European Mathematical Society Magazine

ISSN 2747-7894 / Issue 128 / June 2023

EMS Magazine

Jean-Pierre Bourguignon and Balázs Szendrői
Gene Calabi at 100 – Memorable encounters with Eugenio Calabi

Alessandra Micheletti
A new paradigm for artificial intelligence based on group equivariant non-expansive operators

Thierry Bodineau, Isabelle Gallagher, Laure Saint-Raymond and Sergio Simonella
On the dynamics of dilute gases

Katja Rasch and Mareike Wöhler
Mathematics at the Deutsches Museum: On-site, digital, and to go

EUROPEAN MATHEMATICAL SOCIETY

European Mathematical Society Magazine

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Published by EMS Press, an imprint of the
European Mathematical Society – EMS – Publishing House GmbH
Technische Universität Berlin, Institut für Mathematik,
Straße des 17. Juni 136, 10623 Berlin, Germany

https://ems.press

Typesetting: Simon Winter, Berlin, Germany
Printing: Beltz Bad Langensalza GmbH, Bad Langensalza, Germany

ISSN (print) 2747-7894; ISSN (online) 2747-7908

The cover illustration is a portrait of Eugenio Calabi by António B. Araújo, based on a 2012 photograph by Constantin Raducan. The surface represented at the bottom left of the cover is the Calabi–Yau quintic cross-section $z_1^5 + z_2^5 = 1$.

A message from the president

Photo by Jim Høyer, University of Copenhagen

It's now been nearly half a year since I took over as president of the EMS. It's been a busy time, with lots of new challenges, and I admit that there are many aspects of the EMS that were new to me and with which I have had to become acquainted. During my time as president, I have met and interacted with many research mathematicians and administrative staff from all over Europe who work for the EMS in one capacity or another. It has been truly incredible to see the enthusiasm with which all these people contribute to and support the EMS; our society would not be what it is without them. I would, in particular, like to highlight the work that is being done in all the EMS standing committees.[1] As part of becoming better acquainted with the EMS I have participated in the meetings of several of the committees already and plan to visit them all by the end of the year. It has been a great experience to see all the important work being done and I want to take this opportunity to make sure that we at EMS acknowledge the great efforts invested by the many committee members.

This spring the EMS elected the first group of 30 members of its new European Mathematical Society Young Academy EMYA.[2] I participated in their first meeting and I am looking forward to seeing how they will contribute to and benefit from EMS in the future. I have high expectations for the Academy. The next deadline for nominations by full and institutional members of the EMS is 31 July 2023.[3] Members are elected for four years and EMYA will eventually have approximately 120 members.

The preparations are well under way for the 9th European Congress of Mathematics in Seville in July 2024.[4] Please save the date and join us in Seville. The call for hosting the 10th ECM has been published with a deadline of 30 June 2023.[5]

The long-term EMS Digest editor Mireille Chaleyat-Maurel has decided to retire; we are very grateful for all the great work Mireille has done. We are in the process of remodelling the digest and hope to be able to present it in a new form before too long.

We are also working on our membership database with improved accessibility options, and we hope to be able to present it this fall.

Finally, while EMS has a very strong base of corporate member societies, it is my hope that the EMS can grow its base of individual members. In order for EMS to become a strong united voice for mathematicians across Europe we need continued support and hard work from individuals. We are therefore beginning a campaign to enlist more members. Most of you reading this message are, I hope, individual members of EMS. Please encourage your colleagues to join us.

Moreover, if you are not already following the EMS online I encourage you to please join me in following the EMS on social media: Twitter,[6] Facebook,[7] LinkedIn,[8] and (new!) Mastodon.[9]

I hope you will enjoy reading this issue of the EMS Magazine. It is available online and we hope that most members are happy accessing it in this way. It would save the EMS a substantial amount of money if we do not have to mail a lot of hardcopies of the magazine.

Jan Philip Solovej
President of the EMS

[1] https://euromathsoc.org/committees
[2] https://preview.euromathsoc.org/EMYA-list2023
[3] https://directus.backend.euromathsoc.org/assets/b8a93363-9e3d-490a-a4b2-85cacb1f3633
[4] https://www.ecm2024sevilla.com
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A new paradigm for artificial intelligence based on group equivariant non-expansive operators

Alessandra Micheletti

The recent frantic surge of machine learning and, more broadly, of artificial intelligence (AI) brings to light old and new open issues, and among them, the so-called eXplainable artificial intelligence (XAI) – AI that humans can understand – as opposed to black-box learning systems where even their designers cannot explain AI decisions. One of the major XAI questions is how to design transparent learning systems that incorporate prior knowledge. These topics are becoming more relevant and pervasive as AI systems become more unfathomable and entangled with human factors. Recently a new paradigm for XAI has been introduced in literature, based on group equivariant non-expansive operators (GENEOs), which are able to inject prior knowledge in a learning system. Hence, the use of GENEOs dramatically reduces the number of unknown parameters to be identified and the size of the related training set, providing both computational advantages and an increased degree of interpretability of the results. Here we will illustrate the main characteristics of GENEOs and the encouraging results already obtained on a couple of industrial case studies.

1 Introduction

The use of techniques and architectures of artificial intelligence (AI) is becoming more and more pervasive in a wide range of applications, starting from automation or quality control in industry, to self-driving vehicles, crime surveillance, health monitoring and many others.

As the Oxford Dictionary states, by AI one means *the theory and development of computer systems able to perform tasks normally requiring human intelligence, such as visual perception, speech recognition, decision-making, and translation between languages*. Such systems are quite often based on machine or deep learning techniques, that is, on different types of neural networks, with many layers and thus with a huge number of unknown parameters, which need to be identified on the basis of a training set of data. Even if in many applications AI and deep learning prove to be very effective, two main problems often arise: the limited availability of data in some applications, which prevents the scientists to define a sufficiently large training set, and the 'black-box' nature of deep learning systems, having as a consequence that even its designers cannot explain AI decisions.

Equivariant operators are proving to be increasingly important in deep learning, in order to make neural networks more transparent and interpretable [2–4, 10, 20, 21, 27, 28]. The use of such operators corresponds to the rising interest in the so-called "explainable machine learning" [8, 14, 23], which looks for methods and techniques that can be understood by humans. In accordance with this line of research, group equivariant non-expansive operators (GENEOs) have been recently proposed as elementary components for building new kinds of neural networks [5, 6, 11]. Their use is grounded in topological data analysis (TDA) and guarantees good mathematical properties, such as compactness, convexity, and finite approximability, under suitable assumptions on the space of data and by choosing appropriate topologies. Furthermore, GENEOs allow to shift the attention from the data to the observers who process them, and to incorporate the properties of invariance and simplification that characterize those observers. The basic idea is that we are not usually interested in data, but in approximating the experts' opinion in presence of the given data [12].

More formally, a GENEO is a functional operator that transforms data into other data. By definition, it is assumed to commute with the action of given groups of transformations (equivariance) and to make the distance between data decrease (non-expansivity). The groups represent the transformations that preserve the "shape" of our data, while the non-expansivity condition means that the operator must simplify the data metric structure. Both equivariance and non-expansivity are important: while equivariance reduces the computational complexity by expressing the equivalence of data, non-expansivity guarantees that the space of GENEOs can be finitely approximated, under suitable assumptions. The key point for the use of GENEOs is the possibility of focusing on them in the search for optimal components of neural networks, instead of exploring the infinite-dimensional spaces of all possible operators. The relatively small dimension of the spaces of GENEOs – and their good geometric and topological properties – open the way to a new kind of "geometric knowledge engineering for deep learning," which can allow us to drastically reduce the number of involved parameters and to increase the transparency of neural networks, by inserting information in the agents that are responsible for data processing.

In this paper we will introduce GENEOs and their main mathematical properties and we will show the quite promising results already obtained by their application to two different industrial problems, namely, protein pocket detection and maintenance of electric power lines.

Let us remark that, in rapidly evolving scientific fields like artificial intelligence, which claims for new ideas and mathematical instruments, it is crucial to establish a strict interaction and collaboration between academic mathematical research and industry, in order to focus on the most important mathematical problems that must be addressed to produce technological innovation. Such industry-academia interactions are fostered since many years by the European Consortium for Mathematics in Industry (ECMI)[1], with its many initiatives, and in particular with its biannual conference, whose next edition[2] will be held in Wrocław (Poland) on June 26–30, 2023.

2 GENEOs as models for observers

Observers can be often seen as functional operators, transforming data into other data. This happens, for example, when we blur an image by a convolution, or when we summarize data by descriptive statistics. However, observers are far from being entities that merely change functions into other functions. They do that in a compatible way with respect to some group of transformations, i.e., they commute with these transformations. For example, the operator associating to each regular function $f : \mathbb{R}^n \to \mathbb{R}$ its Laplacian Δf commutes with all Euclidean isometries of $\mathbb{R}^n$. More precisely, we say that this operator is *equivariant* with respect to the group of isometries.

Another important property of observers should also be considered: they are endowed with some kind of regularity. A particularly important regularity property is non-expansivity. That means that the distance between the input data is not smaller than the distance between the output functions. This type of regularity is frequently found in applications, since usually operators are required to simplify the metric structure of data. We can obviously imagine particular applications where this condition is violated locally, but the usual long term purpose of data processing is to converge to an *interpretation*, i.e., a representation that is much simpler and meaningful than the original data. As a consequence, it is reasonable to assume that the operators representing observers, as well as their iterated composition, are non-expansive. This assumption is not only useful for simplifying the analysis of data, but it is also fundamental in the proof that the space of group equivariant non-expansive operators is compact (and hence finitely approximable), provided that the space of data is compact with respect to a suitable topology [5].

[1] https://ecmiindmath.org
[2] https://ecmi2023.org

2.1 Basic definitions and properties of GENEO spaces

Let us now formalize the concept of group-equivariant non-expansive operator, as was introduced in [5].

We assume that a space Φ of functions from a set X to $\mathbb{R}^k$ is given, together with a group G of transformations of X, such that if $\varphi \in \Phi$ and $g \in G$, then $\varphi \circ g \in \Phi$. We call the pair (Φ, G) *perception pair*. We also assume that Φ is endowed with the topology induced by the L_∞-distance $D_\Phi(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_\infty$, $\varphi_1, \varphi_2 \in \Phi$. Let us assume that another perception pair (Ψ, H) is given, with Ψ endowed with the topology induced by the analogous L_∞-distance D_Ψ, and let us fix a homomorphism $T : G \to H$.

Definition 1. A map $F : \Phi \to \Psi$ is called a *group equivariant non-expansive operator* (GENEO) if the following conditions are satisfied:

1. $F(\varphi \circ g) = F(\varphi) \circ T(g)$ for any $\varphi \in \Phi$ and any $g \in G$ (equivariance);
2. $\|F(\varphi) - F(\varphi')\|_\infty \le \|\varphi - \varphi'\|_\infty$ for any $\varphi, \varphi' \in \Phi$ (non-expansivity).

If we denote by F_{all} the space of all GENEOs between (Φ, G) and (Ψ, H) and endow it with the metric

$$D_{\text{GENEO}}(F_1, F_2) = \sup_{\varphi \in \Phi} \|F_1(\varphi) - F_2(\varphi)\|_\infty, \quad F_1, F_2 \in F_{\text{all}},$$

the following main properties of spaces of GENEOs hold true (see [5] for the proofs).

Theorem 2. *If Φ and Ψ are compact, then F_{all} is compact with respect to the topology induced by the metric D_{GENEO}.*

Corollary 3. *If Φ and Ψ are compact with respect to the ∞-metrics D_Φ and D_Ψ, respectively, then for any $\varepsilon > 0$ the space F_{all} can be ε-approximated by a finite set.*

Theorem 4. *If Ψ is convex, then F_{all} is convex.*

Theorem 2 and Corollary 3 guarantee that if the spaces of data are compact, then also the space of GENEOs is compact, and can then be well approximated by a finite number of representatives, thereby reducing the complexity of the problem. Theorem 4 implies that if the space of data is also convex, then any convex combination of GENEOs is still a GENEO. Thus, when both compactness and convexity hold, we have an easy instrument to generate any element of F_{all} starting from a finite number of operators. Additionally, the convexity of F_{all} ensures that each strictly convex cost function in the space of the observers admits a unique minimum, and thus the problem of finding the 'optimal observer' can be solved.

3 Application to protein pockets detection: GENEOnet

We used GENEOs to build *GENEOnet* [7], a geometrical machine-learning method able to detect pockets on the surface of proteins. Protein pockets detection is a key problem in the context of drug development, since the ability to identify only a small number of sites on the surface of a molecule that are good candidates to become binding sites allows a scientist to restrict the action of virtual screening procedures, thus saving both computational resources and time, and fostering the speed up of the subsequent phases of the process. This research is still ongoing, in collaboration with the Italian pharmaceutical company *Dompé Farmaceutici*.

This problem is particularly suitable to be treated with GENEOs, since, on the one hand, there is some important empirical chemical-physical knowledge that cannot be embedded in the usual machine learning techniques, but can be injected in a GENEO architecture, and, on the other hand, the problem enjoys a natural invariance property: indeed, if we rotate or translate a protein, its pockets will undergo the same transformation, coherently with the entire protein. This clearly implies that pocket detection is equivariant with respect to the group of spatial isometries.

For this application we used a subset of more than 10,000 protein-ligand complexes extracted from the PDBbind v2020 data-set [19]. Input data was discretized by surrounding each molecule by a cubic bounded region divided into a 3D grid of voxels. In this way the data are modelled as bounded functions from the Euclidean space $\mathbb{R}^3$ to $\mathbb{R}^d$. We chose $d = 8$, that is, the number of distinct geometrical, chemical and physical potential fields that we computed on each molecule and took into account for the analysis. The potentials are here called 'channels', imitating the nomenclature used in image analysis (see [7] for further details on the specific channels).

3.1 The GENEOnet model

The input data are fed to a first layer of GENEOs chosen from a set of parametric families of operators, each one parametrized by one single shape parameter σ_i, $i = 1, \dots, 8$. These families were designed in order to include the a priori knowledge of the experts of medicinal chemistry in the equivariance properties of the GENEOs. We opted for convolutional operators, whose properties can be completely determined by the nature of their convolution kernels. Moreover, by making the i-th kernel dependent on only one shape parameter σ_i, we have direct control on the action of each operator. We mainly used Gaussian kernels or kernels having shapes of spheres or of spherical crowns, assuming alternatively positive and negative values in different parts of the interior of the sphere or crown, and zero outside. In this way we could detect both spherical voids close to the protein that are surrounded by protein atoms and the change of sign of the measured potentials, since protein cavities which show high gradients of the values of the measured potentials are the most promising to become binding sites. The shape parameter σ_i of each kernel was connected with the radius of the sphere or spherical crown, or with the standard deviation of the Gaussian.

All the chosen operators share a common feature: their kernels are defined through rotationally-invariant functions. This fact, together with the properties of convolution, guarantees that the corresponding GENEOs satisfy the key requirement to be equivariant with respect to the group of isometries of $\mathbb{R}^3$.

In the second step, these operators are combined through a convex combination with weights $\alpha_1, \dots, \alpha_d$ such that $\alpha_i \in [0, 1]$ for all i and $\sum_{i=1}^d \alpha_i = 1$. We then obtain a composite operator $F_\alpha(\cdot) = \sum_{i=1}^d \alpha_i F_i(\cdot)$ whose output is normalized to a function ψ from $\mathbb{R}^3$ to $[0, 1]$. Here $\psi(x)$ can be interpreted as the probability that a point $x \in \mathbb{R}^3$ belongs to a pocket. Finally, given a probability threshold $\theta \in [0, 1]$, we get the different pockets returned by the model by taking the connected components of the superlevel set $\{\psi \geq \theta\} \subseteq \mathbb{R}^3$. The entire model pipeline is depicted in Figure 1.

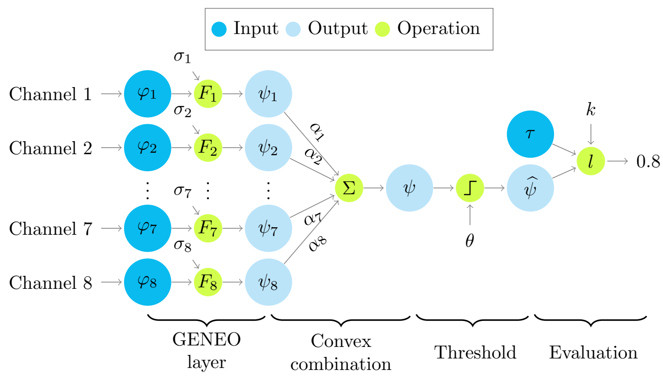

Figure 1. Model workflow. The input channels $\varphi_1, \dots, \varphi_8$ are fed to the GENEOs $F_1, \dots, F_8$ that depend on the shape parameters $\sigma_1, \dots, \sigma_8$; this first layer returns the intermediate outputs $\psi_1, \dots, \psi_8$. Then with these outputs one forms a convex combination with weights $\alpha_1, \dots, \alpha_8$ to get the final result ψ. To obtain pockets, a thresholding operation with a parameter θ is applied to ψ, producing the binary function $\widehat{\psi}$, which finally can be compared to the ground truth τ through the accuracy function.

A discretization into voxels similar to the one adopted for the molecules has been applied also to the GENEOs, which have thus been expressed as discrete convolutional operators. The choice of convolutional operators allowed us to exploit the efficient implementation of discrete convolution, reducing the computational costs.

3.2 *Parameter identification*

The model that was described so far, as shown in Figure 1, has a total of 17 parameters (σ_i, $i = 1, \dots, 8$, a_j, $j = 1, \dots, 8$ and θ). The codes were written using both C and Python. The fact that the model only employs convolutional operators and linear combinations thereof allowed us to set up an optimization pipeline quite similar to a 3D convolutional neural network (CNN), but with two fundamental differences. First, our model has a really tiny set of parameters, if compared to a classical CNN: we estimated that a recent method called DeepSite [15], which implements a classical 3D CNN for pocket detection, has 844 529 parameters; DeepPocket [1], an even newer approach that uses a 3D CNN to rescore fPocket [18] predictions, has 665 122 parameters. Second, the convolutional kernels of the GENEOs are not learned entry by entry as in classical CNNs, since in this way equivariance would not be preserved at each iteration; instead, at each step the kernels are recomputed from the shape parameters that are updated during the optimization. Finally, the estimated values of the parameters a_j, $j = 1, \dots, 8$, can be interpreted as weights giving the relative importance of each considered channel to the final pocket detection.

In order to identify the unknown parameters, we have to optimize a cost function that evaluates the goodness of our predictions. If we denote by $\hat{\psi}$ the output of the model after thresholding, then we must compare $\hat{\psi}$ to the ground truth represented by the binary function τ, which takes the value 1 in those voxels occupied by the ligand and 0 in the other voxels. We adopted the following accuracy function that needs to be maximized:

$$I(\hat{\psi}, \tau) = \frac{|\hat{\psi} \wedge \tau| + k \cdot |(\mathbf{1} - \hat{\psi}) \wedge (\mathbf{1} - \tau)|}{|\tau| + k \cdot |\mathbf{1} - \tau|} \in [0, 1].$$

Here $\hat{\psi} \wedge \tau$ denotes the minimum between the two functions, $|\cdot|$ denotes the volume of the set where the function equals 1 and $\mathbf{1}$ denotes the constant function equal to 1. Note that the function $I(\hat{\psi}, \tau)$ is well defined, since all our functions are defined only on a (voxelized) compact cubic region surrounding the molecule. The hyperparameter k ranges in $[0, 1]$, and when $k = 1$, the accuracy function is simply the fraction of correctly labelled voxels out of the total. We choose $k < 1$, which allows to balance the two terms of the sum in the numerator to obtain more and slightly bigger pockets. In particular, we empirically found that values of k in the interval $[0.01, 0.05]$ give similar and good results, all characterized by a rather small number of pockets of suitable size.

Eventually, pockets are found as the connected components of the thresholded output of the model. In this way we get an array where voxels located in a pocket are labelled with the successive number of the connected component they belong to, while they are labelled with 0 if they do not belong to a pocket. Actually, this representation is not very informative, since in the applications of pocket detection in medicinal chemistry it is desirable to compute also the druggability of the identified cavities, that is, a ranking of the pockets on the basis of their fitness to host a ligand.

To assign a score to each pocket, we went back to the output of the model before the thresholding, that is, to the function $\psi(x)$, which was interpreted as the probability that a voxel x belongs to a pocket. The score of a pocket was then computed as the average value of ψ taken only over the voxels belonging to the pocket, rescaled by a factor proportional to the volume of the pocket so as to avoid giving too high scores to very small pockets. Eventually, the final output of the model consists in a list of pockets, given as the coordinates of their voxels, and the corresponding scores.

Figure 2 displays an example of results of GENEOnet applied to the protein 2QWE. The picture shows a relevant aspect of GENEOnet: the depicted protein is made up of four symmetrical units so that the true pocket is replicated four times. GENEOnet correctly outputs, among the others, four symmetrical pockets that receive high scores. This happens thanks to equivariance, because the results of the model on identical units are the same, with position and orientation coherently adjusted.

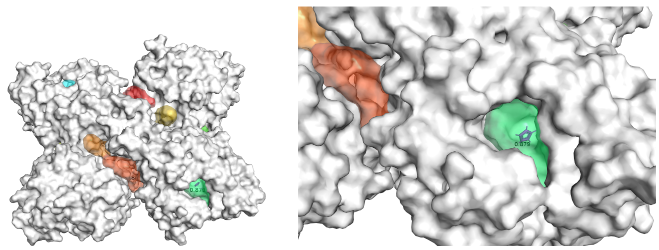

Figure 2. Model predictions for protein 7WIY. Left: the global view of the prediction, where different pockets are depicted in different colors and are labelled with their scores. Right: a zoomed view of the pocket containing the ligand.

3.3 *Comparison between GENEOnet and other methods*

We compared the results of GENEOnet with a set of other recent methods for protein pocket detection based on machine learning techniques. Since the output of such methods can be different from our output, both in terms of the discretization strategy and of the objective function to be optimized, we decided to base our comparison on the scores given by the different methods to the cavities. In this way we can perform a comparison based on the ability of the model to assign the highest scores to pockets that match the true ones. Given our dataset of proteins, having only one ligand, and thus one 'true pocket' each, we can compute the fraction of proteins whose true pocket is hit by the predicted one with highest score, by the one with second highest score, and so on. We say that a predicted pocket A hits the true pocket B if A has the greatest overlap with B. If no predicted pocket has an intersection with the true one, we say that the method failed on that protein. Finally, we computed the cumulative sum of these fractions; in this way we get a curve where the i-th point represents the fraction of proteins whose true pocket has been recognized within the first i highest scored predicted pockets.

In the following we will denote by H_j the proportion of correct recognitions, i.e.,

$$H_j = \frac{\#(\text{proteins whose true pocket is hit by the } j\text{th top ranked})}{\#(\text{proteins})},$$

and by T_j the corresponding cumulative quantities, i.e.,

$$T_j = \frac{\#(\text{proteins whose true pocket is hit within the } j\text{th top ranked})}{\#(\text{proteins})} = \sum_{i=1}^{j} H_i.$$

In this way different methods can be compared directly: if a model has a cumulative curve that stands above all the others, then that model is definitely better. We chose to use this approach to compare our model with the following other state-of-the-art methods:

1. fPocket [16]: a fast geometrical method that employs a detection algorithm based on alpha-spheres.
2. P2RANK [18]: a model that uses random forests to make predictions on a cloud of points evenly sampled on the solvent accessible surface.
3. DeepPocket [1]: a method that performs a re-scoring of fPocket cavities by means of 3D CNNs.
4. Caviar [22]: a model that uses a novel approach to the classical technique of points enclosure.
5. SiteMap [13]: a model that clusterizes site points based on surface distance and how well they are sheltered from the solvent.
6. CavVis [24]: a model that uses Gaussian surfaces to predict pockets based on a visibility criterion.

The results are reported in Figure 3, which demonstrates that GENEOnet has a better performance than all the other methods considered in the comparison.

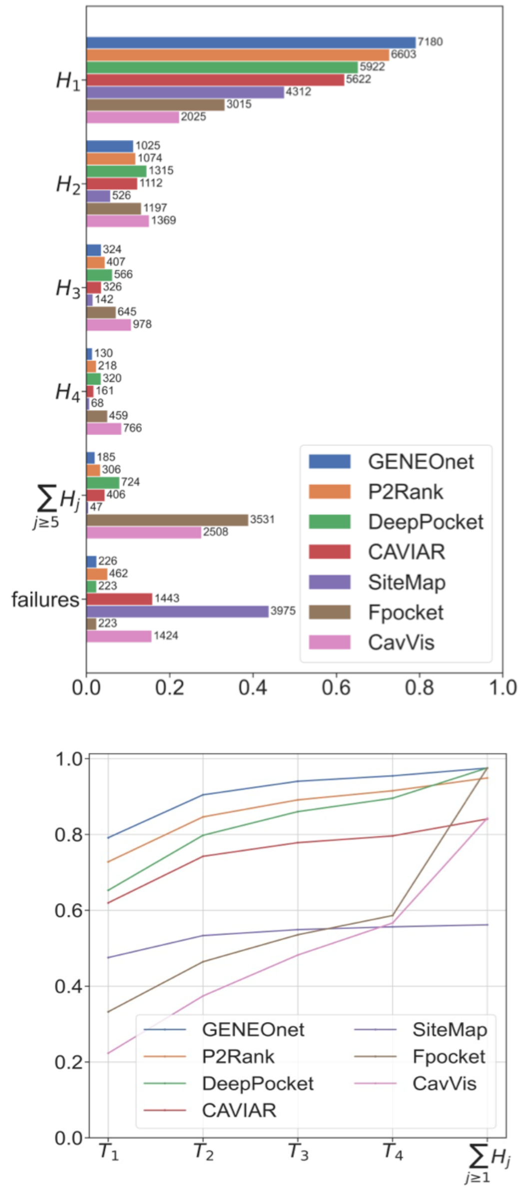

Figure 3. Comparison results. The top figure shows a bar chart of the proportions of correct recognition H_j for the different methods, while the bottom figure shows the corresponding cumulative frequency curves.

4 SCENE-net: application of GENEOs to LiDAR scans segmentation for maintenance of electric power plants

In Portugal, the maintenance and inspection of the energy transmission system is based on LiDAR point clouds. Low-flying helicopters are deployed to scan the environment from a bird-eye view (BEV) perspective and store the results in a 3D point cloud format for further processing by maintenance personnel. This results in detailed large-scale point clouds with high point density, no sparsity and no object occlusion. The captured 3D scenes are quite extensive and mostly composed of arboreal/rural areas, with the transmission line making a small percentage of the LiDAR scans. As a result, maintenance specialists spend the majority of their time manually sectioning and labelling 3D data in order to focus on 3D scenes that encompass the transmission line, for later inspection, to avoid collisions with the vegetation that may cause fires. In order to accelerate this task, we applied GENEOs to the detection of power-line supporting towers and produced a semantic segmentation. These metal structures serve as points of reference for the location of the electrical network. By doing so, the laborious task of manually searching and sectioning the 3D scenes that contain parts of the

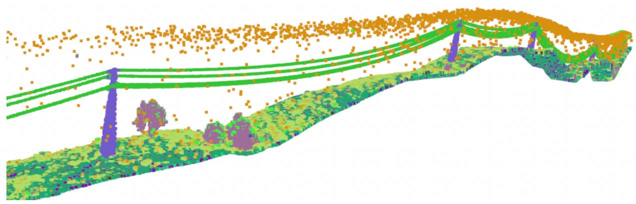

Figure 4. Visualization of TS40K raw point cloud with colored labels.

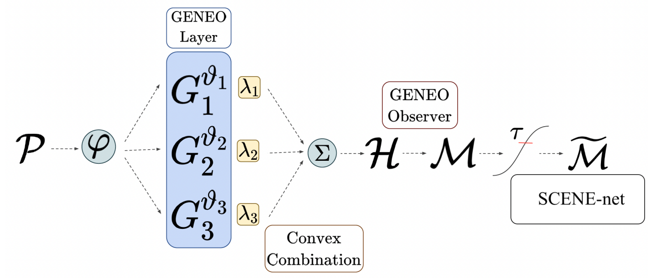

Figure 5. Pipeline of SCENE-net: an input point cloud $\mathcal{P}$ is voxelized and a measurement φ is applied. This representation then is fed to a GENEO layer, where each $G_i^{\vartheta_i}$ separately convolves the input. A GENEO observer $\mathcal{H}$ is then achieved by a convex combination of the operators in the GENEO layer. The function $\mathcal{M}$ transforms the analysis of the observer into the probability of belonging to a tower. Lastly, a threshold operation is applied to classify the voxels. Note that this final step occurs after training is completed.

transmission network can be automated, which grants a significant speed-up to the whole procedure. This research is still going on and is performed in collaboration with the CNET Center for New Energy Technologies SA, of the company EDP, the main electric energy provider in Portugal.

We considered a dataset provided by EDP, using 40 000 km of rural and forest terrain labelled with 22 classes, culminating in 2823 samples describing the transmission system, named TS40K (see Figure 4). Withal, the provided point clouds exhibit noisy labels and are mainly composed of non-relevant classes for our problem, such as the ground. Power-line supporting towers make up less than 1% of the overall point clouds, which makes noisy labelling a major issue for the segmentation task. For instance, patches of ground incorrectly classified as tower amount to roughly 40% of tower 3D points.

One plausible way to approach our problem would be to employ state-of-the-art methods with respect to 3D semantic segmentation. However, most proposals [9, 26, 29] do not account for the existence of ground, as it is usually removed to boost efficiency in urban settings, and this is not possible in rural scenes, due to their irregular terrain. Moreover, the high point density combined with the severe class imbalance and noisy labelling in TS40K are sure to affect the performance of these models in real scenarios.

We then built an architecture called SCENE-net, based on GENEOs [17], whose equivariance properties encode prior knowledge on the objects of interest (such as geometrical characteristics of towers) and embed them into a model still based on convolutional kernels, similarly to the previous application.

Note that also for this problem there is a piece of information that can be injected in a learning system based on GENEOs, exploiting the equivariance property. In fact, the shape of the towers could easily be recognized by a human being, but should be learned by a 'blind' machine-learning system. Therefore, also in this case study, the knowledge injection in the GENEO network results in a simplified and thus more interpretable model.

4.1 SCENE-net: the model

The pipeline used for SCENE-net is quite similar to the one of GENEOnet, used for protein pocket detection. A schematic representation is reported in Figure 5.

The input is a point cloud denoted by $\mathcal{P} \in \mathbb{R}^{N\times(3+d)}$, where N is the number of points and $3+d$ is the number of spatial coordinates and of any point-wise recorded features, such as colors, labels, normal vectors, etc. The cloud $\mathcal{P}$ is first discretized, using a 3D regular grid, or voxel discretization of the considered scene, and then fed to a layer of GENEOs $G_i^{\vartheta_i}$ (GENEO layer), each chosen from a parametric family of operators, and defined by a set of trainable shape parameters ϑ_i, $i = 1, \dots, n$. Such GENEOs are employed as kernels for convolutional operators. The output of the GENEO layer is then combined into another GENEO $\mathcal{H}$ obtained by a convex combination of the $G_i^{\vartheta_i}$, with weights $\lambda_1, \dots, \lambda_n$:

$$\mathcal{H}(x) = \sum_{i=1}^{n} \lambda_i G_i^{\vartheta_i}(x), \quad \lambda_i \in [0,1], \ \sum_{i=1}^{n} \lambda_i = 1,$$

where x is a point of the discretization grid. Because of the properties of GENEOs recalled in Section 2.1, $\mathcal{H}$ is still a GENEO that can be interpreted as an 'expert' observer, and the estimated value of each coefficient λ_i represents the contribution given to the expert observer by the 'naive observer' $G_i^{\vartheta_i}$. The parameters λ_i grant then our model its intrinsic interpretability. They are learned during training and represent the importance of each $G_i^{\vartheta_i}$, and, by extension, the importance of their encoded properties, in modelling the ground truth.

Next, we transform the observer's analysis into the probability $\mathcal{M}(x)$ that x belongs to a supporting tower, as follows:

$$\mathcal{M}(x) = \big(\tanh(\mathcal{H}(x))\big)_+.$$

Negative signals in $\mathcal{H}(x)$ represent patterns that do not exhibit the sought-out geometrical properties. Conversely, positive values quantify their presence. Therefore, tanh compresses the observer's value distribution into $[-1,1]$, and a rectified linear unit (ReLU) is then applied to enforce a zero probability to negative signals.

Lastly, a probability threshold $\tau \in [0,1]$ is defined and applied to $\mathcal{M}$ to detect the points of the discretization grid that lie on the towers:

$$\widetilde{\mathcal{M}} = \{x \in \text{grid} \mid \mathcal{M}(x) \geq \tau\}.$$

In order to recognize the main geometrical characteristics of towers, we used three different kernels for the GENEOs $G_i^{\vartheta_i}$:

- A cylindrical kernel, with main axis orthogonal to the plane of the ground. The corresponding GENEO is thus equivariant under rotations around a vertical axis and is able to identify vertical structures which are much higher than the surrounding landscape, as towers are.
- A cone-cylinder kernel, formed by a cylinder with a cone on the top. The corresponding GENEO is still equivariant under rotations around a vertical axis and is able to distinguish towers from trees, because of the typical shape formed by power lines stemming from the top of the towers.
- A sphere with negative values in its interior, able to detect bushes and tree crowns and to assign them a negative weight.

4.2 Parameter identification

The unknown parameters $\vartheta_i, \lambda_i, i = 1,2,3$ of the model are identified by solving the optimization problem

$$\min_{\underline{\vartheta},\underline{\lambda}} \mathbb{E}[\mathcal{L}(\underline{\vartheta},\underline{\lambda},X)] \quad \text{such that} \quad \vartheta_i \geq 0,\ \forall i, \\ \lambda^T \mathbf{1} = 1, \\ \lambda_i \geq 0,\ \forall i.$$

Here the loss $\mathcal{L}$ is defined by

$$\mathcal{L}(\underline{\vartheta},\underline{\lambda},X) = f_{\mathrm{w}}(\alpha,\varepsilon,y)\,(\mathcal{M}_{\underline{\vartheta},\underline{\lambda}}(X) - y)^2,$$

where $\mathcal{M}_{\underline{\vartheta},\underline{\lambda}}(X)$ is the estimated probability that the voxel X lies on a tower, y is the ground truth probability that voxel X lies on a tower (computed as the proportion of LiDAR scanned points lying in voxel X whose labels belong to a tower) and f_{w} is a weight as proposed in [25] to mitigate data imbalance. The hyperparameter α emphasizes the weighting scheme, whereas ε is a small positive number which ensures that the weights of the samples are positive (see [17, 25] for more details).

Like in the previous case study, the Adam algorithm was applied to solve the optimization problem.

4.3 SCENE-net results and comparison with other methods

In order to limit the unbalanced nature of the dataset in the training phase, the entire TS40K dataset has been sectioned into 2823 subsets, each cropped around one different supporting tower. The samples have then been randomly split into a training set (80% of the total), a validation set (20% of the total), and a test set (10% of the total).

