma_s(G) \tag{3}$$

(see Theorem 5).

For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\, 1 \leq i \leq \ell. \tag{4}$$

Let $P \supset B$ be a standard parabolic subgroup with Lie algebra $\mathfrak{p}$ and let $\mathfrak{l}$ be its unique Levi component containing the Cartan subalgebra $\mathfrak{h}$. Let $\Delta(P) \subset \Delta$ be the set of simple roots contained in the set of roots of $\mathfrak{l}$. Let W_P be the Weyl group of P (which is, by definition, the Weyl Group of the Levi component L) and let W^P be the set of the minimal length representatives in the cosets of W/W_P. For any $w \in W^P$, define the Schubert variety:

$$X_w^P := \overline{BwP/P} \subset G/P.$$

It is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. Let $\mu(X_w^P)$ denote the fundamental class of X_w^P, considered as an element of the singular homology with integral coefficients $H_{2\ell(w)}(G/P, \mathbb{Z})$ of G/P. Then, from the Bruhat decomposition, the elements $\{\mu(X_w^P)\}_{w \in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P, \mathbb{Z})$. Let $\{[X_w^P]\}_{w \in W^P}$ be the Poincaré dual basis of the singular cohomology $H^*(G/P, \mathbb{Z})$. Thus,

$$[X_w^P] \in H^{2(\dim G/P - \ell(w))}(G/P, \mathbb{Z}).$$

Write the standard cup product in $H^*(G/P, \mathbb{Z})$ in the $\{[X_w^P]\}$ basis as follows:

$$[X_u^P] \cdot [X_v^P] = \sum_{w \in W^P} c_{u,v}^w [X_w^P]. \tag{5}$$

Introduce the indeterminates τ_i for each $\alpha_i \in \Delta \setminus \Delta(P)$ and define a deformed cup product $\odot$ as follows:

$$[X_u^P] \odot [X_v^P] = \sum_{w \in W^P} \Big(\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(w^{-1}\rho - u^{-1}\rho - v^{-1}\rho - \rho)(x_i)} \Big) c_{u,v}^w [X_w^P], \tag{6}$$

where ρ is the (usual) half sum of positive roots of $\mathfrak{g}$. By Corollary 16 and the identity (13), whenever $c_{u,v}^w$ is nonzero, the exponent of τ_i in the above is a nonnegative integer. Moreover, the product $\odot$ is associative (and clearly commutative). The cohomology algebra of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[\tau_i], \odot)$ is denoted by $(H^*(G/P, \mathbb{Z}), \odot_0)$. Thus, as a $\mathbb{Z}$-module, this is the same as the singular cohomology $H^*(G/P, \mathbb{Z})$ and, under the product $\odot_0$, it is associative (and commutative). Moreover, it continues to satisfy the Poincaré duality (see [BK$_1$, Lemma 16(d)]). The definition of the deformed product $\odot_0$ (now known as the *Belkale-Kumar product*) came from the crucial concept of Levi-movability in Definition 14. For a cominuscule maximal parabolic P, the product $\odot_0$ coincides with the standard cup product (see Lemma 17).

Now we are ready to state the main result on solution of the eigenvalue problem for any connected semisimple G. For a maximal parabolic P, let α_{i_P} be the unique simple root not in the Levi of P and let $\omega_P := \omega_{i_P}$ be the corresponding fundamental weight.

Theorem 2. Let $(h_1, \ldots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:

(a) $(h_1, \ldots, h_s) \in \bar{\Gamma}_s(\mathfrak{g})$.

(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$[X_{w_1}^P] \cdots [X_{w_s}^P] = d[X_e^P] \text{ for some } d \neq 0,$$

the following inequality holds:

$$I_{(w_1,\ldots,w_s)}^P : \quad \omega_P\Big(\sum_{j=1}^s w_j^{-1} h_j\Big) \leq 0.$$

(c) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$[X_{w_1}^P] \cdots [X_{w_s}^P] = [X_e^P],$$

the above inequality $I_{(w_1,\ldots,w_s)}^P$ holds.

(d) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$[X_{w_1}^P] \odot_0 \cdots \odot_0 [X_{w_s}^P] = [X_e^P],$$

the above inequality $I_{(w_1,\ldots,w_s)}^P$ holds.

The equivalence of (a) and (b) in the above theorem for general G is due to Berenstein-Sjamaar (2000). Kapovich-Leeb-Millson (2009) showed the equivalence of (a) and (c). The equivalence of (a) and (d) is due to Belkale-Kumar (2006). If we specialise the above theorem to $G = \mathrm{SL}(n)$ then, in view of Theorem 10, the equivalence of (a) and (b) is nothing but Horn's conjecture (Corollary 11) solved by combining the work of Klyachko (1998) with the work of Knutson-Tao (1999). In this case, the equivalence of (a) and (c) is due to Belkale (2001) and every maximal parabolic subgroup P is cominuscule; hence, the deformed product $\odot_0$ in $H^*(G/P)$ coincides with the standard cup product and (c) and (d) are the same in this case.

Because of the identification (3), the above theorem allows us to determine the saturated tensor semigroup $\Gamma_s(G)$ (see Theorem 18 for a precise statement).

The following result was proved by Ressayre [R] (2010). As mentioned above, for $\mathfrak{g} = \mathfrak{sl}(n)$ it was proved by Knutson-Tao-Wodward. Ressayre's proof relies on the notion of well-covering pairs, which is equivalent to the notion of Levi-movability with cup product 1.

Theorem 3. The inequalities given by (d) of the above theorem form an irredundant system of inequalities determining the cone $\bar{\Gamma}_s(\mathfrak{g})$ (see Theorem 23 for a more precise statement).

As shown by Kumar-Leeb-Millson (2003), (c) of the above theorem gives rise to 126 inequalities for $\mathfrak{g}$ of type B_3 or C_3, whereas (d) gives only 93 inequalities.

We refer the reader to the survey article of Fulton [F] on the Hermitian eigenvalue problem and, for general G, the survey articles by Brion [Br] and by Kumar [K$_3$].

3 Determination of the Eigencone (A Weaker Result)

Below, we give an indication of the proof of the equivalence of (a) and (b) in Theorem 2.

Definition 4. Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety $\mathbb{X}$ and let $\mathbb{L}$ be an S-equivariant line bundle on $\mathbb{X}$. Any algebraic group morphism $\mathbb{G}_m \to S$ is called a *one parameter subgroup* (OPS) in S. Let $O(S)$ be the set of all OPSs in S. Take any $x \in \mathbb{X}$ and $\delta \in O(S)$ such that the limit $\lim_{t\to 0} \delta(t)x$ exists in $\mathbb{X}$ (i.e. the morphism $\delta_x : \mathbb{G}_m \to X$ given by $t \mapsto \delta(t)x$ extends to a morphism $\widetilde{\delta}_x : \mathbb{A}^1 \to X$). Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \delta)$ as follows. Let $x_o \in X$ be the point $\widetilde{\delta}_x(0)$. Since x_o is $\mathbb{G}_m$-invariant via δ, the fiber of $\mathbb{L}$ over x_o is a $\mathbb{G}_m$-module and, in particular, is given by a character of $\mathbb{G}_m$. This integer is defined as $\mu^{\mathbb{L}}(x, \delta)$.

Under the identification $\varphi : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ (via the Killing form), $\Gamma_s(G)$ corresponds to the set of integral points of $\overline{\Gamma}_s(\mathfrak{g})$. Specifically, we have the following result, essentially following from Mumford [N, Appendix] (see also [Sj, Theorem 7.6] and [Br, Théorème 1.3]).

Theorem 5.

$$\varphi(\overline{\Gamma}_s(\mathfrak{g})) \cap \Lambda_+^s = \Gamma_s(G).$$

Let P be any standard parabolic subgroup of G acting on P/B_L via the left multiplication, where L is the Levi subgroup of P containing H, and $B_L := B \cap L$ is a Borel subgroup of L. We call $\delta \in O(P)$ *P-admissible* if, for all $x \in P/B_L$, $\lim_{t\to 0} \delta(t) \cdot x$ exists in P/B_L. If $P = G$ then $P/B_L = G/B$ and any $\delta \in O(G)$ is G-admissible since G/B is a projective variety. For $\delta \in O(G)$ define the *Kempf's parabolic subgroup* of G by $P(\delta) := \{g \in G : \lim_{t\to 0} \delta(t) g \delta(t)^{-1} \text{ exists in } G\}$.

Observe that, B_L being the semidirect product of its commutator $[B_L, B_L]$ and H, any $\lambda \in \Lambda$ extends uniquely to a character of B_L. Thus, for any $\lambda \in \Lambda$, we have a P-equivariant line bundle $\mathcal{L}_P(\lambda)$ on P/B_L associated to the principal B_L-bundle

$P \to P/B_L$ via the one dimensional B_L-module λ^{-1}. We abbreviate $\mathcal{L}_G(\lambda)$ by $\mathcal{L}(\lambda)$. We have taken the following lemma from [BK$_1$, Lemma 14]. It is a generalisation of the corresponding result in [BS, Section 4.2].

Lemma 6. Let $\delta \in O(H)$ be such that $\dot{\delta} \in \mathfrak{h}_+$. Then, δ is P-admissible and, moreover, for any $\lambda \in \Lambda$ and $x = ulB_L \in P/B_L$ (for $u \in U$, $l \in L$), we have the following formula:

$$\mu^{\mathcal{L}_P(\lambda)}(x, \delta) = -\lambda(w\dot{\delta}),$$

where U is the unipotent radical of P, $P_L(\delta) := P(\delta) \cap L$ and $w \in W_P/W_{P_L(\delta)}$ is any coset representative such that $l^{-1} \in B_L w P_L(\delta)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_s) \in \Lambda_+^s$ and let $\mathbb{L}(\lambda)$ denote the G-linearised line bundle $\mathcal{L}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{L}(\lambda_s)$ on $(G/B)^s$ (under the diagonal action of G). Then, there exist unique standard parabolic subgroups $P_1, \ldots, P_s$ such that the line bundle $\mathbb{L}(\lambda)$ descends as an ample line bundle $\bar{\mathbb{L}}(\lambda)$ on $\mathbb{X}(\lambda) := G/P_1 \times \cdots \times G/P_s$. We call a point $x \in (G/B)^s$, *G-semistable* (with respect to not necessarily ample $\mathbb{L}(\lambda)$) if its image in $\mathbb{X}(\lambda)$ under the canonical map $\pi : (G/B)^s \to \mathbb{X}(\lambda)$ is semistable with respect to the ample line bundle $\bar{\mathbb{L}}(\lambda)$. Now, one has the following fundamental theorem due to Klyachko [Kly] for $G = \mathrm{SL}(n)$, extended to general G by Berenstein-Sjamaar [BS].

Theorem 7. Let $\lambda_1, \ldots, \lambda_s \in \Lambda_+$. Then, the following are equivalent:
(a) $(\lambda_1, \ldots, \lambda_s) \in \Gamma_s(G)$.
(b) For every standard maximal parabolic subgroup P and every Weyl group elements $w_1, \ldots, w_s \in W^P \simeq W/W_P$ such that

$$[X_{w_1}^P] \ldots [X_{w_s}^P] = d[X_e^P], \quad \text{for some } d \neq 0, \tag{7}$$

the following inequality is satisfied:

$$I^P_{(w_1,\ldots,w_s)} : \qquad \sum_{j=1}^{s} \lambda_j(w_j x_P) \le 0,$$

where α_{i_P} is the unique simple root not in the Levi of P and $x_P := x_{i_P}$.

The equivalence of (a) and (b) in Theorem 2 follows easily by combining Theorems 7 and 5.

Remark 8. As proved by Belkale [B$_1$] for $G = \mathrm{SL}(n)$ and extended for an arbitrary G by Kapovich-Leeb-Millson [KLM], Theorem 7 remains true if we replace d by 1 in the identity (7). A much sharper (and optimal) result for an arbitrary G is obtained in Theorem 18.

4 Specialisation of Results to $G = \mathrm{SL}(n)$: Horn Inequalities

We first need to recall the Knutson-Tao saturation theorem [KT], conjectured by Zelevinsky [Z]. Other proofs of their result are given by Derksen-Weyman [DK], Belkale [B$_3$] and Kapovich-Millson [KM$_2$].

Theorem 9. Let $G = \mathrm{SL}(n)$ and let $(\lambda_1, \ldots, \lambda_s) \in \Gamma_s(G)$ be such that $\lambda_1 + \cdots + \lambda_s$ belongs to the root lattice. Then,

$$([\lambda_1] \otimes \cdots \otimes [\lambda_s])^G \neq 0.$$

Specialising Theorem 7 to $G = \mathrm{SL}(n)$, as seen below, we obtain the classical Horn inequalities.

In this case, the partial flag varieties corresponding to the maximal parabolics P_r are precisely the Grassmannians of r-planes in n-space $G/P_r = \mathrm{Gr}(r, n)$, for $0 < r < n$. The Schubert cells in $\mathrm{Gr}(r, n)$ are parametrised by the subsets of cardinality r:

$$I = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, n\}.$$

The corresponding Weyl group element $w_I \in W^{P_r}$ is nothing but the permutation

$$1 \mapsto i_1, \quad 2 \mapsto i_2, \cdots, r \mapsto i_r$$

and $w_I(r+1), \ldots, w_I(n)$ are the elements in $\{1, \ldots, n\} \backslash I$ arranged in ascending order.

Let I' be the 'dual' set

$$I' = \{n + 1 - i, \ \ i \in I\}$$

arranged in ascending order.

Then, the Schubert class $[X_I := X_{w_I}^{P_r}]$ is Poincaré dual to the Schubert class $[X_{I'}] \in H^*(\mathrm{Gr}(r, n), \mathbb{Z})$. Moreover,

$$\dim X_I = \operatorname{codim} X_{I'} = \Big(\sum_{i \in I} i\Big) - \frac{r(r+1)}{2}. \tag{8}$$

For $0 < r < n$, recall the definition of the set S_n^r of triples (I, J, K) of subsets of $\{1, \ldots, n\}$ of cardinality r from Section 2. The following theorem follows from Theorem 7 for $G = \mathrm{SL}(n)$ (proved by Klyachko) and Theorem 9 (proved by Knutson-Tao). Belkale [B$_3$] gave another geometric proof of the theorem.

Theorem 10. For subsets (I, J, K) of $\{1, \ldots, n\}$ of cardinality r,

$$[X_{I'}] \cdot [X_{J'}] \cdot [X_K] = d[X_e^{P_r}], \text{ for some } \ d \neq 0$$
$$\Leftrightarrow (I, J, K) \in S_n^r.$$

For an Hermitian $n \times n$ matrix A, let $\lambda_A = (\lambda_1 \ge \cdots \ge \lambda_n)$ be its set of eigenvalues (which are all real). Let $\mathfrak{a}$ be the standard Cartan subalgebra of $sl(n)$ consisting of traceless diagonal matrices and let $\mathfrak{b} \subset sl(n)$ be the standard Borel subalgebra consisting of traceless upper triangular matrices (where $sl(n)$ is the Lie algebra of $\mathrm{SL}(n)$ consisting of traceless $n \times n$ matrices). Then, the Weyl chamber is:

$$\mathfrak{a}_+ = \Big\{\mathrm{diag}\,(e_1 \ge \cdots \ge e_n) : \sum e_i = 0\Big\}.$$

Define the *Hermitian eigencone*:

$$\bar{\Gamma}(n) = \{(a_1, a_2, a_3) \in \mathfrak{a}_+^3 : \text{there exist } n \times n \text{ Hermitian matrices } A, B, C \text{ with } \lambda_A = a_1, \lambda_B = a_2, \lambda_C = a_3 \text{ and } A + B = C\}.$$

It is easy to see that $\bar{\Gamma}(n)$ essentially coincides with the eigencone $\bar{\Gamma}_3(sl(n))$. Specifically,

$$(a_1, a_2, a_3) \in \bar{\Gamma}(n) \Leftrightarrow (a_1, a_2, a_3^*) \in \bar{\Gamma}_3(sl(n)),$$

where $(e_1 \ge \cdots \ge e_n)^* := (-e_n \ge \cdots \ge -e_1)$.

Combining Theorems 7 and 5 for $sl(n)$ with Theorem 10, we get the following Horn conjecture [Ho], established by the works of Klyachko (equivalence of (a) and (b) in Theorem 2 for $\mathfrak{g} = sl(n)$) and Knutson-Tao (Theorem 9).

Corollary 11. For $(a_1, a_2, a_3) \in \mathfrak{a}_+^3$, the following are equivalent:

(a) $(a_1, a_2, a_3) \in \bar{\Gamma}(n)$.

(b) For all $0 < r < n$ and all $(I, J, K) \in S_n^r$,

$$|a_3(K)| \le |a_1(I)| + |a_2(J)|,$$

where, for a subset $I = (i_1 < \cdots < i_r) \subset \{1, \ldots, n\}$ and $a = (e_1 \ge \cdots \ge e_n) \in \mathfrak{a}_+$, $a(I) := (e_{i_1} \ge \cdots \ge e_{i_r})$ and $|a(I)| := e_{i_1} + \cdots + e_{i_r}$.

We have the following representation theory analogue of the above corollary, obtained by combining Theorems 7, 9 and 10.

Corollary 12. Let $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$, $\mu = (\mu_1 \ge \cdots \ge \mu_n \ge 0)$ and $\nu = (\nu_1 \ge \cdots \ge \nu_n \ge 0)$ be three partitions such that $|\lambda| + |\mu| - |\nu| \in n\mathbb{Z}$, where $|\lambda| := \lambda_1 + \cdots + \lambda_n$. Then, the following are equivalent:
(a) $[\nu]$ appears as an $\mathrm{SL}(n)$-submodule of $[\lambda] \otimes [\mu]$.
(b) For all $0 < r < n$ and all $(I, J, K) \in S_n^r$,

$$|\nu(K)| \le |\lambda(I)| + |\mu(J)| - \frac{r}{n}(|\lambda| + |\mu| - |\nu|),$$

where, for a subset $I = (i_1 < \cdots < i_r) \subset \{1, \ldots, n\}$, $\lambda(I)$ denotes $(\lambda_{i_1} \ge \cdots \ge \lambda_{i_r})$ and $|\lambda(I)| := \lambda_{i_1} + \cdots + \lambda_{i_r}$.

5 Deformed Product

This section is based on the work [BK$_1$] of Belkale-Kumar. Consider the shifted Bruhat cell:

$$\Phi_w^P := w^{-1}BwP \subset G/P.$$

Let $T^P = T(G/P)_e$ be the tangent space of G/P at $e \in G/P$. It carries a canonical action of P. For $w \in W^P$, define T_w^P to be the tangent space of Φ_w^P at e. We shall abbreviate T^P and T_w^P by T and T_w respectively when the reference to P is clear. It is easy to see that B_L stabilises Φ_w^P keeping e fixed. Thus,

$$B_L T_w \subset T_w. \tag{9}$$

The following result follows from the Kleiman transversality theorem by observing that $g\Phi_w^P$ passes through $e \Leftrightarrow g\Phi_w^P = p\Phi_w^P$ for some $p \in P$.

Proposition 13. Take any $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$\sum_{j=1}^{s} \operatorname{codim} \Phi_{w_j}^P \le \dim G/P. \tag{10}$$

Then, the following three conditions are equivalent:
(a) $[X_{w_1}^P] \ldots [X_{w_s}^P] \ne 0 \in H^*(G/P)$.
(b) For general $(p_1, \ldots, p_s) \in P^s$, the intersection

$$p_1\Phi_{w_1}^P \cap \cdots \cap p_s\Phi_{w_s}^P \text{ is transverse at } e.$$

(c) For general $(p_1, \ldots, p_s) \in P^s$,

$$\dim(p_1T_{w_1} \cap \cdots \cap p_sT_{w_s}) = \dim G/P - \sum_{j=1}^{s} \operatorname{codim} \Phi_{w_j}^P.$$

The set of s-tuples in (b) as well as (c) is an open subset of P^s.

Definition 14. Let $w_1, \ldots, w_s \in W^P$ be such that

$$\sum_{j=1}^{s} \operatorname{codim} \Phi_{w_j}^P = \dim G/P. \tag{11}$$

We then call the s-tuple $(w_1, \ldots, w_s)$ *Levi-movable* (or *L-movable*) if, for general $(l_1, \ldots, l_s) \in L^s$, the intersection $l_1\Phi_{w_1}^P \cap \cdots \cap l_s\Phi_{w_s}^P$ is transverse at e.

By Proposition 13, if $(w_1, \ldots, w_s)$ is L-movable, then $[X_{w_1}^P] \ldots [X_{w_s}^P] = d[X_e^P]$ in $H^*(G/P)$ for some nonzero d.

For $w \in W^P$, define the character $\chi_w \in \Lambda$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R_\mathfrak{l}^+) \cap w^{-1}R^+} \beta, \tag{12}$$

where $R_\mathfrak{l}^+$ is the set of positive roots of $\mathfrak{l}$.

Then, from [K$_1$, 1.3.22.3],

$$\chi_w = \rho - 2\rho^L + w^{-1}\rho, \tag{13}$$

where ρ is half the sum of roots in R^+ and ρ^L is half the sum of roots in $R_\mathfrak{l}^+$.

Proposition 15. Assume that $(w_1, \ldots, w_s) \in (W^P)^s$ satisfies equation (11). Then, the following are equivalent:
(a) $(w_1, \ldots, w_s)$ is L-movable.
(b) $[X_{w_1}^P] \ldots [X_{w_s}^P] = d[X_e^P]$ in $H^*(G/P)$ for some nonzero d and, for each $\alpha_i \in \Delta \setminus \Delta(P)$, we have

$$\left(\Big(\sum_{j=1}^{s} \chi_{w_j}\Big) - \chi_1\right)(x_i) = 0.$$

Corollary 16. For any $u, v, w \in W^P$ such that $c_{u,v}^w \ne 0$ (see Equation (5)), we have

$$(\chi_w - \chi_u - \chi_v)(x_i) \ge 0, \quad \text{for each } \alpha_i \in \Delta \setminus \Delta(P). \tag{14}$$

The above corollary, together with the identity (13), justifies the definition of the deformed product $\odot_0$ given in Section 2. This deformed product is used in determining the facets (codimension 1 faces) of $\bar{\Gamma}_s(\mathfrak{g})$.

Lemma 17. Let P be a cominuscule maximal standard parabolic subgroup of G (i.e. the unique simple root $\alpha_P \in \Delta \setminus \Delta(P)$ appears with coefficient 1 in the highest root of R^+). Then, the product $\odot$ coincides with the cup product in $H^*(G/P)$.

6 Efficient determination of the eigencone

This section is again based on the work [BK$_1$] of Belkale-Kumar. The following theorem [BK$_1$, Theorem 22] determines the saturated tensor semigroup $\Gamma_s(G)$ efficiently. Specifically, as proved by Ressayre (see Theorem 23), the set of inequalities given by (b) of the following theorem is an irredundant set of inequalities determining $\Gamma_s(G)$.

For $G = \mathrm{SL}(n)$, each maximal parabolic P is cominuscule and, hence, by Lemma 17, $\odot_0$ coincides with the standard cup product in $H^*(G/P)$. Thus, the following theorem in this case reduces to Theorem 7 with $d = 1$ in the identity (7).

It may be mentioned that replacing the product $\odot_0$ in (b) of the following theorem by the standard cup product (i.e. Theorem 7 with $d = 1$ in the identity (7) – see Remark 8), we get, in general, 'far more' inequalities for simple groups other than $\mathrm{SL}(n)$. For example, for G of type B_3 (or C_3), Theorem 7 with $d = 1$ gives rise to 126 inequalities, whereas the following theorem gives only 93 inequalities (see [KuLM]).

Theorem 18. Let G be a connected semisimple group and let $(\lambda_1, \ldots, \lambda_s) \in \Lambda_+^s$. Then, the following are equivalent:
(a) $\lambda = (\lambda_1, \ldots, \lambda_s) \in \Gamma_s(G)$.
(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$ such that

$$[X_{w_1}^P] \odot_0 \cdots \odot_0 [X_{w_s}^P] = [X_e^P] \in (H^*(G/P, \mathbb{Z}), \odot_0),$$

the following inequality holds:

$$I^P_{(w_1,\dots,w_s)}: \qquad \sum_{j=1}^{s} \lambda_j(w_j x_P) \le 0,$$

where α_{i_P} is the (unique) simple root in $\Delta \setminus \Delta(P)$ and $x_P := x_{i_P}$.

We briefly recall some of the main ingredients of the proof of the above theorem that are of independent interest.

Definition 19. (Maximally destabilising one parameter subgroups.) Let $\mathbb{X}$ be a projective variety with the action of a connected reductive group S and let $\mathbb{L}$ be an S-linearised ample line bundle on $\mathbb{X}$. Introduce the set $M(S)$ of fractional OPSs in S. This is the set consisting of the ordered pairs (δ, a), where $\delta \in O(S)$ and $a \in \mathbb{Z}_{>0}$, modulo the equivalence relation $(\delta, a) \simeq (\gamma, b)$ if $\delta^b = \gamma^a$. The equivalence class of (δ, a) is denoted by $[\delta, a]$. An OPS δ of S can be thought of as the element $[\delta, 1] \in M(S)$. The group S acts on $M(S)$ via conjugation: $g \cdot [\delta, a] = [g\delta g^{-1}, a]$. Choose an S-invariant norm $q : M(S) \to \mathbb{R}_+$, where norm means that $q_{|M(H)}$ is the square root of a positive definite quadratic form on the $\mathbb{Q}$-vector space $M(H)$ for a maximal torus H of S. We can extend the definition of $\mu^{\mathbb{L}}(x, \delta)$ to any element $\hat\delta = [\delta, a] \in M(S)$ and $x \in \mathbb{X}$ by setting $\mu^{\mathbb{L}}(x, \hat\delta) = \frac{\mu^{\mathbb{L}}(x,\delta)}{a}$.

For any unstable (i.e. nonsemistable) point $x \in \mathbb{X}$, define

$$q^*(x) = \inf_{\hat\delta \in M(S)} \left\{ q(\hat\delta) \mid \mu^{\mathbb{L}}(x, \hat\delta) \le -1 \right\}$$

and the *optimal class*

$$\Lambda(x) = \{\hat\delta \in M(S) \mid \mu^{\mathbb{L}}(x, \hat\delta) \le -1, q(\hat\delta) = q^*(x)\}.$$

Any $\hat\delta \in \Lambda(x)$ is called a *Kempf's OPS associated to x*.

By a theorem of Kempf (see [Ki, Lemma 12.13]), $\Lambda(x)$ is nonempty and the parabolic $P(\hat\delta) := P(\delta)$ (for $\hat\delta = [\delta, a]$) does not depend upon the choice of $\hat\delta \in \Lambda(x)$. The parabolic $P(\hat\delta)$ for $\hat\delta \in \Lambda(x)$ will be denoted by $P(x)$ and called the *Kempf's parabolic associated to the unstable point x*. Moreover, $\Lambda(x)$ is a single conjugacy class under $P(x)$.

We recall the following theorem due to Ramanan–Ramanathan [RR, Proposition 1.9].

Theorem 20. For any unstable point $x \in \mathbb{X}$ and $\hat\delta = [\delta, a] \in \Lambda(x)$, let

$$x_o = \lim_{t\to 0} \delta(t) \cdot x \in \mathbb{X}.$$

Then, x_o is unstable and $\hat\delta \in \Lambda(x_o)$.

