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EMS Magazine

Paweł Dłotko
On the shape that matters –
topology and geometry in data science

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The Maison Poincaré –
a maths museum in France

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A woman in a man's world
An interview with Elisabetta Strickland

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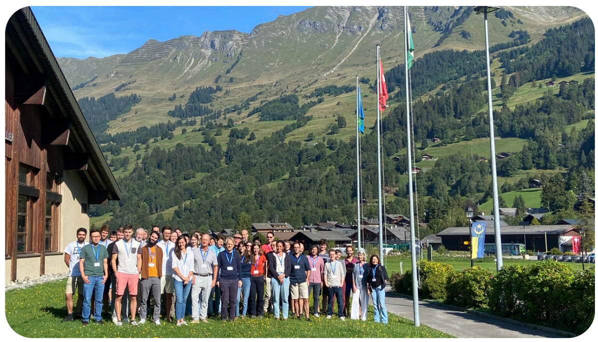

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3 A message from the president
Jan Philip Solovej

4 Brief words from the editor-in-chief
Fernando Pestana da Costa

5 On the shape that matters – topology and geometry in data science
Paweł Dłotko

14 Luis Caffarelli, winner of the Abel Prize 2023
Juan Luis Vázquez

23 The Maison Poincaré – a maths museum in France
Sylvie Benzoni-Gavage

30 A woman in a man's world
An interview with Elisabetta Strickland
Ulf Persson

38 Henry Pinkham (1948–2023)
In memory of a friend and colleague
Ulf Persson

44 Hédy Attouch, in memoriam
Michel Théra

47 On the Balkan Mathematics Conference (BMC) series
Betül Tanbay and Volker Mehrmann

49 African Institute for Mathematical Sciences (AIMS)
Winnie Nakiyingi

54 SUBgroups
Nelson Niu, Katie Waddle, Marquia Williams and Yufei Zhang

57 Youth Commission of the Royal Spanish Mathematical Society
Érika Diz Pita

61 "If, and only if" in mathematics
Omid Ali Shahni Karamzadeh

67 Book reviews

69 Report from the EMS Executive Committee online meeting, 17–20 November 2023
Richard Elwes

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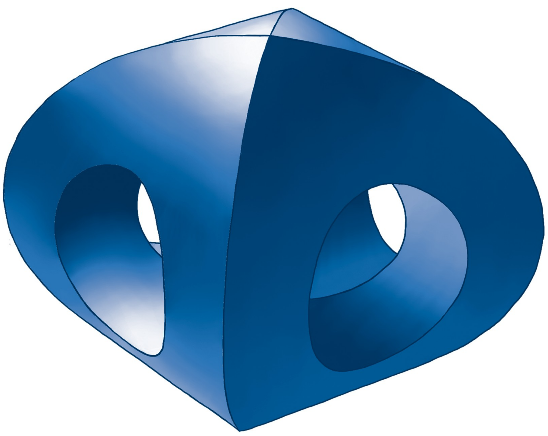


The object depicted in the cover illustration by A. B. Araújo is a variation on the Rulpidon, a series of works by the French sculptor Ulysse Lacoste, and a symbol of the Maison Poincaré in Paris.

Photo by Jim Høyer, University of Copenhagen.

The ninth European Congress of Mathematics (9ECM) in Seville is a few weeks away, and I am excited to meet many of you there. We will be celebrating mathematics and mathematicians.

Just before the congress, the European Mathematical Society is organizing its council in Granada. This is an important event for the EMS. The council is the governing body of the EMS and it has delegates representing all the national member societies as well as delegates representing individual members. This will be my first council as president of the EMS, and I am looking forward to meeting the delegates and to discussing the future directions of the EMS.

Over the years, the EMS has increased its activities and its engagement with the European mathematical community. Our ten standing committees are deeply involved with many issues of great importance to individual mathematicians and to the mathematical community as a whole. The committees oversee research, education, outreach, applications and interdisciplinarity, diversity, ethics, publishing, and equity within mathematics in Europe and beyond. The EMS is deeply committed to the publishing and dissemination through the EMS Press and our involvement with zbMATH open. We have engaged with the early career segment of the community through the establishment of the European Mathematical Society Young Academy (EMYA). The EMS offers many grant opportunities for travel grants and conference support and many other sponsored activities. We celebrate great achievements in mathematics with our many prizes. We will present a number of these at the 9ECM. The last council approved the formation of EMS Topical Activity Groups (TAGs) that foster collaboration within specific mathematical topics across Europe. Currently, we have six active TAGs. The EMS had also previously issued a call for larger-scale activities – the EMS Strategic Activities – but did not fund any. We decided to pause this initiative temporarily while we consider the correct level of funding and structure of the Strategic Activities program.

It is fantastic to see how EMS has increased its level of activities. I cannot personally take much credit for this but should thank many of the previous presidents and EMS councils for having had the vision to help bring the EMS to where it is today. I do think, however, that we need to rethink the way we are going to organize the EMS as a truly professional organization. We need professional support to structure the way we interact with our membership and the society in general. We have therefore decided to create a role of a community manager for both the EMS and EMS Press. I hope soon to introduce you to this new person.

As many of you know, our social media presence and more general news announcements have hitherto been taken care of by our publicity officer Richard Elwes. Richard has done this with great success. It has been a tremendous amount of work, and I am embarrassed to say he has had to do it in his spare time. It was partly seeing what we asked of Richard that made me realize we cannot expect this to be done only *con amore*. Richard's term as publicity officer is coming to an end just before the congress. It is not possible to thank Richard enough for all the work he has put into the EMS. He has probably done more for the visibility of the EMS than anyone else in the organization. After Richard's term is up, we will not appoint a new publicity officer. Instead, the work of the publicity officer will now be with the new community manager.

I have already mentioned in my previous message that the term of Fernando Manuel Pestana da Costa, the editor-in-chief of the magazine you are currently reading, will also end. In fact, this is the last issue under his editorship. I want to thank Fernando for all his hard work with the Magazine and I know that all of you reading this are joining me in thanking Fernando. While saying goodbye to Fernando, we should also welcome our new editor-in-chief, Professor Donatella Donatelli from Università degli Studi dell'Aquila. I am very much looking forward to working with her.

Wishing you all a great summer and hoping to see many of you in Seville in a few weeks.

Jan Philip Solovej
President of the EMS

Brief words from the editor-in-chief

Dear Reader of the EMS Magazine,

With the issue that you are now reading my four-year term as editor-in-chief comes to an end. It has been a great honour to serve the European mathematical community in the editorial board of the Newsletter/Magazine for the last seven and half years (before being editor-in-chief I had been editor responsible for the section "Societies") and some words of thanks are in order: first, to the previous editor-in-chief, Valentin Zagrebnov, and the former EMS president, Volker Mehrmann, for having believed I could take care of the job; and to the current EMS president, Jan Philip Solovej, for his continuing trust and support. Thanks are also due to the staff of EMS Press (past and present) whose professional competence and the always-ready and friendly help has been incredibly important in having the job done with a minimal hassle: in particular I would like to name Apostolos Damialis, Sylvia Fellman, Simon Winter, Theresa Haney, and Vanessa Haazipolo. Finally, but by no means less important, my gratitude goes to all my colleagues in the editorial team, without whom an editor-in-chief can do little…

In these last four years several changes took place in our journal, some of a more superficial nature like the change of name from Newsletter to Magazine, the change in design, and the use of colour, others of a deeper character; of these I would like to point out just one: in the natural renewal of the editorial team it has been possible to increase the female participation from just under 7% to slightly over 31%, thus attaining a gender balance in the editorial committee that mirrors the percentage of overall female participation in the mathematical community as a whole.

Some ideas and projects were not possible (or I was not able) to implement, and there are always things that could have been done better. But overall it has been four fantastic and incredibly exciting years during which I could get a much clearer view of the European mathematics community and I had the privilege of working with, and getting to know, some fantastic people.

It is now time to get back to my sole role as reader of the Magazine: from the September issue onwards the editor-in-chief will be Donatella Donatelli, a colleague of the current editorial team that has done an incredible job in revitalizing the "Book Reviews" section and, I am sure, will do an even better work in the next four years at the head of our Magazine.

To you, the reader, I trust that you will continue to support the EMS and our Magazine for many years to come.

Fernando Pestana da Costa
Editor-in-chief

On the shape that matters – topology and geometry in data science

Paweł Dłotko

The seemingly simple question, "What is the shape of things?", gains precise mathematical meaning when examined through the lens of modern topology and geometry. This paper surveys a few methods of topological data analysis (TDA), a powerful tool for characterising and predicting the shape of a dataset. Extending beyond traditional statistics, we will present various shape descriptors offered by TDA, elucidating their computation and practical applications. Last but not least we will demonstrate the effectiveness of the presented methodology through several real-world examples.

1 Mathematics of numbers

Mathematics is often regarded as the art of numbers, laying the foundational concepts of the discipline. This Platonic view facilitates counting of objects, regardless of their nature. Historically, this journey commenced with natural numbers, later incorporating zero to represent the absence of objects. Subsequently, the need to deal with debts introduced negative numbers, and the necessity for division and sharing led to the creation of fractions. The evolution of more complex mathematical calculations, often originating from geometry, spurred the quest to solve increasingly intricate equations. This positive feedback loop resulted in the development of more sophisticated constructs: rational, irrational, real, complex numbers, and quaternions. For instance, quaternions are employed in modern three-dimensional computer graphics to describe the rotation of three-dimensional objects.

A wide range of disciplines, particularly a substantial part of mathematical modelling, are fundamentally grounded in numbers. Typically, when posing a practical mathematical question, it revolves around a single number or a set of numbers that represent, for instance, a function that is the solution to the problem at hand.

However, this numerical perspective represents just one facet of comprehending the world around us. We possess the ability to count and intuitively recognise the size of objects, but our perception also enables us to abstract away extraneous details and grasp the essential features such as the *shape of objects*, even in the presence of considerable noise. This phenomenon is prominently featured in various artistic movements, ranging from Cubism and Impressionism to abstract art. A tangible illustration of this is the widely recognised drawing "The Horse" by Ali Bati (see Figure 1), showcasing our capacity to perceive evolving forms and concepts amidst significant deformation and noise.

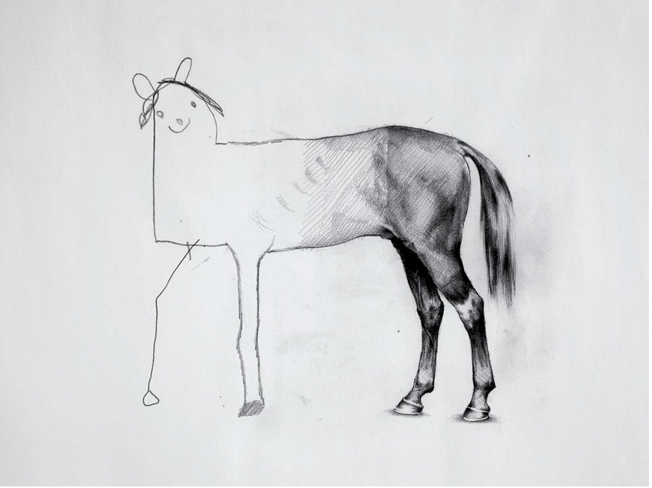


Figure 1. Ali Bati's "The Horse" strikingly demonstrates how the fundamental concept or Platonic ideal of a horse's shape can persist, even in the presence of considerable distortion of its physical form. This artwork showcases the resilience of shapes we perceive despite significant structural changes.

This artwork, which has become a popular meme, encapsulates the Platonic idea of shape. Despite significant deformation, we can still discern the concept of a horse in there. This human observation raises a crucial question: Are there mathematical notions that allow us to recognise a shape amidst substantial noise and distortion? The paper will cover state-of-the-art tools that address this query, delving into the basic formalism and algorithms. Additionally, various applications of the introduced methodology will be discussed.

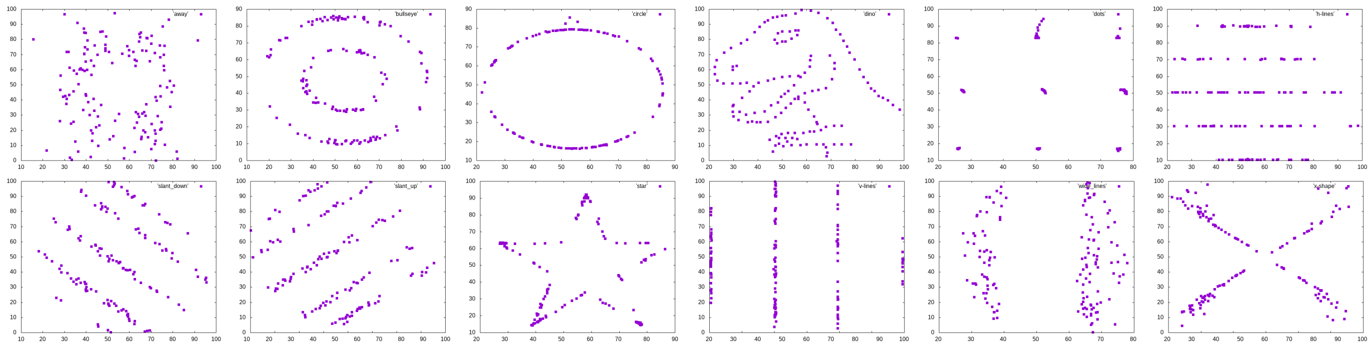


Figure 2. Same statistics, different shapes, the Datasaurus dozen, see [13].

2 Always visualise!

Before driving into topology, it is crucial to acknowledge that standard statistics already provide a range of basic analytical tools for discrete shapes. These include standard statistical moments, correlations between dimensions, and one- or two-sample statistical goodness-of-fit tests, which facilitate the comparison of various samples (a.k.a. point clouds). Although these methods have robust theoretical underpinnings and offer assurances of limit convergence, they encounter a notable limitation: they tend to condense the characteristics of often high-dimensional and complex shapes into a single numerical value, a statistical representation of the shape. From the era of Anscombe's quartet [1] to the more recent Datasaurus dozen [13] illustrated in Figure 2, comes a persistent message urging for visualisation of data. This is because the sole reliance on statistical values may not be sufficient. The Datasaurus dozen, for instance, presents very different datasets that share nearly identical summary statistics.

Visualisation proves to be straightforward when handling two-dimensional samples. Yet, the vast majority of datasets are much higher dimensional, rendering direct visualisation a challenging task. To circumvent this issue, dimension reduction techniques are commonly employed, though they inevitably cause some loss of information. The topological methods discussed below offer a novel pathway for the visualisation and comprehension of data's shape, effectively addressing the complexities associated with high-dimensional datasets.

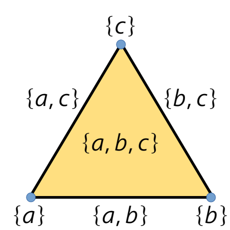
3 Second star to the right and straight on till morning

Many classical topological characteristics of spaces are invariant under continuous deformations. This property has inspired a humorous adage among topologists, stating that they cannot distinguish between a coffee mug and a doughnut, as one can be continuously deformed into the other. This analogy, viewed positively, underscores the robustness of topology against substantial amounts of noise and deformation.

The study of invariants in topology dates back to 1758, when Euler discovered that for a convex polyhedron, the number of vertices (V), edges (E), and faces (F) are interconnected by the formula $V - E + F = 2$, regardless of the specific arrangement of vertices, edges, and faces. Euler's insight precipitated at least two significant breakthroughs: firstly, it facilitated the representation of space using a finite set of information, thus effectively generalising the concept of an abstract graph, also introduced by Euler on occasion of solving the Königsberg bridge problem in 1736. Secondly, it provided one of the earliest topological characteristics of a space, which essentially states that every convex polygon has one connected component and encloses one cavity. Consequently, shapes with varying numbers of connected components or cavities exhibit distinct Euler characteristics.

Traditionally, the Euler characteristic is defined for solids, while typical inputs in data analysis comprise a finite point sample $X \subset \mathbb{R}^n$. Formally, such a point sample represents a discrete collection of points, lacking any inherent geometrical structure. However, a metaphorical "squinting of our eyes" can reveal an underlying shape, particularly evident when examining the upper right panel of Figure 4.

There are many ways in which the process of "squinting eyes" can be formalised. Let us start with the concept of an *abstract simplicial complex* [9] – a collection of sets that is closed under the operation of taking subsets. For instance, the collection $\mathcal{K} = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ satisfies this condition, as every subset of each set from $\mathcal{K}$ is also an element of $\mathcal{K}$. This particular simplicial complex represents a geometrically filled-in triangle. Elements of the complex are referred to as *simplices*. The *dimension* of a simplex $s \in \mathcal{K}$, $\dim(s)$, is defined as one less than its cardinality. The



alternating sum of the counts of simplices of successive dimensions, $\sum_{s\in\mathcal{K}}(-1)^{\dim(s)}$, gives the Euler characteristic of $\mathcal{K}$.

One of the earliest ways of constructing abstract simplicial complexes from point samples is attributed to Vietoris and Rips [17]. Given a finite sample X equipped with a metric d and a proximity parameter r, this approach connects with an edge points in X that are at most r apart. Cliques in the resulting r-neighbourhood graph correspond to simplices in the Vietoris–Rips complex. Since a sub-clique of a clique is itself a clique, we obtain a well-defined simplicial complex for every r.

Although the Vietoris–Rips complex is relatively easy to define, it has a serious drawback. For sufficiently large r, the size of the complex grows exponentially with respect to the cardinality of X. To circumvent this problem, both topology and geometry need to be taken into account. First, take the union of balls of radius r, centred at points from X. Then restrict these balls to the Voronoi cells of their centres. The n-fold nonempty intersections of those restricted balls correspond to simplices in the so-called alpha complex, see [9]. While more challenging to compute, alpha complexes offer the clear advantage of having sizes, under mild assumptions, proportional to the size of X. Moreover, they can be effectively computed using CGAL and Gudhi, European software packages in computational geometry and topology, respectively.

Both of the presented constructions depend on the distance parameter r, which serves as the resolution parameter. Since there is no canonical method for choosing a single value of r, a whole range of radii, typically spanning from zero to infinity, is considered. Given two radii $r < r'$, the complex $\mathcal{K}_r$ obtained at radius r is a subset of the complex $\mathcal{K}_{r'}$ obtained at radius r'. Since both $\mathcal{K}_r$ and $\mathcal{K}_{r'}$ are complexes, we say that $\mathcal{K}_r$ is a *subcomplex* of $\mathcal{K}_{r'}$. In this scenario, each simplex is equipped with the value of r at which it appears for the first time, referred to as a *filtration of the simplex*.

This approach transforms the point sample X into a multiscale combinatorial structure of a *filtered simplicial complex* that effectively summarises the data and formalises the "squinting eyes" process. For an example of filtered simplicial complex depending on a growing proximity parameter, please consult Figure 3.

4 Summaries of filtration: ECC

The classical concepts of Euler characteristics and filtration come together handily. By combining these methods and computing the Euler characteristic for each radius in a filtration we get a function called the *Euler characteristic curve (ECC)*. It assigns to a radius $r \geq 0$ the Euler characteristic of a complex $\mathcal{K}_r$ and constitutes the most fundamental multiscale summary of the shape of the sample. Figure 4 exemplifies the ECC for the point sample at the upper right panel.

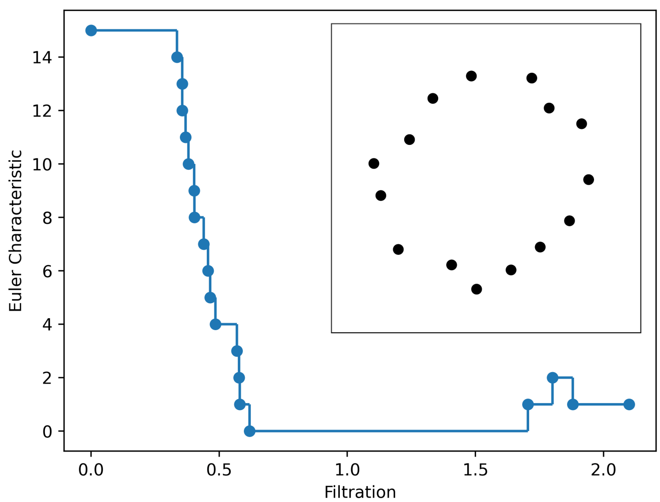


Figure 4. Points sampled from a circle with a bit of uniform noise (upper right) and the corresponding Euler characteristic curve.

The Euler characteristic, advantageously positioned at the confluence of topology, differential geometry (as exemplified by the Gauss–Bonnet theorem), and vector calculus (illustrated by the Poincaré–Hopf theorem), emerges as a versatile and universal tool with a wide array of applications. Among its most notable uses is in cosmology, where it aids in understanding the geometry of both

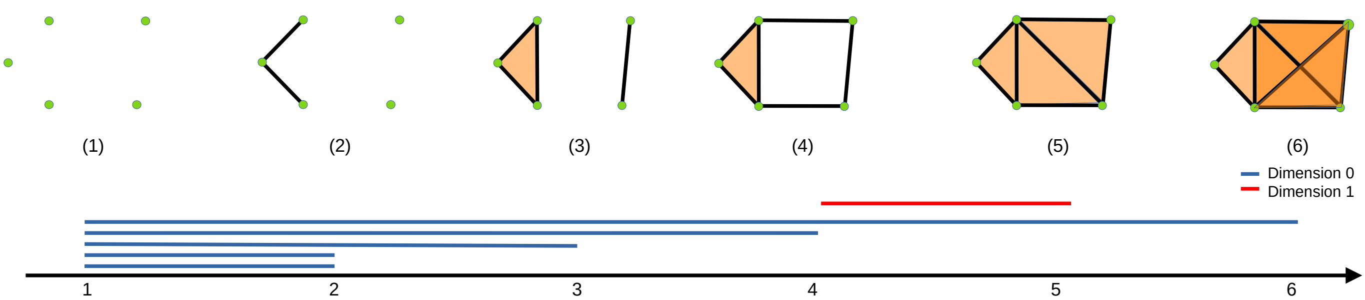


Figure 3. Example of distance-based filtered simplicial complex. As the radius grows in steps 1–6, more and more simplices are added to the complex. At the bottom, the persistence intervals in dimension 0 (blue) and dimension 1 (red) of the considered complex are presented.

the current and early universe [18]. In this study, we shall explore another vital application, particularly pertinent to data analysis: the use of ECC in statistics, with a focus on goodness-of-fit tests.

In the classical statistical framework, we encounter two primary challenges: one-sample and two-sample goodness-of-fit problems. The one-sample test is employed to determine whether a set of points has been sampled from a known and explicitly defined probability density. On the other hand, the two-sample tests involve an additional finite sample, substituting for the probability density. The aim here is to determine whether the two provided samples are derived from the same probability density.

The availability of tools to address one-sample and two-sample goodness-of-fit problems is heavily influenced by the dimensionality of the data. For one-dimensional cases, there is a plethora of tools, including the Kolmogorov–Smirnov, Cramér–von Mises, Anderson–Darling, chi-squared, and Shapiro–Wilk tests, to name a few. These are supported by multiple efficient computational tools implemented in various programming languages. In two-dimensional cases, theoretical results for the Kolmogorov–Smirnov and Cramér–von Mises tests are available, along with some implementations in Python and R. However, for higher-dimensional data, while theoretical results for the Kolmogorov–Smirnov test exists, only a handful of implementations are available.

Standard tests primarily rely on the cumulative distribution function (CDF) of a real-valued random variable X that, at a given point x, determines the probability of X being less than or equal to x. Generalising this concept to higher-dimensional data poses significant challenges with current implementations, as it necessitates considering all permutations of the data axes.

However, topological characteristics of a sample, such as the Euler characteristic curve, are invariant under multiple data transformations, including permutations of the axes and affine transformations. While providing slightly weaker invariants, they do not encounter the same problems as traditional methods including CDF. In our recent work [7], the ECC of a sample is utilised as a surrogate for the cumulative distribution function, yielding an efficient statistical test that surpasses the state of the art. This new family of tests, referred to as *TopoTests*, has proven to outperform existing methods even in low-dimensional and small data samples scenarios.

To illustrate it, consider the matrices in Figure 5. The value at position (i, j) in each matrix indicates the power of the test, defined as the probability that the test successfully recognises that a sample, in this case of size 100, taken from the distribution in the i-th row, does not originate from the distribution in the j-th column. When $i = j$, the value 0.05 on the diagonal represents the confidence level for which the test was designed. It is therefore supposed to reject the true hypothesis in 5% of cases.