The results of SCENE-net have been compared with those of a convolutional neural network (CNN) applied to the same data. The following metrics have been used to compare the two methods:

- Precision = (# true positive)/(# true positive + # false positive). This index tells us from all voxels predicted positively, what percentage did the model classify correctly.
- Recall = (# true positive)/(# true positive + # false negative). This index provides the percentage of voxels lying on a tower that were correctly classified.
- Intersection over union (IoU) = (# true positive)/(# true positive + # false negative + # false positive). This index measures the overlap between the prediction and the ground truth, over the total volume they occupy.

The results are reported in Table 1.

Method	Precision	Recall	IoU
CNN	0.44 ($\pm$0.07)	**0.26** ($\pm$0.02)	0.53
SCENE-net	**0.68** ($\pm$0.08)	0.16 ($\pm$0.05)	**0.58**

Table 1. Comparison metrics between SCENE-net and CNN on TS40K.

Quantitatively, using SCENE-net we observe a lift in Precision of 24%, and of 5% in IoU, and a drop of 10% in Recall. The lower Recall of SCENE-Net is due to noisy labels in the ground truth. As shown in Figure 6, the ground surrounding supporting towers as well as power lines are often mislabeled as tower.

Additionally, from Figure 7 we note that the performance of SCENE-net is comparable to that of CNN when we change the classification threshold τ in the model pipeline, but SCENE-net has in total 11 parameters to be identified, while CNN has about 10^3 unknown parameters and therefore needs to be trained with a much bigger training set.

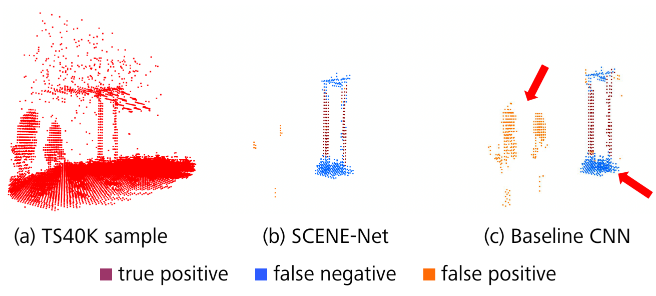

Figure 6. For the TS40K sample shown in (a), SCENE-Net accurately detects the body of the power grid tower (b), while a comparable CNN has a large false positive area in the vegetation (c). Our model is interpretable with 11 trainable geometric parameters, whereas the CNN has a total of 2190 parameters. Note that the ground around the towers and the lines above are mislabeled as towers in the ground truth.

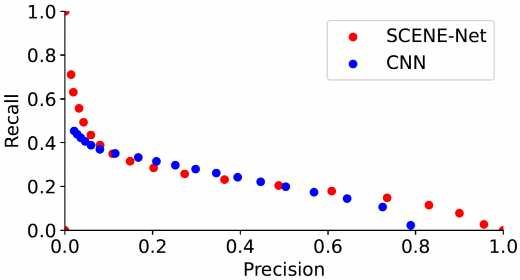

Figure 7. Precision–Recall curve for SCENE-Net and the CNN benchmark, with changing detection threshold. Although our model SCENE-Net has two orders of magnitude less parameters than the CNN, it attains a comparable area under the P–R curve.

Acknowledgements. Thanks to Krzysztof Burnecki for the kind invitation to write this article. Fruitful discussions and scientific collaborations on the subject of the article are also acknowledged to Patrizio Frosini (University of Bologna), Claudia Soares (NOVA University of Lisbon), and to Giovanni Bocchi and Diogo Lavado, who are the PhD students in Milan acting as driving force of this research. The author is also indebted to Alessandro Pedretti (Department of Pharmaceutical Sciences at University of Milan) for the interdisciplinary communication environment that initiated this work, and to the collaborators from EDP CNET Centre (Manuel Pio Silva, Alex Coronati) and from Dompé Farmaceutici (Andrea Beccari, Filippo Lunghini, Carmen Gratteri, Carmine Talarico), whose experience and attitude to interdisciplinary and cross-sectoral collaboration is a strong added value to this research. This work has been partially funded by Dompé Farmaceutici, and the collaboration started during events funded by the EU funded MSCA project BIGMATH (grant number 812912).

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On the dynamics of dilute gases

English translation of the paper “Sur la dynamique des gaz dilués”
published in *La Gazette des Mathématiciens*, Number G174, October 2022

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The evolution of a gas can be described by different mathematical models depending on the scale of observation. A natural question, raised by Hilbert in his sixth problem, is whether these models provide mutually consistent predictions. In particular, for rarefied gases, it is expected that the equations of the kinetic theory of gases can be obtained from molecular dynamics governed by the fundamental principles of mechanics. In the case of hard sphere gases, Lanford (1975) has shown that the Boltzmann equation does indeed appear as a law of large numbers in the low density limit, at least for very short times. The aim of this paper is to present recent advances in the understanding of this limiting process.

1 A statistical approach to dilute gas dynamics

1.1 The physical model: A dilute gas of hard spheres

Although at the time Boltzmann published his famous paper [8] the atomic theory was still rejected by some scientists, it was already well established that matter is composed of atoms, which are the elementary constituents of all solids, liquids and gases. The particularity of gases is that the volume occupied by their atoms is negligible as compared to the total volume occupied by the gas, and there are therefore very few constraints on the atoms' geometric arrangement: they are thus very loosely bound and almost independent. Neglecting the internal structure of the atoms, their possible organization into molecules, and the effect of long-range interactions, a gas can be represented as a system formed by a large number of particles that move in a straight line and occasionally collide with each other, resulting in an almost instantaneous scattering. The simplest example of such a model consists in assuming that the particles are small identical spheres, of diameter $\varepsilon \ll 1$ and mass 1, interacting only by contact (Figure 1). We refer to this as a *gas of hard spheres*. This microscopic description of a gas is explicit, but very difficult to use in practice because the number of particles is extremely large, their size is tiny and their collisions are very sensitive to small shifts (Figure 2). This model is therefore not efficient for making theoretical predictions. A natural question is whether one can extract, from such a complex system, less precise but more stable models suitable for applications, such as kinetic or fluid models. This question was formalized by Hilbert at the International Congress of Mathematicians in 1900, in his sixth problem:

> Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua.

The Boltzmann equation, mentioned by Hilbert and described in more detail below, expresses that the particle distribution evolves under the combined effect of free transport and collisions. For these two effects to be of the same order of magnitude, a simple calculation shows that, in dimension $d \geq 2$, the number of particles N and their diameter ε must satisfy the scaling relation $N\varepsilon^{d-1} = O(1)$, called *low density scaling* [14]. Indeed, the regime described by the Boltzmann equation is such that the mean free path, i.e., the average distance traveled by a particle moving in a straight line between two collisions, is of order 1. Thus, a typical particle should go through a tube of volume $O(\varepsilon^{d-1})$ between two collisions, and on average, this tube should cross one of the $N-1$ other particles. Note that, in this regime, the total volume occupied by the particles at a given time is proportional to $N\varepsilon^d$ and is therefore negligible compared to the total volume occupied by the gas. We speak then of a *dilute gas*.

1.2 Three levels of averaging

Henceforth, it is assumed that the particle system evolves in the unit domain with periodic boundary conditions $\mathbb{T}^d = [0,1]^d$. We consider that the N particles are identical: this is the exchangeability assumption. The state of the system can be represented by a measure in the phase space $\mathbb{T}^d \times \mathbb{R}^d$ called *empirical measure*,

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x-x_i}\delta_{v-v_i},$$

where δ_x is the Dirac mass at $x = 0$. This measure is completely symmetric (i.e., invariant under permutation of the indices of the particles) because of the exchangeability assumption. This first aver-

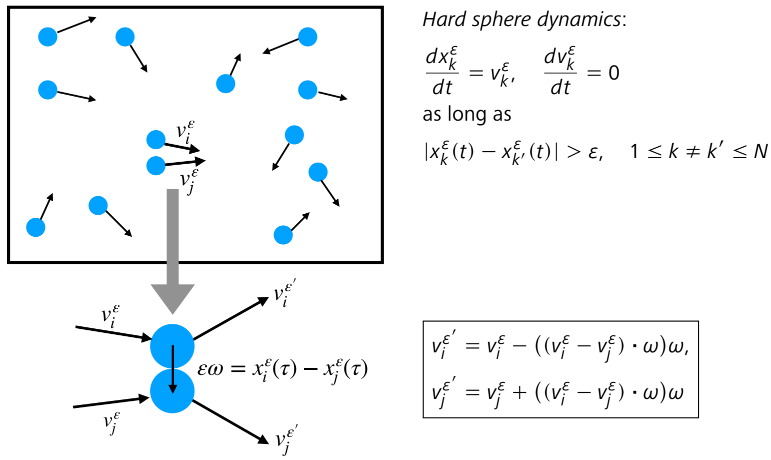


Figure 1. At time t, the system of hard spheres is described by the positions $(x_k^\varepsilon(t))_{k\le N}$ and the velocities $(v_k^\varepsilon(t))_{k\le N}$ of the centers of gravity of the particles. The spheres move in a straight line and when two of them touch, they are scattered according to elastic reflection laws.

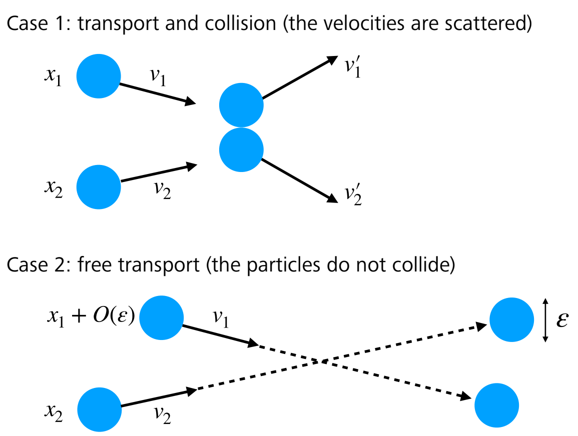


Figure 2. The particles are very small (of diameter $\varepsilon \ll 1$) and the dynamics is very sensitive to small spatial shifts. In the first case depicted above, two particles with initial positions x_1, x_2 and velocities v_1, v_2 collide and are scattered. In the second case, after shifting the position of the first particle by a distance $O(\varepsilon)$, they no longer collide and each particle keeps moving in a straight line. Thus, a perturbation of order ε of the initial conditions can lead to very different trajectories.

aging is however not sufficient to obtain a robust description of the dynamics when N is large, because of the instabilities mentioned in the previous section (Figure 2) which lead to a strong dependence of the particle trajectories on ε.

We will therefore introduce a second averaging, with respect to the initial configurations; from a physical point of view, this averaging is natural since only fragmentary information on the initial configuration is available. We therefore assume that the initial data $(X_N, V_N) = (x_i, v_i)_{1\le i\le N}$ are independent random variables, identically distributed according to a distribution $f^0 = f^0(x,v)$. This assumption must be slightly corrected to account for particle exclusion: $|x_i - x_j| > \varepsilon$ for $i \ne j$. This statistical framework is called the *canonical* setting. It is a simple framework allowing us to establish rigorous foundations for the kinetic theory, i.e., to characterize, in the large N asymptotics, the average dynamics and more precisely the evolution equation governing the distribution $f(t,x,v)$ at time t of a typical particle.

In this paper, our aim is to go beyond this averaged dynamics, and to describe in a precise way the correlations that appear dynamically inside the gas. Fixing a priori the number N of particles induces additional correlations, so to circumvent them, we introduce a third level of averaging by assuming that N is also a random variable, and that only its average $\mu_\varepsilon = \varepsilon^{-(d-1)}$ is determined (according to the low density scaling). To define a system of initially independent (modulo exclusion) identically distributed hard spheres according to f^0, we introduce the *grand canonical* measure as follows: the probability density of finding N particles of coordinates $(x_i, v_i)_{i\le N}$ is given by

$$\frac{1}{\mathcal{Z}^\varepsilon}\frac{\mu_\varepsilon^N}{N!}\prod_{i=1}^N f^0(x_i,v_i)\prod_{i\ne j}\mathbf{1}_{|x_i-x_j|>\varepsilon} \quad \text{for } N = 0,1,2,\ldots, \tag{1.1}$$

where the constant $\mathcal{Z}^\varepsilon$ is the normalization factor of the probability measure. We will assume in the following that the function f^0 is Lipschitz continuous, with a Gaussian decay in velocity. The corresponding probability and expectation will be denoted by $\mathbb{P}_\varepsilon$ and $\mathbb{E}_\varepsilon$.

1.3 *A statistical approach*

Once the initial random configuration $(N, (x_i^{\varepsilon 0}, v_i^{\varepsilon 0})_{1\le i\le N})$ is chosen, the hard sphere dynamics evolves deterministically (according to the hard sphere equations shown in Figure 1), and we seek to understand the statistical behavior of the empirical measure

$$\pi_t^\varepsilon(x,v) := \frac{1}{\mu_\varepsilon}\sum_{i=1}^{N}\delta_{x-x_i^\varepsilon(t)}\delta_{v-v_i^\varepsilon(t)} \tag{1.2}$$

and its evolution in time.

A law of large numbers

The first step is to determine the law of large numbers, that is, the limiting distribution of a typical particle when $\mu_\varepsilon \to \infty$. In the case of N identically distributed independent variables $(\eta_i)_{1\le i\le N}$ of expectation $\mathbb{E}(\eta)$, the law of large numbers implies in particular that the mean converges in probability to the expectation:

$$\frac{1}{N}\sum_{i=1}^{N}\eta_i \xrightarrow[N\to\infty]{} \mathbb{E}(\eta).$$

One can easily show the following convergence in probability:

$$\langle \pi_0^\varepsilon, h\rangle := \frac{1}{\mu_\varepsilon}\sum_{i=1}^{N} h(x_i^{\varepsilon 0}, v_i^{\varepsilon 0}) \xrightarrow[\mu_\varepsilon\to\infty]{} \int f^0 h(x,v)\,dx\,dv,$$

under the grand canonical measure. The difficulty is to understand whether the initial quasi-independence propagates in time so that there exists a function $f = f(t,x,v)$ such that the following convergence in probability holds:

$$\langle \pi_t^\varepsilon, h\rangle \xrightarrow[\mu_\varepsilon\to\infty]{} \int f(t,x,v)h(x,v)\,dx\,dv \tag{1.3}$$

under the grand canonical measure (1.1) over the initial configurations. The most important result proving this convergence was obtained by Lanford [16]: he showed that f evolves according to a deterministic equation, namely the Boltzmann equation. This result will be explained in Section 2.2.

A central limit theorem

The approximation (1.3) of the empirical measure neglects two types of errors. The first is the presence of correction terms that converge to 0 when $\mu_\varepsilon \to +\infty$. The second is related to the probability, which must tend to zero, of configurations for which this convergence does not occur. A classical problem in statistical physics is to quantify more precisely these errors, by studying the fluctuations, i.e., the deviations between the empirical measure and its expectation. In the case of N independent and identically distributed variables $(\eta_i)_{1\le i\le N}$, the central limit theorem implies that the fluctuations are of order $O(1/\sqrt{N})$, and the following convergence in law holds true:

$$\sqrt{N}\Big(\frac{1}{N}\sum_{i=1}^{N}\eta_i - \mathbb{E}(\eta)\Big) \xrightarrow[N\to\infty]{\text{(law)}} \mathcal{N}(0,\mathrm{Var}(\eta)),$$

where $\mathcal{N}(0,\mathrm{Var}(\eta))$ is the normal law of variance $\mathrm{Var}(\eta) = \mathbb{E}((\eta - \mathbb{E}(\eta))^2)$. In particular, at this scale, we find some randomness. Investigating the same fluctuation regime for the dynamics of hard sphere gases consists in considering the fluctuation field ζ_t^ε defined by duality, namely,

$$\langle \zeta_t^\varepsilon, h\rangle := \sqrt{\mu_\varepsilon}\big(\langle \pi_t^\varepsilon, h\rangle - \mathbb{E}_\varepsilon(\langle \pi_t^\varepsilon, h\rangle)\big), \tag{1.4}$$

where h is a continuous function, and $\mathbb{E}_\varepsilon$ the expectation with respect to the grand canonical measure. At time 0, one can easily show that, under the grand-canonical measure, the fluctuation field ζ_0^ε converges in the low density limit to a Gaussian field ζ_0 with covariance

$$\mathbb{E}\big(\zeta_0(h)\zeta_0(g)\big) = \int f^0(z)h(z)g(z)\,dz.$$

A series of recent works [4–7] has allowed to characterize the fluctuation field (1.4) and to obtain a stochastic evolution equation governing the limit process. These results are presented in Section 3.3.

On large deviations

The last question generally studied in a classical probabilistic approach is that of the quantification of rare events, i.e., the estimation of the probability of observing an atypical behavior (which deviates macroscopically from the mean). For independent and identically distributed random variables, this probability is exponentially small, and it is therefore natural to study the asymptotics

$$I(m) := \lim_{\delta\to 0}\lim_{N\to\infty} -\frac{1}{N}\log\mathbb{P}\Big(\Big|\frac{1}{N}\sum_{i=1}^{N}\eta_i - m\Big| < \delta\Big) \quad \text{with } m \neq \mathbb{E}(\eta).$$

The limit $I(m)$ is called the large deviation functional and can be expressed as the Legendre transform of the log-Laplace transform $\mathbb{R}\ni u\mapsto \log\mathbb{E}(\exp(u\eta))$. To generalize this statement to correlated variables in a gas of hard spheres, it is necessary to compute the log-Laplace transform of the empirical measure on deterministic trajectories, which requires extremely precise control of the dynamical correlations. Note that, at time 0, under the grand canonical measure, one can show that, for any $\delta > 0$,

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\mu_\varepsilon\to\infty} -\frac{1}{\mu_\varepsilon}\log\mathbb{P}_\varepsilon\big(d(\pi_0^\varepsilon,\varphi^0)\le\delta\big)\\ = H(\varphi^0|f^0) := \int\Big(\varphi^0\log\frac{\varphi^0}{f^0} - (\varphi^0 - f^0)\Big)\,dx\,dv,\end{aligned}$$

where d is a distance on the space of measures. The dynamical cumulant method introduced in [4, 6] is a key tool for computing the exponential moments of the hard sphere distribution, thus obtaining the dynamical equivalent of this result in short time. We give an overview of these techniques in Section 3.

2 Typical behavior: A law of large numbers

2.1 Boltzmann's amazing intuition

The equation that rules the typical evolution of a gas of hard spheres was heuristically proposed by Boltzmann [8] about a century before its rigorous derivation by Lanford [16], as the "limit" of the particle system when $\mu_\varepsilon \to +\infty$. Boltzmann's revolutionary idea was to write an evolution equation for the probability density $f = f(t,x,v)$ giving the proportion of particles at position x with velocity v at time t. In the absence of collisions, and in an unbounded domain, this density f would be transported along the physical trajectories $x(t) = x(0) + vt$, which means that $f(t,x,v) = f^0(x - vt, v)$. The challenge is to take into account the statistical effect of collisions. As long as the size of the particles is negligible, one can consider that these collisions are pointwise in both t and x. Boltzmann proposed a quite intuitive counting:

- the number of particles of velocity v increases when a particle of velocity v' collides with a particle of velocity v_1', and takes the velocity v (Figure 1 and (2.2));
- the number of particles of velocity v decreases when a particle of velocity v collides with a particle of velocity v_1, and is deflected to another velocity.

The probability of these jumps in velocity is described by a transition rate, called the *collision cross section*. For interactions between hard spheres, it is given by $((v - v_1) \cdot \omega)_+$, where $v - v_1$ is the relative velocity of the colliding particles, and ω is the deflection vector, uniformly distributed in the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$.

The fundamental assumption of Boltzmann's theory is that, in a rarefied gas, the correlations between two colliding particles must be very small. Therefore, the joint probability of having two pre-colliding particles of velocities v and v_1 at position x at time t should be well approximated by the product $f(t,x,v)f(t,x,v_1)$. This independence property is called the *molecular chaos hypothesis*. The Boltzmann equation then reads

$$\partial_t f + \underbrace{v \cdot \nabla_x f}_{\text{transport}} = \underbrace{C(f,f)}_{\text{collision}}, \tag{2.1}$$

where

$$C(f,f)(t,x,v) = \iint [\underbrace{f(t,x,v')f(t,x,v_1')}_{\text{gain term}} - \underbrace{f(t,x,v)f(t,x,v_1)}_{\text{loss term}}] \times \underbrace{((v - v_1) \cdot \omega)_+}_{\text{cross section}} \, dv_1 \, d\omega,$$

with the scattering rules

$$v' = v - ((v - v_1) \cdot \omega)\omega, \quad v_1' = v_1 + ((v - v_1) \cdot \omega)\omega \tag{2.2}$$

being analogous to those introduced in Figure 1, with the important difference that ω is now a random vector chosen uniformly in the unit sphere $\mathbb{S}^{d-1}$: indeed, the relative position of the colliding particles disappeared in the limit $\varepsilon \to 0$. As a result, the Boltzmann equation is singular because it involves a product of densities at a single point x.

Boltzmann's idea of reducing the Hamiltonian dynamics describing atomic behavior to a kinetic equation was revolutionary and paved the way to the description of non-equilibrium phenomena by mesoscopic equations. However, the Boltzmann equation (2.1) was first strongly criticized because it seems to violate some fundamental physical principles. It actually predicts an irreversible evolution in time: it has a Lyapunov functional, called entropy, defined by $S(t) := -\iint f \log f(t,x,v) \, dx \, dv$, such that $\frac{d}{dt} S(t) \geq 0$, with equality if and only if the gas is in thermodynamic equilibrium. The Boltzmann equation thus provides a quantitative formulation of the second principle of thermodynamics. But at first glance, this irreversibility seems incompatible with the fact that the dynamics of hard spheres is governed by a Hamiltonian system, i.e., a system of ordinary differential equations that is completely reversible in time. Soon after Boltzmann postulated his equation, these two different behaviors were considered, by Loschmidt, as a paradox and an obstruction to Boltzmann's theory. A fully satisfactory mathematical explanation of this question remained elusive for almost a century, until the role of probabilities was precisely identified: the underlying dynamics is reversible, but the description that is given of this dynamics is only partial and is therefore not reversible.

2.2 Typical behavior: Lanford's theorem

Lanford's result [16] shows in which sense the Boltzmann equation (2.1) is a good approximation of the hard sphere dynamics. It can be stated as follows (this is not exactly the original formulation; see in particular Section 2.4 below for comments).

Theorem 2.1 (Lanford). *In the low density limit ($\mu_\varepsilon \to \infty$ with $\mu_\varepsilon \varepsilon^{d-1} = 1$), the empirical measure π_t^ε defined by* (1.2) *concentrates on the solution of the Boltzmann equation* (2.1)*: for any bounded and continuous function h,*

$$\forall \delta > 0, \quad \lim_{\mu_\varepsilon \to \infty} \mathbb{P}_\varepsilon \Big(\Big| \langle \pi_t^\varepsilon, h \rangle - \int f(t,x,v) h(x,v) \, dx \, dv \Big| \geq \delta \Big) = 0,$$

on a time interval $[0, T_{\mathrm{L}}]$ that depends only on the initial distribution f^0.

The time of validity T_{L} of the approximation is found to be a fraction of the average time between two successive collisions for a typical particle. This time is large enough for the microscopic system to undergo a large number of collisions (of the order of μ_ε), but (much) too small to see phenomena such as relaxation to (local) thermodynamic equilibrium, and in particular hydrodynamic regimes. Physically, we do not expect this time to be critical, in the sense that the dynamics would change in nature afterwards. In fact, in practice, Boltzmann's equation is used in many applications (such as spacecraft reentrance calculations) without time restrictions. However, it is important to note that a time restriction might not be only technical: from a mathematical point of view, one cannot exclude that the Boltzmann equation presents singularities (typically spatial concentrations that would prevent the collision term from making sense, and that would also locally contradict the low density assumption). At present, the problem of extending Lanford's convergence result to longer times still faces serious obstacles.

2.3 Heuristics of Lanford's proof

Let us informally explain how the Boltzmann equation (2.1) can be predicted from the dynamics of the particles. The goal is to transport via the dynamics the initial grand canonical measure (1.1) and then to project this measure at time t onto the 1-particle phase space. We thus define by duality the density $F_1^\varepsilon(t,x,v)$ of a typical particle with respect to a test function h by

$$\int F_1^\varepsilon(t,x,v)h(x,v)\,dx\,dv := \mathbb{E}_\varepsilon(\langle \pi_t^\varepsilon, h\rangle). \tag{2.3}$$

Theorem 2.1 states that F_1^ε converges to the solution to the Boltzmann equation f in the low density limit. So let h be a regular and bounded function on $\mathbb{T}^d \times \mathbb{R}^d$ and consider the evolution of the empirical measure during a short time interval $[t, t+\delta]$. Separating the different contributions according to the number of collisions, we can write

$$\begin{aligned}
&\frac{1}{\delta}(\mathbb{E}_\varepsilon[\langle \pi_{t+\delta}^\varepsilon, h\rangle] - \mathbb{E}_\varepsilon[\langle \pi_t^\varepsilon, h\rangle]) \\
&\quad = \frac{1}{\delta}\mathbb{E}_\varepsilon\Big[\frac{1}{\mu_\varepsilon}\sum_{\substack{j \\ \text{no collision}}}\big(h(z_j^\varepsilon(t+\delta)) - h(z_j^\varepsilon(t))\big)\Big] \\
&\qquad + \frac{1}{\delta}\mathbb{E}_\varepsilon\Big[\frac{1}{2\mu_\varepsilon}\sum_{\substack{(i,j) \\ \text{one collision}}}\big(h(z_i^\varepsilon(t+\delta)) + h(z_j^\varepsilon(t+\delta)) \\
&\qquad\qquad - h(z_i^\varepsilon(t)) - h(z_j^\varepsilon(t))\big)\Big] \\
&\qquad + \cdots.
\end{aligned} \tag{2.4}$$

To simplify, $z_i^\varepsilon(t)$ denotes the coordinates $(x_i^\varepsilon(t), v_i^\varepsilon(t))$ of the i-th particle at time t. Since the left-hand side of (2.4) formally converges when $\delta \to 0$ to the time derivative of $\mathbb{E}_\varepsilon[\langle \pi_t^\varepsilon, h\rangle]$, we will analyze the limit $\delta \to 0$ of the first two terms in the right-hand side of (2.4), which should lead to a transport term and a collision term as in (2.1). We will also explain why the remainder terms, involving two or more collisions in the short time interval δ, tend to 0 with δ (showing that they are of order δ).

Since the particles move in a straight line and at constant speed in the absence of collisions, if the distribution F_1^ε is sufficiently regular, the definition (2.3) of F_1^ε formally implies that, when δ tends to 0, the first term in the right-hand side of (2.4) is asymptotically equal to

$$\int F_1^\varepsilon(t,z) v\cdot\nabla_x h(z)\,dz = -\int (v\cdot\nabla_x F_1^\varepsilon(t,z))h(z)\,dz.$$

The transport term in (2.1) is thus well obtained in the limit. Let us now consider the second term in the right-hand side of (2.4). Two particles of configurations (x_1, v_1) and (x_2, v_2) at time t collide at a later time $\tau \le t+\delta$ if there exists $\omega \in \mathbb{S}^{d-1}$ such that

$$x_1 - x_2 + (\tau - t)(v_1 - v_2) = -\varepsilon\omega. \tag{2.5}$$

This implies that their relative position must belong to a tube of length $\delta|v_1 - v_2|$ and width ε oriented in the $v_1 - v_2$ direction. The Lebesgue measure of this set is of the order $\delta\varepsilon^{d-1}|v_2 - v_1| = O(\delta\varepsilon^{d-1})$ (neglecting large velocities). More generally, a sequence of $k-1$ collisions between k particles imposes $k-1$ constraints of the previous form, and this event can be shown to have probability less than $(\delta\varepsilon^{d-1})^{k-1} = (\delta\mu_\varepsilon^{-1})^{k-1}$ (again neglecting large velocities). Since there are, on average, μ_ε^k ways to choose these k colliding particles, we deduce that the occurrence of $k-1$ collisions in (2.4) has a probability of order $\delta^{k-1}\mu_\varepsilon$. This explains why the probability of having $k \ge 3$ colliding particles can be estimated by $O(\delta^2)$ and thus can be neglected in (2.4).

It remains to examine more closely the collision term involving two particles in (2.4), in order to obtain the collision operator $C(f,f)$ of the Boltzmann equation (2.1). This term involves the two-particle correlation function F_2^ε. For any $k \ge 1$, we define

$$\begin{aligned}
&\int F_k^\varepsilon(t, Z_k)h_k(Z_k)\,dZ_k \\
&\qquad = \mathbb{E}_\varepsilon\Big(\frac{1}{\mu_\varepsilon^k}\sum_{(i_1,\dots,i_k)} h_k\big(z_{i_1}^\varepsilon(t),\dots,z_{i_k}^\varepsilon(t)\big)\Big),
\end{aligned} \tag{2.6}$$

where $i_1,\dots,i_k$ are all distinct and $Z_k = (x_i, v_i)_{1\le i\le k}$. We can then show that, in the limit $\delta \to 0$,

$$\partial_t F_1^\varepsilon + \underbrace{v\cdot\nabla_x F_1^\varepsilon}_{\text{transport}} = \underbrace{C^\varepsilon(F_2^\varepsilon)}_{\substack{\text{collision} \\ \text{at distance } \varepsilon}}, \tag{2.7}$$

where

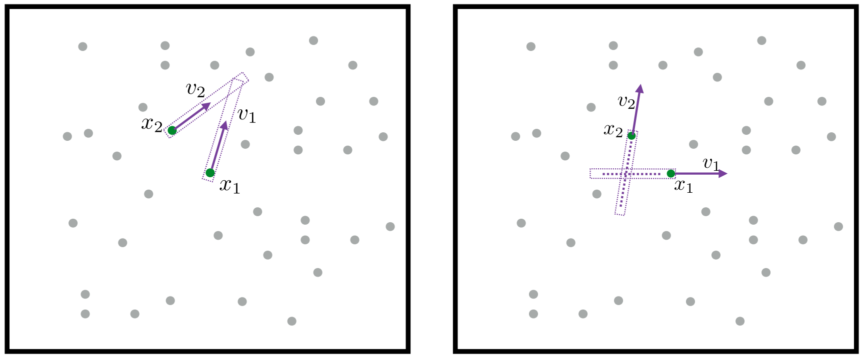
$$\begin{aligned}
&C^\varepsilon(F_2^\varepsilon)(t,x,v) \\
&\quad = \iint [\underbrace{F_2^\varepsilon(t,x,v',x+\varepsilon\omega,v_1')}_{\text{gain term}} - \underbrace{F_2^\varepsilon(t,x,v,x-\varepsilon\omega,v_1)}_{\text{loss term}}] \\
&\qquad\qquad \times \underbrace{((v-v_1)\cdot\omega)_+}_{\text{cross section}}\,dv_1\,d\omega.
\end{aligned}$$

The key step in closing the equation is the *molecular chaos assumption* postulated by Boltzmann, which states that the pre-collisional particles remain independently distributed at all times so that, with the convention (2.5) fixing the sign of ω, we have

$$F_2^\varepsilon(t,z_1,z_2) \simeq F_1^\varepsilon(t,z_1)F_1^\varepsilon(t,z_2) \quad \text{if } (v_1 - v_2)\cdot\omega > 0. \tag{2.8}$$

When the diameter ε of the spheres tends to 0, the coordinates x_1 and x_2 coincide and the scattering parameter ω becomes a random parameter. Assuming that F_1^ε converges, its limit must satisfy the Boltzmann equation (2.1).

Establishing the factorization (2.8) rigorously uses a different strategy, elaborated by Lanford [16], then completed and improved over the years: see the monographs [10, 11, 25]. In the last few years, several quantitative convergence results have been established, and the proofs have been extended to the case of somewhat more general domains, potentials with compact support, or with super-exponential decay at infinity: see [1, 12, 13, 17, 21, 22].



$F_2^\varepsilon(t,x_1,v_1,x_2,v_2) \simeq F_1^\varepsilon(t,x_1,v_1)F_1^\varepsilon(t,x_2,v_2)$ $\quad$ $F_2^\varepsilon(t,x_1,v_1,x_2,v_2) \not\simeq F_1^\varepsilon(t,x_1,v_1)F_1^\varepsilon(t,x_2,v_2)$

Figure 3. In the left figure, particles 1 and 2 will meet in the future; with high probability, they did not collide in the past and we expect the correlation function F_2^ε to factorize in the $\mu_\varepsilon \to \infty$ limit. In the figure on the right, the coordinates of the particles belong to the bad set $\mathcal{B}_2^\varepsilon$, which means that they most likely met in the past. In this case, microscopic correlations have been dynamically constructed and the factorization (2.8) should not be valid.

2.4 *On the irreversibility*

In this section, we will show that the answer to the irreversibility paradox lies in the molecular chaos hypothesis (2.8), which is valid only for specific configurations.

In fact, the notion of convergence that appears in the statement of Theorem 2.1 differs from the one used in Lanford's proof: Theorem 2.1 states the convergence of the $\langle \pi_t^\varepsilon, h\rangle$ observables, i.e., the convergence in the sense of measures, since the test function h must be continuous. This convergence is rather weak and is not sufficient to ensure the stability of the collision term in the Boltzmann equation because this term involves traces. In the proof of Lanford's theorem, we consider all k-particle correlation functions F_k^ε defined by (2.6) and show that, when $\mu_\varepsilon \to \infty$, each of these correlation functions converges uniformly outside a set $\mathcal{B}_k^\varepsilon$ of negligible measure. Thus, the proof uses a much stronger notion of convergence than that stated in Theorem 2.1. Moreover, the set $\mathcal{B}_k^\varepsilon$ of bad microscopic configurations (t, Z_k) (on which F_k^ε does not converge) is somehow transverse to the set of pre-collisional configurations (as can be seen in Figure 3; two particles in $\mathcal{B}_2^\varepsilon$ tend to move away from each other so that they are unlikely to collide). The convergence defect is therefore not an obstacle to taking bounds in the collision term (correlation functions are only evaluated there in pre-collisional configurations). However, these singular sets $\mathcal{B}_k^\varepsilon$ encode important information about the dynamical correlations: by neglecting them, it is no longer possible to go back in time and reconstruct the backward dynamics. Thus, by discarding the microscopic information encoded in $\mathcal{B}_k^\varepsilon$, one can only obtain an irreversible kinetic picture that is far from describing the full microscopic dynamics.

3 Fluctuations and large deviations

3.1 *Corrections to the chaos assumption*

Returning to equation (2.7) on F_1^ε, we can see that, apart from the small spatial shifts of the collision term, the deviations of the Boltzmann dynamics are due to the factorization defect $F_2^\varepsilon - F_1^\varepsilon \otimes F_1^\varepsilon$, a geometric interpretation of which is given below.