Indication of the Proof of Theorem 18: The implication $(a) \Rightarrow (b)$ of Theorem 18 is, of course, a special case of Theorem 7.

To prove the implication $(b) \Rightarrow (a)$ in Theorem 18, we need to recall the following result due to Kapovich-Leeb-Millson [KLM]. Suppose that $x = (\bar g_1, \dots, \bar g_s) \in (G/B)^s$ is an unstable point and $P(x)$ the Kempf's parabolic associated to $\pi(x)$, where $\pi : (G/B)^s \to \mathbb{X}(\lambda)$ is the map defined above Theorem 7. Let $\hat\delta = [\delta, a]$ be a Kempf's OPS associated to $\pi(x)$. Express $\delta(t) = f\gamma(t)f^{-1}$, where $\dot\gamma \in \mathfrak{h}_+$. Then, the Kempf's parabolic $P(\gamma)$ is a standard parabolic. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_j B w_j P(\gamma)$ for $j = 1, \dots, s$. Let P be a maximal parabolic containing $P(\gamma)$.

Theorem 21. (i) The intersection $\bigcap_{j=1}^s g_j B w_j P \subset G/P$ is the singleton $\{fP\}$.

(ii) For the simple root $\alpha_{i_P} \in \Delta \setminus \Delta(P)$, $\sum_{j=1}^s \lambda_j(w_j x_{i_P}) > 0$.

The equivalence of (a) and (d) in Theorem 2 follows easily by combining Theorems 18 and 5.

Remark 22. The cone $\bar\Gamma_3(\mathfrak{g}) \subset \mathfrak{h}_+^3$ is quite explicitly determined: for any simple $\mathfrak{g}$ of rank 2 in [KLM, §7]; for any simple $\mathfrak{g}$ of rank 3 in [KuLM]; and for $\mathfrak{g} = so(8)$ in [KKM]. It has: 12 (6+6); 18 (9+9); 30 (15+15); 41 (10+21+10); 93 (18+48+27); 93 (18+48+27); 294 (36+186+36+36); 1290 (126+519+519+126); and 26661 (348+1662+4857+14589+4857+348) facets inside $\mathfrak{h}_+^3$ (intersecting the interior of $\mathfrak{h}_+^3$) for $\mathfrak{g}$ of type $A_2, B_2, G_2, A_3, B_3, C_3, D_4, F_4$ and E_6 respectively. The notation 30 (15+15) means that there are 15 (irredundant) inequalities coming from G/P_1 and there are 15 inequalities coming from G/P_2. (The indexing convention is as in [Bo, Planche I–IX].)

The following result is due to Ressayre [R]. In the case $G = \mathrm{SL}(n)$, the result was earlier proved by Knutson-Tao-Woodward [KTW].

Theorem 23. Let $s \ge 3$. The set of inequalities provided by (b) of Theorem 18 is an irredundant system of inequalities describing the cone $\Gamma_s(G)_{\mathbb{R}}$ generated by $\Gamma_s(G)$ inside $\Lambda_+(\mathbb{R})^s$, i.e. the hyperplanes given by the equality in $I^P_{(w_1,\dots,w_s)}$ are precisely those facets of the cone $\Gamma_s(G)_{\mathbb{R}}$ which intersect the interior of $\Lambda_+(\mathbb{R})^s$, where $\Lambda_+(\mathbb{R}) := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{R}_+ \forall \alpha_i\}$.

By Theorem 5, the same result is true for the cone $\bar\Gamma_s(\mathfrak{g})$.

Let $\mathfrak{g}$ be a simple simply-laced Lie algebra and let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a diagram automorphism with fixed subalgebra $\mathfrak{k}$ (which is necessarily a simple Lie algebra again). Let $\mathfrak{b}$ be a Borel subalgebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that they are stable under σ. Then, $\mathfrak{b}^{\mathfrak{k}} := \mathfrak{b}^\sigma$ is a Borel subalgebra and $\mathfrak{h}^{\mathfrak{k}} := \mathfrak{h}^\sigma$ is a Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{h}_+$ and $\mathfrak{h}_+^{\mathfrak{k}}$ be the dominant chambers in $\mathfrak{h}$ and $\mathfrak{h}^{\mathfrak{k}}$ respectively. Then,

$$\mathfrak{h}_+^{\mathfrak{k}} = \mathfrak{h}_+ \cap \mathfrak{k}.$$

We have the following result originally conjectured by Belkale-Kumar [BK_2] and proved by Belkale-Kumar [BK_2], Braley [Bra] and Lee [Le] (case by case).

Theorem 24. For any $s \ge 1$,

$$\bar\Gamma_s(\mathfrak{k}) = \bar\Gamma_s(\mathfrak{g}) \cap (\mathfrak{h}_+^{\mathfrak{k}})^s.$$

7 Saturation Problem

In Section 2, we defined the saturated tensor semigroup $\Gamma_s(G)$ (for any integer $s \ge 1$) and determined it by describing its facets (see Theorems 18 and 23).

Define the *tensor semigroup* for G:

$$\hat\Gamma_s(G) = \left\{ (\lambda_1, \dots, \lambda_s) \in \Lambda_+^s : ([\lambda_1] \otimes \cdots \otimes [\lambda_s])^G \ne 0 \right\}.$$

It is indeed a semigroup by [K_2, Lemma 3.9]. The *saturation problem* aims to compare these two semigroups. We recall the following result (see [Br, Theorem 2.1]).

Lemma 25. There exists a uniform integer $d > 0$ (depending only upon s and G) such that, for any $\lambda = (\lambda_1, \dots, \lambda_s) \in \Gamma_s(G)$, $d\lambda = (d\lambda_1, \dots, d\lambda_s) \in \hat\Gamma_s(G)$.

We now begin with the following definition. We take $s = 3$ as this is the most relevant case to the tensor product decomposition.

Definition 26. An integer $d \geq 1$ is called a *saturation factor* for G if, for any $(\lambda, \mu, \nu) \in \Gamma_3(G)$ such that $\lambda + \mu + \nu \in Q$, we have $(d\lambda, d\mu, d\nu) \in \hat{\Gamma}_3(G)$, where Q is the root lattice of G. Of course, if d is a saturation factor then so is any of its multiples. If $d = 1$ is a saturation factor for G, we say that the *saturation property holds for* G.

The *saturation theorem* of Knutson-Tao (see Theorem 9) asserts that the saturation property holds for $G = \mathrm{SL}(n)$.

The following general result (though not optimal) on saturation factors is obtained by Kapovich-Millson [KM_2] by using the geometry of geodesics in Euclidean buildings and Littelmann's path model. A weaker form of the following theorem was conjectured by Kumar in a private communication to J. Millson (see also [KT, § 7, Conjecture]).

Theorem 27. For any connected simple G, $d = k_{\mathfrak{g}}^2$ is a saturation factor, where $k_{\mathfrak{g}}$ is the least common multiple of the coefficients of the highest root θ of the Lie algebra $\mathfrak{g}$ of G written in terms of the simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$.

Observe that the value of $k_{\mathfrak{g}}$ is 1 for $\mathfrak{g}$ of type $A_\ell (\ell \geq 1)$; 2 for $\mathfrak{g}$ of type $B_\ell(\ell \geq 2), C_\ell(\ell \geq 3), D_\ell(\ell \geq 4)$; and 6, 12, 60, 12, 6 for $\mathfrak{g}$ of type E_6, E_7, E_8, F_4, G_2 respectively.

Kapovich-Millson determined $\hat{\Gamma}_3(G)$ explicitly for $G = \mathrm{Sp}(4)$ and G_2 (see [KM_1, Theorems 5.3, 6.1]). In particular, from their description, the following theorem follows.

Theorem 28. The saturation property does not hold for either $G = \mathrm{Sp}(4)$ or G_2. Moreover, 2 is a saturation factor (and no odd integer d is a saturation factor) for Sp(4), whereas both 2 and 3 are saturation factors for G_2 (and hence any integer $d > 1$ is a saturation factor for G_2).

It was known earlier that the saturation property fails for G of type B_ℓ (see [E]).

Kapovich-Millson [KM_1] made the following very interesting conjecture:

Conjecture 29. If G is simply-laced then the saturation property holds for G.

Apart from $G = \mathrm{SL}(n)$, the only other simply-connected, simple, simply-laced group G for which the above conjecture is known so far is $G = \mathrm{Spin}(8)$, proved by Kapovich-Kumar-Millson [KKM, Theorem 5.3] by explicit calculation using Theorem 18.

Finally, we have the following improvement of Theorem 27 for the classical groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(2\ell)$. It was proved by Belkale-Kumar [BK_2, Theorems 25 and 26] for the groups $\mathrm{SO}(2\ell + 1)$ and $\mathrm{Sp}(2\ell)$ by using geometric techniques. Sam [S] proved it for $\mathrm{SO}(2\ell)$ (and also for $\mathrm{SO}(2\ell + 1)$ and $\mathrm{Sp}(2\ell)$) via the quiver approach following the proof by Derksen-Weyman [DW] for $G = \mathrm{SL}(n)$. Further, it has been shown by Hong-Shen [HS] that the spin group $\mathrm{Spin}(2\ell + 1)$ has saturation factor 2.

Theorem 30. For the groups $\mathrm{SO}(n)$ $(n \geq 7)$, $\mathrm{Spin}(2\ell + 1)$ and $\mathrm{Sp}(2\ell)$ $(\ell \geq 2)$, 2 is a saturation factor.

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George Boole and Boolean Algebra

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George Boole (1815–1864) was responsible, in the years 1847–1854, for initiating the revolution in the subject of logic by creating an algebra of logic for classes. This is all the more remarkable because Boole was largely self-educated in mathematics (and several languages), having had to give up attending school at the age of 16 to start his career as a schoolteacher to provide financial support for his parents and siblings. He started publishing mathematical papers, mainly on analysis, in 1841. Three years later, in 1844, at the age of 29, he won the first gold medal awarded in mathematics by the Royal Society.

Boole struck up a friendly correspondence with Augustus De Morgan (1806–1871). Subsequently, De Morgan's noisy feud (over a rather trivial matter in logic) with the respected philosopher Sir William Hamilton (1788–1856) of Edinburgh inspired Boole to write a booklet [2] in 1847 applying algebra to logic. In 1849, at the age of 34, Boole left school-teaching in Lincolnshire, England, for a professorship at Queen's College in Cork, Ireland. For fascinating details on Boole's life, see the excellent biography [23] by Desmond MacHale – including such remarkable details as the fact that Boole's youngest daughter Ethel Lilian (1864–1960) wrote a novel called *The Gadfly* which essentially became the 'bible of the Russian revolution'.

Although Boole was primarily an algebraist and analyst, today he is best known for his work in logic, in particular for his 1854 book *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities*. We will refer to this book as LT. The first two-thirds of LT are on Boole's algebra of logic for classes (which will henceforth be called "Boole's algebra") and the last third on applications of this algebra to probability theory. This article is only concerned with the logic portion of LT. Boole's algebra had equational laws, rules of inference for equational reasoning and a powerful Rule of 0 and 1, which has only recently been deciphered. Indeed, it is remarkable how long it has taken to properly understand Boole's algebra – the breakthrough came in 1976 with the publication of [15] by Theodore Hailperin (1916–2014).[1]

1 Using Ordinary Algebra

One of the distinguishing features of Boole's algebra of logic is the extent to which it looks like ordinary algebra. Indeed, this fact is beautifully summarised in the following two quotes:

> That the symbolic processes of algebra, invented as tools of numerical calculation, should be competent to express every act of

1 For a modern introduction to Boole's algebra of logic, see the author's article [7] on Boole in the online Stanford Encyclopedia of Philosophy, as well as the texts [8], [9] of two recent talks by the author on Boole's algebra of logic.

> thought, and to furnish the grammar and dictionary of an all-containing system of logic, would not have been believed until it was proved. (Augustus De Morgan, in *A Budget of Paradoxes*.)

> The algebra which Boole himself used was simply ordinary numerical algebra as applied to a collection of quantities each of which was assumed to be subject to the quadratic equation $x(1-x) = 0$, and Boole showed how this hypothesis could be applied to the solution of many logical problems. (C. S. Peirce, from the Nachlass *Notes on the list of postulates of Dr Huntington's Section 2*, p. 264 in Vol. IV of Peirce's collected papers.)

The equational theory of commutative rings with unity, with a conveniently chosen axiom set CR_1, is a good place to start when defining ordinary numerical algebra. This is often thought of as the algebra of polynomials. The modern way to build an algebra of logic on CR_1 is to add the idempotent law $(\forall x)(x^2 = x)$, giving the set BR of axioms for Boolean rings. The relevant models of BR are the algebras $\langle P(U), +, \cdot, -, 0, 1\rangle$, where $P(U)$ is the power set of the universe U; addition and subtraction are symmetric difference, multiplication is intersection, 0 is the empty set and 1 is the universe. But this is, indeed, a modern approach and it did not appear until the 20th century, most notably in the 1936 paper [31] of Marshall Stone (1903–1989).

Boole's approach was closer to ordinary numerical algebra than Boolean rings. He (implicitly) added the quasi-identities $(\forall x)(nx = 0 \rightarrow x = 0)$, for $n = 1, 2, \ldots$, to CR_1, giving the set of axioms that will be called NCR_1.[2] These axioms are indeed true of the ordinary number systems, e.g. the integers $\mathbb{Z}$.

Boole also added one non-numerical law, namely $x^2 = x$. This was an unusual law in that it only applied to variables, not to compound terms like $x + y$.

Boole's models $\mathbf{P}(U) := \langle P(U), +, \cdot, -, 0, 1\rangle$ were given by the definitions

$$\begin{aligned} A \cdot B &:= A \cap B, \\ A + B &:= \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined otherwise,} \end{cases} \\ A - B &:= \begin{cases} A \setminus B & \text{if } B \subseteq A \\ \text{undefined otherwise,} \end{cases} \\ 0 &:= \emptyset, \\ 1 &:= U. \end{aligned}$$

Since addition and subtraction were only partially defined, his models were *partial* algebras. Boole's partial algebras can be seen to arise naturally as follows. His definition of multiplication as intersection was a consequence of using composition of selection operators to determine multiplication – here a selection operator such as Red chooses the red elements in a class. (The use of composition of selection operators is clearly stated in his discussion of multiplication on p. 165 of LT.) The definition of multiplication gave Boole his idempotent law $A^2 = A$. Then, the numerical equations $x \cdot 0 = 0$ and $x \cdot 1 = x$ led to his definitions of 0 and 1 as stated above.

Now, suppose $A + B$ is defined; then, it must satisfy $(A + B)^2 = A + B$. From NCR_1 and the fact that $A^2 = A$, $B^2 = B$, one easily derives $A \cdot B = 0$, that is, A and B must be disjoint. This tells us that (1) addition must be a partial operation, and (2) the law $x^2 = x$ cannot apply to terms in general (in particular, not to $x + y$). An easy argument shows that if $A + B$ is defined, it must be $A \cup B$. Likewise, if $A - B$ is defined then one has $B \subseteq A$ and $A - B = A \setminus B$.

The price Boole paid for being so close to ordinary numerical algebra is that his models were partial algebras. This might have discouraged most from continuing but not Boole. In Chapter V of LT he stated his "Principles of Symbolical Reasoning", which essentially said that one can carry out equational reasoning in his system as though the models were total algebras.[3] These Principles are in general false but, thanks to Hailperin's work, we know they hold in Boole's system. Boole's Principles allowed him to claim that an equational argument $\varepsilon_1(\mathbf{x}), \ldots, \varepsilon_k(\mathbf{x}) \ \therefore\ \varepsilon(\mathbf{x})$, with totally defined equational premises and conclusion, could be justified by an equational derivation that involved (partially) uninterpretable terms. He said it was just like using the uninterpretable $\sqrt{-1}$ to derive trigonometric identities.

2 Boole's Rule of 0 and 1

After introducing his laws, rules of inference and partial algebra models, Boole stated a remarkable foundational principle which we will call his Rule of 0 and 1 (LT, p. 37):

> Let us conceive, then, of an Algebra in which the symbols x, y, z, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

This foundation has been essentially ignored until the last decade. What Boole meant is that an equational argument $\varepsilon_1(\mathbf{x}), \ldots, \varepsilon_k(\mathbf{x}) \ \therefore\ \varepsilon(\mathbf{x})$ is correct in his system if and only if when the variables are restricted to the values 0 and 1 then it holds in the integers $\mathbb{Z}$. We write this in modern notation as follows, where $\mathsf{Idemp}(\mathbf{x})$ says the variables in the list $\mathbf{x} := x_1, \ldots, x_m$ are idempotent:

$$\mathsf{NCR}_1 \vdash \mathsf{Idemp}(\mathbf{x}) \rightarrow (\varepsilon_1(\mathbf{x}) \wedge \cdots \wedge \varepsilon_k(\mathbf{x}) \rightarrow \varepsilon(\mathbf{x}))$$

iff

$$\mathbb{Z} \models \mathsf{Idemp}(\mathbf{x}) \rightarrow (\varepsilon_1(\mathbf{x}) \wedge \cdots \wedge \varepsilon_k(\mathbf{x}) \rightarrow \varepsilon(\mathbf{x}))$$

iff

$$\mathbb{Z} \models \bigwedge_{\sigma} (\varepsilon_1(\sigma) \wedge \cdots \wedge \varepsilon_k(\sigma) \rightarrow \varepsilon(\sigma)),$$

where σ is a string of 0s and 1s of the same length as the list of variables $\mathbf{x}$. It would be nearly a century after the publication of LT before the corresponding result for Boolean rings was known, namely:

$$\mathsf{BR} \vdash \varepsilon_1(\mathbf{x}) \wedge \cdots \wedge \varepsilon_k(\mathbf{x}) \rightarrow \varepsilon(\mathbf{x})$$

iff

$$\mathbb{Z}_2 \models \varepsilon_1(\mathbf{x}) \wedge \cdots \wedge \varepsilon_k(\mathbf{x}) \rightarrow \varepsilon(\mathbf{x}),$$

2 The N in NCR_1 is supposed to suggest "no additively nilpotent elements", to use Hailperin's terminology. Some might prefer to say that "the additive group is torsion-free".

3 To justify equational arguments $\varepsilon_1(\mathbf{x}), \ldots, \varepsilon_k(\mathbf{x}) \ \therefore\ \varepsilon(\mathbf{x})$ using purely equational reasoning, one needs to treat the laws $(\forall x)(nx = 0 \rightarrow x = 0)$ as equational rules of inference $\dfrac{nx = 0}{x = 0}$.

where $\mathbb{Z}_2$ is the two-element Boolean ring. The same result holds if we change BR to BA, a set of equational axioms for Boolean algebras, and $\mathbb{Z}_2$ to $\mathbf{2}_{\mathsf{BA}}$, the two-element Boolean algebra.

3 Boole's Four Main Theorems

This brief overview of Boole's work will conclude with his four main theorems. But first we need Boole's notion of a *constituent* $C_\sigma(\mathbf{x})$, where σ is a string of 0s and 1s of the same length as the string $\mathbf{x}$ of variables. A simple example will suffice: $C_{101}(x_1, x_2, x_3) := x_1(1 - x_2)x_3$.

Boole's four main theorems (expressed in modern terminology) are:

1. $p(\mathbf{x}) = \sum_\sigma p(\sigma)C_\sigma(\mathbf{x})$ (EXPANSION).
2. $p_1(\mathbf{x}) = \cdots = p_k(\mathbf{x}) = 0$ iff $0 = \sum_i p_i(\mathbf{x})^2$ (REDUCTION).
3. $(\exists \mathbf{x})p(\mathbf{x}, \mathbf{y}) = 0$ iff $0 = \prod_\sigma p(\sigma, \mathbf{y})$ (ELIMINATION).
4. $q(\mathbf{x}) \cdot y = p(\mathbf{x})$ iff
$$p(\mathbf{x}) \cdot (p(\mathbf{x}) - q(\mathbf{x})) = 0$$
and
$$(\exists v)\left[y = \sum_{\sigma \in J_1} C_\sigma(\mathbf{y}) \ + \ v \cdot \sum_{\sigma \in J_2} C_\sigma(\mathbf{y}) \right],$$
for J_i suitably determined (see [7]) by the equalities and inequalities holding within the triples $(p(\sigma), q(\sigma), 0)$ (SOLUTION).

All of these results can be readily translated into theorems of Boolean rings as well as of Boolean algebras, the latter having been carried out in 1877 by Schröder in [27], where the fourth theorem has a considerably simpler statement.

In the following, for Σ any set of axioms, let Σ^+ be $\Sigma \cup \{0 \neq 1\}$. Boole's theorems, as stated above, are easily proved by strengthening Boole's Rule of 0 and 1 to Horn sentences φ:

$$\mathsf{NCR}_1^+ \vdash \varphi|_{\mathsf{Idemp}} \quad \text{iff} \quad \mathbb{Z} \models \varphi|_{\mathsf{Idemp}},$$

where $\varphi|_{\mathsf{Idemp}}$ means the variables of φ are relativised to idempotent elements (see [10]). The corresponding result for Boolean rings is

$$\mathsf{BR}^+ \vdash \varphi \quad \text{iff} \quad \mathbb{Z}_2 \models \varphi.$$

For Boolean algebras, one has the parallel result

$$\mathsf{BA}^+ \vdash \varphi \quad \text{iff} \quad \mathbf{2}_{\mathsf{BA}} \models \varphi,$$

stated in 1967 by Fred Galvin [13].

4 The Reaction to Boole's Algebra of Logic

Boole's early successors appreciated his main results but, with the exception of John Venn (1834–1923), they found Boole's use of the algebra of numbers with partial algebra models at best unattractive, at worst completely unacceptable. Boole's work was so mysterious that the first attempt at a substantial review [33] of LT did not occur until 1876, 22 years after the publication of LT; it was written by Venn. It provided little insight into why Boole's algebra actually gives correct results and would not be improved upon for the next 100 years. Finally, in 1976, Hailperin [15] showed that *characteristic functions* were the key to justifying Boole's algebra.

Within a decade of the publication of LT, the move was underway to replace Boole's algebra with the modern version. Starting with the 1864 book *Pure Logic* by William Stanley Jevons (1835–1882), there followed the 1867 paper [24] of Charles S. Peirce (1839–1914), the 1872 book *Die Formenlehre oder Mathematik* by Robert Grassmann (1815–1901), the 1877 *Operationskreis des Logikkalkuls* of Ernst Schröder (1841–1902) and the 1880 paper [25] of Peirce. Schröder's *Operationskreis* was the first publication to absorb all of Boole's main theorems into the modern setting.

Boole's translations of propositions into equations also came under fire, and were replaced in the 1890s by equations and negated equations in Schröder's monumental three-volume *Algebra der Logik*. This was possibly the last time anyone seriously read Boole's LT as a fundamental source of information on the algebra of logic for classes; future research in this subject, aside from historical studies, used the modern version. However, as with many new subjects introduced into mathematics, it took decades before mathematicians could agree on standard nomenclature, symbols and axioms for the modern version. The 1904 paper [17] of the Harvard mathematician E. V. Huntington (1874–1952) was one of the most famous papers devoted to describing axioms for the modern algebra of logic for classes – it provided three sets of axioms. In 1933, Huntington [18] returned to this topic, offering three more sets of axioms.

5 Harvard and the name "Boolean Algebra"

The name of the modern version fluctuated for decades, usually being called the *algebra of logic*, and sometimes the *calculus of logic*. Peirce was the only one who occasionally named the subject after Boole, calling it *Boolian algebra* or *Boolian calculus*. At times he referred to those who used equational logic to develop an algebra of logic for classes, such as Jevons and Schröder, as *Boolians*. In 1898, Peirce gave a series of invited lectures at Harvard titled "Reasoning and the Logic of Things". These lectures so inspired the Harvard philosophy Professor Josiah Royce (1855–1916) that he decided, mid-career, to embark on a crash course to learn mathematics and modern logic. Royce's PhD students who benefited from this included Henry M. Sheffer (1882–1964; PhD 1908), C. I. Lewis (1883–1964; PhD 1910) and Norbert Wiener (1894–1964; PhD 1913). In 1913, both Royce and Sheffer published papers [26], [30] using the phrase "Boolean algebra" for the modern version of the subject. Wiener [36] added a paper on "Boolean algebra" in 1917. Sheffer's paper became the more famous and it used "Boolean algebra" in the title – this was the paper with the Sheffer stroke as the sole operation. Bertrand Russell (1872–1970) was so impressed with Sheffer's stroke that it was used in the second edition of *Principia Mathematica*. The name "Boolean algebra" for the modern version became ever more standard; in 1933, Huntington [18] would credit Sheffer with introducing this name. Another Harvard mathematician Marshall Stone (1803–1989) would also use the name "Boolean algebra" in his famous papers of the 1930s. Up till 1940, the name "Boolean algebra" was used mainly in papers published in American journals; but by 1950, it was the worldwide standard for the modern version of the algebra of logic for classes.

6 Boole's Translations

In order to apply his algebra of classes to Aristotelian logic, Boole first needed a translation of categorical propositions into equations. Recall that there are four kinds of categorical propositions: (A) All *x* is *y*; (E) No *x* is *y*; (I) Some *x* is *y*; and (O) Some *x* is not *y*. The following gives Boole's translations of 1847 and 1854, followed by a recommended revision:

	1847	1854	(see [11])
(A)	$x = xy$	$x = vy$	$x = xy$
(E)	$xy = 0$	$x = v(1-y)$	$xy = 0$
(I)	$v = xy$	$vx = vy$	$v = vxy$
(O)	$v = x(1-y)$	$vx = v(1-y)$	$v = vx(1-y)$

Boole viewed the syllogisms as simple applications of the elimination theorem. Traditional logicians of the Aristotelian school argued that Boole offered nothing new since his general eliminations could be achieved by a sequence of syllogisms. Boole eventually agreed (see LT, p. 240) that syllogisms were indeed sufficient to achieve his elimination results but pointed out that the traditionalists lacked a description of how the sequence of syllogisms was to be created. Then he added that (with the Solution Theorem) his system went beyond what the Aristotelians had achieved.

7 Was Boole misguided to have used the algebra of numbers?

Most of Boole's successors found his use of ordinary numerical algebra a totally unnecessary piece of baggage. However, an obvious advantage of being able to use the algebra of numbers was that its notation was standardised; practising mathematicians were expected to be quite fluent in its use. By using the algebra of numbers, it is likely that Boole was able to try out a large number of ideas and examples, quickly noting which ones succeeded, thereby rapidly gaining a sense of the 'big picture'. Perhaps it was because he was using a familiar algebra that he was able to go so far in such short time – [2] was written in a few weeks in 1847. Those opting for a modern algebra of classes spent decades trying to come to agreement on notation and axioms; their best theorems were the ones borrowed from Boole.