Figure 5 showcases the performance comparison between the Kolmogorov–Smirnov test (left) and TopoTests (right) for a three-dimensional sample. In the colouring scale used in the figure, a better test can be recognised as one with more yellow entries in the matrix. In this instance, TopoTests consistently outperform the standard Kolmogorov–Smirnov test, illustrating the potential

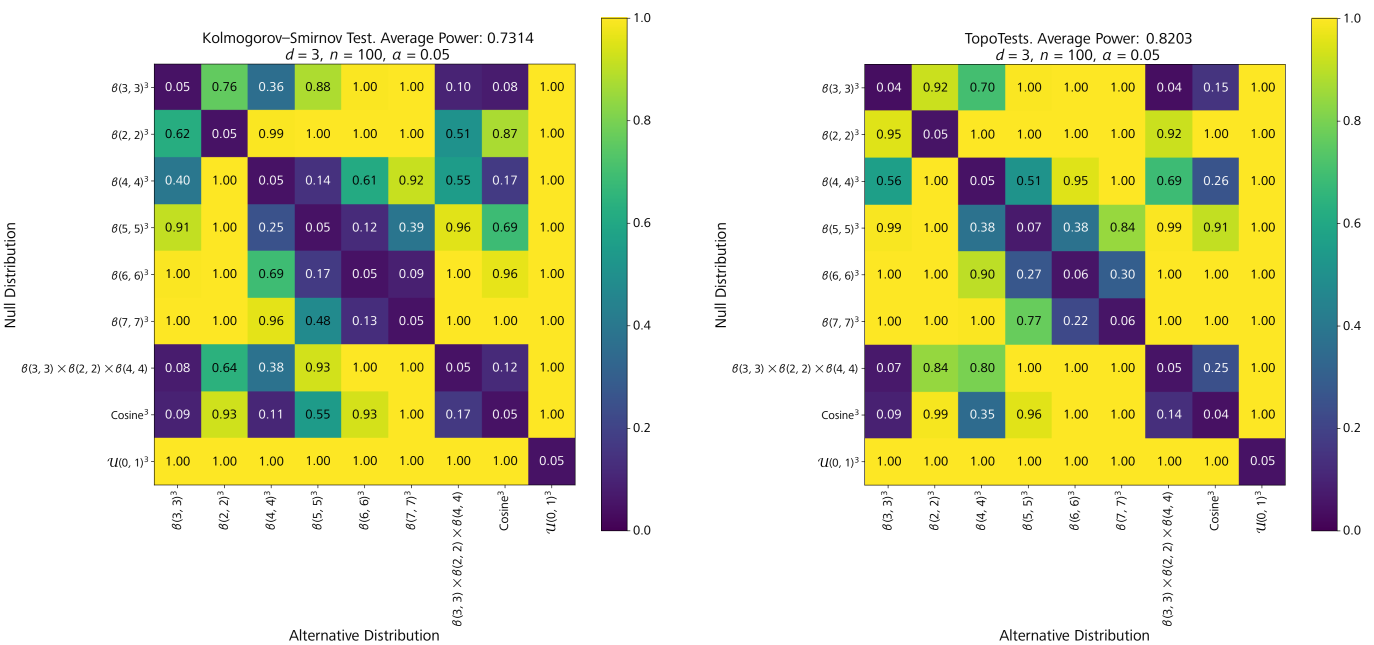


Figure 5. Performance of the Kolmogorov–Smirnov test (left) and TopoTests (right) in the task of distinguishing various three-dimensional distributions (refer to axis descriptions for details). Image obtained from [7].

applicability of topological tools in statistical analysis. As described in [7], TopoTests can be easily accessed and utilised through a public domain implementation.

It is noteworthy that this approach comes with asymptotic theoretical guarantees. It is also important to be mindful that in some rare instances, the Euler characteristic curves of different distributions may be quite similar, leading to less effective performance of the test. These occasional limitations are a trade-off for the additional advantages offered by the proposed TopoTests. Please consult [7] for further details.

Integrating topology with statistics offers an additional significant advantage: it extends the application of statistical goodness-of-fit tests to a broader range of inputs, specifically those for which an ECC can be calculated. For instance, inputs in the form of a scalar-valued function on a bounded domain, such as an image (referenced in Figure 6), can be processed in this framework. In this figure we see a visualisation of three solutions of the Cahn–Hilliard–Cook equation,

$$\frac{\partial u}{\partial t} = -\Delta(\varepsilon^2 \Delta u + f(u)) + \sigma_{\text{noise}} \xi,$$

where $u(0, x) \approx \mu$ for every x in the domain of u (in this case, the unit square). The constant μ is the total mass, see [8] for details. The three solutions illustrated in Figure 6 represent $u(t, x)$ for a fixed $t > 0$ and the initial condition $\mu = 0.2$, 0.2 and 0.12, respectively. It turns out that the information about the initial condition μ as well as the time t at which the solution is obtained can be recovered from the topology of the patterns in Figure 6. In this case, the first two images, corresponding to $\mu = 0.2$, depict "drop-like" formations, while the third image, associated with $\mu = 0.12$, exhibits a snake-like behaviour. For a preliminary study on this, refer to [8]. This example illustrates that the topological approach enables us to broaden the scope of data types that can be analysed using standard statistical methods

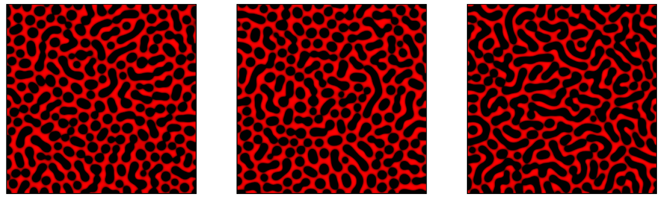


Figure 6. Three solutions of the Cahn–Hilliard–Cook equation. The left and middle image correspond to solutions obtained using the same model parameters, resulting in "drop-like" patterns. The right one is derived from slightly different parameter, resulting in a "snake-like" pattern.

5 Summaries of filtration: persistence

While the ECC is nice and simple, TDA offers a more advanced multiscale invariant of data: *persistent homology*. From persistent homology, also referred to as *persistence*, one can easily obtain the ECC, but not vice versa. Persistent homology is a multiscale version of the homology theory briefly outlined below.

Suppose we have an abstract simplicial complex $\mathcal{K}$ obtained from a sample X for fixed radius r using either the Vietoris–Rips, or the alpha complex construction. Homology recovers information about connectivity of $\mathcal{K}$ in different dimensions. At dimension 0, homology recovers connected components of $\mathcal{K}$. Focusing on the filtration presented in Figure 3, in step (1) there are five connected components (corresponding to points of X), three at the level (2), two at level (3), and a single connected component since after. Homological features of dimensions 1 are represented by classes of one-dimensional cycles that do not bound any collection (or formally, a chain) composed of two-dimensional simplices. Such a cycle can be observed at the step (4) of the filtration. We can intuitively think about them as bounding "one-dimensional holes" in the complex. The story continues for higher dimensions, where homology theory detects features, informally bounding higher-dimensional holes in the considered complex.

Persistent homology enables tracking of the homological features, such as connected components and holes of dimension one or higher, as the filtration of the complex evolves. During this process, homology classes are created and then some of them cease to exist. Consider for example the one-dimensional cycle from step (4) of Figure 3 – it appears (is born) at step (4) and ceases to exist (dies) at step (5), becoming the boundary of two two-dimensional simplices added in step (5). The *persistence interval* $[4, 5)$, presented in red in Figure 3, spans the filtration values in which the one-dimensional topological feature exists. A similar narrative applies to homological features in dimension 0. For instance, consider the three leftmost points of X in step (1) of the filtration. They become connected at step (2), forming a single component. Consequently, two of them cease to exist, giving rise to two persistence intervals $[0, 1)$ in zero-dimensional persistent homology. The collection of persistence intervals for the filtration at the top of Figure 3 is given at the bottom of the figure. Persistence intervals of various filtrations can be compared using for instance distances developed to solve optimal transport problems.

In this short exposition we have barely scratched the surface of persistent homology. For a comprehensive introduction, please consider [9].

6 Questions you did not know you had

"Visualisation gives you answers to questions you didn't know you had" – this famous quote by Ben Shneiderman encapsulates a fundamental desire in various scientific fields: to discern patterns, formulate hypotheses about underlying principles, and subsequently verify them. Topological data analysis offers tools to visualise high-dimensional data through the so-called mapper algorithm. Introduced in 2007 by Gunnar Carlsson and coauthors [16], the mapper algorithm represents a given high-dimensional sample X as an abstract graph, termed a *mapper graph*. Since its inception, the mapper algorithm has had a significant academic and industrial impact, boasting hundreds of successful applications and numerous industrial implementations, including extensive work by Symphony AyasdiAI.

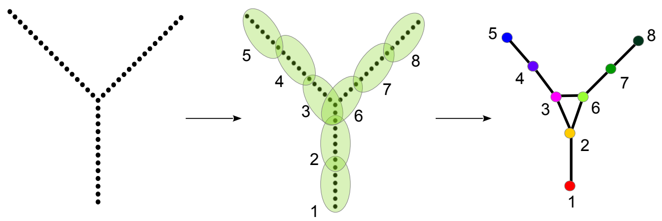


Figure 7. From point cloud to overlapping cover, its one-dimensional nerve and a (colouring) function defined on it – a general scheme of mapper algorithms.

A general method to derive a mapper graph from a point sample X is straightforward: one needs to construct an *overlapping cover of* X, namely, a collection of subsets $C_1, \dots, C_n$ such that $C_i \subset X$ and $\bigcup_{i=1}^n C_i = X$. Subsequently, a graph called the *one-dimensional nerve of the cover* is built. This abstract graph's vertices correspond to elements of the cover, and its edges represent the nonempty intersections of these elements, as illustrated in Figure 7. The mapper graph models a space X upon which a function $f\colon X \to \mathbb{R}$ can be visualised. This can be achieved, for example, by calculating the average value of f for each C_i. The value at the vertex of the graph corresponding to C_i can then be visualised using an appropriate colour scale.

There are two main methods to construct such an overlapping cover. The first one, proposed in [16], originates from the Reeb graph construction. Initially, X is mapped into $\mathbb{R}$ using a so-called lens function $l\colon X \to \mathbb{R}$. The interval $l(X)$ is then covered by a series of overlapping elements $I_1, \dots, I_k$, with each consecutive pair having a nonempty overlap. For each $l^{-1}(I_j)$, a clustering algorithm is then applied, and the obtained clusters are used as cover elements. As $l^{-1}(I_j)$ and $l^{-1}(I_{j-1})$ overlap, this results in an overlapping cover of X.

The second construction, proposed in [5], leading to a *ball mapper* graph, involves a fixed $\varepsilon > 0$ and a metric d on X. An ε-net is built on X, defined as a subset $Y \subset X$ such that for every $x \in X$, there exists $y \in Y$ with $d(x, y) \leq \varepsilon$. Consequently, $X \subset \bigcup_{y \in Y} B(y, \varepsilon)$ and the family of balls $B(y, \varepsilon)$ for $y \in Y$ forms an overlapping cover of X.

Topological visualisation tools, such as mapper graphs, have a plethora of applications across various fields. We have selected two examples, one from social sciences and the other from material design.

In the realm of social sciences, we explore the phenomenon of Brexit – the process leading to the outcome of the UK's 2015 referendum, where a majority of voters decided for the UK to leave the EU. Our analysis, presented in [15], is based on the 2012 census data and superimposed it with the Brexit referendum results at the constituency level.

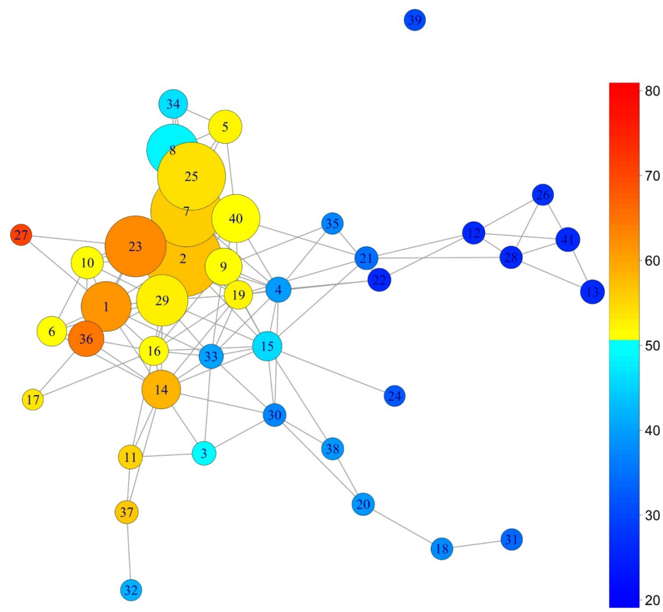


Figure 8. Homogeneity to leave, heterogeneity to remain – multidimensional view on the Brexit phenomena (colours represent Brexit support in UK constituencies) [15].

The ball mapper graph, thoroughly discussed in [15], is depicted in Figure 8. Despite the substantial aggregation of information, several sociological observations emerge. The most notable among them is the relative homogeneity (at a sociological level) of the constituencies supporting Brexit (marked with yellow and orange), contrasted with the vast heterogeneity of those favouring the UK's continued membership in the EU (marked with blue). This observation, along with multiple other conclusions and hypotheses, is elaborated upon in [15].

Our second example, elaborated in detail in [12], pertains to descriptors of three-dimensional porous structures of hypothetical zeolites. Zeolites are chemically-simple nanoporous structures derived from SiO_4 tetrahedra, assembled into hundreds of thousands of different crystal structures. Although primarily used in detergents to soften water, zeolites have the potential for applications in gas capture and storage (such as methane and carbon dioxide), noble gas separation, and other areas. Given their chemical simplicity, the defining factor for the properties of a given material is the shape of its pores. In the study [12], this shape is characterised using the persistent homology of points sampled from the pores' surfaces. The persistence intervals of successive dimensions obtained for a given material serve as the material's features. They allow inducing a distance function between different materials as the distance between the persistence diagrams obtained from them, see [9].

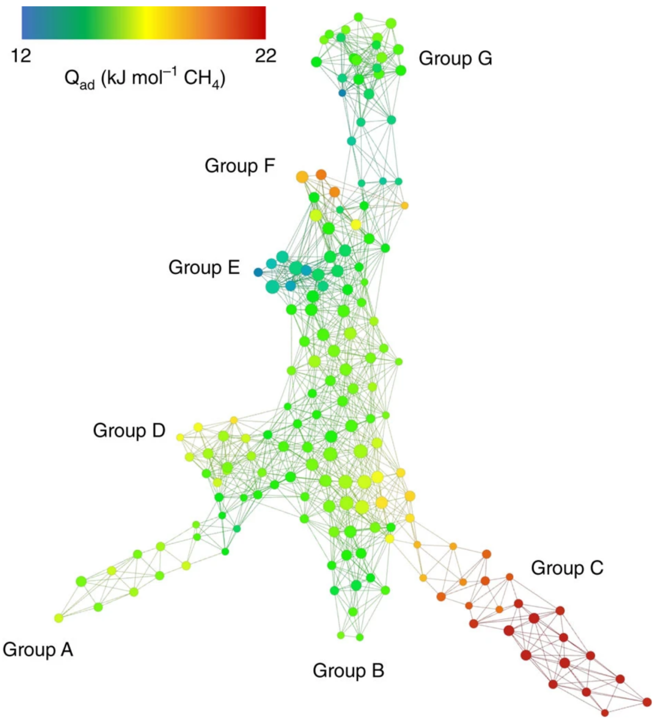
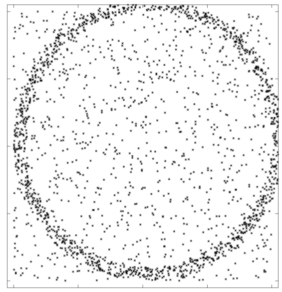


Figure 9. Landscape of more than 140,000 hypothetical zeolites coloured by the heat of adsorption of the materials. Picture taken from [12].

The presented approach gives us a discrete metric space representing the considered database of hypothetical zeolites. A mapper graph, representing the shape of this space, is illustrated in Figure 9. The graph uses a colouring function based on the heat of absorption, which determines the temperature at which a given material can absorb the maximum amount of gas, in this case, methane. We observe that this property appears to be "continuously dependent" on the material's shape. Furthermore, different regions of this space, denoted by distinct groups in the graph, correspond to various humanly-interpretable geometries of pores. This research highlights the synergy between statistics and machine learning-friendly topological descriptors (such as persistent homology), and topological visualisation. This combination provides a comprehensive overview of the shape of the space of hypothetical zeolites. The colouring function enables the identification of regions in the space containing materials with the desired values of properties of interest.

7 Multiple filtrations at once?

The filtrations considered thus far have focused on a single aspect of the data – either the mutual distances between points, or the greyscale intensity values of an image's pixels. However, scenarios exist where examining multiple characteristics concurrently is necessary. For instance, in the context of images, one might consider multiple channels, such as RGB. With point samples,



envision a set of points sampled from a circle but contaminated with a lower density of uniform noise inside the circle. While our brain is adept at discerning the shape of a circle despite the noise, a clear persistent interval might not be obtained with a filtration based solely on distances. The inclusion of additional filtration parameters, like local density, becomes necessary. However, effective generalisations of persistent homology to multiple filtration parameters have proven to be a serious challenge. This is due to problems rooted in representation theory, which pose an obstruction against the existence of a counterpart to persistence intervals, see [3]. This is the case despite the considerable collective efforts of the TDA community.

Addressing the challenges in generalising persistent homology, we return to the foundational concept of the Euler characteristic. Specifically, its parametric version, the Euler characteristic curve, at a given radius r, is essentially an alternating sum of the number of simplices of successive dimensions. Once a simplex s enters the filtration, it remains therein, contributing to the alternating sum and hence the ECC. Therefore, the contribution of simplex s to the ECC is an indicator function; it equals 0 for all arguments below the simplex's filtration value and 1 thereafter. The ECC we have considered so far is the alternating sum of such indicator functions.

This simple observation paves the way to generalise the ECC for the case of multidimensional filtrations. It is conceivable that a simplex s appears at one or several non-comparable points $f_1(s), \dots, f_k(s) \in \mathbb{R}^n$ in an n-dimensional filtration. In such cases, simplex s will contribute a value of 1 at every point that is coordinate-wise greater or equal to any of $f_1(s), \dots, f_k(s)$, and 0 at all other points. Through this method, we obtain a stable invariant of an n-dimensional filtration, referred to as the *Euler characteristic profile*, see [6] for further discussion and properties.

Let us consider a simple example of such a scenario: a problem of analysing prostate cancer features on hematoxylin and eosin (H&E) stained slide images. Our results are based on publicly available 5182 images, each of 512×512 resolution, obtained from

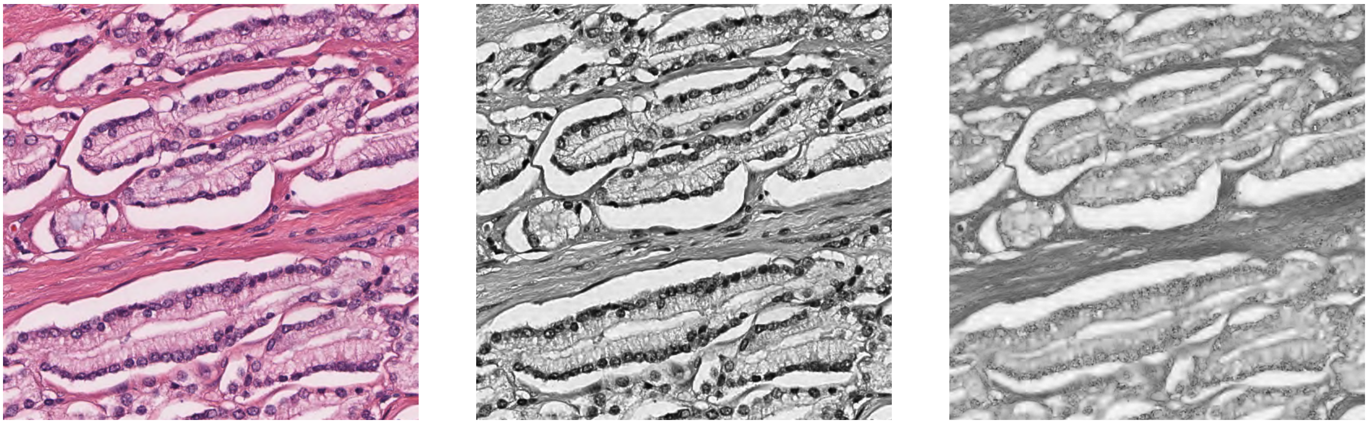


Figure 10. Prostate cancer ROI, left, the raw image; middle, the hematoxylin channel; right, the eosin channel. Image from [10].

the Open Science Framework [11], as analysed in [6]. These images represent various regions of interest (ROIs) from prostate cancer H&E slices, collected from 39 patients. The unique aspect of each image is its annotation with a Gleason score, 3, 4, or 5, reflecting the architectural patterns of the cancer cells. A higher Gleason score is indicative of increased cancer aggressiveness.

Given such an annotated dataset, a natural question arises: can a Gleason scale assigned by a histopathologist be deduced from the shape of the structures visible in Figure 10 using appropriate regression techniques applied to topological characteristics of images? For this purpose, methods of persistent homology and Euler characteristics have been employed, achieving an accuracy of approximately 76%. The utilisation of Euler characteristic profiles utilising both H&E channels further enhanced the accuracy to 82%. In both instances, random forest regression methods were applied. This example illustrates that considering multiple filtrations simultaneously can lead to significant improvements in performance of the data analysis tools.

8 Summary

"Data has shape, shape has meaning, and meaning brings value" – this foundational quote by Gunnar Carlsson encapsulates the essence of topological data analysis: to seek out rich and robust features that summarise the geometric and topological structure of complex and high-dimensional datasets. Now, after nearly 20 years of development, the field boasts many success stories and offers a wealth of tools to the community. In informal conversations, I often refer to the tools we provide as "statistics on steroids" – they go beyond relying on single numbers and embrace much more complex features, while retaining (almost) all the properties of standard statistics. In addition, they are directly applicable to standard algorithms in statistics [2], machine learning [14] and AI [4].

Among other initiatives, my Dioscuri Centre in Topological Data Analysis is contributing new tools to the field and the whole scientific community. We work closely with domain experts in mathematics, medicine, economics, finance, physics, biology, and more, aiming to integrate our new tools into the daily practice of applied mathematicians and researchers utilising mathematics in their fields. If you are interested in our research, please visit our web page. All the theoretical tools described in this paper are implemented and freely available at our GitHub page.[1]

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[2] Please see https://dioscuri-tda.org/members/pawel and https://scholar.google.com/citations?user=-_znDLoAAAAJ for further details.

Luis Caffarelli, winner of the Abel Prize 2023

Juan Luis Vázquez

The Argentine mathematician has been a leading figure in the development of partial differential equations in the last 50 years and was awarded with the 2023 Abel Prize "for his seminal contributions to regularity theory for nonlinear partial differential equations including free-boundary problems and the Monge–Ampère equation." His contributions are not limited to these fields, they include problems in fluid mechanics, optimal transportation, non-local operators, homogenization, and other topics with important connections with other sciences. In a series of landmark contributions Luis used brilliant geometric insights together with ingenious analytical tools to study areas of nonlinear mathematics that were barely explored some decades ago and are now in full bloom. This text combines a description of Luis's main contributions with a glimpse into his life and personality.

Luis Caffarelli, photo from the University of Texas at Austin, USA.

Introduction

The Norwegian Academy of Sciences and Letters has awarded the 2023 Abel Prize to Luis Ángel Caffarelli, a mathematician from the University of Texas at Austin, USA. The prize is generally considered the mathematical equivalent of the Nobel Prizes of the Royal Swedish Academy of Sciences and has been awarded since 2003 to individuals who have made "outstanding scientific work in the field of mathematics." The citation about Caffarelli's work specifically says: "for his seminal contributions to regularity theory for nonlinear partial differential equations including free-boundary problems and the Monge–Ampère equation." But his contributions are not limited to the already mentioned fields, they also include problems in fluid mechanics, optimal transportation, calculus of variations, non-local operators, homogenization, and other topics that have important connections with other sciences. According to the American Mathematical Society: "Some of his most significant contributions are the regularity of free boundary problems and solutions to nonlinear elliptic partial differential equations, optimal transportation theory, and, more recently, results in the theory of homogenization."[1]

His work on the regularity of free boundaries has opened remarkable ways for geometric methods to play a relevant role in the analysis of equations. Free boundary problems appear naturally in very different contexts, ranging from fluid filtration, elasticity, optimal strategies in finance, economics, metal industry, interacting particle systems in physics, to topics in biology and ecology.

As Norwegian mathematician Helge Holden, chair of the Abel Committee, noted: "Combining brilliant geometric insight with ingenious analytical tools" to pioneer a field of mathematics that was barely explored four decades ago. And, later, Holden added: "Caffarelli's theorems have radically changed our understanding of classes of nonlinear partial differential equations with wide applications. The results go to the core of the matter, the techniques show at the same time virtuosity and simplicity, and cover many different areas of mathematics and its applications." Many other prominent mathematicians have pondered on his gifts. Francesco Maggi, a colleague of Luis at the University of Texas at Austin, said: *"Forty years after these papers appeared, we have digested them, and we know how to do some of these things more efficiently.*

[1] Notices of the AMS: https://www.ams.org/notices/201404/rnoti-p393.pdf. See also: https://www.ams.org/journals/notices/201808/rnoti-p1019.pdf

Luis Caffarelli receives the Abel Prize from King Harald of Norway.
© Alf Simensen – NTB / The Abel Prize

But when they appeared back in the day, in the 80s, these were alien mathematics."[2] Many of us have often wondered about his unexpected insights combined with skillful use of analytical tools… plus a touch of deceivingly simple geometry.

Biographic highlights

Luis Caffarelli was born in 1948 in Buenos Aires, Argentina, where he got his PhD degree in mathematics in 1972. He has lived in the US since 1973. His early American years were spent in Minnesota, a beautiful state of the Midwest with quite extreme weather that has been home to brilliant science during the 20th century. The University of Minnesota, where he taught, brings back very fond memories to many Spanish scientists of the generation that began their research careers in the 1970s, especially experts in economics, medicine, and mathematics.

Luis rose to worldwide fame at the end of that decade because of his surprising work on the regularity of the so-called free boundaries, nowadays familiar to a wide scientific public largely thanks to his work. We will see a brief description of these concepts later on. Luis surprised everyone in the mathematical community with the article "The regularity of free boundaries in higher dimensions," published in 1977 [4]. This very remarkable work laid the basis of his future fame. It is where his ingenuity in mixing partial differential equations (PDEs) and geometry first manifested. The novelty and brilliance of his treatment of the *obstacle problem* is already a classic reference in pure and applied mathematics, a paradigmatic example of what we call free boundary problems. It is worth highlighting the contributions that Luis has made to the *Stefan problem*, which mathematically describes solid–liquid phase transitions. It models, among other applications, the evolution of the ice–water system with its fine separation interface, a topic that reaches the general public through the process of melting glaciers or ice cubes melting in water. The substances in the applications need not be ice and water. Thus, in the continuous casting process employed in the steel industry, the free boundary problem concerns the determination of the solid–liquid steel interface.

