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The cover drawing is an azimuthal equidistant spherical perspective ("360-degree fisheye"), representing four stacks of identical cube pairs floating over a uniform grid. This perspective projects the set of rays from the eye onto a closed disc. It captures everything around the viewer, and renders every line with exactly two vanishing points. The golden disc in the picture has half the radius of the full perspective disc and represents the frontal half of the view. Some types of spherical perspectives, such as this one, can be efficiently drawn by hand using ruler and compass operations. The method for thus drawing the 180-degree fisheye perspective was first described in Barre and Flocon's 1968 book "La perspective curviligne". The generalization to 360 degrees was described in "Ruler, compass, and nail: constructing a total spherical perspective" [J. Math. Arts 12, 2–3, 2018]. This perspective superficially resembles a reflection on a sphere, but it verifies radial occlusion (the main axiom of Euclid's optics) while a spherical reflection does not.

António B. Araújo

A message from the president

Dear members of the EMS,

There is a light on the horizon: the hope that the vaccination campaign will normalize the way we do our research and teaching, getting us out of the video conferences and back into the classroom. But there is still a long way to go, and it is possible that our mathematical life may continue to be different from before.

The EMS needs to find new post-pandemic procedures, and learn how to perform research and communication within the mathematics community of the future. The virtual European Congress 8ECM in Portorož will be a major test case for these new procedures. Although we will certainly miss the aspects of personal meeting and communication, it will also open new possibilities, in particular the opportunity for young mathematicians everywhere to participate in the online community life. I am looking forward to the congress, and hope that many of you will join us online.

One of the new initiatives that the EMS executive committee hopes to pursue in the future is the creation of an EMS Youth Academy. The idea is that each year, member societies will propose excellent young mathematicians just before and after their Ph.D., from among which a committee will then select young academy members for a period of four years. The members of the Youth Academy will then organise themselves, initiating new activities and also participating in the already-established EMS committees. This idea will be discussed in detail at the EMS president's meeting on May 29, 2021, and hopefully approved at the next EMS council in 2022.

Let us make the best out of the terrible pandemic experience and move the EMS forward.

Volker Mehrmann
President of the EMS

Brief words from the editor-in-chief

Dear readers of the EMS Magazine,

In this issue of the EMS Magazine you can find two more articles written by 2020 EMS Prize winners, this time by Alexander Efimov and Joaquim Serra. Additionally, as usual, there are articles on a variety of topics, for instance: the one by Emmylou Haffner on the edition of Riemann's collected works, or the article by Quentin Mérigot and Boris Thibert about mirrors, lenses and Monge–Ampère equations, among many other contributions to the usual columns on societies, research centres, or maths education.

The current issue is also the first with Donatella Donatelli as the new editor for book reviews, substituting in this role Jean-Paul Allouche who completed his second term as editor at the end of 2020.

Finally, you may have noticed that the first two issues of the EMS Magazine (Issue 119 and the current one, 120) have arrived to your mail box later than expected. I apologize for this delay and assure you that we (the editors and the EMS Press staff) are doing our best to smooth processes so that the Magazine reaches you on time in future.

Fernando Pestana da Costa
Editor-in-chief

Categorical smooth compactifications and neighborhoods of infinity

Alexander I. Efimov

In this note we give a short overview of some of our results on derived categories of coherent sheaves, in particular on smooth categorical compactifications and on the formal punctured neighborhoods of infinity.

Introduction

This note is devoted to a short overview of some results on derived categories of coherent sheaves concerning smooth categorical compactifications and the formal punctured neighborhoods of infinity.

In Section 1, we discuss the conjecture of Bondal and Orlov about the categorical properties of the resolution of singularities of an algebraic variety with rational singularities (Conjecture 1.1). This conjecture states that the derived pushforward functor on the derived categories of coherent sheaves is a quotient functor (that is, a localization). The conjecture is difficult and still open in general. It turns out that it is possible (Theorem 1.2) to prove a version of such statement for an arbitrary separated scheme of finite type over a field of characteristic zero (the reader may safely assume that we are dealing with quasi-projective schemes). The methods make it possible to prove Conjecture 1.1 for a cone over a projective embedding of a smooth Fano variety (that is, a smooth projective variety with an ample anti-canonical line bundle).

In Section 2, we consider DG categorical smooth compactifications. Here DG stands for "differential-graded". This is a straightforward generalization of the usual algebro-geometric smooth compactification. The following natural question was formulated by B. Toën (Question 2.3 below): is it true that any smooth DG category "of finite type" admits a smooth categorical compactification? The question was considered to be difficult, but most experts expected that the answer should be "yes". However, in [4] we gave a negative answer, obtained by disproving a closely related conjecture of Kontsevich (Conjecture 2.5 below) on the generalized version of the degeneration of the Hodge-to-de-Rham spectral sequence. We also obtained a dual version of these results, in which smooth DG categories are replaced by proper DG categories, and a smooth compactification is replaced by a categorical resolution of singularities.

In Section 3 we outline a certain construction called a "categorical formal punctured neighborhood of infinity". For a smooth algebraic variety X this is obtained as follows: take some smooth compactification $\bar{X}$, consider the formal completion at the infinity locus $\bar{X} - X$, and then take the corresponding punctured formal scheme. The resulting object $X_{\hat{\infty}}$ (considered for example as an adic space) is independent of the compactification, as is the category of perfect complexes on it. In [3] we give a purely categorical construction of $\mathrm{Perf}(X_{\hat{\infty}})$ which generalizes to arbitrary smooth DG algebras and DG categories. A curious special case is the algebra of rational functions on a smooth projective curve. There, our construction gives exactly the ring of adeles.

1 Rational singularities and a conjecture of Bondal and Orlov

Let X be an algebraic variety over a field of characteristic zero. Recall that X has rational singularities if for some (and then any) resolution of singularities $\pi : Y \to X$ we have $\mathbf{R}\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$. Equivalently, the pullback functor $\mathbb{L}\pi^* : D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(Y)$ is fully faithful. The following conjecture is still open.

Conjecture 1.1 ([1]). *With the above notation, the functor $\mathbf{R}\pi_* : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$ is a localization. That is, the induced functor $D^b_{\mathrm{coh}}(Y)/\ker(\mathbf{R}\pi_*) \to D^b_{\mathrm{coh}}(X)$ is an equivalence.*

The following result is a version of such statement which holds in a much more general framework.

Theorem 1.2 ([5]). *Let X be a separated scheme of finite type over a field k of characteristic zero. Then there exist a smooth projective variety Y and a functor $\Phi : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$ such that the induced functor $D^b_{\mathrm{coh}}(Y)/\ker(\Phi) \to D^b_{\mathrm{coh}}(X)$ is an equivalence. Moreover, the triangulated category $\ker(\Phi)$ is generated by a single object.*

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This theorem in particular confirms a conjecture of Kontsevich on the homotopy finiteness of the DG category $D^b_{\mathrm{coh}}(X)$. The proof is based on a certain construction of a categorical resolution of singularities, due to Kuznetsov and Lunts [8].

The methods developed to prove Theorem 1.2 actually also work to prove Conjecture 1.1 in a certain class of cases. In particular, the following result holds.

Theorem 1.3 ([5]). *Let $X \subset \mathbb{A}^n$ be a cone over a smooth Fano variety in $\mathbb{P}^{n-1}$. Let $\pi : Y \to X$ be the resolution given by the blow-up of the origin point. Then the induced functor $D^b_{\mathrm{coh}}(Y)/\ker(\mathbf{R}\pi_*) \to D^b_{\mathrm{coh}}(X)$ is an equivalence.*

2 Categorical smooth compactifications

Theorem 1.2 deals with a special case of a categorical smooth compactification. We first recall some basic definitions.

Definition 2.1 ([7]). 1. A small DG category $\mathcal{C}$ over k is smooth if the diagonal $\mathcal{C}$-$\mathcal{C}$-bimodule is perfect.
2. $\mathcal{C}$ is called proper if for $X, Y \in \mathcal{C}$ the complex $\mathcal{C}(X,Y)$ is perfect over k.

In particular, we have the notions of smoothness and properness for DG algebras (a DG algebra can be considered as a DG category with a single object). When X is a separated scheme of finite type over a field k, then X is smooth (resp. proper) if and only if the DG category $\mathrm{Perf}(X)$ is smooth (resp. proper) ([11, Proposition 3.30], [10, Proposition 3.13]). Hence, these basic geometric properties of X are reflected by the DG category $\mathrm{Perf}(X)$.

We recall the following definition.

Definition 2.2. For a pre-triangulated DG category $\mathcal{A}$, a categorical smooth compactification is a DG functor $F : \mathcal{C} \to \mathcal{A}$, such that:

1. $\mathcal{C}$ is a smooth and proper pre-triangulated DG category;
2. the induced functor $\mathcal{C}/\ker(F) \to \mathcal{A}$ is fully faithful;
3. every object $x \in \mathcal{A}$ is a direct summand of some $F(y)$, $y \in \mathcal{C}$.

The basic geometric example of a categorical smooth compactification is given by the usual one. Namely, let X be a smooth algebraic variety over k, and let $j : X \hookrightarrow \bar{X}$ be an open embedding, where $\bar{X}$ is smooth and proper. Then the restriction functor $j^* : \mathrm{Perf}(\bar{X}) \to \mathrm{Perf}(X)$ is a categorical smooth compactification.

Theorem 1.2 provides a categorical smooth compactification of the DG categories of the form $D^b_{\mathrm{coh}}(X)$, where X is a separated scheme of finite type over a field of characteristic zero.

There is a notion of a homotopically finitely presented (hfp) DG category which should be thought of as a smooth DG category "of finite type" (we refer to [14] for the precise definition). The following general question was formulated by Bertrand Toën.

Question 2.3 (Toën). Is it true that any homotopically finitely presented DG category over a field of characteristic zero has a smooth compactification?

The question is difficult, but the general consensus was that the answer should be "yes". However, in [4] the author gave a negative answer to this question. Here we explain the rough idea of the results of [4].

It turns out that Question 2.3 is closely related with the noncommutative (categorical) Hodge-to-de Rham degeneration. Recall that the classical Hodge theory implies (via GAGA) the following algebraic statement: for any smooth algebraic variety X over a field k of characteristic zero the spectral sequence

$$E_2^{pq} = H^q(X, \Omega_X^p) \Rightarrow H_{DR}^{p+q}(X)$$

degenerates.

The following categorical generalization was conjectured by Kontsevich and Soibelman [7], and proved by Kaledin [6].

Theorem 2.4 ([6, Theorem 5.4]). *Let A be a smooth and proper DG algebra over a field of characteristic zero. Then the Hochschild-to-cyclic spectral sequence degenerates, so that we have an isomorphism $HP_\bullet(A) \cong HH_\bullet(A)((u))$.*

In the special case when $\mathrm{Perf}(A) \simeq \mathrm{Perf}(X)$ for a smooth and proper variety X, Theorem 2.4 gives exactly the usual (commutative) Hodge-to-de Rham degeneration.

The following two conjectures were formulated by Kontsevich for smooth and for proper DG algebras.

Conjecture 2.5 (Kontsevich). *Let A be a smooth DG algebra over a field of characteristic zero. Then the composition*

$$K_0(A \otimes A^{\mathrm{op}}) \xrightarrow{\mathrm{ch}} \big(HH_\bullet(A) \otimes HH_\bullet(A^{\mathrm{op}})\big)_0 \xrightarrow{\mathrm{id}\otimes\delta^-} \big(HH_\bullet(A) \otimes HC^-_\bullet(A^{\mathrm{op}})\big)_1$$

vanishes on the class $[A]$ of the diagonal bimodule.

Here $\delta^- : HH_\bullet(A^{\mathrm{op}}) \to HC^-_\bullet(A^{\mathrm{op}})[-1]$ denotes the boundary map, see [2, Section 3].

Conjecture 2.6 (Kontsevich). *Let B be a proper DG algebra over a field* k *of characteristic zero. Then the composition map*

$$\big(HH_\bullet(B) \otimes HC_\bullet(B^{\mathrm{op}})\big)[1] \xrightarrow{\mathrm{id}\otimes\delta^+} HH_\bullet(B) \otimes HH_\bullet(B^{\mathrm{op}}) \to \mathrm{k}$$

is zero.

Here $\delta^+ : HC_\bullet(B^{op})[1] \to HH_\bullet(B^{op})$ denotes the boundary map, see [9, Section 2.2].

Both conjectures 2.5 and 2.6 hold, roughly speaking, for all DG categories coming from (commutative) algebraic geometry.

Conjecture 2.5 is related to Question 2.3 as follows. Suppose that we have a smooth compactification $\mathcal{C} \to \mathcal{A}$ (hence $\mathcal{A}$ is smooth). Then we have the following commutative diagram:

$$\begin{array}{ccc} HH_\bullet(\mathcal{C}) \otimes HH_\bullet(\mathcal{C}^{op}) & \xrightarrow{\mathrm{id}\otimes\delta^-} & HH_\bullet(\mathcal{C}) \otimes HC^-_\bullet(\mathcal{C}^{op})[-1] \\ \downarrow & & \downarrow \\ HH_\bullet(\mathcal{A}) \otimes HH_\bullet(\mathcal{A}^{op}) & \xrightarrow{\mathrm{id}\otimes\delta^-} & HH_\bullet(\mathcal{A}) \otimes HC^-_\bullet(\mathcal{A}^{op})[1]. \end{array}$$

The left vertical map sends $\mathrm{ch}(I_\mathcal{C})$ to $\mathrm{ch}(I_\mathcal{A})$. Hence, applying Kaledin's Theorem 2.4, we obtain that Conjecture 2.5 holds for $\mathcal{A}$.

A dual argument implies that Conjecture 2.6 holds for proper DG categories which can be fully faithfully embedded into a smooth and proper DG category (such an embedding is called a *categorical resolution* in the terminology of Kuznetsov and Lunts [8]).

However, in [4] we disproved both conjectures.

Theorem 2.7 ([4, Theorem 4.5, Theorem 5.4]). 1. *There exists a homotopically finitely presented DG algebra A for which Conjecture* 2.5 *does not hold. In particular, A gives a negative answer to Question* 2.3*: the DG category* Perf(A) *does not have a smooth categorical compactification.*

2. *There exists a proper DG algebra B for which Conjecture* 2.6 *does not hold. In particular, the category* Perf(B) *does not have a categorical resolution of singularities.*

The DG algebra B from part 2 is quasi-isomorphic to a certain explicit 10-dimensional A_∞-algebra for which the supertrace of m_3 on the second argument is non-zero.

3 Categorical formal punctured neighborhood of infinity

Another subject related to the notion of a smooth categorical compactification is that of a formal punctured neighborhood of infinity. Suppose that we have a usual smooth compactification $j : X \hookrightarrow \bar{X}$ of a smooth algebraic variety X. Then one can take the formal neighborhood $\bar{X}_{\hat{Z}}$, and then "remove" Z. The resulting object $\bar{X}_{\hat{Z}} - Z$ (the so-called generic fiber, considered as an adic space) does not depend on the choice of the compactification $\bar{X}$. Let us set $X_{\hat{\infty}} := \bar{X}_{\hat{Z}} - Z$. The corresponding category of perfect complexes $\mathrm{Perf}(X_{\hat{\infty}})$ also does not depend on Z and it is therefore an invariant of X.

The natural question arises: can we describe the category $\mathrm{Perf}(X_{\hat{\infty}})$ purely in terms of $\mathrm{Perf}(X)$? This question is partially motivated by mirror symmetry since an analogue of $\mathrm{Perf}(X_{\hat{\infty}})$ exists in symplectic geometry in the framework of Fukaya categories. It turns out that the purely categorical construction is possible, and it was described by the author in [3]. Here we give an outline.

First, we describe a "non-derived" version of the construction. Let A be an associative algebra over a field k. Then one can describe the algebra $H^0(A_{\hat{\infty}})$ as follows.

$$H^0(A_{\hat{\infty}}) = \{\varphi \in \mathrm{End}_\mathrm{k}(A) \mid \forall a \in A,\ \mathrm{rk}[\varphi, R_a] < \infty\}/(A^* \otimes A).$$

Here $\mathrm{End}_\mathrm{k}(A)$ is the algebra of k-linear endomorphisms of A (as a vector space) and $A^* \otimes A \subset \mathrm{End}_{\mathrm{k}(A)}$ is the two-sided ideal of operators of finite rank. The commutator is the additive one (the Lie algebra bracket) and $R_a : A \to A$, $R_a(b) = ba$, is the operator of right multiplication by a.

Example 3.1. It is a pleasant exercise to check that for $A = \mathrm{k}[t]$ we have $H^0(A_{\hat{\infty}}) \cong \mathrm{k}((t^{-1}))$. A similar computation shows that $H^0(\mathrm{k}[x^{\pm}]_{\hat{\infty}}) \cong \mathrm{k}((t)) \times \mathrm{k}((t^{-1}))$.

Example 3.2. A less trivial example is the following: let X be a smooth projective connected curve over k. Then we have $H^0(\mathrm{k}(X)_{\hat{\infty}}) \cong \mathbb{A}_X$, where $\mathbb{A}_X$ is the ring of adeles on X. Recall that $\mathbb{A}_X \subset \prod_{x \in X^{\mathrm{cl}}} \hat{K}_x$ is the subring of the product of complete local fields, consisting of elements $(a_x)_{x \in X^{\mathrm{cl}}}$ such that $a_x \in \hat{\mathcal{O}}_x$ for all but finitely many x.

Now let A be a smooth DG algebra. The DG algebra $A_{\hat{\infty}}$ is defined by the formula

$$A_{\hat{\infty}} := C^\bullet(A, \mathrm{End}_\mathrm{k}(A)/A^* \otimes A).$$

Here $C^\bullet(A, -)$ denotes the Hochschild cochain complex. The product on $A_{\hat{\infty}}$ comes from the product on $\mathrm{End}_\mathrm{k}(A)/A^* \otimes A$.

To describe the DG algebra $A_{\hat{\infty}}$ more conceptually, we recall the following notion.

Definition 3.3. 1. Let k be a field. The Calkin (DG) category Calk_k is defined as the quotient Mod-k/Perf(k). More explicitly, the objects of the DG category Calk_k are complexes of k-vector spaces, and the morphisms are given by $\mathrm{Calk}_\mathrm{k}(V, W) = \mathrm{Hom}_\mathrm{k}(V, W)/V^* \otimes W$.

2. More generally, for a DG algebra A the Calkin category Calk_A is defined as the quotient Mod-A/Perf(A).

We can consider A (and any other right A-module) as an object of $\mathrm{Rep}(A^{op}, \mathrm{Calk}_\mathrm{k})$ – suitably defined category of representations of A^{op} in Calk_k. Note that

$$A_{\hat{\infty}} \simeq \mathrm{End}_{\mathrm{Rep}(A^{op}, \mathrm{Calk}_\mathrm{k})}(A).$$

The DG category of topological perfect complexes over $A_{\hat{\infty}}$ is defined as follows.

Definition 3.4. For a smooth DG algebra A we define

$$\mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}}) \simeq \ker\big(\mathrm{Rep}(A^{op}, \mathrm{Calk}_\mathrm{k}) \to \mathrm{Perf}(A \otimes \mathrm{Calk}_\mathrm{k}) \to \mathrm{Calk}_A\big).$$

Here the embedding $\mathrm{Rep}(A^{\mathrm{op}}, \mathrm{Calk}_{\mathrm{k}}) \hookrightarrow \mathrm{Perf}(A \otimes \mathrm{Calk}_{\mathrm{k}})$ comes from the assumption that A is smooth. The functor $\mathrm{Perf}(A \otimes \mathrm{Calk}_{\mathrm{k}}) \to \mathrm{Calk}_A$ is given by the tensor product: $(A, V) \mapsto V \otimes A$ for $V \in \mathrm{Calk}_{\mathrm{k}}$.

Theorem 3.5 ([3]). *Let X be a smooth algebraic variety over a field* k, *and assume that X has a smooth compactification. Let A be a DG algebra such that* $\mathrm{Perf}(A) \simeq \mathrm{Perf}(X)$. *Then we have an equivalence* $\mathrm{Perf}(X_{\hat{\infty}}) \simeq \mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}})$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Perf}(X) & \xrightarrow{\sim} & \mathrm{Perf}(A) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(X_{\hat{\infty}}) & \xrightarrow{\sim} & \mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}}). \end{array}$$

Remark 3.6. It is possible to obtain an extended version of Theorem 3.5 where the category $\mathrm{Perf}(X_{\hat{\infty}})$ is replaced by the category of nuclear modules in the sense of Clausen and Scholze [12, Definition 13.10]). This is more involved (and unpublished), and we will not cover this in the present note.

Remark 3.7. The construction of the DG algebra $A_{\hat{\infty}}$ and the DG category $\mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}})$ is very much in the spirit of Tate's paper on residues of differential on curves [13].

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The geometric structure of interfaces and free boundaries

Joaquim Serra

Interfaces are surfaces that separate two regions of space with different physical properties: molecule A/molecule B, ice/water, charges/void, etc. The understanding of their geometric structure has boosted the development of Nonlinear Elliptic PDEs during the second half of the 20th century, and continues to do so at the beginning of the 21st.

1 Background: Minimal surfaces

Plateau's problem. Given a curve in $\mathbb{R}^3$, is there a surface with minimal area having this curve as boundary? This question, raised by Joseph-Louis Lagrange in 1760, is one of the most classical and influential problems in the Calculus of Variations. It is known as *Plateau's problem*, after the 19th century Belgian physicist Plateau, who experimented with soap films. Due to surface tension, soap films provide natural examples of area minimizing surfaces.

In 1930, Douglas and Radó gave the first solutions of Plateau's problem in the context of immersions. Later, other notions of solution were proposed by De Giorgi, Federer and Fleming, Reifenberg, and Almgren, among others. Heuristically, the weaker a notion of solution is, the easier it becomes to prove its existence. But solutions of Plateau's problem fail to be unique, so how can we be sure of not finding spurious solutions? Are all weak solutions "genuine" ones? *Regularity theory* gives detailed answers to this sort of question.

The regularity theory of area minimizing hypersurfaces. Let $\Omega \subset \mathbb{R}^n$ be some bounded domain, $n \geq 2$. We say that a hypersurface[1] $S \subset \mathbb{R}^n$ is *area minimizing*[2] in Ω if the following holds:

- The boundary of $S \cap \Omega$ is contained in $\partial\Omega$.
- For every hypersurface S' such that the boundaries of $S' \cap \Omega$ and of $S \cap \Omega$ coincide, we have $\operatorname{area}(S' \cap \Omega) \geq \operatorname{area}(S \cap \Omega)$.

Throughout the 20th century, many outstanding geometers and analysts worked on the following question: *Are area minimizing hypersurfaces smooth, or might they have "singularities"?* They arrived at a detailed and complete answer which can be summarized as follows:

(i) Any area minimizing hypersurface is smooth (analytic) in dimensions $n \leq 7$ (Fleming [24], De Giorgi [14, 15], Almgren [2], and Simons [40]).
(ii) In dimensions $n \geq 8$ the Simons cone $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\} \subset \mathbb{R}^n$ is an example of area minimizing hypersurface with a $(n-8)$-dimensional singular set (Bombieri, De Giorgi, and Giusti [7]).
(iii) In dimensions $n \geq 8$ area minimizing hypersurfaces are smooth (analytic) outside of a closed singular set of Hausdorff dimension $\leq n - 8$ (Federer [19]).

The earlier regularity theory, together with Almgren's [3] prodigious extension of it to m-surfaces in $\mathbb{R}^n$ with $2 \leq m \leq n-2$, inspired several other theories for geometric variational problems, interfaces, and free boundaries. We will refer to it a few times in what follows.

Stable minimal surfaces. Consider a soap film between two parallel circles of diameter 1, at small distance. We obtain a catenoid as in the left picture of Figure 1. When the separation (distance) between the two circles is small, the catenoid is an area minimizing surface. However, as we separate the circles more and more, we will reach a *first critical separation*, after which the area of the catenoid will be greater than 2π. Now the catenoidal soap films are no longer minimizers of the area (two flat disks joined with a thin neck would outperform them) but this does not cause any instability. Then, if we continue separating the circles, we reach a *second critical separation*, after which the soap film breaks into two disconnected disks, as shown in Figure 1.

[1] $(n-1)$-dimensional surface.
[2] This is an intentionally imprecise notion: more rigorously, S can be the boundary of a set of minimal perimeter, or a mass minimizing integer rectifiable current.

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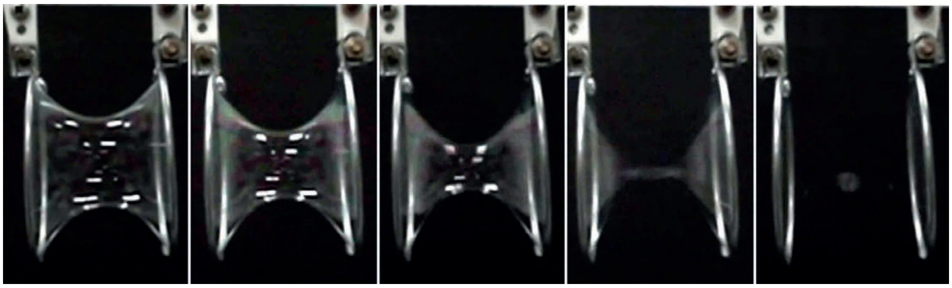

Figure 1. Unstabilizing a soap-film catenoid: Pictures from [25], reproduced with the authors' authorization

What happens between the two critical separations? The answer is given by the notion of *stable minimal surface*: although these catenoids are not "absolute" minimizers of the area, they still have a lesser area than any small variation of them. And this is enough to stabilize them.

As the previous example shows, not only energy *minimizers* are found in nature. Also *stable solutions*, i.e., those outperforming any small perturbation of them, are of physical interest. However, for Plateau's problem, as well as for several other important non-convex variational problems, fundamental questions that are well-understood in the case of minimizers remain completely open in the case of stable solutions. We next give a concrete example that will motivate some of our results described later.

A priori curvature bounds. The nowadays standard regularity theory for area minimizers – see (i) above – implies the following:

Theorem 1. *Let $n \leq 7$ and $S \subset \mathbb{R}^n$ be an area minimizing hypersurface in the unit ball $B_1 \subset \mathbb{R}^n$. Then the curvatures of S inside the half ball $B_{1/2}$ are bounded by dimensional constants.*

It has long been conjectured[3] that

Conjecture 2. *Theorem* 1 *holds replacing "area minimizing hypersurface" by "stable minimal hypersurface".*

By a simple (though clever) scaling and compactness argument of White (see [44]), Conjecture 2 is equivalent to

Conjecture 3. *Let $n \leq 7$ and $S \subset \mathbb{R}^n$ be a connected, complete, stable minimal hypersurface. Then S is an hyperplane.*

The previous conjectures have been proved only in the case $n = 3$ (surfaces in $\mathbb{R}^3$); the earliest proofs date from the 1970's, see [12]. But, unfortunately, their beautiful and relatively short proofs are extremely specific to the case of minimal surfaces in $\mathbb{R}^3$: they cannot be extended to higher dimensions, nor even to other interface models in $\mathbb{R}^3$ which are very similar to minimal surfaces.

2 Interfaces in phase transitions

The Allen–Cahn equation. Consider a *binary fluid*, i.e., a mixture containing two types of molecule: A and B (like oil and water). In many cases, these molecules have an energetic preference to be surrounded by others of their same kind. It undergoes phase separation into A-rich and B-rich regions.

Phase transition and phase separation phenomena – such as the previous one – are modelled by means of the *scalar Ginzburg–Landau energy*:

$$J_\varepsilon(v) := \int_\Omega \Big(\tfrac{1}{2}|\nabla v| + \tfrac{1}{4\varepsilon^2} W(v)\Big)\, dx, \quad \varepsilon > 0,$$

defined on scalar fields $v : \Omega \to [-1,1]$, where $\Omega \subset \mathbb{R}^n$. Here $W(v)$ is a so-called *double-well potential* with "wells" (i.e., minima) at ± 1. Typically one takes $W(v) = (1 - v^2)^2$.

Scalar fields $u_\varepsilon : \mathbb{R}^n \to [-1,1]$ satisfying

$$\frac{d}{dt}\Big|_{t=0} J_\varepsilon(u_\varepsilon + t\xi) = 0$$

for all $\xi \in C_c^\infty(\Omega)$ are called *critical points* (in Ω) of J_ε. They solve the *Allen–Cahn equation*: $-\Delta u_\varepsilon = \frac{1}{\varepsilon^2}(u_\varepsilon - u_\varepsilon^3)$. A critical point u_ε is called a *minimizer* (in Ω) if $J_\varepsilon(u_\varepsilon + \varphi) \geq J_\varepsilon(u_\varepsilon)$, for all $\varphi \in C_c^\infty(\Omega)$.

Let us come back to the binary fluid example to see how the scalar fields u_ε encode A-rich and B-rich regions. The idea is to interpret $\frac{1}{2}(u_\varepsilon(x)+1)$, a number in the interval $[0,1]$, as the relative density of molecules of type A at x. In other words, $u_\varepsilon(x) \in (0.99, 1]$ means that x belongs to a A-rich region while $u_\varepsilon(x) \in [-1, -0.99)$ means that x belongs to a B-rich region.

[3] In the case $n = 4$ this is *Schoen's conjecture* (see [12, Chapter 2]).

When the parameter $\varepsilon > 0$ is small the potential $\frac{1}{4\varepsilon^2}W(v)$ strongly penalizes intermediate states $v \in (-0.99, 0.99)$ and the space essentially splits into two regions, $\{u_\varepsilon > 0.99\}$ (A-rich region) and $\{u_\varepsilon < -0,99\}$ (B-rich region), which are separated by an *interface* $\{|u_\varepsilon| < 0.99\}$ (mixture of both molecules). The interface is a "fat surface" of thickness $\le C\varepsilon$. On the other hand, the Dirichlet term of the energy $\int_{\mathbb{R}^n} \frac{1}{2}|\nabla v|^2$ makes transitions between ± 0.99 costly, so interfaces are energetically expensive.

The zero level set $\{u_\varepsilon = 0\}$ can be thought as the surface which best approximates the interface $\{|u_\varepsilon| < 0.99\}$.

An important family of explicit solutions to the Allen–Cahn equation is given by

$$U_\varepsilon^{e,b}(x) = \tanh\left(\frac{e \cdot x - b}{\sqrt{2}\,\varepsilon}\right), \tag{2.1}$$

where $e \in \mathbb{S}^{n-1}$ and $b \in \mathbb{R}$. Via a calibration argument [4], one can see that $U_\varepsilon^{e,b}$ are minimizers of J_ε in all of $\mathbb{R}^n$.

Connection with minimal surfaces. By the results in [10, 30], if u_{ε_k} is a sequence of minimizers of J_{ε_k}, then the surfaces $\{u_{\varepsilon_k} = 0\}$ *converge locally uniformly*[4], as $\varepsilon_k \to 0$, towards area minimizing hypersurfaces.

It is then natural to ask if the surfaces $\{u_\varepsilon = 0\}$ inherit the regularity properties of the area minimizing hypersurfaces to which they converge. In other words:

Is $\{u_\varepsilon = 0\}$ smooth in dimensions $n \le 7$, with robust estimates as $\varepsilon \to 0$?

This delicate question is nowadays completely understood in the case of energy minimizers. Indeed, Savin established in 2009 the following celebrated result.