Let us first describe the geometric representation of F_1^ε. We look at the history of particle $1^\star$ located at position $x_{1^\star}$ with velocity $v_{1^\star}$ at time t, in order to characterize all initial configurations that contribute to $F_1^\varepsilon(t, x_{1^\star}, v_{1^\star})$. The particle $1^\star$ performs a uniform rectilinear motion $x_{1^\star}(t') = x_{1^\star} - v_{1^\star}(t-t')$ until it collides with another particle, called particle 1, at a time $t_1 < t$. This collision can be of two types: either a physical collision (with deflection), or a mathematical artifact arising from the loss term in equation (2.7)

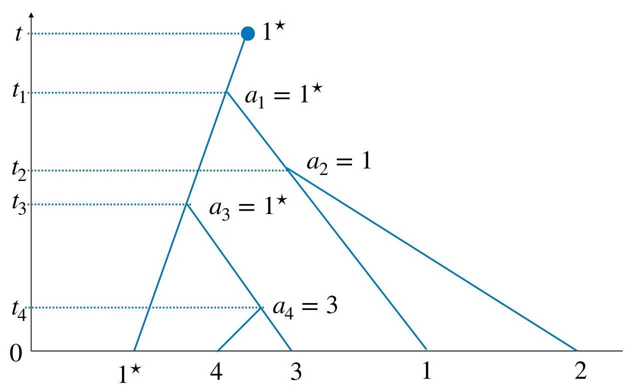


Figure 4. The history of the particle $1^\star$ can be encoded in a tree a, say of size n, whose root is indexed by $1^\star$. The pseudotrajectory is then prescribed by the collision parameters $(t_i, v_i, \omega_i)_{1\le i\le n}$.

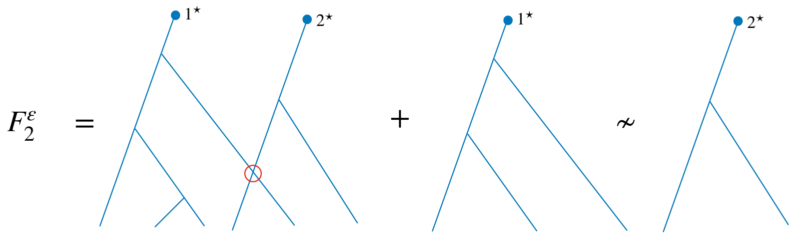


Figure 5. F_2^ε trees are classified into two categories: those involving an (external) collision between the $1^\star$ and $2^\star$ trees, and others for which the particles in the $1^\star$ tree are always at least ε away from those in the $2^\star$ tree (which we denote by $\nsim$).

(the particles touch but are not deflected). From then on, to understand the history of particle $1^\star$, we need to trace the history of both particles $1^\star$ and 1 before time t_1. From time t_1 on, both particles perform uniform rectilinear motions until one of them collides with a new particle 2 at time $t_2 < t_1$, and so on, until time 0. Note that, between the times of collision with new particles, the particles can collide with each other: this will be called *recollision*. The history of the particle $1^\star$ can be encoded using a rooted tree a whose vertices correspond to the different collisions that took place in the history of $1^\star$ and are indexed by the parameters of these collisions. An example is shown in Figure 4. The root of the tree a is indexed by $1^\star$. If n is the total number of collisions, and $0 < t_n < \cdots < t_1 < t$ are the times of the collisions, one can order the particles so that, at time t_i, $1 \le i \le n$, the collision occurs between the i-th particle and the j-th particle, where $j \in \{1^\star, 1, \dots, i-1\}$ (necessarily, $j = 1^\star$ at time t_1). Then the branching of the tree a associated with the i-th collision is indexed by the relation $a_i = j$, where $j \in \{1^\star, 1, \dots, i-1\}$, together with the collision parameters $(t_i, v_i, \omega_i)_{1 \le i \le n}$, where ω_i is the deflection vector. The tensor product $F_1^\varepsilon \otimes F_1^\varepsilon$ is then described by two independent collision trees, with roots $1^\star$ and $2^\star$, and respectively n_1 and n_2 branches.

Now consider the second-order correlation function: F_2^ε can be described by a collision graph constructed from two collision trees with roots $1^\star$ and $2^\star$, and $n_1 + n_2$ branches. The main difference with $F_1^\varepsilon \otimes F_1^\varepsilon$ is that the particles in the $1^\star$ and $2^\star$ trees may interact. We can thus decompose the trees constituting F_2^ε into two categories: those such that there is at least one collision involving a particle from each tree (such a recollision will be called *external*), and the others (Figure 5).

Note, however, that two collision-free trees do not correspond to independent trees, precisely because of the dynamical exclusion condition. This exclusion condition can itself be decomposed as $\mathbf{1}_{1^\star \nsim 2^\star} = 1 - \mathbf{1}_{1^\star \sim 2^\star}$ (Figure 6), where $\mathbf{1}_{1^\star \sim 2^\star}$ means that there is an overlap at some point between a particle from the $1^\star$ tree and a particle from the $2^\star$ tree. This decomposition is a pure mathematical artifact, and the $1^\star \sim 2^\star$ overlap condition does not affect the dynamics (the overlapping particles are not deflected).

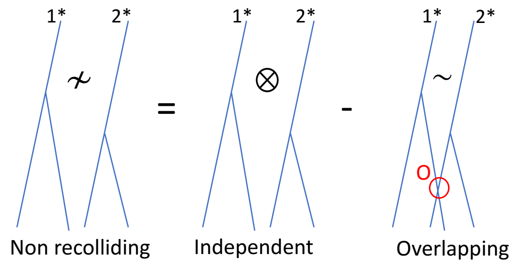


Figure 6. Decomposition of the dynamical exclusion condition.

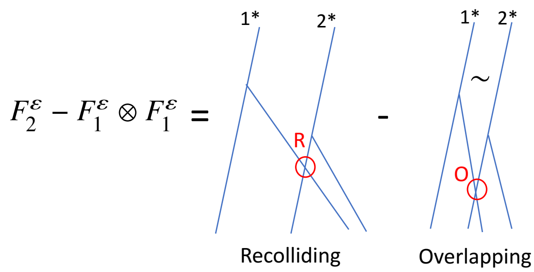


Figure 7. The second-order cumulant corresponds to the occurrence of at least one external recollision or an overlap.

Let us now define the *second-order rescaled cumulant*

$$f_2^\varepsilon := \mu_\varepsilon (F_2^\varepsilon - F_1^\varepsilon \otimes F_1^\varepsilon). \tag{3.1}$$

The previous discussion indicates that this cumulant is represented by trees that are coupled by external collisions or overlaps (Figure 7). In view of definition (3.1) and the discussion in Section 2.3 giving an $O(t/\mu_\varepsilon)$ estimate of the Lebesgue measure of configurations giving rise to a collision, one can expect f_2^ε to be uniformly bounded in L^1 and therefore to have a limit f_2 in the sense of the measures. One can prove in addition that f_2 corresponds to trees with *exactly* one external recollision or overlap on $[0, t]$: any other interaction between the trees gives rise to additional smallness and is therefore negligible.

Remark 3.1. The initial measure does not factorize exactly ($F_2^{\varepsilon,0} \neq F_1^{\varepsilon,0} \otimes F_1^{\varepsilon,0}$) because of the static exclusion condition. Thus, the initial data also induce a small contribution to f_2^ε, but this contribution is significantly smaller than the dynamical correlations (by a factor ε).

3.2 The cumulant generating function

For a Gaussian process, the first two correlation functions F_1^ε and F_2^ε determine completely all other k-particle correlation functions F_k^ε, but in general, part of the information is encoded in the cumulants of higher order ($k \geq 3$)

$$f_k^\varepsilon(t, Z_k) := \mu_\varepsilon^{k-1} \sum_{\ell=1}^{k} \sum_{\sigma \in \mathcal{P}_k^\ell} (-1)^{\ell-1}(\ell-1)! \prod_{i=1}^{\ell} F_{|\sigma_i|}^\varepsilon(t, Z_{\sigma_i}),$$

where $\mathcal{P}_k^\ell$ is the set of partitions of $\{1, \dots, k\}$ into ℓ parts with $\sigma = \{\sigma_1, \dots, \sigma_\ell\}$, $|\sigma_i|$ being the cardinality of the set σ_i and $Z_{\sigma_i} = (z_j)_{j \in \sigma_i}$. Each cumulant encodes finer and finer correlations. Contrary to the correlation functions (F_k^ε), the cumulants (f_k^ε) do not duplicate the information which is already encoded at lower orders. From a geometric point of view, we can extend the analysis of the previous section and show that the cumulant f_k^ε of order k can be represented by k trees that are completely connected either by external collisions, or by overlaps (Figure 8). These dynamical correlations can be classified by a signed graph with k vertices representing the different trees, coding tree collisions (the corresponding edges take a + sign) and overlaps (the corresponding edges take a − sign). We can then systematically extract a minimally connected graph T by identifying $k-1$ "aggregations" of tree collisions or overlaps. We then expect f_k^ε to decompose into a sum of $2^{k-1}k^{k-2}$ terms, where the factor k^{k-2} is the number of trees with k numbered vertices (from Cayley's formula). For each given signed minimally connected graph, the collision/overlap conditions correspond to $k-1$ independent constraints on the configuration $z_{1^\star}, \dots, z_{k^\star}$ at time t. Therefore, neglecting the issue of large velocities, this contribution to the cumulant f_k^ε has a Lebesgue measure of size $O((t/\mu_\varepsilon)^{k-1})$, and we derive the estimate

$$\|f_k^\varepsilon\|_{L^1} \leq \mu_\varepsilon^{k-1} C^k \times 2^{k-1} k^{k-2} \times (t/\mu_\varepsilon)^{k-1} \leq k!\, C(Ct)^{k-1}. \quad (3.2)$$

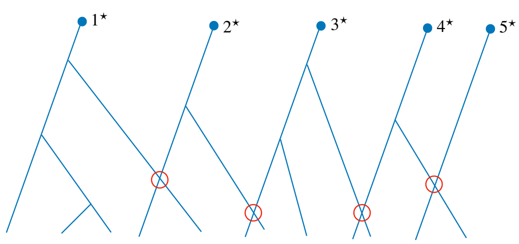
$f_5^\varepsilon = \mu_\varepsilon^4 \times$ 

Figure 8. The cumulant of order k corresponds to trees with roots in $1^\star, \dots, k^\star$ that are completely connected by external collisions or overlaps.

A geometric argument similar to the one developed in Lanford's proof and recalled in the analysis of the second-order cumulant above shows that f_k^ε converges to a limiting cumulant f_k and that only graphs with exactly $k-1$ external collisions or overlaps (and no cycles) contribute in the limit.

Note further that a classical and rather simple calculation (based on the series expansions of the exponential and logarithm) shows that the cumulants are nothing but the coefficients of the series expansion of the exponential moment:

$$\mathcal{I}_t^\varepsilon(h) := \frac{1}{\mu_\varepsilon} \log \mathbb{E}_\varepsilon[\exp(\mu_\varepsilon \langle \pi_t^\varepsilon, h \rangle)]$$
$$= \sum_{k=1}^{\infty} \frac{1}{k!} \int f_k^\varepsilon(t, Z_k) \prod_{i=1}^{k} (e^{h(z_i)} - 1)\, dZ_k. \quad (3.3)$$

The quantity $\mathcal{I}_t^\varepsilon(h)$ is called the *cumulant generating function*. Estimate (3.2) provides the analyticity of $\mathcal{I}_t^\varepsilon(h)$ in short time as a function of e^h, and this uniformly with respect to ε (sufficiently small). The limit $\mathcal{I}_t$ of $\mathcal{I}_t^\varepsilon$ can then be determined as a series in terms of the limiting cumulants f_k,

$$\mathcal{I}_t(h) = \sum_{k=1}^{\infty} \frac{1}{k!} \int f_k(t, Z_k) \prod_{i=1}^{k} (e^{h(z_i)} - 1)\, dZ_k.$$

In a suitable functional setting [5], it can be shown that this functional satisfies a Hamilton–Jacobi equation

$$\partial_t \mathcal{I}_t(h) = \int dz \frac{\partial \mathcal{I}_t(h)}{\partial h} v \cdot \nabla_x h + \mathcal{H}\Big(\frac{\partial \mathcal{I}_t(h)}{\partial h}, h\Big)$$

with initial condition $\mathcal{I}(0, h) = \int dz f^0 (e^h - 1)$ and Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H}(\varphi, h) := \frac{1}{2} \int \varphi(z_1)\varphi(z_2)(e^{\Delta h} - 1)\, d\mu(z_1, z_2, \omega), \quad (3.4)$$

where $\Delta h(z_1, z_2, \omega) = h(z_1') + h(z_2') - h(z_1) - h(z_2)$. We use here notation (2.2) for the pre-collisional velocities and the definition

$$d\mu(z_1, z_2, \omega) := \delta_{x_1 - x_2}((v_1 - v_2) \cdot \omega)_+\, d\omega\, dv_1\, dv_2\, dx_1.$$

The successive derivatives of this functional being precisely the limit cumulants f_k, the successive derivatives of the Hamilton–Jacobi equation provide the evolution equations of these cumulants: for example, differentiating this equation once produces the Boltzmann equation, differentiating it twice produces the equation of the covariance described in the next paragraph.

3.3 Fluctuations

The control of the cumulant generating function allows in particular to obtain the convergence of the fluctuation field defined in (1.4) and thus to analyze the dynamical fluctuations over a time $T^\star$ of the same order of magnitude as the convergence time T_L of Theorem 2.1.

Theorem 3.2 (Bodineau, Gallagher, Saint-Raymond, Simonella [5]). *The fluctuation field ζ_t^ε converges, in the low density limit and on a time interval $[0, T^\star]$, towards a process ζ_t, solution to the fluctuating Boltzmann equation*

$$\begin{cases} d\zeta_t = \underbrace{\mathcal{L}_t \zeta_t \, dt}_{\text{linearized Boltzmann operator}} + \underbrace{d\eta_t}_{\text{Gaussian noise}}, \\ \mathcal{L}_t h = \underbrace{-v \cdot \nabla_x h}_{\text{transport}} + \underbrace{C(f_t, h) + C(h, f_t)}_{\text{linearized collision operator}}, \end{cases} \quad (3.5)$$

where f_t is the solution at time t to the Boltzmann equation (2.1) *with initial data f^0, and $d\eta_t$ is a centered Gaussian noise delta-correlated in t, x with covariance*

$$\mathrm{Cov}_t(h_1, h_2) = \frac{1}{2} \int dz_1 \, dz_2 \, d\omega ((v_2 - v_1) \cdot \omega)_+ \delta_{x_2 - x_1} \\ f(t, z_1) f(t, z_2) \Delta h_1 \Delta h_2 (z_1, z_2, \omega),$$

where $\Delta h(z_1, z_2, \omega) = h(z_1') + h(z_2') - h(z_1) - h(z_2)$.

The limiting process (3.5) was conjectured by Spohn in [25], and this reference also presents a large panorama on the theory of fluctuations in physics. In the context of dynamics with random collisions, a similar result is shown by Rezakhanlou in [24]. In the deterministic setting, the noise obtained in the limit is a consequence of the asymptotically unstable structure of the microscopic dynamics (Figure 2) combined with the randomness of the initial data at small scales.

3.4 Large deviations

The strength of the cumulant generating function becomes really apparent at the level of large deviations, i.e., for very improbable trajectories that are at a "distance" $O(1)$ from the averaged dynamics: roughly speaking, we can show that the probability of observing an empirical distribution close to the density $\varphi(t, x, v)$ during the time interval $[0, T]$ decays exponentially fast with a rate quantified by a functional $\mathcal{F}_{[0,T]}$ which evaluates the cost of this deviation in the low density asymptotics

$$\mathbb{P}_\varepsilon(\pi_t^\varepsilon \simeq \varphi_t, \forall t \leq T) \sim \exp\big(-\mu_\varepsilon \mathcal{F}_{[0,T]}(\varphi)\big). \quad (3.6)$$

The proximity between π^ε and φ is measured in the weak topology on the Skorokhod space of measure-valued functions. A precise formulation of (3.6) and a proof can be found in [6]. The result of [6] can be summarized as follows: for a class of functions φ in a neighborhood of the solution to the Boltzmann equation, there exists a time interval $[0, T^\star]$ where the asymptotic (3.6) is characterized by a functional $\mathcal{F}_{[0,T^\star]}$ obtained by a certain Legendre transform of the Hamiltonian $\mathcal{H}$ defined by (3.4). This functional is identical to the one conjectured in [9, 24], by analogy with stochastic collision models of Kac type [2, 15, 18, 23]. Let us also note that the limiting SPDE (3.5) could be predicted by the same analogy with Kac's model for which collisions are modeled by a Markov process [19, 20]. Thus, the statistical analysis of the fluctuations and large deviations of the empirical measure confirms the robustness of Boltzmann's intuition (cf. Section 2.1): even on exponentially small scales, the behavior of the empirical measure of a hard sphere gas is identical to that of a model of particles with random collisions depending only on the local density. This does not contradict the Hamiltonian structure of the microscopic dynamics. Memory effects persist, but they are encoded in ways that are "transverse" to the empirical measure (or at different spatial scales).

4 Conclusion

Over a short time, Lanford's theorem states the convergence of the empirical measure of a hard sphere gas to the solution to the Boltzmann equation (Theorem 2.1). This result is completed by the analysis of fluctuations (Theorem 3.2) and large deviations (Section 3.4) of the empirical measure. These stochastic corrections are proved on times of the same order of magnitude as Lanford's theorem.

The strategy of the proof consists in tracking how the randomness of the initial measure is transported by the dynamics of hard spheres and how the instability of this dynamics transfers, in the low density asymptotics, the initial randomness into a dynamical white noise (space/time). The convergence time is limited because the current proof gives only rough estimates of the dynamical correlations, obtained by considering that collisions only destroy the initial chaos by forming larger and larger aggregates of correlated particles. An important step to progress in the mathematical understanding of these models would be to show that the disorder is not simply the result of the initial data, but that it can be regenerated by the mixing properties of the dynamics.

A more favorable framework for controlling long time evolution is to consider an initial measure obtained as a perturbation of an equilibrium measure. The stationarity of the equilibrium measure then becomes a key tool to control dynamical correlations. The simplest case consists in perturbing only one particle, which shall be called the tagged particle, and to study its evolution over time. In [3], it is established that this particle follows a Brownian motion for large times. Another case where we know how to use the invariant measure is the study of the fluctuation field at equilibrium. In a series of recent works [5, 7], Theorem 3.2 has been generalized to arbitrarily large, and even slightly divergent, kinetic times. This allows in particular to derive the fluctuating hydrodynamic Stokes–Fourier equations.

Acknowledgements. The *EMS Magazine* thanks *La Gazette des Mathématiciens* for authorization to republish this text, which is an English translation of the paper entitled "Sur la dynamique des gaz

dilués" and published in [*La Gazette des Mathématiciens*, Number G174, October 2022]. The main part of the text is extracted from an article published in the ICM 2022 proceedings.

The authors warmly thank Stéphane Baseilhac for his attentive proofreading and his numerous suggestions. They are also grateful to J.-B. Bru and M. Gellrich Pedra for the English translation of the original paper.

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Stable homotopy groups

Guozhen Wang and Zhouli Xu

1 Introduction

Homotopy theory studies homotopy invariants of topological spaces, i.e., invariants that are stable under continuous deformations. The fundamental problem is to understand the classification of continuous maps between spaces under homotopy.

In most situations, the spaces of interest are cellular, i.e., the spaces built from spheres in various dimensions. In this sense, spheres are the basic building blocks of spaces, and we would like to understand homotopy classes of maps from spheres to general spaces. By taking concatenation of maps, the homotopy classes of based maps from the n-sphere S^n to a space X form a group for $n \geq 1$, which is called the n-th homotopy group of X. When $n \geq 2$, there are different ways to concatenate maps and the resulting homotopy groups are commutative.

When X is a simply connected finite CW complex, Serre [41] proved that all homotopy groups of X are finitely generated abelian groups. So we can localize at a fixed prime p when studying these groups, and once we understand the p-local parts for all p, the structures of the original groups can be recovered.

In this article, we give a survey of the stable part of the homotopy groups of spheres. We will first recall the notion of stable homotopy, and then discuss an interpretation in terms of the framed cobordism and an application to the classification of exotic spheres. In the last part we discuss some methods for computing these stable homotopy groups.

2 Stabilization of homotopy groups

One basic operation in homotopy theory is the suspension. For a pointed space X, its (reduced) suspension ΣX is defined to be the smash product of X with S^1, i.e., the quotient space $X \times S^1 / X \vee S^1$. Roughly speaking, the effect of the suspension operation is to increase the dimension of all cells (other than the based point) of X by one. For example, $\Sigma S^n \cong S^{n+1}$. The suspension operation is functorial, so we have a suspension homomorphism $\pi_n(X) \to \pi_{n+1}(\Sigma X)$. The celebrated Freudenthal suspension theorem says that it is an isomorphism when X is sufficiently connected:

Theorem 1 (Freudenthal [16]). *If X is n-connected, then the suspension homomorphism $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k \leq 2n$.*

In particular, the groups $\pi_{n+k}(\Sigma^n X)$ depend only on k when n is sufficiently large, and we define this group to be the k-th stable homotopy group of X, denoted by $\pi_k^s(X)$. In contrast to the unstable homotopy groups, the stable homotopy groups form a generalized homology theory. This fact makes stable computations much simpler than those in the unstable cases.

The stabilization process can be categorified. We can define the (infinity) category of finite spectra by formally inverting the suspension functor on the category of finite CW complexes. The category of spectra is then defined as the ind-category of finite spectra. (See Lurie [27, Section 9] for details.) From the definition it follows that, for any space X, there is an associated suspension spectrum $\Sigma^\infty X$. The stable homotopy group $\pi_k^s(X)$ is the group of homotopy classes of maps from $\Sigma^\infty S^k$ to $\Sigma^\infty X$ in the category of spectra.

The computation of the stable homotopy groups of the sphere spectrum $\Sigma^\infty S^0$ has a long history. It is easy to see that the group $\pi_n^s(S^0)$ is trivial for $n < 0$, and $\pi_0^s(S^0) \cong \mathbb{Z}$ by the Hopf degree theorem. Using geometric methods, works of Hopf [19], Freudenthal [16], Whitehead [57], Pontryagin [36] and Rokhlin [39] determined $\pi_n^s(S^0)$ for $n \leq 3$. Serre started the study of homotopy groups using algebraic machinery. In [40] Serre computed the homology of iterated loop spaces using the Serre spectral sequence and determined $\pi_n^s(S^0)$ for $n < 9$. Toda [48] introduced the method of secondary compositions, the Toda brackets. By studying the EHP sequence with the composition method, Toda determined $\pi_n^s(S^0)$ for $n \leq 19$.

The introduction of the stable homotopy category by Spanier–Whitehead [44] and Boardman [9] brought to light the analogy between homotopy theory and homological algebra. Adams [1] introduced the Adams spectral sequence, which can be thought of as the descent spectral sequence using the Eilenberg–MacLane spectrum as a cover for the sphere spectrum. Other covers, such as using the complex cobordism spectrum, give a more general Adams–Novikov spectral sequence. May [29], Barratt–Mahowald–Tangora [7], Bruner [12], Nakamura [34], Tangora [47], Aubry [4] and Ravenel

[38] studied the Adams (and the Adams–Novikov) spectral sequence using techniques such as the May spectral sequence, the Massey product, Toda brackets, power operations, and the chromatic spectral sequence, etc., and determined $\pi_n^s(S^0)$ up to $n = 45$ at the prime 2, up to $n = 108$ for $p = 3$, and up to $n = 999$ for $p = 5$. See also [54] for a survey of classical methods. Recently, Isaksen [21] and Isaksen–Wang–Xu [22, 23] made progress by using motivic methods, extending the knowledge of the $p = 2$ component of $\pi_n^s(S^0)$ up to $n = 90$.

3 Framed cobordism

The Pontryagin–Thom construction gives a geometric interpretation of the homotopy groups of spheres.

Suppose we have a smooth map $f : S^{n+k} \to S^n$. Take a generic point $x_0 \in S^n$. Then the pre-image $f^{-1}(x_0)$ is a k-dimensional submanifold of S^n. Moreover, the normal bundle of $f^{-1}(x_0)$ is the pull-back of the normal bundle of x_0 in S^n, so it is a trivial bundle and has a preferred trivialization. Pontryagin showed that $\pi_{n+k}(S^n)$ is isomorphic to the group of cobordism classes consisting of k-dimensional submanifolds of S^{n+k} equipped with a framing on the normal bundle.

The Pontryagin–Thom construction can be stabilized. The special case of the Freudenthal suspension theorem for spheres can be deduced from the Whitney embedding theorem. Once the background space is of sufficiently large dimension, the cobordism classification of k-manifolds becomes independent of the embedding. In particular, we have:

Theorem 2 (Pontryagin [36]). *The stable homotopy groups of spheres classify the cobordism classes of manifolds equipped with framings of their stable normal bundles.*

For simplicity, manifolds with framings of their stable normal bundles will be referred to as framed manifolds (to be distinguished from manifolds with framings on their tangent bundles).

Using Pontryagin's theorem, one can see immediately that $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$. Using the knowledge of $\pi_1(SO(n))$ and the classification of 1-manifolds, one can show that $\pi_3(S^2) \cong \mathbb{Z}$ and $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for $n \geq 3$, generated by the Hopf map (i.e., the attaching map in $\mathbb{C}P^2$) and its suspensions.

The geometric computation of the second stable homotopy group of spheres is more subtle. One has to take care of the framings on the normal bundle of surfaces. Given such a surface and an essential loop on it, the obstruction to filling the loop and extending the framing is an element in $\pi_1(SO) \cong \mathbb{Z}/2$ (where $SO = \mathrm{colim}_n\, SO(n)$). It turns out this obstruction is quadratic in the mod 2 homology class of the loop, and the obstruction for the framed surface to be a boundary is the Arf invariant of this quadratic form. It follows that $\pi_2^s(S^0) \cong \mathbb{Z}/2$. See [36] for detailed arguments.

There is a special class of framed manifolds, consisting of those whose underlying manifolds are the standard spheres. Since all framings on the sphere S^k can be classified by the group $\pi_k(SO)$, we have the J-homomorphism

$$J : \pi_k(SO) \to \pi_k^s(S^0)$$

introduced by Whitehead [56]. The image of J was computed by Adams [3] in terms of the Adams conjecture, which was later proved by Quillen [37] and Sullivan [46]:

Theorem 3. *The image of the J-homomorphism is a direct summand of $\pi_n^s(S^0)$, and is cyclic for all n.*

- *If* $n \equiv 0$ *or* 1 mod 8*, the image of J has order* 2*.*
- *If* $n = 4k - 1$*, the order of the image of J is the denominator of* $B_{2k}/(4k)$*, where* B_{2k} *is the Bernoulli number.*
- *In all other cases the image of J is trivial.*

Recall that the Bernoulli number is defined by the generating function

$$\frac{x}{e^x - 1} = \sum \frac{B_k x^k}{k!}.$$

By the von Staudt–Clausen theorem, it follows that the order of $\pi_k^s(S^0)$ is unbounded as k increases. The following is a list of some Bernoulli numbers:

k	2	4	6	8	10	12	14	16	18
B_k	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$

4 Exotic spheres

The classification of manifolds with the homotopy type of the sphere is a long-standing problem in topology, starting with Poincaré's famous conjecture on simply connected 3-manifolds. By works of Smale [42], Freedman [15] and Perelman [35], all homotopy spheres are homeomorphic to the standard sphere. For the smooth classification, in dimension 2 and 3, any manifold has a unique smooth structure, according to work by Moise [31]. In dimension 4, it is still unknown if there exist exotic 4-spheres. In dimension ≥ 5, we can classify exotic spheres by Kervaire–Milnor theory in terms of stable homotopy groups of spheres.

For $n \geq 5$, we let Θ_n be the set of smooth structures on the spheres. (By the h-cobordism theorem of Smale [43], this is the same as the classification of h-cobordism classes of homotopy spheres.) It forms an abelian group under connected sum. Kervaire–Milnor [26] introduced a two-step strategy to study Θ_n. First we classify the homotopy spheres up to framed cobordism, and then classify the homotopy spheres that bound framed manifolds.

One can prove that all homotopy spheres admit stable framings, and the choices of the different framings are cosets by the image of J. So we get a homomorphism

$$\Theta_n \to \pi_n^s(S^0) / \operatorname{Im}(J).$$

One needs to understand the kernel and the cokernel of this map. We let Θ_n^{bp} denote the kernel, which consists of the homotopy spheres that bound framed manifolds.

The study of the cokernel amounts to the following:

Question 4. What is the obstruction for a framed cobordism class to have a homotopy sphere as a representative?

This question can be studied with the surgery theory, introduced by Milnor [30]. Suppose X is an n-manifold. A surgery on X is to first remove from X an embedded $D^k \times S^{n-k}$, and then to fill its boundary $S^{k-1} \times S^{n-k}$ along the other direction with $S^{k-1} \times D^{n-k+1}$. When X is framed, one needs to pay additional care to extend the framing. The operation of surgery is exactly what happens to the level set of a Morse function when crossing a critical point. So preforming surgery does not change the cobordism class and in fact generates the equivalence relation of cobordism.

For a framed n-manifold, one can perform suitable surgeries to kill all homotopy groups below the middle dimension. By Poincaré duality, in odd dimensions we would end up with a homotopy sphere. For n even, the intersection form in the middle-dimensional cohomology enters the scene. If $n = 4k$, then the obstruction to killing the middle cohomology is the signature of the intersection form. Since our manifold has trivial stable normal bundle, by Hirzebruch's signature theorem, this obstruction vanishes and we end with a homotopy sphere. In the case when $n = 4k + 2$, similar to the situation in dimension 2, we can define a quadratic form on the modulo 2 cohomology, and the obstruction to getting a homotopy sphere via surgery is its Arf invariant. This is called the Kervaire invariant, originally introduced by Kervaire [25] to construct topological manifolds that admit no smooth structures. In summary, a framed cobordism class of dimension $n = 4k + 2$ contains a homotopy sphere if and only if its Kervaire invariant is trivial.

To understand the structure of Θ_n^{bp}, we start with a homotopy sphere which bounds a framed $(n+1)$-manifold X. Then again we try to do surgery on X to make it contractible. If this can be achieved, then by the h-cobordism theorem, the boundary will be the standard sphere when $n \geq 5$. As before we can kill homotopy classes below the middle dimension, and for $n + 1$ odd there are no obstructions, so $\Theta_n^{bp} = 0$. When $n + 1 = 4k$, the obstruction to killing the middle dimension is the signature of the intersection form, which can be any multiple of 8 using the plumbing construction. There is another operation we can perform, namely, taking the connected sum with a framed manifold whose boundary is a standard sphere. The boundaries of these objects are classified by the kernel of the J-homomorphism. Using Theorem 3 and Hirzebruch's signature theorem, the effect of this operation is fully understood. Finally, if $n + 1 = 4k + 2$, then the obstruction for the middle-dimensional surgery is the Kervaire invariant, which can take any value in $\mathbb{Z}/2$. Again we can alter X by taking the connected sum with a closed framed manifold, so this obstruction either becomes trivial, or does not depend on the existence of a closed framed $(n+1)$-manifold of Kervaire invariant 1. In summary:

Theorem 5 (Kervaire and Milnor [26])**.** *Let $n \geq 5$.*

- *When $n \neq 2 \bmod 4$, there is an exact sequence*

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n/J \to 0.$$

- *When $n = 2 \bmod 4$, there is an exact sequence*

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n/J \xrightarrow{\Phi} \mathbb{Z}/2 \to \Theta_{n-1}^{bp} \to 0.$$

- *If n is even, then $\Theta_n^{bp} = 0$.*
- *If $n = 4k - 1$, then*

$$\Theta_n^{bp} \cong \mathbb{Z}/2^{2k-2}(2^{2k-1} - 1)c_k,$$

where c_k is the numerator of $4B_{2k}/k$.

Here Φ is the Kervaire invariant and B_{2k} is the Bernoulli number.

Together with the knowledge of the stable homotopy groups of spheres, we can partially answer the question: In which dimensions does the sphere have a unique smooth structure? Based on Serre's computations [40] and Toda's computations [48], Kervaire and Milnor found that S^5, S^6, S^{12} have a unique smooth structure. Isaksen's computation [21] implies that S^{56} also has a unique smooth structure. The last sphere we know of that has a unique smooth structure is S^{61}, by work of Wang–Xu [55]. This solves the problem in all odd dimensions.

Theorem 6. *S^1, S^3, S^5 and S^{61} are the only odd-dimensional spheres with a unique smooth structure.*

In even dimensions, by Behrens–Hill–Hopkins–Mahowald [8], the only spheres below dimension 140 which have unique smooth structures are S^2, S^6, S^{12}, S^{56} and perhaps S^4. Based on the above results, we have following conjecture:

Conjecture 7. *If S^n has a unique smooth structure, then either $n \leq 6$, or $n = 12, 56, 61$.*

5 The Adams spectral sequence

A basic homotopy invariant is cohomology. Maps inducing nontrivial homomorphisms on cohomology are not homotopic to constant maps. To get finer invariants, we consider cohomology operations.

Cohomology operations are natural transformations of cohomology theories. To understand stable homotopy, we usually consider stable cohomology operations, i.e., the ones commuting with the suspension. The Bockstein homomorphism is such a stable operation. More generally, the Steenrod reduced power operation (see Steenrod–Epstein [45]), which arises from the Spanier–Whitehead dual of the diagonal map, turns out to be stable. Since (ordinary) cohomology theories are represented by the Eilenberg–MacLane spectra, the stable cohomology operations can be classified by the cohomology of these objects, which was computed by Cartan [13].

Theorem 8. *The stable cohomology operations on mod p cohomology form a graded associative algebra $\mathcal{A}^*$ generated by the Steenrod squares Sq^i for $p = 2$, and by the Steenrod reduced powers P^i and the Bockstein β for p odd. They satisfy the Adem relations, which for $p = 2$ are*

$$Sq^i \circ Sq^j = \sum_{0 \le k \le \frac{n}{2}} \binom{j-k-1}{i-2k} Sq^{i+j-k} \circ Sq^k$$

when $0 < i < 2j$.

The algebra $\mathcal{A}^*$ is called the Steenrod algebra.

One can use these cohomology operations to detect non-trivial maps that induce trivial homomorphisms in cohomology. For example, consider the Hopf map $\eta : S^3 \to S^2$ and its mapping cone, i.e., the complex projective plane $\mathbb{C}P^2$. The Steenrod square Sq^2 acts non-trivially on the mod 2 cohomology of $\mathbb{C}P^2$, and consequently η represents a non-trivial stable class in $\pi_1^s(S^0)$.

In general, if there is a map $f : X \to Y$ that induces the trivial map on mod p cohomology, then we have a short exact sequence

$$0 \to H^{*-1}(X) \to H^*(Cf) \to H^*(Y) \to 0.$$

Here Cf is the mapping cone of f and we abbreviate $H^*(\cdot;\mathbb{F}_p)$ by $H^*(\cdot)$. These cohomology operations act on every term, so this is a short exact sequence of $\mathcal{A}^*$-modules, and therefore it corresponds to an element in $\mathrm{Ext}^1_{\mathcal{A}^*}(H^*(Y), H^{*-1}(X))$.