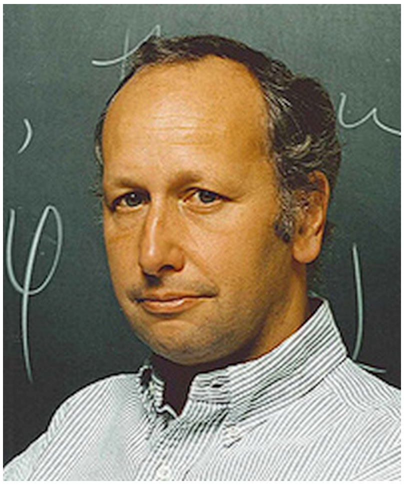
A second work by Luis that was to have a great impact dates back to 1982. It arose as the result of a collaboration with Robert Kohn and Louis Nirenberg and deals with fluids. The paper, "Partial regularity of suitable weak solutions of the Navier–Stokes equation," was published in 1982 [1]. The work was carried out during Caffarelli's first period as a professor at the Courant Institute of Mathematical Sciences in New York. Louis Nirenberg, also an Abel Prize winner (in 2015), was his supporter in those years and remained his friend for life. Years went by and this beautiful result, the theorem known as CKN in honor of its authors, continues to stand out as the last great contribution made in the study of regularity of solutions of the Navier–Stokes equations for viscous fluids. Here we are talking about one of the famous seven "Millennium Prize Problems" of the Clay Mathematics Institute.

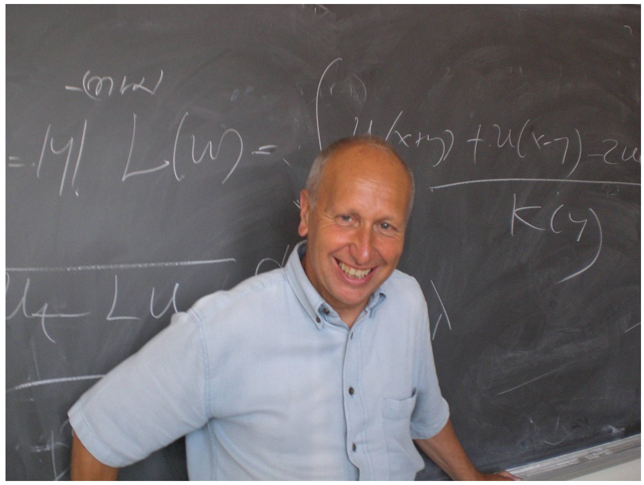
We have specifically cited these two works by Luis because the work of great mathematicians is often linked to some influential articles containing far-reaching results, or to books presenting deep theories. The 1980s were a prodigious time for Caffarelli. A cascade of articles with various coauthors established him as the best worldwide representative of the legacy of David Hilbert (circa 1900) and Ennio de Giorgi (circa 1960) on "how to study the regularity inherent in the solutions of the problems of the calculus of variations." Luis added to the program of the great Ennio the study of free boundary problems that we will deal with shortly. These problems had challenged the best experts during the 1960s and 1970s due to their intricate combination of difficulties from analysis and geometry.

Luis is one of the world's leading experts in the field of nonlinear partial differential equations. Partial differential equations have been studied for hundreds of years and describe almost every sort of physical process, ranging from fluids and combustion engines to financial models. Caffarelli's most important work concerns nonlinear PDEs, which describe complex relationships between the variables of the system and their derivatives. The influence of geometrical thinking permeates his highly original contributions.

From 1986 to 1996 Luis was a permanent professor at the Institute for Advanced Study, Princeton, an institution famous for legendary figures such as Albert Einstein, John von Neumann, and Robert Oppenheimer. After that, Luis went back as a professor to the Courant Institute, and finally moved in 1997 to the University of Texas at Austin, where he is still a professor (Sid W. Richardson Foundation Regents' Chair in Mathematics No. 1). Social recognition for him and his work has been continuous and growing. Since

[2] Quoted from the article in New Scientist, "Mathematician wins Abel Prize for solving equations with geometry," 22 March 2023, by Alex Wilkins.





Luis Caffarelli in younger years.

1991 he has been an outstanding scientist in the US and a member of the National Academy of Sciences. In addition, a series of international awards came to recognize the impact of his scientific contributions: in the present century, the Rolf Schock Prize, from the Royal Swedish Academy of Sciences (2005); the Leroy P. Steele Prize "for lifetime achievement" from the American Mathematical Society (AMS, 2009); the very prestigious Wolf Prize (2012); the Solomon Lefschetz Medal, from the Mathematical Council of the Americas (2013); again, a Leroy P. Steele Prize, this time "for seminal contribution to research" for the landmark article published in 1982 with Robert Kohn and Louis Nirenberg, from AMS (2014); as well as the Shaw Prize in Mathematics, which is considered the "Asian Nobel Prize" (2018). A brief glimpse of his greatest scientific achievements will be shown in the next section.

Mathematical area and contributions in some detail

Caffarelli's scientific work developed inside the mathematical field of nonlinear partial differential equations (NL PDEs), but it is important to highlight the interest that Luis showed for the connections between nonlinear PDEs and other sciences since his pre-doctoral studies. We reproduce some fragments of the citation for the Abel Prize 2023: "Partial differential arise naturally as laws of nature, from the description of the flow of water to the growth of populations. These equations have been the constant subject of intense study since the days of Newton and Leibniz. However, despite substantial efforts by mathematicians over the centuries, fundamental questions related to stability or even uniqueness and the occurrence of singularities remain unresolved for some key equations. Throughout more than 40 years, Luis Caffarelli has made pioneering contributions to rule out or characterize singularities. This is known as regularity theory, and it captures key qualitative features of solutions, beyond the original functional analytic configuration."

The above quote touches on two major themes. The first one is "differential equations," which appear as a very powerful tool to describe the movement of bodies and the variation of geometric figures, as well as their states of equilibrium. These equations came to life as part of the differential calculus in the 17th century. They relate certain magnitudes (called system variables) with their relative rate of variation (i.e., with their derivatives). When the unknown variables depend on several space and/or time coordinates, the resulting equations are called "partial differential equations (PDEs)."

Over the last centuries the number of such equations with relevance in mathematics, physics, biology, and engineering has not stopped growing. Today the study of these equations is one of the most active branches of mathematics, and it is going through one of its golden periods due to the enormous influence of its results and techniques in diverse applications. Indeed, partial differential equations are used to mathematically describe processes that occur in nature and in so many fields of technology. The fundamental PDEs of physics describe phenomena arising in the study of waves and vibrations, the motion of fluids, the behavior of structures, electromagnetism, or the fundamentals of quantum mechanics. In topics closer to daily life, we find them in the study of the propagation of natural phenomena, such as fires or tsunamis, or the dynamics of invasive species, or the evolution of a disease in an organism or pandemic in a population. Let us also mention Euler's equations for fluids, Maxwell's for electromagnetism, Einstein's for gravitation, Schrödinger's and Dirac's for quantum mechanics, or the Hamilton–Jacobi equations…, as major topics known to the public. Within mathematics itself, PDEs have deep connections with other fields such as the calculus of variations, differential geometry, harmonic analysis, probability theory, geometric measure theory, or computational mathematics.

Another important aspect of the equations studied by Caffarelli is "nonlinearity." Let us examine this feature. Although nature has the good taste to resort to linear processes for many of its basic models, such as the transmission of waves or heat, and also the equations of quantum theory, it is well known that many of the most important processes in science are nonlinear, and understanding that added difficulty is part of the glory and the cross of the current mathematical profession.

Nonlinear PDEs are more difficult to solve than linear PDEs, and it may happen that they produce solutions that make no sense in the physical world. Hence, the work of deep analytical minds is needed to sort out the correct concepts and to show how to calculate with them. In the world of nonlinear problems, the superposition principle, which is a basic instrument in linear science, does not apply. *Grosso modo*, in a linear system the response corresponding to the sum of two data (whether input functions or external forces) is the sum of the two individual responses obtained separately with each of the data. This is the case, for example, in the world of Fourier analysis. However, in a nonlinear system, this is no longer true, and as a consequence the problem with general data cannot be reduced to the sum of solutions of certain basic blocks. Typical nonlinear analysis graphs are not flat surfaces or straight lines, but instead involve more arbitrary curves and surfaces.

Nonlinear equations appear in many natural phenomena, from the movement of fluids to phase transitions, from the shape of space-time to the behavior of stocks. These phenomena do not really obey the "principle of superposition of effects" and some of their "key features" (such as turbulence in fluids) are clearly the result of the "nonlinear" character of some underlying PDEs. These key features often show up as "oddities" where the equations that are used no longer hold in the usual way. Furthermore, there is no way to avoid the issue since such *singularities* are inherent to many problems. It is the mathematician's task to understanding them and explaining to the scientist or engineer how to deal with them. Singularities are not a passing nuisance, on the contrary, they contain important, often crucial, information about the process where the model appears.

We may gladly declare that the mathematics of the late 20th century have been excellent in the study of those nonlinear processes that represent a stage of difficulty superior to the study of linear processes. Nonlinearity is an infinite source of complexity, and it is the origin of a many highly innovative theories. Chaos, shock waves, black holes, or climate evolution models are relevant scientific objects originated in the nonlinear worlds.

Caffarelli's theorems have radically changed our understanding of a large class of nonlinear PDEs. The union of geometry and mathematical physics around the analysis of the equations produces pages of great beauty. We describe below the main mathematical contributions of Luis in more detail. The less expert reader may skip the details, at least on a first reading.

A. Regularity theory for free boundary problems

A central part of Caffarelli's work concerns the regularity of the so-called free-boundary problems. These problems occur both in stationary processes and in processes that evolve with time. "Regularity" refers to the functions that appear as data or solutions, and it means in this context that such functions must have as many derivatives as they are required to ensure that they satisfy the equations in which they appear. The problem arises because we usually solve these and similar mathematical problems by resorting to approximate or numerical methods, after which the limit of such approximations is taken as the candidate to be a correct solution. The question is whether this limit is regular enough to satisfy the governing differential equation. This is an essential difficulty that Caffarelli tackles since his early papers, and to the study of which he contributed for many years. It is also remarkable that a lucky situation occurs: there are certain basic problems where the essential difficulties can be thoroughly examined and which have simple statements. It is true that in solving these "model problems" one encounters considerable technical difficulties, but this fact delights the researchers with a strong mathematical mind. Moreover, it was proved by additional research that solving the model problems opened the way for the mathematical study of realistic problems in different applications. Let us mention at this point that in this area there are multiple useful scientific and industrial applications.

We will present next some of these "model problems" where Caffarelli made fundamental contributions: the obstacle problem, the Stefan problem, the porous medium equation.

"Obstacle problem" (OP)

This is the model example for free boundaries of the stationary type.

Suppose we are interested in determining the position of an elastic membrane that is subject to different forces, and that in addition certain conditions are imposed on the border that encloses it (technically called "the boundary"). According to the laws of elasticity, the equilibrium position of the membrane is described by an equation of state that in the simplest case reads

$$\Delta u + f = 0 \quad \text{in } \Omega, \tag{E}$$

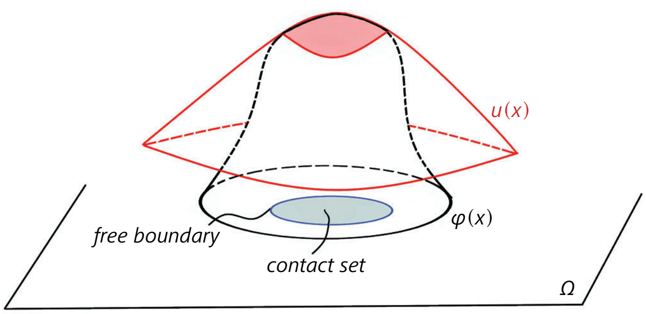
where $u(x,y)$ represents the vertical position of the membrane over a point (x,y) of a domain Ω of the plane, $f = f(x,y)$ represents the possible forces acting on the membrane, and Δ is the Laplace operator. We are also given the function $z = g(x,y)$ that represents the known height on the enclosing set where the membrane is attached to the border (the so-called "boundary conditions"). Then, it is well known in PDE theory that equation (E) together with the boundary conditions allow us to obtain a unique solution of the problem. Furthermore, if $f(x,y)$ is a fairly regular function, then $u(x,y)$ is also regular, even if the boundary conditions are not. These claims have been rigorously proven in the first half of the 20th century and were well known and studied years before.

The problem we have just described changes in difficulty when an "obstacle" is placed under the elastic membrane, $u(x,y) \geq \varphi(x,y)$. Then we have to consider the points inside the membrane (not on the boundary) where the membrane touches the obstacle. Obviously, the previous analysis does not apply. The modified analysis is best done if the domain Ω is divided into two parts: a "*contact set*," in which the membrane rests on the obstacle, and the remaining part in which the position of the membrane is located above the obstacle, the "*non-contact set.*" It then happens that in the latter region the elasticity equations must be satisfied, and it is proved that the equation of state (E) is valid there. In the best-case scenario, there is a clear line of separation between the two sets. It is called the *free boundary*, FB, so named because it is not known *a priori*, it must be calculated as part of solving the problem. Summing up, the whole problem becomes more complicated. It is about achieving two related goals, namely:

(1) Finding the set representing the part of the membrane that does not touch the obstacle.

(2) Solving the previous boundary problem (but now only in the domain where there is no contact) to find the position of the membrane. In the rest of the domain the membrane coincides with the obstacle.

This is the obstacle problem (OP). In short, the OP is a typical free boundary problem, FBP. Solving it requires combining the analysis of equations with differential geometry, two major areas of modern mathematics. Achieving such a combination is what constitutes the difficulty of the problem, which occupied the best minds in the field in the 1960s and 1970s. In 1977 Caffarelli surprised everyone with the seminal article in *Acta Mathematica*, where he considered the weak solutions of the OP (a class of possibly non-regular solutions obtained by functional methods, for example by minimizing the energy of the system), and then he proved that they are actually regular functions far away from the free boundary (provided the obstacle and the forces fulfill some minimal conditions).



Membrane over an obstacle. Notation x instead of (x,y).

And he also proved that this is true not only in two spatial dimensions, but also in all higher spatial dimensions. Furthermore, he showed that the free boundaries are indeed regular (hyper)surfaces, except for a possible "exceptional set" of singular points that are anyway very rare (technically, they form a set of small geometric measure).

The aforementioned singularities are not merely hypothetical; they do occur in practice. Thus, a duality is established: *generic regularity versus exceptional singularity*, a result that set the pace for further mathematical research on related topics ever since. The found regularity is technically called "Hölderian continuity," and this has been the field of excellence of Luis Caffarelli. For details of the theory, see [5].

The result we have described uses quite novel methods that enabled Caffarelli in subsequent years to provide insightful solutions to a whole series of free boundary problems with applications to solid–liquid interfaces, jet and cavitation flows, and gas and liquid flows in porous media, as well as financial mathematics. Caffarelli's regularity results are based on successively expanding the region close to the free boundary (a process called *blow-up*) and classifying the resulting extensions, where non-generic extensions correspond to the singularities of the free boundary. Exceptions to the regularity can occur, but they have a smaller dimension (as sets). The blow-up analysis is a very delicate technique that we will not explain further here.

"Stefan problem" (SP)

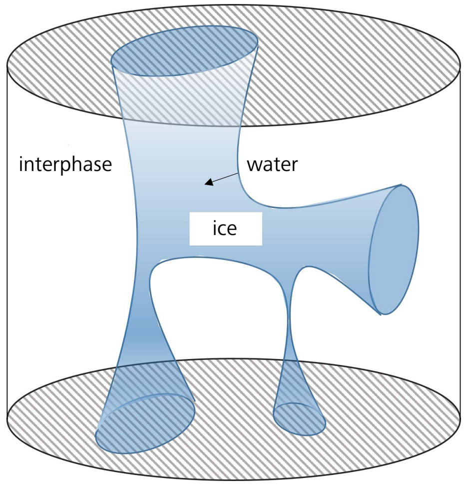
The most typical example of an evolution problem with a free boundary to be determined as part of the solution concerns the process of ice melting into water, called "Stefan problem" after the great mathematician-physicist Josef Stefan. This is a so-called phase transition problem. Here the free boundary is the interface between two phases of a medium, for example, water and ice. That interface is an unknown in the problem, that is, part of the whole solution that must be determined. Its location and properties are perhaps the most interesting features.

As this interface moves over time, it can be more precisely called a "moving free boundary." The SP is the standard problem in the mathematical description of phase transitions, with applications that include ecology, finance, and industry among many others.

There are many stationary and evolution free boundary problems in the theory of nonlinear PDEs that attracted Luis's attention. We may just mention the problems of jet flows and cavities, the dam problem, or flows in porous media. This last problem allowed me to interact with him.

B. My scientific life and friendship with Luis, the porous medium equation

After meeting Luis Caffarelli and Don Aronson at a conference on free boundaries in Italy in the summer of 1981, I spent the academic year 1982–83 as a visitor in Minnesota, where Luis and Don were professors at the time, and we began to work on the



Water–ice system and its separation by a free boundary. The arrow in the right picture indicates free boundary motion from water to ice.

regularity of the free boundary of the nonlinear version of the heat equation, now called PME, porous medium equation:

$$u_t = \Delta u^m, \quad m > 1.$$

The equation with the exponent $m = 2$ models the flow of water that permeates through a porous medium (Boussinesq, 1903). Here, $u(x, t)$ denotes the height of the wet region over a flat impermeable bed. As usual, Δ is the Laplace operator. In this problem, the interface between the wet and dry parts of the medium must be found along with the solution. The equation appears also (with different exponents $m > 1$) in the study of the propagation of compressible gases in porous media following Darcy's law (hence the popular name) and is applied in modeling oil reservoirs since the 1930s (Leibenzon and Muskat). There, the function $u(x, t)$ stands for the gas density. Applications to plasma physics (in the Zeldovich school, 1950, Oleinik et al., 1958) and biology (population dynamics, Gurtin and McCamy, 1974) appeared later.[3] As part of the problem, in all of these applications there is a *moving free boundary* whose location is to be calculated (issue of specific application, say, engineering interest) and whose regularity as a surface is to be determined (topic of high interest in mathematics). Note that the existence of such a free boundary is implicit, i.e., it is not explicitly mentioned as part of the initial problem statement, it happens that it is present as part of the geometry of the solution. In any case, this problem is one of the motivations for a broad theory of degenerate diffusions in which Caffarelli was a dominant figure.

My collaboration with Luis became a lasting friendship. During all these decades I paid regular visits to him at his different locations: Chicago, Princeton, Courant, and Texas, as did many of his longtime collaborators, and the meetings included the many events that Luis (co)organized worldwide, motivated by his scientific leadership and generosity. These events had the added charm of his friendly character and down-to-earth approach.

Since the 1980s we worked on the regularity of free boundaries (1985, 1987), the heat propagation arising in combustion (1995), and the theory of viscosity solutions (1999). When, at the beginning of the new century, I was invited by Oxford University Press to write a book about the theory of the porous medium equation, which was by then well developed, I asked Luis to be coauthor and leader of the project. He replied by encouraging me to do it myself and offering me his help in any difficulties I could encounter, with an invitation to visit Texas when needed. The book [10] took time to be written and appeared in 2007. This is maybe a good place to express once more my gratitude and recall the many pages of the book that contain the master's ideas.

Life went on. In the same year, 2007, Luis invited me to embark on the study of non-local parabolic equations, his passion at the time, as we will see below. Non-local parabolic equations and fractional Laplacian operators have occupied much of my time, and the time of my students and collaborators, ever since. But that is recent history, let us return to the older times and visit other provinces where Luis soon showed his best mathematics.

C. The regularity of viscous fluids, the Navier–Stokes equations

In 2014 the American Mathematical Society (AMS) awarded the Leroy P. Steele Prize, one of the highest honors in mathematics, to Luis Caffarelli along with Robert Kohn and Louis Nirenberg. It did so for his "seminal contribution to research," being honored for his aforementioned 1982 article [1] on the regularity or possible singularity of solutions to the Navier–Stokes equations which govern the motion of viscous fluids. The mention says: "This paper was and remains a milestone in the understanding of the behavior of

[3] Information about these references and further applied literature can be found in [10].

solutions to the Navier–Stokes equations and has been a source of inspiration for a generation of mathematicians." The Navier–Stokes equations are fundamental to the mathematical understanding of fluid dynamics and the existence and smoothness of solutions is one of the seven Clay Mathematics Institute's Millennium Prize Problems.

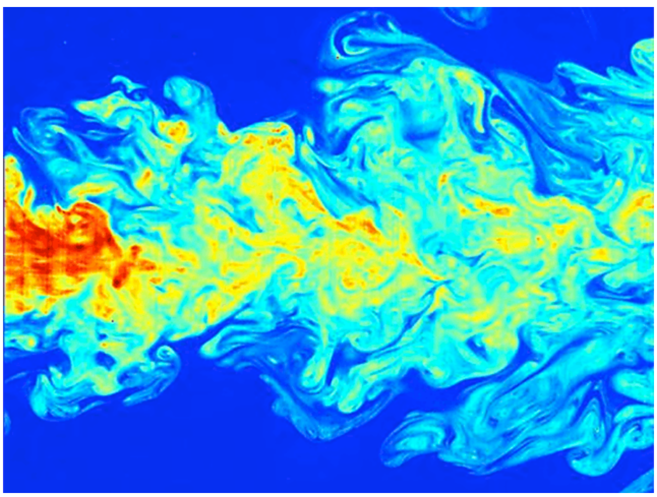
The Caffarelli–Kohn–Nirenberg result deals with partial regularity, something already found in the study of free boundaries, but now in a very different, seemingly unrelated topic: it states that there is still the possibility of singularities appearing in the density of the flow after some time, but in any case the set of singular points (x, t) in space-time is very small, not very dense (technically, it has a very small geometric measure). In particular, it cannot contain any curve pieces. The possible singular Navier–Stokes turbulence would therefore be quite an exceptional event, if it occurs. The result represented a great advance in our understanding of these equations and has motivated many subsequent developments and simplifications. But the advances made from 1982 till today do not have at all the depth and scope of the CKN result. The problem of total regularity is still open.

D. Fully nonlinear problems, Monge–Ampère equation, optimal transportation

Caffarelli's regularity theorems from the 1990s represented a major advance in our understanding in a third direction, again very different from the previous ones. This concerns the Monge–Ampère equation, a highly nonlinear partial differential equation that is used, for example, to construct surfaces of a prescribed Gaussian curvature. The great Russian mathematician Pavel S. Alexandrov obtained important existence results in this direction. Caffarelli, in collaboration with Louis Nirenberg and Joel Spruck, established essential properties of the solutions, with notable additional contributions from Lawrence C. Evans and Nicolai V. Krylov. Caffarelli later closed the gap in our understanding of singularities by showing that certain explicitly known examples of singular solutions are the only possible ones.

Starting from the Monge–Ampère equation, Caffarelli devoted several works to the study of the class of equations called completely nonlinear. His book, written in collaboration with Xavier Cabré, *Fully Nonlinear Elliptic Equations* [6], is a mandatory reference on this topic.

Caffarelli, together with his collaborators, has applied these results to the Monge–Kantorovich "optimal mass transportation" problem, building on earlier work by Yann Brenier. His knowledge of the Monge–Ampère equations enabled him to make fundamental contributions in this area as well.



Example of a turbulent fluid. © C. Fukushima and J. Westerweel

E. Integro-differential operators, anomalous diffusion, and non-local processes

Since the beginning of this century, there has been an enormous interest in research that deals with the interactions between particles or populations, or in the world of information. Recently, this interest has focused on considering actions at a distance, where notable effects are felt far away from the origin of the signal. In particular, these interactions do not follow the well-known Brownian-type decay patterns. The canonical form of these interactions is what is known as *Lévy processes* in probability and as "fractional Laplacian operators" in mathematical analysis and PDEs. The growth of this theory within PDEs has been spectacular in the last two decades. A fundamental reference in the mathematical analysis of this area is the 1970 article by Luis Caffarelli and Luis Silvestre: *An extension problem related to the fractional Laplacian* [3]. It heralded a new era in the research on such diffusive phenomena, and it has become a standard reference for scholars in the area.

The applications of fractional calculus are today varied and numerous. To name a few, we mention quasi-geostrophic flows, turbulence, molecular dynamics, stellar relativistic quantum mechanics, as well as various applications in probability and finance.

In a paradigmatic example, Luis Caffarelli and Alexis Vasseur obtained deep regularity results for the quasi-geostrophic equation, a popular model with application to fluid theory. In their 2010 article, *Drift diffusion equations with fractional diffusion and the quasi-geostrophicequation* [7], Caffarelli and Vasseur demonstrate the total absence of singularities in this model. This result is somewhat related to the possible existence of singularities in the Navier–Stokes problem. Contrary to the current guess about the existence of singularities for the NS problem, here the mathematical analysis is able to eliminate them. May this shed some light on NS?

In another direction, the classical idea of minimal surface, and that of perimeter of a solid have been successfully extended to the context of long-range interactions by Luis Caffarelli and his collaborators, such as Jean-Michel Roquejoffre, Ovidiu Savin and Enrico Valdinoci. These lines of research are supposed to have a great future and reflect the deep interaction of geometry and physics in Caffarelli's mind, with the theory of probability close by.

Finally, the combination of nonlinear diffusion ideas with non-local interactions has given rise to the theories of "nonlinear and non-local diffusion" that have been developed in recent years, particularly in projects of Luis with Spanish collaborators, in a series of influential applications, that are being used in the mathematical modeling of processes in various fields of science. An example of this vitality is, for example, the new book on the subject: *Integro-Differential Elliptic Equations*, by Xavier Fernández-Real and Xavier Ros-Oton [8].

In addition, Caffarelli has made outstanding contributions to homogenization theory, which seeks to characterize the effective or macroscopic behavior of media that have a microstructure, for example, are made of a composite material. A typical problem in this theory concerns a porous medium, such as a hydrocarbon reservoir, where one has a solid rock with pores. The system has a complex and largely unknown structure, through which fluids flow. Nonlinear mathematics has done important inroads in that field.

Final notes

Caffarelli is an exceptionally prolific mathematician: he has written more than 320 mathematical articles in the best journals with more than 130 collaborators, ranging from some of the most distinguished mathematicians to very bright young people from all over the world. He has had more than 30 doctoral students over a period of almost 5 decades and has more than 160 descendants[4]. Luis has always been a very open scientist and a dedicated organizer, who keenly thought about reaching out to bright students everywhere. He has a huge number of friends in the PDE community, and he had a very strong influence on the work of many brilliant scientists. The list of names goes from the old generation—let me just mention Don Aronson, Avner Friedman, and Louis Nirenberg—to the younger generation, which may be exemplified by Alessio Figalli, Fields Medalist 2018, who stayed many years in Austin and added brilliant pages to Luis's heritage.