It is interesting to note that, in 1933, Hassler Whitney (1907–1989) published a paper [35] showing how to convert expressions in the modern Boolean algebra of classes into expressions in the algebra of numbers, noting that this made the verification of Boolean algebraic properties quite straightforward. His method was to map a class A to its characteristic function χ_A, for then one had $A \cup B \mapsto \chi_A + \chi_B - \chi_A \cdot \chi_B$, $A \cap B \mapsto \chi_A \cdot \chi_B$ and $A' \mapsto 1 - \chi_A$. Unfortunately, he saw little connection between his work and that of Boole – he did not notice that the fragment of the ring $\mathbb{Z}^U$ consisting of the characteristic functions was isomorphic to Boole's partial algebra $\mathbf{P}(U)$; otherwise, he would have been able to provide a justification of Boole's main theorems 40 years before Hailperin.

A point of direct contact in the 20th century with Boole's approach came from the Harvard Computation Laboratory under the direction of Howard Aiken (1900–1973). In 1947, he set out to create a mathematics for electronic switching circuits, to be used in the building of the Mark IV computer – the results were published in 1951 in [1]. He said that there were those who found the language of propositional logic a good choice, as well as those who liked the use of Boolean algebra made by Claude Shannon (1916–2001) in his famous 1938 paper [29]; but Aiken decided to use Boole's arithmetical algebra on the grounds that it would be more comfortable working with a familiar algebra. The first examples in the book are the switching function for a triode tube being $t(x) = 1 - x$ and the switching function for a pentode tube being $p(x, y) = 1 - xy$. So it seems that, even in the mid 1950s, there were people who knew that Boole's algebra was based on the algebra of numbers and was not to be confused with modern Boolean algebra. However, it is likely that the majority of modern mathematicians and computer scientists believe that Boole's algebra is Boolean algebra. Hailperin tried to set the record straight with his 1981 publication *Boole's algebra isn't Boolean algebra* but it is doubtful that it has had significant impact.

8 Justifying Boole's Algebra of Classes

The reasons that Boole's algebra of numbers worked so well in his algebra of logic remained a complete mystery until Hailperin's 1976/1986 book *Boole's Algebra and Probability* noted that Boole's partial algebra $\mathbf{P}(U)$ was isomorphic to the partial algebra consisting of the idempotent elements of the ring $\mathbb{Z}^U$. The ring $\mathbb{Z}^U$ satisfied NCR_1 and the uninterpretables of Boole's system became interpretable as elements of the subring of $\mathbb{Z}^U$ generated by the idempotent elements. Still, some issues remained unresolved until recently, namely the deciphering of Boole's Rule of 0 and 1 (see [7], [10]) and a revision of Boole's translations (see the table in Section 6) to allow one to translate particular categorical propositions as equations (see [11]). Finally, we can say that Boole's algebra of logic for classes was essentially sound.

Acknowledgements.
Throughout 2015, University College Cork (UCC) has been celebrating the bicentennial of the birth of Boole, its first and most famous mathematics professor. This article is the direct result of being invited to give a talk in August 2015 "A Primer on Boole's Algebra of Logic" at the Irish Mathematical Society meeting in Cork. The financial support of the European Mathematical Society is gratefully acknowledged. Thanks are also due to Michel Schellekens of UCC for many discussions on Boole's contributions.

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Stanley Burris [snburris@math.uwaterloo.ca] joined the Pure Mathematics Department at the University of Waterloo in 1968 and has published over 70 research papers since then. He was a research associate of Alfred Tarski in 1971. He has carried out research in universal algebra for over two decades, with a primary interest in Boolean constructions and decidability, leading to the 1981 memoir "Decidability and Boolean Constructions" with Ralph McKenzie and the popular textbook "A Course in Universal Algebra" with H. P. Sankappanavar. With increasing interest in computer science, in 1996 he published "Logic for Mathematics and Computer Science". In the 1990s, he started work on asymptotics and logic, leading to the 2001 book "Number Theoretic Density and Logical Limit Laws". He has studied the history of logic (especially of the 19th century) for over two decades, clarifying several aspects of Boole's algebra of logic and writing the article "George Boole" in the online Stanford Encyclopedia of Philosophy. Being an avid fan of Photoshop and of recognition of women in mathematics, he spearheaded the MAA Women of Mathematics poster project, strongly supported by Joseph Gallian, the MAA President at the time (2007–2008).

Interview with Abel Laureate Louis Nirenberg

Martin Raussen (Aalborg University, Denmark) and Christian Skau (Norwegian University of Science and Technology, Trondheim, Norway)

This interview took place in Oslo on 18 May 2015.

Partial differential equations (and geometry/physics)

Firstly, we want to congratulate you on being awarded (with Professor Nash) the Abel Prize for 2015. You will receive the Abel Prize from His Majesty the King of Norway in a ceremony tomorrow.

Your first important achievement in mathematics was solving the so-called Weyl problem in your PhD thesis. Could you tell us what the Weyl problem is about?

The problem was originally stated by Hermann Weyl. You have a two-dimensional sphere with a metric (that is, a way of measuring distance), and connected with the metric is its curvature. If this curvature is positive, the question is whether you can find a convex body in three dimensional space with a mapping to the sphere so that when you measure the distance in Euclidean space, it agrees with the metric? Weyl went quite far toward solving this problem but there were some estimates missing. My contribution was to fill in those missing estimates.

When you express the problem mathematically, it involves partial differential equations. The equations were so-called non-linear partial differential equations and the problem was proving the existence of solutions of these equations. Much of my career has really been devoted to studying partial differential equations in general but also applying them to problems from geometry and complex analysis. I even wrote two papers with a friend in economics involving partial differential equations. In my mind, it is a wonderful field. A big part of the problem is proving that solutions exist because equations can be written down for which one knows there are no solutions.

Many of these problems come from physics so solutions would be expected to exist?

Yes. But, for instance, for equations in fluid dynamics (the so-called Navier-Stokes equations that were introduced 150 years ago), mathematicians have not been able to prove that smooth solutions exist for all time. So that is still an open problem.

Is it true that the best result in that direction is your joint work with Caffarelli and Kohn from 1982?

That result is not about the existence of solutions but about the dimension of singularities if they do occur. They cannot have a high dimension; for instance, they cannot fill a curve. They must fill a set of dimension less than one. You may wonder what the hell that is? It either has dimension zero or it has dimension one. But, no! There are concepts of dimension of any non-negative number. We proved that the one-dimensional measure had to be zero, so the set could not have dimension one. The paper is very technical.

Louis Nirenberg is awarded the Abel Prize by King Harald V of Norway; left: John F. Nash (photo: NTB/Scanpix)

But it is very important in connection with the Navier–Stokes equations?

Well, it is a useful result mathematically whether engineers use it or not. Aeroplanes fly whether we solve the Navier–Stokes equations or not. But it is a big mathematical challenge to show that there are smooth solutions.

Do you often think about the Navier–Stokes equations?

Once in a while. But I don't really have any fresh ideas. I think it is up to younger people.

Start of a career in mathematics

May we ask how your mathematical life started? We were told that a certain teacher of Hebrew played an instrumental role. Is that true?

My father tried to teach me Hebrew. I resisted, stupidly, and now I know no Hebrew at all. He hired a friend to give me lessons in Hebrew. This friend happened to like mathematical puzzles and half the lessons then consisted of these puzzles. I found them quite fascinating but, I must say now, at my age, I am no longer fascinated by puzzles. They are for young people.

That started my interest in mathematics. I also went to a very good high school. This was during the depression and to be a high school teacher was considered a very good job. I had excellent teachers and I must say that the quality of the students was also very good. I particularly enjoyed the mathematics courses and especially geometry and physics. I then decided I would like to study physics.

Were there already clear signs that you had an exceptional talent for mathematics?
The teachers considered me good but I think it became clearer in college that I had some talent in mathematics. When I graduated from university, I actually received a gold medal for my work in mathematics and physics.

You graduated from McGill University in Montreal. Perhaps you could tell us about your experience studying mathematics and physics at university?
I finished high school and applied for a scholarship at McGill, which I didn't get. The high school offered an additional year, equivalent to a first year at college. I did that, applied again to McGill and then got a scholarship. So I was at McGill for three years rather than the usual four. This was during World War II and I graduated in the Spring of 1945, just at the end of the war in Europe.

It was a pleasure to study mathematics and physics. However, that was the only thing I studied. Because I missed the first year, I didn't take any courses in other interesting subjects. I am sorry I didn't.

How did you end up at the Courant Institute in New York?
By pure luck! When I finished at McGill, I had a Summer job at the National Research Council where they did atomic research. A son of Courant[1] had married a young woman from Montreal, whom I knew. They both worked there and one day she said they were going to New York to visit Courant. I asked her to ask him to suggest some place I could apply to do graduate work in physics.

She came back and said that Courant suggested that I come and take a Master's degree in mathematics. I could then go on to do physics, he said. I went down for an interview and was offered an assistantship in mathematics. I got a Master's degree and I just stayed on. I never left New York University.

Courant, Friedrichs and the CUNY

Courant was head of a very famous institute in Göttingen, Germany. He was kicked out when the Nazis came to power but he was offered a position at New York University a year later to set up a graduate programme in the mathematics department. They only had undergraduate training at that time. He came to New York to do that but there were very few students in those first years. The number of students only increased after the war. When I came, just after the war, there were a number of very talented students. Some of them became well known mathematicians. I was part of a very good body of students and it was an exciting time.

Usually, when you get a PhD at some university in America, you then leave. You go to another university for your first job. Courant was different. He kept the good people. If good people got PhDs, he simply offered them jobs.

Did it help being offered a job if you played an instrument?
I didn't play an instrument. But if I had, it may have helped even more. Of course, the rumour was that he hired people who played instruments (unless they played the piano, which he played himself).

Did you meet with him often?
Oh, yes. He often invited the students socially to his home. His wife was completely devoted to music and played a number of instruments. She was the daughter of the mathematician Carl Runge,[2] by the way. They had two daughters who were both very ardent musicians. One of them became a professional musician and is now married to Peter Lax. Courant was wonderful with young people – very encouraging and really exceptional.

Mathematically speaking, your mentor was Kurt Friedrichs?[3]
Yes. Friedrichs was the person I regard as my Sensei (as the Japanese say). I really was most influenced by him. He worked mainly with partial differential equations but he also did other things. He wrote a book on quantum theory and a book, together with Courant, on shock wave theory, which was widely used and translated into many languages.

You mentioned that there was a special atmosphere at the Courant Institute, in part because no distinction was made between pure and applied mathematics…
That's right. Courant insisted there was no difference between pure and applied mathematics. He did both and he encouraged people to do the same. It is just mathematics. When New York University hired him, they asked him what a mathematics department needs and he said: "A library and a coffee room." So we have a very nice lounge that we use all the time.

It is remarkable that you are the fourth Abel Prize Laureate associated with the Courant Institute (after Peter Lax in 2005, Srinivasa Varadhan in 2007 and Mikhail Gromov in 2009). What has made this institute so successful?
Well, partly it is just the warm atmosphere. I think graduate students are very happy there and there is a lot of interaction between the students and the faculty. It is, of course, much bigger now than it was when I was a student there. But the warm atmosphere has prevailed.

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[1] 1888–1972.

[2] 1856–1927.

[3] 1901–1982.

Who were your most important colleagues over your career?
There is Friedrichs but also two other students of Courant: Fritz John,[4] a wonderfully talented mathematician who later became a faculty member (I had the fortune of writing one paper with him) and Hans Lewy[5] (I wrote several papers related to some of his work). Hans left Germany immediately after Hitler came to power. He came to the United States and had a career at Berkeley.

Partial differential equations and inequalities

Your name, often with various co-authors, is attached to many fundamental concepts and theorems in PDEs. If you just look at the citation list, your work has had a tremendous impact. Let's start with Fritz John, with whom you authored a very influential paper about BMO functions (BMO standing for "Bounded Mean Oscillation").
That was his idea. He introduced BMO functions. It came from some work he had done in elasticity theory. He approached me saying: "I have a class of functions and I believe they should have such-and-such a property." I worked on it and was able to prove that property. He then improved it so the final version is better than what I had done. It became a joint paper and I must say a lot of people have referred to it.

Absolutely! It became famous – if we may say so – because of the many applications. For instance, Charles Fefferman got the Fields Medal in 1978 and one of his main contributions was to show that the BMO space is dual to the Hardy space H^1.
Charles Fefferman did many things but in particular, he proved the duality result that you refer to.

Your paper with Fritz John contains the John–Nirenberg inequality. You love inequalities?
I love inequalities. And what we proved in the paper was really an inequality.

Would you explain why inequalities are so important in the theory of PDEs?
When you look at a partial differential equation, you may ask whether a solution exists. Now, you can't write down the solution so you need to know some bounds. It cannot be too big, it cannot be too negative, its derivatives cannot be too big and so on. You try to get estimates of the size of the function and of its derivatives. All these estimates are inequalities. You are not saying that something is equal to something but that something is less than some constant. Thus, inequalities play an essential role in proving the existence of solutions. In addition, you want to prove properties of solutions and, again, inequalities play a central role. Hence, inequalities are absolutely fundamental to studying partial differential equations; for that matter, so are they for ordinary differential equations.

[4] 1910–1994.
[5] 1904–1988.

Let's move on to your joint research with Shmuel Agmon and Avron Douglis.[6] There were two very important papers. Can you explain what they contained?
What we did was to extend some classic work, by the Polish mathematician Schauder, to higher order equations.[7] There is a fundamental paper of Schauder for second-order, so-called elliptic equations. We thought it would be useful for people to be able to deal with higher order equations and systems of equations so we proved the analogues of those results. In the other paper, we proved the results for systems and also for different norms, that is, for different ways of measuring the size of the solutions. We published several different kinds of inequalities and they have been used by many people.

You wrote a paper with Joseph Kohn introducing the important notion of pseudo-differential operators. You are one of the fathers of that concept. Can you explain why this concept is so important and how you came upon it?
Joe Kohn had published a fundamental paper in complex analysis. It involved the regularity of solutions for a certain class of systems up to the boundary – a rather difficult paper! He suggested we should try to generalise this to more general systems of equations. We started to look at it and we had to consider so-called commutators of operators. You apply an operator and then you apply a second one. Then you take the difference of that result with the operator obtained by applying the second one and then the first. We needed properties of the commutator. We were using a certain space, called an L^p-space, and a theory due to Calderón[8] and Zygmund[9] for certain singular integral operators. We needed to extend their result to commutators so we thought: "How do we extend these singular integral operators to make an algebra out of them?"

That led to what we call pseudo-differential operators. The concept came from a very specific problem in systems of partial differential equations but it turned out to be a useful thing in itself. It grew out of Calderón and Zygmund's theory. By the way, Calderón was a wonderful mathematician and he danced the tango, which I admired enormously.

You had a very bright student, August Newlander, with whom you wrote a very important joint paper in 1957. Can you tell us about the results you proved there?
It was a problem I first heard of from André Weil.[10] He said: "Here's a problem in complex analysis. Why don't you people in partial differential equations work on this kind of problem?" I thought: "Why not? Let's try." I took a student who was very bright and I said: "Let's look at the very simplest case, in the lowest dimension." The student, Newlander, had the initial idea, which worked fine

[6] 1918–1995.
[7] 1899–1943.
[8] 1920–1998.
[9] 1900–1992.
[10] 1906–1998.

in low dimensions but, to our surprise and dismay, didn't work in higher dimensions. We had to come up with a completely different proof in higher dimensions. It led from a linear problem to a non-linear problem. It was kind of strange but the non-linear problem was in some ways more accessible.

What was André Weil's reaction when you solved the problem?
He was very happy and so were other people in complex analysis. Many people have used the result. Some years later, Hörmander[11] found a linear proof of the same result. It was very technical but it was purely linear.

Are there any outstanding problems in the enormous field of partial differential equations, apart from the Navier–Stokes problem, that you would like to highlight?
Well, I think almost nothing has been done in so-called over-determined systems, that is, where there are more equations than unknowns. You may have two unknowns and five equations so there have to be some compatibility relations. There's almost no analytic theory of that. There is a theory developed by Cartan[12] and Kähler[13] but that assumes that everything is analytic. Outside analytic category, almost nothing is known about such systems. They often come up in geometry so I feel that this is a big gap in the theory of partial differential equations.

Mathematics and mathematicians all over the world

May we ask you some questions about international mathematics? We know that you travelled to post-war Europe very soon after your graduation.
Yes. I had a fellowship and came to Zurich during the academic year 1951/52. I went mainly to be with Heinz Hopf,[14] who was a geometer and a topologist. Heinz Hopf was a wonderful person – a lovely and extremely kind man. I also spent one month in Göttingen that year. That was arranged by Courant who felt I should go there. During that year, I didn't actually carry out any research. What I did was to write up the things I had done before. I had been very slow at writing them up for publication because I somehow had a block against writing. So during that year I wrote several papers.

Did Courant ever return to Göttingen?
Yes. After the war, he went back to Germany many times. He had many contacts and he wanted to help build up German mathematics again.

He must also have been very bitter?
Well, he was bitter but, at the same time, he had friends and he wanted to encourage and help to develop mathematics in Germany.

[11] 1931–2012.
[12] 1869–1951.
[13] 1906–2000.
[14] 1894–1971.

You also went to the Soviet Union?
Yes. The first time I went was in 1963. It was a joint Soviet-American symposium on partial differential equations, arranged by Courant on one side and the Soviet mathematician Lavrentyev[15] on the other. There were about two dozen American mathematicians and about 120 Soviet mathematicians from all over the Soviet Union. It is one of the best meetings I have ever attended. It was in Novosibirsk, Siberia, which was the academic city that Lavrentyev had helped create. It was like being aboard a ship for two weeks with people you make friendships with immediately. I made friends with Russians that are still friends today. Some have died, unfortunately, but I have had very good friends in Russia since then. I have never collaborated with any of them but they are still very warm friends; we would meet and talk about mathematics, politics and all kind of things.

How about China?
I have been to China a number of times. The first visit was arranged by Chern,[16] a Chinese mathematician who had settled in America. This was in 1975 and the Cultural Revolution was still going on, though I didn't realise it at the time. For instance, I was visiting the Chinese Academy of Science but I was taken to Beijing University. I said I would like to meet the faculty but they said they were busy teaching – which was simply a lie. There was no teaching going on. They showed me the library and then they wanted to take me to some other university but I said: "There's no point. Either I meet the faculty or I don't go."

They had me give many lectures but I said I also wanted to hear what some of the people there were doing. So some young people spoke about some of their research. I learned later that they had to get permission to attend my lectures. I didn't make close friends at that time. It was an interesting experience and, of course, things have changed enormously since then. I did make friends with some who subsequently came and spent a year or two at Courant.

We should also mention that you were awarded the first Chern Medal of the International Mathematical Union.
Yes. That's true. That was in 2010.

You were also awarded the first Crafoord Prize in 1982, together with Arnol'd.[17]

Perhaps it was a tongue-in-cheek comment but Arnol'd once said something like: "Mathematics is the part of physics in which experiments are cheap."
It wasn't entirely tongue-in-cheek. He really felt that the contact of mathematics with physics and the real world was important.

He didn't get permission to go and get the Crafoord Prize. I visited Moscow just before I went to Sweden and had dinner with him in his home. He was waiting until

[15] 1900–1980.
[16] 1911–2004.
[17] 1937–2010.

the last minute to see if he would get permission, but he didn't.

When I went back to America, I got a call from a woman claiming to be Arnol'd's sister. I thought: "How is that possible?" I had just seen Arnol'd a few weeks before and he never mentioned he had a sister in New Jersey. She came to my office and, indeed, it was Arnol'd's sister. He never mentioned a word. It's incredible!

Talking about Arnol'd, on some occasions he expressed frustration that results proved in the West had already been proved in Russia but, because of poor communication during the Cold War, these results were not known. Did he express these feelings to you?

He tended to do that. I remember once he was visiting New York. Someone was giving a seminar talk and he was attending the lecture. During the talk, Arnol'd said: "Oh, that was already proved by such-and-such a Russian." But the person giving the seminar talk then checked and the Russian had never proved it. So Arnol'd was not always correct. He tended to give more credit to Russians than was due.

You may have heard the joke where the Russian says: "What you proved, I proved first. And anyway, the result is trivial."

Problems, collaboration and "Sitzfleisch"

It is striking that 90% of your published papers describe joint work. Can you explain why this is so?

It is just a pleasure! It is just an enormous pleasure talking mathematics with others and working with them. Of course, much of the work you do yourself. I mean, you discuss ideas and work with others but then you go home and think about what you have done. You get some ideas and you get together again and talk about the new ideas. You get reactions to your ideas and you react to their ideas. It is a wonderful experience.

Do you usually start out with a goal in mind?

Usually there is a goal. But somebody once used the expression: "There are those mathematicians who, when they come to a fork in the road, they take it." I'm that kind of mathematician. So, I may be working on a problem with a colleague when we come to something that looks interesting, and we explore that and leave the original problem for a while.

Are you more of a problem solver?

Yes, definitely. There are two kinds of mathematicians. There are those who develop theories and those who are primarily problem solvers. I am of the latter.

Do you come up with interesting problems through discussions with other mathematicians? What kinds of problems are you attracted to? Is there any pattern?

It's hard to say. A graduate student once asked me how I find good research problems. I said to him that I sometimes see a result but don't like the proof. If the problem appeals to me, I start to think if there is a better proof. My ideas may lead to a better proof or may lead to something new. The student said he'd never seen a proof he didn't like and I thought: "He is hopeless!"

May we ask you a question that we have asked several previous laureates? How does one find the proof of a mathematical result?

Some people work with perseverance until a proof is complete but others tell us that insight appears in a sudden flash – like lightning. Do you have experiences of this sort?

Both may happen. But most of the time you are stuck. Maybe you make a breakthrough with some problem as you get some insight and see something you didn't see before. But the perseverance and all the work you carried out before seems to be necessary to have this insight. You need perseverance or, as the Germans say, you need "Sitzfleisch".

Are you the kind of person that gets so involved in trying to solve a problem that you are, so to speak, lost to the world?

Not all the time but it can happen for many hours. Sometimes, I wake in the middle of the night and start thinking about a problem for hours and cannot sleep. When you do that, it is very hard to fall asleep again! If I have an idea, I just follow it. I see if it leads to something. I still try to do that but in the last few years it has not led anywhere. I haven't had any success.

Communicating mathematics

You have had 45 PhD students. That is an impressive number! Can you tell us what your philosophy is? How do you come up with problems for your students?

It's hard to say. Sometimes it is hard to think of a suitable problem. It is easier to think of problems that are too hard, and just not practical, than to think of a problem that is good and can be solved in reasonable time. I can't really answer that question. I don't know how I go about posing problems.

Were there occasions when you had to help students along?

Oh, yes. I meet the students regularly, usually once a week. We discuss their progress and I might make suggestions. I may say: "Look at this paper, this may lead to something."

How would you describe your love for mathematics? What is it about mathematics that is so appealing to you? Is it possible to communicate this love to people outside the mathematical community? Does one have to be a mathematician to appreciate the appeal of mathematics?

Some people are very good at communicating to the general public. I am not so very good at that. But once you are in it, once you are hooked, it's very exciting and fun. I have used the word "fun" before. But it is really fun to do mathematics. It is an enormous pleasure to think about mathematics even though you are stuck 90% of the time, perhaps even more.

That is what people outside mathematics cannot comprehend.

Yes, it is hard to comprehend. You have to be in it and I think it does take some talent to be able to do mathematics. But it also takes, as I said, "Sitzfleisch". You need to be stubborn and have perseverance, and you can't give up. I have been stuck on some problems for years.

But you do think it's important to try to communicate to the general audience?

Yes, I do think that is important: (a) for the development of mathematics, and (b) to show them that it is a pleasure to do mathematics. Courant and Robbins[18] wrote a very nice book: 'What is Mathematics?'. It is a lovely book. There is also a recent book by Edward Frenkel, a mathematician who came as an immigrant from Russia as a young man. It is called 'Love and Math'. He makes a valiant attempt to get the general public interested in the branch of mathematics in which he works (which is also connected to physics). It is very hard to do. He tries but I think it is too hard for the general public. But he makes a real attempt to do it and I must say I admire him for that. I just recently read his book.

Music and movies

We have one final question that we have asked several laureates before. What are you interested in when you are not doing mathematics?

I love music. I love movies. You won't believe this but at the time when I lived in Montreal, in the province of Québec, you could not get into a movie before you were 16. Incredible! Now it's hard to believe. So when I was 16, I went crazy and started to go to movies. When I moved to New York, there were suddenly all these foreign movies: Italian movies, Russian movies, French movies. I went crazy. I went almost every night to the movies. Since then, I have loved movies.

Have you seen "A Beautiful Mind"?

Of course, and I have read the book.

What kind of music do you like?

Mainly classical but I also listen to jazz. My grandson, who will be at the ceremony tomorrow, is a professional jazz drummer. And I love Argentinian tango. I have a large collection of records of Argentinian tango.

Not only on behalf of us but also on behalf of the Norwegian, Danish and European Mathematical Societies, we would like to thank you for a very interesting interview.

Louis Nirenberg (left) interviewed by Christian Skau and Martin Raussen (photo: Eirik F. Baardsen, DNVA).

Martin Raussen is professor with special responsibilities (mathematics) at Aalborg University, Denmark. Christian Skau is professor of mathematics at the Norwegian University of Science and Technology at Trondheim. They have together taken interviews with all Abel laureates since 2003.

[18] 1915–2001.

Interview with Manjul Bhargava

Ulf Persson (Chalmers University of Technology, Goteborg, Sweden)

Were you surprised to get the Fields Medal?
Well, I guess maybe it wasn't the hugest surprise, since people had been talking about it and asking me about it for so long. But, of course, when it actually happens, it is a surprise, and a pleasant one.

There has been a lot of attention this year, more I think than four years ago. It was impossible to reach you by email.
Indeed, within a couple days of the announcement, I had received around 18,000 emails! There was no way I could read through them all. I'm so sorry that I was hard to reach. But I plan to respond to all of them – you will eventually hear from me!

I can understand. I tried to reach you after your talk as well. It was impossible; the crowd around you was incredible. I felt like the paparazzi.
Yes, I have never experienced anything like it. It was fun at first, giving autographs and taking photos. But then the crowd was getting too big and compressing towards the centre, and people were getting crushed. Two security guards came to try and disperse the crowd but they too got crushed. Then they radioed a whole team of security, who made their way in and formed a ring around me and took me away.

I never thought I would experience something like that as a mathematician! I am extremely impressed that mathematics in Korea is held in such high regard. The ceremony was broadcast live on national television and so many from the public watched it, enough so that we would get stopped for pictures and autographs just walking on the streets of Seoul.

So you are experiencing the proverbial 15 minutes of fame.
Yes, or more precisely about a week of fame, until I leave Korea and return to the US!

Will this change your life significantly?
Well, I hope not. I know it will give me new opportunities to disseminate mathematics to the public around the world that I didn't have before, and that is a responsibility that I look forward to and will, of course, not take lightly. At the same time, I know that it is mathematics research and teaching that gave me this platform and I do not wish to compromise too much on my research and teaching and the ongoing work with my students and collaborators. So it will all be about finding the right balance between the two.

To start from the beginning, what is your family background?
I was born in Canada and moved to the United States early on. I grew up in a very Indian home and I also spent a lot of time in India growing up, which is where my extended family is from. Indians tend to have very large and supportive families and I always received a lot of encouragement from them.

My greatest inspiration was my grandfather, who was a Sanskrit scholar. I spent a lot of time learning from him, not just Sanskrit but also history, philosophy, literature and other languages like Hindi and English.