Theorem 4 ([36]). *Assume that $n \le 7$. Let u_ε be a minimizer of J_ε in $B_1 \subset \mathbb{R}^n$ with $u_\varepsilon(0) = 0$. Then $\{u_\varepsilon = 0\} \cap B_{1/2}$ is a $C^{1,\alpha}$ hypersurface, with robust estimates as $\varepsilon \downarrow 0$.*

A "famous" consequence of Theorem 1 and scaling is that *any minimizer of J_1 in all of $\mathbb{R}^n$ must be either* ± 1 *or of the form* (2.1) *with* $\varepsilon = 1$.

Combining Savin's result with the recent $C^{2,\alpha}$ estimates of Wang and Wei [42] we obtain:

Theorem 5 ([36, 42]). *Assume that $n \le 7$. Let u_ε be a minimizer of J_ε in $B_1 \subset \mathbb{R}^n$ with $u_\varepsilon(0) = 0$. Then, the curvatures of the hypersurface $\{u_\varepsilon = 0\}$ are bounded by dimensional constants in $B_{1/2}$.*

Conjectures on stable solutions. As in the case of soap films, it is very natural to ask:

Does Theorem 5 *hold when "minimizer" is replaced by "stable critical point" (i.e., minimizer among small perturbations)?*

Like for minimal surfaces, thanks to the striking results from [42], the previous question can be reduced to the following long-standing

Conjecture 6. *Assume that $n \le 7$. Let u be a stable critical point of J_1 in the whole space $\mathbb{R}^n$ different from* ± 1. *Then u must be of the form* (2.1) *with* $\varepsilon = 1$.

Even in the case of $\mathbb{R}^3$, Conjecture 6 is a very challenging and completely open problem (although the analogous result for minimal surfaces in $\mathbb{R}^3$ is known, its very rigid proof does not generalize to stable critical points of J_ε). The case of $n = 2$, which is already nontrivial, was proven by Ambrosio and Cabré [4] in 2000.

Interestingly, Conjecture 6 is known to imply a famous 1979 conjecture of De Giorgi [16]: for all $n \le 8$ (one dimension more than before) any solution of the Allen–Cahn equation in the whole space $\mathbb{R}^n$ satisfying $\partial_{x_n} u > 0$ must be of the form (2.1), with $\varepsilon = 1$ and $e \cdot e_n > 0$.

"Counterexamples" to Theorem 5 and Conjecture 6 for $n \ge 8$, and to De Giorgi's conjecture for $n \ge 9$ were obtained – via very delicate and involved constructions – in [17, 29].

The Peierls–Nabarro equation. Introduced in the early 1940's in the context of crystal dislocations [32, 33], the Peierls–Nabarro equation also models phase transitions with line-tension effects [1] and boundary vortices in thin magnetic films [27]. It concerns the energy functional

$$I_\varepsilon(v) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\left|v(x) - v(\bar{x})\right|^2}{|x - \bar{x}|^{n+1}}\, dx\, d\bar{x} + \frac{1}{\varepsilon}\int_{\mathbb{R}^n} W(v)\, dx\,.$$

As in the previous section, $v : \mathbb{R}^n \to [-1, 1]$ is a scalar field and $W(v)$ is a double-well potential.

In this context a natural double-well potential is $W(v) := 1 + \cos(\pi v)$, and for this choice of W an explicit family of solutions is given by

$$U_\varepsilon^{e,b}(x) = \frac{2}{\pi}\arctan\left(\frac{e \cdot x - b}{\varepsilon}\right). \tag{2.2}$$

The two functionals J_ε and I_ε behave similarly, and there is an almost perfect parallel between their interface regularity theories. To start with, by [1, 38], if u_{ε_k} is a sequence of minimizers of I_{ε_k} then the interfaces $\{u_{\varepsilon_k} = 0\}$ converge locally uniformly as $\varepsilon_k \to 0$ towards area minimizing hypersurfaces, just as they do for J_ε.

In this context the analogue of Theorem 4 – i.e., a local $C^{1,\alpha}$ estimate for $\{u_\varepsilon = 0\}$ in the case of energy minimizers – was obtained in [37], also by Savin, using similar techniques.

Given the parallel between J_ε and I_ε, it is conjectured that for $3 \le n \le 7$ all stable critical points of I_1 in the whole space $\mathbb{R}^n$ must

[4] In the sense of the Hausdorff distance and up to subsequences.

be of the form (2.2) with $\varepsilon = 1$ (in other words that the analogue of Conjecture 6 replacing J_ε by I_ε holds).

While Conjecture 6 (for J_ε) remains completely open in dimensions $3 \le n \le 7$, Figalli and the author [23] were able to establish it for I_ε in dimension $n = 3$.

Theorem 7 ([23]). *Let u be a stable critical point of I_1 in the whole space $\mathbb{R}^3$. Then u must be of the form* (2.2) *with* $\varepsilon = 1$.

This result finally broke the parallel of known results for J_ε and I_ε, in favour of I_ε. Its proof exploits the "long-range interactions" from the term

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+1}}\,dx\,d\bar{x},$$

borrowing ideas from a paper of Cinti, the author, and Valdinoci [11] on *nonlocal minimal surfaces*.

3 The obstacle problem and Stefan's problem

Pushing an elastic membrane with an obstacle. Given some smooth domain $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, both smooth and satisfying $g \ge \varphi|_{\partial\Omega}$, consider the convex minimization problem

$$\min\left\{\int_\Omega |\nabla v|^2 dx \ : v \ge \varphi,\ v = g \text{ on } \partial\Omega\right\}.$$

For $n = 2$, one can think of $x_3 = v(x_1, x_2)$ as the equilibrium position of an elastic membrane whose boundary is held fixed while it is pushed from below by an *obstacle* (the hypograph of φ).

The function $u := v - \varphi \ge 0$ can be shown to satisfy $\Delta u = (-\Delta\varphi)\chi_{\{u>0\}}$ in Ω. In the "model case" $\Delta\varphi \equiv -1$ one obtains

$$u \ge 0, \quad \Delta u = \chi_{\{u>0\}} \qquad \text{in } \Omega. \tag{3.1}$$

In other words, the domain Ω is split into two subdomains $\{u > 0\}$ and $\{u = 0\}$ and inside the first one we have $\Delta u = 1$. The unknown interface between the two subdomains, denoted $\partial\{u > 0\}$, is called the *free boundary*. Since u must satisfy (3.1) (in the sense of distributions) in Ω, not only u but also $|\nabla u|$ must vanish continuously on $\partial\{u > 0\}$. In this "double constraint" (3.1) encodes the geometric information about the free boundary.

As an interesting fact, solutions u of (3.1) minimize the following convex energy functional:

$$\int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 + \max(0,u)\right) dx. \tag{3.2}$$

A potential theoretic motivation of the obstacle problem. Imagine a cloud made of a very large number of identical point charges in $\mathbb{R}^3$. They interact through the standard Coulomb potential, repelling each other. In absence of external forces the cloud would expand indefinitely, but inside some exterior potential the cloud will reach an equilibrium, occupying only a bounded region of the space. This motivates the introduction of the so-called (Frostman) *equilibrium measure* for Coulomb interactions with an external "field" V (growing at infinity), defined as the unique probability measure μ on $\mathbb{R}^3$ which minimizes

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{|x-y|} d\mu(x)\,d\mu(y) + \int_{\mathbb{R}^n} V(x) d\mu(x). \tag{3.3}$$

Denoting by $v(x) := \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x-y|}$ the potential generated by μ, the equilibrium measure μ is compactly supported and uniquely characterized by the fact that there exists a constant c such that $v \ge c - \frac{V}{2}$ in $\mathbb{R}^3$ and $v = c - \frac{V}{2}$ on the support of μ. In other words u solves the obstacle problem in the whole space with obstacle $\varphi = c - \frac{V}{2}$.

Ice melting in water. Dating back to the 19th century, Stefan's problem [41] aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water.

Its most classical formulation is as follows: let $\Omega \subset \mathbb{R}^3$ be some bounded domain, and let $\theta = \theta(x,t)$ denote the temperature of the water at the point $x \in \Omega$ at time $t \in \mathbb{R}^+ := [0,+\infty)$. We assume that $\theta \equiv 0$ on the ice and $\theta > 0$ in the water. The temperature satisfies the heat equation $\partial_t\theta = \Delta\theta$ inside the water $\{\theta > 0\}$ and the *Stefan condition*[5] $\partial_t\theta = c|\nabla\theta|^2$ on the interface $\partial\{\theta > 0\}$.

Baiocchi and Duvaut [5, 18] introduced the transformation $u(x,t) := \int_0^t \theta(x,\tau)d\tau$ and showed that the new scalar field u satisfies[6]

$$u \ge 0, \quad \partial_t u \ge 0, \quad \text{and} \quad (\Delta - \partial_t)u = \chi_{\{u>0\}}. \tag{3.4}$$

In addition, by definition of u we have $\{u > 0\} \equiv \{\theta > 0\}$ and

$$\partial_t u > 0 \quad \text{inside } \{u > 0\}. \tag{3.5}$$

Interestingly, the evolution (3.4) is the gradient flow of the convex functional (3.2). Thanks to this convex structure, some basic questions such as existence and well-posedness of Stefan's problem – which would be very non-obvious in the original formulation – can be shown via standard Functional Analysis methods.

Other motivations. Stefan's and obstacle problems have other well-known applications in physics, biology, or financial mathematics. Some examples are: the dam problem, the Hele–Shaw flow,

[5] The normal velocity $\vec{V}$ of $\partial\{\theta > 0\}$ is proportional to the flux of heat (which is used to melt the ice). By Fourier's law this flux is proportional to the gradient of temperature, hence $\vec{V} = -c\nabla\theta$. But, since $\theta \equiv 0$ on the moving interface we obtain $\partial_t\theta + \vec{V}\cdot\nabla\theta = 0$ on $\partial\{\theta > 0\}$, from which Stefan's condition follows.

[6] Near points that were inside the ice at initial time and for $c = 1$.

pricing of American options, quadrature domains, random matrices, etc.

Regularity of free boundaries: Main questions and difficulties. Any solution u of (3.4) can be shown to be of class $C^{1,1}$ in space and $C^{0,1}$ in time. This regularity is optimal because the right hand side $\chi_{\{u>0\}}$ in (3.4) forces $(\Delta - \partial_t)u$ to be discontinuous across $\partial\{u > 0\}$.

The most interesting regularity questions concern the *free boundary* $\partial\{u > 0\}$:

- Is the free boundary a smooth hypersurface, or may it have *singularities*?
- If the singular set is nonempty, how "large" can it be?

Classical examples by Lévy and Schaeffer (some known from before the 1970's) show that solutions of the obstacle problem with non-smooth free boundaries exist already in the smallest nontrivial dimension $n = 2$; see [26]. Hence, any positive regularity result on the free boundary must be "conditional".

It was not until 1977, with the groundbreaking paper of Caffarelli [8], that a regularity theory for the free boundaries of solutions of (3.4) was established. Since (3.1) is a particular case of (3.4) – that of constant in time solutions – Caffarelli's results apply at the same time to both the obstacle problem and Stefan's problem.

Caffarelli's breakthrough. The approach of Caffarelli to the regularity of free boundaries of (3.4) – or of (3.1) – has some similarities with the regularity theory of area minimizing hypersurfaces described in Section 1. In Caffarelli's regularity theory (as in minimal surfaces) *blow-ups* are very important actors. Informally speaking, one looks at the free boundary through a microscope, and then tries to infer its "macroscopic properties" from its "microscopic" ones.

For (3.4) the scaling of the problem suggests considering, for given $(x_\circ, t_\circ) \in \partial\{u > 0\}$ and $r > 0$,

$$u^{x_\circ,t_\circ,r}(x,t) := \frac{1}{r^2}u(x_\circ + rx, t_\circ + r^2 t).$$

It is easy to see that $u^{x_\circ,t_\circ,r}$ is again a solution of (3.4). *Blow-ups* are defined as accumulation points of $u^{x_\circ,t_\circ,r}$ as $r \downarrow 0$.

The main results from [8] (combined with [26], [9] and [6]) can be summarized as follows:

Theorem 8. *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ be a solution of* (3.4)*. For every $(x_\circ, t_\circ)$ belonging to the free boundary $\partial\{u > 0\}$ one of the following two alternatives holds:*

(a) *$u^{x_\circ,t_\circ,r} \to \frac{1}{2}\big(\max(0, e\cdot x)\big)^2$ as $r \downarrow 0$, for some $e \in \mathbb{S}^{n-1}$; and the free boundary is a (moving) analytic embedded $(n-1)$-surface near $(x_\circ, t_\circ)$.*
(b) *$u^{x_\circ,t_\circ,r} \to \frac{1}{2}x\cdot Ax$ as $r \downarrow 0$, for some nonnegative definite matrix A with trace equal to* 1*; and the free boundary has a singularity*[7] *at $(x_\circ, t_\circ)$.*

Further known results on singular points. After the results of Caffarelli [8], a natural question is: *what else can be said about singular points?*

For the obstacle problem (3.1) in dimension $n = 2$, Sakai [34,35] used methods in complex analysis to give an extremely accurate description of the possible singularities. In particular, the results of Sakai imply that at every singular free boundary point $x_\circ$ of a solution of (3.1) in $\mathbb{R}^2$ we have

$$u(x_\circ + x) = \tfrac{1}{2}x\cdot Ax + \omega(x). \tag{3.6}$$

with $|\omega(x)| \le C|x|^3$. This significantly improved the qualitative description of Theorem 8(b), which is equivalent to $\omega(x) = o(|x|^2)$, and entailed some interesting consequences. Unfortunately, Sakai's complex analysis methods cannot work in higher dimensions, nor for Stefan's problem (not even for $n = 2$). Thus, improving Caffarelli's result for (3.1) in dimensions $n \ge 3$ required new ideas.

Understanding singularities better. The first new result in this direction for $n \ge 3$ was established by Colombo, Spolaor, and Velichkov in 2017 [13]. By improving and refining the methods of Weiss [43], they proved that at every singular point, the expansion (3.6) holds with explicit logarithmic modulus of continuity $|\omega(x)| \le C|x|^2(\log|x|)^{-\gamma}$, where $\gamma > 0$. Independently and with different methods, Figalli and the author proved in [22] the following:

Theorem 9 ([22]). *Let u be a solution of the obstacle problem* (3.1) *with $\Omega \subset \mathbb{R}^n$. For all singular points outside some "anomalous" set of Hausdorff dimension $\le n-3$,* (3.6) *holds with $|\omega(x)| \le C|x|^3$. Moreover, there exist examples in $\mathbb{R}^3$ of isolated singular points for which $|\omega(x)| \gg |x|^{2+\varepsilon}$ as $|x| \to 0$ for all $\varepsilon > 0$.*

The previous theorem suggests, for one thing, that we may be able to give a very precise quantitative description of most singularities. However, the existence – already in $\mathbb{R}^3$ – of singular points for which $|\omega(x)| \gg |x|^{2+\varepsilon}$ for all $\varepsilon > 0$ tells us that we cannot hope for some analytic structure of singularities as in Sakai's result for $\mathbb{R}^2$: in higher dimensions some singularities may be very complicated.

Another insightful result from [22] is that, for all singular points outside some $(n-2)$-dimensional set we have, after rotation, the

[7] For the evolutionary problem (3.4) singularities are associated to changes of topology of the ice $\{u = 0\}$. For instance, the ice may develop a very thin shrinking neck which eventually breaks into two pieces after producing a singular point.

improved expansion $u(x_\circ + x) = \frac{1}{2}x_n^2 + x_n Q(x) + o(|x|^3)$, where Q is some quadratic polynomial satisfying $\Delta(x_n Q) = 0$. This invites us to investigate higher order expansions that hold at most singular points (although proving this turned out to be quite a delicate task, and the tools needed to complete it were only developed later in [20]).

It is interesting to notice that the methods introduced in [22] for the obstacle problem are closely connected with Almgren's regularity theory [3] for mass minimizing m-surfaces in $\mathbb{R}^n$ with $n \geq m + 2$. In particular, Almgren's frequency formula plays an important (and unexpected) role.

The size of the singular set. An important consequence of Theorem 8 is that, in both the obstacle and Stefan's problems, the singular sets enjoy spatial C^1-regularity, in the sense that they can be covered by $(n-1)$-manifolds of class C^1 (see [6, 9]). Note, however, that this is not a very useful piece of information on the size of the singular set, since the regular part of the free boundary is also $(n-1)$-dimensional and thus, a priori, the singular part could be as large as the regular one.

As explained above, Theorem 8 applies at the same time to both the obstacle problem and Stefan's problem, since (3.1) is a particular case of (3.4). However, when we seek to obtain improved bounds on the size of their singular sets, the two problems need to be treated in completely different ways. On the one hand, in Stefan problem it is natural to try to exploit (3.5) – which was not used in Caffarelli's theory – and to ask if the free boundary is free of singularities most of the time. On the other hand, for the stationary problem (3.1), the previous evolutionary point of view makes no sense. In the absence of time, the only thing one can hope to prove is that for "generic" boundary values, solutions of (3.1) do not have singular points. This is actually something that has been expected to be true since the 1970's [39]:

Conjecture 10 (Schaeffer, 1974). *Generically, solutions of the obstacle problem have smooth free boundaries.*

Until recently Conjecture 10 was only known to hold in the plane $\mathbb{R}^2$ (see [31]).

Generic regularity for the obstacle problem. Building on the methods initiated in [22] we were recently able to obtain a positive answer to Schaeffer's conjecture in low dimensions:

Theorem 11 ([20]). *Conjecture* 10 *holds in* $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

Our strategy towards this theorem is reminiscent of *Sard's theorem* in analysis. By adding $\tau \in \mathbb{R}$ to the boundary values we produce a monotone 1-parameter family of solutions. We then prove that the set of "singular values" of τ has measure zero by improving the order of approximation of certain polynomial expansions at most singular points. This is a long and delicate proof because the singular sets need to be split into several different strata, and in each of them the corresponding singular values have measure zero for very different reasons.

The singular set in Stefan's problem. As said above, in order to investigate the size of the singular set in Stefan's problem, we will use (3.5). In particular, from now on solutions will never be stationary.

Fix $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and let $u : \Omega \to \mathbb{R}$ be a solution of (3.4)–(3.5). It will be useful to define the spatial and time projections $\pi_x(x,t) := x$ and $\pi_t(x,t) = t$.

Let us denote by $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ the set of all singular free boundary points of u.

Caffarelli's regularity theory implies (see [6, 9]) that every "time slice" of $\Sigma \cap \pi_t^{-1}(\{t_\circ\})$ can be locally covered by $(n-1)$-manifolds of class C^1. This may not seem like a very strong piece of information, since the regular part of the free boundary is also $(n-1)$-dimensional. However, it is not difficult to construct solutions of (3.4)–(3.5) with rotational symmetry $u(x,t) = U(|x|,t)$ such that for countably many times t_i the time slice $\Sigma \cap \pi_t^{-1}(\{t_i\})$ contains some $(n-1)$-sphere $\partial B_{R_i}(0) \times \{t_i\}$.

The previous examples show that even for countably many times, the singular set can have positive $(n-1)$-dimensional measure. At those times, the singular set is as large as the regular part of the free boundary. Still, inspection of explicit examples suggests that Σ should be smaller in some sense than the regular part of the free boundary, perhaps as a subset of the "space-time" $\mathbb{R}^n \times \mathbb{R}$.

Until recently, the best results available in this direction, such as [28], could not even rule out $\Sigma \cap \pi_t^{-1}(\{t_\circ\})$ being $(n-1)$-dimensional for every time $t_\circ$!

In the forthcoming article [21], we are able to prove a much stronger result, which gives a precise structure and sharp dimensional bounds on the singular set of Stefan's problem.

Theorem 12 ([21]). *There exist* $\Sigma^\infty \subset \Sigma$ *such that the following holds:*

(i) $\dim_{\mathrm{par}}(\Sigma \setminus \Sigma^\infty) \leq n-2$, *where* $\dim_{\mathrm{par}}$ *denotes the parabolic Hausdorff dimension;*[8]
(ii) $\pi_x(\Sigma^\infty) \subset \mathbb{R}^n$ *can be covered by countably many* C^∞ $(n-1)$-*manifolds;*
(iii) $\pi_t(\Sigma^\infty) \subset \mathbb{R}^n$ *has zero Hausdorff dimension.*

[8] For $E \subset \mathbb{R}^n \times \mathbb{R}$ and $\beta \geq 0$, we say that $\dim_{\mathrm{par}}(E) \leq \beta$ if, for all $\beta' > \beta$, E can be covered by countably many parabolic cylinders $B_{r_i}(x_i) \times (t_i - r^2, t_i + r_i^2)$ making $\sum_i r_i^{\beta'}$ arbitrarily small. This notion of Hausdorff dimension is well-adapted to the parabolic scaling $(rx, r^2 t)$ under which (3.4) is invariant.

This is a very precise result. Recall that in radial examples the singular set can contain some $(n-1)$-sphere countably many times. Such spheres would be covered by the set Σ^∞ in Theorem 12. Now, we cannot prove that in general $\pi_t(\Sigma^\infty)$ is countable as it is in such examples, but we do show that it is a 0-dimensional set (and Hausdorff dimension cannot distinguish between countable and 0-dimensional sets, so the result is sharp in this sense). However, the complement of Σ^∞ inside Σ is a set of "bad" singular points. These "bad" points do not a priori enjoy any extra spatial regularity, but in exchange they are lower-dimensional: their parabolic Hausdorff dimension is bounded by $n-2$. This bound is also optimal, as can be shown by considering any radial solution in $\mathbb{R}^2$ with a singular point at $(0,0)$.

An important consequence of Theorem 12 is the following:

Corollary 13 ([21]). *The set of singular times for Stefan's problem in $\mathbb{R}^3$ has Hausdorff dimension at most $1/2$. In particular, it has measure zero.*

Also, Theorem 12 implies that in $\mathbb{R}^2$ the set of singular times for Stefan's problem has zero Hausdorff dimension (prior to our results it was not even known that in $\mathbb{R}^2$ the set of singular times had measure zero).

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Mirrors, lenses and Monge–Ampère equations

Quentin Mérigot and Boris Thibert

Is it possible to shape a piece of glass so that it refracts and concentrates sunlight in order to produce a given image? The modelling of this kind of problem leads to nonlinear second-order partial differential equations, which belong to the family of Monge–Ampère equations. We will see how semi-discrete methods, that can be traced back to Minkowski's works, allow us to numerically solve such equations.

1 Anidolic optics and Monge Ampère type equations

In anidolic optics, or non-imaging optics, one studies the design of devices that transfer light energy between a source and a target. The general problem is to design the shape of a mirror (or a lens) that reflects (or refracts) the light emitted from a given source towards a target whose geometry and intensity distributions are prescribed (see Figures 1 and 2). Applications of anidolic optics include the design of solar ovens, public lighting, car headlights, and more generally the optimization of the use of light energy and the reduction of light pollution.

Near-field and far-field light sources

There exists many different problems in anidolic optics, depending for instance on the geometry of the light source, the type of optical component, and the target to be illuminated. These problems are distinguished in particular by the spatial position of the target illumination. A problem is called *near-field* when the target is located within a finite distance, i.e., when one wishes to illuminate an area of space such as a screen. In Figures 1 and 2, the target illumination is on a wall, making the problem near-field. Most of the illustrations in this article correspond near-field targets. However, we will first consider the *far-field* case, which is mathematically simpler, and

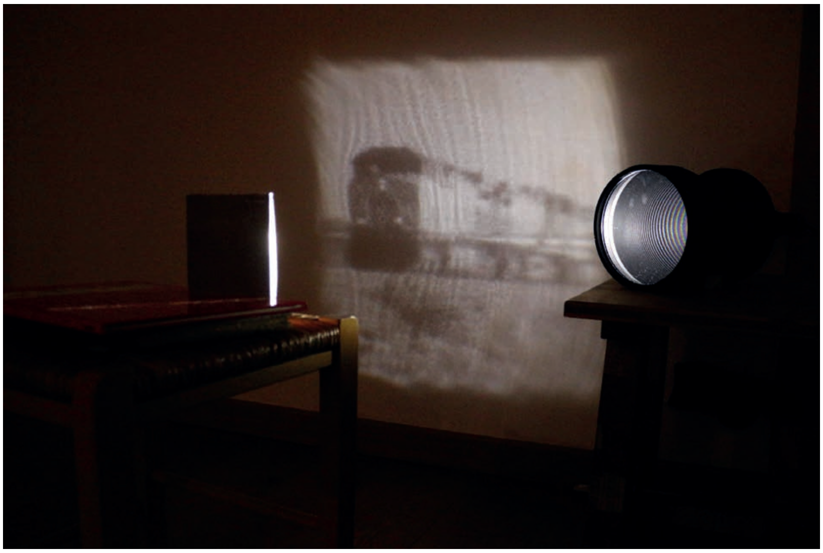


Figure 1. Mirror transforming a parallel, uniform light source into the shape of a train

Figure 2. Lens transforming a parallel light source into a hikari

in which the target lies "at infinity", in the space of directions. In practice, this means that when light is reflected or refracted from a point of the optical component, we can forget the spatial position of the reflected or refracted ray, keeping only its direction, which can then be encoded by a unit vector. Note that if the near-field target illumination is far away from the optical component, each point of the target almost corresponds to a direction, so that the far-field problem is a good approximation of the near-field one in this situation. We will see in Section 4.2 that one can solve a problem involving a near-field target by iteratively solving problems with far-field targets.

We will first present two *far-field mirror problems* in their continuous form, as illustrated in Figure 3. Then, we will explain how these continuous problems can be approached by discrete problems, following the so-called *supporting quadric* method introduced by Luis Caffarelli and Vladimir Oliker. This method can be traced back to work of Hermann Minkowski and Aleksandr Aleksandrov in convex geometry.

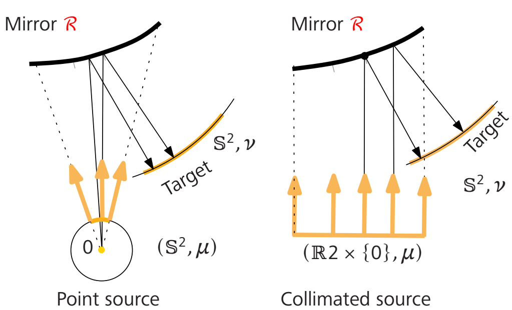


Figure 3. Mirror transforming light from a point source (left) or collimated light (right)

1.1 Mirror for a point light source

In this first problem, light is emitted from a point O, which we assume to be located at the origin of the space $\mathbb{R}^3$. The intensity of the light source is modeled by a probability density μ on the sphere of directions $\mathbb{S}^2$. Let $X \subset \mathbb{S}^2$ denote the support of the measure μ. For example, if the light is emitted in a solid cone, then X is a disc on the sphere. The quantity of light emanating from a measurable set of directions $A \subset X$ is given by $\mu(A)$. For the far-field problem, the target is described by a probability measure ν on the sphere of directions $\mathbb{S}^2$, which then represents the directions "at infinity", i.e., after reflection. Let $Y \subset \mathbb{S}^2$ denote the support of the measure ν.

The inverse problem considered here consists in constructing the surface $\mathcal{R}$ of a mirror which will transport the intensity μ of the light source to the desired light distribution ν at infinity using Snell's law of reflection. For example, if the target measure is a Dirac mass δ_y, meaning that we want to reflect all the light in a single direction y, then the shape of the mirror is given by a paraboloid of revolution.

Let $\langle \cdot | \cdot \rangle$ denote the Euclidean scalar product on $\mathbb{R}^3$. An incident ray $x \in \mathbb{S}^2$ is reflected by a surface $\mathcal{R}$ in the direction $R(x) = x - \langle x | n(x) \rangle n(x)$, where $n(x)$ is the unit vector normal to the surface $\mathcal{R}$ at the point touched by the direction x and oriented so that $\langle x | n(x) \rangle \leq 0$. The surface $\mathcal{R}$ solves the inverse mirror problem if R transports the source measure μ to the target measure ν, in the sense that for any measurable subset B of the sphere one has

$$\forall B \subseteq \mathbb{S}^2, \quad \nu(B) = \mu(R^{-1}(B)).$$

Note that the preservation of overall light quantity was already ensured by having chosen probability measures, i.e., $\mu(\mathbb{S}^2) = \nu(\mathbb{S}^2) = 1$. Now assume that μ and ν are absolutely continuous measures with respect to the area measure on the sphere. Let $\mu(x) = \rho(x)\mathrm{d}x$ and $\nu(x) = \sigma(x)\mathrm{d}x$, where ρ and σ are the densities of μ and ν respectively. The previous equation then reads

$$\forall B \subseteq \mathbb{S}^2, \quad \int_B \sigma(x)\mathrm{d}x = \int_{R^{-1}(B)} \rho(x)\mathrm{d}x. \tag{1}$$

Suppose furthermore that the densities ρ et σ are continuous and that R is a diffeomorphism from X to Y. By the change of variable $y = R(x)$, the last equation is then equivalent to $\sigma(R(x))\det(DR(x)) = \rho(x)$ for any $x \in X$.

Since the mirror reflects rays emitted from the origin, we will assume that the surface $\mathcal{R}$ is radially parametrized by $x \in \mathbb{S}^2 \mapsto u(x)x$, where $u : \mathbb{S}^2 \to \mathbb{R}^+$ is a positive function that must be determined. The unit normal to the surface $\mathcal{R}$ at the point $xu(x)$ and the direction of the reflected ray can both be expressed as a function of x and of the gradient $\nabla u(x) \in \mathrm{T}_x\mathbb{S}^2$:

$$R_u(x) = x - \langle x \,|\, n_u(x)\rangle n_u(x) \quad \text{and} \quad n_u(x) = \frac{\nabla u(x) - u(x)x}{\sqrt{\|\nabla u(x)\|^2 + u(x)^2}}.$$

This allows us to formulate the problem as a system of partial differential equations, i.e., the problem of finding a positive function $u : \mathbb{S}^2 \to \mathbb{R}^+$ of class $\mathcal{C}^2$ which satisfies

$$\begin{cases} \sigma(R_u(x))\det(DR_u(x)) = \rho(x), \\ R_u \text{ be a diffeomorphism from } X \text{ to } Y. \end{cases} \tag{Mir-Ponc-C}$$

The first line of equation (Mir-Ponc-C) involves the determinant of a quantity which depends on the second derivatives of u. This equation belongs to the family of Monge–Ampère equations. Note that the requirement that R_u is a diffeomorphism is non-standard and difficult to handle. In practice, it is replaced by a condition on u which is akin to convexity, and by the so-called second boundary condition $R_u(X) = Y$. These two conditions ensure the ellipticity of the problem. Caffarelli and Oliker proved in 1994 [1] the existence of weak solutions to this equation, i.e., the existence of a locally Lipschitz function u such that the application R defined by the last two lines of (Mir-Ponc-C) satisfies (1). The existence of regular solutions to the problem (Mir-Ponc-C) is due to Wang and Guan [2, 6].