Luis Caffarelli was never very fond of writing monographs, and that is a real pity. However, two long books reflect two areas of his teaching. One of them is the already mentioned book with Xavier Cabré [6], dating from his years at Courant, and the other is the book with his long-time collaborator Sandro Salsa [2] with a very suitable title: *A Geometric Approach to Free Boundary Problems.*

Luis has had and continues to have an enormous impact on the field. In his speech in Oslo last May, Jan Philip Solovej, president of the European Mathematical Society (EMS), recalled the impression made on him as a young man when reading some of Luis's articles: "There was this beautiful idea of controlling or taming the scary difficulties by intuitive geometric ideas of shapes. Taming difficulties is what mathematics is often about, and Professor Caffarelli has, more than anyone else, mastered the art of taming irregularities."

Luis Caffarelli has had a huge influence on the development of partial differential equations in several countries, mainly the US, Argentina, Spain, Italy, Greece, Korea, and China (to keep the list short). His collaboration with Spanish authors dates back to the 1980s and 1990s, after we met in Italy. He collaborated with many Spanish researchers, like Xavier Cabré, Antonio Córdoba, Ireneo Peral, Rafael de la Llave, Fernando Soria, and several younger ones. Luis has participated in numerous courses and schools in Spain, most notably in the series of UIMP Santander Summer Courses that we organized in the wonderful setting of the Magdalena Palace in Santander, thanks to his support and presence. Luis was named *doctor honoris causa* by the Autonomous University of Madrid in 1992. Since 2015, he has been a foreign academician of the Spanish Royal Academy of Sciences in recognition of his fruitful involvement with Spanish mathematicians for so many years. Luis received the Honorary Fellowship from RSME, the Royal Spanish Mathematical Society, and then the Rey Pastor Award (2017), on similar grounds.

Along with all of the above, we must mention his proverbial hospitality and the well-deserved reputation of being a great cook, in the Argentine-Italian tradition. Many of us remember memorable dinners with Luis and his wife Irene Gamba, also a mathematician, in their house in Austin.

Acknowledgements. The author is grateful to a number of friends and colleagues of Luis Caffarelli, who have contributed to improve this text with suggestions and corrections. He is especially grateful to Victoria Otero who coauthored an article with him to appear in Spanish, [9], where, along with many of the above ideas, they talk about his influence in Spain.

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The Maison Poincaré – a maths museum in France

Sylvie Benzoni-Gavage

Beginning of the story

The Institut Henri Poincaré (IHP) has a long history[1] dating back to its inauguration in 1928 by the then President of the Council, Raymond Poincaré, a cousin of Henri Poincaré (1854–1912). Founded by the mathematician Émile Borel (1871–1956), the institute was built thanks to private funds from the International Education Board of the Rockefeller Foundation and the French patron Edmond de Rothschild. It sits opposite the physical chemistry laboratory built by the physicist Jean Perrin (1870–1942) just two years earlier. Perrin's building itself (Figure 1) is adjoining the Institut du Radium, Marie Curie's laboratory, which was completed on the eve of WWI.

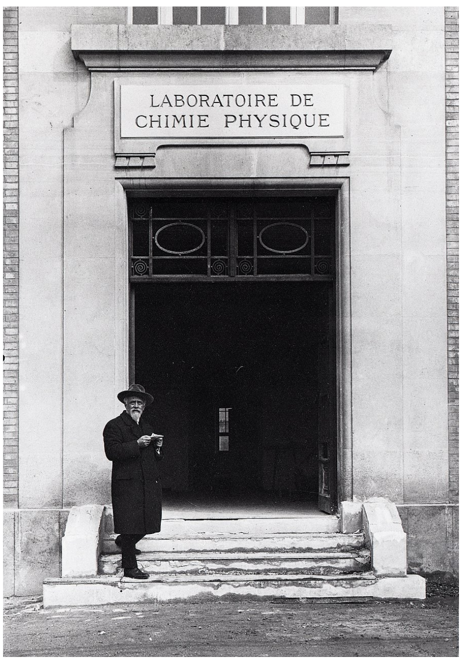

Figure 1. Jean Perrin in front of his laboratory.
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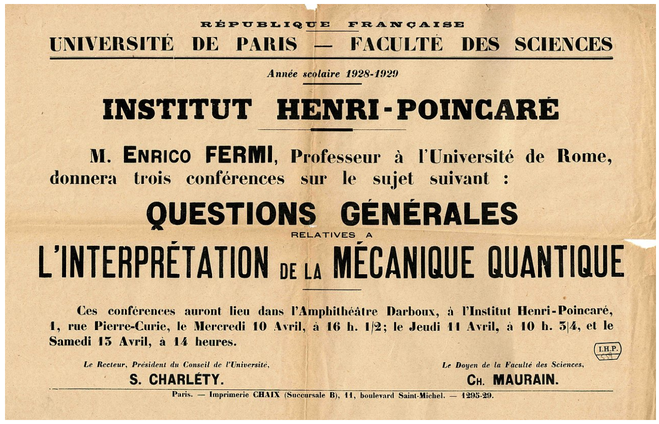

Figure 2. Poster announcing a series of lectures by Fermi at IHP in 1929.
© Institut Henri Poincaré, Paris

All of these brick buildings are located at the heart of Paris on the fields of a former convent – later converted to a reformatory run by nuns – acquired by the University of Paris at the beginning of the 20th century.

In the interwar period, Borel and his assistant Jeanne Fournier, born Ferrier – who held a Ph.D. in mathematics and was appointed as assistant in calculus of probability but actually worked as a secretary – invited to IHP the greatest specialists of the time in analysis, probability, and mathematical physics: Léon Bloch, George Birkhoff, Max Born, Marcel Brillouin, Francesco P. Cantelli, Torsten Carleman, Charles G. Darwin, Paul Dirac, Théophile de Donder, Albert Einstein, Enrico Fermi (Figure 2), Vladimir A. Kostitzine, Paul Lévy, George Pólya, Erwin Schrödinger, Vito Volterra, etc., from whom we can read lectures in the Annales de l'Institut Henri Poincaré.[2]

Over the years, IHP developed as a renowned international research centre for mathematics and theoretical physics. Even though Borel's building was raised by two additional floors in the 1950s, the institute's activities became somehow cramped at the turn of the 21st century. All the more so that it hosted several learned

[1] https://www.ihp.fr/en/history

[2] https://www.ihp.fr/en/library/annals

societies (SMF, SMAI, SFdS, SFP),[3] non-profit associations (Animath, Femmes & mathématiques, MATh.en.JEANS) and the Fondation Sciences Mathématiques de Paris. It had become difficult to accommodate all the people in the offices, along with regular seminars, national or international meetings beside the IHP scientific programmes coordinated by the Centre Émile Borel – a dedicated department created in 1994 when the institute was reborn after a post May 1968 period of uncertainty.

Inception of the expansion

Cédric Villani became director of IHP in 2009. He had great plans for it and looked for possibilities to expand the premises from the start. Perrin's building, which was still hosting the physical chemistry laboratory, renamed Laboratoire de chimie physique – matière et rayonnement (LCPMR), was not up to modern standards. The Université Pierre et Marie Curie (UPMC), who owned the building, was planning to move LCPMR to more suitable premises at "Jussieu" (Figure 3), its main campus.

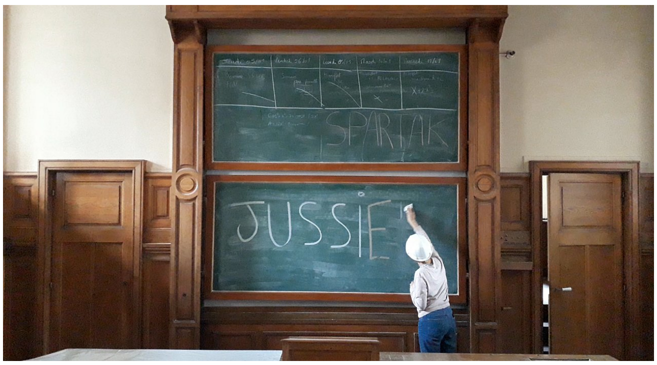

Figure 3. Screenshot of video immortalizing the moment when the word JUSSIEU left by LCPMR members was erased from Perrin's lecture hall, 30 June 2020.

Villani managed to convince UPMC to reallocate Perrin's building to IHP as part of a major centre for mathematics which would be open to the public. This plan was reminiscent of Michel Demazure's ideas, who had defended IHP's future when meeting with Lionel Jospin, the Minister of Education, in the late 1980s. Villani laid the groundwork for his plans with first initiatives such as exhibitions, a film club, public lectures, documentaries, aimed at the general public, school children, companies and society in general.

[3] Société Mathématique de France, Société de Mathématiques Appliquées et Industrielles, Société Française de Statistique, Société Française de Physique.

Figure 4. Entrance of Borel's building seen from Perrin's building door during refurbishment work, 4 March 2022.

Together with his deputy director Jean-Philippe Uzan and with the help of many others supporting these ideas, Villani secured an impressive 14-million-euro public funding for the refurbishment of Perrin's building (Figure 4). This included 8 million from the city of Paris, 3 million from the Île-de-France region, 2 million from the French Government, and 1 million from the CNRS, not to mention the estimated 9 million value of the building per se. As of its completion in 2023, the whole budget eventually approached 17 million, including an additional contribution from the French Government as part of the recovery plan following the Covid-19 crisis. During the construction, a small part of the budget was dedicated to renovate Borel's original building.

The project of an expanded mathematical centre, known internally as IHP+, has also been supported by the IHP Endowment Fund, created in 2016 by Villani on that purpose with the CNRS, UPMC, learned societies and a Circle of Partner Companies as founding members. Outside the mathematical community, it was promoted under the name of "Maison des mathématiques" project,[4] with a mathematics museum as flagship, which would occupy the ground floor (600 m^2) and part of the basement (300 m^2) of Perrin's building.

In 2017, Villani was elected Member of Parliament and resigned from his director position. Before his departure, the expansion project had been launched by UPMC as the project owner. An architect had been chosen, Atelier Novembre, along with a museum

[4] https://www.youtube.com/watch?v=AFLhIUGdBLE, https://www.youtube.com/watch?v=79obSq24tyw

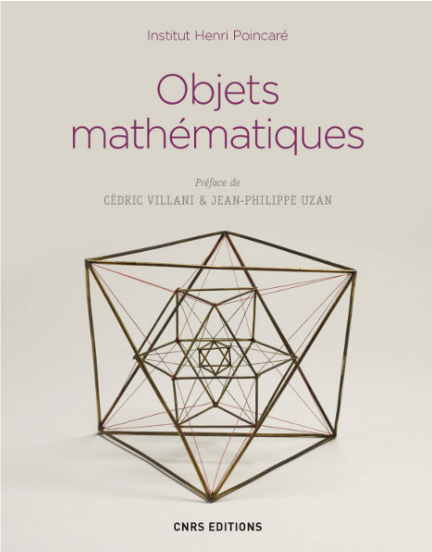

Figure 5. Cover of book published by CNRS Éditions for IHP about mathematical objects. © CNRS Éditions

Figure 6. Experimenting Holo-Math.
© Institut Henri Poincaré, Paris – Thibaut Voisin

designer, du&ma. Their joint project had indeed convinced the jury of the architectural competition that it was the best at enhancing heritage features while complying with modern standards.

On 1 January 2018, UPMC (Paris 6) merged with Université Paris-Sorbonne, also known as Paris 4, to become Sorbonne Université, and I took over as director of IHP. For years, I worked hard with numerous people to make IHP+ a reality. For an extended period, I lived with visualisations of the IHP+ project provided by Atelier Novembre and du&ma, and I was lucky enough to see this dream decor come to life. I got much more involved in the museum project than I had anticipated. I loved this amazing experience, of which the following gives an overview.

Museum project

At first, the museum concept relied on a few rough ideas launched by the former directors, Villani and Uzan. These ideas ranged from celebrating the founders (Borel, Perrin, Rockefeller, Rothschild) and telling the story of the building to showcasing a selection of mathematical objects from IHP's collection of around 600 pieces (Figure 5) and developing a most innovative outreach experience in mixed reality, Holo-Math (Figure 6).[5]

We did implement the original ideas while modifying some as I shall explain later, and many, many more which we developed along the process. A huge amount of work was to be done to design the very first museum in France that would be fully dedicated to mathematics and its applications.

One of the features which made it unique among maths museums in the world is that it was to be installed within an international research centre. Our targeted audience was thought of as middle schoolers, high school students, college students, teachers, and of course the general public. A challenge and goal at the same time would be to foster interactions between the public and the researchers coming to work at IHP, either visiting or dropping by.

It soon became clear that we needed to find a good name. Indeed, "Maison des mathématiques" was not specific enough, as the whole institute had been known as the "House of mathematics and theoretical physics" for a long time. However, we liked the warm connotation of the word Maison. This is how we came up with the name "Maison Poincaré," after a public poll. This name inspired the graphic designers working with du&ma, who proposed a logo (Figure 7) based on a pun in French that was already used by the Poincaré family themselves ("point" meaning dot and "carré" meaning square).

The name choice was settled early 2020. Marion Liewig, who had been appointed by the CNRS as project manager in 2016,

Figure 7. Logo of the Maison Poincaré.
© Institut Henri Poincaré, Paris – Pentagon

[5] https://holo-math.org

Figure 8. Atrium – CONNECTER. © Institut Henri Poincaré, Paris – Atelier Novembre, du&ma, Thibaut Voisin

Figure 9. Gallery – MODÉLISER. © Institut Henri Poincaré, Paris – Atelier Novembre, du&ma, Thibaut Voisin

became the very first head of the newly created department at IHP called Maison Poincaré, which led to a museum bearing the same name [3].

How it started

Back in 2018, we still had to imagine and work out the contents of the permanent exhibition, which we wanted to represent contemporary mathematics in all its immensity and vitality. We reached out to the IHP Outreach Advisory Board, chaired by Olivier Druet at the time, and also invited a number of specialists who we hoped to bring on board. After gathering everyone in a large lecture hall at Jussieu in May 2018, we involved researchers, teachers and science communicators in half a dozen working groups that were set up accordingly with the architectural spaces of the future museum. Marion, Olivier and I were very happy with the outcome and this nourished our energy to go on.

However, there was still a gap in the whole picture. We needed an expert who would be able to interact with these working groups and with the scenographer (the museum designer): a museographer.[6] Only a few days after the meeting at Jussieu, I was contacted by one of them, Céline Nadal, who happened to be originally trained as a physicist and had heard of the project. The connection was facilitated by IHP's deputy director at the time, Rémi Monasson, who had known her as a physicist. Luckily, we were allowed by Sorbonne Université to appoint Céline Nadal as our official museographer. This led to a most fruitful collaboration for the next five years.

[6] A job explained in this video: https://www.youtube.com/watch?v=mfiL3v1hQjo.

By March 2019, Céline Nadal had come up with a detailed programme in which every museum space was described by a word intended to summarize its aim. Choosing topics and words had been a challenging task for every working group. The result was to be showcased by means of giant suspended letters designed by the scenographer: CONNECTER (connecting), MODÉLISER (modelling), VISUALISER (visualizing), DEVENIR (becoming), INVENTER (discovering), PARTAGER (sharing).

Figures 8–10 and 14 are actual pictures of the completed museum. It is amazing how they resemble the 3D simulations that were provided by the scenographer. We felt like we were entering our screen when we saw the real layout of the spaces.

Challenges and achievements

The issue of language came up early in the discussions with du&ma and their graphic designers. It was clear to us that we needed everything to be translated into English at least, so that our international researchers and tourists could enjoy the museum. This meant that we had to optimize the content so that both languages – French and English – would fit on panels, exhibits, hands-on, games, films, etc. The six giant words are among the very few words that are translated separately on panels or sheets.

It was also very important for us to ensure accessibility to people with disabilities. We were guided by a specialist to design adapted exhibits for visually impaired people. They include texts in Braille and figures in relief. It was a challenge for the scenographer to find room for this additional material. While it was a challenge to make everything fit, we are very happy with the outcome which also attracts the attention of people without any disabilities. For people that are deaf or hard of hearing, we included French sign language and subtitles in both French and English in all videos.

Figure 10. Perrin's office – DEVENIR. © Institut Henri Poincaré, Paris – Atelier Novembre, du&ma, Thibaut Voisin

One of the most active mathematicians in the working groups, Clotilde Fermanian Kammerer, had been involved in the project since its inception, representing the CNRS National Institute for Mathematical Sciences and their Interactions as its deputy director for several years. In the autumn of 2019, she took over as chair of the IHP Outreach Advisory Board and played a crucial role in deciding the content of the Maison Poincaré [2]. At this point, I also thank Antoine Chambert-Loir who played a great role as a leader of a working group and who co-authored the mathematical Metro with me.

Quite notable, by 2019, the project management team had become 100% female with Marion, Céline, Clotilde and me. This certainly played a role in the decisions regarding the gender gap in mathematics that we took regarding the Maison Poincaré: First, we quickly abandoned the original idea of welcoming the visitors with portraits of the solely male founders and searched for alternatives. We came to the conclusion that our aim was to showcase an inclusive STEM world, as balanced as it should be. Therefore, we decided to present as many women as men [4], whether they are historical characters or contemporary personalities. This implied in particular to showcase the female physicist Yvette Cauchois (1908–1999), a successful and influential scientist who worked in this very office as director of the Laboratoire de chimie physique for 25 years.

A further big concern was the Holo-Math project, for which there was no pilot available when I took over as director of IHP. My first testing of the prototype was quite disappointing, and I was also put off by the tremendous cost of code development, not to mention the equipment. Nevertheless, we were convinced that this high-tech and innovative experience would be an asset to the museum. Fortunately, we managed to hire a science communicator, Adrien Rossille, early in 2019. He was fascinated by the concept and managed the Holo-Math project [5]: With Adrien, we were able to find enough financial support to fully develop an experience around the Brownian motion, particularly thanks to the IHP endowment fund and the Fondation Sciences Mathématiques de Paris, and also to acquire a fleet of mixed reality headsets.

Serendipity and emotion

During the construction work I had the opportunity to see an ancient shaft (Figure 11) leading to 30-meter-deep experiment galleries which I would not have dared to explore even if I had been allowed to do so. We had very little information on these galleries, but I was originally told that they were used by Marie Curie (1867–1934) herself, which sounded quite exciting. An extensive search of the archives produced only two pictures and a sketch map. In particular, I found out that this shaft had been built to connect the building's basement to underground old quarries… in 1935. Too late for Marie Curie! Despite my disappointment, I find it moving to have seen this shaft before it was sealed up by half a meter thick concrete disk for security reasons. Visitors of the museum can nevertheless see a glass disk indicating its location in the temporary exhibition space, together with a panel displaying the sketch map.

Figure 11. Ancient shaft before definitive closing, lighted by the site manager.

As an anecdote, another view that is lost forever is the painting (Figure 12) I discovered on a basement door in December 2019, after the building was cleaned from radioactivity, asbestos, lead, etc. Even though it was not a Monet, I liked this greeting left by former

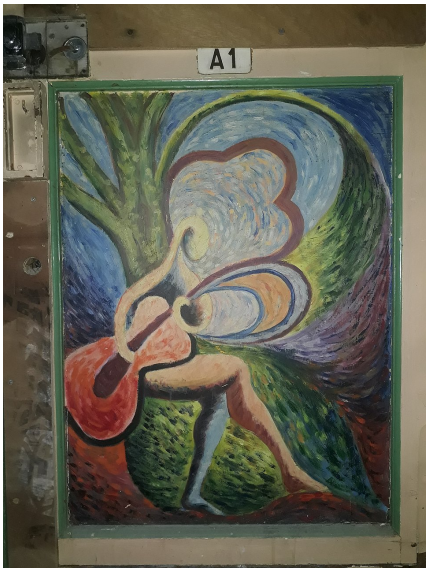

Figure 12. Painting on door in Perrin's building basement before its refurbishment.

occupants even though we were not able to keep it. We were able to preserve another piece of art though: a painting of Émile René Ménard (1862–1930) dated from 1927. It spans over several meters on the rear wall of Perrin's lecture hall, illustrating and surrounded by the first words of his bestseller book "Les atomes."

Speaking of art and science, I have to mention the Rulpidon. This is the name given by the French artist Ulysse Lacoste to a series of metal art pieces. He had handed me a small, bronze version of it as a gift to IHP in 2019 – this is now displayed in an optical theatre in the entrance hall of the building. I was instantly fascinated by the Rulpidon, and in turn found several mathematical stories to tell about it [1]. We ended up choosing it as the symbol of the Maison Poincaré: In addition to the mathematical stories, the Rulpidon fits very well with the logo. A monumental, steel version of it (Figure 13) was later commissioned by the IHP Endowment fund and installed in the garden.

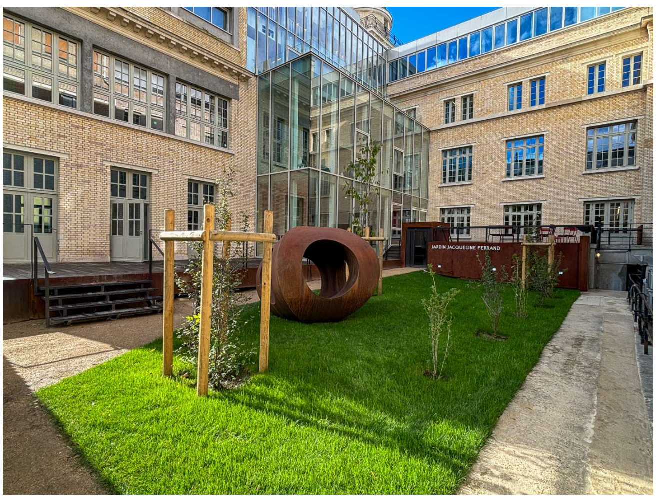

Figure 13. The Rulpidon by sculptor Ulysse Lacoste in the Maison Poincaré garden – Jardin Jacqueline Ferrand. © Institut Henri Poincaré, Paris

A last personal involvement I would like to talk about is the knots panel in the tearoom (PARTAGER) (Figure 14). Even though it is far from my field of research, knot theory is fascinating to me, as being rather recent – more recent anyway than most of the mathematics that people learn at school – and still an active theory opening to wider topics, while being rooted in ancestral knowledge of humanity. When discussing the best use of a large panel in the tearoom, the showcases of which were already planned to host mathematical objects in connection with art, Céline, Clotilde and I quickly agreed that knots were a golden subject. The scenographer supported our idea after we suggested that the knots should be made like bead necklaces to echo of the œuvre of French artist Jean-Michel Othoniel.

Figure 14. Tearoom – PARTAGER. © Institut Henri Poincaré, Paris – Atelier Novembre, du&ma, Thibaut Voisin

Opening and first feedback

After all these years of preparation, the Maison Poincaré eventually opened its doors to the public on 30 September 2023, three days after its official inauguration in the presence of Sylvie Retailleau, the Minister of Higher Education and Research.

This opening was a huge success, which we owe to the hard work of numerous people, starting from the IHP staff and in particular the Maison Poincaré team, managed since 1 January 2022 by Élodie Christophe Cheyrou. She secured additional funding and built a team composed of a cultural projects manager and three science communicators. Over a period of several months, they

Figure 15. Poster of publicity campaign, as shown in the subway.
© Institut Henri Poincaré, Paris – A. Solano/Shutterstock.com

prepared the contents of guided tours to be offered to schools and to the general public, developing these in collaboration with teachers. Élodie also implemented a thorough communication plan with the CNRS and Sorbonne Université (Figure 15). This enabled us to have colossal media impact[7] in newspapers, magazines, on the radio and TV.

The museum's capacity is limited to 200 visitors, for security reasons. This limit has been reached every Saturday and during all holidays since the opening. We also have a lot of visitors during school days. By the end of November 2023, all slots for school groups were booked for the whole school year. Depending on the success of the museum, we will be able to increase the staff number to offer more slots for groups.

Since the start in 2018 we have reached an impressive set of milestones, and achieved this amazing project in ways that could not have been foreseen when I jumped on board. As the next step, we now have to ensure sustainability of the Maison Poincaré, as part of an expanded Institut Henri Poincaré in a magnificent building. More challenges to come!

[7] https://www.ihp.fr/fr/espace-presse

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A woman in a man’s world

An interview with Elisabetta Strickland

Ulf Persson

Elisabetta Strickland and I met at the Institut Mittag-Leffler in Stockholm in the fall of 1980 and have kept in touch ever since. (As a curiosity, we once had a photo exhibit together in Rome.) Hence, it was natural for me to include her in the ongoing series of interviews with women mathematicians, especially as she has taken a very active role in promoting women in mathematics.

Elisabetta Strickland, Rome 2011 (photo by Martina Lanini).

Ulf Persson: *Let us start from the start. Why not give us some family background?*

Elisabetta Strickland: You mean talking about my mother and father?

UP: *Go ahead!*

ES: My father was a Royal Air Force officer in the Second World War, in fact in Africa. Afterward he was sent to Italy when the allies liberated Rome, and they needed to find some Italian who was fluent in English to act as an interpreter. My mother was well-qualified for the job, and she applied. My father had to interview the applicants, and he fell in love with her right away.

UP: *It was mutual?*

ES: Yes, they got married soon thereafter and went to England.

UP: *How romantic. Just like that? No problems?*

ES: There were some forty Englishmen at the time who married Italian girls. And the priests did not demand them to convert to Catholicism as long as their spouses stayed Catholic and most importantly that their children were raised Catholic.

UP: *This makes sense, please continue.*

ES: My mother put up with England for a while, but then she realized that she could not live without the sun and went back to Rome.

UP: *And your father followed suit?*

ES: Naturally, did he have a choice? But then of course he had to face the problem of employment. How could he make a living? He ended up becoming a businessman, importer and exporter, that kind of thing.

UP: *Could you be more specific?*

ES: Why? It had to do with aluminium on the behest of someone who became a partner. Actually it took us to Venice where we lived for five years. At Lido to be more specific.

UP: *What time of life did that happen?*

ES: In my case between five and ten years of age.

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UP: *But you returned to Rome.*

ES: Yes we did. I do not think that my mother could conceive of living anywhere else, nor can I by the way.