I also learned a lot about these subjects from my mother, who is a mathematician (so, of course, I learned a lot of mathematics from her as well).

I didn't like to go to school so much. I liked learning on my own but then I'd show up for exams and, of course, also for after-class activities at my school, which was a running joke among my teachers and friends.

What is your earliest memory of mathematics?
I always liked mathematics, as far back as I can remember. I loved shapes. My favourite toy as a 2- or 3-year-old (my earliest memories are from then) was a cube-shaped puzzle with various shapes cut out of it, and the goal of the puzzle was to quickly identify the shapes and insert them into the slots where they exactly fit. A bit later, I remember trying to work out ways to add and then multiply large numbers in my head using my fingers.

I was always interested in numbers. I really liked big numbers and learned the names of bigger and bigger numbers in both Hindi and English ("mahaashankh" is 10^{18} in Hindi; then I learned the word "vigintillion" in English, which is 10^{63}; then I learned from my grandfather that back before 500 BC, there was the Sanskrit word "dhvajaagranishamani" for 10^{421}, which I found very exciting).

Big numbers are much more mindboggling and conducive to vertigo than mere infinity.
Do you know of the number "Googolplex"?

Of course.
That really used to intrigue me – after coming to terms with mahaashankh, vigintillion and dhvajaagranishamani, I used to try to wrap my head around what *that* [Googolplex] meant. Yes, the feeling of vertigo is quite accurate!

So you were a math prodigy?
No, I would not say that. I did not skip grades or anything like that. But I did like to discover things for myself, encouraged by my mother.

I read that you made algebraic calculations at the age of eight such as computing the number of oranges in a triangular pyramid, given the number of oranges on an

edge of the pyramid. This is impressive. When did you encounter, say, complex analysis?
I always enjoyed doing such things on my own for fun. But, as I said, I never skipped any grades. For example, I did not learn complex analysis until I was a sophomore in college and took a course. So I was never that much more advanced than my peers, particularly when I reached college at Harvard, where so many students were really advanced.

Did you ever read a math book cover to cover?
Yes, I often did in graduate school. Princeton did not offer many courses for beginning graduate students so this was the way I learned much mathematics.

You always were a good boy?
I guess so. I'm told I was quite hyper and naughty when I was a 3- or 4- or 5-year-old but it seems I outgrew that!

Did you have other interests? Reading? The classical Indian epics such as Ramayana and Mahabharata constitute a treasure trove which must be very fascinating to a child.
Yes, these epics, as well as various other works of literature, were frequently the topics of conversation over family dinners at home. In fact, my uncle often made these conversations into contests, as to who knew the most details about various aspects of these works. These contests were a lot of fun and added – for my cousins and me – a further motivation to read!

Later on, did you consider any alternative careers?
I did often toy with the idea of becoming a musician and tabla player as a profession. But at some point, I realized that if I became a mathematician, I'd still find time to keep up music as well but if I became a full-time musician, I'd probably have a tough time keeping up mathematics!

So you were all set to be a mathematician from an early age?
I always loved math and because my mother was a mathematician, I was always aware of that pleasant career option. But I did have brief periods when I changed and thought maybe I'd like to do computer science, or economics, or physics, or linguistics, as I liked these subjects a lot too. But I always realized that mathematics was what was bringing these subjects together, and mathematics was what I liked about them.

How do you do mathematics?
I usually like thinking about mathematics while walking, or jogging, or pacing, or working with someone at the blackboard. I quite rarely sit down with a pen and paper and do mathematics; that usually comes after something has been worked out already.

So you cannot think when you are standing still.
I guess that's a good way of putting it! The reason I stay away from paper is that often when I think about mathematics, it is not in words or in terms of things that can be written down yet. That usually comes later when I try to translate these thoughts into more usual mathematical language.

What about collaboration?
I love working with others. Thinking and talking about and working on mathematics together with others is one of the joys of mathematics. It can also be more productive. Collaboration forces you to think things through by explaining them, and the other person can then give invaluable input, feedback and new perspectives, as you bounce ideas off one another. Sometimes, different collaborators can bring different areas of expertise to a problem, which can be very valuable and help bring some cross-pollination between areas.

Does working with someone not interfere with your thinking process? When you are thinking hard about something, you do not want to be interfered with.
Sometimes. But, in collaboration, one generally decides which parts are more productive or fun when discussed together and which parts each should go home and think about alone. In the latter case, collaborators can go home, think separately and then bring their ideas the next time. So, even in collaboration, there is usually a lot of individual thinking. Actually, sometimes you'll be working with a collaborator where no one speaks for over an hour – individual thinking is going on! But still it's fun to do it together, knowing that there is someone there in case you are ready to share an idea, or problem, or general confusion.

Nothing beats direct human conversation, or should I say interaction. Are there problems you have solved in collaboration that you would not have been able to solve by yourself?
Almost certainly! This is, of course, impossible to know for sure. But I definitely think so, and I suspect this is likely the case in most of my collaborations!

Do you understand everything you are doing in mathematics? I mean, what is important about a theorem is not its precise formulation but the idea of the proof which can be used to prove many different theorems. If you just use the exact formulation, you are actually treating the theorem as a black box.
I used to understand all the mathematics that I do, from the very basics all the way up. There were no black boxes. But in some recent collaborations just in the last year, I have used some theorems I did not understand from first principles but took on trust, using them as black boxes as you say. I must admit that at first it made me quite uncomfortable. It did not seem to bother my collaborators as much though, who were already used to it after many years of it! This is a modern phenomenon in doing mathematics, building on years and years – and pages and pages – of our predecessors' work, to the point where it is not possible for any one person to know every detail. But one must get used to it. In some sense, it is also awe-

inspiring: mathematics is bigger than any one person and, as a mathematical community, we are able to go further than any one person could alone.

I think that this is a bit sad. The Australian aborigines are supposed to be the best trackers in the world. They are personally very connected to their environment; their technology may be primitive but they have an intimate command of it. Modern man, on the other hand, is more alienated; he is using gadgets all the time of which he has not the faintest understanding, such as mobile phones – black boxes in other words. One would think that the life of an aborigine would in some sense be more satisfying, while we become more and more consumers rather than producers. Would you care to elaborate on that?

I do agree in a sense – as I said, until the recent development, everything I did in mathematics till now I understood quite thoroughly from the bottom up and that was, as you say, very satisfying. I felt like I was firmly touching the ground at all times rather than floating in air. But one cannot deny some of the things in mathematics that have been achieved by mathematicians being consumers of other mathematicians' works rather than just being producers.

When you think of mathematics, do you do it systematically, one problem at the time?

No, generally not – I usually like to think about many problems at the same time. That way, if I'm stuck on one, I can always move to another for a little while, and maybe, when I get back to the one I was stuck on, I'll have a fresh perspective. I like to have problems around of different levels – some of them easy, in the sense that I have at least some methods that can be used to make progress on them, while others are much harder, for which I often do not have any clear idea how to solve them. And then there are long-range problems I keep in the back of my mind, hoping that one day I will have a moment of inspiration and find some possible inroad.

Do they range over a wide field?

Well, they definitely range over a slightly larger field every year, as I learn more mathematics and get interested in more problems. But the core of most of the problems I get interested in tend to be related to classical areas of number theory somehow.

The psychologist William James claimed that every interest in adult life can be traced back to one formed in your childhood or youth.

This is very much so with my mathematical interests – much of what I do can be directly traced back to interests I formed when I was young.

By the way, who was your advisor?

Andrew Wiles – he was definitely a great inspiration to me! I didn't end up working directly in his area and tended to work more on my own problems but he was just wonderful to discuss mathematics with. He was always willing to listen and inspire and you learn a lot just through conversation with such a figure. But, above all, he had a wonderful sense of what is important in mathematics.

Manjul Bhargava receiving the Fields medal from the Korean president Mrs Park Geun-hye. (Photo courtesy T. Gowers)

John Conway and Peter Sarnak were also inspirational figures to me during graduate school; I also talked to them quite regularly and learned so much from them as well.

This leads me to another question. Is too much mathematics being published?

What do you mean? In terms of too much paper being wasted – too many trees being cut in the forests?

I meant that there is too much and that the important things are being buried under a heap of garbage.

Strong words! But is the peer review system not taking care of that?

Not very effectively.

As to important things being buried, I think with modern search engines and MathSciNet, we are in a position, as never before, to look and find what is important to us and the things that we need in our work.

So this means that, in the future, everything should be available electronically, a huge database which you can mine. Maybe that will change the forms of publishing dramatically; instead of writing traditionally structured papers, results will be presented much less polished – not an entirely pleasant scenario.

Actually, I do not think this to be so bad. More material is available at our fingertips than ever before and available sooner to us than in times past. We have the option to look at less polished material that is available much sooner; of course, we can still wait for the peer-reviewed and more polished versions as we always did. So we have more options than we used to have, which I feel is a good thing.

The connection between mathematics and music is often pointed out. Do you think there is one and, if so, how do you explain it?

There definitely is. Many mathematicians are musical and I don't think that is a coincidence. What is interesting is that it doesn't quite work the other way round, at least not to the same extent.

If you meet a musician and tell him that he must be good at mathematics, chances are that he or she will look at you blankly.
Precisely.

It also means that musical talent is much more common than mathematical talent.
I guess so – an interest in music certainly tends to be much more common than an interest in mathematics.

But we are not talking about just an interest but about the active sense of playing an instrument. Do you play an instrument?
I did learn several instruments as a child: tabla, sitar, guitar, violin and a bit of piano later on. But tabla is the instrument that I always enjoyed the most and stayed most in touch with over the years.

Do you actually perform to paying audiences?
Yes, I do but definitely less so than in the past. It takes so much time to prepare and practice for such concerts and I'm getting less time every year, unfortunately.

But why is there such a connection? Have you ever given it a thought?
I have given it a lot of thought.

So what are your thoughts?
Well, I think that both mathematics and music are about patterns and how they fit together. Practitioners of both subjects are guided by beauty and elegance, with the aim of telling stories and conveying thoughts that ordinary words cannot express.

I would perhaps put it a bit differently. It is about themes that repeat themselves with subtle variations, not mechanically or predictably. Music is not about perfect symmetry, but almost, just as two different mathematical fields can be very similar without being isomorphic – if the latter were the case, it would be boring, just as algorithmically generated music is supposed to be boring.
I would say that part of the divergence of the two subjects has to do with the difference between the types of thinking in the right and left hemispheres of the brain. There is a formal, logical aspect to mathematics, which is the only one people in general are aware of, but there is also a very artistic side, having to do with creating and discovering patterns and seeing how they can be connected in a coherent and expressive way. The same is true of music as well: there are formal constraints on composition/improvisation dictated by the genre of the music but, beyond that, what remains is the artistic expression given those constraints of the genre. I'd say the formal and logical aspects are slightly more critical in mathematics than in music (in that a piece of mathematics is rejected if it is not completely logical) but the artistic and logical aspects play an important role in both.

But some mathematicians are not musical at all. Does that mean that they are different as mathematicians, maybe even defective?
No – perhaps they have the potential but never had the opportunity to develop it. Not all excellent musicians are also excellent painters or sculptors. It is up to each person to decide how best to apply and develop one's artistic sensibilities and interests.

In order to appreciate mathematics, you have to be a mathematician but you can appreciate music without being able to compose music. I suspect that the emotional impact of music has very little to do with the composing of music, maybe not even with musicality per se. Mathematics does not have this aspect. It does not give the same kind of emotional sustainment. It is very different listening to a math lecture and a piece of music.
Every subject can be appreciated at different levels. I definitely think it is possible for the general public to appreciate the beauty of mathematics, if it is presented in an accessible way. Maybe a general person will not appreciate it at the same level and in the same way that a mathematician would, just as a general person might not appreciate a technical piece of music in the same way that a professional musician would. But I do agree that music tends to appeal slightly more to one's emotional side and mathematics slightly more to one's logical side. But is not the sensation of beauty listening to a lecture by Serre similar to listening to a beautiful piece of music? Serre's lectures are like musical improvisations – he is sensitive to the reactions of the audience and changes his presentation as he goes along.

No – there is, I think, a profound difference. The appreciation of mathematical beauty is not tied to a definite presentation; it can easily be paraphrased.
Well, it depends what kind of music you are talking about. Not all music is tied to a definite presentation. I think elegant chalk-and-blackboard mathematics lectures are more like the improvisations of classical Indian music or jazz, where the speaker/musician's presentation is affected by the atmosphere and mood of the day and the reactions and type of the audience. The main themes are decided in advance but the exact presentation can vary, and can come out differently each time. That is part of the excitement of it.

Misspellings or mispronunciations have no impact on the mathematical beauty but playing the wrong notes has a jarring effect.
Wrong notes, i.e. notes played outside the framework of the musical piece, are similar to mathematical errors in a lecture – and the latter may be equally (if not more!) jarring.

But take a standard piece in the Western classical repertoire, say the Goldbach variations by Bach. They are essentially played in the same way over and over again, although of course I am aware of the subtle variations

from one performance to another, just like differences in the translations of the same novel. You can take a recording and listen to it again and again but if you played the same video of a Serre lecture over and over again, you would get bored.
Well, that's why I was saying that a chalk-and-blackboard math lecture is more similar to an improvisational form of music, such as classical Indian music or jazz, where the themes are predetermined but the exact rendition of the themes varies from performance to performance, depending on the mood and the audience that particular day – that is part of the excitement of those forms of music.

For example, the *raaga* Darbaari is one of the most beautiful raagas in Indian classical music but each time one goes to hear it live, one is likely to hear a different rendition (possibly even on a different instrument or instruments) and that is part of the beauty and appeal of this melodious raaga. One would likely get bored if one went to hear exactly the same performance of Darbaari over and over again, unless it is one of the truly historical and legendary renditions (such as that of Ustad Vilayat Khan or Pandit Bhimsen Joshi).

Incidentally, I am not so sure I would get bored viewing the same lecture by Serre over and over again – no doubt I would always find new insights each time!

But if you want to understand a piece of mathematics, it is better to read different presentations rather than read the same one over and over again. Mathematical ideas are not as tightly tied to their presentations as musical.
Again, the very same is true for improvisational forms of music such as classical Indian music and jazz: listening to several renditions of the same raaga or an improvisational jazz piece by different expert musicians is the very best way to understand it, absorb it and appreciate it.

A further example to illustrate the difference between composing and listening is that Haydn reportedly burst into tears when listening to his own creation (actually `The Creation'), claiming that he could not have composed it; it was too beautiful.
Actually, I truly feel this happens to mathematicians all the time. Who hasn't looked back on one's prior works and asked: "How did anyone come up with that?" It happens to me on occasion. I come across things that I have written in the past and am amusingly impressed. How did I ever think of that? At the time, it might have been obvious but, of course, not later. Looking back at one's artistic thought or creation as a third person can be an extremely different experience than the moment or experience of creating it.

That is true. When you write something down, it is no longer a part of you; it has an independent objective existence. Popper would express it by saying it has been moved from World Two to World Three, from your own personal world to one which can be shared by all thinking people.
Indeed.

Are you interested in mathematical philosophy – the issue of mathematical Platonism?
A bit, but I can't say I give it a lot of thought.

So it means that you are a Platonist.
[Laughing] Sure – I do feel that good and natural mathematics is discovered and not invented.

So you believe in an external mathematical truth, that the theorems we prove are not just figments of our imaginations or mere social constructs.
I do. I think many mathematicians would probably not do mathematics otherwise.

One of the wonders of mathematics is that some mathematical conjectures are actually being solved.
I agree. That is rather amazing! But it is conceivable that some famous conjectures that people work on might be undecidable.

Maybe because they are, as they say in the field of PDEs, not well-posed. The natural questions seem to be amenable to solutions, such as the rather remarkable link between Fermat's conjecture and work in Elliptic curves that your advisor exploited.
That does seem true so far. But there are also several well known and seemingly well-posed problems that are still open – it could be possible that some of these are undecidable…

Do you have any personal opinions on the status, say, of the Riemann Hypothesis or the Birch–Swinnerton–Dyer Conjecture?
I'd certainly love to see a solution to one or both of these in my lifetime. But it's not clear to me in either case how close we are. I've certainly thought more about the latter and feel like it could be within reach.

You have ideas of how to prove it, not just ways of vindicating it, as in your talk, which was very nice by the way.
Thanks. I think I have a few ideas of how to prove it, in the sense that I have some ideas that I have not yet tried and so they have not failed yet!

Which you are not going to expand upon?
It would take us too much afar.

We started our conversation about large numbers and how they exercise a fascination, especially on burgeoning mathematicians. But there are different kinds of numbers; not all numbers seem directly related to cardinality, and some basic cardinalities are very low. When it comes to mathematical abstraction, I believe that there may only be three or four significant levels, just as we can only sense number, so called subitising, when there are very few.
That is certainly true. There are very few levels of abstraction that mathematicians tend to employ. But that may be because mathematics is still in its infancy.

It seems to me that when it comes to very large numbers, they do not really have mathematical significance. The celebrated Skewe's number resulted from a crude estimate, which has, I believe, been drastically improved. Much of real mathematics only concerns the exponential; if you get double exponential bounds, your estimates are bound to be crude.
Well, that is true about Skewe's number. But do you know about the the Paris–Harrington Theorem? It says that the smallest number satisfying the strengthened finite Ramsey theorem with given parameters grows well beyond exponentially in those parameters; it is not even primitive recursive and is beyond what can be defined using Peano arithmetic. It's a pretty natural example where such a huge function naturally arises and cannot be made smaller. I would not be surprised if more examples are discovered in the future in number theory and beyond.

But there are many places where logic and mathematics part ways. In logic, you can formally have arbitrarily long chains of quantifiers but, in practice, those chains are severely limited in length, just like the levels of abstractions. Does anything beyond the cardinality of the continuum enter real mathematics?
It's true that we rarely work with cardinalities beyond the continuum but it does happen, no? Many theorems in topology and model theory (which has also been extensively used in number theory in recent years) make use of ultrafilters, which even on Z have cardinality beyond that of the continuum… I'm sure there are other examples.

There are no known effective bounds on the sizes of minimal rational solutions to algebraic equations over the rational numbers but, if they existed, would they not be exponential with respect to the coefficients?
Well, given the Paris–Harrington theorem, it seems to me that larger-than-exponential functions could potentially arise in this scenario as well, no? I don't see why not. In particular, if there were such general exponential bounds, would that not give an algorithm to determine the solubility of Diophantine equations, which we know cannot exist by work of Matiyasevich, etc.?

What I am aiming at is some formulation along the lines of the unreasonable reasonableness of mathematics. Number theory inevitably touches on it and where it does, it gets hopeless.
Could be. At the moment, though, there are far easier places where it gets hopeless! The fact that, as yet, there is no algorithm that provably determines whether an elliptic curve equation has a finite or an infinite number of rational solutions is intriguing. You did go to my talk?

Yes, I did, and this is why I brought this up. It was very nice, perhaps because I knew a dense subset of it. It is always the completion at infinity that you can bring home with you. If you do not know enough, you could get lost.
True, there is only so much information you can process during a talk.

So what makes a good math talk?
Well, as you say, you should not strive to convey too much information and you must try to build on what your audience already knows. Doing less with clarity and purpose is much better than doing more. I also feel a talk should try to impart a sense of the wonder, the motivation, the connections with other related problems or applications and a glimpse of some of the key ideas. I think that goes for all talks, to the public as well as to one's professional colleagues, although how one goes about it in each case can be quite different.

Thanks so much for your time during this very busy period for you.

It was a pleasure. Thanks very much for your interest, and for the excellent questions!

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Recollection of a Singular School

Sylvie Paycha (University of Potsdam, Germany)*

Some 40 participants are seated in the large seminar room; they have come from 12 different African countries[1] to Ouagadougou, the capital of the landlocked country of Burkina Faso[2] to learn about Fourier integral operators and their many concrete applications inside and outside the realm of mathematics. The school[3], funded by the Volkswagen Foundation, started three days ago and, for the last talk of the day, I am about to explain to them how to use the inverse Fourier transform to build pseudo-differential operators from rational functions, when my colleague Bernard Bonzi, co-organiser of the school, steps into the room calling me to the door.

I follow him along the corridor, where three other colleagues and co-organisers, Marie-Francoise Ouedraogo, Stanislas Ouaro and Hamidou Touré, are waiting with anxious looks; something serious has happened, they tell me. Is it that serious that I cannot have another 10 minutes to conclude my presentation? My colleagues seem reluctant to let me finish but finally nod approvingly, insisting that I should conclude hastily. I still have no idea why there is this sudden tension and I am suspecting a problem with the premises we are using, a three-storey, medium size, cream-coloured building on the outskirts of Ouagadougou, some 20 minute bus drive from the two hotels most of us are staying in and not too far from the campus of the University of Ouagadougou (co-organising institution of the event). Some 10 participants are staying at the guest house of the university, which is near the conference hall but far from the two hotels in the city centre where most of the participants are housed. The local organising committee has rented the conference hall as well as a nearby room where we gather for lunch and coffee breaks from the private University of Saint Thomas d'Aquin.

Back in front of the audience patiently waiting for me, I mumble a few words explaining, with a touch of irony, that we should conclude rapidly before we are required to leave the room. I feel I am legitimately and dutifully finishing the explanations I had started. The students who had got actively involved in the discussion about the correspondence between symbols and operators via the Fourier transform seem somewhat disappointed when we stop soon after the interruption. My colleagues, who still look very preoccupied, take me away into a room, which adds to the mystery of their sudden interruption, and answer my questioning look: "There seems to have been a putsch; it still is not confirmed but considering the potential danger of the situation, we should all get back to the hotels immediately." By the time I have digested the news and further questioned my colleagues, we realise that the participants have already left for the hotels with the hotel van. In the car that is driving us back, I insist that we should go to the hotels to inform the participants and speakers of the situation. My colleagues hesitantly comply as I argue that informing the participants of the situation is the best thing to do at that stage, after which they leave for their respective homes on the other side of the city. Back in our hotel after the sudden interruption of the talks, we gather with other guests staying at the hotel around the television set in the reception hall to hear that "the event" was indeed a putsch by General Gilbert Diendéré.

This was how the school was suddenly interrupted on Wednesday 16 September 2015, a date that probably none of the participants of the school will ever forget.

Diendéré had served for three decades as former President Blaise Compaoré's Chief of Staff. This coup was supported by the presidential guard known as the RSP, the 1,200 strong Regiment of Presidential Security, of which Diendéré was also seen as the figurehead commander despite having retired from the force in 2014. The RSP arrested both President Michel Kafando of the transition government and Prime Minister Isaac Zida. The air and terrestrial borders were closed for some days[4] and a curfew from 7 pm to 6 am (lasting some 10 days) was declared.[5]
Little had I anticipated such a dramatic event during the opening ceremony,[6] which had taken place that very morning in the presence of the President of the University of Ouagadougou.

I had expressed my hopes that this school might help overcome the obstacles that set walls between our respective nations and continents. A putsch was not among the

Place de la Révolution.

* On leave from the Université Blaise Pascal in Clermont-Ferrand

1 Benin, Cameroon, Chad, Congo, Democratic Republic of Congo (RDC), Ethiopia, Ivory Coast, Mali, Morocco, Niger, Nigeria and Senegal.

2 The "Land of people of integrity" ("Le pays des hommes intègres"), formerly Upper Volta.

3 Summer school on Fourier integral operators and applications, Ouagadougou, 14–25 September 2015.

4 The airport was briefly reopened later to let the presidential delegation of Macky Sall (President of Senegal) and Thomas Boni Yayi (President of Benin) land to start the negotiations with General Diendéré.

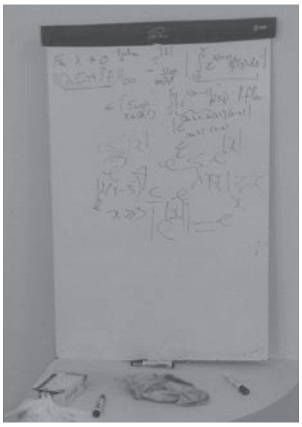
5 The curfew restrictions were to be lessened a week later to 11pm–5am.

6 The ceremony had been postponed to the third day of the meeting, due to my late arrival.

obstacles I had envisaged but now, the school seemed to be doomed to end three days after it had started.

The "Revolution square" (Place de la Révolution[7]) where protests started that very Wednesday evening – nearly a year after the October 2014 protests that had forced the former President Blaise Compaoré to resign following some 27 years in power – was to separate the local organisers and participants in their homes from us foreign organisers, speakers and participants in our hotels on the other side of the city. What was named "the event" for a while was to set up an imaginary yet tangible wall between us. Apart from a very brief visit of a couple of local organisers to one of the hotels three days after the "event", only some eight days later (a day after the Tabaski celebrations[8], seriously hampered by the lack of food in the city) when the city seemed to start getting back to normal life did the local organisers dare to venture back to the hotels. How frustrating and disappointing for them when they had put so much energy into organising the school! Little had they anticipated "the event", especially as, on the request of the Volkswagen Foundation, they had asked the President of Ouagadougou University to confirm that organising such a school in Ouagadougou would be safe.

Gradually measuring the importance of what was happening, and the potential danger of violent confrontations between the population and the RSP, as co-organiser of the school, I began to worry about what to do in such circumstances. What were we to do on the morning after the putsch, and the days to follow? One option was to declare the school over but then what would the participants and speakers now stuck in their hotels do all day; would they not start panicking? An alternative was to try to adapt the organisation of the school to the circumstances, an a priori risky solution considering the instability of the situation. Indeed, the Balai Citoyen (the Civic Broom[9]) had grown in determination and efficiency, so a violent reaction could be expected from the Burkinabé people, who were surely not going to accept Diendéré's diktat. Going back to the conference hall in the outskirts of Ouagadougou was therefore impossible due to potential riots and shooting on the streets. But going from the hotel, where most of the speakers were staying, to the nearby hotel where most of the participants were lodged (a 10 minute walk) seemed feasible. And this is indeed what we did.



The white board on which the talks were given.

[7] Nickname for the "Square of the Nation" (place de la nation), the main square of Ouagadougou after the 2014 protests initiated by the "Balai Citoyen" (see footnote below).

[8] *Eid al-Adha*, "Feast of the Sacrifice", A Muslim celebration honouring the willingness of Abraham to sacrifice his son, as an act of submission to God's command.

[9] A civic organisation, founded by two musicians in 2013, which played a central role in the protest movement that forced Blaise Compaoré to resign in October 2014.

Hardly no one on the otherwise lively streets of Ouagadougou (left) and participants on their way to the improvised conference venue (right).

The hotel manager kindly lent us a small seminar room, which seemed unused. We found a tiny, narrow whiteboard to lean against the wall, a board we had to hold up straight on the table with one hand while writing with the other. In that small room and on that board were held some 40 lectures (four a day over 10 days), thanks to the speakers[10] who gave talks in such difficult conditions. We could only count on those who were fit enough to give a talk – for many fell ill over several days due to the preventative medication against Malaria – and who were ready to take the risk of walking ten minutes from their hotel to the new, improvised conference room in the other hotel nearby. Participants searched their pockets for a few marker pens to give us and, when we ran out of pens on the fourth day, the hotel sent out an employee on a difficult mission (considering the circumstances) to drive through Ouagadougou, where most shops had remained closed since the coup, in search of a stationers who might sell markers. Thus, we could go on covering the white board from top to bottom with semi-groups, distributions, wave front sets, Fourier transforms, pseudo-differential operators, characteristics, singular supports, Lagrangian submanifolds, Bergmann transforms, Fourier integral operators, fundamental groups and Maslov indices. With the "event", singularities, which were the central theme of the school, had become a characteristic of this very singular school.