More generally, suppose a map $f : X \to Y$ can be written as the composition of a sequence of maps

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n = Y \qquad (1)$$

such that each f_i induces a trivial map on mod p cohomology. Then we have an element in $\mathrm{Ext}^1_{\mathcal{A}^*}(H^*(X_{i+1}), H^{*-1}(X_i))$ for each i, and composing them together gives an element in $\mathrm{Ext}^n_{\mathcal{A}^*}(H^*(Y), H^{*-n}(X))$. In contrast to the $n = 1$ case, the decomposition of f is not necessarily unique, and in general different decompositions yield different classes in the group $\mathrm{Ext}^n_{\mathcal{A}^*}(H^*(Y), H^{*-n}(X))$. However, we will see below that we do get an invariant by taking the cosets by certain subgroups of $\mathrm{Ext}^n_{\mathcal{A}^*}(H^*(Y), H^{*-n}(X))$ (which are hit by some Adams differentials).

The method of the Adams spectral sequence introduced by Adams [1] is in some sense taking the universal example of the above decomposition. For a spectrum Y, an Adams tower is a sequence of maps $\cdots \to Y_2 \to Y_1 \to Y$ such that each map induces a trivial homomorphism in mod p cohomology, with its cofiber being homotopy equivalent to a wedge sum of (suspensions of) Eilenberg–MacLane spectra. The spectral sequence associated to an Adams tower is called the Adams spectral sequence. Adams towers always exist, and different towers for the same Y always induce the same spectral sequence from the E_2-page. Moreover, the Adams E_2-page is the Ext groups over the Steenrod algebra.

Theorem 9 (Adams [1]). *Suppose X and Y are finite spectra. Then we have the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{A}^*}(H^*(Y), H^{*-t}(X)) \Rightarrow [\Sigma^{t-s}X, Y]_p^\wedge$$

which converges to the p-completion of homotopy classes of maps from X to Y.

For general spectra that are not necessarily finite, we still have the Adams spectral sequence, but the convergence issue is more subtle. See Bousfield [10] for details.

We say an element $f \in [\Sigma^* X, Y]$ has Adams filtration $\ge n$ if it factors through Y_n in an Adams tower. Then this Adams tower gives a decomposition of f in the form (1). The corresponding element in $\mathrm{Ext}^n_{\mathcal{A}^*}(H^*(Y), H^*(X))$ is the element detecting f in the E_∞-page of the Adams spectral sequence. (Here we adopt the convention that $0 \in \mathrm{Ext}^n_{\mathcal{A}^*}(H^*(Y), H^*(X))$ "detects" elements with Adams filtration $\ge n+1$.)

If X and Y are both the sphere spectrum, then the composition induces a ring structure on $\pi_*^s(S^0)$, which is commutative (in the graded sense). In this case, the Adams spectral sequence is multiplicative. The multiplication on the E_2-page is the Yoneda product on Ext groups, which also turns out to be commutative (in the graded sense).

Let us give some examples of elements in $\pi_*^s(S^0)$ with low Adams filtrations.

The identity map is essentially the only class with Adams filtration 0. For Adams filtration 1, note that $\mathrm{Ext}^1_{\mathcal{A}^*}(\mathbb{F}_p, \mathbb{F}_p)$ is generated by the indecomposable elements in $\mathcal{A}^*$, which turns out to be the vector space with basis $\{Sq^{2^i}$ for $i = 0, 1, \ldots\}$ at the prime 2. We denote by h_i the class that corresponds to Sq^{2^i}.

The multiplication by 2 map is detected by h_0 in the Adams E_2-page. The previous example regarding the Hopf map tells us that η is detected by h_1. Furthermore, the attaching maps in projective planes over the quaternions and octonions give us elements $\nu \in \pi_3^s(S^0)$ and $\sigma \in \pi_7^s(S^0)$ that are detected by h_2 and h_3, respectively. We cannot produce more examples along this way because there are no more division algebras over the real numbers. In fact, Adams [2] proved that all the h_i's for $i \ge 4$ do not survive in the Adams spectral sequence, and consequently there are no more Hopf invariant one classes. As a consequence, S^1, S^3, S^7 are the only spheres that have a trivial tangent bundle.

By computations of Adams [2], $\mathrm{Ext}^2_{\mathcal{A}^*}(\mathbb{F}_2, \mathbb{F}_2)$ is spanned by elements of the form $h_i h_j$ under the relations $h_i h_{i+1} = 0$. Among these classes are the Kervaire classes h_i^2. By Browder's theorem [11], the Kervaire invariant for framed manifolds is trivial in dimensions other than $2^n - 2$, and the existence of framed $(2^n - 2)$-manifold with Kervaire invariant one is equivalent to the statement that h_{n-1}^2 survives in the Adams spectral sequence.

By the existence of η, ν and σ, we deduce that there exist Kervaire-invariant one manifolds in dimensions 2, 6 and 14. In fact, one can take manifolds $S^1 \times S^1$, $S^3 \times S^3$ and $S^7 \times S^7$ with suitable framings. Mahowald–Tangora [28] and Barratt–Jones–Mahowald [6] proved (see also Xu [58]) that the elements h_4^2 and h_5^2 survive in the Adams spectral sequence. Using equivariant methods, Hill–Hopkins–Ravenel [18] proved that for $n \geq 7$, h_n^2 all support non-trivial differentials, and consequently the last dimension where there could exist a Kervaire-invariant one manifold is 126.

6 Motivic homotopy theory

In general, it is hard to determine differentials and hidden extensions in the Adams spectral sequence. Various techniques are used, but none of them can solve all the problems. This phenomenon is described as the Mahowald uncertainty principle; see [24] for more details. Nevertheless, one of the most recent technique involves motivic homotopy theory and it turns out to be very effective.

The original motivation for developing motivic homotopy theory in Morel [32] and Morel–Voevodsky [33] is to construct a homotopy theory in the world of algebraic varieties. From the perspective of topologists whose main focus is classical homotopy theory, motivic homotopy theory is obtained by adding new objects in the world of topological spaces.

In classical homotopy theory, simplices are the basic building blocks. The classical homotopy category is equivalent to the category of simplicial sets, i.e., presheaves over simplices. In general, a category of presheaves can be viewed as the category freely generated from certain building blocks. The motivic category is constructed by first formally adding smooth varieties along with simplices as basic building blocks. In contrast to simplices, smooth varieties are not "independent", in the sense that two varieties can be glued together to form a new one. To incorporate these relations, we consider simplicial sheaves (under certain Grothendieck topology, the most fruitful one being the Nisnevich topology) over the category of smooth varieties, instead of just presheaves. Finally, we invert $\mathbb{A}^1$-homotopy equivalences to get the motivic homotopy category. See Morel–Voevodsky [33] for details of this construction.

An interesting fact in the motivic world is that there are two kinds of spheres, the simplicial sphere $S^{1,0}$ and the multiplicative group $S^{1,1} = \mathbb{G}_m$ (the sheaf represented by the punctured affine line). Taking the smash product of these objects, we obtain motivic spheres $S^{i,j}$, where the first index i indicates the dimension and the second index j is the motivic weight. They are analogs of representation spheres in equivariant homotopy theory. To construct the stable motivic homotopy category, we mimic the construction in the equivariant setting, inverting suspensions with respect to both kinds of spheres. Analogously, we can define the notion of stable motivic homotopy groups, and as a result there are two gradings.

Now suppose we work with the base field $\mathbb{C}$. These two kinds of spheres are related by an element τ constructed as follows. At a prime p, for any n, we take a p^n-th root of unity, which induces a map $S^{0,0} \to S^{1,1}$, representing an element in $\pi^s_{0,0}(S^{1,1})$ of order p^n. So it is the image of some element $\tau_n \in \pi^s_{1,0}(S^{1,1}; \mathbb{Z}/p^n)$ under the Bockstein homomorphism. When we take a compatible system of p^n-th roots of unity for all n, then the resulting τ_n's are compatible, and we define τ to be the limit of τ_n in $\pi^s_{1,0}(S^{1,1}; \mathbb{Z}_p)$, which can be viewed as a self-map of the p-completed sphere of degree $(0, -1)$. Intuitively, τ can be regarded as the Bockstein pre-image of the infinitesimal generator of the multiplicative group. See Hu–Kriz–Ormsby [20] for more details.

By works of Voevodsky [49–52], as in the classical case, we can define motivic cohomology, motivic Steenrod algebra and motivic Adams spectral sequence. Over $\mathbb{C}$, the coefficient ring of mod p motivic cohomology is a polynomial ring $\mathbb{F}_p[\tau]$ generated by τ. The motivic Steenrod algebra $\mathcal{A}^{*,*}_{\mathrm{mot}}$ is generated by a motivic analog of the Steenrod reduced powers, satisfying a motivic analog of the Adem relations.

There is a Betti realization functor from the motivic homotopy category to the classical homotopy category, induced by the functor sending a complex analytic variety over $\mathbb{C}$ to its underlying topological space. Under the Betti realization, the map τ becomes an equivalence and the two kinds of motivic spheres become classically equivalent. Moreover, Dugger–Isaksen [14] proved that the τ-inverted motivic Adams spectral sequence for the motivic sphere recovers the classical Adams spectral sequence for the classical sphere. So intuitively we find that (after p-completion) the classical homotopy theory is the τ-inverted motivic homotopy theory.

From a computational perspective, we can view τ as a deformation parameter of the motivic deformation. The generic fiber is the world of classical homotopy theory. Gheorghe–Wang–Xu [17] discovered that the special fiber lands in the algebraic world:

Theorem 10 (Gheorghe, Wang and Xu [17]). *Let $S^{0,0}/\tau$ be the cofiber of τ. The category of cellular $S^{0,0}/\tau$-modules in the stable motivic homotopy category over $\mathbb{C}$ is equivalent to the derived category of BP_*BP-comodules as stable ∞-categories.*

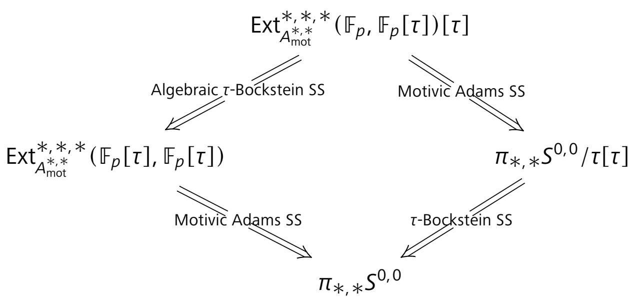
The latter algebraic category can be further identified with the derived category of quasi-coherent sheaves over the moduli stack of p-completed formal groups.

In particular, the Adams spectral sequence in the category of $S^{0,0}/\tau$-modules is also algebraic in nature. In fact, we have the following:

Theorem 11 (Gheorghe, Wang and Xu [17]). *The motivic Adams spectral sequence for $S^{0,0}/\tau$ is isomorphic to the algebraic Novikov spectral sequence.*

Recall that the algebraic Novikov spectral sequence computes the Ext groups of BP_*BP-comodules using the filtration by powers of the augmentation ideal of BP_*. The structure of the algebraic Novikov spectral sequence can be determined effectively with a computer using a minimal resolution. See Wang [53] for an algorithm of this computation.

So in principle we can get information on the special fiber of the motivic deformation as far as we wish. To get information on the classical homotopy theory, we try to propagate the information from the special fiber to the generic fiber of this motivic deformation. In practice, we use the τ-Bockstein spectral sequence. We have a square of four spectral sequences:



One notes that the algebraic τ-Bockstein spectral sequence is equivalent to the motivic analog of the classical Cartan–Eilenberg spectral sequence, and the τ-Bockstein spectral sequence is equivalent to the motivic analog of the classical Adams–Novikov spectral sequence. Hence, our theorem links these classical objects through motivic theory and we are able to compare data obtained from different classical perspectives.

Remark 12. In Bachmann–Kong–Wang–Xu [5], the above motivic square over $\mathbb{C}$ is extended to one over a general base field. In general, we replace the τ-adic tower by the Whitehead–Postnikov tower with respect to the Chow t-structure. Consequently, the motivic Adams spectral sequences of these layers are different, but they are still algebraic.

As an illustration of the method, we compute the first few Adams differentials in stem 15. By Theorem 11 and computer output, there is a motivic Adams differential $d_2(h_4) = h_0h_3^2$ for $S^{0,0}/\tau$. By comparison using the map $S^{0,0} \to S^{0,0}/\tau$, we find that in the motivic Adams spectral sequence for $S^{0,0}$, h_4 must support a non-zero differential of length at most 2. The only possibility is that $d_2(h_4) = h_0h_3^2$ also holds for $S^{0,0}$. By inverting τ, we arrive at the same differential for the classical Adams spectral sequence for the sphere. The differential $d_3(h_0h_4) = h_0d_0$ can be proved similarly. In fact, all non-zero differentials up to stem 45 can be computed in this way, with very few exceptions. See the appendix of Gheorghe–Wang–Xu [17] for more details.

Acknowledgements. The first author acknowledges the support of NSFC-12226002, Shanghai Rising-Star Program 20QA1401600, and Shanghai Pilot Program for Basic Research-FuDan University 21TQ1400100(21TQ002). The second author acknowledges the support of the NSF under grant DMS 2105462.

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Gene Calabi at 100 – Memorable encounters with Eugenio Calabi

Jean-Pierre Bourguignon and Balázs Szendrői

It is rare indeed to be able to celebrate the centenary of a living legend. The last time this happened in mathematics may have been the 100th birthday of Leopold Vietoris in 1991; the spring of 2023 brings up the centenary of Eugenio Calabi. Born in Milan on 11 May 1923, he has lived in the United States since an early age. He completed a PhD at Princeton University in 1950. Following temporary positions, he was appointed to a professorship at the University of Minnesota, and finally settled at the University of Pennsylvania in 1964, where he held for several decades the Thomas A. Scott Professorship of Mathematics. He is Commander of the Order of Merit of the Italian Republic. We wish to celebrate the occasion by collecting personal reminiscences of encounters with Eugenio Calabi.

Jerry Kazdan
University of Pennsylvania

I first met Gene in Fall 1962, when I was a grad student at the Courant Institute of Mathematics at New York University. A group of us, mainly students of Lipman Bers, often met on Fridays for lunch with Bers: "Children's Lunch." Gene was visiting New York and had dropped in to visit Bers. He was directed to the restaurant. He overflowed with mathematics, a pleasure to see.

We next met at the January 1966 Annual Meeting of the AMS. He immediately began telling me about Kähler manifolds and his conjecture concerning what are now called Calabi–Yau manifolds. This was a bit technical. I confess that I did not follow everything and did not appreciate the depth of the ideas he revealed.

That Fall I moved to the University of Pennsylvania. Gene had moved there in 1964 from the University of Minnesota. I had the pleasure of frequent personal interaction. His amazing geometric insight was a gift. He had a deep intuitive sense of what was important and interesting. He always shared his ideas generously. Often he came to my office and began explaining some of his recent thoughts at the blackboard (this sometimes bewildered undergraduates who might have happened to be there for my office hours).

It was refreshing to see Gene's original views on many things. When driving with him, those in the car were perplexed at the sometimes circuitous routes he chose. When asked, Gene said that he was minimizing the number of traffic lights. Someone suggested calling these "Calabi geodesics." The name fit.

Blaine Lawson
Stony Brook University

Eugenio Calabi has certainly been one of the most original geometers of the twentieth century. As a graduate student in the late 1960s, I was fortunate to witness him delivering an early address on one of his deep and beautiful results concerning minimal 2-spheres in the Euclidean n-sphere. That day is one I will never forget!

At that time, I thought I had a sufficient result for my thesis. I had mentioned it to several people on the faculty and had written it up. It said that a non-compact holomorphic curve of constant curvature in complex projective n-space $\mathbb{P}^n(\mathbb{C})$ (with its standard metric) had to have curvature $\frac{1}{k}$ for an integer $1 \leq k \leq n$, and would be locally congruent to a specific rational normal curve in the linear subspace $\mathbb{P}^k(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$. This generalized to spaces of all dimensions. Early on that day, my advisor, Bob Osserman, mentioned this to Calabi, who told him that, unfortunately, that result was part of his thesis. You can imagine my chagrin. Nevertheless, I went to his colloquium that afternoon, and for me it was one of the most beautiful and exciting lectures I had ever heard. I was so inspired, I threw away my old work and wrote my thesis on minimal surfaces in the sphere. Since that time, I have had many conversations with Gene. They have always been fascinating and actually wondrous (that is, the part from him to me). Over time, I realized how generous he has always been to young mathematicians.

Nigel Hitchin
University of Oxford

Thirty years ago, I spoke at a conference in Pisa for Eugenio Calabi's 70th birthday. It is remarkable that today we can celebrate his achievements over a much greater span of time.

My first encounter with him was when I was a postdoc at the Institute for Advanced Study in 1972. One of the first invitations to give a seminar in nearby universities was at Penn. I spoke about positive scalar curvature obstructions, and he was generous in his comments and advice. I was very much aware of bigger issues in differential geometry at the time as Shing-Tung Yau was also at the Institute and we frequently discussed whether the Calabi conjecture was true or false.

The most direct impact that Calabi had on my work was his paper [4], which gave constructions of complete Ricci-flat Kähler manifolds. I had learned of some 2-dimensional examples from Gary Gibbons and Stephen Hawking in Cambridge and also through the work of Penrose in Oxford, but this paper opened up a new world independent of any relativity connections. It is difficult to appreciate that only a few years earlier there were absolutely no examples or existence proofs for complete non-flat manifolds with zero Ricci tensor. The paper also introduced the word "hyperkähler" to describe the differential geometry which previously went under various names associated to the quaternions. It was an inspired choice emphasizing that one should view these manifolds as possessing many Kähler structures and not be led astray by trying to define quaternionic coordinates. It was a language which pointed the way to further discoveries and in particular our hyperkähler quotient construction.

It would be a long task to list the papers of Calabi which foresaw future developments, even those which have influenced my own work. I wish him well on this his 100th birthday.

Jean-Pierre Bourguignon
Institut des Hautes Études Scientifiques

The first time I met Eugenio Calabi was in 1972 in Oberwolfach during one of the biennial meetings *Differentialgeometrie im Großen* organised for many years by Martin Barner, Wilhelm Klingenberg and Chern Shiing Shen. Eugenio attended these sessions regularly; he had been there already in 1962 and 1969. That year, he gave Bures-sur-Yvette as address, so I may have met him already at IHÉS where I had started to attend the Thom seminar.

It is precisely in 1969 that, in a groundbreaking article [3], Eugenio discusses for the first time global properties of groups of diffeomorphisms preserving a symplectic structure. At that time, global symplectic geometry was still in its infancy. He introduces a very ingenious tool, often called the *Calabi morphism*. Very recently, the beautiful result establishing the non-simplicity of the group of homeomorphisms of the 2-sphere preserving the volume uses an extension of the Calabi morphism to the continuous context, showing the remarkable actuality and power of ideas Eugenio introduced more than 50 years ago.

Eugenio also visited Paris many times, lecturing at the Differential Geometry seminar led by Marcel Berger. And, early on in my life as a mathematician, Berger informed me of the importance of the Calabi conjecture, in particular in relation with the existence of metrics with vanishing Ricci curvature, an intriguing case connected to special holonomy groups. This is why, in 1972–1973, Yau Shing Tung and I, both of us assistant professors in Stony Brook, tried hard to find obstructions to the existence of such metrics on K3 surfaces. There was a good reason for us to fail since, as is now well known, Yau proved the Calabi conjecture in 1976, opening the way to the epoch-making 'Calabi–Yau' metrics. This led me to a number of exchanges with Eugenio as the various follow-ups of the solution of the Calabi conjecture occupied me for quite some time. Through these contacts, I could appreciate his kindness as well as his exceptional capacity to use all kinds of tools to address geometric questions, often creating very innovative approaches.

Getting involved in the publication of Eugenio's Complete Works [8] was therefore for me a natural gesture to pay back a little for all what I owe to him.

Dear Eugenio, happy 100th birthday and my best thanks for the inspiration and the support!

Eugenio Calabi with Shiing-Shen Chern in Oberwolfach in 1976
(© Dirk Ferus)

Simon Salamon
King's College London

If the exact dates of most of my encounters with Eugenio Calabi are vague, I can pinpoint with precision one day that I spent in his company. This was Monday 30 March 1981, when he had invited me to give a seminar at the University of Pennsylvania. I must have left Union Station in DC early in the morning, and President Ronald Reagan was to joke some hours later (quoting W. C. Fields) about rather being in Philadelphia. My talk was around lunchtime, and I recall being shocked by the setup: two blackboards on opposite walls with a long table at right angles stretching between them. The audience sat on both sides of the table, possibly eating sandwich lunches, whilst any speaker (without advanced practice) was forced to move to and fro between the boards. My talk went well enough, though I am not sure I succeeded in rousing Calabi, who had assumed his characteristic pose with eyes shut in the middle of the table (though Jim Eells had warned me not to underestimate his perceptive powers in the lecture room.) About the time I was speaking, Reagan was addressing union representatives (no doubt with eyes wide open) in the Washington Hilton. What happened next is well documented, also in movie form. As his team emerged from a side entrance of the hotel, John Hinckley Jr fired six .22 calibre pistol shots. The first was to disable press secretary James Brady for life, and others were injured. The last bullet ricocheted and (as was soon to become apparent) punctured the President's lung. News of the shooting probably reached us around 3pm and immediately put a stop to any mathematics. I recall a frantic search for a transistor radio, which was installed in Calabi's office and became a focal point for colleagues, many of whom will have recalled events in Dallas almost two decades before. But within a couple of hours, it was clear that Reagan was out of danger, though subsequent details revealed how lucky he had been. The rest of the day is a blur, but I do remember Calabi walking me to the station, probably anxious to resume serious scientific discussion. We waited together on the concourse, where I picked up a late edition of the Philadelphia Daily News with a stark 300 point headline "REAGAN SHOT; IS ALERT, STABLE" – it remains in my office as a reminder of Calabi, not Reagan. Despite the drama of that ultimately tragic day, my overriding recollection is of my host's kindness, which was to provide a model of how to treat a visiting speaker.

I am sure that the mathematics I learnt from Eugenio in Philadelphia and on other precious occasions (whether it related to special holonomy, minimal surfaces, twistor spaces, non-Kähler geometry, or Cayley numbers) had a significant influence on my subsequent work. I was especially proud to be present with Calabi at the first joint international meeting of the AMS and UMI in Pisa in 2002, at which Vestislav Apostolov and I conceived of our paper that helped develop closer links between Calabi–Yau spaces and metrics with holonomy G_2.

Claude LeBrun
Stony Brook University

I am delighted to have been recently assured by close mutual friends that Gene Calabi remains intellectually active, in good health, and in excellent spirits, even as he embarks on the second century of his remarkable life. Calabi's work has had an overwhelming impact on my own career, but a large contingent of the geometers of my generation would no doubt say much the same thing. However, I have also had the good fortune to have gotten to know Gene personally, to have had many delightful and informative conversations with him, and to have been able to become a member of his large circle of friends.

Although I first met Gene in Italy in 1980, when I was straight out of graduate school, it was not until half-a-dozen years later that I had a truly life-altering encounter with him. This happened during a brief visit to the University of California at San Diego, where I'd gone to run a few questions past Rick Schoen. Suddenly, Gene materialized in Rick's office, and immediately launched into an extensive lecture on extremal Kähler metrics. I soon realized that I would need to carefully read Gene's groundbreaking papers on the subject, because my own work on anti-self-dual 4-manifolds had recently led me into the realm of scalar-flat Kähler metrics, which represented one tiny piece of Gene's grand vision. Over the following decades, I had the pleasure of discussing related areas of differential geometry with Gene many times, sometimes at University of Pennsylvania, sometimes at Stony Brook, and sometimes at conferences held at other universities. His beneficent influence on my life has continued to exert itself in many ways – directly through ideas, of course, but also indirectly, through mathematicians whose research directions have been molded by Gene's mathematical taste.

I last saw Gene in person in 2019, a few months before the pandemic. The director of the Simons Center for Geometry and Physics proposed that a filmed interview with Calabi would be of great interest to both mathematicians and theoretical physicists, and I had the good fortune to then be asked to go conduct the interview in Philadelphia. The resulting documentary [7], entitled *Quintessentially Science Fiction*, aims to capture Gene's charm and intelligence. The title, incidentally, comes from a comment that Gene made during the interview regarding the nature of mathematics: we invent imaginary worlds, and scientists only decide long afterwards whether any of these places might actually make good homes for genuine scientific theories. That's vintage Gene. Simple, terse, and perhaps a little cryptic; but definitely worth pondering at length!

Gene, I've said it before, but, on this very happy occasion, let me say it again: Your visionary ideas have made our world a richer and more interesting place. Thank you for your ideas, thank you for your kindness, and, above all, thank you for your friendship!

Fabrizio Catanese
University of Bayreuth

I had a few friendly encounters with Eugenio Calabi, first when he was visiting the Scuola Normale Superiore of Pisa as a guest of the late Edoardo Vesentini, and, later on, as I had the honour to be invited as a speaker at the conferences held in Italy in honour of some of his birthdays. On these occasions, I had the opportunity of talking to him, and to get to know about his real and mathematical life through Italy and then in the USA.

He looked to me like an old-fashioned gentleman, a species of mathematician in danger of extinction. Yet, his eyes and quiet speech were sparkling of a deep intellectual life. I did not dare to ask him many mathematical questions, even if at a certain point I had been quite involved with some of his constructions. For me, Calabi was like a grandfather, since I regarded him as a teacher of Edoardo Vesentini, one of my teachers who introduced me to differential geometry back in 1970–71.

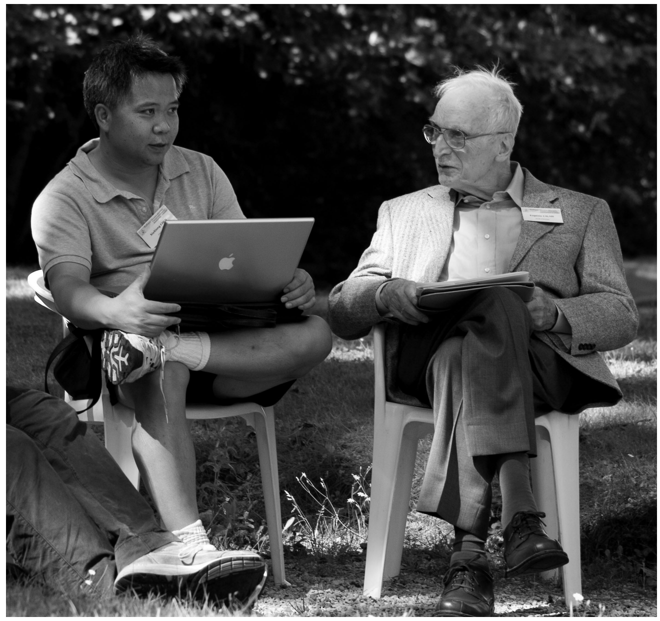
Making difficult things simple and finding elegant solutions has been the great talent of Eugenio Calabi. In the words of Vesentini: "Amidst intimidating theories and theorems which were tormenting me, came the simple explanations by Calabi: everything seemed just straightforward linear algebra, and easy calculations with matrices were yielding the desired curvature results." I later read their paper [10] myself, I loved it, and I fully agree with Vesentini's statement: explicit calculations are easily understood, and concrete mathematics will live longer than awe-generating abstract theories.

Wolfgang Ziller
University of Pennsylvania

Some of my fondest and most important experiences at Penn were my mathematical interactions with Gene Calabi. He would often come by at my office when arriving by train and would explain to me what he was thinking about in the shower that morning. It was always fascinating and I was able to ask him questions about what I was working on. His insight into what the core of a problem was, and his ability of coming up with relevant examples always amazed me.

Let me tell one of my favourite stories. In 1982, I was teaching a course on closed geodesics and asked Gene over tea about a conjecture of Poincaré, which states that for a metric on S^2 with positive curvature any shortest closed geodesic is simple. He thought about it for 5–10 minutes and told me the problem was very subtle. On a bi-equilateral triangle (which can be blown up to a convex surface) there are two closed geodesics of the same length, one simple and one with a self intersection. Thus the answer is no in the non-smooth limit. He was intrigued by the question. A year later he told me how to prove the positive answer in the smooth case by modifying the Birkhoff curve shortening process to 1-cycles, not just simple curves, a technique that foreshadowed developments that are much more recent. He only published the result 10 years later [9] together with a graduate student at the time, Jianguo Cao.

A second example is his discovery, more or less at the same time, of the grim reaper for the curve shortening flow of closed curves on the plane, again over tea at Penn (it is not well known that he discovered, and named, this example). It is the unique simple ancient solution of the flow, foreshadowing that this is a crucial property for the flow, only understood much later. His influence on me and many others was through mathematical conversations, with observations that often became crucial in later developments. His love for mathematics (but not for writing papers) was obvious to everyone. Happy Birthday Gene!



Eugenio Calabi with Xiuxiong Chen at IHÉS, 2007 (© Jean-François Dars)

Xiuxiong Chen
Stony Brook University

Shortly after I arrived at the University of Pennsylvania to pursue my PhD studies, I ran into Prof. Eugenio Calabi. Following a brief introduction, he started to explain something that he believed or hoped I would find interesting. Little did I know that this would become our routine for the ensuing years.

Prof. Calabi would pen his explanations or thoughts on whatever was available in hand or at hand, be it an envelope or a napkin, or a blackboard in a nearby classroom. We would talk hours and end often in his office, but also in the mail room and in the hallways. I would take home those envelopes and napkins (regrettably many of them got lost during our many moves), but most of the time I would jot down on my notepad what he wrote or said, or occasionally my own musings.

Prof. Calabi would ask me to repeat what he said or what I heard the next time we met without consulting any notes. As he explained, "it wouldn't become yours until it's imprinted in your memory". Though not an immediate embracer, it didn't take too long for me to appreciate that advice. Now I am a fervent adherent of the doctrine, and I have been passing it on to my own students.

Ludmil Katzarkov
University of Miami

I met Gene Calabi for the first time in September 1990. His class immediately impressed me: he introduced me to special metrics and multiplier ideal sheaves; later I wrote papers on these subjects. Gene also introduced me to Shing-Tung Yau.

I have had many discussions with Gene on European history – his knowledge of the subject was spectacular.

Gene was also my ride to Princeton. He regularly drove me from University of Pennsylvania to IAS to attend a course on harmonic maps to buildings by Richard Schoen. Later, I used the knowledge acquired in the proof of the Shafarevich conjecture. With the exception of his adventurous Italian driving style (the Honda Civic felt more like a Ferrari), these were memorable drives – really unforgettable scientific, intellectual and cultural experiences.

Happy 100th birthday Gene! Thank you very much for teaching me so many things, in particular that mathematics can be a subject for gentlemen.

Eugenio Calabi with Shing-Tung Yau at École Polytechnique in 2007 (© Jean-François Dars)

Antonella Grassi
University of Pennsylvania and Università di Bologna

When I started on a tenure-track position at the University of Pennsylvania, Eugenio Calabi had just retired. More precisely, he had to retire, having reached the age of 70, the mandatory retirement age at the time. Colleagues were saying that he was the last person to whom the mandate applied, and commented on the irony of it, as Gene did not show any sign of slowing down. He never said a word about this, and he continued his activities as usual. This was typical of Eugenio: quiet, understated, reserved, dignified, and at the same time determined.

I was quite intimidated when he asked me to tell him about the interest of the physics community in certain mathematical objects, whose properties he described in impeccable formal Italian; with a shy smirk he eventually used the words *Calabi–Yau*. He told me how Yau and he, and others, met on a Christmas day at New York University, to discuss Yau's proof of his conjecture (Giuliana, Eugenio's wife, later commented that Eugenio's profession is to create problems for others to solve). Years later, when Yau came to Penn to deliver the Rademacher Lecture in 1999, I spent time with both of them together.

Over the years I grew very fond of Eugenio. He likes mathematics, but above all, from our conversations his passion emerged for justice, art, music and dedication to his family. He told me very proudly of the social accomplishments of his sister the journalist, who had moved back to Italy. He was careful never to mention her name, as in a riddle, and he was delighted when I eventually figured out which important public figure she was (Tullia went by her husband's last name). He would tell me about the current math question he had come up with, then he would give me practical advice for an Italian in the United States. He also shared his very useful method to walk safely on a narrow busy road in the Italian Alps. The last time we spoke in person, after he gave a seminar on the occasion of his 95th birthday, shortly before I moved to the Università di Bologna, he shared humorous, but as always humble, reflections on his career. He then went on to talk about his "nipotini," with an affection that warms the heart.

One winter, years before, I had returned to the Department with short hair. The difference must have been so drastic that several colleagues did not recognize me. Eugenio, after the ever polite and warm greetings, commented that Giuliana had also cut her hair short after the birth of their first child. Ovvio!

Claudio Arezzo
The Abdus Salam International Centre for Theoretical Physics

Calabi's work has had a huge impact on my education and research for the depth and beauty of his results, and for the elegance and simplicity in which he wrote, and spoke about, them. It is well known that his work on the existence of Kähler metrics with prescribed Ricci curvature has changed algebraic and differential geometry, as well as mathematical physics, forever. Three other themes of his work that I find as important are holomorphic isometric immersions of Kähler manifolds [1], minimal surfaces in spheres [2] and extremal Kähler metrics [5, 6], representing Calabi's proposal for a "best metric" on the largest possible space of Kähler manifolds. While these papers in particular have attracted a large amount of attention, the most fundamental existence question remains unanswered.

I want to stress especially the beauty of the presentation Calabi uses in his papers. It is a pure joy to read his works; I still remember with great nostalgia the many nights in the library studying the paper [4], which became famous as "Calabi's Ansatz" … He manages to teach the reader not just about the specific topic, but also how to choose a good problem and how a good idea is born, without tricks or intimidation. I cannot think of a better example of what Plato meant when stating in the Republic that "… the object of education is to teach us to love beauty".

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Balázs Szendrői is an algebraic geometer, specialising in the study of Calabi–Yau spaces. Having studied in Cambridge, and worked in Warwick, Utrecht and Oxford, he is currently University Professor of Algebraic Geometry at the University of Vienna. He is vice-chair of the Committee for Developing Countries of the European Mathematical Society.

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F. William Lawvere (1937–2023): A lifelong struggle for the unity of mathematics

Anders Kock

Francis William Lawvere was one of the most influential figures in the late 20th century and up till now, because of his drive to unify and simplify mathematics, by sharpening the tools of category theory. The following is an attempt of describing some of the milestones and visions in this process.

1 Continuum physics

Lawvere was born in February 1937, as son of a farmer in Muncie, Indiana. He studied physics at the University of Indiana, and there soon felt the need for more useable and explicit foundations for the reasoning employed, in particular in continuum physics. He was in Indiana a student of Clifford Truesdell, the founder of the Springer journal "Archive for Rational Mechanics and Analysis," who had a similar foundational agenda. L. saw already at this time the need for a category-theoretic approach. One first step was to achieve a "categorical dynamics" (some of which was materialized in the late 1960s). A crucial step was his category-theoretic formulation of the formation of function spaces, in terms of universal properties (adjoint functors): Cartesian closed categories.

Truesdell personally contacted Eilenberg to facilitate L.'s entrance into Columbia as Eilenberg's Ph.D. student 1960–63 – with a break 1961–63, where L. went to California, to learn more set theory and logic from experts in the fields (Tarski, Scott and others). In the California period, L. finished his (Columbia) Ph.D. thesis on functorial semantics of algebraic theories, where in particular the notion of algebraic theory was given in a presentation-free way.