1.2 Mirror for a collimated light source

We now present a second inverse problem arising in anidolic optics. This time the light source is collimated, which means that all the rays of light emitted by the source are parallel. We furthermore assume that they are positively collinear to the vertical vector $e_z = (0,0,1)$ and emitted from a domain of the horizontal plane $X \subset \mathbb{R}^2 \times \{0\}$. For convenience, we will identify $\mathbb{R}^2$ and $\mathbb{R}^2 \times \{0\}$. We assume that the surface of the optical component is smooth and parametrized by a height function $u : X \to \mathbb{R}$. The intensity of the light source is modeled by a probability measure μ on X. As in the previous case, the intensity of the target illumination is modeled by a probability measure ν on the sphere of directions at infinity. At each point $(x, u(x))$ of the optical component, the gradient $\nabla u(x)$ encodes the direction of the normal to the surface and we denote by $F(\nabla u(x)) \in \mathbb{S}^2$ the direction of the ray reflected by Snell's law. The reflector defined by u solves the inverse mirror problem between μ and ν if for any measurable set $B \subset \mathbb{S}^2$ one has

$$\nu(B) = \mu\big((F \circ \nabla u)^{-1}(B)\big).$$

Let us introduce the measure $\tilde{\nu}$ defined by $\tilde{\nu}(B) = \nu(F(B))$, which is supported on $\mathbb{R}^2$. We assume that μ and $\tilde{\nu}$ are absolutely continuous with respect to the Lebesgue measure, with continuous densities ρ and σ, and that $x \mapsto \nabla u(x)$ is a diffeomorphism on its image. Then, with the change of variable $y = \nabla u(x)$, the inverse mirror problem for a collimated source can also be phrased as a partial differential equation:

$$\begin{cases} \sigma(\nabla u(x))\det(D^2 u(x)) = \rho(x), \\ F \circ \nabla u \text{ is a diffeomorphism from } X \text{ to } Y. \end{cases} \tag{Mir-Coll-C}$$

We finally note that if u is smooth and strongly convex (or strongly concave), the application ∇u is a diffeomorphism on its image.

1.3 Lenses

The construction of lenses that transform a light source into a target illumination prescribed at infinity are similar and are also described by Monge–Ampère type equations. As with mirrors, when the light is emitted from a point, the equation to be solved is on the sphere and when the light source is collimated, it is on the plane. In these problems, the light source passes through the surface of one side of the lens, either flat or spherical, and the aim is to construct the surface of the other side of the lens such that it refracts the light onto a prescribed target illumination at infinity. We do not detail the modelling of these problems here, but we will show results with lenses at the end.

2 Geometric discretization of Monge–Ampère equations

The two inverse problems in anidolic optics described in the previous section each involve two sets X and Y, on which we have two probability measures μ and ν respectively, representing the light source and the desired target illumination. We saw that when these measures are absolutely continuous, the problems of construction of optical components correspond to partial differential equations of Monge–Ampère type.

The most direct method to solve a partial differential equation numerically is to approximate the domain X with a discrete grid and to replace the partial derivatives with differences of the values of the function at the points of the grid divided by the grid step. In the case of Monge–Ampère equations, the application of these methods is made difficult by the non-linearity of the Monge–Ampère operator and by the diffeomorphism condition. We refer to the work of Adam Oberman, Brittany Froese, Jean-David Benamou and Jean-Marie Mirebeau for this line of research.

In recent years, alternative methods, called *semi-discrete* methods, have been used to discretize and numerically solve Monge–Ampère type equations arising from optimal transport. In order to apply this method, one assumes that one of the two measures μ or ν is absolutely continuous, while the other is finitely supported. Here we assume that μ has a density $\mu(x) = \rho(x)\mathrm{d}x$ on the space X and that ν is a discrete measure on the space $Y = \{y_1, \dots, y_N\}$, i.e., $\nu = \sum_{1 \le i \le N} \delta_{y_i} \nu_i$ where δ_{y_i} is the Dirac mass in y_i.

In this section, we describe the semi-discrete variant of the two far-field mirror problems seen in the previous section, leaving aside the problem of convergence of the solutions to the discrete problems towards those to the continuous problems. These constructions give rise to equations which can naturally be seen as *discrete Monge–Ampère equations*. We also propose an economic interpretation by addressing the *bakeries problem*.

2.1 Mirror for a point light source

Let us go back to the problem of constructing mirrors that transform the light emitted by a point light source (see Section 1.1). As in the previous section, the light source is modeled by a continuous probability density ρ on the sphere of directions $\mathbb{S}^2$, whose support $X_\rho := \{x \in \mathbb{S}^2,\ \rho(x) > 0\}$ corresponds to the set of directions in which light is emitted. This time we assume that the desired target illumination is described by a probability measure $\nu = \sum_{1 \le i \le N} \delta_{y_i} \nu_i$ supported on a set $Y = \{y_1, \dots, y_N\} \subset \mathbb{S}^2$ of distinct directions. The problem is still to find the mirror surface $\mathcal{R}$ that will reflect the measure μ onto the measure ν under Snell's law, but this time the target measure ν is discrete.

Mirror composed of paraboloid pieces

We use the method of *supporting paraboloids* proposed by Caffarelli and Oliker in 1994 [1], which was originally developed to show the existence of weak solutions in the case where both measures are absolutely continuous. Caffarelli and Oliker's idea is based on a well-known property of paraboloids of revolution: a paraboloid of revolution with focal point O and direction y reflects any ray coming from point O to the direction y. It is thus natural to seek to construct a mirror whose surface is composed of pieces of paraboloids, each paraboloid illuminating a direction y_i.

More precisely, we take $\psi = (\psi_1, \dots, \psi_N) \in \mathbb{R}^N$ and denote by $P(y_i, \psi_i)$ the solid (i.e. filled) paraboloid of direction y_i, with focal point at the origin O and focal distance ψ_i. This means that $\frac{1}{2}\psi_i$ is the distance between O and the paraboloid's closest point to O.

We define by $\mathcal{R}_\psi$ the surface bordering the intersection of the solid paraboloids $P(y_i, \psi_i)$:

$$\mathcal{R}_\psi = \partial\Bigg(\bigcap_{1 \le i \le N} P(y_i, \psi_i)\Bigg).$$

For each $i \in \{1, \dots, N\}$ we denote by $\mathrm{V}_i(\psi)$ the set of rays $x \in \mathbb{S}^2$ emitted by the light source and reflected by Snell's law in the direction y_i. This set is called the i-th *visibility cell of the mirror* $\mathcal{R}_\psi$. By construction, it corresponds to the radial projection of $\mathcal{R}_\psi \cap \partial P(y_i, \psi_i)$ onto the sphere (see Figure 4).

A simple calculation shows that the intersection of two confocal paraboloids $\partial P(y_i, \psi_i)$ and $\partial P(y_j, \psi_j)$ is included in a plane curve. Projecting radially onto the unit sphere, this implies that the intersection of two visibility cells $\mathrm{V}_i(\psi) \cap \mathrm{V}_j(\psi)$ is included in a curve on the sphere. We deduce that the set of visibility cells forms a partition of the sphere $\mathbb{S}^2$, up to a set of measure zero.

The paraboloid of revolution $\partial P(y_k, \psi_k)$ can be parametrized radially by the function $x \in \mathbb{S}^2 \mapsto x\rho_k(x)$, where $\rho_k(x) = \psi_k/(1 - \langle x | y_i \rangle) \in \mathbb{R}$. We deduce that x belongs to the visibility cell $\mathrm{V}_i(\psi)$ if and only if the distance $\rho_i(x)$ is smaller than the distances $\rho_j(x)$ for $j \in \{1, \dots, N\}$. Composing with the logarithm to linearize the expression in ψ, we obtain an explicit expression for the visibility cells

$$\mathrm{V}_i(\psi) = \Big\{x \in \mathbb{S}^2 \mid \forall j,\ c(x, y_i) + \ln(\psi_i) \le c(x, y_j) + \ln(\psi_j)\Big\},$$

where $c(x, y) = -\ln(1 - \langle x | y \rangle)$.

By construction, each ray emitted by the point source and belonging to the cell $\mathrm{V}_i(\psi)$ hits the mirror $\mathcal{R}_\psi$ at a point which belongs to the paraboloid $\partial P(y_i, \psi_i)$ and which is reflected in the direction y_i. The quantity of light received in the direction y_i is therefore exactly the quantity of light emanating from the visibility cell $\mathrm{V}_i(\psi)$, i.e., $\mu(\mathrm{V}_i(\psi))$. The desired quantity of light in the direction y_i is ν_i. The equation to be solved is therefore $\mu(\mathrm{V}_i(\psi)) = \nu_i$ for any $i \in \{1, \dots, N\}$. Moreover, note that a paraboloid of revolution is only determined by its focal point, its direction and its focal distance. The free parameter remaining for each paraboloid $\partial P(y_i, \psi_i)$ is the focal distance ψ_i.

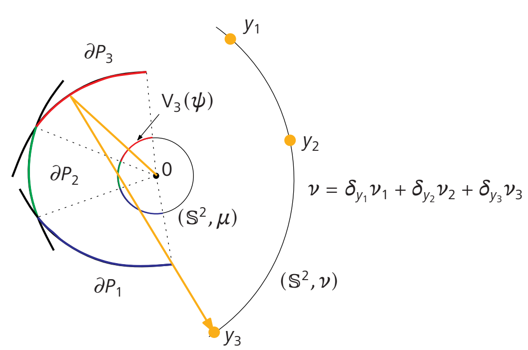


Figure 4. Mirror composed of three pieces of paraboloides reflecting in three directions

Formulation of the problem
The semi-discrete far-field mirror problem for a point source can be formulated as the problem of finding focal distances $\psi = (\psi_1, \dots, \psi_N) \in \mathbb{R}^N$ that satisfy

$$\forall i, \quad \mu(\mathrm{V}_i(\psi)) = \nu_i \qquad \text{(Mir-Ponc-SD)}$$

where $c(x,y) = -\ln(1 - \langle x \,|\, y\rangle)$ and where

$$\mathrm{V}_i(\psi) = \Big\{ x \mid \forall j,\ c(x,y_i) + \ln(\psi_i) \le c(x,y_j) + \ln(\psi_j) \Big\}.$$

We will see in Section 3 how to solve such systems of equations. Note that if $\psi \in \mathbb{R}^N$ is a vector of focal distances solving the mirror problem for a point-like source, then the surface of the mirror is parametrized by

$$\mathcal{R}_\psi : x \in \mathbb{S}^2 \mapsto \min_i \frac{\psi_i}{1 - \langle x \,|\, y_i\rangle} x.$$

In the numerical experiments, we assume that the target illumination ν is included in the half-sphere $\mathbb{S}^2_- := \{x \in \mathbb{S}^2,\ \langle x \,|\, e_z\rangle \le 0\}$, that the support X_ρ of ρ is included in the half-sphere $\mathbb{S}^2_+ := \{x \in \mathbb{S}^2,\ \langle x \,|\, e_z\rangle \ge 0\}$, and that the mirror is parametrized above the domain X_ρ.

Remark 2.1. The mirror surface is by construction the boundary of a convex set, i.e., the intersection of the solid paraboloids $P(y_1, \psi_1), \dots, P(y_N, \psi_N)$. It is also possible to construct a mirror contained in the boundary of the union of solid paraboloids rather than an intersection. This produces mirrors that are somewhat less interesting in practice, as they are neither convex nor concave.

2.2 Mirror for a collimated light source

Let us now consider the mirror problem for a collimated light source, already seen in Section 1.2. As before, the probability measure modelling the light source has a density ρ with respect to the Lebesgue measure on the plane. However the probability measure modelling the target illumination intensity is discrete $\nu = \sum_{1 \le i \le N} \delta_{y_i} \nu_i$, supported on a finite set $Y = \{y_1, \dots, y_N\} \subset \mathbb{S}^2$ of distinct directions. The problem is, again, to find the surface $\mathcal{R}$ of a mirror which reflects the measure μ to the measure ν.

Mirror with planar faces
We choose to construct the mirror surface $\mathcal{R}$ as the graph of affine height functions of the form $x \in \mathbb{R}^2 \mapsto \max_i \langle x \,|\, p_i\rangle - \psi_i$ (see Figure 5). The vector p_i is chosen so that the plane $P_i = \{(x, \langle x \,|\, p_i\rangle) \mid x \in \mathbb{R}^2\}$ reflects vertical rays, i.e., with direction e_z, into the direction $y_i \in \mathbb{S}^2$. We need to determine the heights ψ_i of those planes. Given a family of heights $\psi \in \mathbb{R}^N$, we define the i-th visibility cell as

$$\mathrm{V}_i(\psi) = \{x \in \mathbb{R}^2 \times \{0\} \mid \forall j,\ -\langle x \,|\, p_i\rangle + \psi_i \le -\langle x \,|\, p_j\rangle + \psi_j\}.$$

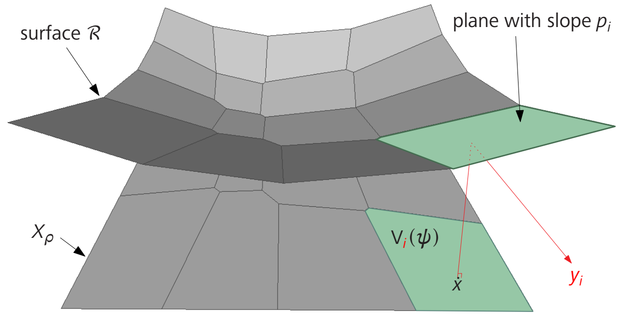


Figure 5. Convex mirror for a collimated light source

By construction, for each $i \in \{1, \dots, N\}$, any vertical ray emitted from a point $x \in V_i(\psi)$ hits the mirror $\mathcal{R}$ at a height $\langle x \,|\, p_i\rangle - \psi_i$ and is reflected in the direction y_i. Thus, the amount of light reflected in the direction y_i is equal to $\mu(\mathrm{V}_i(\psi))$.

Formulation of the problem
Solving the semi-discrete far-field mirror problem for a collimated light source amounts to finding the heights $\psi \in \mathbb{R}^N$ that satisfies

$$\forall i, \quad \mu(\mathrm{V}_i(\psi)) = \nu_i \qquad \text{(Mir-Colli-SD)}$$

where $c(x,y) = -\langle x \,|\, y\rangle$ and

$$\mathrm{V}_i(\psi) = \Big\{ x \mid \forall j,\ c(x,y_i) + \psi_i \le c(x,y_j) + \psi_j \Big\}.$$

A solution of the equation (Mir-Colli-SD) induces a parametrization of the convex mirror $\mathcal{R}$ that reflects μ onto ν:

$$\mathcal{R}_\psi : x \in \mathbb{R}^2 \mapsto \big(x, \max_i \langle x \,|\, p_i\rangle - \psi_i\big) \in \mathbb{R}^3.$$

In practice, we only consider the part of the mirror located above the domain $X_\rho := \{x \in \mathbb{R}^2 \times \{0\},\ \rho(x) \neq 0\}$.

Remark 2.2. The function $\mathcal{R}_\psi$ being the maximum of affine functions, it is convex. The optical component which is parametrized by the graph of this application is also convex. Note that one could have the same construction by replacing the max in the formula by a min. This would result in a concave function $\mathcal{R}_\psi$ and a concave mirror.

Remark 2.3. Problem (Mir-Colli-SD) is very similar (but not equivalent) to Minkowski's problem in convex geometry which is also an inverse problem. Given a set of unit vectors y_i and real numbers $\nu_i > 0$, this problem consists in building a convex polyhedron whose i-th facet has normal y_i and area ν_i – which is possible only under some assumptions on the directions and areas. We also note that Oliker, who was the first to introduce semi-discrete methods for the numerical resolution of Monge–Ampère equations, was a doctoral student of the famous geometer Aleksandrov who is known (among other) for introducing and studying the "continuous" formulation of Minkowski's problem.

2.3 The bakeries problem

We now present an economic analogy which leads to an equation having the same structure as in the two optical problems presented above. We assume that X represents a city whose population density is described by a probability density ρ, that the finite set $Y = \{y_1, \ldots, y_N\}$ represents the locations of the city's bakeries and that ν_i represents the quantity of bread available in bakery y_i. Customers living at a location x in X naturally will look for the bakery minimizing the cost of walking from x to y_i, denoted $c(x, y_i)$. This leads to a decomposition of the city space into *Voronoi cells*,

$$\mathrm{Vor}_i := \{x \in \Omega_X \mid \forall j,\ c(x,y_i) \le c(x,y_j)\}.$$

The number of customers going to a bakery y_i is equal to the integral of the density ρ over Vor_i. Suppose that a bakery y_i receives too many customers in comparison to its bread's production capacity ν_i – this could be the case in Figure 6 for the downtown bakery y_1 where the population density is high. This means we have $\mu(\mathrm{Vor}_1) > \nu_1$, where we denote $\mu(x) = \rho(x)\mathrm{d}x$. The baker y_1 will then seek to increase the price of his bread. This will reduce the number of potential customers, but will increase the baker's profit as long as he manages to sell all his stock. We write $\nu_i \ge 0$ for the proportion of the population that the bakery y_i is able to serve, and ψ_i the price of the bread in the bakery y_i. If we assume that the customers living at point x make a compromise between walking cost and price of bread by minimizing the sum $(c(x, y_i) + \psi_i)$, the city is then decomposed into *Laguerre cells*

$$\mathrm{Lag}_i(\psi) = \Big\{x \in X \mid \ \forall j,\ c(x,y_i) + \psi_i \le c(x,y_j) + \psi_j\Big\}.$$

Note that we do not necessarily have $y_i \in \mathrm{Lag}_i(\psi)$, and that it is even possible to have $\mathrm{Lag}_i(\psi) = \varnothing$: indeed, if the bread is very expensive in a certain bakery, even people living next door may prefer going to a more distant one.

Problem formulation

The bakeries problem therefore boils down to finding a price vector $\psi \in \mathbb{R}^N$ such that each bakery sells all its stock of bread ν_i. This is described by the system of equations

$$\mu(\mathrm{Lag}_i(\psi)) = \nu_i \quad \forall i \in \{1, \ldots, N\},$$

This equation has exactly the same structure as (Mir-Ponc-SD) and (Mir-Colli-SD). We will see in the next section how to solve this class of equations.

3 Numerical resolution

The discrete problems mentioned in the previous section all show the same structure; our focus will now be on their numerical resolution. We start by introducing the semi-discrete Monge–Ampère equation, and show that its solution is equivalent to finding the maximum of a concave function. Subsequently, we present a Newton method that allows us to solve these equations efficiently.

3.1 Semi-discrete Monge–Ampère equation

Let X be a compact subset of the space $\mathbb{R}^2$ or of the sphere $\mathbb{S}^2$, let $Y = \{y_1, \ldots, y_N\}$, and let $c \in \mathcal{C}^1(X \times Y)$ be a cost function. The *Laguerre cell* (which corresponds to a visibility cell in optics) associated with a family of real numbers $\psi = (\psi_1, \ldots, \psi_N) \in \mathbb{R}^N$ is given by

$$\mathrm{Lag}_i(\psi) = \Big\{x \in X \mid \ \forall j,\ c(x,y_i) + \psi_i \le c(x,y_j) + \psi_j\Big\}.$$

Suppose that the cost function satisfies the *Twist condition*

$$\forall x \in X, \quad y \mapsto \nabla_x c(x,y) \text{ is injective,} \qquad \text{(Twist)}$$

which ensures that the Laguerre cells form a partition of the domain X up to a negligible set.

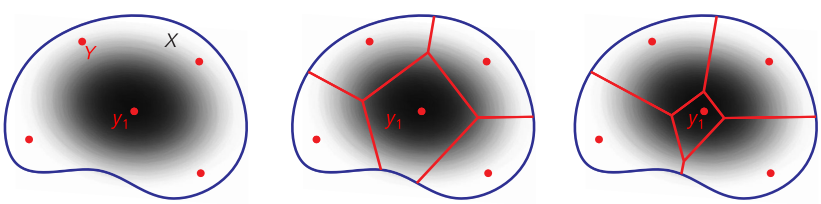


Figure 6. Bakeries: The city X with its boundary drawn in blue is endowed with a probability density pictured in grayscale representing the population density. The set Y (in red) represents the location of bakeries. Here, $X, Y \subseteq \mathbb{R}^2$ and $c(x,y) = |x-y|^2$. We see the Voronoi tessellation of the city (in the middle, uniform price) as well as its Laguerre tessellation (on the right, only the bread price ψ_1 has increased).

Semi-discrete Monge–Ampère equation

Let μ be a probability measure on X with density ρ with respect to the area measure, and let $\nu = \sum_i \nu_i \delta_{y_i}$ be a probability measure on Y. In the following equation, the discrete probability measure ν is conflated with the vector $\nu = (\nu_i)_{1 \leq i \leq N}$. We are seeking $\psi \in \mathbb{R}^N$ satisfying

$$G(\psi) = \nu, \tag{MA}$$

where the function $G : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$G(\psi) = \big(G_1(\psi), \ldots, G_N(\psi)\big) \quad \text{and} \quad G_i(\psi) = \mu\big(\mathrm{Lag}_i(\psi)\big).$$

Remark 3.1. The visibility cells used in optics in (Mir-Ponc-SD) and (Mir-Colli-SD) are Laguerre cells, with

$$c(x,y) = -\log\big(1 - \langle x \,|\, y\rangle\big) \quad \text{and} \quad c(x,y) = -\langle x \,|\, y\rangle$$

respectively. Equation (MA) is a reformulation of equations (Mir-Ponc-SD) and (Mir-Colli-SD). Note that the Laguerre cells are invariant under addition of a constant to ψ, and that the solution of (MA) is therefore defined up to an additive constant. Optical problems have a similar invariance: for example, if a surface $\mathcal{R}$ is a solution of the mirror problem for a point source, then so is $\lambda\mathcal{R}$ for all $\lambda > 0$.

3.2 Variational formulation

The following theorem shows that the function G in the semi-discrete Monge–Ampère equation is the gradient of a concave function.

Theorem 3.1. *We assume that the cost function c satisfies* (Twist)*. Then the function $\mathcal{K} : \mathbb{R}^N \to \mathbb{R}$ defined by*

$$\mathcal{K}(\psi) = \sum_{1 \leq i \leq N} \int_{\mathrm{Lag}_i(\psi)} \big(c(x,y) + \psi_i\big)\rho(x)\,\mathrm{d}x - \sum_{1 \leq i \leq N} \psi_i \nu_i$$

is concave, of class $\mathcal{C}^1$ and with gradient

$$\nabla\mathcal{K}(\psi) = G(\psi) - \nu = \big(\mu\big(\mathrm{Lag}_i(\psi)\big) - \nu_i\big)_{1 \leq i \leq N}.$$

As we will see in the next paragraph, the function $\mathcal{K}$ is related to the Kantorovitch duality in optimal transport theory, and we will therefore call it the *Kantorovitch functional*. Moreover, since a concave function of class $\mathcal{C}^1$ reaches its maximum exactly at its critical points, we obtain the following corollary:

Corollary 3.2. *Under the assumptions of Theorem* 3.1*, a vector $\psi \in \mathbb{R}^N$ is a solution to equation* (MA) *if and only if ψ is a maximizer of $\mathcal{K}$.*

Since the function $\mathcal{K}$ is invariant under addition of a constant, one can choose to work on the set $\mathcal{M}_0$ of vectors whose coordinates sum to zero. It can be shown that the function $\mathcal{K}$ is proper on $\mathcal{M}_0$, i.e., $\lim_{\|\psi\| \to +\infty, \psi \in \mathcal{M}_0} \mathcal{K}(\psi) = -\infty$, which ensures that it reaches its maximum: the problem (MA) thus admits a solution.

3.3 Relation to optimal transport

The variational formulation of the Monge–Ampère equation, i.e., the search for a maximizer of the Kantorovitch functional, corresponds in fact to the dual of the Monge–Kantorovitch problem in optimal transport theory. We discuss this link in detail below in the semi-discrete case. The reader interested in the proofs may refer for instance to the book chapter [4].

Monge's problem

The image of a probability measure μ on X under a measurable application $T : X \to Y$ is the measure $T_\#\mu$ on Y defined by $T_\#\mu(B) = \mu(T^{-1}(B))$. If $T_\#\mu = \nu$, we say that T *transports* μ to ν. Since the set Y is finite, we have $T_\#\mu = \sum_{1 \leq i \leq N} \mu(T^{-1}(y_i))\delta_{y_i}$. Monge's optimal transport problem consists in finding a transport map T that transports μ to ν and that minimizes the total cost $\int_X c(x, T(x))d\mu(x)$. If the cost function c satisfies the Twist condition, Brenier and Gangbo–McCann, relying on Kantorovich duality, proved the existence of a minimizer for this problem when the source μ is absolutely continuous. For example, one can state the following:

Theorem 3.3 (Kantorovitch duality)**.** *Suppose that c satisfies the condition* (Twist) *and that μ is absolutely continuous. Then*

$$\min_{\substack{T:X\to Y\\ T_\#\mu=\nu}} \int_X c\big(x, T(x)\big)d\mu(x) = \max_{\psi \in \mathbb{R}^N} \mathcal{K}(\psi).$$

If moreover ψ is a maximizer of $\mathcal{K}$, then the function $T_\psi : X \to Y$ defined μ-a.e. by $T_\psi|_{\mathrm{Lag}_y(\psi)} = y$ realizes the minimum in Monge's problem.

Remark 3.2. Not all Monge–Ampère equations derive from an optimal transport problem and not all of them admit a variational formulations. These two strong properties come in fact from the very particular structure of Laguerre cells, which follows from the functions $\psi \mapsto c(x,y) + \psi(y)$ being affine.

We saw that the far-field optics problems presented in Section 2 possess this structure. On the other hand, if we consider the mirror construction problems for a target illumination in the near-field (i.e., we are illuminating points in $\mathbb{R}^3$ and not directions at infinity), we still have semi-discrete Monge–Ampère equations to solve, but the Laguerre cells are of the form

$$\mathrm{Lag}_i(\psi) = \{x \in X \mid \forall j,\ G(x, y_i, \psi_i) \leq G(x, y_j, \psi_j)\},$$

where the function G is nonlinear in ψ. These equations do not derive from the optimal transport problem and in fact do not admit a variational formulation. They are called *prescribed Jacobian equations* by Trudinger, and are the subject of recent research both in analysis and in more applied fields (optics, economics).

3.4 *Laguerre cells and derivatives*

Before applying Newton's method to solve the equation $G(\psi) = \nu$, we need to show that the function G is of class $\mathcal{C}^1$ (or equivalently that $\mathcal{K}$ is of class $\mathcal{C}^2$), calculate its partial derivatives and study the (strict) concavity of $\mathcal{K}$. To do this, we need a genericity assumption which is somewhat technical, but which is natural and not restrictive in practice. In the optical cases mentioned in this paper, this assumption is satisfied if the intersection of three distinct Laguerre cells is finite and if the intersection of two Laguerre cells with the boundary of X is also finite. For more details, the reader may refer to the book chapter [4].

Theorem 3.4 (Differental of G). *Suppose that the cost satisfies* (Twist)*, that Y is generic (see above), and ρ is continuous. Then the application $G : \mathbb{R}^N \to \mathbb{R}^N$ is of class $\mathcal{C}^1$ and*

$$\forall j \neq i, \quad \frac{\partial G_i}{\partial \psi_j}(\psi) = \int_{\mathrm{Lag}_{ij}(\psi)} \frac{\rho(x)}{\|\nabla_x c(x, y_i) - \nabla_x c(x, y_j)\|} \mathrm{d}x,$$

$$\forall i, \quad \frac{\partial G_i}{\partial \psi_i}(\psi) = -\sum_{j \neq i} \frac{\partial G_i}{\partial \psi_j}(\psi),$$

where $\mathrm{Lag}_{ij}(\psi) = \mathrm{Lag}_i(\psi) \cap \mathrm{Lag}_j(\psi)$.

The formula for the partial derivatives of G has a geometric interpretation. In the following two figures, which are obtained for the cost $c(x, y) = \|x - y\|^2$ on $\mathbb{R}^2$, we explain why the formula for partial derivatives involves integrals over the interfaces between Laguerre cells and how the singularities of DG may occur depending on the geometry of the points y_i.

Figure 7 illustrates that the partial derivative $\partial G_i / \partial \psi_j(\psi)$ is an integral over the interface $\mathrm{Lag}_{ij}(\psi)$: the value $G_i(\psi)$ is an integral over the Laguerre cell $\mathrm{Lag}_i(\psi)$ (in grey on the left); we increase the value ψ_j by $\varepsilon > 0$ considering $\psi + \varepsilon e_j$; the rate of increase $(G_i(\psi) - G_i(\psi + \varepsilon e_j))/\varepsilon$ is proportional to an integral over the symmetric difference between two Laguerre cells (in grey in the middle); passing to the limit we obtain an integral over the green segment $\mathrm{Lag}_{ij}(\psi)$. The signs that occurs in the formula for the partial derivatives can also be interpreted with the bakeries metaphor: when the price of bread ψ_i increases, the number of customers of the bakery y_i decreases (i.e., the Laguerre cell $\mathrm{Lag}_i(\psi)$ shrinks) and the number of customers for the other bakeries increases, so that $\partial G_i / \partial \psi_i(\psi) \leq 0$ and $\partial G_i / \partial \psi_j(\psi) \geq 0$ for $j \neq i$.

In Figure 8, the genericity condition is not satisfied because y_1, y_2 and y_3 are aligned, and there exists $\psi \in \mathbb{R}^N$ for which $\mathrm{Lag}_1(\psi) \cap \mathrm{Lag}_2(\psi) \cap \mathrm{Lag}_3(\psi)$ is a line segment. The partial derivative $\partial G_2 / \partial \psi_3(\psi)$ is an integral on the (green) segment $\mathrm{Lag}_{23}(\psi)$. If we simultaneously decrease ψ_1 and ψ_2 by the same amount, we can see that the segment $\mathrm{Lag}_{23}(\psi)$ varies continuously and then suddenly becomes empty when the cell $\mathrm{Lag}_2(\psi)$ gets empty (bottom right of Figure 8). Thus, $\partial G_2 / \partial \psi_3(\psi)$ is not continuous. Newton's method requires a certain regularity, and we will see below that it converges under the above genericity assumptions.

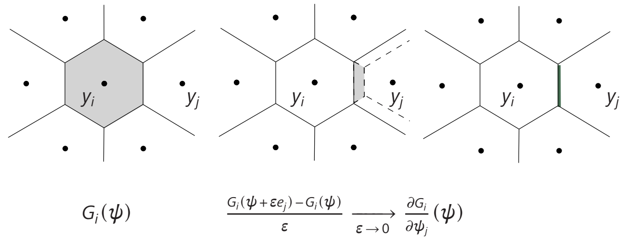


Figure 7. The partial derivatives are boundary integrals

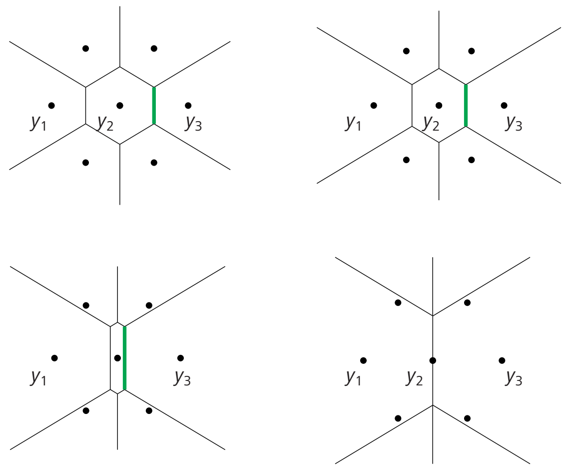


Figure 8. Non-continuous partial derivative: $\partial G_2 / \partial \psi_3$ is an integral on the green segment Lag_{23} which is discontinuous.