Phones would ring during the talks and participants (and even the speakers) would leave the room for a moment to reassure a relative worried by the news of the coup and return to the seminar room with a gloomy look, after having heard that their flight back had been cancelled because of the coup, or sometimes a happy face, having been informed of the new departure time of their plane. The lecturers' moods, looks and tones of voice varied from day to day according to their state of health and the latest news they had received but their faces and voices would invariably look and sound happier and more enthusiastic as their lecture evolved. I had not suspected how far a mathematics talk can pull the speaker and the audience away from the distressing reality around them, an observation shared by both participants and speakers. During the talks, a short silence would follow what we thought might be the sound of guns or any other suspicious sound but no one dared

[10] Michèle Audin (Strasbourg), Viet Ngyuen Dang (Lyon), Julio Delgado (London), Catherine Ducourtioux (Corte), Massimiliano Esposito (London), Matthias Krüger (Göttingen), Gilles Lebeau (Nice), Cyril Lévy (Albi/Toulouse), Michael Ruzhansky (London) and René Schulz (Hannover)

The market closed because of the putsch.

make a comment, so uncertain was the situation. In the midst of this overwhelming tension, one would hear the participants making jokes about the situation, such as the rather repetitive lunch menu: due to the acute food shortage caused by the putsch, the hotel manager had arranged for a relative living in the outskirts of the city to provide him with chickens, which were served for lunch every single day!

Four days after the coup, the negotiations between General Diendéré and the presidential delegation had concluded in favour of an amnesty for General Diendéré and the eligibility of the former CDP (Congrès pour la démocratie et le progrès, Blaise Compaoré's party) members. This was clearly a threat to peace; how could the population accept such a deal? Violent protests were to be expected. On the Monday following the coup, the regular army marched into Ouagadougou, publicly announcing its intention to disarm the RSP while avoiding any fighting. The night before the announced military manoeuvre, rumours had spread that the army would march into the city overnight. I packed my backpack with what I considered important belongings in case I had to suddenly flee from the fighting during the night and woke up at dawn, worried by some voices I could hear outside my bedroom window which led to a large terrace roof. A glimpse through the window reassured me; it was only the Radio France International two man team broadcasting the morning news from the terrace. No military confrontation could have taken place since the army had not yet reached the city. People gathered at sunset cheering on the highway as they waited for the anticipated entry of Burkina Faso's regular army, who vowed to disarm the RSP. That morning,[11] the streets, which had remained silent and empty since the coup, seemed to come back to life. In the afternoon, Diendéré publicly gave rather contradictory and inadequate apologies, asking the people of Burkinabé to forget about the putsch but claiming full responsibility for it and promising to restore civilian government.

But, by the evening, the situation radically changed; we heard that Michel Kafando had asked for protection from the French Embassy. A veil of silence covered the city again. Macky Sall, President of Senegal, who had come over the weekend with Thomas Boni Yayi, President of Benin, to negotiate with Diendéré, had failed to find a resolution to the crisis in spite of his political weight and diplomatic experience. Following an extraordinary summit meeting of ECOWAS (Economic Community of West African States), another delegation of presidents[12] arrived a couple of days later to calm down the situation. This time their intervention had an effect; a week after the coup,[13] an agreement was passed and a peace deal was presented to the Mogho Nabaa, King of Burkina Faso's leading Mossi tribe. Michel Kafando, who had been under house arrest for some days after his first detention, was now free and announcing his return to power.

The school went on running in the midst of the turmoil, a form of resistance to Diendéré's diktat. The applications to climate change and seismology we had planned for the second week were never discussed during the school. The flights of the speakers[14] who were due to arrive at the end of the first week had been cancelled and the airport remained closed until the middle of the second week. Yet, the participants were eager and happy to learn about the fundaments of FIOs and indeed learned a lot of abstract material during the talks and informal discussions with the speakers. A couple of participants from Benin had spent several days on a coach to reach Ouagadougou, having had to wait on the coach for the border to reopen, and were all the more determined to make the most out of the school. One could perceive the anxiety of some of the participants and most of the speakers but all agreed that, under the circumstances, it was best to go on with the talks. Keeping busy with mathematics, claimed many participants, was a very efficient way to dispel the worries, and various speakers asked to give more talks to keep their minds occupied preparing them. The particular circumstances the school was now held in were actually more propitious to informal interactions between the speakers and the participants than the more formal setup the school might have allowed for had the "event" not happened. I am very grateful to all the speakers and participants and admire their courage.

Despite questions raised as to the sincerity of Diendéré's public apologies, eight days after the putsch and one day after the Tabaski celebrations, the tension one had felt on the streets of Ouagadougou melted down and the sun dared to venture back. The preceding days had not been too hot, with sudden wind blasts and strong rain showers, as is to be expected during the rainy season. The city of Ouagadougou was now glowing with the pride of victory over the usurpers. With this coup, we (participants, speakers and organisers of the school) had unexpectedly borne witness to the complex, painful and still ongoing emancipation of the Burkinabé people from 27 years of dictatorial leadership and its ramifications.

Sylvie Paycha is a Professor at the University of Potsdam, on leave from the Université Blaise Pascal, Clermont-Ferrand. Her research topics are pseudodifferential operators, renormalisation techniques and index theory. She was the co-organiser of two schools in Ouagadougou: "Index theory and interactions with physics", 21–29 May 2009 (Research school co-funded by the CIMPA and the University of Ouagadougou, with the help of external funding), and "Fourier integrals and applications", 14–25 September 2015 (School funded by the Volkswagen Foundation and co-organised with the University of Ouagadougou).

[11] On Monday 21 September.
[12] A delegation comprising the Presidents of Ghana and Benin, as well as the Vice-President of Nigeria.
[13] On Tuesday 22 September.
[14] Nicolas Burq (Orsay), David Dos Santos Ferreira (Nancy) and Jérôme Le Rousseau (Orléans).

A Tour of the Exhibition *MadeInMath*

Gilberto Bini (Università degli Studi di Milano, Italy)

On one side, the work of mathematicians is incredibly multi-faceted, ranging from the decoding of emotional biochemistry to isogeometric analysis. On the other side, the layman is more and more astonished by these various facets but still somewhat puzzled. As is well known to the members of the EMS, it is therefore important to raise public awareness and make mathematics more tangible in everyday life.

The first exhibition of *MadeInMath* was curated by Gilberto Bini, Maria Dedò and Simonetta Di Sieno (*matematita* Research Centre) and Renato Betti and Angelo Guerraggio (MATEpristem Research Centre), and organised by Vincenzo Napolano (INFN in Rome). The title was *MaTeinItaly* and was mainly supported by the Università degli Studi di Milano, the Università degli Studi di Milano-Bicocca, the Università Commerciale "L. Bocconi" and Politecnico di Milano.

The *leitmotif* of the exhibition is the recurring question: "What is the profession of a mathematician today?" The curators make an effort to answer this question from an Italian point of view but soon address their curiosity to other realities, as mathematics is, in fact, universal.

The exhibition is designed to take any visitor – even one who scorns mathematics – on a tour [1] and show them that this discipline has always inspired the growth of our society, in connection with other disciplines, such as history and art, medicine and biology, etc.

The tour begins in Ancient Greece, where the debate on the fundamental meaning of numbers and geometric figures was of a philosophical and aesthetic nature. Afterwards, it goes through another important stage: the story of Leonardo Fibonacci in 1200 (see Figure 1) and his contribution to modern mathematics in Europe, such as the famous numbers named after him and their connection with the golden ratio.

Figure 1. An installation on the story of L. Fibonacci.

In modern times, mathematics (and not only numbers) helps scientists describe the quintessence of natural phenomena. In general, mathematicians focus on problems, which may be concrete or abstract. In order to solve them, they formulate models of various types. The *Tabula Peutingeriana* (a map of the world known to Romans) and the *Mappamondo di Fra' Mauro* (one of the last maps before the discovery of the American continent) are displayed as examples of models used to represent the world (see Figure 2).

Figure 2. Some models from the past.

As proved by C.F. Gauss (1777–1855), the curvature of the Earth causes a distortion effect on geographical maps. An installation allows the visitor to choose a projection. Through a touchscreen, the visitor can generate a small Tissot indicatrix and move it around, thus experiencing the distortion effect on the chosen map (see Figure 3).

Figure 3. Interacting with geographical projections and their distortion.

The history of models unfolds in various other contexts. Models arise when mathematics meets, for example, physics or astronomy, or perspective during the Renaissance Age. Indeed, a reproduction of the *La Città Ideale*, attributed to Francesco di Giorgio Martini, is displayed on a big screen and, through an *ad hoc* installation, visitors can interact with the painting. Moving in a specific area opposite to the screen remodels the scene as if it were painted from the visitor's point of view.

Nowadays, the mathematical writing of the world reaches its apex with digital technologies and mathematical models that describe, forecast and even transform everyday life. Think, for instance, of applications to medicine (to prevent cardiovascular pathologies), conservation of the landscape (to reduce pollution and deterioration) and sport competitions (to design aerodynamic cars, comfortable helmets and competitive swimsuits through numeric simulations).

An installation along the exhibition path lets visitors play with some 'mathematical models'. Their movements become figures and vector fields, which alter and take into account the main properties of models describing the progression of a swimmer or traffic flow, as well as a crowd flow simulation or the coordinated movements of flocks of birds (see Figure 4).

Figure 4. Interacting with real models.

Mathematicians do not only work on models from the real world. They often come up with new ones for their research. As an example, the exhibition path hosts various installations, among them some on four-dimensional space, so that visitors can enjoy interactive animations on sections and nets of regular polytopes (Figure 5), as well as a journey in the 120-cell. A merry-go-round of fancy polytopes enters our three-dimensional world and their sections appear on a big screen in a series of colourful polyhedra (see Figure 6 and video [2]).

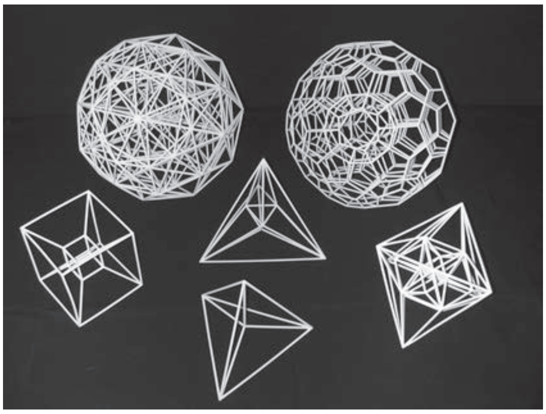


Figure 5. All the regular polytopes in four-dimensional space.

A dozen young mathematicians were interviewed about their future, which they dream to model. Some of them

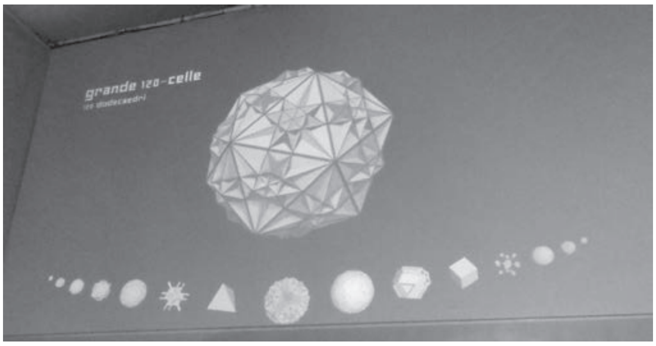


Figure 6. A merry-go-round of polyhedra.

have kept working in their country; some of them have migrated to foreign countries. The visitor can find some of these institutions spread on a map and listen to some videos, or they can follow the flow of migration to another installation, which is based on a map that can be found at the link: http://umi.dm.unibo.it/mappa/.

As you can tell, the exhibition is quite dense. Its success was partly due to numerous trained guides (mainly Master's students in mathematics) that managed to follow school groups of various levels, from primary to high school, as well as pay special attention to numerous individual visitors. Their work really facilitated the appreciation of hidden connections and pointed out possible hints and suggestions for teachers or amateurs as a motivation for further study.

If you missed the opportunity to take a tour of the exhibition during the first set-up last year, do not worry; you will have a second chance in Spring 2016, starting from mid-February, because MadeInMath will be hosted by *Muse – Museo delle Scienze* in Trento.

Websites

[1] https://www.youtube.com/watch?v=62G8yVroWwo
[2] https://www.youtube.com/watch?v=gXqvGl18kS8

Gilberto Bini is an associate professor of geometry at the Università degli Studi di Milano. His research area is algebraic geometry, in particular Calabi–Yau varieties and applications to theoretical physics. He is also active in communication of mathematics as a member of the research centre "matematita" (www.matematita.it) and managing editor of the journal "XlaTangente online" (www.xlatangente.it). Among other activities, he has been one of the curators of the exhibition described in this article.

Explain Your Thesis in Three Minutes

Marie Kreusch (Université de Liège, Belgium)

Think about your research. Have you ever tried to explain it to the average person in the street? You have to use simple words, catch their attention and try to tell a story with a good plot and hopefully a happy ending. It is not that easy! Many PhD candidates and young doctors each year rise to the challenge and explain their research in three minutes to non-specialist audiences.

One of the recent initiatives of universities to promote research, and PhDs in particular, is the creation of the following contests: "Three Minute Thesis (3MT®)" and "Ma thèse en 180 secondes (MT180)".[1] Each participant (PhD candidate or young doctor) gives in English or French (depending on which contest) a generalised, clear, concise and convincing talk on their thesis in three minutes. The exercise develops effective presentation and communication skills.

I met Adrien Deliege, a PhD candidate in mathematics at the University of Liège (Belgium). He won the international final contest MT180 in Paris on 1 October 2015. This young mathematician is studying wavelet transforms and applies this method to climate data such as temperature or time series. Here are his impressions of the competition and his answers to my questions.

How did the idea come to you to take part in this contest? Was it on your own initiative or was it under the initiative of your advisor?
I heard about the contest last year and I found the concept interesting, so I decided to go and see the Belgian final, which was in Liège. I found the candidates very impressive and this motivated me. I noticed it was the kind of contest in which I like to take part. Therefore, a few months later, I thought: 'Ok, let's do this', and here I am.

What is the main difference to a talk given in the context of a 'classical' mathematical conference?
Besides the time limit, which is obviously much longer (and less rigid/stressful), in a 'classical' mathematical conference, I would say that the main difference is the purpose of the talk. On one hand, the aim is to popularise a complicated scientific subject with simple words intelligible to everyone. On the other, the goal is to present the technical details of your work to a scientific audience. These two contexts have basically nothing in common but a talk.

How long did you train for the contest?
The writing of the speech and the preparation of the slides took between 15 and 20 hours, I would say. As far as the training was concerned, I practised a lot the week before the contest to be right on time during the competition. To be honest, I think I repeated my talk at least a hundred times!"

How useful was it for you to participate in the contest?
It forced me to find a simple way to explain the subject of my thesis. It was also an excellent way to learn how to control my voice, rhythm, breath and nerves during a talk. So it helped me develop my communication skills and I guess it could be useful for other talks or during an interview for a job, for example.

What was the biggest difficulty during this contest?
Practising again and again and again! It is hard to stay focused when you repeat the exact same thing for the tenth time in an hour, but it had to be done.

What are your feelings about this contest and did you expect to win? Do you have something to say about the mass media interest that came after the contest?
It is a very interesting contest because it combines a show part and a scientific part, which is something rather unusual. Therefore, as a PhD student, I found it really challenging from both a personal and a professional point of view. Honestly, even though I spent a lot of time preparing for the contest, I didn't imagine I could win in Paris. As a matter of fact, when I saw the presentations of the other finalists, I thought they were so good that I could relax and have fun during mine because I was sure I couldn't do better. So when they called my name for the winner's prize, it came as a total surprise! After the contest, I was astonished by the number of people from the media asking for interviews; that was something quite unexpected as well. It is good that they were interested in this competition because it really helped the popularisation of the scientific content behind it, which is clearly one of the major objectives at stake.

What do you think about the difficulties of "popularising" mathematics in comparison to other subjects?
I think it is really hard to popularise mathematics because it is often extremely abstract and theoretical. Consequently, it is more difficult to touch people because they cannot feel that mathematics could be helpful in their everyday life. The other candidates did a wonderful job; I really enjoyed their presentations and their subjects are of primary importance as well. Nevertheless, everybody has already heard about AIDS or GMOs, and knows more or less what is at stake there, but nobody cares about multifractals or wavelets. These are notions that cannot be explained in a simple way nor clearly illustrated, which is the reason why we struggle in making maths a popular subject. As far as I am concerned, I am lucky to work in the field of applied mathematics, which allows me to switch from theory to practice and

[1] My thesis in 180 seconds.

makes the popularisation of my subject a little bit easier. Anyway, we have to keep trying to make mathematics appealing and understandable to the general public and if this kind of contest can help, it is great!

Would you suggest this contest to PhD candidates? If yes, is this more appropriate at the beginning or at the end of their thesis?
Of course I would advise PhD candidates to participate; it is a thrilling adventure and an exciting personal challenge. It is maybe better in the middle of their thesis. They should take a year or two to watch the previous contests, gain experience in giving talks and get a more global point of view on their work. At the end of their thesis, they will probably have more important things to do, such as writing their thesis!

Thank you Adrien. My last question is: 'What would you advise to future participants?'
Practise again and again and again! And enjoy yourself; it's a game after all!

When I started writing this article, at least two questions came to mind. Why did universities create these contests and why are there so many participants? I would like to provide some clues.

Nowadays, more than 50% of young doctors will NOT pursue their career in academia.[2,3,4] Making a career in another field, which has only been an available option recently, is becoming more and more the norm. At the same time, universities are improving transfers of skills with industry,[5] increasing the visibility of research toward the general public[6] and sharing knowledge with society. These actions have led to changes in doctoral training.

Concerned about the employability of their doctors after their theses, universities have set up a suitable training programme for PhD candidates. Soft skills courses are (most of the time) part of this training programme. The general purpose of these courses is to develop skills that are useful both in academia and in other professional fields. Beside purely scientific tasks that are essential in order to submit a thesis, soft skills and other side activities of PhD candidates are more than ever in the spotlight.

Researchers, and particularly researchers in mathematics, are no longer living in an ivory tower, disconnected from the real world. Many diversified initiatives have come about to popularise mathematics: websites, forums, books, movies, documentaries, exhibitions,[7] public presentations, contests, high school operations research and so on. PhD candidates take part in these interventions and get involved in the visibility of mathematics to the general public.

To conclude, besides completing PhD theses, young researchers are gaining soft skills like communication, education, popularisation and so on. Such skills are styles of the day in addition to being necessary in many professional fields. I think that soft skills courses and initiatives like 3MT or MT180 still have several great years ahead.

The author thanks all those friends who contributed with their careful reading and useful suggestions.

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2 "Want to be a Professor? Choose Math", *Career Magazine*, 24 July 2015.
3 "The impact of doctoral careers", Final report. Leicester: CFE Research, Page 23 (2014).
4 In Belgium, five years after PhD graduation, 33% of doctorate holders are still working at a university. More information can be found in "Careers of doctorate holders: employment and mobility", patterns/STI working paper 2010/4. Paris OECD, Auriol, L. (2010).
5 "Improving knowledge transfer between research institutions and industry across Europe: embracing open innovation", European commission (2007).
6 "Promoting the action – Visibility of EU funding", Article 38 in Horizon 2020.
7 Look at the September 2015 EMS Newsletter for a nice example.

Mathematical Sciences Research Institute (MSRI)

Heike Friedman (MSRI, Berkeley, USA)

The Mathematical Sciences Research Institute (MSRI) is one of the world's pre-eminent research centres for mathematics. Mathematicians from around the world come to MSRI for focused periods of research and collaboration with colleagues in their particular field. The institute is situated in the tranquil hills above the University of California, Berkeley campus, and offers a beautiful retreat from the distractions of academic duties. It is an intellectually stimulating environment that promotes community and strengthens the mathematical sciences by training its next generation of leaders. MSRI's coveted postdoctoral fellowship programme allows selected participants to spend a semester in Berkeley, concentrating on their research and learning from experts in their field. MSRI runs many Summer schools for graduate students and an undergraduate programme MSRI-UP that supports students from under-represented groups. MSRI also plays an active role in K–12 mathematics education.

The institute develops and delivers public outreach events attended by thousands, as well as films and videos seen by an even larger audience. In the Spring of 2015, MSRI launched the first ever National Math Festival in Washington, D.C., in partnership with the Institute for Advanced Study (IAS) in Princeton. The hands-on activities and maths lectures drew well over 20,000 visitors!

History

MSRI was envisioned by three UC Berkeley mathematics professors: Shiing-Shen Chern, Calvin Moore and I. M. Singer, in response to a 1979 call for proposals from the National Science Foundation (NSF). As a result of that competition, the NSF funded two mathematical institutes: MSRI in Berkeley and the IMA in Minneapolis. In September 1982, MSRI began full scientific operation in a temporary building. In 1984, ground was broken for the present building in the hills above UC Berkeley's main campus and MSRI moved into the facility a year later. The building was greatly expanded, with a new auditorium, seminar room, common rooms and library in 2006.

MSRI is located in the hills above the University of California, Berkeley campus.

Governance and funding

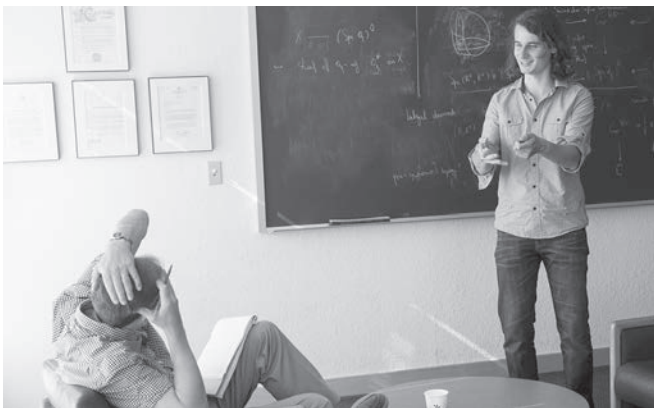
MSRI is a non-profit organisation governed by a Board of Trustees, currently with 32 elected members and seven ex-officio members. The board includes distinguished mathematicians as well as representatives of the worlds of finance and technology who understand the importance of basic research.

The institute operates without permanent faculty. A rotating Scientific Advisory Committee (SAC) creates the scientific agenda. Its 10 regular members, renowned mathematical scientists, serve four-year terms. The SAC selects and plans the institute's programmes and works with the organisers of these programmes on the selection of key participants. This structure ensures that, over time, MSRI's programmes will capture the most significant developments across all mathematical fields.

The Human Resources Advisory Committee (HRAC) augments the SAC's consideration of human resource issues. Its 10 members help recruit participants from under-represented groups and develop new MSRI activities to promote the involvement of these groups in the mathematical sciences.

More than 100 universities form the Committee of Academic Sponsors (CAS). These include most of the top research universities in the United States as well as 20 international institutions. Academic sponsors support MSRI financially through their membership dues and can send up to three students each Summer to one of MSRI's two-week Summer graduate workshops.

MSRI is one of the largest single projects funded by the National Science Foundation's Division of Mathematical Sciences and has received continuous Government support for more than three decades. MSRI has also developed a strong support system of private foundations and individuals. About 40% of the operating budget now comes from non-government sources.



Two of the greatest number theorists of our day, Peter Scholze (Bonn) and Richard Taylor (IAS, Princeton), hard at work at one of MSRI's atrium blackboards.

Scientific programmes and workshops

MSRI runs two topic-centred programmes every semester (occasionally one programme will occupy the entire facility). Programmes are often chosen to exploit an opportunity for cross-pollination between areas of research.

"The programmes that MSRI runs often lead to great mathematical developments," noted Dr David Eisenbud, who returned to MSRI in 2013 for his third term as the institute's director. "One year, we had a programme on *operator algebras*. A postdoc at the time, Vaughan Jones, noticed an amazing connection with *knot theory*, which has led to tremendous developments in mathematics and in physics."

In the Spring of 2016, MSRI will host a programme on *Differential Geometry*, a subject with deep roots and much current activity. Many old problems in the field have recently been solved, such as the Poincaré and geometrisation conjectures by Perelman, the quarter pinching conjecture by Brendle-Schoen, the Lawson Conjecture by Brendle and the Willmore Conjecture by Marques-Neves. The solutions of these problems have introduced a wealth of new techniques into the field.

About 250 members come to MSRI's programmes each year for stays of one to four months. They are joined by more than 1,500 researchers who come for shorter periods, mostly to attend one of the 14–18 workshops offered. Twelve of these workshops are related to the programme, including a *Connections for Women* workshop, which features female speakers and provides networking opportunities for female mathematicians. Though MSRI's programmes are typically planned three years in advance, the annual *Hot Topics* workshop is planned at much shorter notice to catch an emerging subject. For example, the *Hot Topic* in March 2016 will be *Cluster Algebras and Wall-Crossing.* It will highlight a breakthrough connecting ideas originating in theoretical physics (mirror symmetry) and algebraic geometry (geometric invariant theory) with a phenomenon first discovered in a purely algebraic context (cluster algebras) but now appearing as an overarching theme in many areas of mathematics.

More than 1,500 researchers participate in MSRI's programmes and workshops every year

Other workshops explore emerging applications of mathematics in other sciences. "Breaking the Neural Code", for example, involves questions of analysing data of an unprecedented scale that has become available through new ways of monitoring brain activity.

The research at MSRI is shared widely: nearly all lectures are filmed and made available at no cost through the institute's website. The online collection features more than 3,100 lectures.

Edward Frenkel (UC Berkeley) recently gave a series of four popular lectures on the Langlands programme in MSRI's Simons Auditorium. The lectures were filmed by the Japanese National TV company NHK for broadcast on Japanese television.

Postdoctoral fellowships

The postdoctoral fellowship programme provides an unparalleled opportunity for young mathematicians to meet, learn from and collaborate with assembled hosts of experts in their chosen areas. The programme is complementary to the usual postdoctoral positions: most MSRI postdocs come for a semester, on leave from a longer position. The experience is often career-defining. The fellowships are in high demand; more than 280 applicants compete for the 28 available positions each year and the great majority of those chosen accept the position.

Since 1982, MSRI has supported approximately 900 fellows. One of them was Andrei Okounkov, who teaches at Columbia University. He was an MSRI postdoctoral fellow in 1997: "The five months I spent as postdoc at MSRI […] played, without exaggeration, a crucial role in my career as a mathematician." He won the Fields Medal in 2006 and in 2010 and he joined MSRI's Board of Trustees and was elected Co-Chair in 2013.

Terry Tao, the James and Carol Collins Chair in Mathematics at the University of California, Los Angeles, is another former MSRI postdoc and Fields Medallist: "I believe the MSRI postdoctoral fellowships are a wonderful component of an early career, […]. I was fortu-

The postdoctoral fellowship programme provides an opportunity for young mathematicians to meet, learn from and collaborate with experts in their fields.

nate enough to participate in the MSRI programme on harmonic analysis, as well as the introductory workshop in harmonic analysis and PDEs earlier that year. I can credit this programme for starting my own research into PDEs and with starting many collaborations in both harmonic analysis and PDEs, some of which lasted for over a decade." Dr Tao is the lead organiser of the 2017 programme on *Analytic Number Theory*.