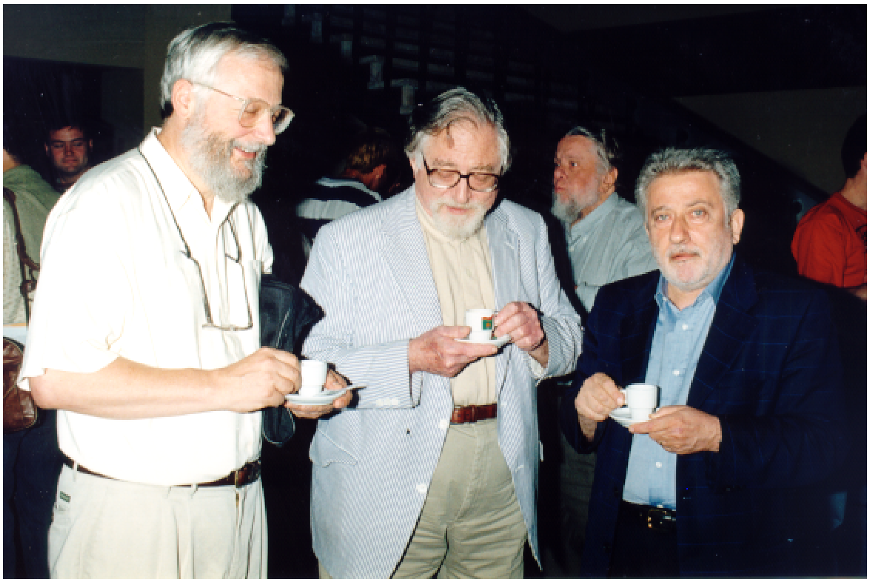
2 The Category of Sets

For L. himself, a turning point in his general search for useable and teachable foundations for mathematics was the year 1963–64 as an assistant professor at Reed College in Oregon. In an extensive interview with L., conducted in 2007 by Maria Manuel Clementino and Jorge Picado in Braga (Portugal) [2], L. says:

F. William Lawvere, Braga, March 2007
(© M. M. Clementino and J. Picado)

> At Reed I was instructed that the first year of calculus should concentrate on foundations, formulas there being taught in the second year. Therefore […] I spent several preparatory weeks trying to devise a calculus course based on Zermelo–Fraenkel (ZF) set theory. However, a sober assessment showed that there are far too many layers of definitions, concealing differentiation and integration from the cumulative hierarchy, to be able to get through those layers in a year. The category structure of Cantor's structureless sets seemed both simpler and closer. Thus, the elementary theory of the category of sets arose from a purely practical educational need.

Many of L.'s mathematical achievements (notions, constructions and theorems) result from efforts to improve the teaching of calculus and of engineering mathematics, and led him to conclude



F. W. Lawvere, A. Heller, R. Lavendhomme (in the back) and A. Carboni at CT99, Coimbra, Portugal (© M. M. Clementino and J. Picado)

that a workable foundation for mathematics, even for a calculus course, cannot be formulated in terms of $x \in y$ (membership), as in ZF, say, but can be formulated in terms of the notion of mappings $f : A \to B$ (and their composition). L. says, in the Braga interview [2] 2007:

> Philosophically, it may be said that these developments supported the thesis that even in set theory and elementary mathematics it was also true as has long been felt in advanced algebra and topology, namely that the substance of mathematics resides not in Substance, as it is made to seem when $\in$ is the irreducible predicate, but in Form, as is clear when the guiding notion is isomorphism-invariant structure, as defined, for example, by universal mapping properties. As in algebra and topology, here again the concrete technical machinery for the precise expression and efficient handling of these ideas is provided by the Eilenberg–Mac Lane theory of categories, functors and natural transformations.

After the year at Reed College, L. went to Zürich, where he was visiting in 1964–66 at Beno Eckmann's Forschungsinstitut für Mathematik. Eckmann had succeeded in attracting several category theorists to participate. Notably, the concept of *monad* ("triple"), and its relationships to algebraic theories and homology were elaborated (as documented in [3]).

From Zürich, it was possible to attend seminars at the nearby Oberwolfach in South Germany. Here, L. met Peter Gabriel and learned from him aspects of Grothendieck's approach to geometry, as expounded in SGA4 [1].

3 Grothendieck

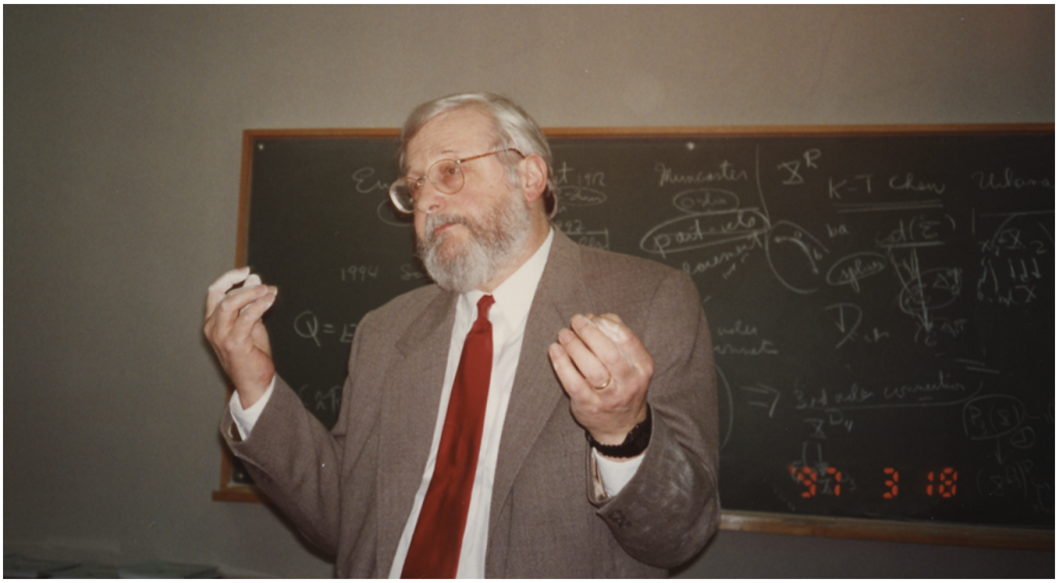
Grothendieck's work had a fundamental influence on L.'s later work. They first met each other at the ICM in Nice (1970), where they both were invited lecturers. L. had here publicly disagreed with Grothendieck in a separate lecture on his "Survival" movement.

In 1973, they were both visiting Buffalo. L. says in the Braga interview:

> I vividly remember his tutoring me on basic insights of algebraic geometry, such as "points have automorphisms." In 1981 I visited him in his stone hut, in the middle of a lavender field in the south of France, to ask his opinion of a project […]. My last meeting was at the same place in 1989 (Aurelio Carboni drove me there from Milano): he was clearly glad to see me but would not speak, the result of a religious vow; he wrote on paper that he was also forbidden to discuss mathematics, though quickly his mathematical soul triumphed, leaving me with some precious mathematical notes.

4 Categorical dynamics and synthetic differential geometry

In most of 1967, L. was assistant professor at the University of Chicago. L. here began to apply Grothendieck's topos theory in an advanced lecture series, centering around the problem of simplified foundations of continuum mechanics, inspired by Truesdell's and Noll's axiomatizations. The series was attended by Mac Lane, Jean



Lecturing in Coimbra, Portugal, March 1997 (© M. Sobral)

Bénabou, Eduardo Dubuc and others, including the present author, who was at that time finishing a thesis, under L.'s supervision. The particular contribution which came out of the seminar was not yet a full-fledged categorical dynamics, but a kinematic basis for it: the idea of having the tangent bundle construction representable, $T(M) = M^D$ for a postulated "infinitesimal" object D (utilizing the Cartesian closed structure of the postulated category of spaces). An aspect of this "kinematic" line of thought was later developed by several people as a more full-fledged "synthetic differential geometry."

The wisdom from algebraic geometry, which was at the basis of this development in categorical dynamics, could also be imported and applied in standard smooth differential geometry; L. uses an algebraic theory (in the sense of his 1963 thesis), namely the theory whose n-ary operations are the smooth functions $\mathbb{R}^n \to \mathbb{R}$ – a theory, for which it is crucial not to ask for a presentation in terms of generators and relations.

5 Elementary toposes, algebraic geometry and logic

L. returned to the Forschungsinstitut in Zürich in 1968–69. At this time, he had become more convinced that toposes were involved not only as a background for categorical dynamics, but also for notions from set theory and logic: boolean-valued models, and forcing (as in Cohen's work (1963) on the continuum hypothesis). In the Braga interview, he says:

> That these apparently totally different toposes, involving infinitesimal motion and advanced logic, could be part of the same simple axiomatic theory, was a promise in my 1967 Chicago course. It only became reality after my second stay at the Forschungsinstitut in Zürich, Switzerland 1968–69, during which I discovered the nature of the power set functor in toposes as a result of investigating the problem of expressing in elementary terms the operation of forming the associated sheaf, and after 1969–1970 […] through my collaboration with Myles Tierney.

This collaboration took place in Halifax (Canada): In 1969, L. had obtained the prestigious Killam professorship at Dalhousie University in Halifax, and was in that context allowed to invite a dozen collaborators (among them Tierney), likewise supported by Killam. This meant that during 1969–1971 Dalhousie became a lively place; mathematically, in particular, the notion of elementary topos gradually crystallized here. Significantly, L. had organized that a preprint version of (exposé I–IV) of SGA4 [1] was handed out to the participants of his seminar (SGA4 is Artin, Grothendieck and Verdier's "Théorie des Topos et Cohomologie Etale des Schémas," not officially published until 1972).

However, in 1971, the dream team at Dalhousie was dispersed; the university administration refused to renew the contract with L., due to his political activities in protesting against the Vietnam war and against the War Measures Act proclaimed by Trudeau, suspending civil liberties under the pretext of danger of terrorism. (But in 1995, Dalhousie hosted the celebration of 50 years of category theory, with participation of L.)

A conference organized by L., on the eve of his stay in Halifax in 1971, carries the significant title: "Toposes, Algebraic Geometry and Logic," and the proceedings from this conference were published in 1972 [6].

After leaving Halifax in 1971, L. became visiting professor in Aarhus (Denmark) 1971–72, and in Perugia (Italy) 1972–73. These were years where the new insights in topos theory, brought about in Halifax, were consolidated and disseminated more widely. Also, after finally settling 1973 in Buffalo (US), L. maintained close contacts, in the form of shorter and longer stays, with his European friends and collaborators; this includes a year 1980–81 at IHÉS (Paris).

The toposes that we studied in Halifax and later, were in particular "gros toposes" (like the topos of simplicial sets), in contrast to the "petit toposes" (like the topos of sheaves on a topological space). This was a distinction made in SGA4, IV.4.10. This distinction was for L. one of the inputs of the study of the *category* of toposes, i.e., toposes in their functorial inter-relationship. Such studies were developed by many researchers, and documented in many mathematical monographs, articles, and in conferences (with or without proceedings). L. was very active in participating in conferences, often as invited keynote speaker; he was less active in getting the wealth of his ideas and visions down in written form. For instance, his seminal talks in Chicago in 1967 on categorical dynamics were not available in written form until in 1978, in the proceedings of a protracted "Open House" summer meeting in Aarhus, on "Topos Theoretic Methods in Geometry" [5].

In 1982, L. (together with his Buffalo colleague Steve Schanuel) organized a conference in Buffalo, "Categories in Continuum Physics," with participation also of many key researchers in continuum physics, like Truesdell and Noll. Three of the articles in the proceedings (published in [8]) deal with the problem of foundations of thermodynamics.

L. was in 1977 in the Scientific Steering Committee of the important and large summer meeting in Durham, "Applications of Sheaves" [4], which marked a breakthrough in exploiting the relative simplicity of toposes in the conceptualization of mathematical and physical theories. L. gave a talk in Durham on "categories in the foundations of thermodynamics," of which, however, I have not been able to find a written account. There does, on the other hand, exist accounts of a talk (with a lively debate) given by L. at this conference, with the title "The Logic of Mathematics," where L. stated his view on the philosophy and development of mathematics. I include it here, since an obituary of L. would be incomplete, if it did not reflect the uncompromising character of his political/philosophical life and work:

In this Durham debate, L. says in the beginning of the talk (according to my notes and memory):

> Mathematics is the science of space forms and quantitative relationships. What is the purpose of mathematics? Its purpose is to clarify this relationship in order to act as a basis of unity of people in solving problems (not mathematical problems) in the struggle for production, and in the conscientiousness of this struggle, which is scientific experimentation.

Already at this early stage of the talk came an interrupting question (possibly rhetoric) from a member of the audience: "What is the purpose of production?" L. thought for quite some time before answering: "To bring you here!"

Later on in the talk, L. stated:

> The purpose of the logic of mathematics; to clarify and simplify the learning, use and development of mathematics. […] In a dialectical way: there is also a counterpurpose: to obscure, complicate and prevent the learning, use and development of mathematics. In particular, to freeze the development by promoting instead: thinking about forcing everything into a preconceived framework […]. Both of these purposes are fighting with each other inside each of us. […] Often the counterpurpose wins over the purpose. This is because the counterpurpose is in the interest of the ruling class. This is a thing which has changed drastically over the last 100 years. The interest of the monopoly capitalist class is against the development of production.

6 Axiomatic cohesion

This is not the place to give (nor would I be able to give) a complete survey of all the aspects of L.'s mathematical and philosophical work. Just some further key-words: probability, categorical logic, indexed/fibered categories, metric spaces as enriched categories, linguistics, extensive vs. intensive quantities, category of physical quantities, Grassmann, axiomatic cohesion.

The idea of axiomatic cohesion, as introduced by L. 2007 [7], has in particular led to recent new developments.

The following is a quotation from this 2007 publication:

> An explicit science of cohesion is needed to account for the varied background models for dynamical mathematical theories. Such a science needs to be sufficiently expressive to explain how these backgrounds are so different from other mathematical categories, and also different from one another, and yet so united that they can be mutually transformed. An everyday example of such mutual transformation is the weatherman's application of the finite element method (which can be viewed as analysis in a combinatorial topos) to equations of continuum thermomechanics (which can be viewed as analysis in a smooth topos, where smooth functions and distributions live).

F. W. Lawvere with the author at Cafe Odeon, Zürich, Fall of 1966 (© A. Kock)

The basis for this axiomatic science of cohesion is a string of four functors

$$p_! \dashv p^* \dashv p_* \dashv p^!,$$

each one in the string left adjoint to the next one. An example of such a string is familiar in topology: $p_!$ associating to a (sufficiently nice) space its set of connected components, p^* associating to a set the discrete space structure on that set, p_* associating to a space its set of points, and finally $p^!$ associating to a set the codiscrete space structure on that set. In the category of toposes, properties of such strings formulate many of the distinctions asked for in the above quotation.

Only some of the many the ideas that L. launched have reached written, let alone published, form, but exist only in the form of seeds in minds and notes of people who have been around.

Probably, many fruitful plants will emerge in the future from these seeds. The germination of the seeds would be enhanced if they were more accessible in some archive. Some activity in creating such archives is taking place, notably in www.acsu.buffalo.edu/~wlawvere.

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"My sincere condolences"

After the death of Henri Poincaré (July–December 1912)

Laurent Rollet

> Science has lost the greatest mathematician of the century […]; but I have lost my dearest friend (27 years of continuous, charming relations), who had always been extremely indulgent and kind to me! I have also lost the best and greatest supporter of a scientific organisation that I had created with so much effort. You know, moreover, that 15 of his most important memoirs were published in the *Rendiconti* [*del Circolo Matematico di Palermo*], among them that of 1894 (On the equations of mathematical physics), which is considered a classic, immortal work of this great genius. The first one (from 1888) was in the form of a letter addressed to me. And the last one… alas! was a farewell!

So said the mathematician Giovanni Battista Guccia (1855–1914) to Louise Poincaré, the mathematician's widow, on 18 September 1912 [16]. Henri Poincaré died on 17 July 1912 of an embolism following a bladder operation. He had clearly been declining for months, even going so far as to mention his 'decrepitude' in a letter to the mathematician Edgar Odell Lovett [28].

For weeks, the French and foreign press emphatically recalled the memory of the man who was a 'poet of the infinite' (Jules Clarétie), a 'modest Titan' (Ernest La Jeunesse) or the 'consulting brain of human science' (Paul Painlevé). These public sources follow a highly codified rhetoric, and their analysis allows us to observe the mechanisms involved in building Poincaré's scientific and cultural heritage [22]. The more intimate sources, such as letters of condolence, provide an overview of the family, friendship, social, scientific and professional networks in which Henri Poincaré was involved, as well as networks of scientific filiation.

The *Archives Henri Poincaré* in Nancy have been publishing Henri Poincaré's scientific, administrative, and private correspondence for many years and are therefore mainly interested in the *direct* epistolary exchanges between the mathematician-philosopher and various correspondents.[1] This undertaking sheds light on the development of his scientific work and opens biographical horizons that still need to be explored.[2] In addition to these direct sources, there are *indirect* ones in which Poincaré's personality and activity are mentioned, sometimes in a very meaningful way. The *Archives* possess a file of letters of condolence collected by his descendants from July to December 1912. They contain many clues about unsuspected networks of sociability and allow us to analyse the mechanisms involved in the early building of his posterity.

This article will present a first exploration of this largely unpublished corpus. It will be organised in three sections. The first part will briefly present the nature of this corpus and the authors of the letters it contains. Then a second part will give an overview of its content by concentrating on three subjects: firstly, the tributes and condolences, focusing on the most personal and less formal passages; secondly, the expressions of thanks sent by several mathematicians; and finally, the events organised in honour of Poincaré following his death.

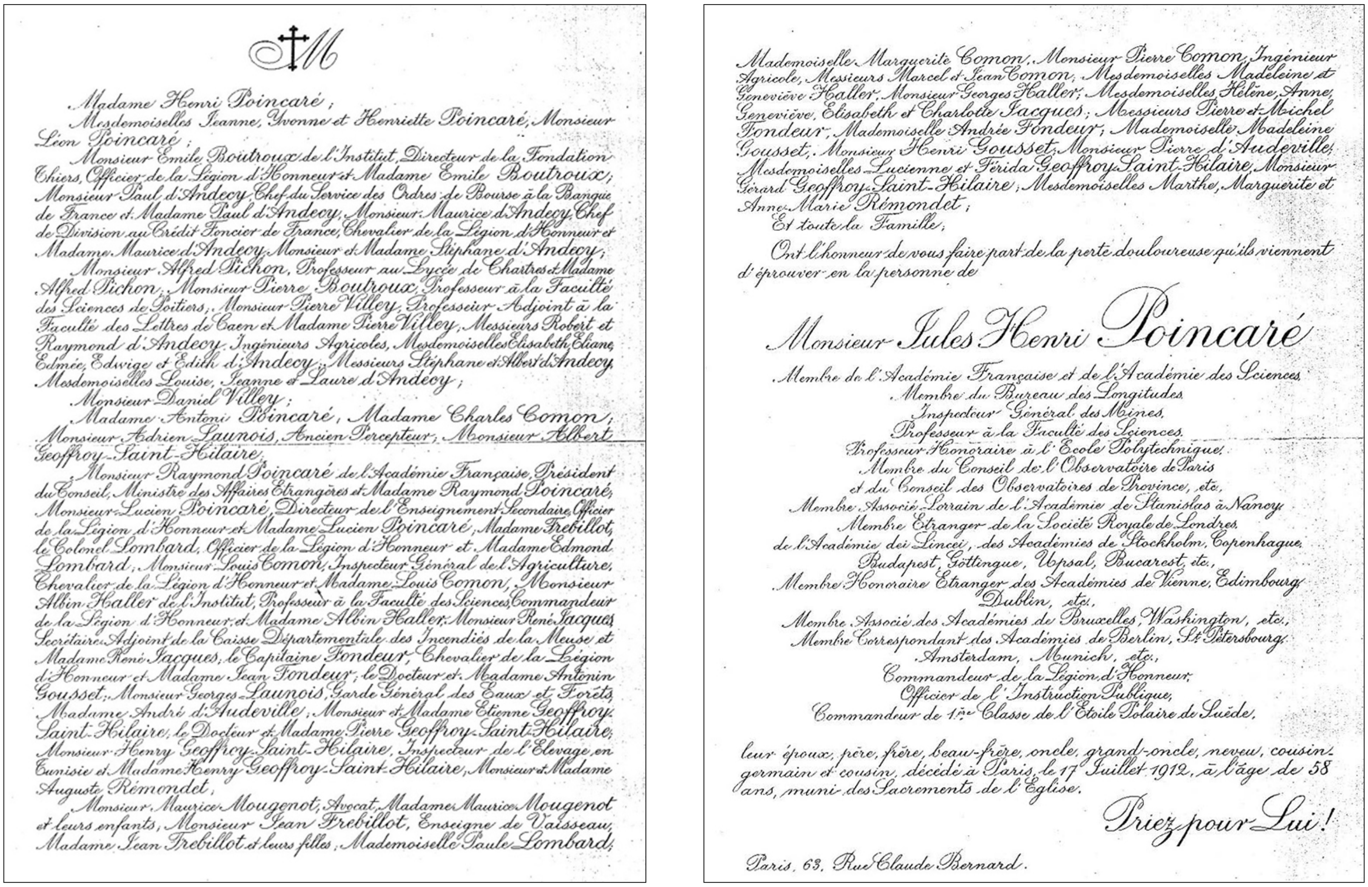
1 Presentation of the corpus and the correspondents

After Poincaré's death, his wife and some other family members kept a record of the expressions of sympathy they had received. This file was kept by the family's descendants and has never been used until now; it was put in a folder labelled "1912. Letters of condolence kept after sorting. March 1955," which seems to indicate that it was compiled in the wake of the celebrations of the centenary of Poincaré's birth [3].[3] This corpus contains a total of 38 letters: 32 were addressed to his wife Louise Poincaré (1857–1934) [21],

[1] The volumes published are devoted to correspondence between Poincaré and Gösta Mittag-Leffler [13], correspondence with physicists [26], correspondence with astronomers, engineers and geodesists [27] and youthful correspondence [23]. The forthcoming volumes are devoted to correspondence with mathematicians [15] and family, private and administrative correspondence [24]. This correspondence is gradually being put online at http://henripoincare.fr/s/accueil/page/accueil

[2] It is important to point out that the two most recent biographies of Poincaré published by Jeremy Gray [7] and Ferdinand Verhulst [25] only focus on his scientific work, leaving out part of his intimate, family, and personal life.

[3] The collection has been digitized by the *Archives Henri Poincaré* and all the letters will be published in volume 5 of the Poincaré correspondence [24].



M

Madame Henri Poincaré ;
Mesdemoiselles Jeanne, Yvonne et Henriette Poincaré, Monsieur Léon Poincaré ;
Monsieur Émile Boutroux de l'Institut, Directeur de la Fondation Thiers, Officier de la Légion d'Honneur et Madame Émile Boutroux ;
Monsieur Paul d'Andecy, Chef du Service des Ordres de Bourse à la Banque de France et Madame Paul d'Andecy, Monsieur Maurice d'Andecy, Chef de Division au Crédit Foncier de France, Chevalier de la Légion d'Honneur et Madame Maurice d'Andecy, Monsieur et Madame Stéphane d'Andecy ;
Monsieur Alfred Pichon, Professeur au Lycée de Chartres et Madame Alfred Pichon, Monsieur Pierre Boutroux, Professeur à la Faculté des Sciences de Poitiers, Monsieur Pierre Villey, Professeur Adjoint à la Faculté des Lettres de Caen et Madame Pierre Villey, Messieurs Robert et Raymond d'Andecy, Ingénieurs Agricoles, Mesdemoiselles Elisabeth, Eliane, Edmée, Edwige et Edith d'Andecy, Messieurs Stéphane et Albert d'Andecy, Mesdemoiselles Louise, Jeanne et Laure d'Andecy ;
Monsieur Daniel Villey ;
Madame Antoni Poincaré, Madame Charles Comon ; Monsieur Adrien Launois, Ancien Percepteur ; Monsieur Albert Geoffroy-Saint-Hilaire ;
Monsieur Raymond Poincaré de l'Académie Française, Président du Conseil, Ministre des Affaires Étrangères et Madame Raymond Poincaré, Monsieur Lucien Poincaré, Directeur de l'Enseignement Secondaire, Officier de la Légion d'Honneur et Madame Lucien Poincaré ; Madame Frebillot, le Colonel Lombard, Officier de la Légion d'Honneur et Madame Edmond Lombard ; Monsieur Louis Comon, Inspecteur Général de l'Agriculture, Chevalier de la Légion d'Honneur et Madame Louis Comon, Monsieur Albin Haller de l'Institut, Professeur à la Faculté des Sciences, Commandeur de la Légion d'Honneur et Madame Albin Haller, Monsieur René Jacques, Secrétaire Adjoint de la Caisse Départementale des Incendiés de la Meuse et Madame René Jacques ; le Capitaine Fondeur, Chevalier de la Légion d'Honneur et Madame Jean Fondeur ; le Docteur et Madame Antonin Gousset ; Monsieur Georges Launois, Garde Général des Eaux et Forêts, Madame André d'Audeville ; Monsieur et Madame Etienne Geoffroy-Saint-Hilaire, le Docteur et Madame Pierre Geoffroy-Saint-Hilaire, Monsieur Henry Geoffroy-Saint-Hilaire, Inspecteur de l'Elevage en Tunisie et Madame Henry Geoffroy-Saint-Hilaire, Monsieur et Madame Auguste Rémondet,
Monsieur Maurice Mougenot, Avocat, Madame Maurice Mougenot et leurs enfants, Monsieur Jean Frebillot, Enseigne de Vaisseau, Madame Jean Frebillot et leurs filles ; Mademoiselle Paule Lombard ; Mademoiselle Marguerite Comon, Monsieur Pierre Comon, Ingénieur Agricole, Messieurs Marcel et Jean Comon, Mesdemoiselles Madeleine et Geneviève Haller, Monsieur Georges Haller ; Mesdemoiselles Hélène, Anne, Geneviève, Elisabeth et Charlotte Jacques ; Messieurs Pierre et Michel Fondeur, Mademoiselle Andrée Fondeur ; Mademoiselle Madeleine Gousset, Monsieur Henri Gousset, Monsieur Pierre d'Audeville, Mesdemoiselles Lucienne et Férida Geoffroy-Saint-Hilaire, Monsieur Gérard Geoffroy-Saint-Hilaire ; Mesdemoiselles Marthe, Marguerite et Anne-Marie Rémondet ;
Et toute la Famille,

Ont l'honneur de vous faire part de la perte douloureuse qu'ils viennent d'éprouver en la personne de

Monsieur Jules Henri Poincaré

Membre de l'Académie Française et de l'Académie des Sciences,
Membre du Bureau des Longitudes,
Inspecteur Général des Mines,
Professeur à la Faculté des Sciences,
Professeur Honoraire à l'École Polytechnique,
Membre du Conseil de l'Observatoire de Paris
et du Conseil des Observatoires de Province, etc.,
Membre Associé Lorrain de l'Académie de Stanislas à Nancy,
Membre Étranger de la Société Royale de Londres,
de l'Académie dei Lincei, des Académies de Stockholm, Copenhague,
Budapest, Göttingue, Upsal, Bucarest, etc.,
Membre Honoraire Étranger des Académies de Vienne, Edimbourg,
Dublin, etc.,
Membre Associé des Académies de Bruxelles, Washington, etc.,
Membre Correspondant des Académies de Berlin, St Pétersbourg,
Amsterdam, Munich, etc.,
Commandeur de la Légion d'Honneur,
Officier de l'Instruction Publique,
Commandeur de 1re Classe de l'Etoile Polaire de Suède,

leur époux, père, frère, beau-frère, oncle, grand-oncle, neveu, cousin-germain et cousin, décédé à Paris, le 17 Juillet 1912, à l'âge de 58 ans, muni des Sacrements de l'Eglise.

Priez pour Lui !

Paris, 63, Rue Claude Bernard.

Figure 1. Death announcement of Henri Poincaré (© Archives Henri Poincaré)

two to his son Léon, two to his future son-in-law Léon Daum[4] and two to the physicist Lucien Poincaré, the mathematician's cousin and brother of Raymond Poincaré, the President of the French Republic.

The history of the building of this file is difficult to determine. It is conceivable that the family received much more than 38 letters of condolence after Poincaré's death, so this collection probably represents only a portion of the letters received after 17 July 1912. Although small, its main interest lies in the fact that it allows us to discover new connections between Poincaré and other actors. And it turns out that many of the letters contained in this collection show correspondents for whom no trace of epistolary exchanges was available until now. The table below gives a broad overview of the authors of the letters along with biographical information.[5]

If we look at the places where the letters were sent from, we can see that a large proportion were sent from France (22 letters), with Germany in second place (5), followed by the United States (3), Japan (2) and Argentina (2). Such a geographical distribution is obviously not representative of the influence of Poincaré's thought in 1912, but it opens new ways for thinking about the building of his posterity. It is worth mentioning that many correspondents listed here were trained, like Poincaré, at the *École Polytechnique*.

[4] Henri Poincaré and Louise Poulain d'Andecy had three daughters and one son. Jeanne Poincaré (1887–1975) married Léon Daum (1887–1966) in 1913. A graduate of the École Polytechnique, a mining engineer and heir to a family of Nancy crystal makers, he had a brilliant career as an industrial administrator and was even president of the European Coal and Steel Community from 1952 to 1953. Yvonne Poincaré (1889–1939) remained single and lived all her life at her mother's side. Henriette Poincaré (1891–1970) married Edmond Burnier (1890–?) in 1921; the couple had four children and divorced in Annecy in 1955. Finally, Léon Poincaré followed in his father's footsteps at the *École Polytechnique* (class of 1913), joined the engineering corps and ended his career as an Air Force engineer. He married Emma Motte in 1920 and had a child, François Poincaré (1920–2012).

[5] In bold the names of correspondents for whom there are no known epistolary exchanges with Poincaré.

Identity of correspondents	Place of dispatch of the letter	Brief biographical information
L. Barthélémy	Spincourt (France)	Partially illegible signature. It was probably a female cousin of Henri Poincaré's mother, Eugénie (1830–1897).
Marie Bonaparte (1882–1962)	Paris (France)	She was the daughter of the geographer and patron Roland Bonaparte (1858–1924), Princess of Greece, a friend of Poincaré and the introducer of psychoanalysis in France. Poincaré was a regular visitor to her salon at the end of his life.
Élie Cartan (1869–1951)	Paris (France)	Mathematician, trained at the *École Normale Supérieure*. In 1912, he had just been appointed professor at the Sorbonne.
Clément Colson	Paris (France)	Poincaré's classmate at the *École Polytechnique*, engineer, specialist in political economy and member of the French *Conseil d'État.*
Victor Crémieu (1872–1935)	Rodié (France)	French physicist trained at the Sorbonne. He had presented a doctoral dissertation under the supervision of Gabriel Lippmann [4] and Poincaré was the author of the report on this dissertation. Crémieu's work had been at the centre of a controversy concerning the interpretation of Henry Augustus Rowland's experiment on the magnetic effects of a charged rotating disk. Poincaré had for a time sided with Crémieu against the interpretation of Harold Pender [8]. This episode is documented by a correspondence between Poincaré and Crémieu [26].
Maurice d'Ocagne (1862–1936)	Etretat (France)	Engineer and mathematician trained at the *École Polytechnique* (class of 1880). He is at the origin of an original method for the graphical solution of algebraic equations using scaled diagrams, called nomography.
Henri Deslandres (1853–1948)	Paris (France)	Engineer, military officer, and astronomer, trained at the *École Polytechnique* (class of 1872), where he probably met Poincaré for the first time. Deslandres was the director of the Meudon Observatory at the time of Poincaré's death. Deslandres and Poincaré were members of the *Bureau des Longitudes.*
Jane Dieulafoy (1851–1916)	Montgiscard (France)	Archaeologist, novelist, journalist, and photographer. She was the wife of Marcel Dieulafoy (1844–1920), a mining engineer trained at the *École Polytechnique* (class of 1863) and at the *École des Mines de Paris*. He became a well-known archaeologist and a member of the *Académie des Inscriptions et Belles Lettres*. The tone of the letter reveals a certain closeness with the Poincaré family.
The director or a teacher at the *École alsacienne*	Paris (France)	Illegible signature. He wrote a letter of comfort to Poincaré's son, Léon, which suggests that Léon had done part of his secondary education at that school.
Francis Foullioux	Égletons-les-Roses (France)	A Bachelor of Science, who declares himself to be a "modest pupil of the late Master".
Tsuruichi Hayashi (1873–1935)	Sendai (Japan)	Professor of mathematics at Tōhoku Imperial University in Sendai, founder in 1911 of the *Tôhoku Mathematical Journal* [9] and translator of the Japanese edition of *Science and Hypothesis* in 1909.
Felix Klein (1849–1925)	Hahnenklee (Germany)	Mathematician, professor at the University of Göttingen.
Johann Robert Lenz	Paris (France)	Lenz was probably a woodcarver who held the position of administrator-treasurer of the *Université populaire du Faubourg Saint-Antoine* in Paris. This institution published a journal, *Les cahiers de l'Université populaire*. Its editor was the anarchist sociologist Henri Dagan (1870–1912). Poincaré, who figured, like his two cousins Raymond and Lucien, among the subscribers of this journal, had given at least two lectures within this institution, one on chance and one on wireless telegraphy.
Max Lenz (1850–1932)	Berlin (Germany)	Historian, rector of the *Friedrich-Wilhelms-Universität* Berlin.
Xavier Léon	Combault-Pontault (France)	Philosopher, founder of the *Revue de métaphysique et de morale* and of the *Société française de philosophie*. Poincaré was asked, along with Henri Bergson and Émile Boutroux, his brother-in-law, to be one of the leading authors for the launch of the journal in 1893. During his career, Poincaré published about twenty articles in this journal. He was also a member of the *Société française de philosophie*. Xavier Leon's family was close to Poincaré's; they met during summer holidays at Houlgate in Normandy. This relationship is documented by a fairly long correspondence [24].