UP: *Anything else you could add?*

ES: My father was compelled to travel a lot, so in fact I did not see too much of him in my childhood. But he was not an absent father, he really took care of us.

UP: *And your mother was a present mother?*

ES: Very much so.

Elisabetta Strickland standing in Rome 1952 with her father Reginald Charles Strickland and her mother Esperia holding her baby sister Sara.

UP: *Your background does not explain your interest in mathematics.*

ES: I was a school girl, in fact an exemplary one, very studious, very serious and very successful. I won so many prizes, I was very proud of that.

UP: *So mathematics was only one among other subjects you excelled in?*

ES: You can say that, but the truth is that I had an excellent math teacher, very clever, it was a pity that he was stuck in life teaching in high school and not at university.

UP: *The intellectual quality of teachers at that time was typically quite high, after all there were few university positions, so sad as his case may have been, it was far from being unusual. But he clearly made a difference, in particular to you.*

ES: Yes, he taught me so much so when I started out at the university I already knew so much. What I particularly appreciate about him was his encouragement to solve problems in unconventional ways.

UP: *So you start university.*

ES: I started out as an engineering student.

UP: *Really?*

ES: Yes, I had this idea of wanting to build bridges. But it only lasted for a week. I was the only girl among one hundred and fifty or so students, and once when I dropped my pencil, all of them dived down to pick it up, begging me for my phone number, so we could get together. That was too much, and I fled to the math department where there were at least some women and hence some semblance of sanity.

UP: *So how was that?*

ES: First it was a four-year program, the first two years devoted to become mathematically literate, and then you had three options. Either doing research, with an academic career, or teach, or do applied mathematics and go out in the supposedly real world. There was never a question in my mind to consider anything beside the first alternative.

UP: *This is nice to know, but it does not address my question.*

ES: Fair enough. 1968 was as you must remember a special year when it came to students, and I was fully swept up in the movement. I was convinced that the educational system was rotten and had to be reformed, and I discovered in myself an affinity, maybe even a talent, for speaking publicly. I enjoyed it immensely, as I enjoyed that period. I flourished.

UP: *You also met your future husband Corrado de Concini at that time?*

ES: Not so fast. Yes, there was this guy de Concini who never bothered to show up at lectures, but had the audacity to ask me for notes. No doubt a very lazy guy.

UP: *So unlike the case of your parents, there was no love lost.*

Her husband Corrado de Concini in Ireland 2006.

ES: Love? It did not enter my mind. But I have to admit that when he handed back the notes he had annotated them with remarks that revealed that he was a very clever guy.

UP: *And you find cleverness in guys very attractive, trumping physical attraction?*

ES: Now you are at it again. Let me just point out that my opinion of the guy became more nuanced, and I saw him in a more favorable light. But that happens to most people when you get to know them better. And to put the topic to rest, de Concini's love interest was clearly elsewhere; he ended up married for some years, and I had a boyfriend who studied engineering, and we were supposed to get married. But I changed ideas.

UP: *Yes, I have some vague memory of that, as you bring it up. So what happened after the four years of basic university education?*

ES: In Italy there was not a tradition of obtaining Ph.D.'s. You just continued on your own writing papers and if you were lucky you obtained a university position, or else taught at school, or went out in the so-called 'real world.' But things started to change a little after 1968, e.g., de Concini decided to go to Warwick and get a Ph.D., actually with Lusztig as an advisor.

UP: *Let's talk about your career instead.*

ES: My mathematical career was rather straightforward. I spent my sabbaticals using a National Council of Research grant and a NATO grant. Then I applied successively for the three different levels of professorships, i.e., assistant, associate and full ones. I became full professor in 1987. The noteworthy thing was that I played according to the rules, much to the frustration of the Italian mathematical community. The rules being that each position was to be publicly announced and decided upon by a scientific community ostensibly arguing on strict scientific merits. The tradition in Italy, as opposed to Northern Europe, was that the appointments were made on a national basis, the powerful clique at each department deciding whom to get, the professor typically choosing their own successor. Traditions die slowly, and I was often approached and asked to withdraw my application, but I persevered as I had the legal right to apply and I stuck to it, and the appointed committees clearly took their duties seriously and did not bow down to departmental pressures.

UP: *You had to qualify for this positions writing papers and I doubt you were sitting put in your study in Rome. What did you do, and more specifically what kind of mathematics did you do? What kinds appeal to you, and what leaves you cold?*

ES: That was a lot of questions. Where should I begin? About mathematics, I must say that I do not like computations, so thus I was never interested in analysis…

UP: *Sorry, but what is the connection? Analysis is not about computations…*

ES: You should of course not take 'computations' literally, but proofs in analysis are very intricate involving clever estimates, you more often than not have no idea what is going on, the trees hiding the forest.

UP: *Yes, in analysis you get your hands dirty, and that does not appeal to you?*

ES: I prefer to keep my hands clean. I like abstract proofs based on concepts rather than, let me say, logical computations which do not leave you any wiser.

UP: *You prefer slick proofs.*

ES: You can put it that way. But algebra and geometry I find much more congenial than analysis.

UP: *Even commutative algebra?*

ES: Yes, I used commutative algebra in my work. But early on I got fascinated by group theory, in particular finite groups. But there was no group theory in Rome, so I commuted to Padua for some time. I was also recommended by Claudio Procesi to get into representation theory if my interest in groups was serious. Very good advice.

UP: *How come you were drawn to algebraic geometry?*

ES: How could I resist, I am Italian after all! As to your other question, I did go abroad to widen my horizons. In particular, I went to the Boston area. And I also had some interaction with Joe Harris, all of which gave me a nudge towards algebraic geometry. Griffiths in particular urged me to think of something in algebra which could be helpful for the varieties I was interested in at that times. I worked very hard, studying in the library day and night, I would not be able to summon such single-minded devotion now.

UP: *We have talked a lot about doing mathematics, but almost nothing about the mathematics which you have been doing. I think we have touched on this before, but when you do mathematics, is it always with a publication in mind, or do you think of something just for fun regardless whether it has a fair chance of 'paying' off or not.*

ES: I certainly take a professional attitude, I do not fool around, I always have a goal in mind, I feel I cannot waste time.

UP: *Do you collaborate a lot?*

ES: Not at all. I very much prefer to work alone, to talk mathematics to other people often gets me confused. This does not mean that I do not interact with other mathematicians, asking questions, getting new ideas, but the idea of actually working with someone solving a specific problem does not appeal to me at all. Too much of an interference.

UP: *So could we be a bit specific about your work without getting into technical details?*

ES: I will try. I did start out in finite group theory, but I left it for algebraic geometry, as I already told you.

UP: *What about finite group theory?*

ES: You are coming back to it. True, my first published paper, back in 1972, concerned semigroups, but it was just a short note. But throughout the 70s all my mathematical papers concerned finite groups, but getting into the 80s my future husband Corrado de Concini started to have a more significant influence. This was a period I have definitely put behind. I was just too isolated in Italy, to say nothing about Rome.

UP: *So back to algebraic geometry. What did you study?*

ES: More specifically I was studying special varieties with a lot of structure and hence there being much to play around with. And also with some interesting applications, inter-mathematical I should add.

UP: *Varieties as such?*

ES: Varieties of complexes, of projectors, flag-varieties, the conormal bundle of the determinantal variety, varieties related to symplectic vector spaces. Other examples are varieties given by G/P where G is a Lie group and P is a parabolic subgroup. The latter may not at first look very geometric, but you can talk about lines on them. This might give you an inkling of my taste.

UP: *What do you do with those varieties?*

ES: All kinds of things, like finding their defining equations. And then of course my interest in groups has not abated, much of what I do centers on groups, I am doing invariant theory.

UP: *Do you do it from a complex analytic point of view, working with truly continuous groups?*

ES: Not at all, I am very much interested in positive characteristics, and have generalized results by Hermann Weyl and George Kempf in 0 characteristic to positive. The methods are quite different and involve a lot of combinatorics.

UP: *Could you elaborate on the Weyl bit?*

ES: Sure. Am I allowed to be a little bit technical?

UP: *Go ahead.*

ES: Let V be a symplectic vector space. Consider for an integer N the algebra of endomorphisms of the tensor algebra $V^{\otimes N}$. Inside it, you find the spin algebra $Sp(V)$ and you are interested in its commutator. This can be described as the algebra generated by the symmetric group S_N acting on the tensor algebra by permutation on the factors, which turns out to be isomorphic to $K[S_N]/I$, where I is the ideal generated by the anti-symmetries on $n+1$ letters, where $2n$ is the dimension of V.

UP: *That was quite a mouthful. As you do things also in finite characteristics, are there interesting connections to finite group theory, in particular the work you used to do, and which might have been of some help?*

ES: It has more to do with representations of classical groups over finite fields of arbitrary characteristic.

UP: *What is the work you have been most proud of?*

ES: I do not know about being proud of, but the hardest work was in connection with so-called wonderful compactifications.

UP: *Which are not so wonderful, I gather. So without being technical could you elaborate?*

ES: No. Let me just say that those compactifications concern adjoint semisimple groups, in arbitrary characteristic mind you. There is nice geometry involved, some aspects of which I have been exploring. To be more specific, I have been looking at wonderful compactifications of symmetric varieties and their Chow rings and trying to compute some invariants thereof. The so-called ring of conditions, if you have heard about that?

UP: *No.*

ES: Anyway. It involves a lot of combinatorics.

UP: *Are you still active?*

ES: Yes, in fact I have just started a research project with a colleague M. Lanini studying the cohomology of a flag variety under a Poincaré–Birkhoff–Witt degeneration.

UP: *So you are after all collaborating.*

ES: I am not a fanatic, sometimes it is after all quite natural and convenient.

UP: *Could you tell me a little of your life as you became an established professor? When was that?*

ES: This was in connection with the birth of my son Guglielmo, who was born in 1988, as you should know…

UP: *Yes, on the very same day as my youngest daughter Alina…*

ES: I had got a full professorship at Roma Tor Vergata, a new university in the city, which had been founded just a few years earlier. The facilities were temporary and primitive, but a few years later they improved, though they were placed in the outskirts of Rome some 25 km from the center. I had to commute by metro and bus, which took me an hour and a half each way.

UP: *It must have been a nightmare? With a baby and everything.*

ES: Not at all. In many ways it was one of my most productive times. I sat and worked hard while in transit. I read papers, I wrote articles, you name it. Sometimes I was so immersed in what I was doing that I forgot to get off at the last stop, and had to be rescued in the dark by the cleaning staff of the Metro. From there on there was a bus to the university, and I was lucky if I got a seat, but was fun nevertheless mingling with students and colleagues.

UP: *Now we have not touched upon the issue of women and mathematics, as far as it is an issue at all.*

ES: What about it?

UP: *You have been active in it, sitting on many committees as I understand, such as the Women in Mathematics Committee of the European Mathematical Society. Do you enjoy such work?*

ES: It has nothing to do with enjoyment at all, it is just very important. I have really put in much effort, and it has been a lot of work during many years. So many committees, so many meetings. Of course, I have to admit that arranging meetings could be quite fun, but it amounts to a lot of work. But meetings are very important, and this of course goes beyond the issue of women in mathematics.

UP: *What is the major issue when it comes to women and mathematics?*

ES: It is representation, they are simply underrepresented in all kinds of respects, as editors of journals, in prize committees, in short whenever it comes to female influence and power. How come until Seoul there were no single female Fields medalist? I remember discussing the matter with Ragni Piene a year or so before.

UP: *As for underrepresentation, women are in great demand to sit on committees in order to meet the requirements, but that is not necessarily welcomed by women, many who feel obliged to participate unduly in such work which I assume consumes a lot of time and effort and thus is not always welcome, maybe even in some cases hampers their careers. Is not the basic problem that so few women go into mathematics, and actually why is that a problem?*

ES: It goes without saying this is an important cause. I would say that France has a long tradition of women doing mathematics and provides an example to all other countries. It is also true that a lot of women go into mathematics in Italy, but the majority does not have the ambition to go beyond teaching in high school; but that might change. The academic situation is much tougher now than forty years ago when we were young, and this holds both for women and men. As to committee work, as I said, I have committed a lot of time to it, and out of duty, not out of real pleasure. And I feel that it has not had the effect I had hoped for, so I have to admit to a great disappointment. I will not put the blame on some specific institutions, at least not publicly.

UP: *A question I often pose to women, at least to successful women, is whether they personally have felt any suppression; for feminists outside mathematics this is taken for granted due to* *the above-mentioned underrepresentation, but so far I have not met anyone who has admitted it. Would you be an exception?*

ES: Let me say that I am a tough lady, I might not look it, but I am. I am quite persistent, not to say stubborn, and I stick up for my rights, and I can argue forcefully for them, so maybe I am not the right person to ask.

UP: *Let me change track and ask you a personal question: Have you seen mathematics as a career or a calling? In other words, have you seen it as important to get ahead, or has this just been a nice but unintended consequence?*

ES: Mathematics is a competitive subject and I quickly realized what was needed. I have worked hard, being by necessity involved in competitions for work, you and your readers are probably familiar with the Italian system of *Concorsi* when you actually compete for positions nationwide, submitting your work to committees. There are three levels, roughly corresponding to assistant, associate and finally full professor. I have been very lucky that I was able to stay in Rome all the time. Only after reaching the upper level could I relax and have a child, it would have been impossible before that. And by the way, the matter of bearing children really puts women at a disadvantage, I am full of admiration for those successful women mathematicians who have managed to have had many children. Of course husbands can be helpful and supporting, but women cannot escape the concomitant duties unlike men, and even if the men go out of their way they cannot relieve you of more than half of the burden. But this is a digression, where were we…

UP: *Talking about the hard work involved in your career.*

ES: Oh yes. It also involves a lot of travel.

UP: *But that is the fun part, to be cynical for many of us, the most fun part.*

ES: Speak for yourself. Travel is unavoidable, you simply have to meet people, to get new ideas and outside stimulations; few indeed are those lone geniuses who are entirely self-sufficient. This is why I travel. And conferences are fun of course, at least the social part, but talking mathematics to people, that is what counts. And of course solving a problem is one thing, indispensable of course, but then you also have to sell your problem and solution and to convince people that what you have done is both clever and important. It is of course great if you have a mentor who explains those things to others, but for most of us you have to do that yourself, and that is quite a challenge.

UP: *So the social aspects of mathematics are important?*

ES: Very much so, as in all other ways of life. It is very important who you know and in what environment you work. Even if you may not be aware of it, you do absorb so much by mere osmosis. Getting ahead just on your own locomotion is not an option for most people. You need outside ideas, otherwise you dry up. This is something you just have to acknowledge.

UP: *So just reading the literature is no substitute?*

ES: Of course not, you need to know what to read, and what it really means. You may have an informal conversation with someone and learn more in half-an-hour than you would only learn from reading a paper for months if even that is enough.

UP: *Why is that? Is it because a paper is too complete, for the sake of documentation you have to include everything, which of course is a good thing, but not for conveying ideas? For the latter you need some motivation, given the right motivation and some key insight, things will fall into place, because most of a proof is just a matter of transportation, and as such rather tedious to read, as opposed to fill in the details.*

ES: Very true. I cannot but agree completely. However, I would like to add something important. You need to meet people in the flesh, it is not enough by e-mail, phone conversations and even zoom. I want to really hear the voices of the others, to become aware of gestures, expression of faces. All of those seemingly peripheral matters actually contribute and can make a difference. And not to forget only by engaging with a person face to face can you tell whether you are dealing with somebody intelligent or not.

UP: *Our common friend Fabrizio Catanese used to say that you can only gauge the worth of a mathematician by collaborating with him/her on a paper.*

ES: So true! I am so glad that you bring up Fabrizio. Discussing mathematics with him has been one of my most rewarding experiences. It is not so easy to have an interesting and fruitful conversation with a mathematician.

UP: *There has to be a common ground, and that is becoming harder and harder with the increased specialization of mathematics.*

ES: Of course you need common ground, that is obvious, but also some kind of chemistry. I have learned a lot of algebraic geometry from Fabrizio, because for one thing he knows a lot obviously, but what is much, much more important is that when he talks mathematics he is not only precise but also clear, and I find this a great help. Then of course from the perspective of algebra I have learned a lot from my husband Corrado and also from Procesi,

Fabrizio Catanese, in Bayreuth 2022 at the home of his colleague Jörg Rambau (photo by Jörg Rambau).

Her friend and colleague Claudio Procesi as member of the Abel Prize committee around 2006.

but I want to emphasize Fabrizio, because most people would not realize that.

UP: *We have now spoken a lot on math and your mathematical career, but people may be curious about the individual Elisabetta Strickland, something that mathematical achievements seldom throw much light on. Was mathematics the only subject that really interested you at school?*

ES: By no means. Literature was my favorite subject. And also remember I was interested in all subjects in principle. I should also mention that I very much liked physics, but between that and mathematics my choice was mathematics.

UP: *So mathematics was special?*

ES: Only to the extent that I was lucky having such a wonderful math teacher.

UP: *But that also applied to your fellow schoolmates, so it must go deeper. What about literature, have you pursued that?*

ES: I am a reader of course, but most of all a writer; I have always loved to write, and so do you, I know, and I have kept on writing all my adult life. I have published several books, and I regularly contribute articles to various journals. One particular example, is that I edit a Newsletter for Women in academics and have to regularly come up with something. As to interests, I have always loved sailing and at one time in my life I submitted articles to various sailing magazines. They paid well.

UP: *What are examples of your non-mathematical books?*

Elisabetta Strickland sailing off the coast of Sardinia, 1977.

Elisabetta Strickland in Edinburgh.

ES: I have written on female scientists, such as Mary Somerville, who among other things 'checkmated' Maxwell, but also on Emmy Noether who stood up to Einstein, and also Sophie Germain, who I claim, provocatively of course, founded mathematical physics and many more not so well known. And then of course on Maryam Mirzakhani, the first female Fields Medalist whose early death was so tragic and such a loss. If you want something less academic and more literary I can refer to *I numeri nel cuore* in which I, along with my colleagues Ciro Ciliberto and Fausto Saleri, collected some of our short stories, albeit with mathematical themes.

UP: *What about your interest in photography?*

ES: That you know about of course, did we not have a photo exhibit together?

UP: *That was ages ago. But what about your interest?*

ES: I combined it with my sailing activities. I loved to take photos of sailboats, especially engaged in racing. One particular case was when I was hard up trying to shoot a really unconventional picture. I approached a guy with a helicopter and was able to convince him to give me a ride. I entered, he strapped me to the seat and took off. I was a bit apprehensive of course, never having been in a helicopter before, and so he opened the door on my side and turned the helicopter sideways, so I was staring into the void below me. Normally I would have been terrified, but I had a job to do, and I leaned out with my camera and shot several pictures from my unique vantage point. They made a splash, some of them even ended up on the cover of sailing magazines and I got a golden plate from Kodak for one of them.

UP: *You got to be famous.*

ES: I would not say that, but it certainly had the sweetness of fame, I was very proud. I have another anecdote to give you and your readers a taste of that life. One morning I decided to do something special coming to cover a sailing event. There was the Whitbread race in England. Somehow I managed to get there, I was a research scholar, so with little money at the time; and once there I had the problem of lodging. I discovered a boat with some young Danes and asked them whether I could sleep over in their boat. They responded with due enthusiasm, and we had a wonderful evening laughing, talking and drinking beer. Wonderful, magical times, nothing untoward happened. In the morning they took off for the race, and I was fit for fight.

UP: *Do you have any other hobbies?*

ES: What do you mean? Is not two such enough? True, I am interested in many things, but if I had not pursued a career in mathematics, I might have been an interior decorator. Such things amuse me a lot. Recently Corrado and I bought some property in Tuscany, which we have set up quite nicely, if for no other reason than to entice our son and his girlfriend, who are living in London, to visit us more often. It seems to work.

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The helicopter photo taken during the Sardinia cup, 1981.

Henry Pinkham (1948–2023)

In memory of a friend and colleague

Ulf Persson

Henry Pinkham and I arrived at Harvard in the fall of 1971 to start graduate school in mathematics. For me, it was a momentous step, only two years out of high school and for the first time in my life being on my own. For Henry, it was not, I assume, such a big deal, being about two years older than me, and having already been an undergraduate at Harvard and in addition having spent a mathematical year in Paris just prior. He was thus far more seasoned than me, and ended up taking the rôle of an elder brother, although none of us may have thought in such terms at the time. While some of us may have suffered from unrealistic hopes and ambitions combined with a lack of confidence, Henry struck me as remarkably level-headed. He had a realistic sense of his capabilities and knew what it took. Henry was three-quarter French, his mother French as well as his paternal grandmother, while of course his paternal grandfather was a Pinkham, and that was the family which counted, among other things connected to the Pinkham Notch up in the mountains of New Hampshire. He and his siblings grew up in New York, where his father was a lawyer. When his father got offered a position at the Paris branch of his firm, Henry and his elder brother were put in a French-speaking school in New York in preparation for the move. When in France, Henry attended the lycée (roughly corresponding to American high school as to age, but academically more demanding). However, he admitted later on, he was not fully accepted, but was known as *l'americain*, and so it was natural for him to return to the States to attend college. His father envisioned a professional future for his son, and was of the conviction that it was only law, medicine and engineering that really counted. Consequently, Henry started out as a pre-med at Harvard, but soon his burgeoning interest in mathematics (for which he had already received prizes at school – and incidentally for Latin essays as well) took over. His father also introduced him to the opera during their Parisian sojourn, to which he often referred fondly. Already as a graduate student Henry had acquired an impressive collection of classical records, especially of opera, for which he had developed a passionate interest. I remember a huge portrait of Verdi hanging prominently on the wall.

Henry Pinkham in academic garb at a commencement at Columbia around 2010 (courtesy of Sophie Pinkham).

Henry and I attended together the introductory course on Algebraic Geometry given by David Mumford (who would soon become our common thesis advisor) during our first year. But our social interaction with each other and other students would not really pick up until our second year, when the department moved from the cramped quarters on 2 Divinity Avenue to the newly built Science center. Now for the first time every graduate student was assigned a little cubicle. The cubicles were arranged in groups of ten or so around so-called talk areas. Henry managed to claim the best cubicle in one of the talk areas, almost the size of a small office, with a window looking out and no window looking in, as was the case with the other cubicles. I and fellow graduate students of Mumford, such as Avner Ash and Linda Ness, moved into the same talk area, and we became a closely knit group. All of this reflected a change of attitude of the faculty towards its students, who now

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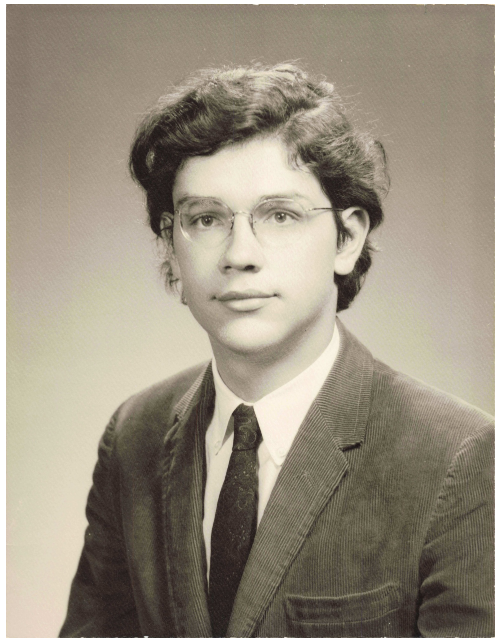
had a more tangible connection to the department and were not as marginal as before. What is cause and what is effect in all of that is hard to know, and most probably a little of both. In all fairness the previous set-up was more or less the norm for most prestigious departments at the time. As Philip Griffiths pointed out at the time, graduate students learn as much from their fellow students as from their professors. Be that as it may, it certainly encouraged learning among peers.

Henry was given a problem on deformation theory after his first year. He worked hard on it, and already by his second year he had a paper submitted to Journal of Algebra. I recall how proudly he showed us the proofs. At this time there was no $\TeX$, so it was only in a journal (if even that) that you could see your work properly typeset, and as my father used to say, seeing your name in print is a big boost. To put things in perspective, deformation theory was at the time still in its infancy, and while deforming complete intersections by varying the equations is immediate, the situation is significantly harder in general. A machinery to do so in general had been worked out, but to apply it in non-trivial situations tended to lead to intractable computations. Mumford had realized that looking at cones over curves should make the calculations much easier, and Henry was advised to look at cones over rational normal curves. The task was not only to find non-trivial deformations, but to describe them all in terms of being parametrized by a variety, the so-called versal deformation space (incidentally the term 'versal' is derived from the well-known term 'universal' when the ultimate ambition of uniqueness in a technical sense, could not generally be achieved). Already at the case of the rational normal quartic, a surprise turned up. The versal deformation space had two components, one 3-dimensional as expected, and one unexpected 1-dimensional (I believe I have seen it referred to as the 'Pinkham component' in the literature, but experts assure me that this is but a figment of my imagination; I think he deserves it, maybe by this remark it leaves my imagination and enters the literature). Anyway, examples are only significant as far as they open eyes and new vistas, and the calculation he had performed turned into a method applicable to a much larger family of examples which got incorporated in the thesis (titled *Deformations of algebraic varieties with G_m-action*) he defended a year later (May 1974), and in the fall of that year it appeared in the Astérisque, the journal of the French Mathematical Society. Not typeset, however, but nicely typed by Laura Schlesinger, who typed all the theses of Mumford's students. Henry had made it, and he landed a junior position at the Columbia Math Department, and we were all impressed. Little did he know, or even suspect, that he would be attached to that department for the rest of his life. But what was really important to him was that it was in New York, the city of his childhood (he used to wax on about walking along a Manhattan avenue, as in a canyon formed by skyscrapers). But most of all it was a world-center of culture, in particular when it came to music with its world-renowned opera.



Henry Pinkham in his youth, probably at Harvard College around 1966 (courtesy of Sophie Pinkham).