While the postdoctoral programme is mainly funded by the NSF, five fellowship semesters are permanently endowed by private sources and a sixth is funded through a multi-year gift.

Summer graduate schools

Every Summer, MSRI organises a number of Summer graduate schools. Deputy Director Dr Hélène Barcelo remarked: "Attending one of these schools can be a very motivating and exciting experience for a student; participants have often said that it was the first experience where they felt like real mathematicians, interacting with other students and mathematicians in their field."

The topics for the Summer schools are often chosen to prepare graduate students to return to MSRI for a related workshop or programme. Last Summer, MSRI ran a Summer school on *Gaps between Primes and Analytic Number Theory*, which ties into the programme on *Analytic Number Theory* that will take place in the Spring of 2017.

The Summer schools are in high demand. MSRI has reached its capacity and has started collaborating with institutions in Italy, Spain, Canada, Mexico, Korea and Japan to hold Summer schools abroad. This has been a very successful model.

MSRI-UP

MSRI-UP is a comprehensive research, enrichment and mentoring programme for under-represented groups of American undergraduates. The six-week Summer programme supports 18 selected students every year. In the first eight years of its existence, MSRI-UP's 134 undergraduate participants have come from 81 different universities representing almost all the states and Puerto Rico. The group includes 62 females, 62 Latinos, 41 African Americans, four Native Americans, three Pacific Islanders and two Filipino Americans.

The goal of the programme is to involve students in small, highly collaborative and supportive research groups, led by a faculty of distinguished mathematicians. The groups work hard at exposition as well as research and give a formal presentation at the end. After the Summer, the groups are encouraged to present their work at national scientific conferences. Some of the groups eventually publish papers based on their research at MSRI. The MSRI-UP programme focuses on long-term mentoring, which continues after the programme and through the transition to graduate or professional schools.

MSRI-UP's first participant to receive a PhD was Talea Mayo, a 2007 MSRI-UP alumna. Dr Mayo was an undergraduate at Grambling State University, received her PhD at the University of Texas, Austin, and became a postdoctoral fellow at Princeton. There have been four more PhDs from the programme since Dr Mayo and many more are now in graduate school.

MSRI-UP involves undergraduate students in small, highly collaborative and supportive research groups, led by a faculty of distinguished mathematicians.

K-12 Maths education

MSRI hosts an annual conference on *Critical Issues in Math Education* (CIME) to engage mathematicians, K-12 teachers and mathematics educators and administrators in a conversation about mathematics education and the contributions that different professional communities make to this work. Recent topics include: *The role of the mathematics department in the mathematical preparation of teachers, Assessment of Mathematical Proficiencies in the Age of the Common Core, and Developmental Mathematics: For whom? Toward what end?*

MSRI has been nurturing informal mathematics education through "Math Circles", a longstanding Eastern European tradition brought to Berkeley by an MSRI postdoc from her native Bulgaria: Zvezdelina Stankova started (and still runs) the Berkeley Math Circles. MSRI was instrumental in spreading the idea of Math Circles across the country through its National Association of Math Circles (NAMC). Currently, there are over 145 Math Circle programmes registered on the NAMC's website (www.mathcircles.org). NAMC's primary activities include supporting new Math Circles through seed funding, providing resources in the form of a book series (the Math Circle Library), maintaining a website and organising workshops to spawn new Circles and support the Math Circle community.

Public Events & Initiatives

As a publicly funded institution, MSRI's leadership feels strongly about giving back to the public and sharing the beauty and power of mathematics with a larger audience. MSRI events have featured such well known figures as Alan Alda, the late Robin Williams, Tom Stoppard, Phillip Glass and Steve Martin. In the Spring of 2015, MSRI organised the first ever National Math Festival in Washington, D.C. The hands-on activities and lectures drew more than 20,000 over just one day, encouraging MSRI to extend the programme to two days for the next festival in 2017.

In the United States, it is still socially acceptable "not to be good at math". MSRI uses film, video and children's books to try to change the perception of mathematics and mathematicians. In the Spring of 2015, MSRI launched a book prize honouring the most inspiring maths-related fiction and nonfiction books. *Mathical: Books for Kids from Tots to Teens* aims to help foster a love and curiosity for maths among readers from pre-kindergarten through to the twelfth grade.

MSRI has produced about a dozen feature-length movies, making accomplishments in mathematics accessible to a larger audience. The newest film, "Counting from Infinity: Yitang Zhang and the Twin Prime Conjecture" by George Csicsery, tells the unlikely story of an unknown mathematician proving a theorem that there are infinitely many pairs of prime numbers that differ by 70 million or less.

MSRI supports and advises the video project Numberphile: former BBC journalist Brady Haran has produced more than 250 videos popularising mathematical concepts and ideas. Numberphile has become one of the most popular YouTube channels with 1,400,377 subscribers and 149,938,107 views (as of 19 October 2015 – the numbers are going up every day.)

More than 20,000 visitors of all ages discovered that maths is fun and relevant to everyone during the first National Math Festival in Washington, D.C.

How to participate

MSRI invites the submission of proposals for full- or half-year programmes as well as *Hot Topics* workshops and Summer graduate schools to be considered by the Scientific Advisory Committee. Planning of programmes is generally done about three years ahead. The deadlines to submit proposals are 15 October and 15 December.

The next application period to participate in a 2017–2018 programme as research professors, research members or postdoctoral fellows will open on 1 August 2016.

MSRI strives to include a diverse community of mathematicians in its programmes. Recognising that family issues can present barriers to participation, MSRI is committed to maintaining family-friendly policies and, where possible, facilitating appropriate arrangements for partners and children of programme members. The institute employs a Family Services Coordinator who provides help in locating schools and other services for mathematicians who are considering coming to the institute with their families.

Maths with a view: many members enjoy the beautiful sunset over the San Francisco Bay from their offices.

Registrations for workshops open on an ongoing basis. Participants who seek funding for their visit are advised to apply at least three months before the workshop.

Future programmes

11 January 2016–20 May 2016: *Differential Geometry* (Jumbo Programme)
15 August–16 December 2016: *Geometric Group Theory* (Jumbo Programme)
17 January–26 May 2017: *Analytic Number Theory*; and *Harmonic Analysis*
14 August–15 December 2017: *Geometric Functional Analysis and Applications*; and *Geometric and Topological Combinatorics*
16 January–26 May 2018: *Group Representation Theory and Applications*; and *Enumerative Geometry beyond Numbers*
For more information, including a list of planned workshops, please visit msri.org.

The Portuguese Mathematical Society (SPM) at 75

Fernando P. da Costa (Universidade Aberta, Lisbon, Portugal, President of the SPM)

The Portuguese Mathematical Society (SPM) was founded in December 1940 thanks to the efforts of a small but very active group of young Portuguese mathematicians determined to stir the then stagnant waters of Portuguese scientific life.

This informal group, which went down in history as the "*Movimento Matemático*" (Mathematical Movement), together with the state body "*Instituto para a Alta Cultura*", had a remarkable influence on the mathematical community in Portugal over the 1930s and 1940s.

Among its many initiatives, the *Movimento* founded in 1937 the research journal *Portugaliæ Mathematica* (nowadays a title property of the SPM and edited by the EMS Publishing House) and in 1940 the *Gazeta de Matemática* (currently also owned and published by the SPM). The SPM was, from the beginning, a focal point for a considerable number of Portuguese researchers, students and teachers interested in the modernisation of mathematical studies and activities in the country. By virtue of its 1940 statutes, the society was, from the start, actively involved in three main areas of intervention: research, teaching and the popularisation of mathematics.

Figure 1: From left to right: Maurice Fréchet, Pedro José da Cunha (first president of SPM), and António Aniceto Monteiro (first Secretary-General of SPM and one of the main boosters of the *Movimento Matemático*). Lisbon, 1942 (photo source: http://antonioanicetomonteiro.blogspot.pt/2011_04_01_archive.html).

To get an idea of the formidable task facing the society at the time, remember that, even by the 1940s, Portugal was still an extraordinarily backward country: with about 8 million inhabitants, half of whom were illiterate, it had a total of about 9000 university students (0.1% of the total population!) in its three universities.

The challenge of raising the cultural level of the population under such difficult initial conditions was further hampered by the political regime at the time: since the second half of the 1920s, Portugal was ruled by a right-wing dictatorship that was very close to the Fascist regime in Italy and also quite averse to raising the educational level of the population above the bare minimum skills provided by the officially compulsory four years of elementary schooling.

The tense relations of the regime with the young generation of scientific researchers and democratic activists of the 1930s and 1940s came to a dénouement in the political repression following the rigged 1945 general election. This resulted in the expulsion of a large number of university staff and students, followed either by their imprisonment or exile, which caused the disappearance of much of the activities of the *Movimento*. A somewhat peculiar consequence of these troubled times was the fact that the SPM had a semi-clandestine existence for the next 30 years: it was never duly recognised by the Government as a collective body within the juridical order of the country until after the termination of the dictatorial regime by the 1974 revolution.

Since 1974, the SPM has been completely reborn in its three statutory fields of action.

In addition to the reinvigoration of its periodicals *Portugaliæ*, *Gazeta* and *Boletim da SPM* (first published in 1951), the activities of the society have also expanded and acquired an appreciable visibility in contemporary Portugal.

Figure 2: The current periodicals of SPM: *Portugaliæ Mathematica*, *Gazeta de Matemática*, and *Boletim da SPM*.

Foremost among the SPM interventions has been a strong public action for the raising of mathematics teaching and assessment standards in Portugal at the elementary and secondary school levels. This is reflected not only in the many regular interventions in debates and public statements about the educational system, national exams, school curricula and training of teachers but also in the participation of the SPM as a consulting member of the official government body charged with production of the end of cycle national exams at the elementary and secondary school levels. Through its Life Long Learning Centre, the SPM provides a number of activities targeting the mathematical upgrade of teachers at all levels of pre-university teaching, from 1st year elementary school onwards. Finally, the SPM is one of the officially credited entities for the evaluation of mathematics school manuals and thus has an important role in the evaluation of the school manuals used in Portugal.

Nurturing young people's interest in mathematics and involvement in science and technology is also one of the goals of the SPM. The society has been active in organising the Mathematics Olympiad in Portugal for more than 30 years, as well as training the Portuguese teams for international competitions. The SPM also collaborates with other scientific and academic associations, schools, museums and other institutions in organising youth events, such as maths fairs and maths games championships.

In the area of mathematical research, in addition to being responsible for its research journal *Portugaliæ Mathematica*, the SPM has two regular meetings: the National Meeting in even years and the Summer School in odd years. Besides these, a regular biannual meeting is jointly organised with our Spanish counterpart, the *Real Sociedad Matemática Española*, and a few more sporadic initiatives exist, such as the joint meeting with the EMS and the American Mathematical Society (the first such event between the EMS and the AMS) that took place in Porto in the Summer of 2015 with more than 900 participants; this was the largest meeting ever organised by the SPM. In the context of science policy, the SPM is a member of the "*Comissão Nacional de Matemática*", a body that advises the Portuguese government about matters concerning mathematics research. This body includes the relevant scientific societies (the SPM and the Portuguese Statistics Society), as well as all the research centres in mathematics and statistics housed in Portuguese universities.

Finally, in the popularisation of mathematics, the SPM has been very active in a variety of ways. A series of public lectures for general audiences have regularly taken place in several cities all over the country, in schools, museums, science centres, bookshops and shopping malls.

Regarding publishing for the general public, in addition to the *Gazeta*, agreements with some Portuguese commercial publishers have allowed the production of a number of books, although in recent years, dire economic constraints have forced a strong curtailment of this activity.

In spite of the difficult economic situation, over the last few years, an important SPM initiative in raising public awareness of mathematics was possible due to a project funded by the Portuguese Government and the European Union. The "*Isto é Matemática*" (This is Mathematics) series of 91 short movies (http://www.spm.pt/istoematematica/) was broadcast weekly over two years on a Portuguese TV cable channel and became a very popular show. The first 13 episodes have been translated into English and disseminated by the RPA Committee of the EMS (http://www.mathematics-in-europe.eu/). An agreement with the Universidad de El Salvador for translation into Spanish of the first 13 episodes is currently under negotiation. The popularity of the award winning first series led to the financing of a new series of 52 episodes by Vodafone Portugal and these are currently in production and will start being broadcast in November.

Figure 3: "Isto é Matemática" (This is Mathematics) the popular SPM TV series.

Summing up, in all areas of activity its founders envisioned as important 75 years ago, the SPM has been able to imprint its mark on Portuguese society and, in some cases, its presence and intervention has extended far beyond the most optimistic expectations. Currently with 954 individual members and 10 institutional members (mainly, but not exclusively, mathematics departments of Portuguese universities), the SPM is a strong voice in defence of research, teaching and popularisation of mathematics in Portugal.

In recent years, the financial difficulties experienced in the country (with reflections felt in economic life, both at a personal level and in the curtailment of funding for higher education research and training programmes, as well as financial support for activities by scientific societies) has resulted in a situation with potential nefarious consequences for Portuguese science, a perspective that deeply worries all Portuguese scientific societies, including the SPM.

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ICMI Column

Jean-Luc Dorier (University of Geneva, Switzerland)

Espace Mathématique Francophone in Algiers

The EMF2015 colloquium was held in Algiers, 10–14 October 2015. The theme for the 6th Espace Mathématique Francophone (EMF) colloquium was "Cultural pluralities and universality of mathematics: issues and challenges for their teaching and learning".

This colloquium brought together 150 participants from 18 nations, with around 90 submissions divided into 10 working groups and three special projects. This makes EMF2015 one of the most important international conferences related to mathematics ever organised in Algeria.

The 10 working groups that worked on five tracks, representing 9.5 hours of debate, were really the heart of the system, which aims, through colloquiums organised every three years, to allow long-term collaborations in the francophone mathematical space. Three special projects can launch more innovative themes.

The lecture by Christine Proust, French historian of mathematics and Director of Research at CNRS, was entitled "Mathematics in Mesopotamia: strange and familiar". In the context of the theme of the colloquium, this contribution on the history of Mesopotamian mathematics, dating back 3,000 years, was a fascinating opportunity to plunge the participants into the reality of a most ancient civilisation, from which all mathematicians of the world have inherited.

Ahmed Djebbar, honorary professor of history of mathematics in Lille and former Minister of Education of Algeria, showed the intercultural aspects of Arab mathematics developed between the 8th and 15th centuries, and their connections with India, China, Sub-Saharan Africa and Europe, as well as with former cultures of Mesopotamia, Greece and Egypt. Like a storyteller of the 1001 nights, he was able to make accessible a large account of his vast knowledge of recent work in the history of mathematics.

Professor Benali Benzaghou, from the Houari Boumediene University, gave an interesting panorama of the teaching of mathematics in Algerian universities with a historical perspective.

Finally, two plenary sessions were devoted to the presentation of the results of the answers of around 1,400 teachers to a survey conducted by Maha Abboud-Blanchard (France), France Caron (Canada), Jean-Luc Dorier (Switzerland) and Moustapha Sohkna (Senegal) on the use of resources by secondary school teachers and their cultural specificities.

This sixth EMF showed the vitality of this community and its rich scientific production, in a partnership that, using the French language, is based on an exemplary North-South collaboration. Besides the specific language, the EMF has established itself as a privileged communication between different stakeholders concerned with issues affecting the teaching of mathematics, including mathematicians, mathematics educators, researchers, trainers and teachers of different levels.

A new executive bureau of the EMF was nominated at the beginning of EMF2015 with a perfect parity between North and South and genders. Its members are: Teresa Assude (France), Faiza Chellougui (Tunisia), Jean-Luc Dorier (Switzerland), Judith Sadja Njomgang (Cameroon), Ahmed Semri (Algeria), Moustapha Sokhna (Senegal) *President*, Laurent Theis (Canada) and Joelle Vlassis (Luxemburg/Belgium). Its new status is to be approved by the Executive Committee of the ICMI.

From left to right: Judith Sadja Njomgang (Cameroon), Faiza Chellougui (Tunisia), Joelle Vlassis (Luxemburg/Belgium),Ahmed Semri (Algeriea), Moustapha Sokhna (Senegal), Laurent Theis (Canada), Teresa Assude (France) and Jean-Luc Dorier (Switzerland).

The proceedings of this meeting should be published in Spring 2016. The conference website (http://emf2015.usthb.dz/) is still accessible and presents a more detailed view of the event.

The next EMF will be held in 2018, in Paris, with Professor Maha Abboud-Blanchard as President of the Scientific Committee, continuing what has become a rule of alternating North-South in hosting the conference.

Professional Development Centres as Levers for Change in Mathematics Education

Katja Maaß (University of Education, Freiburg, Germany), Günter Törner (University of Duisburg-Essen, Duisburg, Germany), Diana Wernisch, Elena Schäfer (both University of Education, Freiburg, Germany)

We report about a recent initiative to build up a network of professional development centres in Europe. It is a goal of the EMS Committee for Education to intensify efforts in promoting continuous development of mathematics and science teachers.

Professional development centres – why?

One primarily associates 'teacher education' with prospective teachers' studies at institutions of higher learning (i.e., colleges and universities). Meanwhile, there is growing awareness that in-service teachers need to refresh their existing competences and obtain new ones in an ongoing manner. This type of 'continuous upgrading' requires that teachers receive support throughout their careers, especially in the form of quality professional development (PD).

However, the question of how to reach all teachers with PD courses and how to best scale up teacher PD, remains largely open. A sign that indicates some progress is being made towards answering these questions is that many responsible educational administrations in various countries have delegated this task to a national institution and have initiated some kind of a professional development (PD) centre.

One of the first of these centres dedicated to teacher PD, and an ongoing initiator in this process, is the *National Centre for Excellence in the Teaching of Mathematics*[1], (NCETM) in England. Furthermore, the NCETM's concepts are spreading and have, for example, acted as a model for the *German Centre for Mathematics Teacher Education*,[2] (DZLM) in Germany, which was founded through a grant of the *Deutsche Telekom Foundation*.[3] This centre supports various educators in their efforts to raise appreciation and enthusiasm for mathematics.

The DZLM's focus lies in PD for teacher educators and on those who are supporting other teachers in their pedagogical development. Furthermore, the centre offers continuous professional development courses and material for out-of-field teachers, mathematics teachers and educators working in elementary education.

Another encouraging sign is that scientists all over the world involved in the issue of in-service teachers' PD are becoming familiar with the relevant term *Continuous Professional Development* (CPD).

However, CPD concepts vary from country to country and depend heavily on educational and systemic traditions. Fortunately, CPD is a growing field of research and first handbooks have been devoted to CPD-related areas.

Linking mathematics and science – why?

Interdisciplinary teaching and learning is on the rise. This positive development requires a shift in teachers' thinking and initial training but especially in their PD. For example, mathematicians need to recognise a further insight, namely, that restricting their work (whether in classrooms, research or PD courses) exclusively to the mathematical domain is artificial. We strongly encourage mathematics educators to seek coalitions with colleagues involved in science teaching. An example of what we mean here is the educational initiative STEM (Science, Technology, Engineering and Mathematics).

One such project is *Mascil*[4] (mathematics and science for life). This FP 7 project is being coordinated by Katja Maaß from the University of Education in Freiburg, Germany, and involves over 50 actors from science and mathematics education research, policymaking and teaching practice. Together, they are working to achieve a widespread implementation of inquiry-based learning (IBL) with a particular focus on connections to the world of work.

Joining forces at the start of a promising story

In December 2014, the European project Mascil and the German centre DZLM hosted a conference entitled *Educating the Educators: International approaches to scaling up professional development in maths and science education*[5] at the University of Duisburg-Essen in Germany.

In the course of the conference, a meeting of European professional development centres involved in maths and science education took place for the first time.

Giving PD centres a European voice: towards establishing a network

Recognising the potential of such mutual learning, the professional development centres expressed their wish to have further joint meetings. Therefore, the University of Education Freiburg organised the *second meeting of European Professional Development Centres in maths*

[1] https://www.ncetm.org.uk/.
[2] http://www.dzlm.de/dzlm/international-visitors.
[3] http://www.telekom-stiftung.de/dts-cms/en.
[4] http://www.mascil-project.eu/.
[5] http://educating-the-educators.ph-freiburg.de/.

and science education, which was held 6–7 May 2015 in Vilnius, Lithuania, in parallel to a Mascil project meeting. Participants from 12 European countries representing approximately 20 maths/science PD centres and other organisations with similar aims (i.e., ministries of education) attended the meeting (see the meeting programme[6]).

Another important issue for future collaboration is the drafting of common standards and guidelines for PD planning, delivery, implementation and evaluation. These should be based on teacher PD research.

For the participants, it was very important to find ways of sustaining the series of meetings that grew out of the Educating the Educators conference. The University of Education Freiburg, in its role of coordinating the network, is therefore currently taking meeting outcomes forward and organising more meetings. There is also an aim to further enlarge the PD centre network in the future. A PD centre network homepage is in development and will be up and running soon.

A third PD centre meeting will take place in Sofia (Bulgaria) in December 2015. The European Mathematical Society Committee for Education will also use this platform to bring in its expertise and and widespread international contacts.

And the future?

The promising development of the professional development centres shows that we need to think outside the box to make mathematics and science (education) attractive to students. Initiatives like the DZLM and projects such as Mascil point the way forward by taking up existing trends (such as that of establishing more PD centres), by linking mathematics and science and by effectively joining forces. We will go on doing so…

[6] http://educating-the-educators.ph-freiburg.de/2ndpd centres meeting/programme.

Katja Maaß (center) is a professor at the University of Education in Freiburg (Germany). She is supported in coordinating mascil by Diana Wernisch (left), project manager, and by Elena Schäfer (right), academic assistant.

Günter Törner, a research mathematician and mathematics educator, is Chair of the EMS Committee for Education. He is also a member of the EC of the German national centre for continuous professional development (DZLM) in Germany.

Both Katja and Günter have recently initiated the European network for professional development centres with about 20 members in Europe.

Connecting Old and New Information: zBMATH as a Hub Connecting Digital Resources

Olaf Teschke (FIZ Karlsruhe, Berlin, Germany)

Over the last few decades, the digital age has put forward the creation of many electronic resources in mathematics. While it is usually easy to find specific information knowing that it exists, a framework to facilitate answers involving distributed information is challenging. We outline some features of zbMATH, some of which have been added recently, to support the connection of mathematical knowledge.

Enhancing access to digital libraries

Digital mathematical libraries of research papers, comprising millions of pages (either scanned or already TeX), still form probably the largest electronic collections of mathematical content on the web, e.g. in the form of academic digitisation efforts (most notably in the EuDML framework, which has been presented frequently in this newsletter), via publishers or on preprint servers. From a technical viewpoint, the creation of electronic full-texts has become extremely efficient, even if the scan of historical volumes is involved. The creation of appropriate metadata (even limited to a minimal level containing author, title and source information) is already more costly by a magnitude and prone to various error sources. For historical publications, OCR will create as many ambiguities as changing attitudes to publication standards over the decades. This raw information is usually what is (at most) loaded into generic information systems, since this can be done cheaply; it is, however, only of limited use. While academic retro-digitisation does quite well from a quality viewpoint compared to commercial efforts (e.g. for many historical publications, full-text links via EuDML seem to be more stable than the doi system, and less OCR artefacts appear), inherent deficits can hardly be addressed on a technical level. This especially concerns issues of content analysis and author disambiguation.

Content analysis is classically provided by zbMATH reviews, keywords and classification – additions which require expert knowledge and are applied to recent as well as historical publications.[1] For historical entries, a simple but non-trivial aspect is the language: while English prevails today as the language of mathematics, French, German and Italian dominated in the 19th century (there was even a relevant share of important Latin papers). These papers would be hardly discoverable by usual search strategies, but some features greatly enhance discoverability, such as English title translation (like most non-English zbMATH entries), language-independent classification,[2] MathML enabling formula search,[3] embedding into a reference network[4] and proper author assignment[5].

The database adds manifold information to the digitised EuDML entry: title translation, keywords, classification, citations and MathML formulae, as well as the linked author profile.

Linking and matching various sources

A basic but essential feature of the zbMATH database is that it both defines a comprehensive scope and unique identifiers beyond the various collections. Consequently, it aims to connect various available sources for documents, including linking to, for example, doi, EuDML, ElibM, Project Euclid, or recently arXiv preprint versions.[6] This obviously requires a precise document

[1] See also B. Wegner, "Dynamic reviewing at Zentralblatt MATH", *Eur. Math. Soc. Newsl.* 78, 57–58 (2010).

[2] P. Ion & W. Sperber, "MSC 2010 in SKOS – the transition of the MSC to the semantic web", *Eur. Math. Soc. Newsl.* 84, 55–57 (2012).

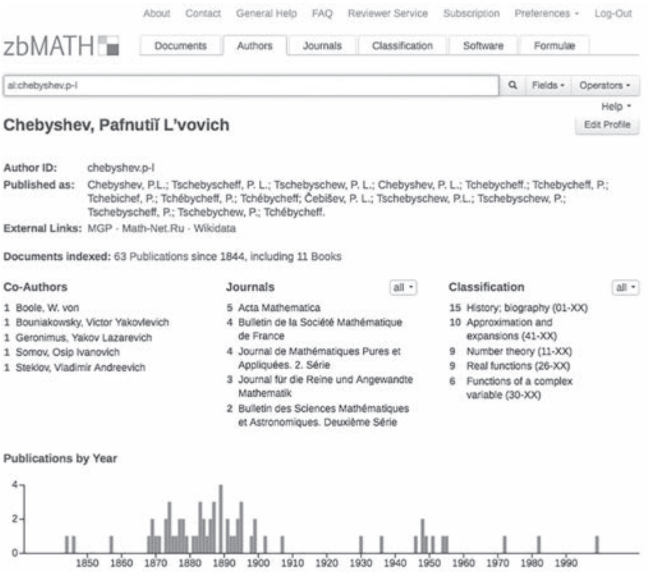
[3] M. Kohlhase et al., "Mathematical formula search", *Eur. Math. Soc. Newsl.* 89, 56–58 (2013).

[4] O. Teschke, "Citation profiles in zbMATH", *Eur. Math. Soc. Newsl.* 96, 62–63 (2015).

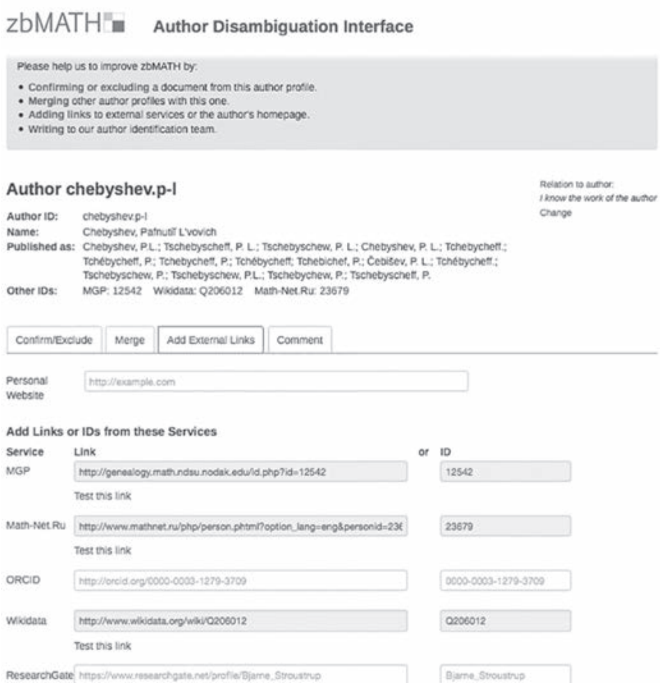
[5] O. Teschke, "On authors and entities", *Eur. Math. Soc. Newsl.* 71, 43–44 (2009).