Identity of correspondents	Place of dispatch of the letter	Brief biographical information
Paul Lévy (1886–1971)	Paris (France)	Mathematician, trained at the *École Polytechnique* (class of 1904). On 24 November 1911, he had presented a doctoral dissertation on integro-differential equations defining line functions before a jury composed of Émile Picard, Jacques Hadamard and Henri Poincaré (Poincaré wrote the report on the dissertation).
Edgar Odell Lovett (1871–1957)	Houston (United States)	American mathematician, founder and first president of the Rice Institute of Houston. He had invited Poincaré to participate in the inauguration of this institution in 1912 – without success, due to Poincaré's precarious state of health.
Juraj Majcen (1875–1924)	Zagreb (Croatia)	Croatian mathematician, professor at the University of Zagreb
Camilo Meyer (1854–1918)	Buenos Aires (Argentina)	Mathematician and physicist born in Verdun. Meyer emigrated to Argentina in 1895 where he became a professor of mathematical physics at the University of Buenos Aires. Meyer was also a close childhood friend of Poincaré.
P. Millon	Sauvagnat (France)	Millon seemed close to the anthropologist and sociologist Gustave Le Bon (1841–1931). Le Bon was the director of the book collection *Bibliothèque de Philosophie Scientifique* published by Flammarion; it was on his initiative that Poincaré had published in this collection his well-known philosophical books, such as *La Science et l'Hypothèse*. Poincaré was quite close to Le Bon and regularly participated in the social dinners he organized.
Hantarō Nagaoka (1865–1950)	Tokyo (Japan)	Professor of physics at the University of Tokyo. He had participated in the first International Congress of Physics in Paris in 1900.
Heike Kammerlingh Onnes (1853–1926)	Leiden (Netherlands)	Professor of experimental physics at Leiden University.
Max Planck (1858–1947)	Berlin (Germany)	Professor of physics at the University of Berlin.
Henri Salomon (1861–?)	Paris (France)	Teacher of history and geography of Poincaré's son, Léon, at the *Lycée Henri IV* in Paris.
Ludwig Schlesinger (1864–1933)	Giessen (Germany)	German mathematician, professor at the University of Giessen.
S. Frankfurter	Vienna (Austria)	The letter mentions a meeting in Vienna with Poincaré. One of Poincaré's last trips abroad seems to have taken place in Vienna in May 1912, on the occasion of a celebration of the Friends of the Gymnasium.
Ernest Vessiot (1865–1952)	Paris (France)	Mathematician trained at the *École Normale Supérieure*. In 1912 he was a lecturer at the Sorbonne. He was also an *examinateur d'admission* at the *École Polytechnique* where his student was Poincaré's son, Léon. The latter would enter the *École Polytechnique* in 1913.
Victorine	Nancy (France)	No last name. The deferential tone suggests that it may have been an employee who had been in the service of the Poincaré family in Nancy.
Jean Vassilas-Vitalis	Athens (Greece)	Professor at the Military School of Athens. He was a member of the Société mathématique de France since 1899.
Alexander Wilkens (1881–1968)	Kiel (Germany)	Astronomer at the Kiel Observatory in 1912. He later became director of the Breslau Observatory and then of the Munich Observatory.
Emily Wilson, born Newcomb (1869–1948)	New York (United States)	Daughter of the mathematician Simon Newcomb (1835–1909), a correspondent of Henri Poincaré. Emily Wilson was both a psychologist and a photographer. She was the wife of Francis Asbury Wilson (1861–1943), an illustrator who worked on advertisements for the R. J. Reynolds tobacco company. She seemed close to the Poincaré family, which she had obviously met at the Congress of Mathematicians in Rome in 1908, when Poincaré had fallen seriously ill.
Paul Xardel (1854–1933)	Rupt-sur-Moselle (France)	Childhood friend of Poincaré – his father was a colleague of Poincaré's father at the Faculty of Medicine of Nancy –, a military officer trained at the Military School of Saint-Cyr. In 1912 he was a colonel in the infantry. We owe him an autobiographical testimony long remained unpublished on his friendship with Poincaré [29].

2 An overview of the corpus

2.1 Tributes and condolences

On 18 July, an unidentified teacher from the *École alsacienne*[6] wrote to Poincaré's son, Léon, to offer his condolences. Léon had obviously done part of his studies there before joining the special mathematics class at the *Lycée Henri IV*. He wrote to him: "I pity you with all my heart and would like to be able to tell you so in person. Keep alive in your soul the memory of the great scholar, the man of such integrity that was your father: he will hover over your existence, beneficent, comforting." The mathematician Ernest Vessiot showed the same concern for Léon Poincaré in August by writing to his mother. After expressing his sorrow at the death of the man who had become his colleague after his appointment to the Sorbonne in 1910, he was careful to let her know that he was ready to postpone the examinations her son had to take as part of the entrance exam to the *École Polytechnique*.

Other personal testimonies are addressed to the family by voices close to Poincaré. Thus Paul Xardel, his childhood friend, wrote to Louise Poincaré on 18 July: "If I were not so far from Paris, I would have liked to join Henri's friends and admirers tomorrow, who will come in droves to attest to the greatness of the loss you have just suffered, you and your children and with you France and the Universe. I am perhaps the oldest of his friends and, among his oldest admirers, who have always proclaimed his genius and predicted his glory. His genius will be celebrated by his followers and his pupils, and his glory is immortal. I would have liked piously to praise his heart, his loyalty to the friends of his childhood and youth, and I would have mourned with you the one you understood and helped so well." Likewise, Marie Bonaparte, who wrote in July: "He was – as you know better than anyone – not only the greatest thinker, the most powerful genius of our time – but also a deep and incomparable heart; and having been close to him remains the precious memory of a whole life."

The same testimony comes from Jane Dieulafoy, who apparently was an intimate of the family: "For me, I will always have the memory of the great mind who seemed to know and understand everything, even the secret of being kind and attentive to the thoughts of those who, compared with him, were only ignorant and puny." (21 July). Or this testimony from a cousin from Poincaré's maternal branch, Madame Barthélémy, who prays for the salvation of his soul: "I pray for you with all my heart, as well as for this beautiful soul. Perhaps, however, it hardly needs it. It seems to me that God must have received him as his child and that Henri is now in infinite happiness, knowing everything, understanding everything, immersed in beauty, in the eternal and shadowless goodness to which we ourselves will perhaps go one day." (22 November 1912).

Perhaps just as moving, Francis Fouillioux, a Bachelor of Science and former student of the "Master," wrote to his widow: "I am only a humble student, a modest pupil of the late Master whom you mourn. I mourned him with you because he was for me the personified glorification of human intelligence. He loved Science and was a philosopher: I owe him the best and greatest joys that a miserable life has allowed me to know." (undated).

On the side of the scientists, the tributes are just as poignant, sometimes tinged with shyness. This is the case of the physicist Victor Crémieu, a student of Poincaré, who did not dare to write to his widow and sent his condolences to another member of the family, perhaps Lucien Poincaré. He evoked the almost filial relationship he had with him, remembering his doctoral dissertation and the scientific controversy that pitted him against Harold Pender: "It is only this morning that I learnt the distressing news of the death of the man I call my scientific father, and for whom I have always had feelings of filial affection. Intellectually I owe him everything, and morally a lot. It is simply out of discretion that since I have been living in the country, I stopped keeping in touch with him." (20 July).

The next day, the astronomer Henri Deslandres wrote to Louise Poincaré and painted a scientific and moral portrait of the man he had worked with at the *Bureau des Longitudes* for many years: "Your husband was exceptional in terms of both his moral and scientific values. He was truly good and an honest man, in the broadest sense of the word; he inspired admiration, esteem and affection in all those who approached him. I had him as a neighbour for ten years at the *Bureau des Longitudes*, and I was able to appreciate him well. When a scientific difficulty arose, it was always to him that one turned, and if the solution was possible, he gave it at once. In matters of elections, he knew how to rise above party or chapel interests, and his opinion and his vote were dictated by justice alone or by a broad spirit of conciliation."

Other scientists evoked the memory of their meeting with Poincaré at various events. For example, the Dutch physicist Heike Kammerlingh Onnes, referring to the great Solvay Congress of 1911 in Brussels: "I will never forget the great honour I had to sit next to Mr Poincaré at the Brussels Council. Who would have thought then that we would so soon experience the loss of his genius. The kindness that the great scientist showed me with his well-known gentleness will remain a beautiful memory for the rest of my life." (27 August). Max Planck, who also attended the congress, recalled his meeting with Poincaré and his daughter (probably Jeanne Poincaré): "Although your husband was known and familiar to me for years in his wisdom, it was only last autumn in Brussels, when I had the honour of making his and your daughter's personal acquaintance, that I had an idea of what you, dearest Madam, have lost in him. He did not work for time, but for eternity, and he lives on in the memory of all those who had the good fortune to approach him."

[6] The *École alsacienne* was founded in Paris after the Franco–Prussian War of 1870. It was a renowned private institution. It still exists today.

Max Lenz, historian and rector of the University of Berlin, recalled Poincaré's lecture on new mechanics in October 1910 at the institution's centenary celebrations [18]: "It touches me all the more painfully today, on behalf of my colleagues, Madam, to have to express our deep participation in the indescribably great loss that you personally have suffered […]. His name, which will last as long as the theorems with which he enriched the mathematical sciences, will always find a place of veneration at the University of Berlin."

Finally, in a long letter dated 9 August, the mathematician Felix Klein evoked the memory of his one-time rival in the 1880s over the naming of Fuchsian functions:

> Please count me among those who are most directly affected by the death of your husband and who best understand how much science and his family have lost in him.
> It is more than thirty years since I encountered your husband and witnessed, so to speak, from week to week, the rise of his mathematical genius. As for me, I quickly collapsed under the weight of the work I had to do and was never able to reach the level of productivity I used to have. He, on the other hand, went from triumph to triumph, working out in a fast and victorious race what the rest of us considered a distant goal, namely full validity in the field of applications in addition to all the achievements in the field of pure mathematics. It is now an abrupt end!
> I do not know how long your husband suffered, but I have read and re-read with pensive interest the words with which he begins his last publication in the *Rendiconti di Palermo*. The rest of us also have enough reason to reflect on the passage of time.[7] I myself have had to take a leave of absence since New Year's Day and I have been living here in a sanatorium ever since. The many unfinished projects that I have undertaken with others over the years have the advantage that I can devote myself to them in detail. Thus, the courses on the theory of automorphic functions, which I started 30 years ago with my brother on the icosahedron and which Rob. Fricke, from Braunschweig, and myself have recently completed have been published.[8] I assume that your husband received the delivery and that he was pleased to learn of the conclusion that our common field of work owes to the research of the younger generations.

2.2 *Expressions of gratitude*

Three letters – written by Maurice d'Ocagne, Élie Cartan and Paul Lévy – show the role played by Poincaré in supporting their careers. They were all addressed to Poincaré's widow.

Maurice d'Ocagne, who was eight years younger than Poincaré, spoke, on 18 July, of his gratitude for the latter's action in his favour when he was appointed professor of geometry at the *École Polytechnique* in 1912. He was also glad to have benefited from the mathematician's support when he first applied for membership of the *Académie des Sciences* (although he was not elected until 1922). Poincaré and d'Ocagne had been in contact for several decades in various mathematical spheres, notably within the *Société Mathématique de France* and at the *École Polytechnique*, where d'Ocagne had been appointed as a *répétiteur* in 1893.

> It was with a painful shock that I learned the awful news for which nothing had prepared me, and it was with a heart gripped by poignant emotion that I had the honour of addressing you a first telegram of condolences, regretting that distance did not allow me to express to you my feelings in person.
> For some thirty years I had the honour of enjoying the kindly friendship of your illustrious husband. There were so many occasions on which he gave me testimony of this that I cannot recall them all here. I will never forget the part he played in my appointment as a professor at the *École Polytechnique*, nor the encouragement he gave to my first attempt at becoming a candidate for the Institute. The fact that I received his vote on this occasion will remain one of the most precious honours of my scientific career. But what I want to remember most of all today is the cordial welcome I was always assured of from him and the camaraderie with which he tried to reduce the enormous intellectual distance between him and me.

On the same day, Élie Cartan, who had been from 1904 to 1909 professor of differential and integral calculus at the Faculty of Sciences of the University of Nancy [14] and then lecturer at the Sorbonne until 1912, claimed to owe his appointment to a professorship to Poincaré's benevolence:

> To the general consternation produced by the news of Henri Poincaré's death, to the grief felt by those who had the privilege of seeing and approaching the master, is added for me a more poignant pain. Perhaps the last act of his life as a professor and scholar was to come to the Sorbonne to read

[7] In this article, devoted to "A new theorem of geometry" linked to the periodic solutions of the three-body problem, Poincaré wrote: "I have never presented to the public such an unfinished work; I therefore believe it necessary to explain in a few words the reasons which determined me to publish it, and first of all those which had engaged me to undertake it […]. It seems that under these conditions I should refrain from any publication until I have resolved the question; but after the useless efforts I have made over many months, it seemed to me that the wisest thing to do was to let the problem mature, resting on it for a few years; this would be very good if I were sure of being able to take it up again one day; but at my age I cannot answer for it." [19, p. 375]

[8] [6]

for me the report he had just made on my work. We will take pride, my family and I, in never forgetting him. It is my bitter regret to think that I will never be able to express my gratitude to him. The thanks that I could not give him, allow me, Madam, to give it to you and to your children.

Finally, on 9 August, the mathematician Paul Lévy recalled Poincaré's warm welcome when he submitted his doctoral dissertation to him in 1911 [10], even going so far as to advise him to defend it earlier than he had hoped. Lévy's family was in acquaintance with Poincaré's since the time when Levy's father, Lucien, was a colleague of Poincaré at the *École Polytechnique*.

On learning of the loss that French science had just suffered in the person of Monsieur Poincaré, I did not dare at first to write to you to tell you how much I was sharing in your grief. The number of those who knew and admired the great scientist who has just passed away is so great that it seemed to me that they could not all let you know individually how much this misfortune affected them.
However, I had reason to tell you how grateful I was to Monsieur Poincaré. More than a year ago, I came to him with a thesis that I asked him to read, and I will never forget how kindly he received me. He immediately started reading this work, and it is to him that I owe the fact that I was able to defend my thesis much earlier than I had hoped, and I also owe him advice which was invaluable for the future and whose significance I appreciate more and more.
It was on receiving the letter of announcement sent to me, and even more on seeing in your letter to my mother your sympathy on the occasion of the mourning which has just affected us in our turn by the loss of my father, Mr. Lucien Lévy,[9] that I decided to offer you the expression of my very respectful sympathy.

2.3 Two commemorations and a missed opportunity

On 30 December 1912, the Japanese physicist Hantarō Nagaoka wrote to Henri Poincaré's cousin Lucien. A physicist trained at the *École Normale Supérieure*, Lucien Poincaré was then Director of Higher Education at the Ministry of Public Instruction. In 1912, Nagaoka was at the Imperial University of Tokyo and he had received international recognition for his work on magnetostriction, earthquake wave propagation, electromagnetic wave transmission and, above all, atomic theory. In 1904, he had developed a planetary model of the atom based on an analogy with the planet Saturn, and some of his predictions had been confirmed by Ernest Rutherford in 1911. Nagaoka studied at the University of Tokyo and then in Europe, in Berlin, Munich and Vienna. He had participated in the first International Congress of Physics in Paris, during which he had undoubtedly been able to listen to and meet Lucien and Henri Poincaré. A theoretician and experimenter, he was also to become the first president of the Imperial University of Osaka and president of the Imperial Academy in 1939 [20].

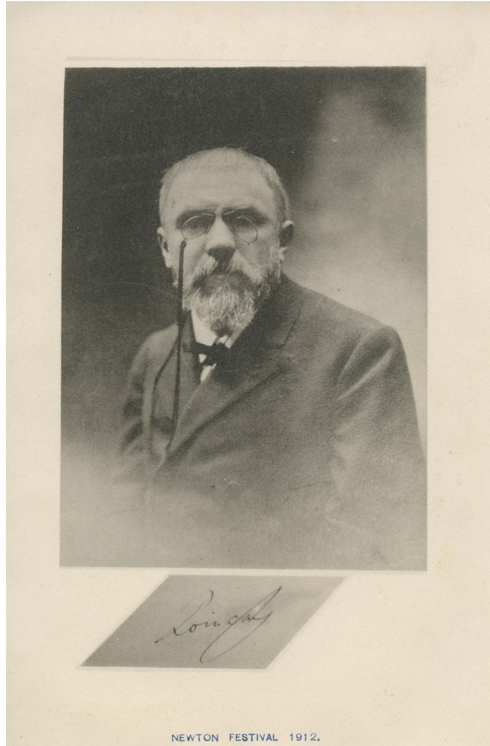


Figure 2. Postcard distributed on the occasion of the Newton Festival at the Imperial University of Tokyo in 1912 (© Archives Henri Poincaré)

In his letter to Lucien Poincaré, Nagaoka told him that the science students at the University of Tokyo had decided to organise a commemoration in Poincaré's honour as part of an annual celebration, the *Newton-Sai*: "I have the pleasure of making you the following communication. The students in mathematics, astronomy and physics at the Tokyo Imperial University hold every year what they call *Newton-Sai* (literally translated, celebration in honour of Newton) on the birthday of Newton. The object of the assembly is simply to celebrate the deeds of great men in the domain of science, in which they are interested. This year they had to lament the death of your cousin, and they have prepared the endorsed postal card with the likeness of the illustrious dead

[9] Lucien Lévy (1854–1912) died on 2 August. Trained at the *École Polytechnique* (class of 1872), he had been in contact with Henri Poincaré, who was a member of the following class. For a while he was a professor of higher mathematics at the *Lycée Henri IV*, and then became head of the *Collège Sainte-Barbe* in Paris. In 1890 he was appointed as an *examinateur d'admission* at the *École Polytechnique*, a position he held until 1910. During the last two years of his life, he was to be an *examinateur de sortie* in mechanics [2].

for private circulation. It gives simple proof how he was admired and how his death was lamented in the Far East." He enclosed the postcard that was distributed on that occasion.

Another example of a commemorative event appears in a correspondence between the Franco-Argentine physicist Camilo Meyer and Poincaré's widow. Born in Verdun, Camilo Meyer was a childhood companion of Poincaré in Nancy [5]. Both had been pupils at the city's high school and Meyer apparently came regularly to the Poincaré's home. He was even a patient of Poincaré's father, Émile Poincaré, who had a well-known medical practice in the city. After obtaining a degree from the Faculty of Science in Nancy, he apparently obtained a doctorate in Law. He moved to Argentina in 1895. Between 1910 and 1915 he taught a free course in mathematical physics at the University of Buenos Aires, based on the courses given by Poincaré at the Sorbonne. He was responsible for the first presentation of quantum theory in Argentina [12, 17]. The long letter he wrote to Louise Poincaré on 1 November 1912 gives an account of the action he took to honour the memory of his childhood friend by delivering, as early as 1 August 1912, a eulogy on the career and work of Poincaré at the Sociedad Científica Argentina of which he was an active member [11].

> Madam,
>
> Although I am a stranger to you, I believe I am authorised by my long-standing relationship with the illustrious scientist, whom the intellectual universe has been mourning for three months, to send you a copy of the lecture which the Argentine Scientific Society requested of me as soon as the terrible news reached us.
>
> This homage, paid to the memory of the great scientist, was for me the fulfilment of a duty all the sweeter because, in seeking to revive the departed genius and to describe his gigantic work, I saw him again in my heart as I had always known him: an excellent comrade, modest, kind to everyone, and seeming to forget with each person the abyss that his unequalled genius opened up between his superhuman intelligence and the mind of ordinary mortals.
>
> I saw him again as a high-school companion, then as a fellow student, in those family gatherings at his home in *rue de Serre* in Nancy; I saw again his father the physician, my physician, also so kind, so helpful. These imperishable memories sustained me in the difficult task that was imposed on me, 3500 leagues away from Paris, only a few days after the telegraph had informed us of the catastrophe; although we were unaware of the details at the time, I was able, in a memorable meeting of the Scientific Society, in the midst of the general mourning of my colleagues, to condense in a talk all that I personally knew of the life and colossal work of my former comrade.
>
> I still do not know whether in other scientific centres similar ceremonies inspired by grief and admiration were organised with such promptness and ardour. In any case, I am proud to think that in this country, where I have lived for so many years, this unexpected death, which necessarily produced general consternation, was able to arouse an echo and provoke a demonstration of mourning for which I was the humble and unworthy spokesman.
>
> Would it be an abuse of your indulgence, Madam, to ask you timidly for the slightest souvenir of the man with whom I was already a close friend more than forty years ago? whom I followed step by step in his triumphal march, and whose works form the main element of my library?
>
> We were only a few days apart in age; despite a separation dating back a long time, I still had the resource of writing to him sometimes, and he always replied to me despite an overwhelming daily workload. You would not believe how happy I would be to possess any object, *the most insignificant of all*, which had belonged to him: it seems to me that I would thus find less bitter the few years left to me to live.
>
> Please excuse my boldness, which is great, and this long letter, and accept, Madam, my highest regards.
>
> *C. Meyer*
> *Professor of Mathematical Physics at the Faculty of Sciences of Buenos Aires*
> *Calle Independencia 1241, November 1912*

A third example of an epistolary exchange allows us to document precisely an episode at the end of Poincaré's life. In 1912, he exchanged several letters with the American mathematician Edgar Odell Lovett. The latter wanted to invite him to give a lecture at the inauguration of the Rice Institute in Houston, which was to take place from 10 to 12 October 1912. Poincaré declined the invitation to travel because of his health and died in July. He had nevertheless promised to send the manuscript of an article to be published in the proceedings of this event. Consequently, Lovett wrote to Louise Poincaré on 4 September 1912 to ask if her husband had had time to write the manuscript before his death.

> As you will recall, the Trustees of the Rice Institute had done themselves the honour of inviting your late distinguished husband, Professor Henri Poincaré, to lecture at the formal opening ceremonies to be held October tenth, eleventh, and twelfth. He feared that his health would not permit him to make the long journey to Houston, but expressed his willingness, on our repeating the invitation, to send us a manuscript in October for publication in the proceedings of our first academic festival.
>
> It has occurred to me that you may find such a manuscript among his papers. If this should be the case, we should be most happy to receive it. In this event we should of course expect to pay to the estate the honorarium which had been proposed.

Louise Poincaré replied to Lovett by saying that she had found no trace of any manuscript and that her husband was probably planning to write it during the summer holidays. Émile Borel, who had been Poincaré's student and a close colleague, was part of the French delegation to the inauguration of the Rice Institute and gave a poignant account of his last exchanges with him on that occasion: "When Henri Poincaré was invited by President Edgar Odell Lovett to deliver an address at this scientific celebration, his acceptance was conditional on the state of his health. A few months later, he finally declined the invitation, promising, however, to send his lecture in writing. I cannot remember without emotion the last conversation I had with him on that subject. I was still hoping that his decision was not final; but, after giving me some friendly advice about my lectures and the journey, he told me with what deep regret he had to give up the thought of ever visiting the United States again, and I felt, for the first time, how serious was the condition which justified his refusal. A few weeks afterward he was gone." [1]

3 Conclusion

As stated above, this corpus of carefully preserved letters makes it possible to discover new facets of Henri Poincaré's relationships with family, friends, social and professional contacts. It also provides an opportunity to analyse the processes transforming his work into a heritage in France and abroad; in this respect, the episodes of commemoration in Argentina and Japan are particularly enlightening. Finally, and above all, these new or poorly documented networks of relations open interesting biographical perspectives insofar as they offer the possibility of discovering both actors who were important to Poincaré and others for whom Poincaré himself was important. Such a source makes it possible to re-discover a personality who was not only a mathematician and a scientist, but also a social actor and a leading intellectual.

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Mathematics at the Deutsches Museum: On-site, digital, and to go

Katja Rasch and Mareike Wöhler

1 Challenge: An abstract science as an exhibition topic

Even the earliest plans for the Deutsches Museum, founded in 1903, included rooms for mathematics. In 1906 the first mathematics exhibition opened in Munich in the former building of the Bavarian National Museum located on Maximilianstraße (today: Museum Fünf Kontinente). In a museum the great challenge to the exhibition makers was, and still is, to identify the topics of mathematics that can be communicated well, as well as to present them to the visitors. Over the decades and with increasing experience, the focus and the way of communication have changed. When the Deutsches Museum was founded, it did not yet have any collections the curators could base the exhibition on. Thus, for the planning of all exhibitions, "lists of desired objects" were prepared by referents. For the mathematics section this was done by Walther von Dyck (1856–1934), mathematician and co-founder of the museum. Based on these wish lists, the collection was first established. The museum received considerable support from various donations and the production of replicas by well-known manufacturers of mathematical instruments and models. After 10 years, the collection of "Mathematische Instrumente, Analoggeräte und -rechner" already counted 500 items. Today it includes more than 4600 objects.

2 Mathematics is fun: The new permanent exhibition "Mathematics"

In July 2022, after nine years of preparation, the Deutsches Museum's new permanent exhibition "Mathematics" opened (Figure 1).[1] Its concept was developed by computer scientist Anja Teuner, who was curator of Computer Science and Mathematics from 2011 to 2018. The hands-on stations, which performed well in the former mathematics exhibition at the Deutsches Museum, the "Mathematisches Kabinett," were planned to be placed in a larger context and enriched with historical mathematical instruments and models. The intention was to create an exhibition that would satisfy mathematics and at the same time would offer fun, but also provide a place for exhibit lovers to experience and enjoy mathematics.

Figure 1. Glimpse into the new permanent exhibition "Mathematics" of the Deutsches Museum. Picture credits: Deutsches Museum / Hubert Czech.

The exhibition that finally emerged focuses on visualizations of popular geometrical objects and phenomena. A playful approach to mathematics was deliberately chosen – from the descriptive to the abstract. The focus is not on the exhibition objects as such, but rather on active participation, trying things out, and experiencing mathematical structures on one's own. By these means, mathematics becomes accessible not only for the experienced, but also for the younger visitors and all other interested people. In doing so, the exhibition reduces not only the immense breadth of mathematics, but also the extreme distance that its concepts and ideas have taken from everyday life and vividness. Von Dyck, the first curator of mathematics at the Deutsches Museum, stated as early as 1925: "To present the essence, content, and aims of mathematical research in its entirety cannot be the task of a museum."[2]

[1] More insights on https://www.deutsches-museum.de/museumsinsel/ausstellung/mathematik.

[2] "Wesen, Inhalt und Ziele der mathematischen Forschung in ihrer Gesamtheit vor Augen zu führen, kann nicht Aufgabe eines Museums sein." [4, p. 192] (translated from German by K. Rasch)

The Mathematics exhibition is based on the formula

games + exhibits + applications = fun to the power of three.

The basic elements here are cubes which appear throughout the exhibition in the thematic areas of Introduction, Dimension, Perspective, and Symmetry. "Hanging" cubes contain key information as well as polyhedral interactive media stations, while cube-shaped experiment tables are spread throughout the exhibition. In this way, the interactive concept is placed in a geometric frame of reference, which is complemented by historical exhibits and insights into the practical applications of mathematics. A wide variety of games helps visitors to understand mathematical ways of thinking in a playful way, and richly illustrated exhibition texts reveal how such thinking is put into practice, for example in everyday life or in architecture. Particularly striking exhibits include a wooden pantograph from 1782 for reducing silhouettes, gold-plated multipurpose dividers from 1586, or finely crafted drawing instruments from 1775 made of silver.

From form to formula and back: mathematics makes everything easier – even if a sequence of numbers, letters and symbols may not suggest this at first glance. For example, a shark's dorsal fin, a triangle in music, and a house gable are just triangles from a mathematical point of view. All triangles have certain common characteristics. These generally valid characteristics then help to create, to comprehend, to describe new items, buildings or technologies. Mathematics – ancient Greek: the "art of learning" – is thus not taught as a fine art, but quite simply as an important aspect of our everyday lives.

3 Gyroid, catenoid, onduloid: Mathematical models on the Deutsches Museum Digital portal

A relatively short and striking equation:

$$\sin x \cos y + \sin y \cos z + \sin z \cos x = 0.$$

But what does it describe? One recognizes that it contains x, y and z and perhaps assumes that a mathematical surface could hide behind the equation. To solve the mystery of what this surface looks like, computer software is used today that makes the surface visible on a screen – or a section of the surface is created using 3-D printing. It can then be inspected from all sides, such as the model of a spherical section of the gyroid[3] (Figure 2), which was made of purple colored plastic. This model was purchased for the new mathematics exhibition. Once you hold it in your hand, you can imagine more easily that two labyrinths have been interwoven into each other.

[3] Inv. No. 2018-285, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/2018-285

Figure 2. The mathematical models object group on the Deutsches Museum Digital portal, with preview image of the gyroid, IMAGINARY, Berlin 2018 (Design: Oliver Labs), Deutsches Museum, Inv. No. 2018-285. Picture credits: Deutsches Museum / Konrad Rainer.

Already in the 19th century, there was the wish to visualize mathematical surfaces. At that time, there were no means of digital representation available. People began to construct models of surfaces. Only sections were made, however, since some surfaces extend to infinity. Von Dyck established a collection which was supplemented and expanded over the years and now comprises more than 250 models. Some of them can already be studied in object datasets with high-resolution photos on the online portal Deutsches Museum Digital[4] in the section "Discover" by clicking on the photo of the gyroid; many more will be added.

The collection of the Deutsches Museum does not only comprise classic models made of plaster, whose prototypes were created in modeling cabinets at universities and were offered for sale in extensive catalogues, such as a catenoid[5] (Figure 3, left) that entered the collection as early as 1906. The museum also preserves fascinating models made of paper or cardboard that change their shape by applying slight pressure, for example an ellipsoid made of 30 cardboard circles[6] (Figure 3, middle).

There are also thread and rod models preserved, for example a thread model for the visualization of a hyperboloid of one sheet by its generatrices[7] (Figure 3, right), which was produced by the Verlagshandlung Martin Schilling in Halle (Saale) around 1900 based on a design by the mathematician Alexander von Brill (1842–1935) [1].

Probably worldwide unique are the "women's tights models" that came to the museum in 2011. The series consists of 42 models,

[4] https://digital.deutsches-museum.de/en

[5] Inv. No. 6449T1, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/6449T1

[6] Inv. No. 6437, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/6437

[7] Inv. No. 6483, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/6483

Figure 3. Left: Plaster model of a catenoid, Verlagshandlung Martin Schilling, Halle (Saale) c. 1900, Deutsches Museum, Inv. No. 6449T1. Middle: Cardboard model of an ellipsoid, Verlagshandlung Martin Schilling, Halle (Saale) c. 1900, Deutsches Museum, Inv. No. 6437. Right: Thread model of a hyperboloid of one sheet, Verlagshandlung Martin Schilling, Halle (Saale) c. 1900, Deutsches Museum, Inv. No. 6483. Picture credits: Deutsches Museum / Konrad Rainer.

each of which represents a section of a minimal surface. As the sections are extended to infinity in the three spatial dimensions, a surface is obtained that does not intersect itself and divides the space into two congruent labyrinths. Some of these surfaces are used to describe the geometrical arrangement of atoms in particular crystal structures. All the models were laboriously handcrafted from metal wire and women's tights by the crystallographer Elke Koch in the late 1980s, for example a minimal surface model with a metal base[8] (Figure 4, left).

Especially difficult to realize are locations in which a surface tapers off to a point. For these singularities, older models had to use supports – as for example this model of a Plücker complex surface (Figure 4, middle). It is made by using painted elements of a lead alloy fixed on a wooden plate. These days, modern manufacturing techniques such as 3-D printing or laser-in-glass technology are used for such complex shapes. The glass models made by mathematician and designer Oliver Labs, such as an Endrass surface[9] (Figure 4, right), really glow by their bottom illumination.

It is also planned to integrate the mathematical models of the Deutsches Museum into the Digital Archive of Mathematical Models (DAMM), which until now only shows models from university collections.[10]

[8] Inv. No. 2012-987T38, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/2012-987T38

[9] Inv. No. 2018-365T4, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/2018-365T4

[10] https://mathematical-models.org/en

4 Proportional dividers and universal scales: Mathematical instruments on the Deutsches Museum Digital portal

Not only the mathematical models, but all objects of the collection of "Mathematische Instrumente, Analoggeräte und -rechner" are to be successively put online on the Deutsches Museum Digital portal.[11] Already today, many scientific instruments such as dividers, slide rules, and planimeters can be studied in detail there with object data, descriptive texts, and object photos that can be enlarged. The latter can be downloaded directly and used under the CC BY-SA 4.0 license. The object data records were enriched with norm data from the Gemeinsame Normdatei (GND) of the Deutsche Nationalbibliothek (for persons and corporate bodies), and with GeoNames (for places), as well as specialist literature, historical sources on instruments, and patents from the German Patent Information System (DEPATISnet). In this way, it is possible to access linked information on several levels if one wants to know more precisely. In addition, the conversion of the object data into the LIDO XML format (developed for the exchange of metadata) will enable the connection to a future national research data infrastructure.

The manufacturers of historical instruments from the early modern period in particular are sometimes more difficult to determine than one might assume at first glance. This fact is shown by the case

[11] https://digital.deutsches-museum.de/en/digital-catalogue/hitlist/?filterCollection=true&filterArchive=false&collection[name]=Mathematische%20Instrumente,%20Analogger%C3%A4te%20und%20-rechner&collection[value]=415

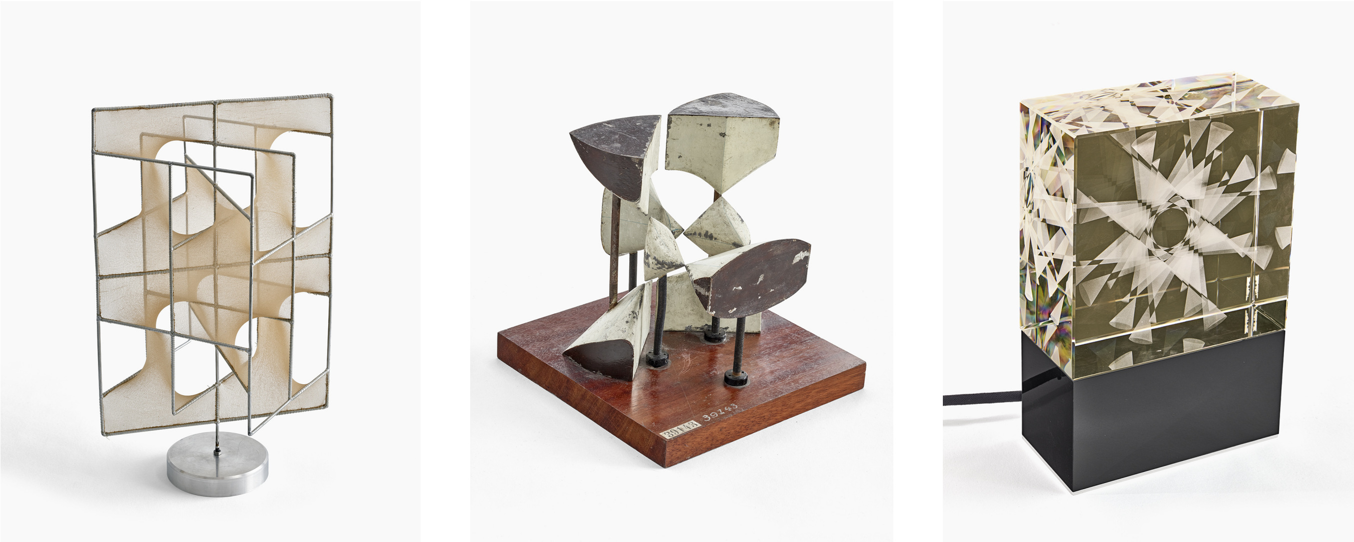

Figure 4. Left: Minimal surface model with metal base, Elke Koch, Universität Marburg, Marburg c. 1986, Deutsches Museum, Inv. No. 2012-987T38. Middle: Model of a Plücker complex surface, Johann Eigel Sohn, Cologne c. 1880, Deutsches Museum, Inv. No. 39143. Right: Laser-in-glass model of an Endrass surface, Oliver Labs, Ingelheim 2018 (Design: Stephan Endraß), Deutsches Museum, Inv. No. 2018-365T4. Picture credits: Deutsches Museum / Konrad Rainer.

of a pair of proportional dividers[12] (also: reduction compass) made of gilded copper alloy, probably before 1616 by the clockmaker Heinrich Stolle (Figure 5, left) [7]. His principal Thomas Ferdinand Teuffel von Zeilberg was previously thought to be the manufacturer. In difficult cases, in addition to knowledge of Latin, so-called historical auxiliary sciences such as heraldry help just as much as the investigation of the object biography and the acquisition history of objects in the archives of the Deutsches Museum, for example in the files of the administrative archives [5]. The search for comparative instruments in other digital object catalogues of museums, such as the catalogues of the British Museum and the Science Museum in London[13], or the catalogues of the Germanisches Nationalmuseum in Nuremberg[14] often supports dating and localization of objects. It also makes it possible to assess whether an instrument is singular or whether it was made often – with individual variations or in machine serial production, depending on the century.