To establish the convergence of Newton's method, we also need to study the concavity of the Kantorovitch functional $\mathcal{K}$, or equivalently the monotonicity of its gradient $\nabla \mathcal{K} = G - \nu$. The functions $\mathcal{K}$ and G are invariant by addition of a constant vector (i.e., $\mathcal{K}(\psi + C(1, \ldots, 1)) = \mathcal{K}(\psi)$), which can be seen in the definition of Laguerre cells. Thus, we can only hope to establish strong concavity of $\mathcal{K}$ in the directions orthogonal to constant vectors, i.e., belonging to the set

$$\mathcal{M}_0 := \Big\{ v \in \mathbb{R}^N \mid \sum_{1 \leq i \leq N} v_i = 0 \Big\}.$$

Another reason for the lack of strong concavity of $\mathcal{K}$ is that if ψ_i is very large, then $\mathrm{Lag}_i(\psi)$ is empty and remains empty in a neighborhood of ψ. In this case, $G_i(\psi)$ is constant equal to zero, and the Hessian matrix $D^2 G(\psi) = DG$ has a row of zeros. We can therefore hope to establish strong concavity only if ψ belongs to the set

$$\mathcal{C}_+ := \{ \psi \in \mathbb{R}^N \mid \forall i,\ G_i(\psi) > 0 \}.$$

The next theorem shows, in a nutshell, that these are the only two obstructions to strong concavity.

Theorem 3.5 (Strict concavity). *We assume the hypotheses of the previous theorem hold. If the set* $\{\rho > 0\}$ *is connected, the function* $\mathcal{K}$ *is locally strongly concave on* $\mathcal{C}_+$ *in the direction* $\mathcal{M}_0$*:*

$$\forall \psi \in \mathcal{C}_+,\ \forall v \in \mathcal{M}_0 \setminus \{0\}, \quad \langle DG(\psi)v \,|\, v\rangle < 0.$$

Remark 3.3 (Uniqueness). We saw above that the function $\mathcal{K}$ has a maximum, and thus equation (MA) has a solution. The previous theorem implies that this maximum is unique if we impose that $\psi \in \mathcal{M}_0$, i.e., ψ has zero average, since a strictly concave function admits at most one local maximum.

We will see in the next paragraph how these results of regularity and monotonicity allow us to iteratively construct a sequence $(\psi^{(k)})_{k\geq 0}$ converging to the unique zero-average ψ^* satisfying $G(\psi^*) = \nu$.

3.5 Newton's method

Newton's method in 1D

We begin by recalling Newton's method for solving the equation $g(x) = 0$, where $g : \mathbb{R} \to \mathbb{R}$ is a real function. Newton's method starts from $x_0 \in \mathbb{R}$ and constructs the sequence $x^{k+1} = x^k - g(x^k)/g'(x^k)$ by induction. If we assume that g is of class $\mathcal{C}^1$ and that there exists $a \in \mathbb{R}$ such that $g(a) = 0$ and $g'(a) \neq 0$, then one can show, using Taylor–Lagrange formulas, that for x^0 sufficiently close to a, the sequence $(x^k)_{k\geq 0}$ converges to a. The convergence is then said to be local. Thus, under a regularity hypothesis ($g \in \mathcal{C}^1$) and monotonicity (g' has constant sign in a neighborhood of a), Newton's method converges locally.

Newton's method (local)

Assume that we are given a zero-average vector $\psi^0 \in \mathcal{M}_0$ such that the mass of all Laguerre cells is strictly positive:

$$\varepsilon_0 := \frac{1}{2} \min \left[\min_{y\in Y} G_i(\psi^0),\ \min_{1\leq i\leq N} \nu_{y_i} \right] > 0.$$

We define ψ^{k+1} in the following way: we start by calculating the Newton direction d^k, i.e., the vector d^k satisfying

$$DG(\psi^k)d^k = -(G(\psi^k) - \nu) \quad \text{and} \quad d_i^k \in \mathcal{M}_0,$$

which exists and is unique by according to Theorem 3.5. The second equation enables us to overcome the invariance of G and thus the non-invertibility of $DG(\psi^k)$. We then define $\psi^{k+1} = \psi^k + d^k$. As in the 1D case, it can be shown that the method converges locally: if ψ^0 is chosen close enough to the ψ^* solution, then the sequence (ψ^k) converges to ψ^*.

Globally convergent Newton's method

However, the condition ψ^0 is close to the solution ψ^* is impossible to fulfill in practice. Fortunately, a very simple modification of the method allows to ensure a global convergence, allowing us to drop this closeness assumption. To do this, one must construct ψ^{k+1} in such a way that the kernel of the Jacobian $DG(\psi^{k+1})$ remains equal to constant vectors, so that the system defining the direction d^{k+1} admits a unique solution. For this purpose, we define the *step* τ^k as the largest real of the form $2^{-\ell}$ (with $\ell \in \mathbb{N}$) such that $\psi^{k,\ell} := \psi^k + 2^{-\ell} d^k$ satisfies

$$\begin{cases} \forall i \in \{1,\dots,N\}, \quad G_i(\psi^{k,\ell}) \geq \varepsilon_0, \\ \left\| G(\psi^{k,\ell}) - \nu \right\| \leq (1 - 2^{-(\ell+1)}) \left\| G(\psi^k) - \nu \right\|. \end{cases}$$

Finally, we define $\psi^{k+1} = \psi^k + \tau^k d^k$.

By using the regularity and concavity results on $\mathcal{K}$, the step τ^k can be bounded from below, thus ensuring the convergence of the sequence constructed above to a solution of the optimal transport problem [4]:

Theorem 3.6. *Under the assumptions of Theorem* 3.5*, there exists* $\tau^* > 0$ *such that*

$$\left\| G(\psi^{k+1}) - \nu \right\| \leq \left(1 - \frac{\tau^\star}{2}\right) \left\| G(\psi^k) - \nu \right\|.$$

In particular, the sequence $(\psi^k)_{k\geq 0}$ *converges to the unique solution* ψ^* *of* (MA) *satisfying* $\sum_i \psi_i^* = 0$*.*

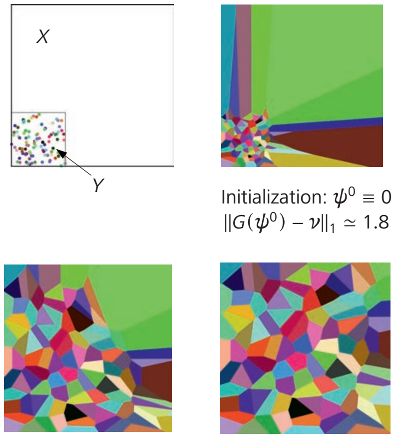
Remark 3.4 (Quadratic convergence). The above theorem shows that the convergence of Newton's method is globally exponential. This convergence is actually called *linear convergence* in optimization. When the cost c satisfies the *Ma–Trudinger–Wang* (MTW) condition that appears in the theory of optimal transport regularity, and the density ρ is Lipschitz, then the convergence is even locally quadratic [3]: for sufficiently large k, we have

$$\left\| G(\psi^{k+1}) - \nu \right\| \leq \frac{1}{2} \left\| G(\psi^k) - \nu \right\|^2.$$

In practice, the convergence is very fast and the basin where quadratic convergence occurs seems to be quite large. This last observation is empirical, and not mathematically explained yet. In Figure 9, $X = [0,1]^2$ is the large white square and Y is a set of points in the lower left corner and $c(x,y) = \|x-y\|^2$. With $N = 100$ points, after three iterations the error $\|G(\psi^3) - \nu\|_1$ is already of order 10^{-9}. Even difficult examples of size $N = 10^7$ in dimension $d = 3$ can be solved to high numerical precision with less than 20 iterations!

4 Applications to anidolic optics

In this section we present the adaptation of semi-discrete methods to the practical resolution of inverse problems in optics. These results were obtained in the PhD thesis of Jocelyn Meyron and the images are taken from the article [5].

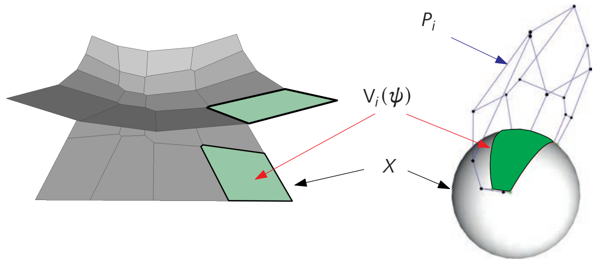


Initialization: $\psi^0 \equiv 0$
$\|G(\psi^0) - \nu\|_1 \simeq 1.8$

Step 1:
$\|G(\psi^1) - \nu\|_1 \simeq 0.6$

Step 3:
$\|G(\psi^3) - \nu\|_1 \simeq 10^{-9}$

Figure 9. Convergence of the sequence (ψ^k). On images 2, 3 and 4 we see the Laguerre cells $\mathrm{Lag}_i(\psi^k)$ for k=0,1,3.



Collimated source

Point source

Figure 10. Visibility cell structure

4.1 Far-field problems

We saw in Section 2 that in several far-field problems, i.e., when the target illumination is at infinity, solving the Monge–Ampère equation (MA) allows us to construct an optical component. This involves modelling mirrors or lenses, with a point or collimated light source, and in each case there are two components that may be produced (one of which is convex), so that in all we have formulated eight different near-field optical problems.

The main difficulty in implementing Newton's algorithm to solve (MA) lies in the evaluation of the function G and its differential DG at point ψ^k, and more precisely in the calculation of the set of Laguerre cells $\mathrm{Lag}_i(\psi^k)$. For cells from non-imaging optics problems, also called visibility cells, it is possible to perform this calculation in almost linear time in the number N of Dirac masses. Take for example the mirror problem for a point source. The visibility cells are obtained by projecting radially onto the sphere an intersection of "solid" confocal paraboloids, and we have already seen that the intersection of two confocal paraboloids is included in a plane. Another simple calculation shows that the radial projection of such an intersection is also included in a (different) plane. This shows that the visibility cells are separated by hyperplanes. In fact, it can be shown that there exists a partition of $\mathbb{R}^3$ into convex polyhedra $P_1, \ldots, P_N$ – called a *power diagram* in computational geometry – such that each visibility cell is of the form $V_i(\psi) = \mathbb{S}^2 \cap P_i$ (Figure 10). A similar property holds for each of the eight problems. The point of this reformulation is that there are powerful libraries – for example CGAL or GEOGRAM – that allow us to compute power diagrams in dimensions 2 and 3, and thus also the Laguerre cells associated with the optics problems. It is therefore possible to implement the damped Newton algorithm, and to use it to construct – numerically and even physically – mirrors and lenses for far-field targets in anidolic optics.

4.2 Near-field problems

It is also possible to deal with more realistic target illuminations in the *near-field* – i.e., when illuminating points at a finite distance rather than directions – with an iterative method that solves a far-field solution at each step [5]. The convergence is very fast, requiring only a few iterations, as illustrated in Figure 11.

In all the experiments presented below, the light source is assumed to be uniform, so that the light source μ has a constant density on its support. The reflection or refraction of this light on a wall is simulated in the computer by the physically realistic rendering software LuxRender.

Generic method

The different problems of anidolic optics having the same structure (point or collimated light sources, mirrors or lenses, convex or concave components, near-field or far-field), it is possible to solve them in a unified, precise and automatic manner with the same

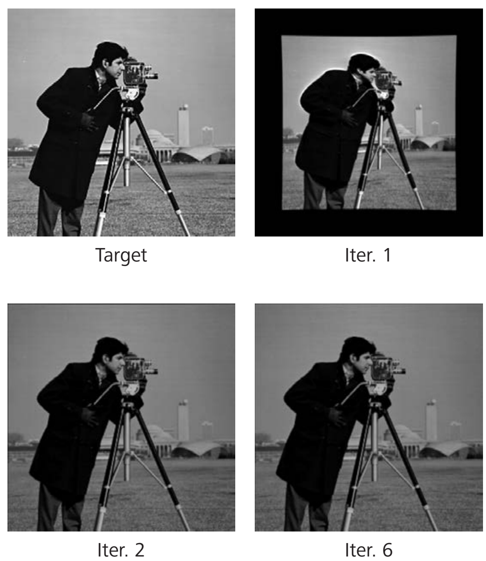


Figure 11. Convergence of far-field mirrors to near-field mirrors

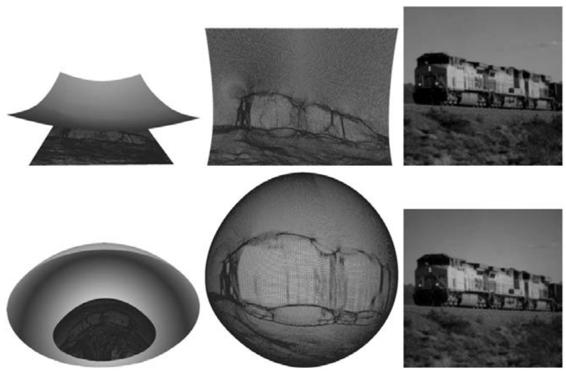


Figure 12. Mirrors for collimated (top) and point (bottom) light; visibility cells (left), component mesh (middle) and rendering with LuxRender (right)

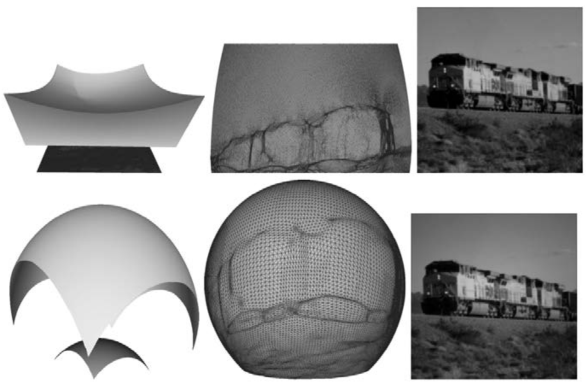


Figure 13. Lenses for collimated (top) and point (bottom) light; visibility cells (left), component mesh (middle) and rendering with LuxRender (right)

Figure 14. A point light (not visible) is placed in front of the mirror and the path of the light is simulated by the computer using the physically realistic renderer LuxRender

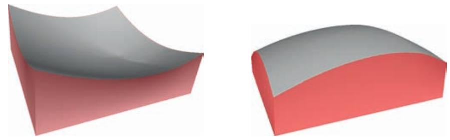


Figure 15. Concave and convex lenses

algorithm (Figures 12, 13 and 14). In Figures 12 and 13, the visibility cells on the sphere or plane are shown on the left, above which is the surface of the optical component. Each surface is represented in the computer by a mesh (a set of triangles) which is shown in the middle. The simulation of the reflected or reflacted light with LuxRender is on the left.

Convexity/concavity of the components

Some applications require the construction of optical components with convexity properties. This is the case in the automotive industry for the construction of mirrors and/or lenses. The reason for this is both practical, as it is easier to build a convex component, and aesthetic. In the case of collimated light sources, mirrors or lenses can always be convex or concave, as can be seen in Figure 15.

Singularity of solutions

The optical components are by construction objects with only a $\mathcal{C}^0$ regularity. Indeed, they are surfaces composed of pieces of planes, paraboloids or ellipsoids (in the case of a mirror for a point light source) which are joined together in a manner that is continuous but not $\mathcal{C}^1$. However, as the discretization of the target illumination becomes finer and finer, the surface tends towards an object that has greater regularity. In Figure 16, we observe a $\mathcal{C}^1$ regularity, except at points on the surface that correspond to black areas in

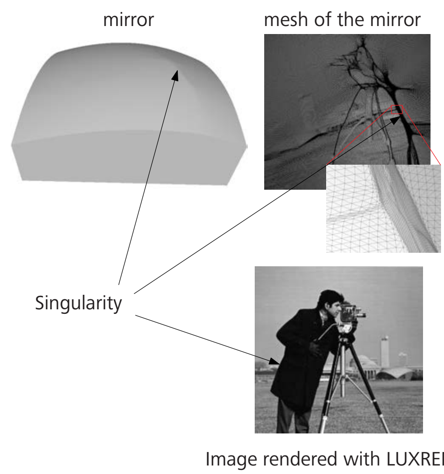


Figure 16. Singularity and meshing (surface singularities correspond to black areas)

the target image. Intuitively, the lack of regularity comes from the fact that the light must avoid the black areas, which results in a jump in the vector field normal to the surface.

Pillows
The inner part of a car headlamp is typically made up of "pillows", i.e., several small components. Each patch is intended to illuminate a fairly wide range of directions, and the lights sent out by each pillow overlap. This ensures a certain robustness in the lighting. If for instance a bird flies past the headlights, not too close, it does not obstruct all the light and the road remains fully illuminated. In Figure 17, the target illumination for each pillows is the cameraman's image. When the calculations are done in the far-field, i.e., when illuminating directions, the images are superimposed, but with an offset due to the size of the pillows. To obtain a clear image, it is necessary to make the calculations in the near-field, so as to illuminate exactly the desired points. Note that the target is always illuminated even if an obstacle, for example a red monkey head, is placed in front of some of the pillows.

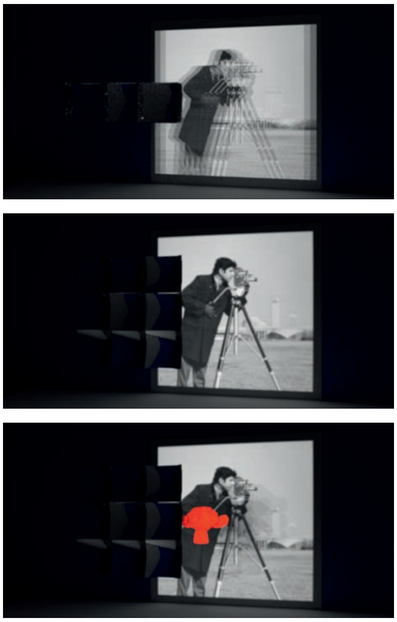


Figure 17. Lens composed of 9 pillows: In the far-field (top); in the near-field (middle); with obstacle (bottom)

Colored target illumination
Similarly, solving the near-field problem makes it possible to illuminate a target in color. Indeed, one can build an optical component for each channel (red, green and blue). Then each of the three lights is sent to its associated component and the colors are added to the target to form a color image. This is done in Figure 18 with three lenses.

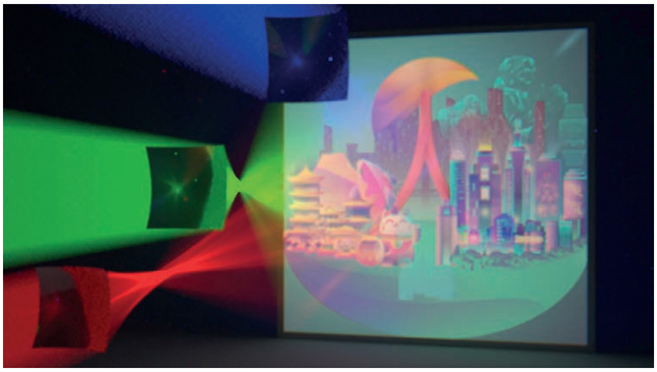


Figure 18. Color image

Construction of mirrors and lenses
We also built optical components. The lenses and mirrors in Figures 19, 2 and 21 were milled by the GINOVA technology platform in Grenoble on a 3-axis CNC (*computer numerical control*) machine with 10 mm radius milling cutters. The path of the milling cutter creates irregularities on the optical components (Figure 22). Note that the convexity of the optical components allows the use of arbitrarily large milling radii, which reduces machining irregularities. In any case, it is necessary to grind under water and then polish the optical components (Figure 23). Of course, this affects the optical quality and tends to whiten the black areas in the target image.

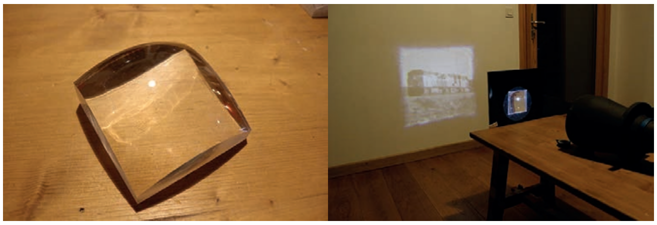


Figure 19. Lens transforming collimated light into the picture of a train

Acknowledgements
The authors thank Damien Gayet for his meticulous proofreading of the French version of this article, and for much good advice. The *EMS Magazine* thanks *La Gazette des Mathématiciens* for its authorisation to publish this text, which is an English translation of the original paper [*La Gazette des Mathématiciens*, Number 166, pp. 6–21, October 2020].

Figure 20. Lens transforming collimated light into the picture of a cameraman

Figure 21. Mirror transforming collimated light into the picture of a cameraman

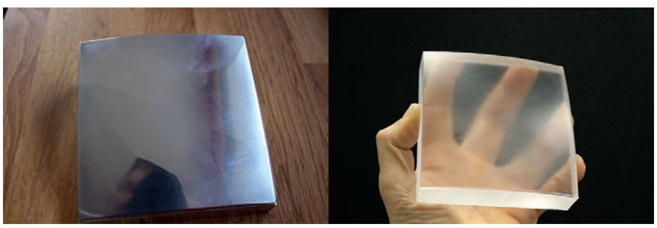


Figure 22. Mirrors and lenses after machining

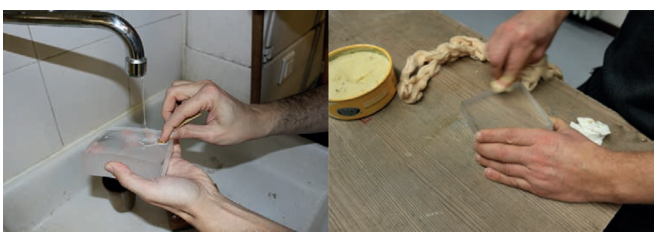


Figure 23. Sanding and polishing by hand

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The edition of Bernhard Riemann's collected works: Then and now

Emmylou Haffner

Bernhard Riemann's collected works were published for the first time in 1876 by Richard Dedekind and Heinrich Weber. The editors' correspondence and the available archive tell us that the process of editing Riemann's collected works was a hands-on process, which is itself of historical and mathematical significance. In this paper, we show how the editors shaped the published texts, and how this can influence our reading of them.

A complex history and a wealth of archive

In 1876 were published Bernhard Riemann's (1826–1866) *Gesammelte mathematische Werke und wissenschaftlicher Nachlass* (Collected mathematical works and scientific archive). These collected works were edited by Heinrich Weber (1842–1913) and Richard Dedekind (1831–1916) and published by B. G. Teubner.[1]

Riemann and Dedekind met while they were Gauss' students in Göttingen. They defended their doctoral dissertation within a year of each other (Riemann in 1851 and Dedekind in 1852), and their respective *Habilitation* with only a few days difference in 1854. Following this, they both worked as *Privatdozenten* in Göttingen, during which time Dedekind followed Riemann's classes. In 1858 Dedekind was offered a position in Zürich and Riemann a post in Göttingen, and they remained friends until Riemann's untimely death in 1866.

It was Riemann's wish that Dedekind would be the editor of his collected works and in charge of his scientific archive after his death. Struggling with this difficult editorial enterprise, in early 1872, Dedekind accepted to work with Alfred Clebsch (1833–1872), who had taken Riemann's chair in Göttingen. Seven of the most complete of Riemann's unpublished works were first published posthumously in various mathematical journals.[2] Clebsch did not wish to publish more of Riemann's manuscripts as he felt the edition, as he wanted it to be, was nearing completion (according to his letters to Dedekind, published in [8], and to Dedekind's first letter to Weber in [32]). Clebsch's sudden death in 1872 put the edition in some difficulty. Dedekind's teaching duties kept him from handling the project by himself. Eventually, upon meeting Heinrich Weber in Zürich in 1873,[3] Dedekind offered him the responsibility of the edition, which he accepted. At this stage, Dedekind wished to retreat from the project, but eventually became more involved in the edition of some of the manuscripts. Both Weber and Dedekind wished to publish more of Riemann's unpublished archive, and it took them two additional years to complete the edition, during which time they also had help from Hermann Schwarz (1843–1921) in working on [20].

The final product of this ten-year editorial endeavour, Riemann's *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*, is one volume divided into three parts and two appendices: the first part contains the 11 papers published by Riemann in his lifetime; the second part contains the 7 papers published

[1] At this time, a considerable number of projects of publishing collected works were launched in France, Germany, Italy, the United Kingdom … The publisher B. G. Teubner, created in 1811 in Leipzig, which specialised in scientific editions (broadly construed, i.e., philology, history, mathematics, physics, etc.), was one of the leading publishers for this type of book in Germany. – Steven W. Rockey from Cornell University published a very complete list of collected works in mathematics: mathematics.library.cornell.edu/about-collected-works/.

[2] [22] was edited by Karl Hattendorff (1834–1882), [21] by Ernst Schering (1824–1889) and Friedrich Henle (1809–1885), the other texts presumably by Dedekind. – A note on the dates of the publications: when it is possible to date Riemann's texts, these are the given dates; when it is not, the dates are that of the first publication.

[3] Maybe a less famous name than Riemann, Clebsch and Dedekind, Heinrich Weber was a prominent mathematician throughout his career. He studied in Heidelberg, Leipzig and Königsberg. He taught in Heidelberg, Zürich, Königsberg (where he taught number theory to Hilbert and Minkowski), Berlin, Marburg, Göttingen, and Strasbourg. He worked extensively on complex function theory, number theory, and algebra. Among several important contributions to the latter, his *Lehrbuch der Algebra* was to be the main reference for teaching algebra in the German speaking world until the publication of Van der Waerden's *Moderne Algebra* in 1930. He also made contributions to mathematical physics, and published *Die partiellen Differentialgleichungen der mathematischen Physik nach Riemann's Vorlesungen*, which was, for a long time, the only reference for Riemann's mathematical physics. Weber was also actively involved in the mathematical community, for example he was a member of the editorial committee of the *Mathematische Annalen* and a founding member of the *Deutsche Mathematiker-Vereinigung*.

posthumously in journals as mentioned above; and the third part contains 12 unpublished texts from Riemann's archive. The two appendices are a selection of Riemann's philosophical writings, and a biography written by Dedekind on the basis of letters from Riemann's widow, Elise Riemann.

Riemann's collected works were republished in 1892, by Weber. In the preface, he explained that Riemann's texts were still very relevant in 1892. Two important changes in the edition should be mentioned. Firstly, the text *Verbreitung der Wärme im Ellipsoid* (Diffusion of heat in an ellipsoid) [28], which was briefly discussed and eventually excluded from the 1876 edition, was published. There are no indications or correspondence that indicate why it was initially excluded (in fact, the letters suggest that it was going to be published in 1876), nor why it was finally published in 1892. Secondly, the notes and commentaries by the editors were revised (following feedback on the first edition) and completed. In 1876, 4 texts were commented (30 pages of commentaries), while in 1892, 10 texts were commented (for a total of 60 pages of commentaries). A third edition was published in 1902 by Max Noether and Wilhelm Wirtinger. The sole but very notable change here is the addition of over a hundred pages of notes from Riemann's lectures (on Abelian, elliptic, hyperelliptic functions, hypergeometric series, etc.) which had only recently become known.[4]

Only for the 1876 edition do we have, rather exceptionally, extensive documentation on the process of editing Riemann's collected works. This is one reason why my focus in this paper will be this first edition.[5] A second reason is that a core interest, here, is how the editorial work shaped Riemann's text, which was largely accomplished in the first edition.

Dedekind and Weber's editorial work was meticulous, mindful and even devoted, according to Elise Riemann. Their collaboration for this publication, which marked the beginning of almost forty years of friendship, was largely carried out in letters written from November 1st 1874 to the end of 1876. These letters have been preserved in Riemann's archive (Cod. Ms. Bernhard Riemann, Niedersächsische Staats- und Universitätsbibliothek Göttingen) and in Dedekind's (Cod. Ms. Richard Dedekind, Niedersächsische Staats- und Universitätsbibliothek Göttingen, and G 98:11–13, Archiv der Universitätsbibliothek Braunschweig),[6] and published in 2014 [32]. As most of their discussions appear in these letters, we have an extensive and detailed vision of the editorial process. Weber and Dedekind discussed every aspect of the edition, from the practical (e.g., the contract with Teubner, the copyrights, the advertisement of the book) to the scientific and philological (e.g., the choice of which texts to publish, their difficulties in understanding Riemann's manuscripts, what kind of corrections or completions should be made before the publication). Indeed, a number of modifications were made to Riemann's texts, from orthographical and typographical changes to the redaction of missing passages.

The process of editing Riemann's *Werke* was thus a hands-on process, in which the editors were deeply involved in both the mathematical and philological aspects. Weber and Dedekind – and Hattendorff and Schwarz for some texts – engaged in a systematic verification of each and every one of Riemann's texts, including those that had already been published. Some texts were, in fact, written by several hands: Riemann's and the editor's (for example, some parts of [20] are marked as being explicitly written by Schwarz). This raises questions on the genesis of the text and on the authorship.

After the publication of the Dedekind-Weber correspondence, it became clear to me that there was, here, material to study how the edition of Riemann's collected works was crafted. It also provides an opportunity to unfold parts of their mathematical activity which have been largely overlooked (until now!), and indeed to understand important aspects of Riemann's influence on both mathematicians.[7] It also allows us to make connections with research in the history of text – how did the editing process shape the texts published? how did it shape the book itself? – and with the history of mathematical publishing.

Common interests in these questions led to collaboration between the History of Science, History of Text research group in the Laboratoire SPHERE (Université de Paris) and the Interdisziplinäre Zentrum für Wissenschafts- und Technikforschung (Bergische Universität Wuppertal) with the organisation of an ongoing series of workshops and seminars on the history of collected works as an editorial and scientific practice. Among our observations (some of which I will return to towards the end of this paper), the most relevant to the case of the Riemann edition are the following: texts published in collected works often bear the traces of the editorial work – maybe in more ways than we would expect – and for the editors, this was not solely an editorial or philological undertaking, but also a scholarly endeavour, and indeed one we seem to have overlooked so far.

The ongoing analysis of the edition of Riemann's collected works is made possible by the documents available in Riemann's archive,[8] whose origins are described in [14], and in which most

[4] In 1990, the 1902 edition was reprinted along with additions. A French translation appeared in 1898, translated by Léonce Laugel and published by Gauthiers-Villars (see p. 37). The first English translation appeared in 2004 [29].

[5] Unless stated otherwise, "edition" will refer to the first edition, from now on.

[6] Heinrich Weber's archive seem, however, to have been lost [32, p. 16].

[7] In 1882, Dedekind and Weber published *Theorie der algebraischen Funktionen einer Veränderlichen* (Theory of algebraic functions of one complex variable) in which they transfer Dedekind's concepts of field, module and ideal from number theory to function theory to give a new definition of the Riemann surface and related notion, such as the genus.