Our relationship might have ended here, as it often does upon graduation, when everyone is dispersed by the wind, pursuing their separate lives. Instead of ending, the second part of it, even more important than the first, would start. In fact, much to my surprise, I would follow him to Columbia the next year. Unlike Henry, I had no sentimental attachment to New York; although it did have since childhood a certain romantic appeal due to the abovementioned skyscrapers, it did on the contrary scare me a bit, and there were actually good reasons for that at the time. Nevertheless, I chose Columbia, and in retrospect it must have been his presence there that tipped the scales. Our first year we shared a nice corner office with a view of Broadway below. I learned to have take-out lunches from the local delis and get my taste of cultural life through becoming a member of both MoMA and the Metropolitan. However, my immersion in the city life was somewhat hampered by regular weekend commutes on Amtrak to my wife in the Boston area. And this brings us to the strange parallelisms of our lives we had started to notice.

In the summer of 1973 I was invited with my then girlfriend Mindy to his wedding to Wendy (both names somewhat similar) down in Long Island, and the next summer he was one of the exclusive guests to my own wedding at Arnold Arboretum in Boston. In the spring of 1976 we both applied for leave from Columbia; I to join my wife in Cambridge where she was doing a medical internship, he to accompany his wife to Paris, where she was going to be posted as a lawyer. And at the end of the same year I was divorced, and the following summer so was he, and we

Henry Pinkham with his youngest daughter Jess around 1986 (courtesy of Sophie Pinkham).

both experienced a sense of elation at the perceived liberation we both shared when we met in Paris the following summer. He had bought a ten-speed bicycle (something his ex-wife supposedly had not allowed) and told me excitedly about his recent bike trips with his new girlfriend in the countryside, sleeping over in barns. Henry was something of an athlete. We had both started running at the Columbia indoor track during our common year, and as noted, he liked to bicycle. Furthermore, he told me about his wind-surfing board down at his family's summer place by the dunes at Cap Ferret, close to Bordeaux (the location of his French roots). This was the first time I heard about such things. His two years in Paris had not been wasted. His thesis had steered him into surface singularities, and he had written some very nice notes on rational surface singularities and simultaneous resolutions of rational double points, delivered at the surface singularity seminar he had run together with the two seniors Demazure and Teissier at École Polytechnique, and published in the Springer Lecture Notes Vol. 777. He had also been pushing the results of his thesis further, developing the so-called Pinkham method of constructing smoothings of so-called negative weight, which would become standard techniques of deformation theory. Clearly he was already establishing himself as one of the young upcoming algebraic geometers.

In the fall of 1978 we were once again united at Columbia. Henry was very happy being back in the States again. Although his identity had, as already noted, a prominent French component, he had come to realize that it was in the States he felt most at home, at least academically, and where his future lay. He also reconnected with his future wife Judy whom he had known a long time before. At the end of that academic year Henry had come up with a very unconventional idea for attacking the problem of degenerations of (polarized) K-3 surfaces, namely to show that either all the components of the degeneration were birationally ruled, or that there was a birational K-3 surface among them, and after a base-change (to get rid of multiplicities of components) all the others could be blown down. The second case, corresponding to finite monodromy, had as a significant corollary that the period map for K-3 surfaces was surjective. I had been given the problem by my advisor Mumford as a graduate student and treated the easy part, namely the case when there were no triple points (meaning that by Hironaka reducing to the case when the degenerate fiber consisted of smooth components meeting transversally there would not appear any triple points) but had been stymied by the intractable combinatorics in the general case, involving complicated birational transformations on three-folds. The strategy was to reduce the canonical divisor of the 3-fold to the special fiber. The Russian mathematician Kulikov had at the time presented a head-on attack along those lines, which had been the subject of some western seminars to understand. However, the formidable birational combinatorics turned out to have been very hard to follow, and there was a need for a more transparent approach. This was exactly what Henry's approach was intended to do, evading that formidable combinatorics of blow downs and blow ups. He discussed it with me, which was natural as I was after all the local expert by virtue of my previous exposure to it. After a brief discussion during which I must have made some relevant technical comments, he very generously invited me to become his co-author, an opportunity at which I jumped. We wrote up the paper, Henry very much being the senior partner, and he suggested that we go big and submit it to the Annals. He talked about it at a meeting in Athens, Georgia a few weeks later, and in the summer we met up again at another conference, now in Angers, France, and this time it was my turn to present it. I guess it was not a success, I lost everyone, but at this stage of my career I thought of that as rather being an accomplishment. As our colleague Fabrizio Catanese has remarked, you can only properly judge the quality of a mathematician by working with him, and this experience of mine bears it out (incidentally, we would co-write a second paper a few years later, this time on my initiative, but – in the words of Henry – it sank like a stone).

Henry was not a flashy mathematician, who thinks quickly on his feet, and impresses, as well as intimidates everyone, with poignant remarks, especially during seminars and such occasions. Brilliancy, at least not in this sense, was not something you associated with Henry, instead it was solidity which may last longer. Our paper was published, and there was some predictable controversy with the Russian school. Henry's unconventional approach immediately found resonance among our younger colleagues, and what more can you expect from a paper? A few years later Henry would write another Annals paper, now with Ian Morrison, a paper which had connection with algebraic number theory.

And then, after such a successful start of his mathematical career he ceased to publish. What happened? I can only speculate as coincidentally there was a twenty-year hiatus in our relations on which there is no need to dwell. My guess is that he realized, as you will eventually do at a party, that it is not going to get any merrier. He was satisfied with what he had done, and felt it was time to move on. Many mathematicians in his position might have stuck with it for want of better things to do, but not Henry. He had so many other interests and talents, obvious ones as well as hidden ones it would turn out. He took charge of running the Columbia math department, and revived it in the process. Although this had been intended as an administrative interlude, part of the responsibility of any successful academic, he carried it beyond the call of duty and pursued it passionately, eventually ending up as a dean. He later told me that he had found in himself unsuspected talents, such as fundraising and generally interacting socially with people in ways most mathematicians, by temperament or professional upbringing, have little experience of. He also impressed me, at a lunch he treated me to, how he had arranged a meeting for Columbia alumni to a grand party in Paris renting le Musée d'Orsay. So this is what is going on behind the scenes? I thought. After twenty years we met again (at Provincetown to be precise) at the 70th birthday party of Mumford. We had of course both changed a little, in his case he had gotten a little slimmer, but we hit it off as if the hiatus had never occurred. It transpired that we both had daughters who studied Russian, another sign of the parallelism of our lives (that the daughters shared the same first name did not hurt). Later on we got briefly together in New York, visiting an exhibition at MoMA and having the abovementioned lunch, then to retire to uptown and his apartment (incidentally supplied with new bookcases, courtesy of the studio which had temporarily used the apartment for shooting the film *The Mirror has Two Faces*, directed by its principal star, Barbra Streisand). We caught up on our lives, and he reminisced on his mathematical career, revealing that he had nursed higher ambitions than he had ever let on. He regretted somewhat that he had gone into algebra, doing analysis instead might have been more fruitful, at least as to applications. In fact, he had already got involved in applications devising algorithms for Adobe PostScript fonts together with Ian Morrison. They even set up a company, which they later sold with a profit, enabling Henry to buy a property in upstate New York, a retreat which many years later would come in good stead during the Covid-pandemic.

When we met in New York, he did not have a cell-phone, whether that was only temporary I do not know. But I definitely sensed it was not a modern gadget of which he approved, just as he did not approve of many features of the prevailing Internet culture

Henry Pinkham with his wife Judy at his summer place at Cap Ferret (courtesy of Sophie Pinkham).

(it is hard for me to imagine that he would spend much time on Facebook). All of that I can heartily sympathize with, but I suspect that Henry carried his antagonism even further than I did, and certainly more consistently. This was also very much in character with his preference for writing with a fountain pen, something I was always struck with, being already by then a trifle old-fashioned. His handwriting was strikingly neat, and even more remarkably, he never seemed to make a single false stroke. In the days before e-mail his handwritten letters were for this reason alone a delight to receive. We both shared a passion for reading omnivorously, but unfortunately we never interacted on this; and just as we may regret many things we never did in our youth, we may also regret that we never fulfilled all the potentials of a friendship. When it came to music I was too incompetent to interact with him at all, although he was quite aware of my sympathetic attitude; more than once he supplied me with tickets to a concert or a dress rehearsal of an opera.

In view of the passion Henry had for music, this might be the place to insert yet another digression. Mathematical talent is often considered to be related to musical talent (but seldom the other way around). I would not say it is a myth, but I suspect that in most cases it is rather a question of sympathy, and then exclusively concerning classical music. When it comes down to it, there is concretely little that the two have in common. The formal mathematical description of music pertains only superficially to either music or mathematics. A visual diagram is much more conducive to mathematical understanding than a melody. However, one talent

Henry Pinkham at Bassin d'Arcachon, Cap Ferret (courtesy of Sophie Pinkham).

does not exclude the other, and when talent in both is present, they may reinforce each other, but still in ways which are mysterious. In fact, the mysterious way both are created shows some hidden affinity, just as on a more abstract level themes occur and reoccur reinforcing each other in mathematics as well as in music. Henry was not only a connoisseur of music, although that is the only aspect I encountered, he once wrote me during his first extended Parisian sojourn, that he had resumed playing the piano and that it had meant a lot to him. And much later on at Columbia he was given the opportunity to teach music, which must have gratified him a lot (one may be tempted to speculate that this would involve more appreciative students than is common in mathematics; after all, people supposedly study music at this level out of pleasure rather than obligation).

After we resumed our contact some fifteen years ago we kept in touch, but only through e-mails. Our contacts were not as frequent nor as urgent as they once had been in our youth, but that was only natural. It came as a shock to me this past August to learn that Henry had passed away in Paris a month or so earlier, a city which he loved. I had hoped to be able to host him down here in southern France whereto I had moved upon my retirement, and looked forward to future conversations. But that was not to be. Our meeting in New York in the summer of 2007 would turn out to be the last time we met in the flesh. When a lifelong friend dies, you are diminished because shared memories are no longer shared and hence lose substance and start to fade and wilt.

Fortunately, the work of a mathematician remains, and in fact, as a common colleague has noted, references to his work have even increased more in recent years.

Henry leaves a widow – Judy Moore, and their two daughters Sophie and Jessica, as well as a grandchild.

Acknowledgements. I would like to thank Eduard Looijenga and Jonathan Wahl for having read an initial draft and supplied invaluable comments and advice not only on the mathematical part (this applies particularly for Looijenga) and thus saved me from potential embarrassment more than once. For this I am deeply grateful.

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Hédy Attouch, in memoriam

Michel Théra

Hédy Attouch.

Hédy Attouch, a brilliant student at a Saint-Germain-en-Laye high school, at one point hesitated between continuing higher studies and a career as professional footballer, a field in which he excelled in the junior and youth teams of Stade Saint-Germain. At the insistence of his father, director of a primary school in Versailles, he finally chose the studies which would lead him, after the Louis-le-Grand high school, to the École normale supérieure of Cachan. With his aggregation in hand, Jacques Deny, specialist in potential theory, encouraged him to follow a series of lectures given in Paris VI by Haïm Brézis. In his biography [2], Hédy writes: *I got fascinated by his* (i.e., Brézis's) *course, where functional analysis was married with such mysterious objects as multivalued monotone operators and lower semi-continuous convex functions, and which allowed to solve large classes of nonlinear PDEs*. Haïm Brézis then jointly proposed to Hédy and Alain Damlamian (whom I did not know at that time, but with whom I subsequently had relations both scientific and friendly) a common research subject whose main idea was to combine, within the same differential inclusion, the monotonicity methods developed by Ky Fan with an application to economics proposed by Claude Henry (a researcher at the Laboratory of Econometrics of the École Polytechnique). This collaboration between Hédy and Alain led them both to complete a doctorate in 1976 at the University of Paris VI (Pierre and Marie Curie) and to publish several joint articles, including a seminal work concerning evolution equations governed by time-dependent subdifferential operators [6].

Hédy's academic career began in 1971 by obtaining a position as Maître de Conférences at the University of Paris Sud, Orsay. In 1976, he defended his Thèse d'Etat in Paris VI, and his second thesis subject (compulsory at that time) on the theory of cooperative games. The subject had been proposed to him by Jean-Pierre Aubin. After his Thèse d'Etat, he obtained a sabbatical leave during which he went to the United States, where he met several renowned mathematicians: Louis Nirenberg at the Courant Institute, Michael Crandall at University of Wisconsin, Felix Browder at University of Chicago, Terry Rockafellar at University of Washington, Seattle, and Roger Wets at University of Kentucky. It was with the latter that Hédy began active and fruitful cooperative research focused on several interconnected areas such as epigraphic analysis, quantitative analysis of stability of variational systems, variational convergence of functions, multivalued random processes, and the epigraphic law of large numbers. It is from this collaboration with Roger Wets that Hédy obtained one of his famous results with which his international reputation began: *Given a sequence of lower semi-continuous and proper convex functions, the sequence epi-converges to a convex function if and only if the graphs of their subdifferentials converge (in the sense of Mosco) to the subdifferential of the limit function, modulo a normalization condition which fixes the integration constant*. For readers who are interested, the numerous results obtained during these years of fruitful cooperation with Roger Wets are recorded in Hédy's book [1] as well as in an article we co-authored on the occasion of an international conference organized at the CIRM [8]. Another very important result obtained by Hédy, which is central in convex analysis, is the Attouch–Brézis theorem, which gives a sufficient condition for the Fenchel conjugate of the sum of two lower-semi-continuous convex functions to be equal to the exact inf-convolution of their conjugates.

At the very beginning of the 1980s, Hédy was invited to Pisa, where a decisive meeting took place with Ennio De Giorgi, the founder of the topological theory of Γ-convergence, a theory that broadens considerably the framework of variational convergences. Hédy took advantage of this stay and the emergence of homogenization theory to redirect his research towards the study of laws of the physics of composite materials from a macroscopic point of view. In 1983, he became a professor at the University of Perpignan, where he was welcomed by Alain Fougères, who was interested in developing new ideas around integral functionals and variational analysis. The latter suggested creating a new laboratory centred on these themes. This is how the AVAMAC (Variational Analysis and Applications to Mechanics, Automatics and Control) laboratory was born, with Hédy as director. Then, in 1988, he joined the University of Montpellier and more precisely the *Convex Analysis Laboratory*, where he was welcomed by Jean Jacques Moreau, Charles Castaing and Michel Valadier, all well known for their work on convex analysis, measurable set-valued mappings and Young measures. It was in this laboratory and in this environment, where Lionel Thibault and many others later joined, that Hédy continued his entire career until his retirement. During these years in Montpellier, he trained numerous students, wrote numerous papers, and participated in the transformation of the journal Travaux du Séminaire d'Analyse Convexe into a high-level international journal, the Journal of Convex Analysis.

Hédy with his wife Annie and Bernard Cornet.

Hédy sharing a discussion after the Convex Analysis Seminar in Montpellier with Jérome Bolte and Michel Théra.

I had the pleasure of welcoming Hédy to the Séminaire d'Analyse Non-linéaire et Optimisation that I had created in Limoges in 1987; he was one of my first guests. The chance to know him better and appreciate him came a few years later during a joint stay at UCD (University of California Davis), when we formed a very friendly and mathematically very fruitful relationship. I came alone with my children, my wife not having received authorization from her employer to be on leave for a long period. We rented apartments next to each other and spent a lot of time together, and Hédy and his wife helped me look after my children; we shared meals and excursions, and we played tennis. It was during this stay that Hédy convinced me to try golf at the Davis Golf Course. We both taught, linear programming for him and a calculus course for me. This stay would constitute for me a turning point in my approach to mathematics, which until now had remained abstract. Hédy, who had, more than me, a very good knowledge of physics and mechanics, pushed me to look at mathematics from an angle linked to potential applications. We wrote several articles together. One of them, an article for which we invited Jean-Bernard Baillon to participate, concerns the notion of variational sum for maximally monotone operators [3]. In [4], we exhibited for an evolution equation the link between the concept of weak solution in the sense of Benilan and Brézis and the concept of variational sum. In addition, we gave an application of the latter to the parabolic evolution equation involving the Schrödinger operator. In relation with Hédy's early work, we also obtained in uniformly Banach spaces a positive result linking the convergence of subdifferentials with the bounded Hausdorff topology. As far as I know, the complete equivalence in a general Banach space is still an open problem [7]. Another article, [9], concerns a general concept of duality for nonlinear problems, a concept that encompasses all classical duality relations like Fenchel's, Toland's and Clarke–Ekeland. This result shows that the dual of the dual problem is the primal problem, and the primal problem has at least one solution if and only if this is the case for the dual problem. This article, very well cited in the literature as the source of the Attouch–Théra duality principle, was completed during a joint visit at the University of Montreal organized by Francis Clarke.

Hédy with Jean-Baptiste Hiriart-Urruty at an international conference (NVA 2007) in Limoges.

Busy with various important administrative tasks, I regretted (and I still regret) not having accepted his offer to participate for the part which concerned duality in the writing of his book [5] co-authored by Giuseppe Buttazzo and Gérard Michaille.

Hédy then became passionate about new themes, with an impressive number of different collaborators: Samir Adly, Felipe Alvarez, Radu Boț, Luis Briceño-Arias, Jérôme Bolte, Alexandre Cabot, Zaki Chbani, Patrick Combettes, Roberto Cominetti, Marc-Olivier Czarnecki, Xavier Goudou, Juan Peypouquet, Patrick Redont, Hassan Riahi, Marc Teboulle and many others. This research was devoted to continuous- and discrete-time approaches and the development of fast algorithmic methods for convex and non-convex optimization problems. More precisely, with his collaborators, Hédy obtained very important results concerning the proximal point algorithm, forward-backward methods, the rapid convergence of inertial dynamics and algorithms with zero asymptotic viscosity, an inertial proximal method for maximally monotone operators via the discretization of a nonlinear oscillator with damping, minimization methods and methods of alternating proximal projections based on the Kurdyka–Łojasiewicz inequality, the convergence of descent methods for semi-algebraic problems, and tame and regularized Gauss–Seidel methods. Despite Hédy's various invitations, I was not able to participate in this work because I was myself involved with different collaborators in the theory of error bounds; but I closely followed the impressive development of his latest results.

In 2021, Hédy received the George B. Dantzig Prize (awarded every three years by SIAM and MOS), jointly with Michel Goemans, *for his fundamental contributions to modern variational analysis and nonsmooth optimization, including new notions of variational convergence, the introduction of novel topologies for the study of quantitative stability of variational systems, and their application in algorithm design and analysis, dynamical systems and partial differential equations*.

To conclude, how can we talk about Hédy without mentioning his kindness, his tolerance, his goodwill towards his colleagues, his ability to communicate and share. He contributed in an exceptional way to international scientific life by enthusiastically sharing his knowledge, and he leaves a significant mathematical work that his numerous students and collaborators will continue.

Hédy left us on 14 October 2023. Farewell, Hédy! Annie, your wife, who accompanied you to all the conferences to which you were invited, misses you very much, and so does our mathematical community.

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On the Balkan Mathematics Conference (BMC) series

Betül Tanbay and Volker Mehrmann

BALKAN
MATHEMATICS
CONFERENCE

The EMS supports several types of regional conferences, some of which have a long tradition, like the Nordic Congress of Mathematicians. While mathematicians from the Nordic countries work together for decades, the situation for mathematicians in other parts of Europe is not as easy. The colleagues have a lot of difficulties, often due to a lack of funding or other support. Moreover, the political situation does not always make regional conferences straightforward. And yet mathematicians should have equal opportunities all over Europe, and the EMS sees it as one of its tasks to bring colleagues together to further develop mathematics and the interchange of ideas between different mathematical communities. These observations were the motivation to start in 2014 the series of Caucasian Mathematical Conferences (CMC). In 2021, the EMS Executive Committee decided to support a new biennial series, the Balkan Mathematics Conference (BMC).[1]

The BMC is planned as a regional conference organised under the auspices of the EMS with the purpose of bringing together mathematicians from the Balkan and neighbouring countries, biennially in one of these countries. These conferences equally welcome mathematicians from all over the world. The BMC is also supported by the Mathematical Society of South Eastern Europe (MASSEE).

Planning was ready in theory, but the pandemic prevented to concretise the first meeting immediately. A cooperation between the EMS and the Romanian mathematical community was established, and the first Balkan Mathematics Conference took place in July 2023 in Pitești, Romania, jointly with the 10th Congress of Romanian Mathematicians.

It was supported by the Romanian Mathematical Society, the Stoilow Institute of Mathematics and the University of Pitești, as well as by the EMS. It was a great success, with six excellent talks and a great organisation by the Romanian colleagues. The opening lecture was given by Dan Voiculescu, which had a symbolic meaning for the Romanian mathematical community, where the topic of operator algebras has a long tradition.[2]

In addition to a local organising committee, two EMS committees, a Steering Committee and a Scientific Advisory Committee support the organisation of a BMC meeting. The Steering Committee consists of three ex-officio members – the EMS president, the immediate past president of the EMS, and the chair of the EMS Committee for Solidarity in Europe – as well as one regional coordinator. The Scientific Advisory Committee consists of the invited speakers of the previous BMC and a representative from the Steering Committee. The Steering Committee is responsible for the scientific program of the conference and for the selection of the next edition of the BMC. It is also responsible for preparing a list of potential invited speakers and a list of potential early-career invited mathematicians. The task of the Scientific Advisory Committee is to select the six invited speakers and the twelve invited early-career mathematicians from the list proposed by the Steering Committee.

[1] https://euromathsoc.org/regional-conferences

[2] http://imar.ro/~bmc

EMS President Jan-Philip Solovej, past EMS Vice President Betül Tanbay and past EMS President Volker Mehrmann at 1st BMC Pitești.

In Fall of 2023, a general call was made for the organisation of the second BMC to take place in June 2025. The Steering Committee has decided to approve the bid of Aristotle University of Thessaloniki. As one of the most ancient cities of Europe, from the Kingdom of Macedonia to the Byzantine Empire and Ottoman Empire, to present day Greece, Thessaloniki has a huge historical heritage, since it has notably given access to the international maritime trade routes to the entire region of the Balkans. Its choice is thus perfectly appropriate for hosting the wide range of mathematicians the EMS aims to reach.

Betül Tanbay is a professor of functional analysis at the Boğaziçi University in Istanbul. She was founder and first co-director of the Istanbul Center for Mathematical Sciences. She was the first female president of the Turkish Mathematical Society, and she has also served and serves in many committees of the IMU or EMS. Betül Tanbay received her undergraduate degree from the Louis Pasteur University, Strasbourg in France, and her graduate degrees from the University of California, Berkeley. She is the chair of the BMC-II Steering Committee.

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Volker Mehrmann is a retired professor at Technische Universität Berlin. He received his diploma, teacher's exam, PhD and habilitation from the University of Bielefeld, Germany. He spent research years at Kent State University, at the University of Wisconsin, and at the IBM Scientific Center in Heidelberg. After spending the years 1990–1992 as a visiting full professor at the RWTH Aachen, he was a full professor at Chemnitz University of Technology from 1993 to 2000 and until 2023 full professor for numerical mathematics at Technical University of Berlin where he was from 2008 to 2016 the chair of the Research Center Matheon in Berlin. He is a member of Acatech (German Academy of Science and Engineering), the Academia Europaea, and the European Academy of Sciences. He is fellow of the AMS and SIAM and he was president of GAMM (Association of Applied Mathematics and Mechanics, Germany) and the European Mathematical Society. His research interests are in the areas of numerical mathematics and scientific computing, applied and numerical linear algebra, control theory, and the theory and numerical solution of differential-algebraic equations. He is an associate editor of several journals and the editor-in-chief of *Linear Algebra and its Applications*. Volker Mehrmann is the chair of the BMC-II Scientific Advisory Committee.

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African Institute for Mathematical Sciences (AIMS)

Winnie Nakiyingi

The African Institute for Mathematical Sciences (AIMS) is a pan-African network of centres of excellence, strategically positioned to contribute to scientific and technological advancement through education, research and innovation. Its mission is to cultivate Africa's talented youth into creative leaders of science and technology. AIMS selects the brightest young minds across Africa, shaping them into independent thinkers and problem solvers with the capacity to innovate, ultimately steering Africa toward economic prosperity. Positioned at the heart of the innovation and transformation ecosystem, AIMS offers a comprehensive array of academic and non-academic programs meticulously designed to provide learners with a distinctive postgraduate training experience.

A glimpse into AIMS

As the first and largest network for postgraduate training in mathematical sciences across Africa, AIMS assumes a pivotal role in sculpting a prosperous Africa, driven by innovative education through its five training centres in South Africa, Senegal, Ghana, Cameroon, and Rwanda, complemented by a research and innovation centre in Rwanda. Since its establishment in 2003, AIMS has evolved into the STEM secretariat for Africa, with over 200 partners, more than 250 researchers, and over 500 world-class lecturers. This pivotal role extends beyond borders, as AIMS actively collaborates with academic institutions, research organizations, and industry partners on a global scale. These collaborative efforts serve as conduits for knowledge exchange, resource sharing, and the seamless integration of African scientists into the broader global scientific community.

AIMS offers a comprehensive array of full-time programmes, including a one-year Structured Master's in Mathematical Sciences, an 18-month Co-operative Education, a Master's in Mathematical Sciences for Teachers (MMST) programme, and PhD programmes in subjects including data science. The master's programmes commence at the end of August and follow a three-semester academic structure, aligning with the distinct phases of skills development, review, and research. The skills and review phases are organized into three-week blocks of courses, with the skills phase comprising three mandatory blocks designed to equip students with essential computational and mathematical problem-solving skills. The subsequent review phase encompasses six blocks, requiring students to successfully complete a minimum of eleven courses, a maximum of two per block. This structured approach ensures a robust and thorough preparation for students to maximize their learning experience throughout the programme.

AIMS graduates an average of 300 master's students every year, with the skills needed to pursue successful careers as university lecturers, research scientists, financial analysts, public health advisors, entrepreneurs, and data scientists among others. By August 2023, AIMS master's alumni constituted a vibrant community of 3177 youth, 34.6% of whom are women. AIMS alumni have demonstrated remarkable employability globally, spanning the private, public, and academic sectors. As of December 2022, the transition rate into employment and further studies stood at an impressive 88%, with African Master's in Machine Intelligence (AMMI) graduates leading with the highest transition to employment at 81%, followed by Co-op Master's at 74%, and Structured Master's at 59%.