Also from a mathematical point of view, the exploiting of objects and their comprehensive explanation for a publication in the digital domain are challenging for museums: A pair of proportional dividers[15] (Figure 5, middle) made in Augsburg in 1586 allows geometric operations such as the transfer of line segments as well as the division, multiplication, and transformation of line segments, areas, and bodies as an analog computer [6].

The question of processed materials – such as ivory and their plastic surrogates developed since the beginning of the 20th century – is relevant for understanding the objects in terms of their manufacture, users, and contexts of use. However, the materials used do not only determine the price of the object at the time of purchase. Ethical questions about colonial loot and the decimation of animal species also play an increasing role in museum discourse for restitutions and with regard to the resources remaining to humanity in the future. (See the catalogue of the correspondent exhibition in the Humboldt Forum [3].) An example of this thematic complex from the mathematical instruments collection is a universal scale[16] made in England at the beginning of the 20th century (Figure 5, right).

5 Exhibition to go: The new math catalogue

The catalogue accompanying the new permanent exhibition, entitled "Mathematik – Vom Anschaulichen zum Abstrakten," was recently published by the museum's own publishing house [2]. The visual approach to mathematics taken in the exhibition is continued here. At the same time, exhibit lovers can enjoy a wide range of

[12] Inv. No. 10505, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/10505

[13] https://collection.sciencemuseumgroup.org.uk and https://www.britishmuseum.org/collection

[14] https://objektkatalog.gnm.de/recherche

[15] Inv. No. 64022, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/64022

[16] Inv. No. 1980-182T3, https://digital.deutsches-museum.de/en/digital-catalogue/collection-object/1980-182T3

Figure 5. Left: Adjustable proportional dividers, Heinrich Stolle (?), Prague before 1616, Deutsches Museum, Inv. No. 10505. Middle: Adjustable proportional dividers, Christoph II Schissler, Augsburg 1586, Deutsches Museum, Inv. No. 64022. Right: Universal scale made of ivory, Technical Manufacturing Corporation, England 1905, Deutsches Museum, Inv. No. 1980-182T3. Picture credits: Deutsches Museum / Konrad Rainer.

finely crafted drawing instruments, delicate surface models and precise mathematical instruments. In addition, numerous essays and thematic pages illuminate the diversity – and beauty – of mathematics from different angles, thus complementing the exhibition. Many models, dividers and other types of instruments can be studied in these more profound texts and in the catalogue section and can be compared with one's own prior knowledge or other collections.

In addition to interested readers of the catalogue, we are looking forward to welcome expert visitors to the new permanent exhibition. The new museum app also provides relevant background information in advance and, if desired, accompanies you in a relaxing manner through the exhibitions.[17]

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[17] Free download for Android and Apple via https://www.deutsches-museum.de/en/museum-island/visit/app-audioguide.

Ascending peaks of knowledge

Jan Overney

Located on the picturesque shores of Lake Geneva, the freshly inaugurated Bernoulli Center for Fundamental Studies at the École Polytechnique Fédérale de Lausanne (EPFL) hopes to become a beacon for the advancement of fundamental research in mathematics, theoretical physics, and theoretical computer science.

November 11, 2022 was a big day at EPFL's new Bernoulli Center for Fundamental Studies. The ceremonial cutting of the ribbon that took place under blue skies and the bucolic harmony of an alphorn choir symbolically ended a brief but intense period of renewal. It was a fresh start for a center with two intense decades of scientific advancement to look back on, in new facilities, building on a solid foundation and many lessons learned.

Undergraduates, PhD students, post-docs, faculty members, and researchers from EPFL and surrounding universities filled the lecture hall at the EPFL Rolex Learning Center. On the agenda: a series of keynote lectures from a star-studded, international speaker panel, including Maryna Viazovska, EPFL's very own Fields Medalist, Abel Prize laureate Avi Wigderson from Princeton's Institute of Advanced Study, Nobel Prize laureate Duncan Haldane from Princeton University, and Fields medalist Hugo Duminil-Copin from the neighboring University of Geneva.

It was a day dedicated to celebrating science, understanding, and excellence. From the opening remarks in the morning to the panel discussion in the late afternoon, there was palpable excitement about the new center's role in bringing together researchers with different backgrounds to meet, learn from each other, and dream up new collaborations. Pushing at the boundaries of knowledge and betting on the "unreasonable effectiveness of theoretical studies" to inspire, fascinate, and, with a bit of luck, spawn a whole new field of scientific discovery.

A proven model of scientific advancement

This model of scientific advancement – creating a venue for the great and aspiring scientific minds of the day to congregate – has a fertile history to look back on, epitomized by the Bernoulli family, an eminent dynasty of Swiss scientists. Bathed in an academic milieu, constantly in touch with leading thinkers of their time, the Bernoullis made major contributions to mathematics, physics, mechanics, and other areas of basic science. Jacob derived the law of large numbers in probability theory, Johann developed the calculus of variations, and Daniel discovered the Bernoulli principle in fluid mechanics, to name just a few.

Founded as the *Centre Interdisciplinaire Bernoulli* (CIB) by Professors Tudor Ratiu and Gérard Ben Arous, the center was born of the desire to bolster the reputation of the engineering-focused school in the theoretical sciences. "The key event that paved the way for the CIB was the fusion of the basic science departments of EPFL and the University of Lausanne. To raise the level of theoretical sciences at EPFL, I had the idea to form a research center focusing on mathematics, physics, and chemistry," recalls Ben Arous. He then set out to develop a center inspired by the Institut des Hautes Études Scientifiques (IHÉS) in Paris.

When the *Centre Interdisciplinaire Bernoulli* opened, it did so with a reduced scope compared to its ambitious aspirations, serving primarily as a visitors center for mathematicians. Nonetheless, it filled an important gap for theoretical mathematicians, explains Nicolas Monod, who directed the center from 2014 to 2021. "Theoreticians don't have labs or experimental facilities such as the CERN's Large Hadron Collider. That's why having what I like to call a Large Brain Collider is so essential. If you are a student, you probably won't spontaneously call up a famous researcher for a chat. But if you can collide with that researcher in front of a blackboard with a strong cup of tea, they might ask you what you are currently thinking about, and the magic can happen."

Under Monod, the Bernoulli Center's attractive force grew to the point that, for the first time, EPFL became closely associated with Fields medalists. "Every living Fields medalist visited EPFL. Abel Prize winners came to give lectures. Wolf Prize recipients and Nobel laureates participated in the center's activities. There was a constant stream of decorated scientists coming to campus, learning to appreciate the surroundings, and, perhaps, hopefully, envisioning a career here," says Monod.

Hosting over 500 visitors per year every year before the pandemic and with hundreds of applications coming in from research-

EMS MAGAZINE 128 (2023) — DOI 10.4171/MAG/131

Figure 1. Cutting the ribbon at the new Bernoulli Center for Fundamental Studies. From left to right: Martin Vetterli, Anna Fontcuberta i Morral, Duncan Haldane (Nobel Prize ’16), Emmanuel Abbé, Maryna Viazovska (Fields Medal ’22), Hugo Duminil-Copin (Fields Medal ’22), João Penedones, Philippe Michel, Martin Hairer (Fields Medal ’14), Avi Wigderson (Abel Prize ’21). © EPFL 2022

Figure 2. The new building of the Bernoulli Center for Fundamental Studies. © EPFL 2022

ers interested in participating in semester projects and shorter workshops, the *Centre Interdisciplinaire Bernoulli* firmly solidified its position on the map in fundamental mathematics. Despite that, with the many competing needs for space on campus and the disruptions caused by the COVID pandemic, the Bernoulli Center was briefly suspended in 2021 due to a lack of dedicated facilities.

Reaching for new heights

The Bernoulli Center’s second incarnation, this time as the Bernoulli Center for Fundamental Studies, was the fruit of a grassroots effort by an interdisciplinary group of theoretical researchers at EPFL that had formed around Emmanuel Abbé. Holding the Chair of Mathematical Data Science, straddling the divide between theoretical mathematics and computer science, Abbé was cut out for the job ahead. His career had been shaped by time spent in similar centers, including the Simons Institute for the Theory of Computing at the University of California, Berkeley, and the IAS at Princeton. “Our goal was to be inclusive and ambitious, and in about six months, we had put together a project backed by around 50 researchers on campus.”

At least in spirit, the new center is a continuation of its predecessor, despite its expanded scope covering theoretical physics and computer science in addition to mathematics. “At the new center’s core is an expanded program, building on three pillars. The first, the scientific program, involves semester-long research projects in which experts from around the world are invited to work with the local scientific community, picking up where the former center left off,” explains Abbé.

“The second pillar strengthens interactions across campus and neighboring institutions by offering a forum for theoretical researchers from different backgrounds to meet, run seminars and collaboration groups, and host other activities.”

"Finally, the third pillar reaches out to young talents – high school students and undergraduates hungry for challenges beyond the scope of their standard curriculum. This program includes a local mathematics competition for bachelor students, a new computer science class for high school students, public lectures, and more."

The fresh start was also an opportunity to revise the center's governance. "To be inclusive and sustainable in the long run, we set up an executive committee representing all three areas of research and overlapping directors with four-year term limits. An internal scientific advisory committee will be charged with curating the center's short-term scientific programs. Meanwhile, an external scientific committee comprising eminent scientists from around the world will support the selection of long-term research programs, underwriting the high quality of the proposed research programs," says Abbé.

The EPFL Bernoulli Center: key dates

2002	Founding of the *Centre Interdisciplinaire Bernoulli* (CIB) by Gérard Ben Arous and Tudor Ratiu
2002–2014	Direction of the center by Tudor Ratiu
2014–2021	Direction of the center by Nicolas Monod
2021	Suspension of the CIB
2021	Reincarnation as the Bernoulli Center for Fundamental Studies. Direction of the center by Emmanuel Abbé
2023	Addition of Martin Hairer to the center's governance

The new Bernoulli Center takes shape

The new center kicked off its activities well in advance of its November 11 inauguration. A Bernoulli Month on "Modern Trends in Combinatorial Optimization," organized by Friedrich Eisenbrand, who heads EPFL's Chair of Discrete Optimization, and Ola Svensson, from EPFL's Theory of Computation Laboratory, brought around 100 international PhD students to campus and featured workshops that attracted leading researchers from the world's most competitive institutions. Its success bodes well for the new center's future.

"The summer school was an excellent opportunity to raise the knowledge and skill of our own EPFL PhD students to the next level and let them compare themselves to international peers," says Eisenbrand. Many international participants reached out to Eisenbrand and Svensson, enquiring about the possibility of spending more time at EPFL for a project or even PhD studies. "It's through these types of activities that we can still increase the prestige and the research performance of our school," he says.

"I'm very confident in the new Bernoulli Center's ability to attract people from around the world, in part due to the high quality of local researchers."

With the official inauguration of the new facilities, the center has entered a new defining phase. "There is a strong feeling of achievement after this first intermediate step. We had to clear many hurdles to get here. But now the real work begins, with much higher expectations," says Philippe Michel, member of the center's executive committee and head of EPFL's Chair of Analytic Number Theory.

The mood is optimistic, even from Ben Arous's vantage point in the United States: "I'm very confident in the new Bernoulli Center's ability to attract people from around the world, in part due to the high quality of local researchers." Martin Hairer, head of EPFL's Chair of Probability and Partial Differential Equations, echoes his sentiment: "The center has a great location, making it easy to attract great mathematicians to the area." Add to that the potential for new partnerships with the neighboring University of Geneva, the SwissMAP Research Station in the nearby Diablerets mountain resort, and the Forschungsinstitut für Mathematik (FIM) at ETH in Zurich.

Future outlook

The inauguration celebrations presented a research center that is off to a strong start. And, says Emmanuel Abbé, the center's current director, the chemistry is right: "All executive committee members get along amazingly well, with everyone pulling in the same direction." His vision: to grow the center through applications from the outside world, making it a major European meeting point for fundamental researchers. Getting there, he predicts, will depend on consistently selecting high-quality research programs. "Excellence will always prime over cross-disciplinarity," Abbé says.

Ultimately, the center's impact will reach beyond EPFL. The global scientific community will gain a uniquely positioned center dedicated to the interface between mathematics, theoretical physics, and computer science. Local researchers will benefit as the center of gravity of theoretical research in Switzerland moves closer to Western Switzerland. And the local youth get a learning accelerator, allowing them to broaden their horizons, potentially catapulting them up to higher echelons of fundamental research.

Come join us

As physicist and Nobel Prize laureate Duncan Haldane put it in his keynote lecture during the inaugural festivities, it takes a lot of luck to stumble upon something that could potentially become a great discovery. Luck alone is, however, rarely enough. It takes

Figure 3. Aerial view of the EPFL campus with Lake Geneva and the Alps in the background. © EPFL 2022

preparedness – deep knowledge – to recognize something unusual for what it is. And finally, it takes a firm commitment to pursue the problem and have the result accepted by the scientific community.

You can cultivate preparedness and commitment. Luck is far more difficult to summon up. By bringing together great minds from a variety of academic, geographic, and personal backgrounds to collectively contemplate the fundamental challenges of our time, the Bernoulli Center is doing its part to increase the odds of that chance encounter, that serendipitous conversation, that, if correctly nurtured, could blossom into something truly remarkable.

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The integration of OEIS links in zbMATH Open

Dariush Ehsani, Matteo Petrera and Olaf Teschke

The transition towards an Open Data Platform enabled zbMATH Open to build a network of open resources. Important components in the evolving information system are mathematical research data, which are of quite heterogeneous nature. For their interlinking, zbMATH Open provides Application Programming Interface (API) solutions to offer mathematical research data to the community. Among other APIs recently implemented at zbMATH Open, the so-called Links API is aimed to document interconnections between our database and external platforms which display mathematical literature indexed at zbMATH Open. The Digital Library of Mathematical Functions (DLMF) has been our first partner and their data have been integrated in our database in 2021. Recently we interlinked with the second platform, the Online Encyclopedia of Integer Sequences (OEIS), a renowned digital database of number sequences that cites a lot of mathematical literature, especially from number theory and graph theory. The purpose of this short contribution is to announce and discuss the links to OEIS data in zbMATH Open.

1 Introduction

As outlined in [3, 4], zbMATH Open[1] is currently transformed into an information system that connects a broad variety of resources relevant for mathematics research, facilitated by the new opportunities provided in the framework of Open Access and Open Data. Mathematical Research Data are an essential additional information layer in such a network. Our commitment, in addition to being diversified on various development fronts, always aims to be exhaustive and publicly accessible, thus offering a complete, easily usable and open service. Our ultimate goal is to consolidate zbMATH Open as a solid reference point and a modern research tool for the entire scientific community.

Recently we developed Application Programming Interface (API) solutions to facilitate and optimize the open access to our mathematical research data. We already documented some of our APIs in previous publications, see, e.g., [5, 6]. At present we have three distinct APIs, an Open Archives Initiative Protocol for Metadata Harvesting (OAI-PMH) API,[2] designed to harvest the entire zbMATH Open dataset or some specific subsets of it, a Representational State Transfer (REST) API,[3] still in the staging phase, and the so-called Links API,[4] designed to document the interconnections between zbMATH Open and external digital platforms with mathematical contents.

The motivation behind the implementation of APIs at zbMATH Open is twofold. On the one hand, we want to provide the community with machine-readable tools to benefit from the open access of our data. On the other hand, we wish to expose and document the dynamic interactions between our data and those coming from other digital resources used by the community.

Among the potential users of our API endpoints are

1. bibliographic consumers (e.g., MathOverflow, Wikimedia) displaying references to scientific publications;
2. aggregators (e.g., research data infrastructures, SemanticScholar) extracting data;
3. archives (e.g., research/software digital archives) storing flows of data;
4. search engines (e.g., Google) implementing the OpenSearch standard;
5. researchers consulting literature for research purposes.

In this short note we announce the integration into the Links API of our second partner, the Online Encyclopedia of Integer Sequences (OEIS). In Section 2 we briefly discuss the main purpose and functionalities of this API. Section 3 will be focused on the role of the OEIS in our database. The concluding Section 4 is devoted to a discussion of future perspectives.

[1] https://zbmath.org

[2] https://oai.zbmath.org

[3] https://donald.zentralblatt-math.org/zbmath-api-test/docs

[4] https://donald.zentralblatt-math.org/linksapi-test/links_api

2 The Links API

The Links API has been developed between 2021 and 2022 to document the interconnections (more specifically, the *links*) between zbMATH Open and external platforms (called *partners*) which display and use mathematical bibliographic data (see Figure 1). As a matter of fact, there exist several well-established digital services serving the mathematical community which cite documents indexed in our database. Among such platforms we mention MathOverflow,[5] the Online Encyclopedia of Integer Sequences,[6] the Digital Library of Mathematical Functions[7] and arXiv.[8] In addition to having retrieved the bibliographic data of the matched documents from the platforms themselves, we have considered making the links to these digital resources directly accessible from our web page. We take this as a valid support for anyone using zbMATH Open for research purposes. All of this has been made possible thanks to the implementation of the Links API.

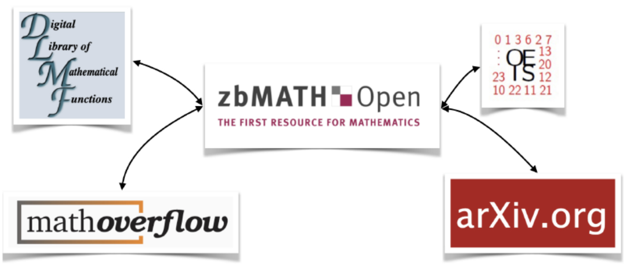

Figure 1. zbMATH Open interconnected with external platforms.

The current partners of the Links API are:

1. The Digital Library of Mathematical Functions (DLMF); the DLMF is a well-established web resource that enlarges and digitally translates the classical "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,"[9] edited by M. Abramowitz and I. A. Stegun in 1964. The DLMF presents its contents in 36 chapters, and the bibliography currently consists of almost 2,800 references,[10] of which about 75 % directly link to zbMATH Open.[11] For more details about the integration of the DLMF data into zbMATH Open we refer to [1, 5].
2. The Online Encyclopedia of Integer Sequences (OEIS); the OEIS is an online database of sequences of numbers. Please see Section 3.

The datasets coming from our partners are automatically updated in our database every six months. In the future we plan to integrate into the Links API our already existing datasets for MathOverflow and arXiv.

2.1 *Response body and endpoints of the Links API*

The JSON response body of the Links API is modeled on the Scholix metadata schema.[12] The models used to pack the data are explicitly reported in the API web interface. Scholix is a well-established framework to exchange information between data and literature links. The architecture of the schema is designed to allow for bulk exchange of link information, which contains all necessary data to keep track of bibliographic parameters identifying scholarly links.

The current version of the API offers twelve endpoints (seven GET routes, one DELETE route, two POST routes, one PATCH route and one PUT route):

- `GET /link`. It retrieves links for given zbMATH objects.
- `DELETE /link/item`. It deletes a link from the database.
- `POST /link/item`. It creates a new link related to a zbMATH object.
- `GET /link/item`. It checks relations between a given link and a given zbMATH object.
- `PATCH /link/item`. It edits an existing link.
- `GET /link/item/{doc_id}`. It retrieves links for a given zbMATH object.
- `GET /partner`. It retrieves data of a given zbMATH partner.
- `PUT /partner`. It edits data of a given zbMATH partner.
- `POST /partner`. It creates a new partner related to zbMATH.
- `GET /source`. It produces a list of all links of a given zbMATH partner.
- `GET /statistics/msc`. It shows the occurrence of primary MSC codes[13] (2-digit level) of zbMATH objects in the set of links of a given partner.
- `GET /statistics/year`. It shows the occurrence of years of publication of zbMATH objects in the set of links of a given partner.

These allow a diversified and targeted search thanks to the use of appropriate filters.

In Figure 2 we see an example of an excerpt of the JSON response body. It corresponds to the output coming from the search with input field `author=Levine` in the endpoint `GET /link`. As a result, it turns out that in the OEIS sequence A104246,[14] called

[5] https://mathoverflow.net
[6] https://oeis.org
[7] https://dlmf.nist.gov
[8] https://arxiv.org
[9] https://zbmath.org/0171.38503
[10] https://dlmf.nist.gov/bib
[11] The remaining 25 % of publications not linked to zbMATH Open refers to documents not indexed in the zbMATH Open database.
[12] http://www.scholix.org/schema/3-0
[13] Mathematics Subject Classification scheme 2020, https://msc2020.org
[14] https://oeis.org/A104246

"Minimal number of tetrahedral numbers ($A000292(k) = k(k+1)(k+2)/6$) needed to sum to n," has a document whose author is N. Levine among its references. This document is indexed as Zbl 0083.04002.

2.2 *Usage*

The Links API represents a tool that can be used in various ways, both from machines and humans. Here, we present some instances where a user of either a given partner or zbMATH Open can benefit from the service:

- A DLMF/OEIS user can easily access all bibliographic resources indexed at zbMATH Open relating to a specific topic of interest (e.g., a special function in the DLMF, a sequence in the OEIS).
- A researcher interested in a publication indexed at zbMATH Open can use our API to verify if and possibly where that publication is cited in DLMF/OEIS. A search of this type can also be diversified thanks to the filters that our routes offer. For example, one might be interested in identifying which DLMF/OEIS links are related to a particular Mathematical Subject Classification (MSC) code or a particular author.
- A researcher more interested in the history of mathematics can use the Links API to trace the bibliography related to a certain topic covered in DLMF/OEIS and observe the historical development of the topic itself in terms of the literature related to it.

Last but not least, thanks to the Links API, we are offering a new service directly visible at our public web site. Precisely, we are displaying all URLs linking a given document to the external web page of the corresponding partner (see Figure 3).

At present, we are displaying 6,312 links from zbMATH Open to the DLMF and 67,436 links from zbMATH Open to the OEIS. From these numbers we can therefore conclude that, as a byproduct, we are also increasing the visibility of our partners.

3 OEIS references in zbMATH Open

3.1 *Some OEIS figures*

The OEIS is an online database of sequences of numbers and most people use it to get information about a particular number sequence. It was launched online in November 2010, based on the book "A Handbook of Integer Sequences" by N. J. A. Sloane published in 1973.[15] This digital platform is very well maintained,[16] daily updated and currently contains over 358,000 sequences. The entry for each sequence gives among other metadata, the beginning of the sequence, its name or description, formulas, references to books and articles where the sequence has appeared.

[15] https://zbmath.org/0286.10001

[16] https://oeis.org/wiki/Welcome

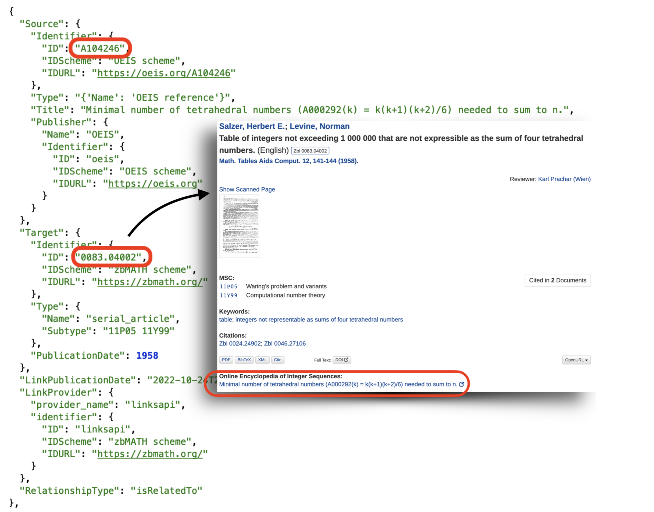

Figure 2. The JSON response body of `GET /link` with `author=Levine` and the corresponding document at zbMATH Open.

Figure 3. A book reviewed in zbMATH Open and linked to both the DLMF and the OEIS (as well as to a MathOverflow thread and mathematical software packages).

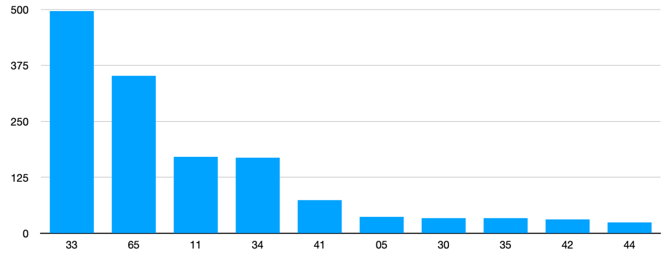

Figure 4. Distribution of primary 2-digit MSC codes (top ten) in the DLMF bibliography.

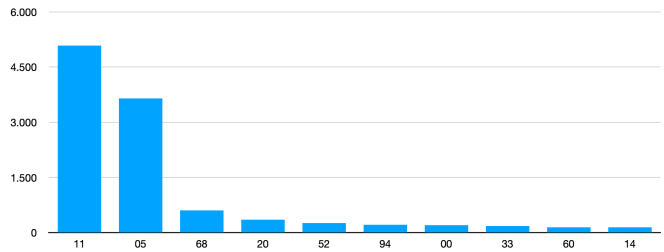

Figure 5. Distribution of primary 2-digit MSC codes (top ten) in the OEIS bibliography.

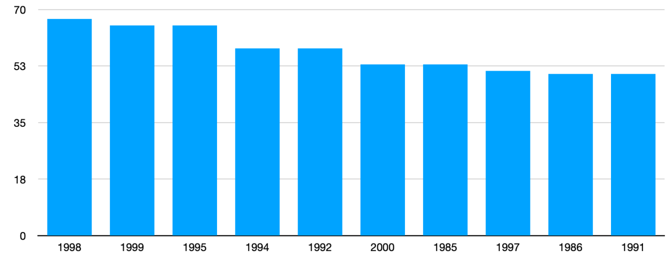

Figure 6. Distribution of years of publication (top ten) in the DLMF bibliography.

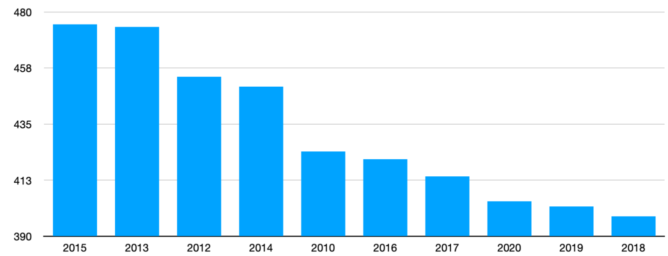

Figure 7. Distribution of years of publication (top ten) in the OEIS bibliography.

This last piece of data is what allowed us to match all bibliographic references in the OEIS with those in our database, thus constructing a dataset of all documents indexed at zbMATH Open cited by the OEIS. It turned out that there exist almost 60,000 references in OEIS matched to about 14,000 documents indexed in zbMATH Open (excluding the original printed version of Sloane's handbook which is, naturally, cited additionally for the majority of sequences). As a comparison, only about 2,000 documents (out of about 2,800) cited by the DLMF are indexed at zbMATH Open. As the figures indicate, the distribution is quite skewed: Several (38, as of January 1, 2023) publications are referenced in OEIS more than 100 times, while almost half (6567) documents have been cited just once in OEIS. A first analysis of their chronological and thematic distribution is given in the next section.

It is worth mentioning that the OEIS is very popular, and not only within the mathematical community, thanks to their commitment to disseminate the fascination and the ubiquity of number sequences. They are visible also on Facebook[17] and YouTube.[18]

3.2 Analysis of available data

Based on our current DLMF/OEIS dataset, it is possible to draw a simple statistical analysis about the documents referenced by our partners. In fact, the two statistics routes of the Links API show the distribution of both primary MSC codes (2-digit level) and years of publication of the references for a given partner.

As one may expect, the most frequent cited primary MSC codes for DLMF and OEIS are 33 (Special functions) and 11 (Number theory), respectively. Even though this result is not at all surprising, this confirms the consistency over time of the MSC assignment to documents. In Figures 4 and 5 we can see the distribution of the 10 most frequently cited primary MSC codes from DLMF and OEIS, respectively.

A similar analysis provides information about the most frequent years of publication of cited references. This analysis is summarized in Figures 6 and 7. From the comparison of data between DLMF and OEIS it may be argued that OEIS seems to pay more attention than DLMF to more recent publications (2010–2020).

4 Concluding remarks

The interlinking of research data resources like DLMF and OEIS provides an additional information layer which enables navigation and analysis beyond the classical publications. Both platforms where a somewhat natural starting point for interlinking, due to their relevance, widespread use, and ample literature references. But the ecosystem of mathematical research data is much more diverse and granular, as discussed earlier in this column [2]. The interlinking of further resources will continue, but may not always as straightforward. For example, the *L*-functions and modular forms database

[17] https://www.facebook.com/OEISF

[18] https://www.youtube.com/watch?v=LCWglXljevY

(LMFDB),[19] is a highly relevant and widely used database in number theory and algebraic geometry, but does not come along naturally with references to the literature. Vice versa, identifying its objects (e.g., modular forms, varieties, or representations) in the literature is something that is currently beyond the reach of automatization, so the only hope is that LMFDB entries are properly referenced when used (which is increasingly, but not systematically, the case). Further mathematical research data resources are currently standardized and integrated within the MaRDI initiative,[20] supported within the German National Research Data Infrastructure (NFDI) framework.

On the other hand, interlinking with zbMATH Open is clearly only a first step: In an ideal world one would, e.g., like to have an integrated formula search that would include full texts as well as functions or sequences. This is currently far from becoming reality due to a variety of issues, such as copyright restrictions for full texts, heterogeneous formats, or the lack of semantic information. Technical progress will surely help to overcome some obstacles, but the mathematical community can also support this by a variety of activities, e.g., by further voluntary engagement in the platforms, or appropriately referencing research data in publications. By interlinking the platforms, we hope to increase the awareness of the mathematical community of these services, and hopefully also motivate a further engagement.

[19] https://www.lmfdb.org
[20] https://www.mardi4nfdi.de

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ICMI column

Núria Planas

The ICMI Database Project: Mathematics curricula all over the world

The term curriculum has its roots in the ancient Roman culture and in the classical Latin verb *currere*, whose closer meaning in modern English could be 'to run.' The running or race metaphor is certainly inspiring. It suggests many ideas, from that of competition with those most rapid winning the race, to that of movement with learners, teachers, families and curriculum developers experiencing together the running. A certain meaning of *currere* can thus expand the educational experience of the curriculum beyond the syllabus, the course, the materials or the objectives of teaching and learning. *Currere* is not then the race to be won, but rather the race or path to be run. In the context of mathematics education and conceived broadly, *currere* can be viewed as a collaborative move towards the progressive understanding and learning of mathematical concepts and structures to be used in practice. For this to happen, educational systems all over the world need to select and interpret mathematical and pedagogical contents, human, symbolic and material resources …, which is far from trivial.

In 2011, the International Commission on Mathematical Instruction (ICMI) launched the Database Project.[1] The ultimate goal of this project is to build and update a database of the mathematics curricula at different levels of instruction over the world. Across the pre-tertiary stages, most mathematics curricula are largely regulated under country policies, whereas at the tertiary stage these curricula are often decided at the more local level of each particular university. The future consideration of university curricula, including mathematics in courses for prospective mathematics teachers, will need a multi-case approach within countries, differently from the common single-case approach for pre-tertiary curricula. At present, pre-tertiary mathematics curricula of 37 countries listed in alphabetical order from Argentina to United Kingdom are documented. The collection of data for each entry is organized through summaries – provided by country representatives – and links to institutional local webpages with curricular texts and guidelines. With the valuable support of the country representatives, this information remains updated over time and, when possible, expanded. *Currere* is, of course, more than links to curricular texts and guidelines. Nonetheless, as we learn about these data, we learn about mathematics curriculum too.

Two more related accounts of the strong interest of ICMI in issues of curricula are (1) the concluded ICMI Study 24, 'School mathematics curriculum reforms: Challenges, changes and opportunities' (for the discussion document, the study conference, and the conference proceedings, visit the website[2]), and (2) the forthcoming Springer volume, 'Mathematics curriculum reforms around the world: The 24th ICMI Study',[3] edited by Yoshinori Shimizu (Japan) and Renuka Vithal (South Africa). The ICMI Study 24 and the Database Project both adopt a defining cross-cultural lens in the approach to the mathematics curriculum. Considering these ICMI projects and their particular similarity in this regard, we may wonder: Why is the collection and presentation of cross-cultural curricular data important for mathematics education? I will next argue that it is important for two reasons at least, and I will illustrate these for the specific case of the Database Project.

A first reason is that collecting and presenting cross-cultural curricular data is important in order to learn from and through diversity. The Database Project reflects and represents an enormous diversity of curricula – and curricular cultures – covering mathematical contents that exist with respect to both student age and country variation. Representations of diversity are always important, because they allow us to foresee and discuss alternatives other than those initially imagined. If mathematics teachers, curriculum developers, stakeholders and researchers have the opportunity to situate their perspectives on curricula in relation to other perspectives in a larger context, they also have the opportunity to learn by contrast and perhaps infer some common lessons. A second reason is that collecting and presenting cross-cultural curricular data is important in order to understand the cultural nature of the mathematics curriculum, and likewise any other subject curriculum. The mathematics curriculum is cultural, not only because it is produced within institutionalized sites, but also because beliefs,

[1] https://www.mathunion.org/icmi/activities/icmi-database-project

[2] http://www.human.tsukuba.ac.jp/~icmi24

[3] https://www.springer.com/series/6351

values and unwritten rules mediate decisions about what to design, teach, learn and assess, and by whom.

Notwithstanding the foregoing, the collection and presentation of cross-cultural curricular data is indeed challenging. Again, I will argue that it is challenging for two reasons at least, which I will illustrate for the singular case of the Database Project. A first reason, especially present in this project, is the language for communication of curricular data. While some countries have English as one of the languages of their curricular texts, hence links to webpages in English are possible, these are exceptions. English summaries, prepared in collaboration with the country representatives, precede texts in the respective official languages. It is still feasible, however, that country-based teams consulting the Database Project may have some people who know one or two more languages other than those official in their context. When this is not the case, there can be other options. A Spanish team has easy linguistic access to the data from Argentina, Colombia, Costa Rica, Cuba, Paraguay, Peru …, although there may be different meanings attached to the same words in use across countries. The Spanish word *evaluación* is a clear example, with some countries using the word in institutional documents to express a focus on assessment of learning and some others to involve rating and performance of programs and educators. A second reason that makes the cross-cultural nature of the Database Project challenging is the necessary caution and concern over the course of any comparison or association. Cultural and societal differences between Eastern and Western approaches to pedagogy, mathematics and mathematics education, for instance, cannot be disregarded in the cross-reading of some entries like those from France and China. As said above, a common language at the level of words and sentences does not imply common interpretations.