[8] The catalog is available here: hans.sub.uni-goettingen.de/nachlaesse/Riemann.pdf

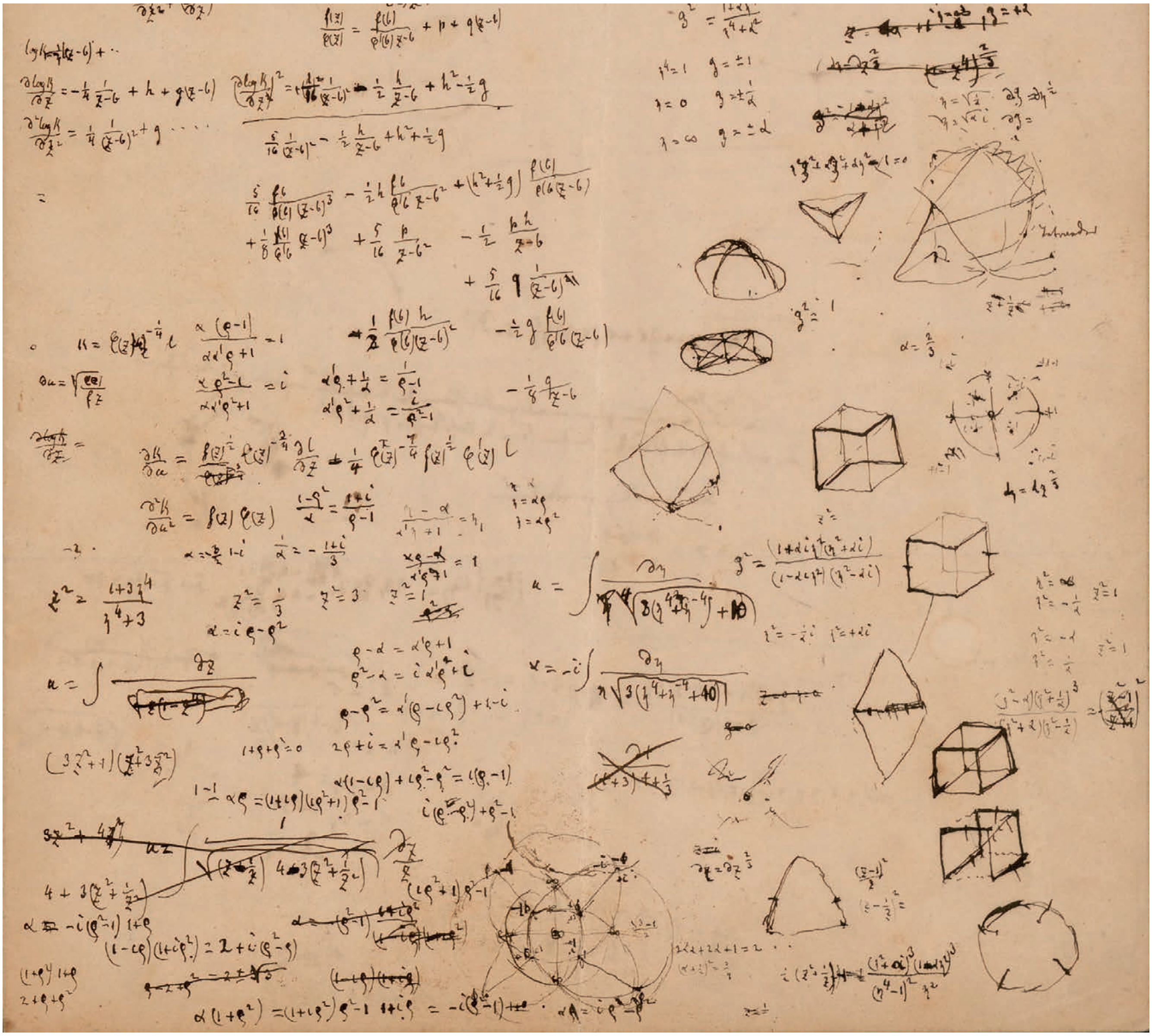

Figure 1. Cod. Ms. Riemann 34 I, p. 4r: extract from the manuscript on minimal surfaces (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

of the documents used by Weber and Dedekind are available.[9] The only exception are the manuscripts that were the basis for [24, 27] which are in Schwarz' archive at the *Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften*. Dedekind's archive also contains interesting material on his work as an editor (see below).

Since the 1970s and the great work done by Erwin Neuenschwander, many interesting historical works have been published using Riemann's archive, a number of which will certainly be useful to the present project. The goal of this project is solely a critical analysis of the process of *editing* Riemann's collected works, which comes along with a comparison of the original manuscripts

[9] See [30] for details on the development of Göttingen as an archive center.

and the published texts. The files in Riemann's archive relating to the published texts contain thousands of pages (and around 500 pages in Dedekind's archive). Most of the files contain several copies of the texts (usually by the editors, more rarely by Riemann), Riemann's original texts and many of his drafts. Using a (semi-)automated approach to the transcription and comparison of the manuscripts with digital tools for handwritten text recognition and the tools developed by the CollEx-Persée project AMOr (www.collexpersee.eu/projet/amor/) should help manage these relatively large files.[10] In some of these files, the most challenging task might be to identify which documents were indeed used by the editors to produce the published text.

Shaping the individual texts

Heinrich Weber wrote an announcement of Riemann's collected works for Koenigsberger and Zeuner's *Repertorium der literarischen Arbeiten aus dem Gebiete der reinen und angewandten Mathematik*, in which he mentions the extent of the editorial work:

> We only corrected some slight inaccuracies which were made known to the editor and could be seen as certain. Some additions, written according to Riemann's manuscripts, and some necessary clarifications were placed in final notes. […] [T]he majority of [Riemann's] posthumous writings contain only formulae with very little indications to find what link them. Hence, a lot of passages written only in a very fragmentary form had to be established as well as we could, and many others are still buried in his archive, for want of being deciphered. [36, pp. 7–8]

A similar statement can also be found in Weber's preface in [25, p. iv].

There are several types of modifications of Riemann's original texts: the local, more or less significant changes to the texts, e.g., correcting an error, which are mentioned in notes; a number of such local changes, which are *not* mentioned in notes; and texts extracted from Riemann's archive which are completed to a greater or lesser extent by the editor.[11] While the reader could expect to be able to identify clearly what was changed or added by the editors, this is not always the case. A number of changes are not clearly identified in any way, and can only be recognised as such by reading the editors' correspondence or comparing the published texts with the manuscripts.[12]

Of course, Weber and Dedekind were cautious with their corrections. In a letter from July 8, 1875, as he was proofreading Riemann's famous *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the number of primes less than a given magnitude) [19], Weber wrote to Dedekind:

> Do you have any remarks on the work on primes? I have come to a difference from Riemann's formula in the calculation, namely to the same one which Scheibner already noticed in his analysis of this work in Schlömilch's journal. [[33]] Despite this, I am far from taking Riemann's result to be incorrect, whose actual proof, as can be seen from a fragment of a letter, is not contained in the work at all. I do not dare to make any changes or additions. [32, p. 71]

We don't have any answer from Dedekind, but Weber later wrote again that he was finding "$-\log 2$ instead of $\log \xi(0)$" (as Scheibner had) but still didn't dare to make any change or note, assuming that "it is probable that Riemann is right" but that he was missing the proof. The 1876 edition does not contain any correction or note, but there is a note by Weber in the 1892 edition, stating that

> If one continues the computation indicated by Riemann, one finds in the formula $\log \frac{1}{2}$ instead of $\log \xi(0)$. It is very likely that this is but a typographical or printing mistake of $\log \xi(0)$ in place of $\log \zeta(0)$, indeed $\zeta(0) = \frac{1}{2}$ [sic].[13] [25, p. 155, 2nd edition 1892]

Changes can be even more important in texts extracted from Riemann's archive. For some of them, the editors decided to write entire paragraphs themselves to complete Riemann's original text before publication. Such changes raise questions as to the authorship of the texts, and the extent to which some of their content could be a result of edition as a collaborative enterprise. Some mathematicians in the years following the publication of Riemann's collected works seemed to keep this aspect in mind, as suggested by a letter from Felix Klein to Henri Poincaré, sent on April 3rd, 1882, following a discussion on Riemann's possible anticipation of some of Friedrich Schottky's results:[14]

> [Regarding] Schottky, I would like to draw your attention to a posthumous essay in Riemann's collected works, p. 413, where exactly corresponding ideas are

[10] Of course, for parts of this archive, in particular the letters, transcriptions are already available.

[11] I have considered these questions in [11, 13]. An in-depth analysis of the edition of [17] is in progress and, as mentioned, so is a critical edition of Riemann's texts.

[12] It is the case with [18], whose edition I presented in [12].

[13] This seems to be a typo correcting the typo, as $\log \zeta(0) = -\log 2 + \pi i = \log \frac{1}{2} + \pi i$, Weber meant to write "indeed $\log \zeta(0) = \log \frac{1}{2}$".

[14] Klein is, here, referring to [34] and which was published in 1877 in the *Journal für die reine und angewandte Mathematik*, 83: 300–351, in which he studied conformal mappings of multiply connected domains, which he was the first to analyse systematically.

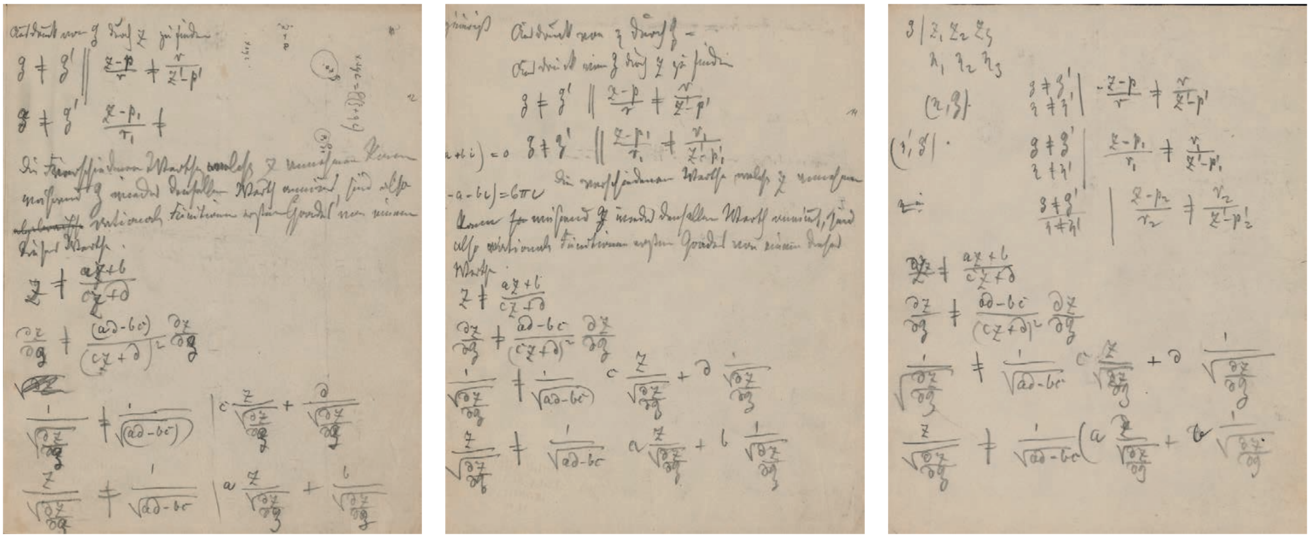

Figure 2. Pages 12r, 14r and 15v of Cod. Ms. Riemann 5 (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

> developed. However, it will be difficult to establish how much the editor, Prof. Weber, has put into it. Riemann's collected works appeared in 1876, Schottky's dissertation in 1870, later as an essay in Borchardt's Journal, 1877. (Letter from Klein, in [15, p. 53])

Klein is referring here to *Gleichgewicht der Electricität aut Cylindern init kreisförmigem Querschnitt und parallelen Axen* (Equilibrium of electricity on cylinders with circular crosssection and parallel axes) [26], which indeed deals with conformal mappings on a multiply connected surface. In a footnote, Weber states that

> [t]here are no completed manuscripts of this and the following works by Riemann. They are composed of pages which, apart from a few hints, contain only formulae. [25, p. 413]

Early in his correspondence with Dedekind, Weber mentioned that he would be "very interested" in being able to "decipher" the manuscripts on "the distribution of electricity on three spheres" [sic], which he hoped to be able to achieve since "on one of the sheets the results seem to be essentially in place" [32, p. 62, letter from March 22, 1875]. As this last remark suggests, Riemann's manuscripts in Cod. Ms. Riemann 5 contain many sheets with various states of development of his investigation. There are 27 pages by Riemann's hand, for 4 pages of text by Weber, and certainly the material differences of each mathematician's handwriting and use of paper do not account for such a large difference. In fact, many of Riemann's notes contain similar computations, see Figure 2.

In addition, to put it bluntly, Weber's version of Riemann's research contains a lot more sentences and far fewer calculations. It is fairly easy to identify which formulae Weber included in his text. However, the *sentences* present in the published text are quite difficult to find in Riemann's manuscripts. Thus, it seems that most of the redaction is by Weber, who completed and clarified Riemann's text. He did not, here, correct or complete Riemann's formulae – rather, he selected the relevant ones. It is, without a doubt, a text written by both Riemann and Weber.

In the available correspondence, Weber did not himself mention Schottky's works. However, Schwarz wrote to Weber about Schottky's dissertation on November 11, 1875 [32, p. 362].[15] Weber's letters to Schwarz have been lost, and we do not know what he answered to this mention of Schottky's paper. In January 1876, this text had, with seven others, already been sent to Teubner [32, p. 95].

Another – and one of the most striking – examples of an extensive mathematical and editorial investment is the work done by Dedekind on "*Fragmente über die Grenzfälle der elliptischen Modulfunctionen*" (Fragments on the limit-cases of elliptic modular functions) [17].

Dedekind started working on these manuscripts in February 1876. The lack of clarity of the notes, both from a material and a mathematical viewpoint, was so bad that editing them took

[15] Schwarz wrote: "On Saturday and Sunday of last week, I was in Berlin and learned from Prof. Weierstrass of the dissertation of one of his students, a certain Schottky: '*Über die conforme Abbildung mehrfach zusammenhängender Flächen*'; if you do not not already know about this dissertation, please allow me to draw your attention to it. The results which are presented in this essay, are of great interest and scientific value; I myself will seek to obtain the dissertation in order to possess it."

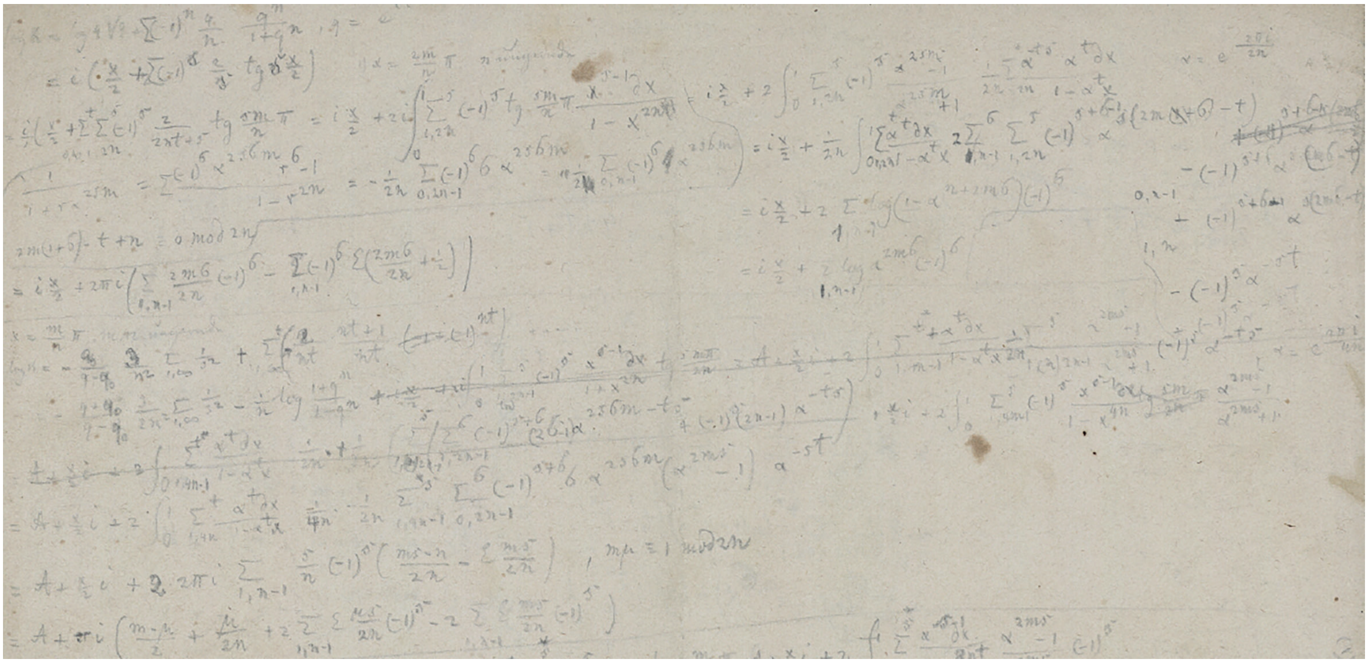

Figure 3. Cod. Ms. Riemann 14, p. 18v: Excerpt from Riemann's "very pale manuscript" (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

Dedekind several weeks and led him to fear having nightmares (see [32, pp. 101–104]). Over a dozen letters were exchanged between Dedekind and Weber from December 1875 to April 1876. Dedekind confided to Weber his difficulties in understanding and editing Riemann's text (which he nicknamed, in his letters and in his own archive, "*sehr blasses Manuskript von Riemann*", Riemann's very pale manuscript, because it was written in pencil and had faded badly). Eventually, Dedekind made it through his deciphering of the manuscript, and was able to produce a complete transcription of Riemann's notes. He verified each formula and corrected them when necessary, but did not make any additions. In addition, he wrote a 10-page commentary containing original research [4].

These *Fragments* consider properties of Jacobi series in elliptic function theory. Without entering into any detail, Dedekind interpreted Riemann's formulae as the study of the logarithm of some modular functions at the limits of their domain of definition. In the collected works, he stated:

> The time of writing of the first of the two fragments (September 1852) makes it likely that Riemann, while working on his memoir *On the representation of a function by a trigonometric series*, was looking for examples of functions with infinitely many discontinuities in each interval. Perhaps the second investigation, which occurs on the barely legible sheet, has the same object.[16] [4, p. 438]

In Cod. Ms. Riemann 14, we find the 15 pages of Riemann's original manuscript, two handwritten transcriptions, the handwritten text for Dedekind's 1876 commentary and the version sent to the editor, some notes written by Weber, the 1876 letters between Dedekind and Weber relating to that text, and one of Dedekind's early works on elliptic functions, which he intended to use to understand Riemann's ideas and likely sent to Weber with one of his letters.

The most exceptional documents can be found in Dedekind's archive. In Cod. Ms. Dedekind XI 11-1, XI 11-2, XII 4,[17] we find several hundred pages of notes written solely by Dedekind. There, we see the progression in his understanding of Riemann's texts and of the writing of his 1876 and 1892 commentaries, as well as continuations of his research on the subject. These pages show the breadth

[16] [1] disagrees with this interpretation. Hopefully, the manuscripts hold some elements to answer this question.

[17] Cod. Ms. Dedekind XII 4 is mistakenly listed as referring to [23] in the Göttingen catalog, because Dedekind refers to the text using the numbering in the table of contents in the 1892 reedition of Riemann's collected works. The contents of the file are, however, undoubtedly related to [17]. In exploring these documents, I have greatly benefited from Walter Strobl's help.

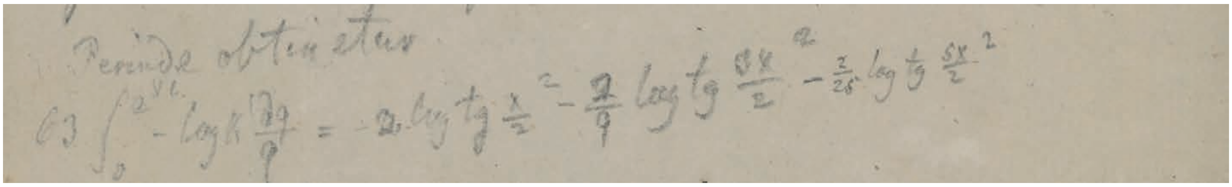

Figure 4. Cod. Ms. B. Riemann 14, p. 12v: Riemann's original manuscript (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

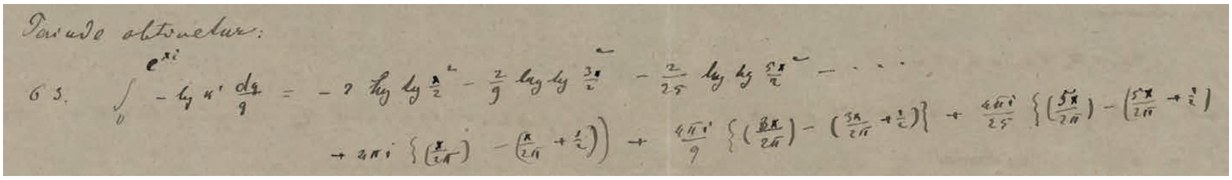

Figure 5. Cod. Ms. B. Riemann 14, p. 2r: First transcription (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

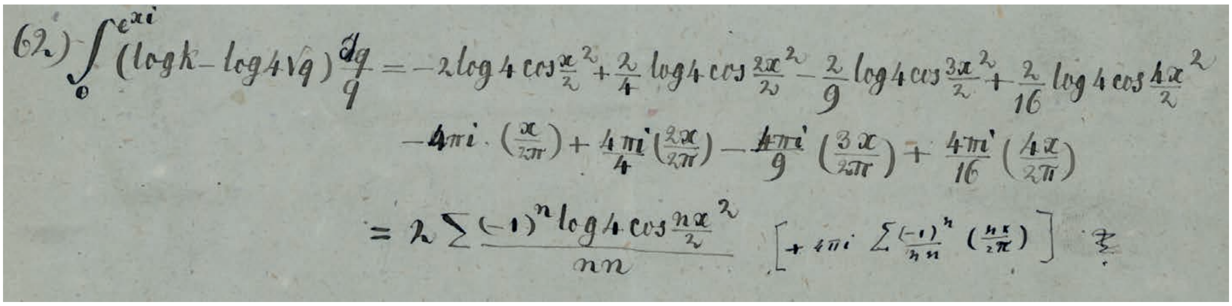

Figure 6. Cod. Ms. B. Riemann 14, p. 20v: Second transcription (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

and depth of the mathematical reflections developed by Dedekind for his editorial work. In addition to computations following Riemann's manuscripts and trying to obtain again Riemann's results, Dedekind developed his own approach to the subject, which ended up being his only way to verify Riemann's results. For this, he drew comparisons between both approaches, at some points relying only on the correspondences between numerical examples, and eventually systematically exploring the correspondences between his and Riemann's results. This research was also the basis for his commentaries, of which we find several drafts in the archive. Both Dedekind's commentaries, although entitled "Explanation on the preceding fragments" do not actually explain what Riemann was trying to do, rather they present:

> a very interesting application related to the so-called theory of the infinitely many forms of the theta-functions, namely the determination of the constants appearing via transformations of first degree, which as is known, were reduced by Jacobi and Hermite to Gauss sums, and thus to the theory of quadratic residues. The following commentary illustrates these relationships. [4, p. 438]

In particular, it is there that Dedekind introduced what we today call the Dedekind eta function.[18]

Shaping the book and shaping the image of the editee

The way in which a book such as a mathematician's collected works is constructed – which texts are chosen to be in this publication; whether unpublished manuscripts are selected and if so, which ones, and how they are published; whether a critical apparatus is

[18] The Dedekind eta function is a modular form defined on the upper-half part of the complex plane by $\eta(\omega) = e^{\frac{\pi i \omega}{12}} \prod_{n=1}^{\infty}(1 - e^{2n\pi i\omega})$ [4, p. 438].

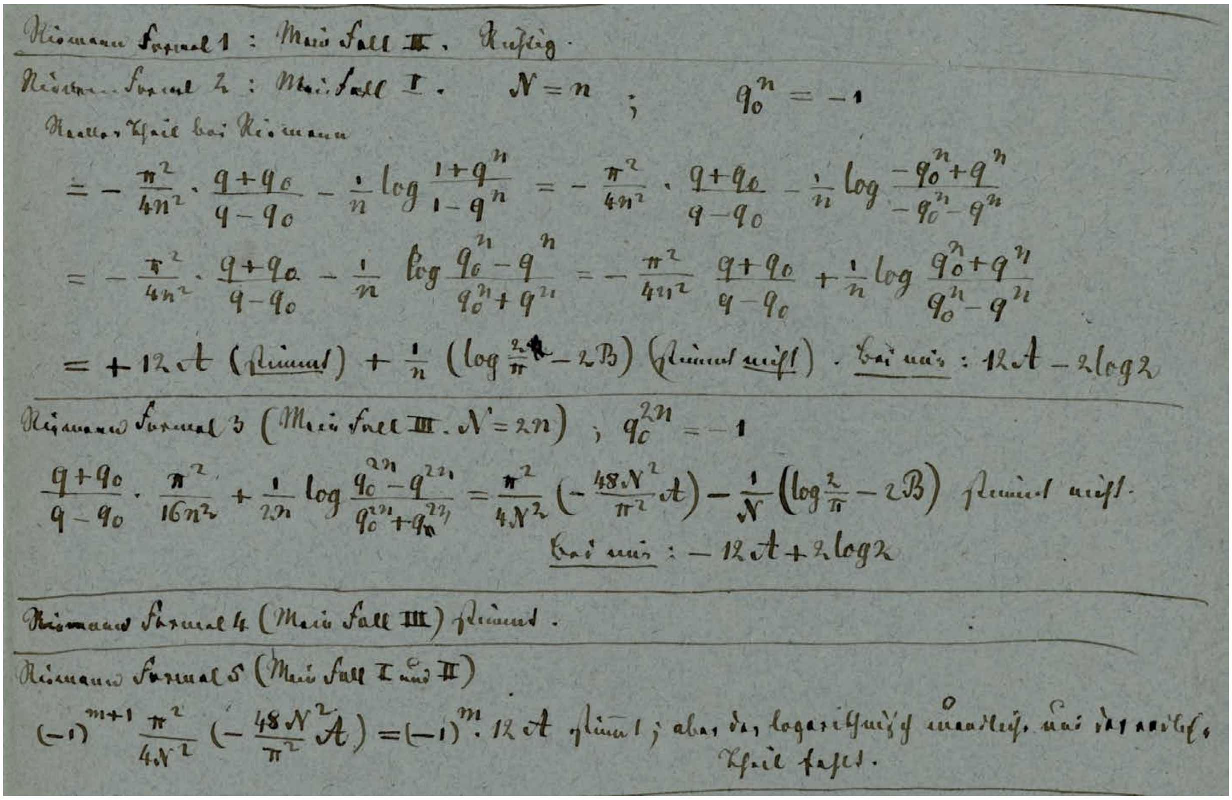

Figure 7. Cod. Ms. R. Dedekind XI 11-1, p. 19r: Summary of Dedekind's comparison of Riemann's results with his own (Niedersächsische Staats- und Universitätsbibliothek Göttingen)

added and which one; how texts are organised and, when applicable, how the multiple volumes are themselves organised, etc. – shapes the image of the editee presented to the readers. Indeed, such choices are a reflection of the editor's own idea of the editee's work, and of what they want to showcase of it.[19] The shaping of the book is, in fact, the shaping of the vector of circulation of the editee's works. Without undue generalization on the possibility of biases on the part of editors, the history of mathematics gives us several examples in which mathematicians works were largely reconstructed by the editors.

The selection of which texts are deemed suitable for publication plays a significant role in such a reconstruction of the works of the editee. Through these choices, the editors impose their own criteria and their own values on the editee's texts. And it is all the more pregnant regarding the choice of excerpts from the author's archives, as there are few ways of knowing whether the author had any intention of publishing these texts, or why they didn't. As such, our vision of the editee's work can be restricted to the editors' reading of it. And this contributes, to a certain extent, to a mythologised history of mathematics.

Let me give three examples, which are not Riemann's collected works, in which this happened. A first, and very striking, example is the edition of Leibniz's works, which was mentioned in David Rabouin's recent paper in the archive series on Leibniz's archives [16], in which he explains how "some texts edited by Gerhardt and Couturat have turned out to be mere artefacts". A second example is the edition of Gauss' collected works, a gigantic enterprise that took several decades, first directed by Ernst Schering, then by Felix Klein (see [10, pp. 67–68] and [30]). Maarten Bullynck showed, in a talk at the Laboratoire SPHERE (Université de Paris), how the edition of Gauss' collected works was one of the

[19] Note that this is also an important point regarding the role that individuals play in editing their own work or in supervising such an edition (e.g., Poncelet, Weierstrass).

elements of Klein's retrospective reconstruction of the so-called Göttingen tradition in mathematics. My third example is the edition of Dedekind's collected works by Emmy Noether, Øystein Ore and Robert Fricke in 1930–1932. The three volumes are organized as follows: the first two contain Dedekind's mathematical papers arranged in chronological order with some extracts from his archive, the third contains his foundational essays on real and natural numbers, partial reproductions of his algebraic number theory[20] and more extracts from his archive. This arrangement creates two illusions. First, that of a difference of status between Dedekind's 'mathematical' and his 'foundational' papers, a distinction he did not make himself. Second, the *partial* reproduction of his algebraic number theory completely disconnects this research from its number-theoretical context and, in fact, excludes its more traditional parts. These choices were likely guided by the editors considering Dedekind as a precursor of the modern structural algebra and certainly participated in perpetuating this retrospective reading of his work.

While in many of these cases we can only observe the choices made by the editors and make assumptions about their intentions, for Riemann's collected works, the letters exchanged between Dedekind and Weber offer us a considerable amount of information on these questions, making it a rather exceptional case study. Their exchanges indeed tell us which texts were dismissed as not 'worthy' of being published, and the criteria that presided on their choices. Let me sum up the main criteria for Dedekind and Weber's choices:

- A text had to be (of course) scientifically sound and generally correct – as correct as possible but the scientific interest came first;
- it had to be understandable – even if this sometimes meant that the editors had to make the text more understandable than the way it was left by Riemann;
- it had to be representative of Riemann's research, it had to have a recognisable place in his overall intellectual production;
- it had to (of course) give a flattering image of Riemann;
- it had to fit into the scientific and philosophical context of the time, to ensure that it would be well received by the scientific community.

Any process of choice is subjective – it would be difficult to think of any editions that are completely unbiased. But in the case of Riemann's collected works, we can pinpoint some of the effects that the editors' choices had. The question of whether the texts in Riemann's collected works can be attributed solely to him, raised by Klein, is one of them.

Another issue is the extent to which the image of Riemann provided by the collected works might have been shaped by what the editors thought it should be. Dedekind was very vocal about seeing Riemann as the best representative of how mathematical definitions and proofs should be grounded on conceptual, fundamental characteristics rather than on computations and notations. He considered himself as following these methodological guidelines. Thus, was born the narrative of a tradition of "conceptual mathematics" in Göttingen, which was later largely continued by Klein and Hilbert's group. These highly influential mathematicians developed a culture in Göttingen which has been described as largely relying on "nostrification" (see [2]), a tendency to reinterpret other people's thoughts so that they would fit their own current picture of the domain. The desire, strongly expressed by Klein, to create a new kind of scientific institution might have led to the reconstruction of a history, an inheritance, which selected and overemphasized some isolated ideas (see [10] and [31]).

This goes, of course, beyond the mere publication of collected works. It is however tangible in the French translation of Riemann's collected works, published in 1898, edited and translated by Léonce Laugel. He chose to exclude not only the papers published in French and Latin, but also most of the papers not related to mathematics (i.e., all papers on physics and the philosophical fragments). He replaced Weber's preface with a preface by Charles Hermite (1822–1901) and added the translation of a talk given by Klein, both of which embrace the idea of Riemann as avoiding computations and relying solely on concepts and a "brilliant power of thought and [an] anticipatory imagination [which] led him frequently to take very great steps that others could not so easily follow", as Dedekind wrote of Riemann in his biography.