The majority of employed AIMS alumni (78%) choose to contribute their skills and expertise within Africa. Their impact is significant across various fields, with tertiary education (such as university lecturers), research (in academic institutions and both public and private sectors), and information and communications technology (ICT) emerging as the top three areas of employment. Diving deeper into their contributions, AIMS alumni are actively engaged in fields crucial for Africa's development, including climate sciences, epidemiology, and data science – machine learning – AI. Furthermore, their influence extends beyond immediate employment, as they play pivotal roles in advancing STEM education and research. This includes roles as secondary school teachers, university lecturers, and researchers, both within Africa and on a broader international scale. The impact of AIMS alumni resonates not only in their individual success stories but also in their collective efforts towards the progress of STEM fields and education.

Many alumni choose to continue their studies post-AIMS and pursue a research master's degree or a PhD, the majority of whom receive competitive full scholarships. A total of 35% of AIMS alumni are pursuing further studies, of which 80% are pursuing a PhD and 20% are pursuing a research master's degree. The top fields of study are mathematics, physics, health, and finance. Of all the alumni who have completed or are pursuing a PhD, 50% of them were accepted directly from the AIMS master's programme, thereby reflecting the calibre and rigour of the AIMS program.

In March 2024, AIMS celebrated 20 years of impact under the theme *Siyakhula: Growing Mathematics in Africa*.[1] The festival brought together all mathematicians, including representatives from universities, research institutions, governments, the private sector, civil society organizations, and the general public.

SIYAKHULA
Growing Mathematics in Africa

AIMS centres

AIMS South Africa

AIMS South Africa[2] was established in 2003, in Cape Town, as the first centre born out of a partnership among six universities: Cambridge, Cape Town, Oxford, Paris-Sud XI, Stellenbosch, and Western Cape. The centre collaborates with local universities to integrate specialized courses into regular honours and master's programmes, fostering innovation through research. In 2008, the AIMS South Africa Research Centre was launched to contribute to advancing African science and academia within a rich multicultural context.

Areas of specialization include cosmology and astrophysics, mathematical and physical biosciences, mathematical finance, foundations of mathematics and scientific computing, and data science and information systems. In September 2023, AIMS South Africa launched the AI for Science Master's programme,[3] in partnership with Google DeepMind. The centre also empowers teachers across Africa through AIMSSEC,[4] amplifying the pipeline of maths and science students through training programs. In alignment with its commitment to showcasing the transformative power of maths and science, AIMS South Africa initiated the House of Science[5] public engagement program which capacitates teachers, elevates public science literacy, and propels mathematics and scientific communication throughout South Africa.

AIMS South Africa Centre in Muizenberg, Cape Town.

AIMS Senegal

In 2011, with support from the Government of Senegal, AIMS Senegal[6] became the second centre within the AIMS Network, with a master's programme accredited by the National Authority for Quality Assurance in Higher Education (ANAQ-Sup), Senegal. Since its inception, AIMS Senegal has graduated students through three distinctive pathways: a structured 10-month master's programme, an 18-month Co-operative Education programme (more than 80% of students following this programme find jobs within the 6 months after graduation), and the African Master's in Machine Intelligence (AMMI)[7] programme, which delves into the realm of machine intelligence, catering to the evolving demands of the digital era.

In 2013, the AIMS Senegal Research Centre was established with the appointment of Prof. Mouhamed Moustapha Fall (a member of the Scientific board of EMS-CDC) as the German Research Chair. AIMS Senegal researchers have won international mathematics prizes like the Ibni Prize, the Abbas Bahri excellence fellowship of Rutgers University, and the DST-ICTP-IMU Ramanujan prize, in geometric analysis, stochastic PDEs, and biomathematics.

[1] https://www.siyakhula.aims.ac.za
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[6] https://aims-senegal.org/our-story
[7] https://aimsammi.org

PhD students during the training school on malaria modelling at the AIMS Senegal Centre in Mbour.

AIMS Ghana Centre in Accra.

AIMS Ghana

Established in 2012 with the Government of Ghana's support, AIMS Ghana[8] is recognized as a UNESCO Category II Centre of Excellence and is accredited by the National Accreditation Board (Ghana). Its programmes include a structured 10-month master's programme and a specialized master's programme for maths teachers (MMST).[9] The centre also runs a Girls in Mathematical Sciences programme (GMSP)[10] a fully-funded 9-month initiative for bright female Senior High School students which mentors participants for STEM careers at the highest levels in research, training, and industry, cultivating a new class of leaders in research and innovation.

The AIMS Ghana Research Centre opened in 2014 with a focus on quantitative biology and epidemiology. In 2016, a German research chair "Mathematics and its applications" was also recruited. The centre's research focuses on stochastic analysis and applications, singular stochastic differential equations, optimal control, controllability, integro-differential equations, game theory, dynamic programming, climate change, and quantum algebra.

AIMS Cameroon

Located in Limbe, AIMS Cameroon[11] was established in 2013 with the support of the Government of Cameroon, as the fourth centre. AIMS Cameroon has a structured 10-month programme and an 18-month Co-operative (Co-op) Education programme with a direct link to industry through work placements, a Teacher Training Program (TTP)[12] designed to improve learning outcomes in mathematics for secondary school students, and a research centre.

MSc graduation at the AIMS Cameroon Centre in Limbe.

The AIMS Cameroon Research Centre, started in 2017, is supported by Alexander von Humboldt Foundation, DAAD, and the German Federal Ministry of Education and Research, with research areas like mathematical analysis, nonlinear analysis, climate sciences, shape optimization, control theory, deep learning, and big data, which address real-world challenges such as environmental pollution and urban networks.

8 https://aims.edu.gh
9 https://aims.edu.gh/aims-master-of-mathematical-sciences-for-teachers-mmst
10 https://aims.edu.gh/girls-in-mathematical-sciences-programme
11 https://aims-cameroon.org
12 https://aims-cameroon.org/teacher-training-program-2

AIMS Rwanda

Established in 2016 with the support of the Government of Rwanda, AIMS Rwanda[13] is the fifth centre of the AIMS Network. The centre offers a structured 10-month master's programme, along with an 18-month Co-operative (Co-op) Education programme with industrial attachment in various fields including data sciences and malaria modelling, and a training programme in mathematics and sciences for high school teachers in Rwanda.

MSc graduation at the AIMS Rwanda Centre in Kigali.

The centre, in collaboration with Rwandan government institutions, provides short courses leading to certificates for professionals in several fields of interest, including climate sciences, big data analytics and software testing. The AIMS Rwanda Research Centre has two research chairs who work with postdoctoral fellows and PhD candidates in mathematical optimization, data-driven and machine learning applications, discrete mathematics and theoretical computer science.

AIMS Research and Innovation Centre

The AIMS Research and Innovation Centre (AIMS RIC)[14] is the youngest centre in the AIMS ecosystem, established in 2023. The centre is home to Quantum Leap Africa (QLA) which seeks to prepare Africa for the coming quantum revolution. Research focuses on data science (including machine learning and AI), quantum science, and mathematical modelling, with team members comprising research chairs, resident researchers, postdoctoral fellows, PhD and MPhil students, and interns.

The centre is positioned to respond to current and emerging challenges such as pandemics, climate change, and food insecurity. For example, the centre's president, Prof. Wilfred Ndifon,

AIMS staff after a meeting at the AIMS Research and Innovation Centre in Kigali, Rwanda.

worked with the Rwanda Biomedical Centre (RBC) to lead a team that developed a COVID-19 pool testing method which was successfully used in Rwanda and beyond to track people infected with the SARS-CoV-2 virus. AIMS RIC operates a doctoral training centre currently providing training in data science and malaria modelling.

Other programmes

AIMS Industry Initiative

The AIMS Industry Initiative[15] builds and leverages industry partnerships to identify win-win opportunities for AIMS students and alumni as well as industry partners. These include short-term work placements, internships and employment. AIMS works with organizations to fill top vacancies with proven talent, selected from a base of AIMS graduates and students with world-class training in mathematical sciences and applied research, in addition to other technical and life skills necessary to succeed in performance-driven organizations.

Impact and alumni success

AIMS has made a significant impact in producing a new generation of African scientists and researchers. Its alumni have gone on to pursue successful careers in academia, research, industry, and policymaking, contributing to the advancement of science and technology in Africa. Some success stories can be found on individual centre websites or on the AIMS blog[16] under the #AlumoftheWeek series.

[13] https://aims.ac.rw
[14] https://research.nexteinstein.org
[15] https://nexteinstein.org/industry-initiative
[16] https://nexteinstein.org/meet-our-alumni

AIMS students interacting with industry partners.

Conclusion

This article aims to inspire mathematicians across Africa by highlighting the transformative impact of AIMS on the continent's mathematical landscape. With a focus on nurturing experiments, stimulating conjectures, and establishing standards for exploring new mathematical frontiers, AIMS provides an inviting environment for mathematicians to actively engage in exploration and experimentation. Beyond being an institution, AIMS serves as a catalyst for mathematicians to embark on journeys of discovery, pushing the boundaries of knowledge and venturing into unexplored territories. This article extends an invitation to mathematicians to join the vibrant community cultivated by AIMS, contributing to the ongoing narrative of growth, exploration, and advancement in mathematics throughout Africa.

For more information about AIMS, please visit the AIMS website.[17]

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[17] https://nexteinstein.org
[18] https://wordsthatcount.org

SUBgroups

Online peer groups for first-year math graduate students (gradsubgroups.org)

Nelson Niu, Katie Waddle, Marquia Williams and Yufei Zhang

SUBgroups

online peer groups for first-year math graduate students

share uplift bridge

The beginning of graduate school can be an isolating, confusing, and overwhelming experience. SUBgroups are online peer support groups that bring together first-year math graduate students, primarily from U.S. institutions. Participants meet biweekly over video conferencing platforms during the first semester of their master's or PhD programs. Established in 2019 and running two cohorts per year, SUBgroups has now supported hundreds of students to navigate these first difficult months.

Beginning as a student in a math graduate program creates new challenges along several axes simultaneously. We will describe four of these challenges and then say how SUBgroups peer groups work to address them. First, students are likely moving to a new city far from their support networks, and setting up a new home, finding a new grocery store, and figuring out a new commute. Students may be in a new country and navigating a language they speak but have not had to use full-time before. Most graduate programs are in large universities with byzantine logistical obstacle courses involving visas, health insurance, and payroll systems. A graduate program likely has intricate requirements, often involving a series of difficult tests, coursework, and finding an advisor.

While most U.S. graduate programs are structured so that a group of students matriculate in a cohort, these cohorts can be small, and it is common for people with minoritized identities to be extremely isolated. Women, people of color, queer folks, etc. often arrive in a graduate program to find they are one of a small handful, and people who have multiple of these identities can find themselves completely alone. Support across difference can be affirming, and allyship is essential for continuing to move towards a more diverse math community. At the same time, being able to talk about what you are experiencing with someone who shares your identity is sometimes essential. And while many of us have supportive friends and family that share our identities, they are often unfamiliar with the world of higher mathematics.

The early years of math graduate programs can feel siloed. Often students are not yet attending conferences or summer schools, and may not yet know many students in other programs. Without knowledge of what exists elsewhere, self-advocacy is difficult. For example, students in one program may struggle through a particularly difficult high-stakes qualifying exam process, while students in another are allowed to use course grades to show proficiency. Without knowing about other systems and realities, students normalize and accept unnecessary hardship.

A final challenge is the hierarchical nature of academia. In those first few semesters, students are not yet attached to an advisor, or see them infrequently. In contrast to high school or college where you hear from mentors frequently and (hopefully) receive positive feedback often, in the first few months of graduate school most students do not yet have anyone championing their success. Professors are over-worked and have little time for students before they pass their initial qualifying exams. In a time when students are taking the most challenging courses they have ever taken, or embarking on real math research for the first time, they are also receiving few signals that they are heading in the right direction.

Graduate programs can and should work to alleviate some of these challenges, and many are already aware of them and do excellent work to combat the isolation students feel. But there is still much room for improvement, and some of these challenges are impossible for a single graduate program to solve on their own. One attempt to work on these problems outside the auspices of a single graduate program is our initiative SUBgroups. In 2019 Drs. Marissa Loving and Justin Lanier established SUBgroups with the goal of bringing together students from different math graduate programs to share the experience of their first semester. The goal of SUBgroups is to leverage the scale of the larger math graduate school community to combat the isolation and siloization of individual programs.

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Challenge #1: Navigating a stressful transition. SUBgroups was designed around the U.S. academic schedule. We coordinate two cohorts of students per year, one that starts meeting in September and another in October (while many schools in the U.S. start their year in late August, some schools begin in late September). The goal is to provide just-in-time support for students as they are beginning their new lives as graduate students. Students may be wondering where to go for academic or emotional support, how to decide which classes to take, how to think about possible advisor options, or be teaching for the first time. In SUBgroups students meet every other week with a group of students facing the same challenges. Group members can share resources, pass along good advice, and act as a sounding board. Most importantly, as a group of sympathetic peers, a SUBgroup can offer moral support during a stressful and potentially emotional time.

We have designed SUBgroups to alleviate the stress, not add to it. We know this is a busy time for students, so we make the application and group setup process as simple as possible. We do not require any essays or letters of recommendation, the application is a short survey that gives us enough information to put together groups that will hopefully have interesting things to talk about. Setting up the group meetings is as simple as using an online tool to find a workable time-slot and setting up a plan for video conferencing. We compose groups with participants who have similar time availability to make coordination easier. Groups only meet once every other week, since the first semester can be so overwhelming, and it can be difficult to find times that work for folks in different time zones with different schedules.

Challenge #2: Isolation of folks with minoritized identities. Our main approach in combating isolation is to cast a broad net, and then leverage our size. We advertise broadly, by reaching out to graduate program directors, putting notices in the AMS Headlines and Deadlines e-newsletter, and posting on Facebook and Twitter both to make use of our networks and to reach underrepresented people. We are as inclusive as possible, and welcome future academics as well as people intending to go on to teach or to work in industry, folks in statistics, and students in bridge-to-PhD programs. A single graduate program may be too small, but there is power in numbers.

To form groups, we collect demographic information in our application process, and ask participants what they are hoping to get out of the experience. Students say things like "I'd really like to connect with other non-traditional students,[1] I feel very alone in my cohort in that sense" or "It would be awesome to have peers who are also trans or nonbinary, or (more broadly) identify as queer." We use this information to put together groups of four to six participants, often that share aspects of their identity. Our recent September cohort included a group of people who identify as queer, a group of Latino students, and a group of students who are first in their families to go to graduate school and wanted to talk to others in a similar situation. To make up for a lack of diversity in small graduate cohorts, SUBgroups can do the legwork to connect folks facing similar challenges across programs.

Challenge #3: Siloization. In some ways our participants are very similar to each other. They are all beginning math graduate programs, almost all at large universities in the U.S., and they are taking courses, exploring their math interests, and beginning the search for an advisor. Before each week of SUBgroups meetings we send out an email with a few suggested prompts. These prompts start by asking participants to share the basic structure of their program—what classes and exams are required, which classes are they taking the first semester, etc. We move on in future weeks to ask about how they are managing the added workload of graduate school, connections they are making with their cohort and classmates, and with faculty. Towards the end of the semester when groups have gotten to know each other, we ask about personal well-being—how they are doing and what strategies they are using to manage the stresses of their new life. Many of our participants comment in our feedback forms that they are grateful to hear that others are experiencing the same things as them. Not only the same types of tasks and hurdles to overcome, but the same stresses and doubts as well.

Sharing these commonalities also helps to reveal differences in helpful ways. For example, one of the authors, Katie, was trying to find community through her school's chapter of the Association for Women in Math, but finding one-hour events in the middle of the day lacking in depth. After talking with a member of her SUBgroup whose AWM chapter was hosting occasional happy hours, Katie went back to her own chapter and suggested an evening event that turned out to be much more personal. From small things like finding out about funding sources, to big structural things like learning that other schools have options to pass coursework instead of exams, learning how things are done elsewhere can be invaluable.

Challenge #4: Positive feedback. One of our hopes for SUBgroups is that it can be a support for students that might not have one otherwise. At the beginning of graduate school in the U.S. system you are not yet attached to an advisor, and so no one is yet invested in your success. First-year graduate courses are extremely challenging, but often no one is checking in to see if you are doing ok. We all need a pat on the back, a gold star, and the encouragement to keep going, and math graduate programs are often not set up to provide that positive feedback. In seeing the same people every

[1] The term "non-traditional student" in the U.S. usually means a student with a less common path to doctoral study. They may have completed two years of community college before transferring to a four-year undergraduate institution, or have worked for a number of years before pursuing graduate studies. It also may mean they have extra responsibilities, such as care-taking for children or parents.

other week, the hope is that this small group can cheerlead each other through the first semester. Whereas a cohort at an individual university may feel disconnected, or worse, competitive, we invite SUBgroups to simply be a small group with a shared understanding about what they are going through that cares about each other's success. Of course graduate program cohorts can provide this support and often do; SUBgroups is a place to seek that support if it is lacking.

The future of SUBgroups. When Marissa and Justin formed SUBgroups in 2019, their intention was to create a sustainable program that was efficient and effective. As they moved on in their careers, they transitioned leadership to the authors, all former SUBgroups participants, who run the program as a service to the math community. While SUBgroups should remain a peer support community of grad students supporting grad students, we would welcome institutional support to help manage logistics.

As a leadership team, we are always trying to learn about and improve SUBgroups, and we try to incorporate feedback regularly. In addition to sending out surveys partway through and at the end of the program, we reach out to some minoritized participants directly to make sure that we are meeting the needs we set out to address.

There are a couple areas we are looking to improve next year. We hope to continue to streamline the logistics of setting up groups at the beginning of the semester—our biggest point of group collapse is still at the initial setup stage when groups fail to find a common time to meet, or somebody drops the ball on communication. We do not have moderators in the groups, because we believe that small groups of peers are the most effective unit to provide the kind of support we are trying to give. But this can be a challenge, as we do not know what exactly is happening in each group, and occasionally we hear only later about uncomfortable group dynamics. We continue to think about how to give participants tools to facilitate effective conversations that feel safe and comfortable for everyone to share.

Finally, SUBgroups has primarily been a U.S.-focused program, and expansion beyond the U.S. comes with new challenges. In particular, the U.S. is highly coordinated around an academic year that starts in late August or late September, and around five to six years long PhD programs that include challenging coursework before students find an advisor. We would be excited to talk to people interested in starting a similar program in Europe that is adjusted to a European academic calendar and European-structured master's and PhD programs. A huge component of SUBgroups is the support of talking to people experiencing something very similar to what you are experiencing, and we have found that folks in the United States and in European graduate programs have a harder time finding commonalities across the many differences. In a different direction, a group of junior physicists has replicated the SUBgroups model with a similar program called SU(5), which is aimed at supporting first-year physics and astronomy graduate students. Their success suggests potential to expand to other fields of study, as needed. We would be interested to hear about any similar initiatives elsewhere, or ideas and suggestions for further developing SUBgroups.

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Youth Commission of the Royal Spanish Mathematical Society

Érika Diz Pita

The Youth Commission of the Royal Spanish Mathematical Society aims to look after the interests of young Spanish mathematicians working in any professional field.
The commission carries out multiple actions, some aimed at analyzing the situation of young mathematicians and detecting possible problems within this collective, and others aimed at finding solutions to them, such as providing information on access to predoctoral and postdoctoral contracts and other jobs or organizing workshops that offer cross-disciplinary training, or that address specific problems such as mental health.

The Royal Spanish Mathematical Society (RSME) is a scientific society whose main objective is to promote and disseminate mathematics and its applications, as well as to encourage research and teaching at all educational levels.

In this society there are different commissions that are responsible for the fulfillment of its different objectives. Each of these commissions has a more specific field of action, for example, one focuses on education, one on outreach, one on the role of women, and one on youth. In what follows, we will try to give an overview of some of the activities carried out by the RSME Youth Commission.

The Youth Commission aims to look after the interests of young mathematicians working in any field and location, in Spain as well as abroad. It is a transversal commission composed of about 10 young mathematicians coming from different professional fields and at different stages of their professional career, including PhD students, young PhDs, high-school teachers, or mathematicians working in the business sector. This composition changes every year, and members may not remain for periods longer than three years, in order to ensure that the commission is diverse and that new opinions and points of view are constantly being received.

1 Analysis of the current situation of young mathematicians

In general, among the young population, some of the main concerns are finding a stable job, having economic independence, or satisfactory working conditions that allow for an adequate balance between personal and family life. Although in recent years mathematics has gained more and more presence in all sectors of society, young mathematicians face difficulties related to achieving these goals at the beginning of their working or research careers. From the Youth Commission we try to identify these problems and determine their seriousness and extent. For example, over the past year, we have sought to obtain testimony about the mental health issues faced by young mathematicians, and to determine the extent to which young researchers consider abandoning their research career, or for what reasons they consider this option.

Leaving the research career

In Spain, in recent years, there has been a growing serious concern about a possible loss of young researchers. In certain regions, some postdoctoral researcher or assistant professor positions are not receiving the necessary candidates to fill the positions offered. In addition, there is an increase in the number of people who, after defending their doctoral thesis, decide to work in the business world instead of continuing their academic career. The Youth Commission designed a survey to detect whether abandoning a research career was a possibility that many young mathematicians were considering. The survey was shared with all the faculties and research centers in Spain, and was disseminated through social networks and the official publications of the RSME. Almost 250 responses were obtained from young mathematicians. A summary of the most significant results was presented in [4, 5] and some examples are included in Figure 1.

Among these results, it is worth noting, firstly, that more than half of the respondents stated that they considered abandoning their thesis before its completion. The respondents include people who are still working on their thesis and also others who have already completed their thesis. If we consider only those respondents who had already defended their doctoral thesis, more than 70% of them considered abandoning at any time.

We asked what factors they consider to be determining when considering alternatives other than academic work: more than 90% highlight the lack of job stability as the main cause, followed by low salaries, mentioned by almost 70% of those surveyed. Almost 60%

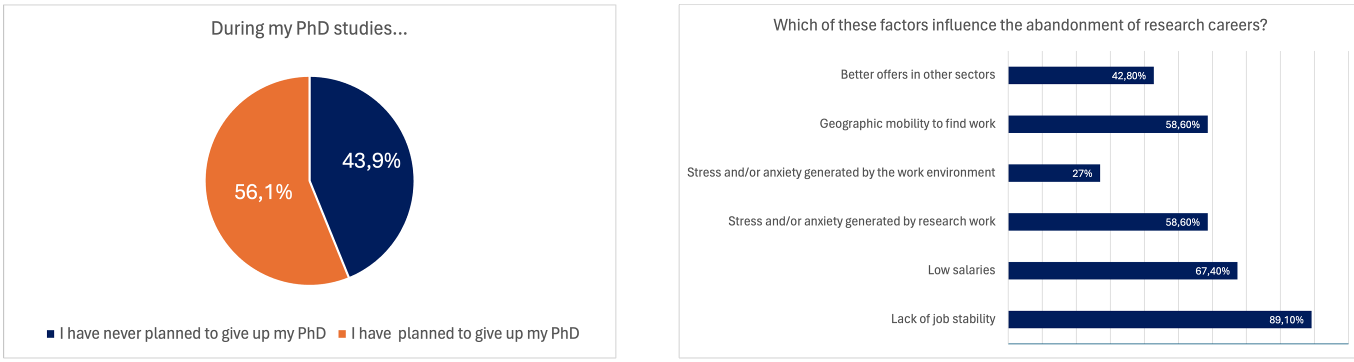

Figure 1. Some results obtained in our survey on the abandonment of the research career.

mention two other problems: the geographical mobility required for many research jobs, and the stress and anxiety generated by working in this field.

All these factors negatively affect the research landscape in Spain, and for this reason, we at the commission actively advocate that society must work to find collective solutions to these problems faced by young researchers.

Mental health

Mental health problems appear as one of the causes that lead young researchers to consider a new future away from academia. Therefore, one of our objectives in the commission is to provide support and guidance in this area. To this end, several brief surveys were also conducted in an attempt to obtain some feedback on the acceptance of our activities and whether the young people consider them necessary and useful. We obtained some remarkable data, for instance that 80% of the respondents say that their work as a researcher causes them some disturbance in their emotional well-being, such as stress or anxiety. More than 70% consider that they do not have all the necessary tools to deal with possible stressful situations at work. More than 90% were willing to participate in activities that would provide them with these tools and resources.

Information on job opportunities

In view of the opinions collected from young mathematicians, achieving job stability and a decent salary is one of the issues of most concern to young people. That's why from the Youth Commission we have compiled a collection of information on possible funding options for the pre-doctoral stage [1, 2], that can serve as an orientation for those who are starting in the world of research. In addition, to try to facilitate the job search process, the RSME together with Infojobs have launched a portal of career opportunities for mathematicians.[1]

[1] https://rsme-talento.infojobs.net

Contact us!

We believe that the relationship between the RSME, especially the Youth Commission, and young mathematicians can be very beneficial for all of us, as we have stated in [3]. We want to be part of a RSME that serves as a meeting point with people who share a similar situation to ours, or who have gone through it before, and who can listen to us, guide us and give us advice. We also believe that society as a whole can benefit if it listens to the voice of young people, everything they have to say, and pays attention to their vision of the world. That is why we believe it is essential to have direct and fluid communication between young mathematicians and the Youth Commission, as a link to the RSME. To this end, we always have a communication channel open through our email, jovenes@rsme.es, where we are happy to receive your opinions and suggestions.

2 Workshops

One of the objectives of the Youth Commission is to provide learning and professional development opportunities for young mathematicians beyond the technical knowledge acquired during their bachelor's and master's degree studies. For this reason, we try to organize different workshops that serve to provide the participants with complementary and transversal tools and skills that will be useful for their professional career.

All the activities we carry out are free of charge, and since we try to accommodate people from all over Spain, the activities always have the option of online attendance. Figure 2 shows the posters of two of these activities.