Be it a challenge, a strength or both, the very conception of the Database Project makes it an ongoing project that is never complete, because of continuous expansion and updating. This is a 'match' with *currere* itself. The mathematics curriculum is also an ongoing project, never finalized regardless of the country. Any set of decisions, texts and actions is alive. It is regularly assessed and it will be revised after some years for redesign and, hopefully, improvement and adjustment to societal changes and to newer research findings.

If you are interested in adding to the Database Project, or if you have comments about some of its entries, please let me know at nuria.planas@uab.cat. I would love to hear from you! You may have experience of other projects, either ongoing or completed, that share some common challenges with the Database initiative. Just this past December, Tomás Recio sent written reflections gained over the course of his participation in European projects with researchers from different countries, all considered as belonging to Western traditions, who mentioned rather different aspects of one curricular mathematical content but named them the same. That message from Tomás resonated with my current participation in a mathematics education research and developmental project with colleagues in England and Germany. When I talk about the Thales theorem as a curricular content in the secondary school mathematics in Spain, I will support my talk with the drawing of two lines in a plane, two segments in one of them, some parallels … to conclude around a proportionality between ratios. If it is my colleague in Germany, for instance, who talks about Thales theorem in her context of secondary school mathematics, she will support her talk with the drawing of a circumference, three distinct points on it with two of them representing the extremes of a diameter … to conclude around the creation of a triangle with a right angle. Thales, born in the wealthy Greek Miletus, is definitely credited with more than two theorems, all beautiful and basic in geometry education. But then again, the Database Project is for connecting and making meaning of mathematical curricular choices that are cultural.

Acknowledgements. This ICMI column is an expanded version of the text published in the ICMI Newsletter – December 2022.[4] I have introduced ideas from an email by Tomás Recio after his reading of the text in that Newsletter. My heartfelt thanks to Tomás.

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[4] https://www.mathunion.org/icmi/icmi-newsletter-december-2022#on-page-6

ERME column

regularly presented by Jason Cooper and Frode Rønning

In this issue, with a contribution by
Mario Sánchez Aguilar, Linda Marie Ahl, Morten Misfeldt and Boris Koichu

CERME Thematic Working Groups

We continue the initiative of introducing the CERME working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for people working in pure and applied mathematics. Our aim is to enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introducing CERME Thematic Working Group 23 – Implementation of Research Findings in Mathematics Education

Mario Sánchez Aguilar, Linda Marie Ahl, Morten Misfeldt and Boris Koichu

Over five decades, the field of mathematics education research has generated a wealth of innovations aimed at improving the teaching and learning of mathematics. However, while mathematics education research has produced solid findings related to fundamental phenomena in teaching and learning mathematics (see Dreyfus [2] for an overview of solid findings published on the pages of the EMS Newsletter), has constructed elaborate theoretical frameworks to investigate and analyze teaching and learning, and has developed rich and consistent suggestions for didactic design, the effect of all these on teaching on a large scale has nevertheless been weak. How the innovations that work well in a research laboratory could be applied in practice remains an open and challenging problem to solve. Addressing this problem, Thematic Working Group 23 "Implementation of Research Findings in Mathematics Education" (TWG23) is a forum dedicated to presenting and discussing studies focused on elucidating the enablers and general conditions that favor or inhibit the implementation in practice of research findings and innovations generated in our research field.

There are obvious reasons for focusing on implementation and implementability in mathematics education. Indeed, many researchers and practitioners have identified issues related to scaling up and making work and results of mathematics education research available to a larger audience. Even though these issues are omnipresent in mathematics teaching, they are rarely addressed as a stand-alone issue. This disparity was the outset of establishing TWG23 in 2017. Hence TWG23 elevates covert concerns about mathematics education research as being "usable" and "making a difference in practice" to a more overt phenomenon named "implementation."

The papers presented in TWG23 illustrate experiences of implementation of research findings in practice – either small or large scale – where the object of the implementation and the implementation process are clearly identified. For instance, there are studies informing how particular treatments, interventions, or didactic designs work in different contexts and with different populations. Likewise, theoretical papers addressing diverse characterizations of implementation in mathematics education have been presented. Overall, the discussions in the group have evolved around the question: how can we bring the accumulated research knowledge into practice?

We posit that the work of the TWG23 is relevant to the readers of the EMS Magazine – particularly those mathematicians involved in tertiary mathematics education and in projects aimed at enhancing teaching mathematics at school level – because of the difficulties that students, including high-achievers, experience in moving from secondary to tertiary education. Such a transition has been identified as a major issue for mathematics departments across Europe and their students (Koichu–Pinto [5]), and described as an educational crisis (Gregorio et al. [3]). This situation suggests a reconsideration and reform of both school and university mathematics teaching practices. Familiarity with research findings on implementation of educational innovations, and even participation in mathematics education implementation research, may be instrumental for mathematics departments and university mathematics teachers interested in improving their students' educational experiences. The interest in implementing innovations in mathematics teaching at the tertiary level, and reporting the results in mathematics didactic journals, can be seen in an ongoing systematic review of the field of implementation research. (The systematic review is done in the project *Implementation research as an emerging field of mathematics education,* financed by the *Swedish Research*

Council.) Preliminary results show that the studies that address the tertiary level are few, but that most have been written in recent years, which we interpret as increased interest.

Evolution of TWG23

TWG23 is one of the newest thematic working groups of the CERME congress. Its first appearance was at the CERME10 congress (2017), led by Uffe Thomas Jankvist (Aarhus University, Denmark), Mario Sánchez Aguilar (National Polytechnic Institute, Mexico), Jonas Bergman Ärlebäck (Linköping University, Sweden) and Kjersti Wæge (Norwegian University of Science and Technology, Norway). At CERME10, the working group undertook initial attempts to make sense of "implementation" as a phenomenon. In the call for papers for TWG23 at CERME10, the construct "implementation research" was considered rather broadly, as systematic exploration of different kinds of didactical design, from task design, lesson design, teaching modules, and courses, to the design of entire programs at all educational levels. Furthermore, "implementation research" was inclusively operationalized as research on aspects of developmental projects, intervention projects, as well as research on aspects of the development and use of educational media (e.g., textbooks, software, and computer-enhanced learning platforms). Fourteen papers and one poster were presented in TWG23 at CERME10. Most of them reported on small-scale studies addressing aspects of how adapted research results and findings can inform practices in schools or other educational settings.

For CERME11, held in 2019, the focus of TWG23 drifted from discussing particular small-scale projects to an effort to articulate what implementation research in mathematics education actually is or can be. Twelve papers and two posters were presented and served as a basis for theorizing implementation research. At this point in the development of the TWG, several bibliographical references were put forward for clarifying and organizing the key notions related to implementation research in mathematics education. In particular, the work of Rogers [6] on diffusion of innovations, of Century and Cassata [1] on conceptualizing implementation of innovations, and of Stein et al. [7] on stages of implementation were brought to the center of the discussion. A first collective attempt within TWG23 to formulate a chain of definitions of the key concepts of "innovation," "implementation" and "implementation research in mathematics education" was undertaken. At the end of this collective discussion, the group formulated the following definition:

Implementation is a change-oriented process of adapting and enacting a particular resource (e.g., an idea, a tool, an innovation, a framework, a theory, an action plan, a curriculum, a policy) that occurs in partnership of two communities, *a community of the resource proponents* (CRP) and *a community of the resource adapters* (CRA). These communities differ but can intersect. At the beginning of the process, the CRP has the ultimate agency over the resource. The process of adapting a resource by CRA includes some of the following: (1) constructing an agency over the resource, (2) changes in ways of communicating, and (3) changes in practice. Accordingly, *implementation research in mathematics education* is research that focuses on aspects of implementation, as specified above, in the context of mathematics education.

After the CERME11 congress, Uffe Thomas Jankvist and Jonas Bergman Ärlebäck left their positions as TWG23 leaders, being replaced by Ana Kuzle (University of Potsdam, Germany) and Morten Misfeldt (by this time affiliated to Aalborg University, Denmark). Two editorial projects related to TWG23 and led by some of its members emerged. First, a new research journal entitled *Implementation and Replication Studies in Mathematics Education* (IRME) was established; Uffe Thomas Jankvist, Mario Sánchez Aguilar, Morten Misfeldt, and Boris Koichu assumed the positions of the editors. IRME focuses on implementation and replication research that communicates and investigates initiatives aiming to improve the teaching and learning of mathematics by using knowledge from mathematics education research and by re-implementing it in new contexts. Second, the thematic issue "Implementation and implementability of mathematics education research" in the research journal *ZDM – Mathematics Education* was conceived, with Boris Koichu, Mario Sánchez Aguilar, and Morten Misfeldt as guest editors [4].

The most recent meeting of TWG23 took place in the online congress CERME12 (2022). At this stage, Ana Kuzle and Kjersti Wæge stepped down from their positions as group leaders and were replaced by Boris Koichu (Weizmann Institute of Science, Israel) and Rikke Maagaard Gregersen (Aarhus University, Denmark). Rikke Maagaard Gregersen participated in the planning of TWG23 at CERME12, but was unable to participate in the congress so that Linda Marie Ahl (Uppsala University, Sweden) stepped in. The focus of the group's discussions was broadened and deepened at this online meeting, thus reflecting the fact that the participants of the TWG have gained more experience in implementation research. Notably, the occurrence of papers on large-scale projects increased significantly, which paved the way for broad and deep discussions. TWG23 at CERME12 received 18 contributions (15 papers and three posters). The contributions were organized in five thematic categories:

- Implementation of problem-solving and problem-posing approaches.
- Implementation of teaching models and teachers' perspectives on implementation.
- Conditions for sustainable implementations.
- Diagnostics tasks, instructional sequences, and curriculum design.
- Implementation of programming, computational thinking, and other digital technologies.

Recent discussions in TWG23

The most recent discussions within TWG23 focused on issues of scale and upscaling, particularly on the purposes that small-scale and large-scale implementation-related studies can attain under a theoretical umbrella of implementation research. There has also been a focus on the conceptualization of "stakeholder" and how this notion can be used to refine different types of analysis of implementation projects. Another recent discussion has been related to the notion of "change" in implementation research, and the need for theories of change that could be used to design, understand, and evaluate implementations.

In connection to scale and upscaling, the participants of TWG23 at CERME12 reinforced the need for both small-scale and large-scale studies because they play different roles in the accumulation of knowledge about implementation in mathematics education. The group pointed out the need to further discuss the strategies required to make decisions about which types of studies can provide the most useful information for different parts of the implementation process. Also, it is necessary to further clarify the concept of "stakeholders" and progress our knowledge base for involving more stakeholders, including mathematics teachers, mathematicians and mathematics education researchers in implementation projects. In relation to "change," the TWG23 participants at CERME12 agreed that the tension between "intended change" and "achieved change" in an implementation project is a delicate question of interest for our research field. We thus see a continuing need to discuss the question of how program theory and theory of change can be used to design, understand, and evaluate implementations.

These discussions about the notions of scale, stakeholder, and change will hopefully continue when TWG23 meets again at CERME13 in Budapest in 2023. We are expecting to have rich discussions in this new meeting of the TWG23 which could allow us to further develop key notions of implementation research and broaden our knowledge about the factors that influence the implementation of educational innovations based on mathematics education research. We invite everyone with an interest in implementation research to contribute their ideas to this vibrant and continually developing thematic working group.

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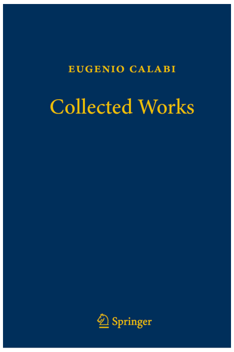
Eugenio Calabi – Collected Works by Jean-Pierre Bourguignon, Xiuxiong Chen and Simon Donaldson (Eds.)

Reviewed by Joel Fine



Eugenio Calabi is a hugely important figure in modern differential geometry, whose ideas have strongly influenced the field for over 70 years and counting. This volume brings together nearly every paper he wrote, together with a collection of commentaries and personal reminiscences on his career.

Of course, putting Calabi's papers in a single volume was always going to be a good idea, but what raises this collection to an even higher level is the accompanying commentaries. They are written by mathematicians who are experts in the areas that Calabi revolutionised, but just as importantly who also know the man himself. They give skilful surveys of several of the topics which Calabi worked on. They also contain some beautiful anecdotes which bring the telling of Calabi's work to life.

The human side of mathematics is, naturally, completely absent from research texts; and yet, the doing of mathematical research is deeply intertwined with personal interaction. For example, I was fascinated to learn from Claude LeBrun's commentary about an exchange between Calabi and Louis Nirenberg. "I am telling you, and repeat after me:" said Nirenberg, "One can't prove existence theorems without a priori estimates." In this single quote, one can see the genesis of modern Kähler geometry! Reading about moments like these, and there are several in the commentaries, was an absolute delight.

The mathematical content of the commentaries is also exemplary. Like most mortals who are interested in Calabi's work, I only knew one of the areas he worked in in any detail (for my part, canonical metrics in Kähler geometry). It was fabulous to be guided through Calabi's contributions in other fields. To be frank, holding the collected works of a mathematician of the stature of Calabi is quite intimidating. Without the commentaries, it might have been difficult to start reading papers on less familiar topics. Even though Calabi writes with ease and elegance, an outsider to a field might not know where to start or how the papers relate to each other. The commentaries, however, make it simple. They summarise Calabi's results and put them in context. They are a beautiful invitation to read the papers themselves, and not just the more familiar ones. In my case, I spent several happy hours learning about affine differential geometry, something that would certainly never have happened if I had not picked up this volume.

The commentaries also put Calabi's work in historical context. In reading them, one thing that shines out above all else is how Calabi's mathematics was frequently many years ahead of its time. His approach to canonical metrics in Kähler geometry, beginning in the 1950s, gave birth to an entire field of research that is perhaps more active today than at any point previously. Some of the questions Calabi asked about Kähler metrics are only now beginning to be fully understood. Indeed, as LeBrun writes "[…] Gene's visionary early work truly seems like a piece of twenty-first century mathematics that somehow landed in the middle of the twentieth century." Another example of Calabi's pioneering style is his notion of weak subsolution for linear elliptic partial differential equations which, in the words of Lawson, "presages modern viscosity theory more than twenty years before its emergence as a major branch of analysis."

Alongside these theoretical paradigm shifts, Calabi also specialised in the construction of examples. Again, the commentaries help put these examples in context. For instance, in joint work with Beno Eckmann, Calabi found the first examples of compact simply-connected complex manifolds which are not algebraic; as Lawson says, this was a transformational result at the time. I was also surprised to learn that Calabi was the first person to build hyperkähler manifolds; indeed, as Bourguignon, Chen and Donaldson point out, he was the first person to consider them at all.

The collected works of Eugenio Calabi are worthy of a place on the bookshelf of any person with a serious interest in differential geometry. Having said that, mine hasn't made it back on to the bookshelf since I opened it!

Jean-Pierre Bourguignon, Xiuxiong Chen and Simon Donaldson (Eds.), *Eugenio Calabi – Collected Works.* With contributions by Shing-Tung Yau, Blaine Lawson, Marcel Berger and Claude LeBrun. Springer, 2020, 843 pages, Hardback ISBN 978-3-662-62133-2, eBook ISBN 978-3-662-62134-9.

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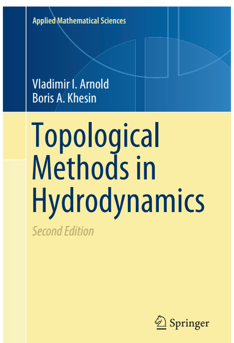
DOI 10.4171/MAG/146

Topological Methods in Hydrodynamics
by Vladimir I. Arnold and Boris A. Khesin

Reviewed by Daniel Peralta-Salas



The theoretical study of fluid flows is a vast area of research that involves many different mathematical disciplines, ranging from the theory of partial differential equations to dynamical systems and differential geometry. More than 250 years after their formulation, the Euler equations (which describe an ideal incompressible fluid) and their viscid counterpart, the Navier–Stokes equations (introduced independently by Claude-Louis Navier and Gabriel Stokes during the first half of the 19th century) still contain a wealth of fundamental open problems. While there are numerous and excellent monographs focusing on the analytic aspects of the equations that govern fluid motions, until recently one can hardly find textbooks on the geometric and topological aspects of fluid flows (which are very rich and significant for understanding hydrodynamics). In 1998 this important gap was filled by the first edition of the book "Topological Methods in Hydrodynamics" by V. I. Arnold and B. Khesin. It is difficult to overestimate the impact this monograph had on those mathematicians who are interested in understanding the equations of fluid mechanics from a geometric viewpoint, as it provides a comprehensive introduction to most of the more remarkable achievements in the area. This includes Arnold's geodesic formulation of the Euler equations, the structure of steady Euler flows, the topological interpretation of helicity and other asymptotic invariants, the effects of the curvature of the group of volume-preserving diffeomorphisms on hydrodynamic instabilities or the fast dynamo problem. More than twenty years later this important book has seen its second edition, the most remarkable novelty being the addition of a very valuable fifty-page appendix that introduces the most significant developments in the area since the publication of the first edition of the book.

Arnold and Khesin's book is the only monograph that presents a thorough introduction to topological fluid mechanics, a young area of research that flourished after the foundational works of Arnold and Moffatt in the 1960's. The interest in the topological and geometric aspects of fluid dynamics probably dates back to Lord Kelvin, who developed an atomic theory in which atoms were understood as knotted vortex tubes in the ether and showed that vorticity is transported by the fluid field in the context of ideal flows, thus implying the preservation of all the vortex structures. In modern times, topological hydrodynamics was considerably developed after the works of Arnold and Moffatt. Arnold realized that the Euler equations of hydrodynamics can be understood as geodesic motions on the infinite-dimensional group of volume-preserving diffeomorphisms and Moffatt unveiled the connection between helicity and the entangledness and knottedness of the fluid. The book under review covers a vast panorama of developments and results in the area, and is an indispensable reference for any researcher interested in fluid mechanics from the geometric, topological or Hamiltonian perspectives. It is impossible to summarize the contents of this book in a few lines, so next I aim to present its chapters, highlighting some landmarks that are introduced in each chapter (paying the price of losing many other interesting results in this short presentation):

Chapter I. Group and Hamiltonian structures of fluid dynamics. This chapter is mainly focused on the study of the Euler equations of ideal fluids from the viewpoints of group theory and Hamiltonian mechanics. A significant part is devoted to developing Arnold's theory relating the Euler equations with the geometry of the infinite-dimensional Lie group of volume-preserving diffeomorphisms of the fluid flow domain. In an important article published in 1966 Arnold showed that the dynamics of an ideal fluid flow can be described by the geodesics on the aforementioned Lie group endowed with the right-invariant metric given by the kinetic energy. This chapter provides a detailed presentation of this result and how it fits within the general framework of the Euler–Poincaré equations for Hamiltonian systems on Lie groups whose action is (right-)invariant, other remarkable examples being the dynamics of the rigid body or the KdV equation. Using this geometric formulation, Ebin and Marsden proved in 1970 the local-in-time existence of solutions to the Euler equations on compact manifolds, both in Sobolev and Hölder classes. The chapter also deals with conserved quantities of the Euler equations (mainly the Casimirs of the adjoint action of the group of volume-preserving diffeomorphisms) and the group setting of ideal magnetohydrodynamics.

Chapter II. Topology of steady fluid flows. This chapter is concerned with the stationary solutions of the Euler equations. It presents in a very detailed way two gems proved by Arnold in the mid 1960s. The first one, nowadays known as Arnold's structure theorem, describes the topological and dynamical structure of analytic 3D fluid steady states in bounded domains whose Bernoulli function is not constant. Under these assumptions, this theorem shows that the Euler flows exhibit the same properties as integrable Hamiltonian systems with two degrees of freedom on an energy hypersurface: presence of subdomains covered by invariant tori or invariant cylinders supporting dynamics that is conjugate to a linear one. In the context of 2D steady states, the second result presented here is Arnold's stability theorem, which provides a sufficient condition for a planar stationary solution to be Lyapunov stable with respect to the L^2-norm of the vorticity. This remarkable result exploits a new variational characterization of steady states discovered by Arnold (in terms of the critical points of the energy functional on the coadjoint orbits of the group of volume-preserving diffeomorphisms) and the Hamiltonian formulation. The topology of the famous Arnold–Beltrami–Childress (ABC) flows, properties of the linearized Euler equations, and Nadirashvili's surprising construction of wandering solutions to the 2D Euler equations on annular regions are also discussed.

Chapter III. Topological properties of magnetic and vorticity fields. In this chapter the authors review several results on the topology of solenoidal fields and how it affects energy relaxation in physical processes, such as ideal MHD evolution. This topology is described using functionals on the space of vorticity fields, most of them of "asymptotic type," which means that the functional is defined using a knot invariant, the integral curves of the field and suitable averages. The chapter presents the helicity functional and its topological interpretation in terms of the linking number discovered by Moffatt in 1969, as well as the connection with the asymptotic linking number introduced by Arnold in 1973. Arnold proved a beautiful theorem asserting that these two apparently very different quantities (the former defined using the Riemannian metric and differential forms, and the latter using the flow of the field and a limit process) coincide, thus extending Moffatt's topological interpretation to arbitrary solenoidal fields. Other remarkable theorems covered in this chapter include lower bounds on energy under ideal relaxation using the helicity and Freedman–He's asymptotic crossing number, and Freedman's remarkable proof of the Sakharov–Zeldovich energy minimization conjecture.

Chapter IV. Differential geometry of diffeomorphism groups. This chapter deals with the geometry (from a Riemannian viewpoint) of the infinite-dimensional group of volume-preserving diffeomorphisms, endowed with the right-invariant metric given by the L^2-norm of the velocity field. It pays special attention to the curvature of the group and how it is related to instabilities of the Euler dynamics. Under suitable assumptions, the curvature of the group of volume-preserving diffeomorphisms is negative along many directions, which in view of the geodesic nature of the Euler flow on that group leads to exponential separation of the Lagrangian trajectories of the fluid. An appealing consequence of this claim is that the weather forecast becomes unreliable after a sufficiently long time, a striking consequence of the Riemannian geometry of the diffeomorphism group! Other interesting studies, such as the existence of conjugate points on the aforementioned Lie group, Shnirelman's description of the diameter of the diffeomorphism group, and Brenier's theory of generalized flows are also discussed.

Chapter V. Kinematic fast dynamo problems. This chapter deals with the equation describing the evolution of magnetic fields in magnetohydrodynamics, i.e., the kinematic dynamo equation. When the fluid is a perfect conductor, the magnetic diffusivity is zero and the magnetic field is transported by the velocity-field flow; in the general case the diffusivity appears as a diffusion term of heat type. The chapter describes several results on the existence of fast dynamos (both for the dissipative and non-dissipative models), which are solenoidal fields that give rise to exponential growth in time of the L^2-norm of the magnetic field. This includes a thorough presentation of the connections between the exponential dynamo growth and the Lyapunov exponents, the topological entropy and homoclinic intersections of the velocity field. The authors also present some discrete models (mainly area-preserving diffeomorphisms on surfaces) of fast dynamos, highlighting a very detailed discussion of Arnold's cat map (a paradigmatic model of an Anosov diffeomorphism on the 2-torus). The antidynamo theorem proved by Cowling and some of its generalizations are also discussed.

Chapter VI. Dynamical systems with hydrodynamic background. The final chapter of the first edition of the book is a survey of various partial differential equations that can be studied from the group-theoretic viewpoint presented in Chapter I, i.e., as geodesics of an infinite-dimensional Lie group of symmetries endowed with a right-invariant metric. This includes the KdV equation (related to the Virasoro group), the equations of gas dynamics and compressible fluids, and the filament and nonlinear 1D Schrödinger equations. While the material covered in this chapter is not directly related to the Euler or Navier–Stokes equations, it is very valuable in the sense that it shows the power of the general framework of geodesic motions on Lie groups for studying the evolution of some PDEs.

Appendix. Recent developments in topological hydrodynamics. This chapter is the main new addition to the second edition of Arnold–Khesin's book. It contains an update of the new developments in the area of topological fluid mechanics since 1999 (so we can strictly speak about XXI century mathematics). The material covered by this chapter is huge, so necessarily not very detailed, but with a vast number of references and indications that certainly help the reader to find further results on each subject. The chapter summarizes new remarkable achievements in all the topics of previous chapters, a non-exhaustive list including: the recently obtained classification of Casimirs for 2D and 3D vorticities, the extension of Arnold's geodesic framework to the context of weak solutions of the Euler equations (exhibiting vortex sheets), the realization theorems for knotted vortex lines and tubes in Beltrami flows, a KAM type approach to study ergodicity and mixing properties of the Euler flow, and the connection between problems of optimal mass transport and the evolution of ideal fluids.

Overall Arnold and Khesin's book is a beautiful and extensive introduction to fluid mechanics from a geometric viewpoint. It is a pleasure to read and each chapter contains very valuable material not only for those mathematicians working with the equations of hydrodynamics, but for any researcher interested in the connections between analysis, geometry and topology. I am sure that any professional mathematician can find food for thought in some of the gems that are presented in this monograph. Certainly this was my case as a graduate student in Madrid twenty years ago.

Vladimir I. Arnold and Boris A. Khesin, *Topological Methods in Hydrodynamics.* Second edition, Applied Mathematical Sciences 125, Springer, 2021, 475 pages, Hardback ISBN 978-3-030-74277-5, eBook ISBN 978-3-030-74278-2.

Daniel Peralta-Salas is a senior scientific researcher at the Institute of Mathematical Sciences in Madrid and chair of the group "Differential Geometry and Geometric Mechanics." His research lines concern the connections and interplay between dynamical systems, partial differential equations and differential geometry. This includes different topics in geometric and topological hydrodynamics.

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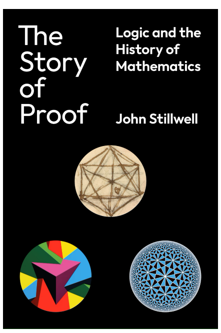
DOI 10.4171/MAG/136

The Story of Proof by John Stillwell

Reviewed by Adhemar Bultheel



Technically, we could reduce the essence of mathematics to the derivation of new theorems from previous knowledge in a coherent and logical way. There has been a tremendous shift in what and how things are proved. The *Elements* of Euclid had an elementary geometric axiomatic system allowing to give visually attractive proofs. Twenty-three centuries later Turing machines and the incompleteness theorems of Gödel had abstract proofs about what could be obtained within a logic system. With this evolution in mind, Stillwell sketches the history of mathematics and how the foundations of proofs, i.e., the axiomatic systems, were sharpened. There is definitely mutual influence of the formalisms and techniques used in the proofs and the emergence of new disciplines like algebra, calculus, topology, graphs, and others. Therefore, describing the history of what is proved and how it is proved, is just a history of mathematics itself, with proof as the binding tissue and that is what Stillwell does in this book.

The 16 chapters are essentially chronologically organized, but this is not a classic history book, with data and bios of mathematicians, and it is not the usual popularizing mathematics book either. For obvious reasons many proofs are included when not too complicated. The title is well chosen, it is an entertaining story being told.

The Pythagoras theorem was long known before Pythagoras. In the Western world as a geometric construction, while for Chinese and Indians, it was more computational and arithmetical (a set of Pythagorean triples). Euclid's *Elements* was the first attempt to formalize the knowledge of his time with an axiomatic system and formal geometric proofs. This remained a standard for many centuries, until a complete geometric axiomatic system was set up in Hilbert's time. Along with the invention of perspective, projective geometry was the first to raise doubts about the parallel axiom.

When the Muslims introduced algebra, new techniques became available, and algebraic equations described polynomials and geometric curves. In search of the roots of polynomials, complex numbers became a natural extension of the number system, and with a modification of the computational rules of addition and multiplication came also the generalization to algebraic structures like fields, rings, and vector spaces. The latter with a strong geometric interpretation, which naturally leads to the algebraic geometry of Fermat and Descartes.

The computation of tangents, length of a curve, area of a surface, or volumes, involved infinite sums and limits, which made the introduction of calculus necessary. With the controversy whether or not computing with infinitesimals is allowed, people started to rethink the notion of real numbers and other aspects of number theory like finite number fields, and the use of complex functions in number theory. Calculus also allowed to define continuity, enabling the mean value theorem and the fundamental theorem of algebra.

But geometry was not forgotten. Spherical geometry was not new, but removing the parallel axiom allowed, for example, hyperbolic geometry, and calculus also allowed differential and Riemannian geometry. Riemann also proposed to see algebraic curves as (Riemann) surfaces. Surfaces with cuts and holes can be modelled with graphs and the invariants are studied in topology. Tilings and graphs initiate excursions in group and knot theory.

As all these tools became available, attention converged on set theory, introducing notions of countability and ordinal numbers, which eventually resulted in sound axiomatic systems for numbers, geometry and set theory, including a foundation of induction proofs which had been used before on an intuitive basis. Also, just picking an infinite set of elements requires in certain situations the axiom of choice or equivalent assumptions like well ordering, or Zorn's lemma, with an important impact on numbers, analysis, graphs, measures, algebra, and sets.

With this level of abstraction and excavation of the very foundations of mathematics, wondering what is indisputably true and what not, the formalization of the logic used in the proofs became a natural step to take. Starting with Boolean algebra, the computations with propositions and predicates became mechanical so that a machine could do it. But are these formal systems complete? Will it be possible to prove every true theorem in this system? And if a machine can do it, will the machine find out whether the theorem is true or false after a finite number of operations? Disappointing negative answers came in the 20th century from Gödel and Turing.

This illustrates that there is a strong interaction between the way mathematics evolved, and the way things are proved. Stillwell does include many proofs. In the earlier chapters, they are very simple, but towards the end, where abstraction has taken over, they require some mathematical maturity. Another tool that is used a number of times are mathematical models, not only when it comes to abstract formalities at the end, but also in the beginning in connection with non-Euclidean geometry, complex numbers, etc.

The first half of the book is certainly readable with some basic mathematical knowledge, but towards the end, it requires a reader that is really interested in the foundations to appreciate the discussion. Proof is essential in mathematics because it is not only important to know what is true and what not, but also why it is true. What and why mathematical statements are true has evolved considerably over the centuries. That we arrived at the current state of mathematics is not a coincidence. There is some logical, natural evolution, and that being clearly explained is what even professional mathematicians will appreciate in this entertaining story book.

John Stillwell, *The Story of Proof. Logic and the History of Mathematics*. Princeton University Press, 2022, 458 pages, Hardback
ISBN 978-0-691-23436-5, ebook ISBN 978-0-691-23437-3.

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DOI 10.4171/MAG/137

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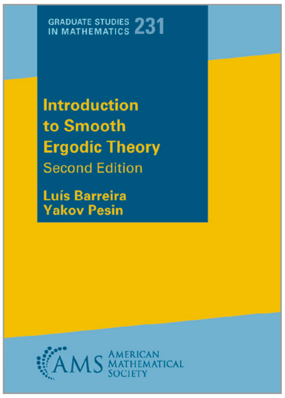

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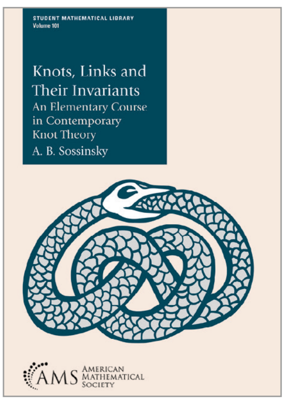

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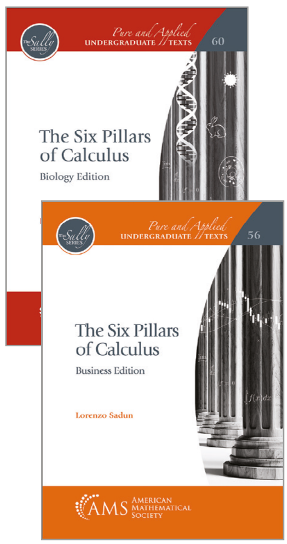

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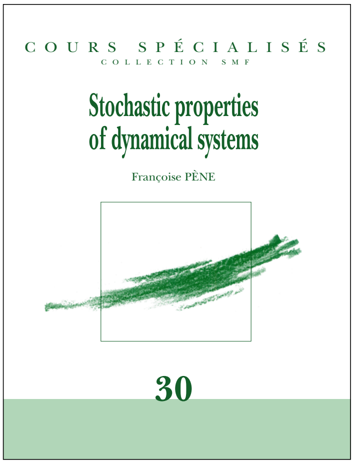
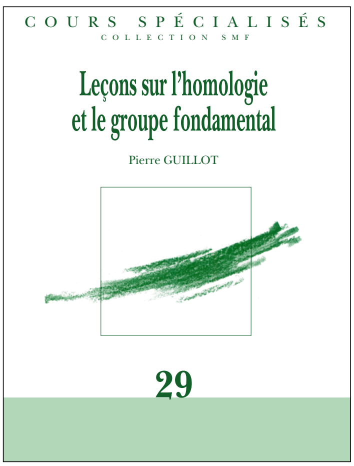


Stochastic properties of dynamical systems

F. Pène (Université de Brest, France)

Vol. 30
ISBN 978-2-85629-967-8
2022 – 276 pages – Hardcover. 17 × 24 – Public*: 54 € – Members*: 38 €

This book provides an introduction to the study of the stochastic properties of probability preserving dynamical systems. Only the usual knowledge of the first year of a Master's degree is required. Many reminders are given. The definitions and results are illustrated by examples and corrected exercises. The book presents the notions of Poincaré's recurrence, of ergodicity, of mixing. It enlights also existing links between dynamical systems and Markov chains. The final objective of this book is to present three methods for establishing central limit theorems in the context of chaotic dynamical systems: a first method based on martingale approximations, a second method based on perturbation of quasi-compact linear operators and a third method based on decorrelation estimates.



Leçons sur l'homologie et le groupe fondamental

P. Guillot (Université de Strasbourg, France)

Vol. 29
ISBN 978-2-85629-965-4
2022 – 334 pages – Hardcover. 17 × 24 – Public*: 60 € – Members*: 42 €

Cet ouvrage reproduit, en les complétant, des notes de cours donnés par l'auteur en M1 et en M2 à l'université de Strasbourg en topologie algébrique. Après des préliminaires concernant l'homotopie, le groupe fondamental, les catégories et les foncteurs, on y aborde l'homologie des complexes simpliciaux puis des espaces topologiques généraux. Les applications classiques sont traitées (théorème de Brouwer, théorème de la boule chevelue, caractéristique d'Euler des solides platoniciens...) et on donne une introduction à la dualité de Poincaré. Dans une troisième partie plus avancée, l'algèbre homologique est étudiée plus en profondeur, avant que la théorie des faisceaux ne soit développée. Le cours se conclut sur la démonstration du difficile théorème dû à Georges de Rham qui fait le lien entre homologie et formes différentielles. Le cours s'adresse aux élèves de M1, et suppose simplement une connaissance des espaces métriques, ainsi que le bagage algébrique usuel vu en licence.

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