Later commentators did not all agree with the image of Riemann that this narrative participated in popularizing among mathematicians. Carl Siegel, who famously discovered the Riemann–Siegel formula in Riemann's archive wrote:

> The legend according to which Riemann found his mathematical results through grand general ideas without requiring the formal tools of analysis, is not as widely believed today as it was during Felix Klein's lifetime. Just how strong Riemann's analytic technique was is especially clearly shown by the derivation and transformation of his asymptotic series for $\zeta(s)$. [35, p. 276] (translated in [9, p. 67])

This was also defended by the historian Harold M. Edwards, in [9], who argued for a strong – albeit maybe hidden in drafts – algorithmic component in Riemann's mathematics. Edwards showed how Riemann, while he may have been "primarily interested in

[20] [3, 6] which were respectively published as Supplements to the 1871 and 1894 editions of Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie* and [5] which was published in French and later as a Supplement to the 1879 edition of Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie*. In these papers, Dedekind introduced and developed the concepts of field and ideal.

grand general abstract concepts," on several occasions "did not venture into these higher realms without doing a lot of serious computation to lay the groundwork for his flights." [9, p. 64].[21] These observations are confirmed by Riemann's archive, in particular by the many parts that remain unpublished – which Carl Siegel, of course, knew very well.

This leads me to one last potential issue, or more exactly to a limitation any editor would face with the manuscripts of a mathematician such as Riemann: understanding their content. As Carl Siegel's work on the Riemann–Siegel formula has shown, Riemann's archive contained, and maybe still contains, important unpublished (even if not fully developed) results that escaped Weber's and Dedekind's attention. This shows the extent to which it can be useful and fruitful to revisit mathematicians' archives.

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Impact factor, an inadequate yardstick

Mohammad Sal Moslehian

Our aim is twofold. On the one hand we discuss the limitations of the impact factor as a criterion for assessing mathematical journals, and suggest substituting a set of different types of indicators including the SCImago Journal Rank. On the other, we state that scientometrics such as the impact factor cannot be used alone in evaluating researchers' work: one must have both a package of metrics as an objective measure and peer review by human beings as a subjective judgement.

In the 1960s, the notion of impact factor was introduced to assist libraries in deciding which journals to purchase. Since the late 1990s, it has been employed as a metric for measuring the quality of scholarly journals.

The Web of Science (WOS), a bibliographical database created by Clarivate Analytics, computes the journal impact factor (JIF) to recognize the relative importance of each journal. To be assigned a JIF, a journal first needs to satisfy certain quality criteria in order to be included in the Journal Citation Report (JCR). The JCR is a selective list consisting of more than 11,000 journals. The (2-year-) impact factor of a journal in a specific year measures the average number of citations from that year of the papers published in that journal during the previous 2 years. More precisely, the 2-year-impact factor of a journal in a year n is computed by the formula

$$JIF_n = \frac{C_n}{P_{n-1} + P_{n-2}},$$

where C_n denotes the number of citations in the year n of papers published in the journal in the years $n-1$ and $n-2$, and P_m stands for the number of papers published in the journal in the year m. A citation of a paper given by the author(s) of the paper is called a self-citation.

The SCImago Journal Rank (SJR) of a journal is a 3-year-impact factor reflecting the influence of the journal supported by Scopus. It depends not only on the number of citations of its published papers but also on the prestige of the journals in which the citations appeared; see [4]. A drawback, however, has been reported regarding Scopus: namely, the database of Scopus journals with assigned SJR includes about 30,000 journals, which is a very large number of journals of varying quality.

Furthermore, WOS provides the indicator Eigenfactor (EF) that ranks journals in the same manner as that used by Google to rank websites. Based on 5-year citation data, it adjusts for citation differences through various disciplines. Thus the SJR and EF seem to be well-suited for evaluation of the quality of a journal; see [7].

Each subject category of JCR journals is divided into four quartiles: Q1, Q2, Q3, and Q4, where Q1 denotes the top 25 percent of all journals in terms of their JIF. There are analogous quartiles for the journals in Scopus according to their SJRs.

Replacement for the impact factor

The JIF has received serious criticism for various reasons, such as: lack of statistical significance [9, 10], poor representativeness and robustness [5], insensitivity to field differences [6], insensitivity to the weight of the citing articles [2] and manipulability by editorial strategies [8]. Here is a list of some of the most significant limitations:

- it counts citations of articles that are not included in the denominator of the above formula;
- its analysis period is 2 years, which is not suitable for evaluation of mathematical research;
- it merely counts citations, without considering their quality. Therefore the JIF may force some mathematicians to do research in topics on which a lot of people are working, who can potentially cite their papers. It is easy to find evidence that such topics are mostly outside the mainstream of mathematics;
- it includes self-citations;
- it is relatively easy to manipulate JIFs and some other scientometrics. There are "mutual citation groups" in which researchers in a certain circle heavily cite each other's work in order to enhance the JIF of a certain journal and artificially inflate the impact of their own papers.

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Table 1. Scientometrics indices as found in databases in the year 2020

	JIF	SJR	MCQ	EF		JIF	SJR	MCQ	EF		JIF	SJR	MCQ	EF
Acta Math.	2.458	5.77	3.95	0.007	*J. Funct. Anal.*	1.496	2.42	1.61	0.035	*Amer. J. Math.*	1.711	3.28	1.67	0.009
Iran. J. Fuzzy Syst.	2.276	0.51	0.11	<0.001	*J. Funct. Spaces*	1.896	0.46	0.43	<0.001	*Mathematics*	1.747	0.3	NA	NA

The SJR aims to fix the above problems by providing a more effective computation formula, including a longer period of 3 years for counting citations, attributing different weight to citations, and limiting self-citations. Some studies show that using the SJR can improve the situation to some extent. It is at any rate a first step towards avoiding some of the limitations of JIF; see [1, 3].

To illustrate the drawbacks and inadequacy of JIF in mathematics, let us take a closer look at the JIF numbers. There are mathematical journals in the 2019 list of JCR-Q1 whose impact factors are "unexpectedly large". For instance, the *Iranian Journal of Fuzzy Systems* is ranked 15 in the category of Mathematics of the JCR list, while the very prestigious journal *Acta Mathematica*, launched in 1882, is ranked 13; also, *American Journal of Mathematics* and *Transactions of the American Mathematical Society* are ranked 32 and 60, respectively.

However, the SJR for *Iranian Journal of Fuzzy Systems* is 0.51 but for *Acta Mathematica*, it is 5.77. Similarly, the Mathematical Citation Quotient (MCQ), a 5-year-impact factor computed by MathSciNet (an online publication of the American Mathematical Society), for *Iranian Journal of Fuzzy Systems* and *Acta Mathematica* are 0.11 and 3.95, respectively.

This pattern can be seen in other journals. For example, *Journal of Function Spaces* is ranked 24, while the leading journal *Journal of Functional Analysis* is ranked 47! Again both the SJR and the MCQ of *Journal of Functional Analysis* are much greater than those of *Journal of Function Spaces*.

There is a similar situation regarding the *American Journal of Mathematics*, established in 1878, and a recently launched JCR journal named *Mathematics*.

Some important reasons for such unexpected JIFs are as follows:

- a high rate of publication on a topic. For instance, "fixed point theory" is a popular topic that a lot of mathematicians work on;
- a considerable number of researchers working on a topic. For example, the number of mathematicians who are working on "fuzzy mathematics" is much greater than those working on "K-theory", and hence the general rate of citations in such topics is high.
- the open accessibility of a journal.
- Non-ethical ways to increase JIF used by a few journals. While the term "predatory journal" is arguable, the mere appearance of this term shows that the problem does exist.

The backlog between acceptance and publication in some mathematics journals may exceed two years. Journals with such large backlogs, which are usually good journals, may have unexpectedly low JIF. Nowadays, some journals have moved to the continuous article publishing (CAP) model in which every article, after acceptance, is published immediately within the current issue.

We think that Clarivate Analytics should improve its formula for computing JIF. Until then, we suggest that scientific committees should consider a package of indicators such as the JIF, SJR, Citescore, Eigenfactor together.

The scientometric indicators developed for journals, essentially based on citations, should not be applied as a tool to assess the work of individual researchers. In fact, as citation occurs after research, the direction of research should not be affected by any demand for citation. The scientometric data reflect to some extent the quality of a journal, but not so much the actual quality of a single paper, since not all papers in a journal are cited equally.

As we explain in the next section, when a scientific committee uses only scientometric data to evaluate a mathematician's achievement, without any human assessment, they are using a flawed approach that may result in an unfair judgement.

The role of human assessment

A large number of universities around the world use scientometrics tools to evaluating the research of academic members, postdoctoral researchers, and Ph.D. candidates for promotion, employment, or funding. It seems that such universities have no other reliable sources, and possibly suffer from lack of any peer-reviewed system in which the content of papers is expected to be evaluated by professional mathematicians. In addition, dealing with scientometric data is much easier than reading papers and assessing their content.

There are mathematicians who believe that scientometric data such as the SJR are reliable instruments for judgments, since they make assessments more objective and free them from the crude or biased judgements of human beings. They argue that quantitative indicators help funding organizations, publishers, and policymakers to gain strategic intelligence that leads toward fairer outcomes and ensures that their budget is spent in the most effective way.

However, there are others who are against using scientometrics to measure scientific publications, due to the lack of transparency. Scientometrics may cause distortions that have detrimental effects on the development of scientific fields. For example, some supporters of the JIF subscribe to the idea that every paper published in a high-ranked journal must contain excellent mathematics, which is not entirely true in general; one can easily find some counterexamples in the literature. Some mathematicians propose that citations are relevant only when dealing with large numbers. In small numbers, they can be a misuse of statistics. These mathematicians continue to trust in evaluation by human beings, even though it may be subjective in the sense that it is influenced by the human dualities of love and hate, good and bad, as well as true and false. They believe that metrics put the worth and livelihood of our young mathematicians at risk and have undesirable impacts on the scientific life of all mathematicians.

Although citations do not show all the good qualities of a paper, they (in particular, non-self-citations by reputed researchers in prestigious journals) may help experts in evaluating and documentating research work. Papers with no citation over a 'long period of time' cannot be regarded as high-level papers. For that matter, not all highly cited papers are necessarily high level papers. However, abuse of scientometric data such as the JIF and games with numbers can happen, and may mislead people instead of being an indicator.

Conclusion

Scientometrics tools can be used, provided that one keeps their disadvantages and distortions in mind, and they are considered together with the judgement of experts based on depth and extent of papers. Such experts could be asked to look at a candidate's self-selected best papers, research programs, and statements of major achievements. No assessment is complete without a peer review. Furthermore, we need a modification of the policies of universities, funding organizations, and so on to support human assessments.

We hope that the various ideas discussed in this note may help not only mathematicians but the whole of the scientific community to improve their point of view and their assessment guidelines.

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Mathematical (online) meetings reimagined?

Ross J. Kang

In our daily professional lives, we have become accustomed to – or wearied by – changes brought on by restrictions that started in the spring of 2020. Many of our mathematical research meetings have been transferred directly online, with varying degrees of success. Could we imagine other, better ways of doing things? In this note, I discuss an experimental initiative launched to investigate this question, and call on others to embrace the challenge.

The enemy of art is the absence of limitations.

Orson Welles

A preface

It is clear that issues like the ones raised here are on the minds of many. Indeed, in an earlier edition of the "Young mathematicians' column" of this periodical, we saw two related pieces from the perspective of early-career researchers. This letter gives the viewpoint of mid-career researchers, ones with plenty of familiarity with the organisation of pre-COVID Mathematical events. It describes early experiences in the conception and setup of an "online research guild" devoted to bespoke, scientifically high-level, interactive workshops, specifically designed for these times, and perhaps beyond. Though it might require a bit of courage, the model could be of interest to other mathematical communities.

Introduction to the issues

It's been over a year now. With all the tragedy and turbulence we've witnessed or endured, we're keen for vaccines to bring us back to some semblance of our previous "normality". For many of us, our professional lives have continued to roll along – we read, think, write, talk, teach, advise, collaborate – albeit subject to the rigours of social distancing.

In mid-2020, it was astonishing how rapidly subject-area seminars bloomed into a dynamic panorama of ideas instantly accessible from anywhere, in digital perpetuity. One might wonder if this marks a lasting change in how we learn about, discuss, and explore new research. (Surely yes!)

One might then also wonder if we could make better use of the communication technologies at our fingertips. Is it enough to do nearly exactly what we used to do, except virtually?

Over a few emails, a colleague of mine, Jean-Sébastien Sereni, and I considered these questions and, like many, noticed some shortcomings of the direct transition of meetings online. One, the natural intimacy and informality in talks has become more limited. Two, the exchange of information is now quite one-directional, especially when talks are intended for online video clips. Three, making new acquaintances and connections is more awkward in large groups. Four, and most importantly, these changes have affected younger, more isolated, or less established researchers more. After all, how brave must such a researcher be to raise their voice in a crowded, recorded, virtual seminar room shared with the world's foremost experts in a given topic?

Through our correspondence, the essential question we eventually arrived at was this: setting aside for a moment existing seminar/workshop/conference series, what is the best way to set up online mathematical meetings?

(Perhaps pause to meditate on this, before reading on about how we analysed the question.)

An analysis and a possible approach

Let's break the problem down. What are the main scheduling constraints? Rather than coordination of travel arrangements, it is the participants' ongoing care/service/teaching obligations and the intersection of their timezones. Scope and scale? Rather than large meetings covering many topics, it's very sensible to focus on one specialised topic at a time in smaller gatherings. Which format encourages intimacy and multilateral interaction? Instead of only showcasing talks with the latest technical results, we can give extra weight to expert surveys or tutorials, and set aside time for reflection, discussion and problem-solving in small groups. How to spark new connections? As it is more difficult to have meaningful chance encounters online (as we used to have at coffee/drinks, over meals, on walks), we can make use not only of common research interests but also of our existing networks to stimulate new links.

Based on these thoughts, we decided to try the following setup. We gathered a broad group of European researchers in our field (graph theory, currently an online research guild of a few dozen members, cheekily dubbed 'A Sparse (Graphs) Coalition'), with the shared aim of curating and organising a diverse series of small-scale, high quality, interactive, online workshops. The goals are to

An obligatory group photo from the first workshop

learn, prove and conjecture. The workshops have focused on open problem-solving in loosely-organised breakout groups, with tutorials, surveys and update/social meetings lightly interspersed. The meetings have deliberately been planned with current workflows in mind, with sparing but strategic use of virtual contact time, to let participants think about the mathematics independently according to their personal schedules and ongoing obligations.

(The wiki at https://sparse-graphs.mimuw.edu.pl/doku.php has more detail on the formats we tried out. By no means do we claim to have found an optimal construction! If others have begun similar initiatives, we'd be happy to get in contact to share best practices.)

To our delight and surprise, however, this method turned out to be very effective. In both pilot sessions, which took place in late-February (on generalised colouring numbers, organised by Piotr Micek and Michał Pilipczuk) and early-March (on the entropy compression method, organised by Jean-Sébastien and myself), there was high interest, engagement and satisfaction. Participation, especially by younger researchers, was eager and committed: they not only enjoyed their experiences greatly, but also learned and achieved a great deal, while forming close new contacts. Several of the working groups have begun preparing manuscripts for publication – already two have been posted: arXiv:2103.17094, arXiv:2104.09360. With all of the online tools available these days, organisation was exceptionally light and pure, and without the usual worries about finances, travel, bookings, and administration.

It went so well that we found ourselves asking why we hadn't done this years ago.

(One can even imagine, a little outside of the traditional structures, whether more creative ideas for increasing the potential of modern internet conveniences in science generally are long overdue – think of journals, societies, and training, for instance.)

Future work

Now this brings me to the main point.

Of course it is natural to take the stance that this is all temporary and we can soon return to our earlier, pleasant, and high-flying ways of discussing mathematics with our distant colleagues. But while this predicament lasts, why not creatively experiment with and get used to something versatile and more sustainable?

There is clearly much to gain, even for when restrictions are relaxed. If remote practices prove sufficiently effective and advantageous in the long run to complement and partially replace our earlier methods, consider the savings in research funding and carbon footprints, or the accessibility regardless of participants' grant status or childcare responsibilities, or the ease and flexibility of organisation, or the rapid responsiveness to developments in the field. With some extra thought on their design, could online workshops be a fast, cost-effective, convenient, accessible, sustainable, engaging, and powerful mode of mathematical cooperation?

I leave this as an open problem.

Acknowledgements. I am sincerely appreciative of some chats with Riccardo Cristoferi, Eoin Hurley, Jean-Sébastien Sereni, Oscar Treffers, and, most especially, my wife Rei during the process of composing this letter.

Ross J. Kang is a Canadian mathematician based in the Netherlands. He obtained his doctorate in Oxford in 2008 and has been based at Radboud University Nijmegen since 2014. His research lies at the interface of algorithms, combinatorics and probability. Recently he has begun actively nurturing his interests towards diversity and openness in science.

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The Weierstrass Institute for Applied Analysis and Stochastics (WIAS) in Berlin

Michael Hintermüller

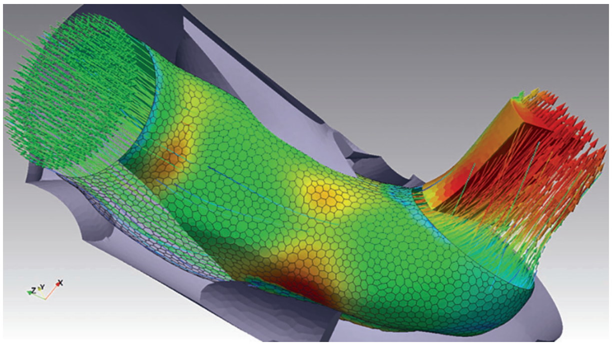
Motivated by current challenges in technology, economics and business, medicine, science, and society at large, the Weierstrass Institute for Applied Analysis and Stochastics (WIAS) conducts project-oriented research in applied mathematics. In its daily routine, WIAS addresses the entire solution cycle from mathematical modelling, analysis, and simulation to optimization – always in close interaction with practitioners or scientists from other disciplines. Moreover, one of its strengths is the interplay of applied analysis and stochastics, which puts it right at the center of the state-of-the-art in applied mathematics. Many of the analytical findings at WIAS lead to the development of solution algorithms and subsequently software packages. In this respect, WIAS's software engineering and licensing strategy targets both academic partners and industry-based users. The sustainable dedication to problem solving and software development is only possible due to WIAS's extraordinarily motivated scientific staff with different career directions, located in a unique working environment near the *Gendarmenmarkt* right in the center of Berlin, Germany.

The various research activities of WIAS unfold around certain main research areas, which address current societal challenges such as the sustainable use of energy, the advance of medical technology, the development of next-generation materials, and the reliable extraction of information from data. Driven by its operation at the forefront of mathematical science and in close interaction with its scientific advisory board and other stakeholders in the German Federal Ministry for Education and Research as well as Berlin's Senate Chancellery, WIAS continuously scrutinizes the relevance of its main research areas and develops adjustment strategies accordingly. Currently, the specific focus areas are:

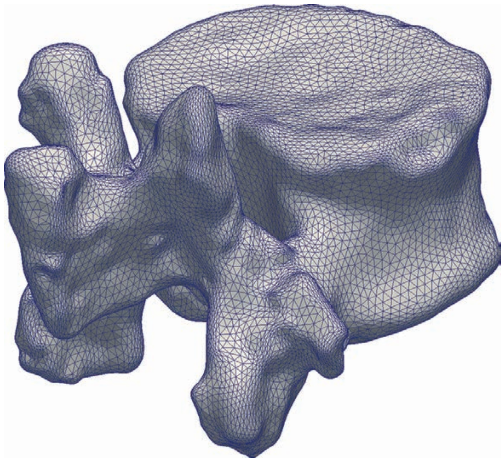
- Conversion, storage, and distribution of energy;
- Flow and transport;
- Material modelling;
- Nano- and optoelectronics;
- Optimization and control in technology and economics;
- Quantitative biomedicine.

The institutional scientific staff organization is structured into research groups, which are typically oriented by mathematical or physical field. Currently, eight groups are installed, with the following respective focal points:

- Partial Differential Equations;
- Laser Dynamics;
- Numerical Mathematics and Scientific Computing;
- Nonlinear Optimization and Inverse Problems;



Shape-optimised exhaust pipe



Mesh of a spinal bone calculated with WIAS mesh generator software TetGen

- Interacting Random Systems;
- Stochastic Algorithms and Nonparametric Statistics;
- Thermodynamic Modelling and Analysis of Phase Transitions;
- Nonsmooth Variational Problems and Operator Equations.

While no specific time period is fixed, these research groups are typically expected to operate on a mid- to long-term basis. In order to flexibly explore novel topics and to advance young scientists into group leader positions, WIAS has installed the Flexible Research Platform, which also helps to foster and host temporary research projects, financed, e.g., by the European Research Council, the Leibniz Association, or similar.

Within Berlin, WIAS has been a reliable partner for excellence projects such as the DFG Research Center MATHEON, the Einstein Center for Mathematics Berlin, or the current Cluster of Excellence MATH+ financed by the German Excellence Strategy. These activities have always been complemented by participation in DFG collaborative research centers established for up to twelve years at one or several of the Berlin universities or the University of Potsdam. In addition, WIAS structurally collaborates closely with Berlin's three major universities, the Freie Universität, Humboldt-Universität, and Technische Universität Berlin, for example in terms of joint appointments, supervision of students and research-oriented teaching.

On a national scale, the Institute is a member of the Leibniz Association, an umbrella organization currently connecting 96 independent research institutions ranging from natural, engineering and environmental sciences to economics, spatial and social sciences and the humanities. Consequently, WIAS is subject to a transparent, independent evaluation procedure installed by the Leibniz Senate. Such evaluations take place routinely every seven years. Among other things, the evaluation focuses on how the institution has developed in the intervening years, in particular in terms of overall scientific content, structure, and future plans. Because of its importance and system-relevance for Germany and its international competitivity, like other Leibniz institutions, WIAS receives its core-funds jointly from Germany's central and regional governments. The Institute actively participates in collaborative structures of the Leibniz Association such as the Leibniz Network *Mathematical Modelling and Simulation* (coordinated by the WIAS) or the Leibniz Research Alliance *Health Technologies*.

Supplementing its basic public funding, the Institute successfully raises funds from a variety of competitive funding programmes as well as from industry and economy. In particular, in recent years it won one ERC Advanced Grant, one ERC Consolidator Grant and three ERC Starting Grant projects with the corresponding research carried out at WIAS. Also on the national scale, the Institute is actively involved in many special research activities, priority programmes, research training groups such as the German Research Foundation (DFG), and programmes funded by the Federal Ministry of Education and Research (BMBF), etc.

In addition to the actual mathematical research, the Institute also actively engages in overarching challenges such as the handling and sustainable use of mathematical research data following the FAIR-principles. In this respect, WIAS is currently coordinating the *Mathematical Research Data Initiative* which is contributing to the German "National Research Data Initiative".

While the FAIR (= findable, accessible, interoperable and reusable) handling of research data is of importance for all sciences dedicated to a modern open research and access policy, WIAS will continue striving to prolong and possibly expand its position in excellence of mathematical research. In particular, the interplay of analysis, stochastics, simulation, and optimization in mastering the transition from a fossil fuel-based energy system to one with a vast portfolio of renewable energy carriers along with

IMU President Ingrid Daubechies inaugurates the IMU Secretariat in Berlin in January 2011 (Photo: Kay Herschelmann)

optimal distribution and storage will be one of the future target areas of WIAS. This is also true for the extraction of information from data of various kinds and the incorporation of data-driven models into its research workflow, as well as for challenges in quantitative biomedicine, in particular medical imaging. In the field of "Material, Light and Devices", research on quantum-technological aspects will be pursued.

WIAS is a part of high calibre national and international institutional activities. First and foremost, it proudly hosts the Secretariat of the International Mathematical Union (IMU) as one of its program units and supports the IMU in its globally important, fascinating activities to the best of its abilities. The main office of the German Mathematical Society (DMV) is located at WIAS. Within the EMS, the Weierstrass Institute is a member of ERCOM (European Research Centres on Mathematics). It is likewise an active member of ECMI (European Centres on Mathematics in Industry).

Some historic facts and current figures

WIAS was established on January 1, 1992. It originated from the former Karl Weierstrass Institute for Mathematics of the GDR Academy of Sciences. The founding committee was headed by Karl-Heinz Hoffmann. The back-then new Institute was provisionally directed by Herbert Gajewski until Jürgen Sprekels was appointed as its director in April 1994. Since the beginning of 2016, it has been headed by Michael Hintermüller. The Institute currently employs around 150 people, among them 120 scientists at various career levels, ranging from PhD students to senior scientists, and jointly appointed professors. The daily work of WIAS benefits from its well-trained administrative and IT staff.

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Michael Hintermüller is the director of the WIAS and holder of the Chair of Applied Mathematics at the Humboldt-Universität zu Berlin. For many years, he has played a major role in shaping the international math hub of Berlin. For example, he assumed the role of Spokesperson of the Berlin Mathematics Research Center MATH+ in 2019. He acted as the Spokesperson of the Einstein Center for Mathematics Berlin (ECMath) from 2016 to 2019. He joined the Humboldt-Universität zu Berlin as a MATHEON Research Professor in 2008, and was appointed Member of the Council of the DFG Research Center MATHEON in 2011. The Austrian mathematician has received multiple awards for his scientific achievements. He is a Fellow of the SIAM (Society for Industrial and Applied Mathematics), and received the "Start-Preis" – Austria's most prestigious award for young scientists – in 2005.

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Slovak Mathematical Society – a short overview of history and present

Martin Kalina

The Slovak Mathematical Society (SMS) is a small society with about 150 members. Among our members, there are university staff, academicians from the Slovak Academy of Sciences, researchers from other research institutes, and also teachers at primary and secondary schools. Our programme is focussed on science, education, and young talents in mathematics.

Our roots

The Slovak Mathematical Society was officially founded on January 1, 1969, as the mathematical branch of the Union of Slovak Mathematicians and Physicists. However, the real root of the organisation of scientific life of Slovak mathematicians lies in the Union of Czechoslovak Mathematicians and Physicists, or, more precisely, in the Union of Czech Mathematicians which was renamed Union of Czechoslovak Mathematicians and Physicists in 1921. The year 1929 saw the beginning of the first regular seminar for mathematics and physics, at the Faculty of Medicine of the Comenius University in Bratislava. The year 1951 was also important, marking the decision to create the Slovak Committee of the Union of Czechoslovak Mathematicians and Physicists. The Mathematical Olympiad has been organized in Slovakia since that time.

What we do

Among Slovak mathematicians there are research groups dealing with

- discrete mathematics: graph theory, combinatorics, …;
- algebra: semigroups, set theory, quantum structures (i.e., orthomodular lattices and Hilbert spaces as the most important representatives of orthomodular lattices);
- theory of chaos;
- mathematical and functional analysis;
- numerical analysis;
- applications of PDE in engineering and in some other areas (e.g., medicine);
- probability and statistics;
- uncertainty modelling: time series, fuzzy logic, aggregation of information, generalised measure theory (capacities, i.e., monotone set-functions) and integrals with respect to capacities.

Mathematics and music

The SMS organizes various series of regular seminars, each of them devoted to a different field of mathematics, corresponding to the research groups listed in the previous section. Among these, the seminar on "Mathematics and Music" is particularly worth mentioning. In this seminar, which originated in the 1970s and lasted, with some breaks, till the 1990s, mathematicians whose hobby was music met with musicians – professional composers and interpreters. One of the main organisers of the seminar was late Professor Riečan (1936–2018). Music was his great passion. Whenever he organised a mathematical conference, there would be one evening session devoted to a concert by the participants. The first of the three photos shows Professor Riečan performing with a Ph.D. student during one of these concerts.

Annual conference of Slovak mathematicians

Another event that I would like to mention is the annual conference of Slovak mathematicians. This is a conference where university staff members and academicians meet teachers. In other words, it is not a high-level scientific conference where the participants present their latest results; rather, there are overview talks on a level that is understandable even for primary school teachers. There is a competition of young mathematicians (under 30 years of age) who compete by submitting series of papers. The winner is awarded the Academician Schwarz Prize (named for Štefan Schwarz 1914–1996, whose main area of interest was the theory of semi-groups). Apart from this, there is also the Peter Pavol Bartoš Prize for teachers and university staff dealing with didactics of mathematics. The prize is awarded for a nice textbook or for long-lasting excellent results in teaching mathematics (Peter Pavol Bartoš 1901–1975 was a high school teacher known for his textbooks and the problems he set for the Mathematical Olympiad).

Czech-Slovak student conferences

A long-standing tradition was the organisation of student scientific conferences in Czecho-slovakia as competitions. Nowadays, we still have such a conference-competition, organized in cooperation with the Czech Mathematical Society in mathematics and informatics, and another one in didactics of mathematics organized

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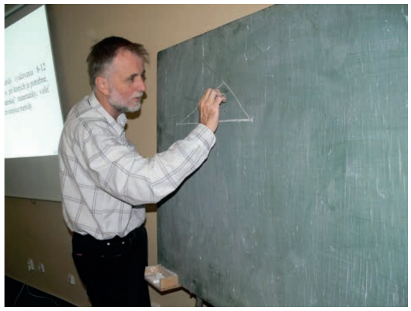
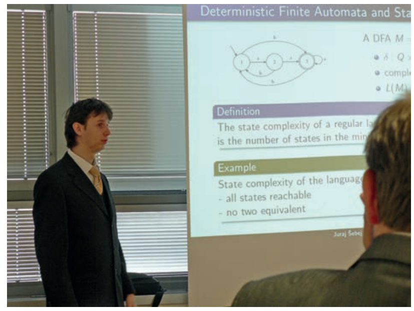
Prof. Riečan with his PhD student during the concert of participants in 2010



František Kosper, high school teacher, the winner of the Bartoš Prize in 2013, giving his talk



Juraj Šebej, one of the participants in the Student Conference in Mathematics and Informatics in 2012

in cooperation with SUMA (Society of Teachers of Mathematics of the Czech Republic). These events have a three years period – they always take place twice at universities in the Czech Republic and once in Slovakia. Only in 2020, due to the coronavirus pandemic, the conference in didactics of mathematics was taken online and the conference in mathematics and informatics was cancelled altogether.

Some other scientific events

Cooperation with mathematical societies of other European countries is pursued thanks to the organisation of several conference series. Let us name a few.

Equadiff is a series of biannual conferences. Its scope is mathematical analysis, numerical mathematics and applications of differential equations. The history of this series is quite long. It rotates between the Czech Republic, Slovakia, and Western Europe. It has already taken place 14 times in the Czech Republic and Slovakia. In 2017, Equadiff was hosted by the SMS.

In cooperation with the Czech, Slovenian, Austrian and Catalan Mathematical Societies, a series of CSASC conferences was organized. In 2018, CSASC was hosted by the SMS. The CSASC conference consists of several mini-symposia. The 1st CSASC was held in Krems, Austria, in 2011.

EUROCOMB is the European Conference on Combinatorics, Graph Theory and Applications. The 1st EUROCOMB was held in Barcelona, Spain in 2001. In 2019, this conference was hosted by the SMS.

Martin Kalina is a professor of applied mathematics at the Slovak University of Technology in Bratislava, Slovakia. Since 2017 he is the president of the Dlovak Mathematical Society and since 2008 the president of the Union of Slovak Mathematicians and Physicists.

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MathsWorldUK: Creating the UK's first National Mathematics Discovery Centre

Katie Chicot

MathsWorldUK's mission is to create a world-class Mathematics Discovery Centre, celebrating the mathematics at the heart of the patterns and structures of our world. A first for the UK, the centre will combine best practice from successful partners across the world with ingenious new ideas and innovative environments developed in the UK. It will be a magnet visitor attraction, drawing visitors from across the UK and from overseas.
We aim to advance the public understanding of mathematics and its applications so that everyone, regardless of age, gender, background and ability, can access and delight in the joy and power of mathematics.