[6] More details will be the subject of a forthcoming column.

matching, which is also necessary for proper reference assignments. On the other hand, such tools are also applied to generate links into zbMATH,[7] thus forming a huge network of mathematical information.



A zbMATH author profile enriched by links to Wikipedia, the Mathematics Genealogy Project and Math-net.ru.



Additional external links can be added via the 'Edit Profile' button in the user interface.

Additional benefits on author information

Issues of the zbMATH author database have been discussed frequently in this column. With a view toward digital resources, the benefits of precise network information are probably largest for author information at the moment, since most resources pertain to this facet. Author disambiguation for historical publications has special challenges – not only that one has to expect less contributions from crowd-sourcing[8] but that one has to take into account missing or wrong surnames, OCR errors, creative transliterations or even sometimes verbal translations of names.[9] On the other hand, one can employ various resources like Wikipedia, the Mathematics Genealogy Project and Math-Net.Ru, which provide valuable information – like biographical details or PhD students – that do not usually come with publications but are worth being integrated as additional links.

Together with more recent information, like ORCID identifier, homepage and social network data, this information can also be added via the user interface, and contribute to an even more reliable disambiguation.

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7 The recently much improved zbMATCH version is available via the interface https://zbmath.org/citationmatching/ or API.

8 H. Mihaljević-Brandt and N. Roy, "zbMATH author profiles: open up for user participation", *Eur. Math. Soc. Newsl.* 93, 53–55 (2014).

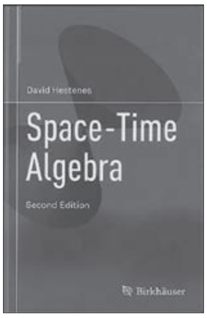
9 Additional historical knowledge may be helpful, e.g. to identify the innocent initials O.L.B.D. in an early Crelle issue as the title of "(königl.) Ober-Landes-Bau-Director", not as part of the name, cf.doi:10.1515/crll.1826.1.5, https://eudml.org/doc/183044, https://zbmath.org/02751806.

Erratum

At the end of the article "EMIS – 20 Years of Cooperation of the EMS with FIZ Karlsruhe/zbMATH" [EMS Newsletter 97, 18–29], an outdated biography of Bernd Wegner from an earlier contribution was reprinted. We give an updated biography here.

Bernd Wegner [wegner@math.tu-berlin.de] has been a professor at TU Berlin since 1974 and was Editor-in-Chief of Zentralblatt MATH and MathEduc until 2012, when he also gave up the scientific supervision of EMIS. He has participated in several electronic library and mathematics knowledge management projects. He has organised several meetings on these subjects. He has been a member of the electronic publishing committee of the EMS for a longer period.

Book Reviews



David Hestenes

Space-Time Algebra (second edition)

Birkhäuser, 2015
xxiv + 102 pages
ISBN 978-3-319-18412-8

Reviewer: Sebastià Xambó-Descamps

The 6th Conference on Applied Geometric Algebras in Computer Science and Engineering[1] (AGACSE) was dedicated to David Hestenes (Arizona State University) in recognition of his masterly and sustained leadership for half a century, particularly at the interface of mathematics and physics. The dedication was celebrated with the launch[2] of a second edition of his *Space-Time Algebra* (Gordon and Breach, 1966). David was present during the whole week and the standing ovation after his keynote lecture,[3] culminating in his recitation of the stirring call to action from Tennyson's Ulysses,[4] was a very moving moment for all participants. The David Hestenes Prize was established for the best work submitted by a young researcher and was awarded to *Lei Huang* (Academy of Science, Beijing, China) for "Elements of line geometry with geometric algebra".[5] His work shows how to bring the power of geometric algebra to bear on 3D projective geometry, thus linking new mathematical theory with very practical applications in computer science. *Pierre-Philippe Dechant* (University of York, UK) and *Silvia Franchini* (University of Palermo, Italy) were finalists with the works "The E_8 geometry from a Clifford perspective" and "A family of embedded coprocessors with native geometric algebra support",[6] respectively. The conference was preceded, for the first time, by a 2-day Summer School to better prepare the less experienced and it was attended by two thirds of the conference participants. The next AGACSE will be in Campinas, Brazil, in 2018.

From left to right: David Hestenes, Lei Huang, Silvia Franchini, Pierre-Philippe Dechant, Sebastià Xambó-Descamps, Eduardo Bayro-Corrochano.

David Hestenes (July 2015, during his keynote lecture).

Space-Time Algebra (STA) is actually a reprint of the first edition, but with two precious new items: a foreword by Anthony Lasenby[7] and a preface by the author "after fifty years". It was a landmark in 1966 and it is as fresh today as it was then in its "attempt to simplify and clarify the [mathematical] language we use to express ideas about space and time", a language that "introduces novelty of expression and interpretation into every topic" (the quotations are from the preface to the first edition). This language is usually called *geometric algebra* (GA), a term introduced by W.K. Clifford in his successful synthesis of ideas from H. Grassmann and W.R. Hamilton. In Part I of STA, GA is advanced and honed into a resourceful mathematical system capable of expressing geometric and physical concepts in an intrinsic, efficient and unified way. Two special cases are worked through in detail: the geometric algebras of 3D Euclidean space (Pauli algebra) and 4D Minkowski space[8] (Dirac algebra). These geometric algebras are then used, in a real tour-de-force, to elicit the deep geometric structure of relativistic physics. This takes the remaining four parts of the book: Electrodynamics, Dirac fields (including spinors and the Dirac equation), Lorentz transformations and Geometric calculus (including novel principles of global and local relativity, gauge transformations and spinor derivatives). There are also four short appendixes,

1 AGACSE 2015, 27–31 July, Barcelona, Spain: http://www-ma2.upc.edu/agacse2015/
2 Suggested as an 80th birthday gift by Leo Dorst, cooperatively backed by Eduardo Bayro-Corochano, Joan Lasenby, Eckhard Hitzer and the author of this review, and enthusiastically embraced by Springer, each participant received a copy by courtesy of the Catalan Mathematical Society and the Royal Spanish Mathematical Society.
3 *Fifty Years with Geometric Algebra: a retrospective.*
4 "Made weak by time and fate, but strong in will / To strive, to seek, to find, and not to yield" (last two verses).
5 Joint work with Hongbo Li, Lei Dong, and Changpeng Shao.
6 Co-authored by Antonio Gentile, Filippo Sorbello, Giorgio Vassallo and Salvatore Vitabile.
7 Professor of Astrophysics and Cosmology at the Cavendish Laboratory, Cambridge University. Co-author of the superb treatise [1].
8 A real vector space with a metric of signature (+, –, –, –).

A to D, which amount to a supplement of the GA part. We will come back to them below.

Although the presentation of GA in STA, and in later works of Hestenes and many others, is framed in a set of quite natural algebraic axioms, it turns out that the approach may come across as unusual for some tastes, which perhaps explains why the book is not as well known among theoretical physicists as it surely deserves. For example, in an otherwise meritorious paper,[9] E.T Jaynes declares (an admittedly extreme view that may be saying more about himself than about STA):

> *It is now about 25 years since I started trying to read David Hestenes' work on space-time algebra. All this time, I have been convinced that there is something true, fundamental, and extremely important for physics in it. But I am still bewildered as to what it is, because he writes in a language that I find indecipherable; his message just does not come through to me. Let me explain my difficulty, not just to display my own ignorance, but to warn those who work on space-time algebra: nearly all physicists have the same hang-up, and you are never going to get an appreciative hearing from physicists until you learn how to explain what you are doing in plain language that makes physical sense to us.*

Fortunately, STA was 'discovered' in the late 1980s by people like Stephen Gull, Anthony Lasenby and others, in Cambridge and elsewhere (see [2], the references therein, and [3]), an eventuality which led to a flourishing of new ideas, results and applications in many fields (see, for example, [1, 4, 5, 6]).

GA, as espoused in STA, seems not to be very well known in mathematical circles either, this time because it may perhaps be perceived as a closed, short-range structure, or even because its presentation may be found not to follow the formal strictures of the trade. As avowed by the vast existing literature, the first perception is untenable, even if one takes into account only its service to mathematics, or even only to geometry. Concerning formalities, there is no doubt that a mathematically minded approach may extract a meaningful and satisfying picture of GA, as this does not (logically) depend on the physics. Assuming basic knowledge of the Grassmann (or exterior) algebra, here is a possible sketch of such a picture. The geometric algebra $\Lambda_g E$ of a (real) vector space E of finite dimension n, equipped with a symmetric bilinear form g (the *metric*), is the exterior algebra

$$\Lambda E = \Lambda^0 E \oplus \Lambda^1 E \oplus \Lambda^2 E \oplus \cdots \oplus \Lambda^n E \ (\Lambda^0 E = \mathbb{R}, \Lambda^1 E = E)^{10}$$

enriched with the *inner product* $x \cdot y$ and the *geometric product* xy, which in turn can be explained as follows. To define the inner product $x \cdot y$, we may assume that

$$x = e_1 \wedge \cdots \wedge e_r \in \Lambda^r E,\ y = e'_1 \wedge \cdots \wedge e'_s \in \Lambda^s E$$
$$(e_1, \ldots, e_r, e'_1, \ldots, e'_s \in E,\ r,s \geq 1).$$

If $r = 1$ (say $x = e \in E$) then $e \cdot y$ is defined as the left contraction of e and y, namely

$$e \cdot y = \Sigma_{k=0}^{k=s} (-1)^k g(e, e'_k)\, e'_1 \wedge \cdots \wedge e'_{k-1} \wedge e'_{k+1} \wedge \cdots \wedge e'_s.$$

For example,

$$e \cdot e' = g(e, e') \text{ and } e \cdot (e'_1 \wedge e'_2) = g\,(e, e'_1)\, e'_2 - g\,(e, e'_2)\, e'_1.$$

If $1 < r \leq s$ then $x \cdot y$ can be defined recursively by the relation

$$x \cdot y = (e_2 \wedge \cdots \wedge e_r) \cdot (e_1 \cdot y).$$

In the case $r \geq s$, analogous formulas using the right contraction $x \cdot e$ lead to the rule

$$x \cdot y = (-1)^{rs+s}\, y \cdot x).$$

We see that $x \cdot y \in \Lambda^{|r-s|} E$ for $x \in \Lambda^r E$, $y \in \Lambda^s E$. If $r = s$, then $x \cdot y = g(x,y)$, where we use the same symbol g for the natural extension of the metric to ΛE, so that, in particular, $x \cdot y = y \cdot x$, as required by the formula above. But note that if $r \neq s$ then $g(x,y) = 0$, whereas $x \cdot y$ may be non-zero and may be the opposite of $y \cdot x$ (precisely when s is odd and r even).

The geometric product xy may be characterised as the only bilinear *associative* product such that

$$ex = e \cdot x + e \wedge x,$$

for any $e \in E$ and any $x \in \Lambda E$.[11] Note that $e^2 = e \cdot e = g\,(e,e)$, so that e is invertible if it is non-isotropic ($g\,(e,e) \neq 0$), $e^{-1} = g(e,e)^{-1} e$, which means that, in GA, *division by non-isotropic vectors is a legal operation*. This fact, together with the associativity of the geometric product, explains why operating with (multi)vectors is so natural and agile. Note also that $e\,e' = e \wedge e' = -e'e$ if (and only if) e and e' are orthogonal ($g\,(e,e') = 0$). With this approach, all the GA formulas in STA, and others obtained afterwards, can be established. Here are some examples. For $e \in E$ and $x \in \Lambda E$, the relation $xe = x \cdot e + x \wedge e$ also holds. If $x \in \Lambda^r E$, $y \in \Lambda^s E$ and $z = xy$ then $z_k \neq 0$ implies that $k = |r-s| + 2j$, $j = 0, \ldots, \min(r,s)$. In particular, the minimum and maximum possible degrees are $|r-s|$ and $r+s$. In fact, it happens that $(xy)_{|r-s|} = \tilde{x} \cdot y$ if $r \leq s$, $x \cdot \tilde{y}$ if $r \geq s$, and $(xy)_{r+s} = x \wedge y$, where $\tilde{x}$ is the result of reversing the order of the factors of x. Thus, we see that the geometric product determines the inner and outer products.[12] Another very useful fact is that if the vectors $e_1, \ldots, e_r$ are pairwise orthogonal then

[9] E.T. Jaynes: Scattering of light by free electrons as a test of quantum theory. In *The electron: New theory and experiment* (D. Hestenes and A. Weingarthofer, eds.), 1–20. Kluwer, 1991.

[10] Its product $x \wedge y$ is the *exterior* or *outer* product. Its elements, which are called *multivectors*, have the form $x = x_0 + x_1 + \cdots + x_n$, with $x_r \in \Lambda^r E$ (the r-vector part of x).

[11] This fundamental formula is what most upset Jaynes, who vehemently objected to its non-homogeneous character: $e \cdot x \in \Lambda^{r-1} E$ and $e \wedge x \in \Lambda^{r+1} E$ when $x \in \Lambda^r E$.

[12] Notice, however, that for these relations to make sense, we need to know the grading, which is not naturally defined by means of the geometric product. Here the grading is taken to be a lower level structure, as the definition of the graded algebra ΛE only depends on the vector space structure of E. Actually this fact is one of the great ideas bequeathed by Hermann Grassmann.

$$e_1 \cdots e_r = e_1 \wedge \cdots \wedge e_r.$$

Let us turn back to the foreword and the new preface of STA. In the foreword, A. Lasenby states that:

> *This small book started a profound revolution in the development of mathematical physics, one which has reached many working physicists already, and which* stands poised to bring about far-reaching change in the future.

The roots of this potential are clearly delineated in Hestenes' preface, "with the confidence that comes from decades of hindsight", by asserting four "**Claims** for STA as formulated in this book" (boldface emphases in the original):

*(1) STA enables a unified, **coordinate-free** formulation for all of relativistic physics, including the Dirac equation, Maxwell's equation and general relativity.*
*(2) Pauli and Dirac matrices are represented in STA as **basis vectors** in space and spacetime respectively, with no necessary connection to spin.*
*(3) STA reveals that the **unit imaginary** in quantum mechanics has its origin in spacetime geometry.*
*(4) STA reduces the mathematical divide between classical, quantum and relativistic physics, especially in the use of **rotors** for rotational dynamics and gauge transformations.*

Before briefly commenting on these claims, here is a simple geometric example that neatly illustrates the core aspects of (3) and (4). It is about the representation in GA of rotations in the ordinary Euclidean space as explained in STA, Appendix C. Let E_3 be the Euclidean 3-space and $e_1, e_2, e_3 \in E_3$ an orthonormal basis. Denote $e_j e_k = e_j \wedge e_k$ by e_{jk}, with a similar meaning for e_{jkl}. Then $i = e_{123}$ commutes with vectors and hence with any element of ΛE_3, and $i^2 = -1$.[13] Since $ie_1 = e_{23}$, $ie_2 = e_{31}$ and $ie_3 = e_{12}$, we have a linear isomorphism $E_3 = \Lambda^1 E_3 \to \Lambda^2 E_3$, $u \mapsto b = iu$. The inverse isomorphism $\Lambda^2 E_3 \to E_3$ is given by $b \mapsto u = -ib$.[14] Now, let $u \in E_3$ be a unit vector and $\alpha \in \mathbb{R}$ (an angle). Then the linear map $\rho_{u,\alpha}: E_3 \to E_3$ defined by

$$\rho_{u,\alpha}(x) = e^{-1/2\alpha iu}\, x\, e^{1/2\alpha iu}$$

is the rotation about the vector u of angle α. Indeed, by the usual expansion of the exponential we get

$$e^{\pm\alpha iu/2} = \cos(\alpha/2) \pm iu \sin(\alpha/2),$$

and the formula follows on noting that u is fixed, as it commutes with either exponential, and that if x is orthogonal to u then $e^{-\alpha iu/2} x\, e^{\alpha iu/2} = x e^{\alpha iu}$ (as $ux = -ux$), which is the result of rotating x in the plane $u^\perp$ by an angle α in the sense of the orientation given by u. To see this, let u_1, u_2 be an orthonormal basis of $u^\perp$ such that $-i(u_1 \wedge u_2) = u$, i.e. $iu = u_1 u_2$ (Hodge duality). Then $(u_1 u_2)^2 = -1$, $e^{\alpha iu} = \cos(\alpha) + u_1 u_2 \sin(\alpha)$, and the claim follows by a simple calculation of $u_1 e^{\alpha iu}$ and $u_2 e^{\alpha iu}$. Note that the GA formula for $\rho_{u,\alpha}(x)$ greatly facilitates the computation of the composition $\rho_{u',\alpha'} \rho_{u,\alpha}$ of two rotations, for it is reduced to the (brief) GA computation of $e^{\alpha iu/2} e^{\alpha' iu'/2}$. This yields, as shown in Appendix C, the remarkable formulas for the angle and axis of the composite rotation.[15]

As the example above shows, complex numbers appear in GA not as formal entities but with a surprising geometric meaning. The significance of point (3) is that this also happens in physics, where the i appearing in, say, the Schrödinger and Dirac equations is revealed to be subtly and significantly related to GA entities. The GA form of the E_3 rotations also illustrates interesting aspects of (4). Expressions such as

$$R = e^{-\alpha iu/2} = \cos(\alpha/2) - iu \sin(\alpha/2) \in \Lambda^0 E + \Lambda^2 E$$

(this is the *even* subalgebra of ΛE) are called *rotors* and the rotation $\rho_{u,\alpha}$ is given by $x \mapsto RxR^{-1}$.

As proved in STA, Part IV, Lorentz transformations (rotations of Minkowski's space) may also be described by rotors $R = e^{-b/2}$, where b is a bivector. With respect to an inertial frame, the rotor can be resolved as a product of one spatial rotor, which gives a rotation in the Euclidean 3-space associated to that frame, and a time-like rotor, which gives a Lorentz boost in that frame. The main resource here is the marvellous way in which the Pauli algebra of that Euclidean space is embedded in the Dirac algebra.

As for claim (2), note that in all these interpretations and calculations, the customary matrix representation of the Pauli and Dirac algebras plays no role, and work with coordinate systems and coordinates is unnecessary. The notion of spin, and its role in particle physics, is also greatly clarified and improved.

Paraphrasing a quote from [2] devoted to physicists, let me finish by expressing the hope that also mathematicians not yet knowing STA will find a number of surprises, and even that they will be surprised that there are so many surprises!

The reviewer thanks Leo Dorst for the improvements made possible by his comments, suggestions and corrections after reading a first draft.

References

[1] C. Doran, A. Lasenby: *Geometric Algebra for Physicists.* Cambridge University Press, 2003.
[2] S. Gull, A. Lasenby and C. Doran: Imaginary numbers are not real – the geometric algebra of spacetime. *Foundations of Physics* 23/9 (1993), 1175–1201. "[…] a tutorial introduction to the ideas of geometric algebra, concentrating on its physical applications".
[3] P. Lounesto: *Clifford algebras and spinors*. LMS LNS 239. Cambridge University Press, 1997.

[13] With the familiar interpretation of the exterior algebra, the element i is the unit volume, so that this relation infuses geometric meaning to the 'imaginary unit'.

[14] These are examples of the ease with which Hodge duality is managed with GA.

[15] These formulas were first obtained by Olinde Rodrigues by means of extensive analytical calculations: *J. Math. Pures Appl.* 5 (1840), 380–440.

[4] L. Dorst, D. Fontijne and S. Mann: *Geometric algebra for computer science: An object-oriented approach to geometry.* Elsevier/Morgan Kaufman, 2007. Revised edition 2009.
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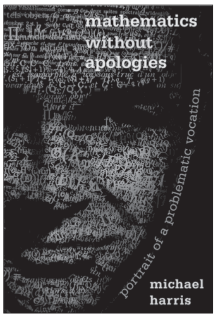
Sebastià Xambó-Descamps is a professor at the Universitat Politècnica de Catalunya (UPC). He is the author of Block-Error Correcting Codes – A Computational Primer. *Since 2011, he has been in charge of the portal ArbolMat (http://www.arbolmat.com/quienes-somos/). He has been: President of the Catalan Mathematical Society (1995–2002) and the Executive Committee for the organisation of 3ecm (Barcelona, 2000); UPC Vice-Rector of Information and Documentation Systems (1998–2002); Dean of the Facultat de Matemàtiques i Estadística UPC (2003–2009); and President of the Spanish Conference of Mathematics Deans (2004–2006). He was also Chair of the AGACSE 2015 Organising Committee.*



Michael Harris

Mathematics Without Apologies
Portrait of a Problematic Vocation

Princeton University Press, 2015
xxii, 438 p.
ISBN 978-0-691-15423-7

Reviewer: Javier Fresán

What drives mathematicians? Why do some of the best minds, generation after generation, leave for seas of thought so far from their "first and authentic" lives? Standard answers to these questions are based upon on three apologies: mathematics is *good* because, regardless of how abstract a theorem seems to be today, it might well have unexpected applications in the future; mathematics is *true*, as it provides "timeless certainty" in a fast-moving world; and it is *beautiful*, although this art form is often hidden to the untrained eye. Avoiding clichés, or rather delving more into them, *Mathematics without apologies* argues that these may be motivations for the romanticised Mathematician but they are quite absent from the everyday life of working (small-m) mathematicians. Which takes us back to the initial question. In this playful, erudite, iconoclast essay, Michael Harris points to a few alternative solutions, including the sense of belonging to "a coherent and meaningful *tradition*", the participation in a *relaxed field* "not subject to the pressures of material gain and productivity" and the pursuit of a certain kind of pleasure. Its main merit, however, lies less in offering new answers than in seriously asking the right questions, perhaps for the first time. Inevitably, some readers will find the result irritating, a mere exercise of quotation dropping, while others will see a genuine piece of cultural criticism which reaches, at its best, the level of Bourdieu's *La distinction* or Foucault's commentary of *Las meninas*.

The book is divided into 10 chapters, together with a series of interludes around the easier question of "How to explain Number Theory at a Dinner Party". Here, Harris introduces the necessary background to state Hasse's bound for the number of points of elliptic curves over finite fields – the inspiration for Weil's use of trace formulas which "converted" the author to number theory – and give a rough idea of the Birch-Swinnerton-Dyer conjecture, which served as a "guiding problem" of his early career. Short explanations about prime numbers, congruences and polynomial equations are followed by an amusing, highly unlikely dialogue between two characters, a Performing Artist and a Number Theorist, who tease each other with quotations from Aristotle, Kronecker, Musil, Levinas and Stoppard. Had *Mathematics without apologies* consisted only of these pages, it would have already been an original work of popular science, with clever findings like the Galois group of Chekhov's *Three Sisters*. But they are simply intended as a complement to an inquiry of much larger scope, which can be skipped without detriment to the reader.

Far from seeing his discipline as a closed paradise to non-experts, Harris defends the fact that outsiders have contributed with valuable insights into what it means to live as a mathematician. What makes them especially relevant for his purposes is that they couldn't help but be conditioned by the public self-image that mathematicians project. A recurring theme of the book is how intentions are misrepresented, starting from the autobiographical writings of mathematicians themselves. For instance, most accounts of the vocation's awakening seem to overestimate the quest of certainty as a driving force. A typical example of how this and other commonplaces notions are turned around is the beginning of Chapter 2: "How I acquired charisma". In a breathless prose, Harris explains that his "mathematical socialisation" began the year when the Prague Spring, the May 1968 events and the riots after the assassination of Martin Luther King shook the foundations of the world he had known before. Luckily, he writes, mathematics was there "to take their place". However, if one continues reading the footnote afterwards, it becomes clear that it didn't really happen that way.

Chapter 6: "Further investigations of the mind-body problem" takes a closer look at how mathematicians are perceived; it is not by chance that the first and the last sentence contain the word "mirror". Some features of the reflected image, like absent-mindedness, persist through the centuries, whereas others have dramatically changed. Following science historian Amir Alexander, Harris illustrates the latter by contrasting the Enlightenment ideal of the mathematician as a "natural man" – exemplified by the encyclopaedist d'Alembert or the fictional geometer of Potocki's novel *The Manuscript found in Saragossa* – with the romantic archetype of the lonely, self-destructive hero, largely inspired by the myths surrounding Galois' death. If the young Stendhal could still think of mathematics as "the royal road to Paris, glory, high society [and] women", a few years later, mathematicians would not be considered good lovers anymore and painters would portray them as figures "absorbed by [their] own inner flame", with characteristic gleaming eyes, while physicists would keep their reputation of "successful men of the world". The chapter also links Edward Frenkel's film *Rites of Love and Math* to the historical search for a love formula and briefly evokes Hypathia's martyrdom and some cases of madness, to conclude that "our readiness to sacrifice our minds and bodies to our vocation is the ultimate proof that what we are doing is important".

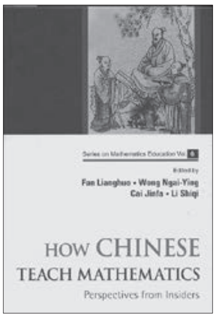
A form of melancholy distinct from the sentimentality of the romantic mathematician traverses Chapter 7: "The habit of clinging to an ultimate ground", which addresses the fascinating question: "How can we talk to one another, or to ourselves, about the mathematics we were born too soon to understand?" The first paragraph describes the vertigo that Grothendieck's dream of a category of motives or the Langlands programme may cause. A quotation from André Weil, "one achieves knowledge and indifference at the same time", serves as a leitmotiv. In the article from which it comes, he explains that 18th century French mathematicians used to employ the word "metaphysics" to refer to vague, hard-to-grasp analogies, which nevertheless played an important role in mathematical creation. The term is nowadays replaced by "yoga" or "avatar", as well as a distinctive use of quotation marks or the word "morally" as an explicit "invitation to relax one's critical sense". Harris also calls attention to the peculiar use of the verb "exist" in mathematics and to the problem of understanding *uniqueness* once *existence* has been established. This leads him to discuss higher categories and Voevodsky's Univalent Foundations, in a train of thought which is at times challenging to follow.

Chapter 8 (one of the most interesting chapters) examines the role of tricks in mathematical practice. The goal is to decide what makes Cantor's diagonalisation trick or Weyl's unitarian trick (to cite a few examples) "trick[s] rather than some other kind of mathematical gesture[s]". In an illuminating archaeology, Harris tracks the first occurrences of the word in scientific writing and exposes the nuances of its translation into various languages. The matter is not so simple because different mathematicians give different meanings to trickiness; otherwise, how could one reconcile Grothendieck's dismissal of Deligne's solution to the last of the Weil conjectures because "the proof used a trick" with the commonly held view that it is one of the most outstanding results of the twentieth century? Harris enriches the classical dichotomy between theory builders and problem solvers with the figures of the *strategist* and the *technician*, who, in contrast to mythology, do not represent a functional division of labour but coexist in most individual mathematicians. The trickster may be seen "as a *bridge* between *high* and *low* genres", which brings the author to the question of why mathematics is systematically classed as a high genre. Or, to put it succinctly: "Why so serious?" The tentative answers form Harris' *Appocalitici e integrati.*

In Chapter 9: "A mathematical dream and its interpretation", the author narrates a dream about the cohomology of unramified coverings of Drinfel'd upper-half spaces, which changed his life "in more ways than [he] care[s] to name". It is hard to imagine what the general reader gets out of the actual content of the dream but I truly enjoyed this personal side-note to a literature where human aspects of creation tend to be reduced to their minimum expression. In most scientific dreams, the unconscious comes to the rescue only after the dreamer has relentlessly tried to solve a problem; this was the case of Thomason – as analysed in another brilliant essay by Harris[1] – who had worked for three years on the extension problem for perfect complexes before the simulacrum of his deceased friend Trobaugh offered the key to the solution. On the contrary, the author's dream sketches a strategy for making progress on a subject "to which [he] had devoted no passion". The question then arises of how those new ideas found their way into his dreams. Harris' interpretation, which is remarkably sincere, exemplifies another aspect of the mathematical pathos: competitiveness, which could be summarised in a quotation from a hedge-fund manager who quit research: "It is hard to do mathematics and not care about what your standing is."

Until now, this review has concentrated on the second and third parts of Mathematics without apologies. The first chapters raise a fundamental question: if mathematics cannot be justified as useful, true and beautiful then how can it be justified, especially when it comes to ask for funding? Harris criticises the use of mathematics to legitimise certain economic policy decisions, as well as other Faustian bargains related to the financial crisis; he also notices an increasing role of private philanthropy in mathematical research, which could end up jeopardising "the professional autonomy to which we have grown attached". A more interesting challenge to this autonomy, although less immediate, is the "paradigm shift" that an extended use of computer-assisted proofs and automatic proof-checkers could introduce. Despite his love for science fiction, Harris does not seem to worry too much about the possibility of machines substituting human mathematicians, as he understands that "the goal of mathematics is to convert rigorous proofs to heuristics". But we readers have got nothing to lose in adding a new question: will apologies still be needed in post-human mathematics?

[1] M. Harris, "Do androids prove theorems in their sleep?". In A. Doxiadis and B. Mazur (eds.), *Circles Disturbed. The interplay of mathematics and narrative*. Princeton: Princeton University Press, 2012, pp. 130–182.



How Chinese Teach Mathematics. Perspectives from Insiders

Lianghuo Fan, Ngai-Ying Wong, Jinfa Cai and Shiqi Li (Eds.)

World Scientific, Singapore, 2015
756 p.
ISBN 978-981-4415-81-1

Reviewer: Mariolina Bartolini-Bussi

In Issue 89 (pp. 60–63), there were reviews of two books on Chinese mathematics education: *How Chinese Learn Mathematics. Perspective from Insiders* (Fan L., Wong N., Cai J. & Li S., 2004, published by World Scientific in Singapore; and *How Chinese Teach Mathematics and Improve Teaching* (Li Y & Huang R., 2013, published by Routledge in New York).

The volume *How Chinese Teach Mathematics. Perspective from Insiders* (the companion book to the earlier title) was announced for February 2014. It was actually published one year later in Spring 2015, with the same editorial team as the first one. As promised, the review process is completed here with this third volume.

It contains 21 chapters (plus an introduction and an epilogue), divided into three sections. The first section introduces readers to an historical and also a contemporary perspective, on traditional mathematical teaching in ancient China and on how modern Chinese mathematics teachers teach and pursue their pre-service training and in-service professional development. The second section presents studies investigating a wide range of issues at both the macro- and micro-levels on how Chinese mathematics teachers teach mathematics. The third section focuses on Chinese mathematics teachers, investigating issues about their knowledge, beliefs, teacher training and professional development. Like its predecessor, this volume is a must for educational researchers, practitioners and policymakers who are interested in knowing more about mathematics teaching, teachers, teacher education and professional development concerning Chinese teachers and learners. It is surely impossible to report on the full richness of its contents. Hence, the chapters will be listed below with partial reviews.

After a short introduction, the first part opens with an historical chapter (*The Wisdom of Traditional Mathematical Teaching in China*, by Dai Qin and Cheung Ka Luen), which is offered by the publisher as a sample for free download.[1] Although in recent years, more and more informed papers have been published on the history of the tradition of mathematics education in China, here there is a thorough discussion of some very interesting "new" issues, such as the combination of logical deduction and intuitive analysis in geometry, illustrated by the joint use of commentaries with words and visual arguments with figures and colours. Colours in geometric proofs are actually mentioned in very ancient Chinese books (although not used in the printed copies due to technical problems) in order to make treatments concise and highly explainable. It is worth mentioning that the first use of colours in the West, with educational intention, dates back to Byrne's edition of Euclid's Elements (Byrne, 1847). Manipulatives and games are also mentioned. A large part of the chapter is devoted to the procedural approach, also mentioning the suggestion of time allocation in the early introduction of arithmetic algorithms. The process of memorising is introduced through mathematical poems, which are still used in Chinese classrooms. The conclusion of the chapter concerns the link between the historical tradition and the present teaching of mathematics in China. This part is interesting and offers some information that is not so easily found in Western literature on mathematics education. However, Western scholars must always be aware that, in spite of the care put in the translation of the Chinese words into English, a gap (écart in French) between the Chinese and the English words is always unavoidable.

The second chapter (by Fan Lianghuo, Miao Zhenzhen and Mok Ah Chee Ida) concerns *How Chinese Teachers Teach Mathematics and Pursue Professional Development: Perspectives from Contemporary International Research Understanding the Chinese Ways of Teaching Mathematics*. The chapter opens with a survey of the literature on Chinese mathematics education, drawing on sources obtained by scanning the most important international research journals published up to 2012 that are peer-reviewed, well-established, truly international and highly reputable. The number is very limited, with some increase in the first decade of the 21st century. The rest of the chapter concerns the description of the models of mathematics teacher education and professional development in China. One of the most interesting issues is teachers' professional development in schools. In each school (starting from primary school), there is usually a mathematics teaching research group, which is a unique organisation originally introduced from the Soviet Union in the 1950s.

> *Working as a team, teachers prepare lessons together, observe each other's lessons, reflect and comment on observations collectively, and conduct open lessons regularly.* (p. 52)

In this way, teachers become reflective practitioners (Schön, 1983) and promote continuous development of the whole teaching team.

The second part of the volume is a collection of studies developed by teams of Chinese and Western scholars. The simple list of topics is impressive and covers many issues (the complete table of contents is published in the volume website mentioned above): coherence in the mathematics classroom, teaching measurement, teaching geometric proofs, teaching calculation of time intervals, teaching number sense, teaching using letter

[1] http://www.worldscientific.com/worldscibooks/10.1142/8542.

to represent number in primary school and exemplary lessons under the curriculum reform.

It is impossible to review all the chapters, hence information will only be given about one of the chapters: *Achieving Coherence in the Mathematics Classroom: Toward a Framework for Examining Instructional Coherence* (by Wang Tao, Cai Jinfa and Hwang Stephen). The chapter has three Chinese authors who live in the West. Hence, they combine the knowledge of Chinese culture and Western traditions of educational research. The abstract reads:

> *Coherence has been identified as an important factor in fostering students' learning of mathematics. In this chapter, by applying classroom discourse theories, we propose a framework for examining instructional coherence through a fine-grained analysis of a video-taped lesson from China. The lesson was chosen because it has been recognized as a model lesson for instructional coherence. Based on a careful analysis of instructional coherence on multiple levels of classroom discourse, we explored discourse strategies the teacher used to achieve instructional coherence in the classroom, as well as the features of classroom instruction in China.* (p. 111)

The lesson is about circles in the 6th grade. One can imagine that this lesson is the outcome of a cycle of "preparing, observing, reflecting and commenting" within a mathematics teaching research group (see above). Actually, coherence is a typical feature of a mathematics lesson in China. Having observed several cases of lessons in China, I have noticed that nothing happens by chance and that a careful design has been prepared. The chapter reports carefully about the global coherence of the overall narrative of the lesson, the episodic coherence of small episodes and the local coherence, about individual sentences or utterances and the sense given by the participants. The structure of a mathematics lesson in China is fixed: reviewing, teaching new content, student practice and assigning homework. Although Western observers may by critical about this fixed structure which may leave no space for improvisation (and creativity), it cannot be doubted that Chinese students perform better than Western students in international comparisons (e.g. PISA or TIMSS).

The third and last part of the volume focuses on Chinese mathematics teachers, teacher education and teacher professional development. In this part, some studies are reported about teachers' beliefs and professional development.

As above, the offering is very rich and it is not possible to review all the chapters. In this case, the selection is: *What Makes a Master Teacher? A Study of Thirty-One Mathematics Master Teachers in Chinese Mainland* (by Fan Linghuo, Zhu Yuan and Tang Caibin). A master teacher is typical of the Chinese instruction system (and of other Far East systems like in Japan and Singapore).

> *'Master teacher' as an honorary title has been used to recognize teachers' outstanding performance in the Chinese mainland since 1978 when the system was initiated. This title does not belong to the official career rank system for teachers, which was not established until the mid-1980s.* (p. 493)

Now, three levels characterise the official career rank system: "second grade" (or junior grade), "first grade" (or middle grade) and "senior grade" (or higher grade). Master teachers are regarded as models for other teachers. They are believed to have a systematic understanding of mathematics, know how to integrate mathematics education theories and psychology into classroom teaching, pay attention to mathematics as a culture and be able to analyse textbooks with deep understanding. The figure of master teacher is important in the Chinese school, as the number of studies (in Chinese) mentioned in this chapter shows. The main study reported in this chapter aims to investigate the reasons behind the success of mathematics master teachers in their acclaimed teaching career on the Chinese mainland. The study investigates the master teachers' beliefs in analysing their own professional success. From the interview of 31 mathematics master teachers (mostly from primary school), dedication to education, inner quality and true professional care toward students appear to be the three most important factors. Reflection (like that carried out in the open classes mentioned above) was considered the most important pathway for teachers to seek professional development, while various formal training and short-term training was considered to be less effective. This result is consistent with the second chapter reviewed above.

Finally, in the epilogue of the volume: *'Why the Interest in the Chinese Learner?'*, the volume editors summarise the motives for publishing the two twin books *How Chinese Learn Mathematics* and *How Chinese Teach Mathematics?* to meet the public's interest in the big success of Chinese mathematics education in international assessments. In their opinion, these books serve at least the following purposes:

> *(1) Letting the researchers on Chinese mathematics education discuss their own views, experiences and interested issues in mathematics education.*
> *(2) Telling the world the 'Chinese story' and responding to their queries.*
> *(3) Having Chinese and non-Chinese, who also have interest and passion in Chinese mathematics education, join hands to discuss issues in mathematics education worldwide.* (p. 706)

I belong to the last group as a non-Chinese scholar with an interest and passion in Chinese mathematics education. My personal interaction with Chinese mathematics education, either in personal visits or in reading reports like the ones of this volume, has convinced me that I have better understood my system by looking at the different choices made in the Chinese system. Following Jullien, a famous French philosopher and Sinologue, we might say that this experience is "not about comparative philosophy or about paralleling different conceptions but about a philosophical dialogue in which every thought, when coming towards the other, questions itself about its own unthought" (Jullien, 2008, my translation). This position is shared by the editors of the volume, who claim:

> *By reflecting on the practices of these regions, one reflects on one's own culture, understand oneself more, and forms a basis of moving forward in one's own way. (p. 707)*

This reflection and not the copy of other's practices is what is really important because:

> *The Chinese maxim that "those stones from other hills can be used to polish the jade" suggests that practices in other countries can serve as food for the improvement of one's own practice.* (p. 707)

This is the reason why I consider the trio of volumes reviewed in Issue 89 and the present issue really useful for Western researchers in mathematics education.

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Letters to the Editor

Reaction to a decision of Sofia University

Ivan Ilchev, Rector,
Members of the Academic Council,
Sofia University "St. Kliment Ohridski"
Sofia, Bulgaria

Dear Professor Ilchev,

Several months ago, Sofia University decided to prematurely retire Professor Emil Horozov, contrary to Bulgarian laws and the practice of the university. The Bulgarian court has decided that the act of Sofia University violates the law and that Professor Horozov must be returned to his job [1]. This, however, has not happened yet.

Sofia University has not explained its decision. Moreover, in the discussions, a professor in law has claimed that the Academic Council is not obliged to give an explanation. Unfortunately, the real incitement is very well known to the scientific community, not only in Bulgaria but also to many prominent scientists abroad.

The reason is that Horozov uncovered large-scale corruption in the Bulgarian Science Foundation. He formed a group of scientists who investigated the situation in the Bulgarian Science Foundation and produced a detailed report with the violations. The report was submitted to the Minister of Education and Science (at that time). After several months without any action, Professor Horozov made the report public in 2011. This was the starting point of the repression from institutions and newspapers. Some of them wrote articles in such a manner that the reader would get the impression that Horozov himself was the offender. As a consequence, Horozov decided to write a book *The project business and the robbing of science (Manual for the managing scoundrel)*. The book describes to the general public the mechanisms of the corruption in Bulgarian scientific financing.

Some of the most prestigious scientific journals in the world like *Nature* and *Science* [2, 3] have discussed the situation in the Bulgarian Science Foundation, unlike the Bulgarian newspapers. In fact, he has been invited by some TV channels to speak on this but, essentially, either the host did not allow him to speak or, in the cases of recorded interviews, they were 'well processed' to skip some of his criticisms.

Another step that Professor Horozov undertook was to inform the Chief Prosecutor of Bulgaria on the violations. After waiting more than a year, he got a decision on several pages that did not answer any of his allegations, instead 'answering' topics that were not discussed in his signal. His appeal pointed out these facts and, once again, he received the same comments on matters not discussed by Horozov.

At the same time, Professor Horozov informed The European Commission (EC) about the above facts, including his letter to the prosecutor. And again, as with the scientific journals, the EC reacted quickly in defence of Bulgarian science, contrary to Bulgarian institutions. After some protests of scientists and eventually the pressure from Europe, the Minister of Education and Science at that time was sacked.

It is no wonder that some of the antiheroes of his book and, in particular, the Sofia University rector and others from its governing body, were looking for revenge. They found it in the fact that Professor Horozov has reached 65 and they decided to retire him, contrary to the rules and the traditions of the university. Essentially, all professors at Sofia University keep their jobs until 68. In particular, last year over 90% kept their positions, but not one of the best scientists of the university and the country. Maybe they did not notice (or intentionally) that they had also violated the law, which explicitly states that the members and corresponding members of the Bulgarian Academy of Sciences have the right to keep their positions until 70, which is the case with Professor Horozov.

There were mass protests against the decision of the rector, signed by more than 1,000 people [4]. A group of leading scientists from the USA, France, Belgium, the Netherlands, Russia, Israel, Bulgaria, etc., who several years ago participated in a conference (sponsored by the American NSF) in honour of Professor Horozov, wrote a protest letter [5] to the Rector of the University, to the Chairwoman of the National Assembly, to the Minister of Education and Science, etc. Most of the newspapers and TV channels were informed too. No reaction came from any institution nor from any mass media.

We hope that the international scientific community will not stay silent when facing the picture described above.

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[1] Decision No. II-56 from 16.02.2015 of the Sofia Regional Court (Bulgarian) http://tinyurl.com/p8gopye.

[2] Bulgarian funding agency accused of poor practice, Nature, 6 April 2011, http://www.nature.com/news/2011/110406/full/472019a.html.

[3] Top Bulgarian Science Officials Sacked, Science, 29 January 2013, http://news.sciencemag.org/europe/2013/01/top-bulgarian-science-officials-sacked.

[4] http://www.peticiq.com/emil_horozov.

[5] http://tinyurl.com/p4gd4wj.

Bojko Bakalov, North Carolina State University, USA
Lubomir Gavrilov, University of Toulouse, France
Iliya Iliev, Institute of Mathematics, Bulgarian Academy of Sciences
Plamen Iliev, Georgia Institute of Technology, USA
Kamen Ivanov, Institute of Mathematics, Bulgarian Academy of Sciences
Milen Yakimov, Louisiana State University, USA

Dear Editor,
Professor **Ivan Vidav** (1918–2015), a leading Slovenian mathematician, has passed away at the age of 98. He was a student of Josip Plemelj and founder of the Slovenian school of mathematics at the University of Ljubljana. Vidav dominated Slovenian mathematics from the fifties through the eighties. He worked mainly in functional analysis, where he is best known for his metric characterization of self-adjoint operators and the related Vidav-Palmer theorem, which characterizes C^* algebras among all Banach algebras.

Tomaž Pisanski

Personal Column

Please send information on mathematical awards and deaths to newsletter@ems-ph.org.

Awards

The **Hausdorff Medal** was awarded by the European Set Theory Society to **Ronald Jensen** (Humboldt-Universität zu Berlin) and **John Steel** (UC Berkeley) at the fifth European Set Theory Conference in Cambridge, 24 August 2015.

The following members of the Real Sociedad Matemáticas Española have received the following prizes:
Ildefonso Díaz (Universidad Complutense de Madrid, Spain) has been awarded the **Jacques-Louis Lions Prize** of the French Academy of Sciences;
Juan Carlos López Alfonso (Technische Universität Dresden, Germany) has been awarded the **Reinhart Heinrich Award** by the European Society for Mathematical and Theoretical Biology;
Francisco Santos (Universidad de Cantabria, Spain) has been awarded the **Fulkerson Prize 2015** of the Mathematical Optimization Society and the American Mathematical Society;
Nuno Freitas (Max Planck Institute for Mathematics, Germany) has been awarded the 2014 **José Luis Rubio de Francia Prize** for young researchers of the Real Sociedad Matemática Española;
Alejandro Castro Castilla (Uppsala Universitet, Sweden), **Jezabel Curbelo Hernández** (Universidad Autónoma de Madrid, Spain), **Javier Fresán Leal** (ETH Zurich, Switzerland), **Rafael Granero Belinchón** (University of California, Davis, USA), **Luis Hernández Corbat** (IMPA, Brazil) and **Xavier Ros Otón** (University of Texas at Austin, USA) have received the 2015 **Vicent Caselles Research Award** of the Real Sociedad Matemática Española and Fundación BBVA; and
José Luis Fernández Pérez (Universidad Autónoma de Madrid, Spain), **Marta Macho Stadler** (Universidad del País Vasco, Spain) and **Antonio Martínez Naviera** (Universidad de Valencia, Spain) have been awarded the 2015 **Medals of the Real Sociedad Matemática Española**.

The Unione Matematica Italiana has awarded the following prizes for 2015:
the **Book Prize** to **Francesco Russo** for his manuscript "On the geometry of some special projective varieties";
the **Gold Medal Guido Stampacchia** to **Alessio Figalli**;
the **Mario Baldassarri Prize** to **Margherita Lelli Chiesa**;
the **Renato Caccioppoli Prize** to **Camillo De Lellis**;
the **Stefania Cotoneschi Prize** to **Fabio Brunelli**;
the **Bruno De Finetti Prize** to **Paola Domingo**;
the **Ennio De Giorgi Prize** to **Aldo Pratelli**;
the **Federigo Enriques Prize** to **Giulia Saccà**;
the **Enrico Magenes Prize** to **Stefano Bianchini**;
the **Franco Tricerri Prize** to **Elia Saini**; and
the **Calogero Vinti Prize** to **Ulisse Stefanelli**.

Jonas Ilmavirta from the University of Jyväskylä, Finland, is the laureate of the 7th "**International Stefan Banach Prize** for a Doctoral Dissertation in the Mathematical Sciences", awarded by Ericpol – a leading Polish IT company – and the Polish Mathematical Society.

Sławomir Kołodziej, from the Jagiellonian University, has been awarded the **Bergman Prize** by the American Mathematical Society.

AMASES has awarded its prize for the best paper by a young scholar to **Lucio Fiorin** at the 39th AMASES Congress, held in Padua, Italy, September 2015.

The **Shaw Prize in Mathematical Sciences** 2015 has been awarded in equal shares to **Gerd Faltings** and **Henryk Iwaniec**.

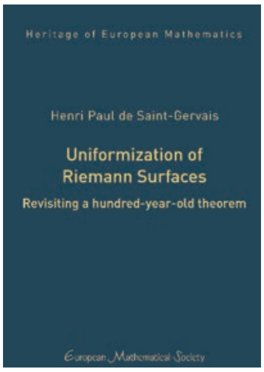
Deaths

We regret to announce the deaths of:

Michael Barratt (5 October 2015, Duluth, USA)
Giuseppe Malpeli (29 October 2015, Parma, Italy)
Czesław Olech (1 July 2015, Warsaw, Poland)
Stefan Rolewicz (9 July 2015, Warsaw, Poland)
Czesław Ryll-Nardzewski (18 September 2015, Wrocław, Poland)
Kazimierz Szymiczek (20 July 2015, Katowice, Poland)
Rainer Vogt (12 August 2015, Osnabrück, Germany)

New books published by the European Mathematical Society

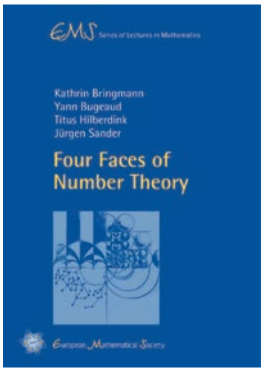
Individual members of the EMS, member societies or societies with a reciprocity agreement (such as the American, Australian and Canadian Mathematical Societies) are entitled to a discount of 20% on any book purchases, if ordered directly at the EMS Publishing House.



Henri Paul de Saint-Gervais
Uniformization of Riemann Surfaces. Revisiting a hundred-year-old theorem (Heritage of European Mathematics)

ISBN 978-3-03719-145-3. 2016. 512 pages. Hardcover. 17 x 24 cm. 78.00 Euro

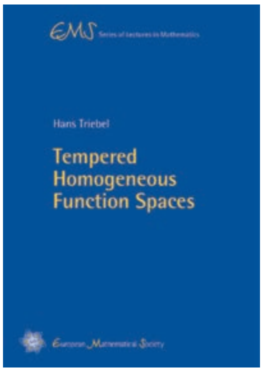
In 1907 Paul Koebe and Henri Poincaré almost simultaneously proved the uniformization theorem: *Every simply connected Riemann surface is isomorphic to the plane, the open unit disc, or the sphere.* It took a whole century to get to the point of stating this theorem and providing a convincing proof of it, relying as it did on prior work of Gauss, Riemann, Schwarz, Klein, Poincaré, and Koebe, among others. The present book offers an overview of the maturation process of this theorem.
The evolution of the uniformization theorem took place in parallel with the emergence of modern algebraic geometry, the creation of complex analysis, the first stirrings of functional analysis, and with the flowering of the theory of differential equations and the birth of topology. The uniformization theorem was thus one of the lightning rods of 19th century mathematics. Rather than describe the history of a single theorem, our aim is to return to the original proofs, to look at these through the eyes of modern mathematicians, to enquire as to their correctness, and to attempt to make them rigorous while respecting insofar as possible the state of mathematical knowledge at the time, or, if this should prove impossible, then using modern mathematical tools not available to their authors.
This book will be useful to today's mathematicians wishing to cast a glance back at the history of their discipline. It should also provide graduate students with a non-standard approach to concepts of great importance for modern research.



Kathrin Bringmann (University of Köln, Germany), Yann Bugeaud (IRMA, Strasbourg, France), Titus Hilberdink (University of Reading, UK) and Jürgen Sander (University of Hildesheim, Germany)
Four Faces of Number Theory (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-142-2. 2015. 198 pages. Softcover. 17 x 24 cm. 32.00 Euro

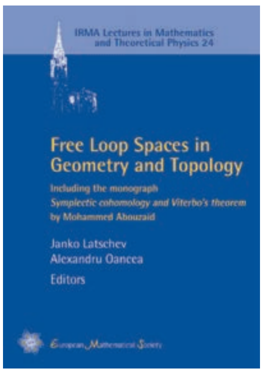
This book arose from courses given at an International Summer School organized by the number theory group of the Department of Mathematics at the University of Würzburg. It consists of four essentially self-contained chapters and presents recent research results highlighting the strong interplay between number theory and other fields of mathematics, such as combinatorics, functional analysis and graph theory. The book is addressed to (under)graduate students who wish to discover various aspects of number theory. Remarkably, it demonstrates how easily one can approach frontiers of current research in number theory by elementary and basic analytic methods.
Kathrin Bringmann gives an introduction to the theory of modular forms and, in particular, so-called Mock theta-functions, a topic which has obtained much attention in the last years. Yann Bugeaud is concerned with expansions of algebraic numbers. Here combinatorics on words and transcendence theory are combined to derive new information on the sequence of decimals of algebraic numbers and on their continued fraction expansions. Titus Hilberdink reports on a recent and rather unexpected approach to extreme values of the Riemann zeta-function by use of (multiplicative) Toeplitz matrices and functional analysis. Finally, Jürgen Sander gives an introduction to algebraic graph theory and the impact of number theoretical methods on fundamental questions about the spectra of graphs and the analogue of the Riemann hypothesis.



Hans Triebel (University of Jena, Germany)
Tempered Homogeneous Function Spaces (EMS Series of Lectures in Mathematics)

ISBN 978-3-03719-155-2. 2015. 143 pages. Softcover. 17 x 24 cm. 32.00 Euro

This book deals with homogeneous function spaces of Besov–Sobolev type within the framework of tempered distributions in Euclidean *n*-space based on Gauss–Weierstrass semi-groups. Related Fourier-analytical descriptions and characterizations in terms of derivatives and differences are incorporated afterwards as so-called domestic norms. This approach avoids the usual ambiguities modulo polynomials when homogeneous function spaces are considered in the context of homogeneous tempered distributions.
These notes are addressed to graduate students and mathematicians having a working knowledge of basic elements of the theory of function spaces, especially of Besov–Sobolev type. In particular it might be of interest for researchers dealing with (nonlinear) heat and Navier–Stokes equations in homogeneous function spaces.



Free Loop Spaces in Geometry and Topology. Including the monograph *Symplectic cohomology and Viterbo's theorem* by Mohammed Abouzaid (IRMA Lectures in Mathematics and Theoretical Physics, Vol. 24)
Janko Latschev (University of Hamburg, Germany) and Alexandru Oancea (Université Paris 6, France), Editors

ISBN 978-3-03719-153-8. 2015. 500 pages. Hardcover. 17 x 24 cm. 78.00 Euro

One of the main purposes of this book is to facilitate communication between topologists and symplectic geometers thinking about free loop spaces. It was written by active researchers coming to the topic from both perspectives and provides a concise overview of many of the classical results, while also beginning to explore the new directions of research that have emerged recently. As one highlight, it contains a research monograph by M. Abouzaid which proves a strengthened version of Viterbo's isomorphism between the homology of the free loop space of a manifold and the symplectic cohomology of its cotangent bundle, following a new strategy.
The book grew out of a learning seminar on free loop spaces held at Strasbourg University and should be accessible to a graduate student with a general interest in the topic. It focuses on introducing and explaining the most important aspects rather than offering encyclopedic coverage, while providing the interested reader with a broad basis for further studies and research.

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