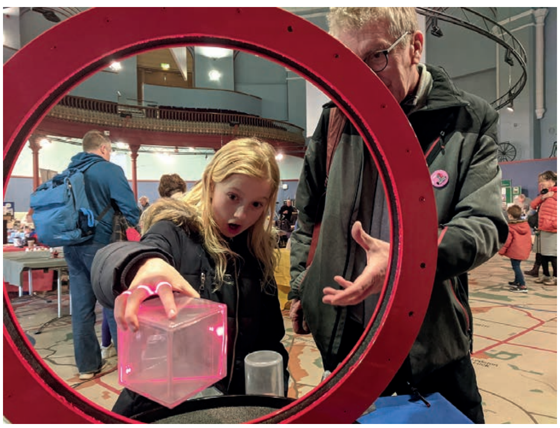
MathsWorldUK works with partners across the UK and internationally to spark public excitement, curiosity and engagement with the wonders of maths. We create events and challenges at science fairs, in schools and public settings, as far afield as Greenwich (Greenwich Maths Time), Birmingham (the Big Bang Young Scientists' and Engineers Fair) and Bradford (International Mathematics Day around the country), aiming to reach a range of groups including those whom might otherwise be excluded from such opportunities.

Our approach is informal and non-threatening, playful and inspiring – inviting children and adults alike to discover the fun of maths and enjoy their own inherent mathematical and problem-solving abilities. As our project continues to develop and we have more space for our exhibits we will offer visitors many hands-on experiences in several different areas of mathematics – geometry, logic, spatial awareness, statistics, mechanical paradoxes and other cheerfully perplexing things, as well as displaying mathematical art and aesthetics.

Our plans have the support of all professional associations in mathematics and mathematics education in the UK and of many Fellows of the Royal Society. Over 40 ambassadors have rallied behind our vision including: Professor Sir David Spiegelhalter; Dr Hannah Fry; Bobby Seagull; Professor Sir Martin Hairer; Tim Harford OBE; and Johnny Ball.

Why do we need a Mathematics Discovery Centre in the UK?

A Mathematics Discovery Centre plays at least two important roles. It goes without saying that the centre should celebrate the major cultural contribution maths has made throughout human history



Families interact with the 'Ring of Fire' at Leeds Science Festival 2020

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Dr Katie Chicot, CEO, Geoff Wain, Co-Chair, and Prof Margaret Brown, Co-Chair (from left to right)

and how it has helped us understand our world. But there is also an important social dimension.

Mathematics is a life skill which is central to one's life prospects and wellbeing. This is regarded as obvious in most countries, but it may surprise European readers that it is not widely believed in the UK (despite urgent government reviews and actions). The UK has one of the lowest uptakes of mathematics post-16 of any developed nation and this is partly a reflection cultural attitudes to mathematics, that it is boring, difficult and irrelevant.

No one doubts the necessity of being literate. We need to generate the understanding that a lack of general mathematical awareness has a similarly strong impact on life chances as the abundant evidence shows.

It isn't only the individual who is affected by a poor culture around mathematics. Society and the economy need a strong base of mathematically qualified citizens. The role of the mathematical sciences is increasing in the workplace. This is due, in part, to increased computing power, big data and greater use of the modelling of complex phenomena in decision making.

A cultural intervention is needed to address the UK's (particularly England's) attitudinal relationship to mathematics. Looking at the UK data we can see that a person's and family's culture is having the strongest impact on their study choices which is why our work focuses on a family-oriented visitor centre. Nevertheless, Our centre will be of interest to all, including professional users of mathematics, It will be an interesting, challenging and fun place to visit, and a place to return to on many an occasion.

Who's involved?

MathsWorldUK was established by a few enthusiastic people, encouraged and supported by the mathematics organizations concerned with public engagement with Mathematics. It became a registered charity (no. 1155010) in 2013.

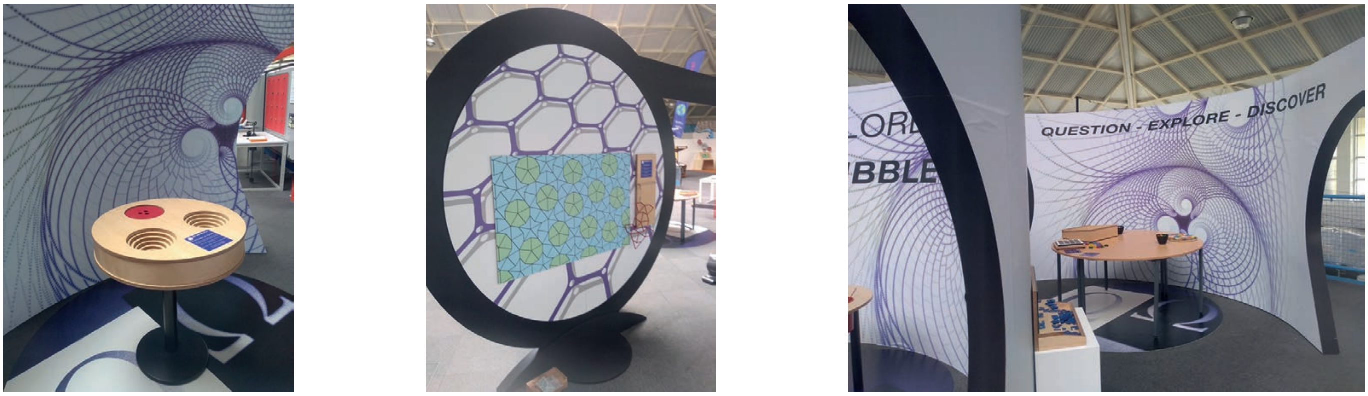
Co-chair, Margaret Brown, OBE, is an Emeritus Professor of Mathematics Education at King's College London and is a previous Head of the School of Education and member of the Senior Management Committee at King's. She has served as President of the British Educational Research Association and of the Mathematical Association, and acted as chair of the Joint Mathematical Council of the UK.

For 10 years Margaret chaired the Trustees of the School Mathematics Project, a charity which was concerned with provision of school textbooks and curriculum materials, and with teacher professional development. She was instrumental in setting up the King's College London Mathematics School for 16–18 year olds. In 2013 Margaret was awarded the prestigious Kavli Medal of the Royal Society for her services to mathematics education.

Co-chair, Geoff Wain, spent 40 years in education, as a teacher and then a teacher trainer. For 24 years he was at the University of Leeds in the Centre for Studies in Science and Mathematics Education, including time as the director of teacher training and as Dean of the Faculty of Education. Geoff was a co-director of the PopMaths Roadshow, a major inter-active exhibition of mathematics which toured 24 cities in the UK and Ireland.

CEO, Katie Chicot, is a Senior Lecturer, Staff tutor in Mathematics at the Open University. Katie has used all sorts of means of communicating mathematics, including co-creating the series Patterns of life for the Open University's YouTube channel, captaining a team on BBC2's Beat the Brain, as academic consultant to BBC Radio 4's More or Less, and the creation of a maths/brain teaser app called Perplex, which is available on the App Store and Google Play.

We have a dedicated executive committee behind the project and a body of volunteers who enable us to carry out our work with the public.



MathsWorldUK's touring exhibition 'Explore Maths' installed at Winchester Science Centre

Pre-Covid-19 strategy

Many Mathematics Discovery Centres around the world have grown out of touring exhibitions. These exhibitions demonstrated the huge appetite for physical maths activities, gradually building momentum for a physical discovery centre. To date, there are over 50 such permanent maths centres around the world, but the UK has none. MoMath, the National Museum of Mathematics in New York has an annual footfall of about 200 000 visitors and has changed for the better the perceptions of numerous people about the nature and importance of mathematics. Other centres have had similar success.

By the start of 2020 we had matched our funding offer of £125,000 (from an anonymous American donor). With this we created our first touring exhibition 'Explore Maths'. We installed exhibits in readiness for a March 2020 launch in Winchester Science Centre (WSC); we ran a training event with the staff and prepared accompanying materials for teachers and families. In the event, owing to the COVID-19 crisis, Winchester Science Centre closed its doors in March, as did attractions across the country. Lockdown was much longer than all had anticipated, clearly putting the rest of our planned tour under threat.

In response to the effect of the Coronavirus on our activities we have developed new plans to maintain our momentum towards creating the UK's first Mathematics Discovery Centre.

Activities during Lockdown

During lockdown we wanted to continue our work with the public and offer mathematics engagement which had an element of delight to counteract what the world was experiencing and to support those learning from home. We obviously couldn't do this in person and we didn't want to lose any of the momentum we had built up. We created a set of videos that have the general title *Maths at Home with MathsWorldUK* and feature all the great maths communicators that we know. The videos were directed by Dr James Grime and are for parents, children and anyone with a bent for exploring mathematics to while away the time spent under Covid restrictions, lockdown or otherwise. There is a prominent link to these videos on the Home Page of MathsWorldUK mathsworlduk.com

The quality of the presenters speaks for itself. You can engage in cutting Mobius Strips with Bobby Seagull, or colouring in 'maps' or regions with Katie Steckles, or learn about measuring the Earth through looking at the historical contributions of some ancient Greek mathematicians with Johnny Ball, or you can enjoy curious ideas like 'Diffy Squares' with Rob Eastaway. You can learn some card trick magic with Zoe Griffiths and experience many more such delights. You can share in the enthusiasm of Nira Chamberlain, the President of the Institute of Mathematics, as he discusses the Gambler's Ruin Problem. At the time of writing the latest video is an exploration of international number systems with Alex Bellos with Danish Numbers as the starting point (youtu.be/yHcdM2MLuLE).

Alongside the Maths at Home videos we made a series of videos highlighting important mathematics used to study the Coronavirus pandemic. Our own Kit Yates, who is a mathematical biologist and Senior Lecturer in Mathematics at the University of Bath, explains the basic mathematics behind modelling the spread of a deadly virus like the Coronavirus or the Zika virus, and how by looking at members of the population who are Susceptible to the virus, those who are Infectious and those who have Recovered, we can develop a basic mathematical model known as the SIR model. We can then use this model to make predictions about how the numbers in each category compare to actual samples taken from the real world to decide whether the SIR model is a good one or not.

David Spiegelhalter's video is a wonderful explanation of the concept of a false positive in a diagnostic testing regime. He explains why even the Health Secretary dealing with the Coronavirus pandemic became confused about how to interpret the significance and meaning of a false positive test result. Starting with the current pandemic, David introduces the terms prevalence (the probability of someone having the disease), sensitivity (the probability of someone having the disease and testing positive) and specificity (the probability of someone not having the disease testing negative) and then uses a probability tree to discuss the relative numbers of true positives of a medical test (when the test correctly indicates that you have the disease when you do) and false positives (when the test indicates you have the disease when in fact you do not). At the end of his exposition David also makes reference to Breast Screening tests and Facial Recognition software as used by the police. In each case it is essential to understand how the number of false positives are related to the samples being investigated.

The Government has repeatedly stressed the idea of social distancing and that the safe distance apart between two people is 2 metres. In her interesting video, Aiofe Hunt uses some familiar mathematics, including the Theorem of Pythagoras, to show that designing spaces to hold numbers of people all safely socially distanced from each other is far from obvious and requires much more space than most people realize.

Professor Jen Rogers was previously the Director of Statistical Consultancy Services and an Associate Professor at the University of Oxford She is now the Vice President for Statistical Research and Consultancy at PHASTAR. Jen's video gives an excellent and reassuring look at vaccine trials and explains how the time for the vaccine development was cut without any reduction in the robust testing procedure or even any reduction in the actually number of man-hours spent developing the vaccine.

Post Covid-19 strategy: MathsCity Leeds

We are now working to establish a pop-up mathematics discovery centre in the centre of Leeds, which we are calling *MathsCity*. We have the support of Leeds City Council and the Leeds Business Improvement District. We are in talks to secure, for at least one year, a suitable rent-free property in a popular shopping area.

MathsCity will start by housing the original touring exhibition, with its problem-solving theme, and also a Shape and Space Zone. One of the many benefits of this organic way of growing our mission is that we can develop the content for the future centre and test it robustly with the public. In this way the best content will be ready for the future National Mathematics Discovery Centre. We will be changing the content of MathsCity in 9–12 month intervals. The second set of contents are planned to be a Codes and Code-breaking Zone and a Zone on the Mathematics of Pandemics.

It will be difficult to launch a centre under the shadow of the pandemic, but we will be able to learn the lessons of other science discovery centres to run a COVID safe environment. Visitors, including school parties, will be asked to pre-book their timed visits, and we will have gaps between visits to allow for cleaning.

Simon Norton Legacy

Our mission now has the generous support of Michael Norton who has pledged £1.3 million to the mathematics discovery centre in memory of his brother, the mathematician Simon Norton. Part of this funding can be used towards the establishment of *MathsCity*, but the majority is pledged towards the ultimate centre.

Simon Norton who passed away in February 2019, was a child prodigy in mathematics. He represented Britain at the International Mathematical Olympiads three times in the 1960s, scoring the top grade each time, once with 100 %, another time with 99 %, and winning a special prize for the elegance of his solutions.

He did his PhD under John Conway in Cambridge and with John Conway he produced the seminal Atlas of Finite Groups.

Simon became the world expert on the Monster Group and its connection to Modular Functions and with John Conway coined the term Monstrous Moonshine. "I can explain what Monstrous Moonshine is in one sentence," said Simon. "It is the voice of God."

Simon was fascinated by the huge number of symmetries associated with the Monster Group. This group has order of roughly 8×10^{53}, that is the number of elements in the set that defines the group.

The Legacy left by Simon Norton is administered by his brother, Michael Norton, and a group of Trustees. Michael Norton is now taking an active role in helping MathsWorldUK to move towards its ultimate goal.

Michael is the Director of the Centre for Innovation in Voluntary Action (CIVA). Michael has decades of experience in developing major innovative projects each of which has the theme of social justice. We thank him for his efforts so far and know that through his generosity we are making good progress to our ultimate goal.

Michael Norton

MathsWorldUK at various festivals and community centres

The UK's first National Mathematics Discovery Centre

The UK is almost alone in not having an informal maths space in which to meet, explore and admire mathematical achievement. We would like the entire maths community to be behind this project and to be involved in shaping its content and direction. We have been fortunate to have guidance from European Mathematics Discovery Centres, most notably Mathematikum in Giessen and MMACA in Barcelona. We would welcome further involvement from supporters in Europe and elsewhere.

If you would like to support to our mission, then visit www.mathsworlduk.com/join-us/ and see the ways you can get involved. Or you can email our CEO on katie.chicot@mathsworlduk.com. We look forward to hearing from you.

MathsWorldUK is registered in the UK as a charity (number 1155010) and as a company (number 837040).

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Katie Chicot is the CEO of MathsWorldUK whose aim is to create the UK's first Mathematics Discovery Centre. Alongside this Katie is a Senior Lecturer, Staff tutor in Mathematics and Statistics at the Open University. This involves working with students, tutors and creating teaching materials. Both roles involve mathematics outreach in many forms. Previous outreach has included free courses, a maths app, competitions, videos, radio, and events. The next step of creating a physical centre is now very close indeed.

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Romania helps Uganda on its way to the International Mathematical Olympiad

Sergiu Moroianu

This article reports on an ongoing privately-financed project started in 2019, promoted by members of the Romanian mathematical community, supporting Uganda's participation in the International Mathematical Olympiads.[1]

About the IMO

The International Mathematical Olympiad (IMO) is arguably the most prestigious scientific event for high-school students worldwide, often copied but never surpassed. The IMO model was adopted for other disciplines: Physics, Computer Science, Chemistry, and also for regional competitions: the Balkan Mathematical Olympiad; the European Girls' Mathematical Olympiad; Olimpiada Iberoamericana de Matemática; the Asian Pacific Mathematics Olympiad; the Pan-African Mathematical Olympiad (PAMO), and many others. The IMO is an intellectual competition akin to the athletic Olympics. It evaluates a certain ability to solve elementary level problems quickly. Competitors fight to surpass themselves, not against other participants. Between the ages of 12 and 18, aspiring IMO participants dedicate most of their free time to preparing for the IMO. They are trained by professional trainers, and their work is mainly driven by inner motivation.

Many successful mathematicians never participated in Olympiads, while many former IMO participants choose various other professions later in life. Nevertheless, the IMO – and elementary math competitions in general – play an outstanding role in raising awareness about mathematics and about our profession among young students during the decisive years of their intellectual formation.

A brief history of the IMO

The IMO is an Eastern European cultural product. The first IMO took place in 1959 in the mountain resort of Sinaia, Romania, involving students from just seven countries from the former Eastern European bloc.[2] Since 1959, it has grown steadily to the point where no less than 115 countries were represented at the 60th edition in England. A notable forerunner of the IMO was the yearly competition organized by the *Gazeta Matematică*, the oldest periodical publication in elementary mathematics worldwide, published uninterruptedly in Romania since 1895, even during the two world wars. In the first twenty years, Eastern European bloc countries dominated the competition, but today students from Australia, Canada, France, Iran, Italy or Vietnam compete successfully with the Eastern European students from Bulgaria, Hungary, Poland or Ukraine. The explanation for the continuous success of some countries lies in the professional methods of selection and training described below. These methods yield spectacular results wherever they are adopted.[3] The countries with the strongest overall performance at the IMO are China, Russia, and the US.

Training and selection for the IMO: The case of Romania

Romania selects future IMO participants from all middle schools around the country. Starting in fifth grade, students are encouraged to solve four Olympiad-type problems each month from the *Gazeta Matematică*. The first two stages of the annual math Olympiad are attended by thousands of the best students in the country. Secondary school teachers receive favorable evaluations when their students qualify for the higher stages of the competition. In most cities, extracurricular excellence clubs are organized weekly, supported financially by the Ministry of Education or by private foundations. Teachers in top schools train their gifted students far beyond the official curriculum. Numerous training camps, clubs, and online training programs take place every year. Coaches for the Olympiad are often themselves former IMO participants. Through this inclusive approach, not only do we select native talents from the maximal pool of 200,000 students in each age group, but we then train them over a long period of time. Participation and training costs are covered by the state. Olympiad winners receive prizes as well as scholarships and admission offers to universities.

[1] www.imar.ro/~sergium/mathuganda/new_mU/index.html
[2] imof.co/about-imo/history/
[3] www.ams.org/notices/200810/fea-gallian.pdf

Like in sports, results at the IMO are directly correlated with the material effort invested by society in young competitors. I need not argue here the merits of having a mathematically-literate workforce. I only mention that in recent years a growing share of Romania's GDP is produced by the vibrant IT industry, made possible in part by the public's 125-year-old obsession with elementary math competitions.

The IMO in developing countries

Developing countries have taken a more sinuous path in adapting to the IMO culture. Some – Thailand, Korea, Singapore, but also Peru, Colombia or Brazil – now have a solid tradition, and their rise in IMO rankings follows that of their GDP/capita. Others have been less successful. Until 2010, no country from "Black Africa" – that is, outside of the Maghreb and South Africa – had ever participated in the IMO. Only 10 out of the 54 African countries participated in IMO 2020, but 6 of these sub-Saharan: Nigeria, Ghana, Kenya, Tanzania, Botswana and Uganda.

There are obstacles these days for a new country to start attending the IMO. Firstly, IMO problems become harder every year. Here is the first problem from the first IMO in 1959: *Show that the fraction* $(21n+4)/(14n+3)$ *is irreducible for any natural number n.* This exercise is today accessible even to a good sixth grade student! As years go by, it becomes more and more problematic for a country lacking a pool of former IMO participants to obtain good results fast enough in order to justify further participation.

Other difficulties are of a more practical nature, hard to understand in privileged countries. The leadership of some poorer countries does not see any financial benefit in supporting the Olympiads; others simply cannot afford the expense, while in some extreme cases, there can be countries which invest in selection, train their team, and pay for plane tickets, only to find themselves unable to attend the IMO for administrative reasons. This was the case for the Nigerian team, which in 2019 had a student on whom they placed high hopes of winning at least a bronze medal. Sadly, due to bureaucratic issues, the team's visas for the United Kingdom were not issued until after the competition was over.

The mathUganda project

In 2018, I was one of the coordinators at the 59th IMO in Cluj-Napoca, Romania. I spoke to half of the team leaders from all over the world, taking the opportunity to inquire about the Olympiads in their countries. One of these leaders was Jasper Okello, the initiator of Uganda's participation in the IMO. Before 2018, Uganda had received a Honorable Mention at the IMO twice. The best ranking of the team had been in 2017, where it was in the bottom 14.55 %. I learned that the Ugandan state does not support students' training or participation in the IMO. I was particularly struck by the fact that the Uganda Mathematical Society had not even been able to obtain support for participation from any international bodies such as the IMU. Together with Jasper, we began sending out funding applications to various charities or learned societies. I also contacted acquaintances in academia, but with no success. It seemed that public funding from rich countries is simply not aimed at talented young people in countries like Uganda.

I finally opened a private online donation list. I estimated the total participation cost for the team as around 10,000 euros, of which I expected to raise 10 %. The response was overwhelming. Donations started to pour in from family, friends, colleagues, and even strangers who learned about our project. We reached the initial target in under three weeks. The Romanian Society for Mathematical Sciences became involved in the project. Colleagues with solid experience in IMO training offered online lessons. By June 2019, we had transferred $8,000 to the Uganda Mathematical Society. Thanks to our help, Uganda was able to send a complete team of six students to the IMO 2019. They purchased their plane tickets early enough to submit visa applications on time, unlike the less fortunate Nigerian team.

At the IMO 2019, Uganda presented a team of three girls and three boys, which had won three bronze medals at the Pan-African Mathematical Olympiad (PAMO). This team ranked 102nd out of 112 participating countries. Out of just 5 points obtained by Team Uganda, 2 were due to Eva Kakyo, who was initially a reserve. Eva's trip to IMO 2019, and consequently the team's result, were possible thanks to the generosity of our Romanian sponsors.

Kampala training in 2020

Encouraged by our project, Jasper Okello applied and succeeded in getting Uganda included in an MIT IMO-training program already implemented in Ghana in 2019. A team of three MIT students, including a former gold medalist at IMO 2018, conducted the IMO selection camp in Kampala in January 2020.

In February 2020 I traveled in Uganda for two weeks at the invitation of the Mathematics Department of Makerere University, the oldest higher education institution in East Africa. I returned home just one week before the borders closed during the pandemic.

Most of my time in Kampala training the IMO and PAMO teams placed emphasis on synthetic geometry: similarity, the circle, cyclic quadrilaterals, intersecting secants, polar lines. We also touched on recurrent sequences, inequalities (Cauchy-Schwarz, AM-GM), number theory, and functional equations.

Traditionally, geometry is Uganda's strong field. To achieve excellence on this topic, I reviewed the whole theory starting from the axioms – the three cases of congruence and parallelism. I had already noticed in 2019 that students tended to learn results "by rote". They needed several good minutes to re-discover the proofs for the sum of the angles in a triangle, the properties of isosceles triangles and of the parallelogram, concurrence of important lines, similarity, and the Thales theorem. We continued with the properties of angles inscribed in a circle and with cyclic quadrilaterals. From that point, the students took off! We began to solve problems in the "IMO training" format as I know it: I would hand

Uganda's PAMO and IMO teams, Makerere University training camp, February 2020

them a list of 2–3 problems, the student who solved one of them had to explain it to the others, and if needed I would rephrase the proof with more details. The first lists of problems were at the level of middle-school Olympiads in Romania, then we advanced to problems from PAMO, the Balkan Olympiad and even the IMO itself. There was a moment of catharsis when I first (casually) told them that the problem they had just solved was from some IMO back in the 2000s. The light in their eyes was priceless!

Most of the students came from middle-class families. The exception was Jesse Enkanya, the son of a former member of Parliament. At no time did I detect any attitude of superiority from him. In fact, all the Ugandan students left a very positive personal impression on me. It was also impressive to observe them solving hard problems from 8 to 5 every day for two weeks in a row, without ever showing any sign of fatigue. This is a most encouraging indicator of what their younger colleagues might achieve if they start training for the IMO at an early age.

Uganda's results at IMO 2020

Jesse Enkanya was the first competitor from Uganda ever to place in the second tier of competitors (i.e., better than 35,93 % of the participants). Richard Ayebare (who will attend MIT starting next fall) and Jonathan Ngabirano also obtained Honorable Mentions. Together, Jesse, Richard and Jonathan solved three problems, surpassing the cumulative performance of Team Uganda from 2012 to 2019.

The team ranked 87th out of 105 participating countries, better than Algeria, Morocco, Chile or Costa Rica, countries significantly richer and mathematically more advanced. Compared to 2012 or 2013, the qualitative leap is impressive.

Due to the pandemic, IMO 2020 was organized online in September. Jesse took the exam in an accredited examination center in the United States, where he is currently enrolled as a freshman at the University of Illinois. The other five students attended the competition from Kampala. I was able to use the balance of $1,500 on the project's account to purchase surveillance equipment in line with the security conditions imposed by the organizers.

I myself tried out Problem 1. As soon as I managed to draw an acceptable picture, I had a happy premonition: that problem was going to be approachable with the cyclic quadrilateral methods we had thoroughly covered during the Kampala training camp! From my direct experience with the team, I believed they would be perfectly able to solve it. And indeed, the final result was in line with what we knew about their abilities.

Conclusions

The *sine qua non* success factor in any competition is participation. Uganda's participation in the IMO is thanks to Jasper Okello, a mathematics teacher at Nabisunsa Girls School in Kampala. Jasper has been the driving force behind this project for 10 years. The first attempts were a bit frustrating, as Uganda's team scored close to zero points. The prohibitive cost of travel was not covered by the state, while sponsors from a developing country are understandably reluctant to finance a contest where the team does not have good prospects. But in the long run, his efforts paid off. After ten years of hard work, Uganda established itself as a regional powerhouse in elementary math competitions at IMO 2020.

As a former IMO participant, I know from first-hand experience how motivating it is to compete as part of a team with a strong track record. But how demoralizing failure can be … How tempting it is to throw in the towel, to admit that you stand no chance against the Europeans or the Chinese, that you will make a fool of yourself! Kudos to Jasper Okello and to Uganda's students for braving this risk.

I should mention the role of Andrew Tugume (honorable mention at IMO 2017) in preparing the team. Andrew, currently an Engineering student in Kampala, delivered an excellent Geometry lesson in my presence. He was the team's main coach in the months before the contest. Although Uganda cannot count on many former IMO participants, having Andrew is precious. I hope he will stay involved.

The mathUganda project continues. I am in contact with mathematicians planning to organize a joint IMO training network for East African countries. We have already raised more than $5,000 for the IMO 2021. Radu Bumbăcea, Dragoș Manea, Flavian Georgescu, Liviu Păunescu and Lucian Țurea, trainers of Romania's IMO team, offered online lessons. I take this opportunity to thank them and our donors for their generosity. It seems that the idea of helping smart students from a distant country, for an intelligent purpose, touched a secret chord in our community.

Promoting mathematical education is a credible strategy for lifting countries like Uganda from poverty, and the IMO is an excellent ambassador for our discipline among the young generations. Our project demonstrates that Europe has the expertise and the will to spread the passion for mathematics in developing countries.

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Survey on Early Childhood Mathematics Education at ICME-14

Iliada Elia, Anna Baccaglini-Frank, Esther Levenson, Nanae Matsuo and Nosisi Feza

This paper is about a survey to be presented at the 14th International Congress on Mathematics Education, which addresses the latest developments on Early Childhood Mathematics Education. The relevance of early mathematics learning and teaching in mathematics education research is explained and an overview of the work done by the Survey Team on this specific theme is described.

At the 14th International Congress on Mathematics Education (ICME-14), which will take place in Shanghai from July 11 to 18, 2021, four surveys will be presented, addressing the latest developments on four particular themes of mathematics education, which are considered important for the ICMI community. The theme of one of these surveys is Early Childhood Mathematics Education (up to age 7). Our aim is to briefly present the relevance of this theme in mathematics education research as well as an overview of the work done by the Survey Team on this specific theme, which may be of interest for mathematics educators and also for mathematicians.

Research in early childhood mathematics education has experienced increasing growth over the last years. The interest in this field is induced mainly by the strong emphasis given on early childhood education in many countries and by the well documented, positive relation between children's early mathematical knowledge and their later success in mathematics learning [2]. The high importance of early childhood education is acknowledged by countries all over the world. This is evident by the increase of their expenses and investments in early childhood education, and by their access to pre-primary education. The emergence of new curricula and higher demands in the quality of early childhood education staff is also manifest in a number of countries [4].

Regarding the association between children's early mathematical skills and their later mathematical achievement, there is clear evidence that when children enter school with high levels of knowledge they maintain these high levels at least through the end of primary school [5]. Research has also provided evidence for statistically significant links between mathematics ability in preschool and mathematics performance in adolescence [7]. Thus, establishing a solid foundation for children's mathematical development before they even enter school plays a crucial role in their future learning.

The quality of early childhood mathematics education also affects children's later mathematical dispositions. Particularly, when approaches to mathematics education are meaningful and enjoyable for children, it is more likely that they will appreciate and engage in mathematics education later on [6]. Considering the decline in attitudes towards mathematics over the school years, starting already in the first years of school, and considering the fact that young children's mathematical knowledge and abilities influence their mathematical affect and dispositions [3], the need for high quality mathematics education in the early years deserves strong emphasis.

In the past few years, a great deal of attention within the field of mathematics education has been given to research on learning and teaching mathematics in early childhood. This is highlighted by the numerous publications on early childhood mathematics education, and by the many special interest or study groups in international mathematics education conferences devoted to this field that focus on the study of the learning and teaching processes in early childhood mathematics education and the environment in which these processes take place [1]. A vast amount of research has been undertaken for an even longer time in the related domains of developmental and cognitive psychology. This research has investigated early-year mathematics with a particular focus on the relationship between children's cognitive abilities (e.g., working memory, visuo-spatial abilities) and their early mathematical skills.

This survey has been designed to establish an in-depth and comprehensive review of the state-of-the-art of the most important developments and contributions since 2012, and of current tendencies, new perspectives and emerging challenges in early childhood mathematics education. The survey drew from a broad range of sources, including peer-reviewed journal articles in the above-mentioned disciplines, as well as international peer-reviewed conference proceedings, including ICME, the Conference of the International Group for the Psychology of Mathematics Education (PME), the Congress of the European Society for Research in Mathematics Education (CERME), ICMI Study Conferences and prominent research handbooks in the discipline of mathematics education. An annotated bibliography listed the papers that have been identified as relevant, leading to a comprehensive analysis of the issues raised by this research literature and to a synthesis of the pertinent findings.

The survey focuses on six major research threads that have been identified in recent literature on early childhood mathematics education. Three of these threads are content-oriented: number sense and whole number development, geometry education, and children's competences in other content domains. A twofold cognition-oriented thread focuses on cognitive skills and special education, respectively, in early childhood mathematics. The role of technologies in early mathematics teaching and learning is another important research thread that is systematically reviewed. Finally, a teacher-oriented thread presents a synthesis of results of recent studies on early childhood teachers' knowledge, education and affective issues in mathematics.

The review of research on the content-oriented threads reveals a common threefold focus across these threads: firstly, on offering insights into young children's competences and development in these content domains, secondly, on identifying influences of certain abilities into children's development, and thirdly, on proposing and investigating the effectiveness of programs or interventions on children's learning. The review of literature on cognitive skills involved in mathematical learning has a particular focus on the learning of numbers and arithmetic from as early as toddler stage, and reveals that processing quantities can be done very early in life through non-verbal innate mental systems. Moreover, visuo-spatial abilities, working memory, finger gnosis, or cognitive flexibility are only some of the key cognitive skills in young children that have been found to be predictive of or associated with mathematical performance. The review of research on the use of technology in early childhood mathematics education highlights how specific forms of interactivity available in multi-touch technology or with programmable robots can be used to enhance mathematical learning. Regarding the teacher-oriented thread, studies on the professional development of early childhood teachers in mathematics focus on enhancing teachers' knowledge of children's mathematical abilities and reasoning, thus influencing teachers' beliefs regarding young children's mathematics learning.

Overall, our work on this survey has shown that there is a plethora of research on early childhood mathematics education and that there will be continued growth and important progress in this field in the years to come. Among the six threads of our survey review, some, e.g., whole number development, cognition-oriented threads, have been studied more extensively than others, e.g., geometry education, other content domains, teacher-related issues. All these threads reflect new areas of development, e.g., use of technological tools, embodied learning, interventions for teachers, comparative studies, as well as more ordinary research topics, e.g., mathematical competences, problem solving, language. We expect that there will be continued growth in all these areas with specific emphasis on the under-researched and more recent areas of study.

The findings of the survey will be presented and discussed in more detail at ICME-14, on July 18, 2021.

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ICMI column

Frederick Leung and Susanne Prediger

Greetings from the new ICMI President, Frederick Leung

January 2021 marks the beginning of the new four-year term for the Executive Committee (EC) and the new ICMI president. We print here parts of his greetings to the worldwide mathematics education community.

"May the year 2021, and the next three years, be healthy, peaceful and productive for you! I am taking up the Presidency of ICMI with awe and humility, fully aware of the challenges that lie ahead of me and the rest of the international mathematics education community. In facing these challenges, I am blessed with the firm and sound foundation laid down by our predecessors – the past Presidents and EC members of ICMI. […]

Let me share with you briefly my vision for mathematics education in the coming years. 'The International Commission on Mathematical Instruction is a worldwide organization devoted to research and development in mathematical education at all levels [and] to promoting international cooperation in mathematics'. In achieving this mission of ICMI, I believe our first and foremost task is to establish mathematics education more solidly as an academic discipline. We as an organization should encourage rigorous research and promote high standards in research methodology. Based on scholarly research, we should facilitate and encourage sharing of best practices and cross-fertilization of ideas, while focusing on capacity building. And in the course of doing this, we should be sensitive to contextual and cultural differences in different countries. […]

As we all know, COVID-19 is affecting school education all over the world, and normal routine classroom teaching is seriously hampered. But the pandemic also brings opportunities. Many teachers around the globe are exploring the use of ICT for mathematics teaching and remote learning. […] Unfortunately, the effective use of ICT for remote teaching and learning, especially for the underprivileged, still remains a potential at the moment. In fact, the evidence so far is that COVID-19 has led to even greater inequality. […] We do not yet have much concrete data on how mathematics education of the underprivileged has been affected by COVID-19, and a Discussion Group in ICME-14 will be devoted to discussing this issue. In this time of crisis, ICMI as a community must reaffirm its mission of bringing about more equity in (mathematics) education opportunities for children in all corners of the world.

One way of achieving more equity is to support mathematics education in disadvantaged countries, and ICMI has been attempting to do this through our Capacity and Networking Project (CANP). One lesson we learned from COVID-19 is that we are living in an interdependent world. Just as immunity in one country is not enough to contain the pandemic, merely improving the quality (and quantity) of mathematics education in our own country is not enough for the whole of humanity to advance in mathematics learning. And in supporting mathematics education in different countries, we need to take their different cultural contexts into consideration. We must understand and respect different cultural traditions: we have much to learn and benefit from cultural diversity. […]

Another important stakeholder is of course the community of mathematicians. I must reiterate here that ICMI is a Commission of IMU, and we have received tremendous support from IMU in our work, professionally, logistically and financially. In promoting mathematics education, we have benefited immensely from the input of mathematicians, and we are thankful to IMU for its support and input. We should consider how we can tap this source of support from mathematicians more deeply.

I understand that serving as the President of ICMI is a huge undertaking, but it is also an honorable and meaningful endeavor. In fulfilling my role, I truly need your support and cooperation to meet the challenges ahead."

Frederick Leung

ICME-13 monographs – A window into worldwide research on mathematics education

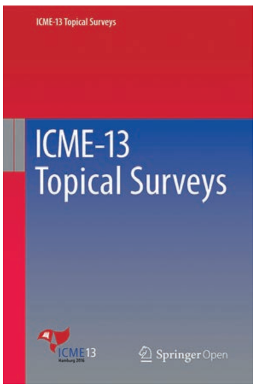
Only a few weeks remain until the hybrid conference ICME 14 (in Shanghai and the virtual space) in July 2021. This is the right

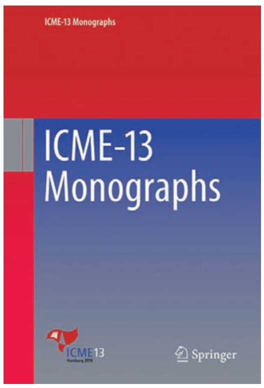
EMS MAGAZINE 120 (2021) — DOI 10.4171/MAG-26

moment to look back on ICME 13 in 2016, when the world met in Hamburg to discuss research in mathematics education. What remains after 5 years from the more than 45 topic study groups, survey teams, many invited lectures and plenary talks?

A lot! With several special issues in international journals and two Springer ICME-13 series, the international communication on multiple research topics is well documented and available to all readers. The series can be found here: www.springer.com/series/15585 and 14352.

Open access is granted to two main proceedings, with all plenary and awardee lectures, survey teams, and invited lectures. As the series are completed, they provide an interesting insight into the wide range of problems and research topics in mathematics education, including multiple research methods and theoretical perspectives. We invite all readers to search for the topic best matching their interests, and get the feeling before diving into ICME 14.





Frederick Leung is chair professor and Kintoy professor in mathematics education at the University of Hong Kong, and the new ICMI president since January 2021. He is widely acknowledged as an expert in comparative studies of mathematics education, including student achievement and classroom practices as well as the influence of culture and language on mathematics teaching and learning. In 2013, he received the Hans Freudenthal Medal, one of the ICMI awards for outstanding scientific achievements.

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ERME column

regularly presented by Jason Cooper and Frode Rønning

In this issue, with a contribution by
Viviane Durand-Guerrier, Reinhard Hochmuth, Elena Nardi and Carl Winsløw

Report on the book *Research and Development in University Mathematics Education. Overview Produced by the International Network for Didactic Research in University Mathematics.*[1] Edited by V. Durand-Guerrier, R. Hochmuth, E. Nardi, and C. Winsløw

This book emerged from the activities of the research project INDRUM (International Network for Didactic Research in University Mathematics, hal.archives-ouvertes.fr/INDRUM). INDRUM is a network that developed out of ERME, and the network aims to contribute to the development of research in didactics of mathematics at all levels of tertiary education, with a particular concern for the development of early-career researchers in the field and for dialogue with university mathematicians. The INDRUM network has been initiated by scholars strongly involved in CERME conferences, and the INDRUM conferences have been labelled ERME Topic Conferences.

The aim of the book is to provide a deep synthesis of the research field as it appears through two INDRUM conferences, which took place in 2016 and 2018. The book addresses seminal theoretical and methodological issues and reports on substantial results concerning the teaching and learning of mathematics at university level, including the teaching and learning of specific topics in advanced mathematics across a wide range of university programmes.

The first part, ***Achievements and current challenges***, contains four chapters based on the two plenary lectures and two plenary panels at the two conferences. Chapter 1 (Artigue) reflects *achievements and challenges of research in mathematics education at university level*, pointing at the strengths of this research, and the promising developments as well as the challenges it faces. Chapter 2 (Lawson and Croft) presents *lessons for mathematics higher education from 25 years of mathematics support*, relying on the authors' extensive experience in the *centres for excellence in university-wide mathematics and statistics support*. Chapter 3 (Bardini, Bosch, Rasmussen, and Trigueros) presents three case studies of interactions between mathematicians and researchers in didactics of mathematics and points out directions that seem important to strengthen. Chapter 4 (Winsløw, Biehler, Jaworski, Rønning, and Wawro) focuses on the *education and professional development of university mathematics teachers.* New ideas and practices for discipline and context-specific teacher preparation and for identifying and rewarding quality teaching are proposed.

The second part, **Teaching and learning of specific topics in university mathematics**, contains five chapters. Chapter 5 (Trigueros, Bridoux, O'Shea, and Branchetti) addresses *challenging issues in the teaching and learning of Calculus and Analysis,* covering research on one variable functions and multivariable functions as well as research on more advanced topics. Chapter 6 (Vandebrouck, Hanke, and Martinez-Planell) presents the various theoretical perspectives which underpin studies on *task design in calculus and analysis*. The authors call for further exploration, documentation and discussion on assessment and for incorporation of technologies, beyond current research, on the formalization of basic notions. Chapter 7 (Chellougui, Durand-Guerrier, and Meyer) explores the relationships between discrete mathematics, computer science, logic and proof. The authors demonstrate the need to deepen epistemological analysis and interdisciplinary didactical engineering in this area. Chapter 8 (Hausberger, Zandieh, and Fleischmann) presents a unified approach to the didactics of *abstract and linear algebra* in terms of structural and discursive characteristics, aiming to overcome the

[1] www.routledge.com/Research-and-Development-in-University-Mathematics-Education-Overview-Produced/Durand-Guerrier-Hochmuth-Nardi-Winslow/p/book/9780367365387

fragmented status of current research. Chapter 9 (González-Martín, Gueudet, Barquero, and Romo-Vázquez) focuses on *mathematics for engineers, mathematical modelling and mathematics in other disciplines,* and addresses the challenges of defining, designing, motivating and assessing mathematics teaching and learning for students who are not specializing in mathematics.

The third part, **Teachers' and students' practices at university level**, contains three chapters. Chapter 10 (Hochmuth, Broley, and Nardi) addresses issues on *transition to, across and beyond university*, including the transition from university to workplace, with an emphasis on the need for more substantial research on the last two types of transition. Chapter 11 (Rasmussen, Fredriksen, Howard, Pepin, and Rämö) focuses on *students' in-class and out-of-class mathematical practices*, use of resources out-of-class, roles in assessment practices and responses to active learning initiatives, in relation to interactions with other students, the teacher, the mathematics, and resources. Chapter 12 (Grenier-Boley, Nicolás, Strømskag, and Tabchi) focuses on *mathematics teaching practices at university level*, with particular emphasis on teacher learning and teacher knowledge, especially with regard to instructional design for inquiry-based learning. The authors conclude with calling for stronger synergy between the communities of mathematics and mathematics education.

We hope that this book will contribute to the development and dissemination of research in the teaching and learning of university mathematics and to bringing together researchers in didactics of mathematics and the whole community of university mathematics teachers.

Viviane Durand-Guerrier is professor emerita of Didactics of Mathematics at the University of Montpellier, France. Starting with a PhD from the University of Lyon (France) her research includes internationally recognized works on argumentation, proof and proving with a focus on the teaching and learning of logic in university mathematics education. She is a former President of ERME (2013–2017), and the coordinator of the INDRUM Network since its creation.

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Carl Winsløw is professor of Didactics of Mathematics at the University of Copenhagen, and (from 2021) president of the European society for Research in Mathematics Education. His research fields include von Neumann algebra theory (the subject of his PhD from Tokyo University, 1994) and didactics of university mathematics, especially in the domain of Analysis. He was among the founding members of INDRUM (International Network for Didactical Research on University Mathematics).

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Connecting Islands: Bridging zbMATH and DLMF with Scholix, a blueprint for connecting expert knowledge systems

Howard S. Cohl, Moritz Schubotz and Olaf Teschke

This paper reports on the recently launched zbMATH Links API. We discuss its potential based on the initial link partner, the National Institute of Standards and Technology Digital Library of Mathematical Functions. As the API provides machine readable data in the links, we show how one can use data from both sources for further analysis. To exemplify the simplicity, we also show how one can use zbMATH's link data in Jupyter notebooks.

1 Introduction

As reported in the last EMS Magazine (formerly Newsletter of the European Mathematical Society) [7], zbMATH Open provides application programming interfaces (API) to make zbMATH data machine accessible. We described the OAI-PMH API which enables the harvesting of zbMATH Open metadata. In contrast, this issue focuses on links between zbMATH Open and third parties. Our zbMATH Links API, available from http://purl.org/zb/14, provides a machine-readable interface for links between academic literature and other resources. To make this API interoperable with various information systems, we rely on the Scholix API standard [2]. Scholix, which is short for *A Framework for Scholarly Link eXchange*, is a long-running initiative supported by partners such as the Research Data Alliance, Crossref, and DataCite amongst many others, which aims to exchange information on research data and related scholarly articles. By exporting our data in a Scholix-compliant manner, we ensure that our data gets integrated into the worldwide ecosystem of open data. In this regard, it is not only important to export individual data sets, but also to explicitly annotate the links between different data sets in a standardized, machine readable format.

The zbMATH Open team is currently in the process of linking zbMATH Open reviews and abstracts with various partners such as

1. NIST Digital Library of Mathematical Functions (DLMF) https://dlmf.nist.gov [3, 5],
2. The On-Line Encyclopedia of Integer Sequences https://oeis.org [3],
3. The arXiv https://arxiv.org[1] [8],
4. MathOverflow https://mathoverflow.net [4],
5. and many others.

The first step, establishing links between zbMATH Open and DLMF has now been completed.

In [6] we described the details of the zbMATH Links API interface and analyzed the current links in the DLMF; statistical analysis of metadata was obtained by combining both data sources. For instance, we can analyze the distribution of Mathematics Subject Classification (MSC) classes in DLMF chapters, or the average age of the referenced publications. Let us now explain how one can proceed to generate any similar analysis in a very short time, using simple tools.

2 Jupyter notebook demonstration

One way to use data from the zbMATH Links API is via Jupyter notebooks jupyter.org. Jupyter notebooks are interactive notebooks that can be run in the browser and thus do not require any setup or configuration. In contrast to other interactive notebooks by commercial publishers, Jupyter notebooks are based on free and open source software, which implies that one is not bound to a specific vendor. Recently, Jupyter notebooks have become increasingly popular and are being used to create easily reproducible scientific workflows [1]. For this demonstration, we use Jupyter with Python and employ the library pandas pandas.pydata.org for data aggregations as well as plotly to create plotly.com interactive visualizations. In Figure 1, we create an interactive version of the MSC distribution of the articles linked in the DLMF as described in [6]. As shown, the visualization can be created in eight lines of code and fetches

[1] The mention of specific products, trademarks, or brand names is for purposes of identification only. Such mention is not to be interpreted in any way as an endorsement or certification of such products or brands by the National Institute of Standards and Technology, nor does it imply that the products so identified are necessarily the best available for the purpose. All trademarks mentioned herein belong to their respective owners.

EMS MAGAZINE 120 (2021) — DOI 10.4171/MAG-35

```
import pandas as pd
pd.options.plotting.backend = "plotly"
msc = pd.read_json(
    'https://zblink.formulasearchengine.com/links_api/statistics/msc/')
msc = msc.rename(columns={msc.columns[0]:'msc',msc.columns[1]:'count'})
msc = msc.set_index('msc')
msc.head(2)
```

msc	count
33	491
65	351

```
fig=msc.plot(y='count',kind='bar')
fig.show()
```

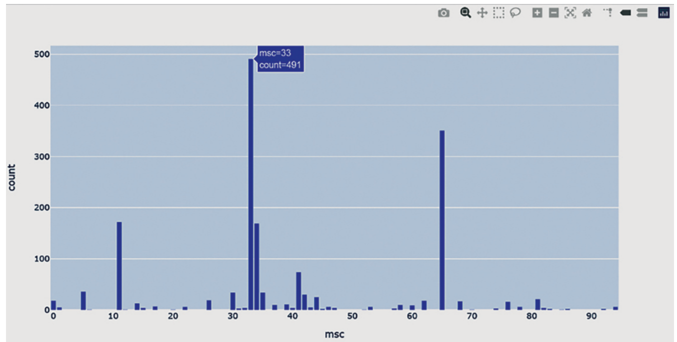

Figure 1. Jupyter notebook running on https://mybinder.org fetching and visualizing data from the zbMATH Links API

data online from our API. In pandas there is built-in support to read from our API endpoint. Thus one can use real-time results from our API without any more effort than loading any other resource from the local file system. The source code with additional examples and further links to the interactive visualization is available from https://github.com/zbMATHOpen/LinksApiJupyterDemo.

3 Conclusion and outlook

We have shown how easy it is to use the data obtained from our zbMATH Links API. While currently, DLMF links are only accessible via this API, additional links are currently in the process of being generated. Moreover, trusted third parties will be able to add new links to their respective services. Additionally, conformity with the Scholix scheme ensures that content aggregators such as OpenAIRE, DataCite, and others can integrate our data into their systems and workflows.

As of the publication of this article, the zbMATH Links data is not yet displayed on the zbMATH Open user-interface. The integration of the data and the API within our user-interface is scheduled for the second half of 2021.

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Howard S. Cohl is a mathematician in the Applied and Computational Mathematics Division at the National Institute of Standards and Technology in Gaithersburg, Maryland, USA. He is the Technical Editor for the DLMF project and the project leader for the NIST Digital Repository of Mathematical Formulae seeding and development projects. Cohl is a research mathematician whose main interests lie in the areas of special functions and orthogonal polynomials. He also has been exploring mathematical knowledge management and the digital expression of mostly unambiguous context-free full semantic information for mathematical formulae.

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Olaf Teschke is Managing editor of zbMATH and Vice-chair of the EMS Committee on publications and electronic dissemination.

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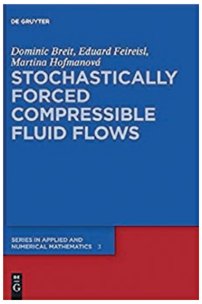
Book review

Stochastically Forced Compressible Fluid Flows by Dominic Breit, Eduard Feireisl and Martina Hofmanová

Reviewed by Donatella Donatelli



The book is focused on systematically developing a consistent mathematical theory of compressible fluids driven by random initial data and stochastic external forces in the context of classical continuum fluid mechanics.

The theory of continuum fluid mechanics is derived from basic physical principles under the assumption that all quantities – fields – are smooth, and the Navier–Stokes system became a well-established model working as a reliable basis of investigation for both theoretical and applied aspects. Built on the foundation of conservation laws, fluid mechanics helps to describe the flow and interactions of gases, liquids and/or plasmas, as well as the forces acting on them. Until fairly recently, these forces have largely been considered to be deterministic. This means that they are functions of microscopic space and time parameters, so that at any given instant of time the fluid position in space is expected to be known. There are still many important open problems, but the literature concerning the deterministic case is very well-established and extensive; see for example the monographs [E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004] or [P. L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol. 2: *Compressible Models*, Oxford Lecture Series in Mathematics and its Applications, vol. 10, The Clarendon Press, Oxford Science Publications, Oxford University Press, New York, 1998].

However, this description is a fairly weak idealisation, which is obvious already from the fact that we are still unable to model extreme fluid mechanic events like turbulence to a sufficient level of accuracy. In fact, the modelling of turbulence can be considered as the prime motivation for the introduction of stochasticity in the study of fluids. Turbulence is frequently associated with an intrinsic element of randomness, and furthermore, experimental studies of turbulence lead more to a statistical approach than to a deterministic one. Moreover, the addition of stochastic terms to the basic governing equations is often used to account for other numerical, empirical or physical uncertainties. Therefore it becomes important, in the framework of partial differential equations, to set up a stochastic PDE theory for fluid flow.

Nowadays there exists a large amount of literature concerning the dynamics of incompressible fluids driven by stochastic forcing. The first results can be found in the pioneering work by Bensoussan–Temam (1973). See also the lecture notes [A. Debussche, Ergodicity results for the stochastic Navier–Stokes equations: An introduction, In *Topics in Mathematical Fluid Mechanics*, volume 2073 of Lecture Notes in Math., pages 23–108, Springer, Heidelberg, 2013], [Flandoli, An introduction to 3D stochastic fluid dynamics, In *SPDE in Hydrodynamic: Recent Progress and Prospects*, volume 1942 of Lecture Notes in Math., pages 51–150, Springer, Berlin, 2008]. Nevertheless, far less is known in the case of compressible fluids. Important questions of well-posedness and even mere existence of solutions to problems dealing with stochastic perturbations of compressible fluids are largely open, with only a few rigorous results available. This monograph is an exhaustive and up-to-date overview of the most recent results by different authors on stochastic compressible fluids.

The book contains eight chapters and is divided into three parts. It starts with Part I, a very didactic introduction providing the necessary background. In a very clear manner, Part I provides the non-expert readers in the field with all the basic results of the theory and, at the same time, a description of more advanced tools in the theory of stochastic PDEs. Part II is the core of the book, containing all that is really new and original compared to the existing literature. The most recent existence results on compressible stochastic fluids are described. This part consists of five chapters, which guide the reader step by step towards the proof of the existence of solutions. Each chapter is devoted to one of the main aspects of the existence theory: the setup of the model, approximation schemes and their convergence, energy inequalities, relative energy inequality, and weak strong uniqueness. In particular, it starts with the existence of local strong solutions defined on a maximal time interval bounded above by a positive stopping time that may depend on the size of the initial data; then, because all real world problems require

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solutions defined globally in time, one has to switch to the notion of weak solutions. This approach is based on the idea of including some form of the energy/entropy balance as an integral part of a weak formulation, and goes back to Dafermos (1979) concerning conservation laws and to Germain (2011) who introduced a similar concept in the context of the deterministic compressible Navier–Stokes system. Therefore, the solutions constructed in this part of the book are the so-called dissipative martingale solutions, which are weak martingale solutions also satisfying a variant of the energy balance.

Finally, Part III of the book is focused on applications such as singular limits. Indeed, by scaling the equations by means of appropriately chosen reference units, the parameters determining the behaviour of the system become evident. Asymptotic analysis and/or singular limits provide a useful tool in situations where these parameters vanish or become infinite. In this part, the authors describe a rigorous mathematical approach to asymptotic analysis in the case of incompressible and inviscid–incompressible limits for the compressible Navier–Stokes system with stochastic perturbations.

To conclude, this is the first book in which one can find a complete description of the available theory on compressible stochastic fluid equations. Compared to the previous literature, this is a new point of view that makes the book original and of very high quality. It is a really valuable and much-needed contribution to the literature in the domain. This monograph is built in a masterly manner, in such a way as to provide not only a complete and up-to-date overview of the problems under consideration, but also a detailed introduction to the topic for the uninitiated reader. The book is very well and rigorously structured, having the excellent attribute of being valuable to both experienced researchers in the domain and to graduate students who wish to explore the different topics in this challenging area of research. Overall, it constitutes an ideal book for researchers (in the broadest sense) who want to enlarge their mathematical knowledge of fluid mechanics.

Dominic Breit, Eduard Feireisl and Martina Hofmanová, *Stochastically Forced Compressible Fluid Flows*. De Gruyter Series in Applied and Numerical Mathematics 3. De Gruyter, 2018, 330 pages
ISBN 978-3-11-049050-3. eBook ISBN 978-3-11-049255-2.

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New editor appointed

Donatella Donatelli is full professor in mathematical analysis at the Department of Information Engineering, Computer Science and Mathematics of the University of L'Aquila, Italy. Her research interests cover partial differential equations of hyperbolic type, relaxation limits for nonlinear hyperbolic systems, incompressible and compressible Navier–Stokes equations, and mathematical models of fluid dynamical type. In particular, her main achievements are on scale limit analysis for fluid dynamic equations with relevant results in the related acoustic waves analysis. In 2011, Donatella was awarded the Marisa Bellisario Prize in the section "Special recognition of young talents in research" in Mathematical and Computer Sciences (this is a national prize under the auspices of the Italian President, lnx.fondazionebellisario.org/online/2011-2/).

She has been the Coordinator of the LLP-EU Intensive Programs Fluid2Bio 2012–2011, contract N.2012-1-IT2-ERA10-38827, 2011, contract N.2011-1-IT2-ERA10-27088 and the Scientific Coordinator of the L'Aquila node of the European Project Marie Curie Actions-MSCA-ITN-2014-ETN, Horizon 2020 "ModCompShock-Modelling and Computation of Shocks and Interfaces", 2016–2020.

Victims of positive discrimination

Valentin Ovsienko

They will never tell you, they suffer silently.

Our colleagues, talented mathematicians, those who are willing and able to make major achievements, are deprived of this opportunity; the only reason is that they are women. Discrimination?! Oh yes, the "positive" one! I claim that women suffer terribly from the thing called "positive discrimination".

The road to hell is paved with good intentions. How many great ideas produce an opposite result, as nuclear power transforms into atomic bomb …

To be more concrete, let me give one example. I know this example very well, because this is my very own dearest wife. She is an actively working mathematician, she loves research, but she is also a member of an uncountable number of committees at her prestigious Sorbonne Université and elsewhere. She spends all her time at meetings, evaluations, distributions of grants, *primes d'excellence et encadrement*, etc. She is a member of hiring committees in Jussieu and everywhere in the large Hexagon. The major problem is that she cannot always answer "no" when asked to participate; she knows exactly who will be asked to do the job if she refuses, and believe me, all of these poor ladies are already overwhelmed with administrative tasks.

France was spared from this hysteria for a long time, but the "thing" is contagious. The Great Idea crossed the Atlantic, and now there is a quota in France: 1/3 of the members of every committee must be women. As we all know, the real proportion of ladies in mathematics is something like 1/10, so the consequence is obvious. Of course, the "Idea" was to protect women from discrimination but the result is exactly the opposite: many female colleagues, active researchers, have no time to do research and high quality teaching any more.

One of the results of the situation described above is the underrepresentation of women on the Editorial Boards of scientific journals,[1] prize juries, academies, etc. Just leave them in peace, let them work normally, and the balance will be renormalized!

Ironically, since men keep the 2/3 majority in all the committees that I know, if we make the (paranoid) assumption that the only goal of the male members is to discriminate against female colleagues, we can continue our dark deeds with ease – positive discrimination is not going to stop us!

Valentin Ovsienko is Senior Researcher at the French National Center for Scientific Research (CNRS), in the Laboratoire de Mathématiques de Reims and Université de Reims Champagne-Ardenne.

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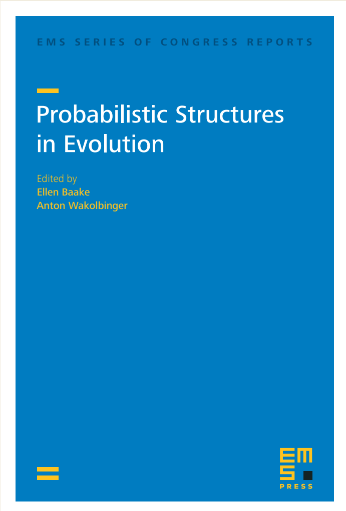
[1] A popular subject, frequently discussed in politically engaged pseudo-mathematical literature. The author prefers not to give precise references here; they are numerous and can be found easily.

New EMS Press books

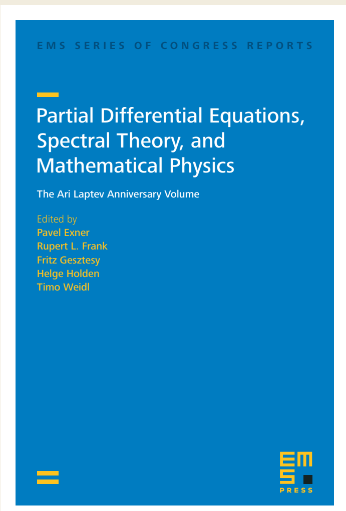


Probabilistic Structures in Evolution
edited by Ellen Baake and Anton Wakolbinger

EMS Series of Congress Reports
ISBN print 978-3-98547-005-1, ISBN online 978-3-98547-505-6
May 2021. Hardcover. 502 pages. €89.00

The present volume collects twenty-one survey articles about probabilistic aspects of biological evolution. They cover a large variety of topics from the research done within the German Priority Programme SPP 1590.

The collection is centred around the stochastic processes in population genetics and population dynamics. On the one hand, these are individual-based models of predator-prey and of coevolution type, of adaptive dynamics, or of experimental evolution, considered in the usual forward direction of time. They lead to processes describing the evolution of type frequencies, which may then be analysed via suitable limit theorems. On the other hand, one traces the ancestral lines of individuals back into the past; this leads to random genealogies. Beyond the classical concept of Kingman's coalescent, emphasis is on genealogies with multiple mergers and on ancestral structures that take into account selection, recombination, or migration.



Partial Differential Equations, Spectral Theory, and Mathematical Physics: The Ari Laptev Anniversary Volume
edited by Pavel Exner, Rupert L. Frank, Fritz Gesztesy, Helge Holden, and Timo Weidl

EMS Series of Congress Reports
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This volume is dedicated to Ari Laptev on the occasion of his 70th birthday. It collects contributions by his numerous colleagues sharing with him research interests in analysis and spectral theory.

In brief, the topics covered include Friedrichs, Hardy, and Lieb–Thirring inequalities, eigenvalue bounds and asymptotics, Feshbach–Schur maps and perturbation theory, scattering theory and orthogonal polynomials, stability of matter, electron density estimates, Bose–Einstein condensation, Wehrl-type entropy inequalities, Bogoliubov theory, wave packet evolution, heat kernel estimates, homogenization, d-bar problems, Brezis–Nirenberg problems, the nonlinear Schrödinger equation in magnetic fields, classical discriminants, and the two-dimensional Euler–Bardina equations. In addition, Ari's multifaceted service to the mathematical community is also touched upon.

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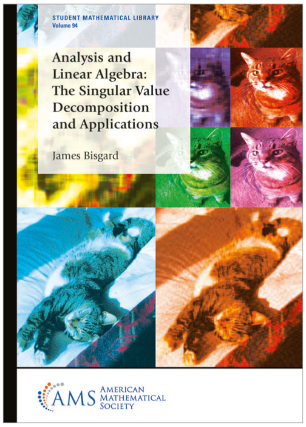

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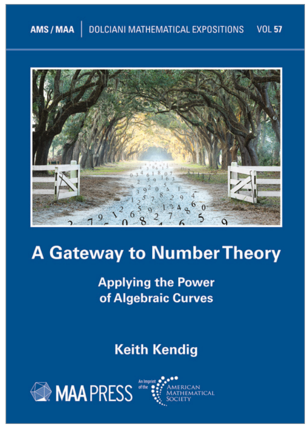

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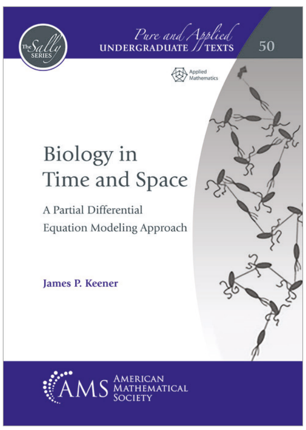

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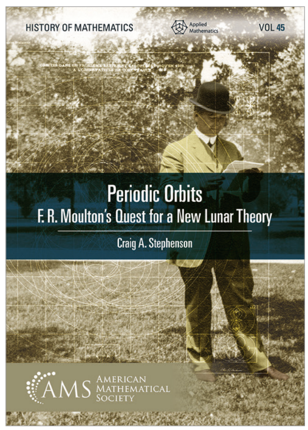

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