Workshop on mental health

The Mental Health Workshop is one of the activities that has become a regular feature of our commission. Three editions of this workshop have already been organized and have been a success in terms of attendance and reception. The latest one, held in

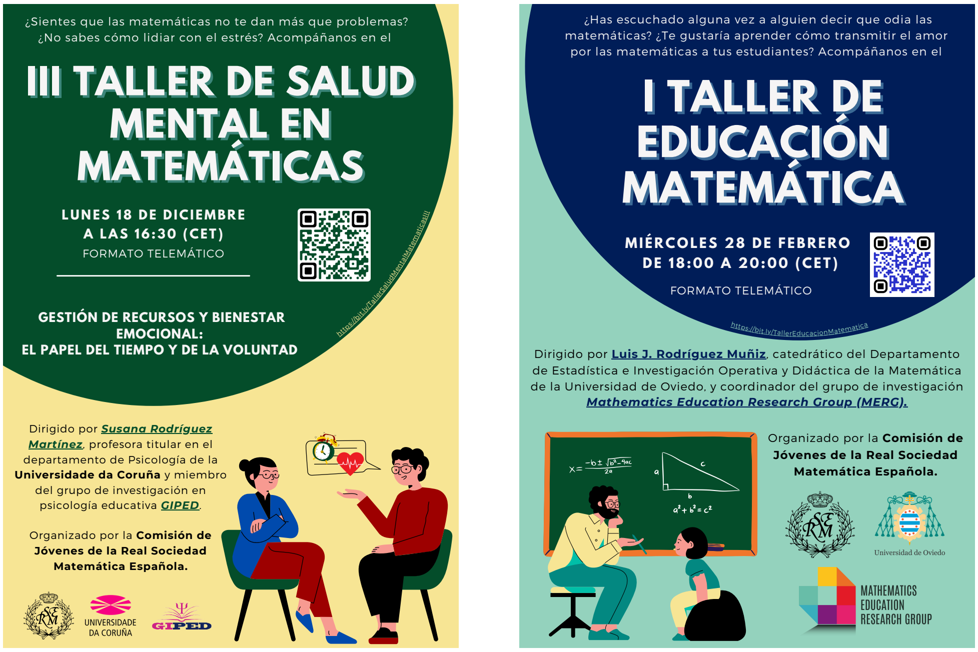

Figure 2. Workshops on mental health and mathematical education organized by the RSME Youth Commission.

December 2023, focused on resource management and emotional well-being [6]. Doubts and problems proposed by the attendees were answered, and tips and tricks were given that can help to manage work in a more efficient and healthy way.

For example, to face difficulties with time management, the psychologist in charge of the workshop explained how to divide tasks into categories: high and low demand tasks, and how to distribute our time alternating one and the other with breaks. Work was also done on how to detect limitations that one imposes on oneself, how to manage some common problems such as the impostor syndrome, or how to achieve an adequate relationship and communication with other people involved in our work, such as PhD advisors, scientific reviewers or senior colleagues.

We believe that this type of activities serve to slightly reduce the impact of some problems young mathematicians are facing. However, we are sure that we need to strengthen this initiative, and provide young people with many more resources for achieving an optimal mental health.

Mathematical Education Workshop

This year (in 2024) we have launched another workshop, in this case with a more didactic orientation, focused on mathematical education. One of the great challenges of mathematicians has always been to achieve a correct transmission and teaching of mathematics to students, who have historically shown, in general, special difficulties and rejection of our subject. If we look, for example, at the results of the latest PISA report, we see that Spain has obtained the worst results in mathematics in history. It seems quite evident, therefore, that improving the teaching and learning of mathematics should be of priority.

The workshop, which took place last February, served to reflect on the way in which mathematics is taught at different educational levels, on which practices could be improved, which should be eliminated because of their poor learning results, or on what other innovative techniques we can use to achieve meaningful learning.

In our country, the didactic training of secondary-school mathematics teachers is reduced in almost all cases to a one-year master's degree, which in many cases is insufficient to deal with the great variety of realities existing in high schools and the needs of a very diverse student body. For university teachers the situation is even more serious, since in many cases they do not have any didactic training before teaching their first classes. In our objective of providing young people with this complementary and transversal training, this is a first small step, which we consider insufficient, and which must be reinforced institutionally, but which can help in some cases to overcome certain fears or to overcome small problems that arise in the day-to-day life of many teachers.

3 Future work

We are proud of the work we do, but we want to do more. Our desire is to be able to help all young mathematicians in the different aspects of their academic and professional careers. To this end, we will continue with many of the initiatives that are already in place, such as the various workshops, but we have many improvements and new proposals in mind.

We would like, for example, to take forward initiatives that are more attractive to young mathematicians in the private sector, or to set up a mentoring program for the youngest people. To achieve more and better results, we are willing to continue working hard, but we believe that two fundamental pillars are necessary: collaboration, with more people and societies, and funding.

What is the situation in other parts of Europe and the world?

Working at the local level offers certain advantages, as each country or region has its own particularities, which is why the existence of national or regional societies is important and valuable. However, from the RSME Youth Commission, we are also interested in knowing what the situation is like in other regions of Europe or the world. Within our commission we have people who are working from other countries, or who have stayed or worked abroad, and they can give us a first version of the differences and similarities with our country.

However, for the future, it seems interesting to collaborate with other institutions and organizations in other regions of the world, such as the EMS Young Academy, and to try to compare the situation in different places, e.g., the conditions under which young people work in each case, and above all, what we can learn from what is being done in other parts of the world, and is currently achieving good results.

Imagine if we had funding!

One of the major problems that science is facing in all of its fields is the lack of funding. So far, all the activities developed from the RSME Youth Commission have been carried out without funding, and therefore by resorting to volunteers who selflessly offer to share their knowledge and enthusiasm in their area of expertise, be it mathematics, didactics or psychology, among others.

At present, and in order to defend our belief that any work should be adequately recognized and remunerated, we are trying to obtain funding from research centers or official calls for proposals. We believe that having sufficient resources will favor our ability to continue organizing activities that are of interest to young people, and will open up new possibilities, achieving a greater reach and impact of all our actions.

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“If, and only if” in mathematics

Omid Ali Shahni Karamzadeh

1 Why should we consider the converses of our results more seriously?

Although mathematics, and sciences in general, owe a lot to our predecessors, it appears that in mathematics some of them are also responsible, unintentionally, for our possible negligences with regard to the forms of the statements of some results. Most of the results in Euclidean geometry, number theory, and on the subject of inequalities, and so on, in the literature are of the form “if P then Q.” In particular, almost all the exercises and problems in the elementary textbooks on Euclidean geometry are of this form (see, e.g., [1, 6, 7, 18]). In the results presented in this form, Q is not asserted, only that Q is a necessary consequence of P. This style of thinking (called here briefly *pq-style*), which refers generally to thinking in the form “if P then Q” for presenting results, is somehow deep-seated in some of us for a long time. Perhaps, from the time of our learning secondary school mathematics. Later, it has also led some of us on, and made us so overwhelmed with this kind of thinking, that we are still generally attempting to draw necessary conclusions.

This is perhaps why Peirce, defining mathematics, begins his lengthy article [20], with “Mathematics is the science which draws necessary conclusions.” And more explicitly, Bertrand Russell, in [23, Chapter 1], essentially says: “Pure Mathematics is the class of all propositions of the form ‘p implies q,’ where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants.” At the same time, there are also some results in the literature which, even though not stated in the “if, and only if” form, are, in fact, intrinsically of this form. However, the actual forms are not emphasized on.

For example, in a triangle with two different angles we can state that an angle is bigger if, and only if, it has a shorter bisector, or equivalently if, and only if, the altitude which goes through the vertex of that angle is shorter, and finally if, and only if, the median which goes through the vertex of that angle is shorter (for the latter, less familiar fact, see the expression of the lengths of medians in terms of the lengths of the sides in [1, Theorem 106]). Most of the related books which contain some of these results state them in pq-style, see for example [1, Theorem 114], [7, Lemma 1.512], [6, Exercise 7, p. 9]. If we were used to presenting our results in the “if, and only if” style, perhaps the notorious old query concerning a direct proof of the Steiner–Lehmus theorem could not stay open in the literature for such a long time. Some students and their teachers, throughout the world, still have some trouble with this question, see [4, 13], for more details. Or, if the Morley’s trisector theorem had been stated in the “if, and only if” style, perhaps this theorem would not have seemed mysterious to so many authors, see [12, Corollary 1] for a statement of the theorem in the latter form, and also see the recently published books on the philosophy of mathematics, [21, pp. 41–43, and p. 46] and [17, p. 253]. Even in most elementary textbooks on geometry, the fact that Euclid’s fifth axiom is valid if, and only if, the sum of the angles in any triangle is 180°, is not emphasized on. It goes without saying that this fact could have helped the kids in high school to learn at an early age that in non-Euclidean geometry this sum is certainly not 180°, even without knowing anything about this geometry. Or, after Andrew Wiles’s proof of Fermat’s Last Theorem (FLT), we could state the theorem as: The equation $x^n + y^n = z^n$, where x, y, z, and $n > 0$ are integers, has nontrivial solutions if, and only if, $n = 2$ (note that, in the case $n \geq 2$ the solutions with $xyz = 0$ are considered trivial, and all the solutions in case $n = 1$ may be considered trivial, too). In this form, FLT also gives a unique characterization of 2, which is not its evenness. However, apparently everyone still prefers to state FLT in the pq-form.

In what follows we will define a concept of “good complete” theorem (briefly, *gc-theorem*), not only for Euclidean geometry, but also for all parts of mathematics, wherever possible. Before this, let us explain the rationale for the “converse” behind our initial question. Clearly, the converse of a mathematical result is not always necessarily true. In that case, the author may provide counterexamples and then, by studying the properties of these counterexamples, she/he might be motivated to add some extra properties to the statement of the original result, in order to get an eligible new result whose converse is also true. This might look as a drawback that causes a loss. However, the author may present the original result without any converse, separately, as an immediate corollary of the above new result, and thus easily overcome the loss (note, certainly the proof of the original result is a part of the proof of the above new result, hence the above corollary, which is in fact the original result, is indeed immediate). We should also emphasize that if the converse of a result in pq-style holds, then stating it the "if, and only if" style not only does not cause a loss, on the contrary, it is all about gaining advantages. Of course, we admit that turning a result into "if, and only if" form is not always easy, and even providing counterexamples in some particular cases may remain an open problem for years. An example is provided by FLT in the above form. However, we should admit that when a result is stated in the "if, and only if" form, a better understanding of the result is reached. Even more, in this case, each part of the result might give a clue for the proof of the other part.

There are some results in our field of expertise (topology and algebra) which are stated in the latter form, where none of the parts can be proved completely without invoking something from the other part, see for example [16, proof of Theorem 1.3]. This means that no part could exist separately. There are also some results which can be stated in the "if and only if" form, in which case they need no proof at all, but if stated in pq-form one runs into difficulties, see [12, Problem 2, Corollary 4], and [12, Problem 1, Corollary 3] for some examples of this kind. Let us recall briefly that [12, Problem 2] was originally stated in pq-style, and I believe that this form of the problem might have misled Terence Tao to explain and justify, in three pages, his trigonometric solution of the problem, and to give up any hope for presenting a possible geometric solution of the problem; see the comment preceding [12, Problem 2]. Whereas if we state this problem in "if, and only if" form, as in [12, Corollary 4], then by invoking a simple geometric fact we notice that the latter corollary needs no proof, a fortiori, the latter problem needs no solution at all. However, see the proof of the corollary in [12], which is given there to justify our latter claim. It is folklore that the fuller statement of any result in mathematics is easier to understand and be proved than a restricted one, let alone if this fuller statement is also stated properly in the "if, and only if" form.

In my opinion, in general presenting a result properly in "if, and only if" form causes no serious drawbacks or any losses, as far as mathematics is concerned, except possibly that we have to devote much time to either guessing a proper converse, or to finding appropriate counterexamples in case the converse is not true. It is also possible that this will slow down the speed of our publications – but this is not a significant loss for most people, and I am one of them, for this is not a loss in terms of mathematics. So one should generally attempt to examine seriously the question of the validity of converse results.

In what follows we give some appropriate examples to show that when the converse of a result fails, we might be able to resolve it in the way described above. To start, we all know that if a triangle $\triangle ABC$ is isosceles with apex A, then in $\triangle ABC$ the altitude, the bisector, and the median that go through the vertex A coincide. The converse is also obvious. We may recast the previous fact by asserting that the bisector of the angle A is also the bisector of the angle between the altitude and the median of this triangle that go through the vertex A. However, in this case, unfortunately, the converse is not true in general (note that the right triangles also have the latter property and may serve as counterexamples). Now let us consider a converse for the latter case and present the next interesting and useful theorem in the "if, and only if" form to justify our claim that considering the converses of our results is important.

Theorem 1.1. *In any triangle $\triangle ABC$, the bisector of the angle $\angle A$ is also the bisector of the angle between the altitude and the median which go through the vertex A, if and only if, the triangle $\triangle ABC$ is either isosceles with apex A, or is a right triangle with $\angle A = 90^\circ$.*

We can easily provide a proof by invoking the well-known fact that in every triangle $\triangle ABC$ the bisector of the angle $\angle A$ and the perpendicular bisector of the side BC either coincide, or meet each other on the circumcircle of the triangle. Is it not interesting that, by just considering a converse for the previous recast statement of an obvious fact we have obtained a nontrivial result which shows that isosceles triangles and right triangles have, in fact, a common characterization? A fact which seems to have been overlooked in the literature, see [12, Corollary 3] for another new common characterization of this kind. Despite their common characterizations, the following new fact in the "if, and only if" form which shows that a certain kind of these two triangles cannot coexist gives a new natural geometric proof of the irrationality of $\sqrt{2}$, see [5, 19] for various proofs of this irrationality. Should we not admit that we owe this to thinking in the "if, and only if" style?

Theorem 1.2. *$\sqrt{2}$ is irrational if, and only if, there exists no isosceles right triangle with rational sides.*

The proof of the theorem is evident in view of the fact that all the isosceles right triangles are similar to each other and, in particular, they are all similar to the one with the side lengths $1, 1, \sqrt{2}$. However, to infer from this theorem that $\sqrt{2}$ is indeed irrational, we may prove the second part of the statement of the theorem without invoking the irrationality of $\sqrt{2}$. To this end, just notice that if there is such a triangle, then by multiplying the sides by an appropriate integer we get isosceles right triangles with integer sides. Hence, there is the smallest isosceles right triangle among the latter ones (i.e., with the integer sides). But the bisector of the right angle in this smallest triangle divides into two congruent smaller isosceles right triangles and still with the integer sides (note, the length of the hypotenuse is even in all the isosceles right triangles with integer sides), which is a contradiction.

Also, see [14, Introduction] for the non-familiar converse of an obvious fact which leads us to some useful nontrivial and overlooked results in the "if, and only if" form in elementary number theory.

Definition 1.3. A theorem is called a *gc-theorem*, if its statement is of the form of "if, and only if" with no part that consists of facts that are too obvious to mention, unless some part is just the definition of a mathematical object.

Although, generally there should not be any restriction to certain kinds of methods (except the correctness of the methods) for the proofs of mathematical results, in my opinion in the case of gc-theorems, the proof of at least one part of the theorem should be motivated.

Needless to say, if a definition contains a statement that is a personal opinion, then one should provide enough explanations to justify this opinion. Accordingly, in what follows, some comments are given to clarify the definition as much as possible, and make it unambiguous. By considering almost any significant result of the form "if, and only if" (or, if not of this form, it can be easily turned into this form) in the literature, which might be regarded naturally by anyone as a gc-theorem, one immediately notices that the proofs of at least one part of such a result is usually motivated. We should emphasize that a proof is motivated if each step in the proof can be done by some reasoning and if it is also free of any kind of *deus ex machina*; see [12, p. 298], for some non-motivated proofs, and see [21, pp. 41–43] for a discussion of a motivated proof of Morley's trisector theorem. An appropriate example of latter kind is Hilbert's weak Nullstellensatz, which asserts that every maximal ideal M of $R = K[x_1, \dots, x_n]$, where K is an algebraically closed field, is of the form $M = (x_1 - a_1, \dots, x_n - a_n)$ for some point $x = (a_1, \dots, a_n)$ in K^n. But, we know that this theorem of Hilbert is nothing but a generalization of the fact that over any field F, every non-constant polynomial has a root if, and only if, F is algebraically closed. Therefore, the weak Nullstellensatz can be easily turned into a gc-theorem, by claiming that M is of the above form if, and only if, K is algebraically closed (note, every non-constant polynomial in R is contained in a maximal ideal).

We should bear in mind that most authors are usually motivated by the existing results in the literature to obtain new results, whether in the form of gc-theorems or not. And in some cases they might also use proofs similar to those of the existing results. When the proofs of the two parts of an "if, and only if"-style result are simultaneously non-motivated and are also too complicated, it seems that the result may be artificially created. For example, see [11, Theorem 3], for such an artificial result, which is given there for some purpose. One can easily notice that the proof of the latter theorem is not motivated, and it is extremely difficult, if not impossible, to be given by anyone who does not already know the side lengths of the required triangles in the statement of the theorem. Incidentally, the statement of this artificial theorem can be easily turned into "if, and only if" form. We must admit that we know of no useful such results in the literature. In a nutshell, if a result in the form of "if, and only if" is to be considered as a genuine gc-theorem, it is reasonable to expect that at least one part of it should be stated in a natural way (i.e., not described in a convoluted way) with a rational motivated method for its proof. Can we call an "if, and only if" result a good complete theorem (i.e., a gc-theorem), if the proof of each part of the result is non-motivated and consisting of, say, more than 100 pages? Should we not naturally ask in this case, how on earth, could anyone have arrived at the idea of guessing such a result? Fortunately, there do not seem to be such results in the literature yet (at least, not to my knowledge). Let us, as a last remark in this regard, recall Fermat's Last Theorem, in the "if, and only if" form suggested above. Is it not true that the short, simple, and motivated proof of its first part, i.e., the case $n = 2$, had a major role in motivating Fermat to further work on the theorem and also for his correct guessing of the statement of its converse? And should we not emphasize that it was this motivated and simple proof of the first part that attracted the attention of so many mathematicians and students alike (not to mention the mathematical cranks) to the theorem, when it was still unresolved? Finally, it is this motivated and simple proof of the first part of FLT, and the simplicity of its statement, that is the source of its popularity, even among the high school kids.

We should emphasize that not all the results in the "if, and only if" form are genuine gc-theorems. For example, Fermat's little theorem in the form: "A prime number p divides $a^{p-1} - 1$, where a is a natural number, if, and only if $(a, p) = 1$," and similarly Wilson's theorem (i.e., an integer p is prime if, and only if, p divides $(p - 1)! + 1$) are not genuine gc-theorems, because one part of their statements is too obvious (note that although these theorems are very useful, they are rarely used to recognize prime numbers). Also, the Steiner–Lehmus theorem in the form: "In a triangle, angle bisectors are equal if, and only if, they bisect equal angles" may not be considered as a gc-theorem either, for the same reason. Since the obviousness of a result might differ from individual to individual, in order to make the statement of our previous definition more precise, I suggest that although the form of results similar to the previous theorems might be improved whenever possible, in my opinion it is still preferable to formulate these three non-genuine gc-theorems in pq-style, and perhaps refer to them as *almost gc-theorems*.

For example, for the converse of Fermat's theorem one may claim that if n divides $a^{n-1} - 1$ and $n - 1$ is the smallest among the natural numbers m with the property that n divides $a^m - 1$, then n must be prime (note, if n is not prime, then $\phi(n) < n - 1$, where $\phi(n)$ is Euler's totient function). However, since there is no to place the phrase "if, and only if" between the statement of Fermat's theorem and the latter statement for its converse, we cannot get a gc-theorem by just combining the two statements. Otherwise, we could present Fermat's theorem as a genuine gc-theorem, too. As for the Steiner–Lehmus theorem, we have already stated it as a gc-theorem, namely, in a triangle with two different angles say, an angle is bigger if, and only if, it has a shorter bisector. We should emphasize that in this form only one direction needs a proof; the reader is referred to [4] and [1, Theorem 114] for the same proof of the theorem, which uses the similarity concept. In [4], the reader may also consult some other different proofs of this theorem, see also [7, Lemmas 1.511, 1.512] and [6, p. 420] for the history and more proofs. In particular, the notorious old query of finding a direct proof of this theorem is discussed in [4]; see also [13, last paragraph] to see a contrast between a very short proof of the theorem in its latter form and its possible formal, too lengthy, direct proof. It goes without saying that if a statement of a theorem consists of several equivalent statements, then the result can be considered as a gc-theorem. However, we should also emphasize that it seems that by inserting the phrase "if, and only if" between any two statements, A and B, say, with the same truth-value, we might get a result which looks like a gc-theorem. But here we must remind ourselves that by the equivalence of these kinds of statements we should mean that a proof of B must be deduced from A and vice versa, see [14, last theorem]. Therefore, this prevents us from interpreting any such artificial gc-theorems as genuine ones. We should also remind the reader that there are many important mathematical results that are mutually non-artificially equivalent, in the sense explained above (e.g., the axiom of choice, Zorn's lemma, Tychonoff's theorem, the fact that every set of independent vectors in a vector space can be extended to a basis of the space, the fact that every ring with unity has a maximal ideal). Indeed, it is well known that all these statements are equivalent, see [10]; for another example of a gc-theorem of this kind, see [14, last theorem].

Fortunately, many of the important results in the literature are genuine gc-theorems. We should remind the reader that the biconditional logical connective phrase "if, and only if" is used commonly enough in mathematics that it has its own abbreviation "iff." Apparently this abbreviation appeared first in John L. Kelly's 1955 book *General Topology*, where its invention is credited to Paul R. Halmos, see the last four lines in the preface of this book. However, in the literature there are still thousands of useful theorems, propositions, corollaries, lemmas and problems, which are not even in the "if, and only if" form, let alone gc-theorems. In what follows we like to consider some of these results and turn them into gc-theorems. Naturally, their selection is somewhat personal. We may deal with many important results, whether elementary or not, and try to turn them into gc-theorems. However, because of the scope of this essay only a few apparently non-elementary cases will be treated, to demonstrate that the literature abounds with non-elementary non-gc-theorems, too. Let us also emphasize that our discussion here is not intended to be a pq-style vs. gc-theorems argument: rather, our aim is to show that among the aforementioned important and useful non-gc-theorems many are eligible to be presented as gc-theorems.

2 Some gc-theorems in topology, algebra, and analysis

There are thousands of useful non-gc-theorems in the three fields listed above, where most of them are eligible to be reconsidered as gc-theorems, see also [8, Section 6.1]. As prototypes, in the following subsections I mention only three of them (one from each field, in reverse order), which are also in the textbooks.

2.1 When does an infinite set in $\mathbb{R}^n$ have a limit point?

Without further ado, we believe the following classical and very important non-gc-theorem is a good candidate to begin with, for it can, immediately, be turned into a gc-theorem which settles the question in the title.

Theorem (Bolzano–Weierstrass). *Every infinite bounded subset of $\mathbb{R}^n$ has a limit point.*

This result appears in every introductory textbook on analysis, see, e.g., [2, Theorems 3.13, 3.29] and [22, Theorem 2.42]. It seems that the existence of some unbounded countable subsets such as $\mathbb{N}$ or $\mathbb{Z}$ in $\mathbb{R}$ is responsible for the above boundedness constraint. At the same time, the set of rational numbers, the set of irrational numbers, and many other infinite sets in $\mathbb{R}$, which are not bounded, have limit points, but not directly as a consequence of the above theorem. However, with a little thought we may restate and record the above theorem as follows, which takes care of these sets, too.

Theorem 2.1 (Bolzano–Weierstrass). *Let $A \subseteq \mathbb{R}^n$. Then A has a limit point if, and only if, A is either uncountable, or it has a countably infinite bounded subset (or equivalently, if, and only if, A has the latter property).*

Proof. The proof for the case $n = 1$, which follows, can be imitated word-for-word for the general case. If A has a limit point, say $x \in \mathbb{R}$, then let $x \in (a, b)$, where $a, b \in \mathbb{R}$. Obviously, $A \cap (a, b)$ is an infinite set that contains an infinite countable subset of A which is clearly bounded. For the converse, we may only show that if A is uncountable without a limit point, we get a contradiction. Put $B = \{(r, s) : r, s \in \mathbb{Q}\}$. Since A has no limit point, for each $a \in A$ the set $F_a = \{G \in B : G \cap A = \{a\}\}$ is nonempty. Now, by the Axiom of Choice, for each $a \in A$ we can choose an element $G_a \in F_a$ and put $E = \{G_a : G_a \in F_a\}$. Obviously, $E \subseteq B$, hence E is countable. But the function $f : A \to E$, where $f(a) = G_a$ for each $a \in A$, is clearly one-to-one, which implies that A is countable too, a contradiction. ∎

Let us comment on what one might lose if we do not care about the converses of our results. Sure, the set theoretic distinction of cardinality was not, perhaps, on the mind of anybody in those days. At the same time, we believe that, had the above two outstanding mathematicians of their time cared about a converse for their result, they could have easily, somehow, formulated one. And then, as a consequence, they could have had some understanding of the concept of the cardinality of a set before Cantor, and even they could have become the inventors of set theory instead of Cantor.

2.2 Rings in which every maximal ideal is generated by an idempotent

Let us first emphasize Cohen's theorem, which says that a commutative ring R is Noetherian (i.e., all ideals in R are finitely generated) if, and only if, every prime ideal in R is finitely generated, is a very useful gc-theorem in commutative ring theory, see, e.g., [3, Theorem 12–6]. Incidentally, it is well known that if in this theorem prime ideals are replaced by maximal ideals, its assertion is no longer true, see, e.g., [9]. However, as a corollary to this theorem it is proved in [3] that if every maximal ideal of R is generated by an idempotent, then R is Noetherian too. Clearly, this corollary is not a gc-theorem. In fact, a much stronger result holds if we try to make it into a gc-theorem. We may say that: A commutative ring R is a finite direct product of fields (in particular, it has only a finite number of ideals, so a fortiori it is Artinian, which in turn implies it is Noetherian too) if, and only if, every maximal ideal in R is generated by an idempotent. Its proof is as easy as the proof of the corollary. Indeed, since every maximal ideal of R is a direct summand, we readily infer that the sum of its minimal ideals (i.e., the socle of R) cannot be contained in a maximal ideal, M say. For if $M = eR$, where e is idempotent, then $(1-e)R$ is a minimal ideal contained in the socle of R, which is not in M. Hence, the socle of R must be equal to R, and we are done. We should remind the reader that a generalization of the latter gc-theorem with the same proof holds in more general (not necessarily commutative) rings, see [15].

2.3 What are the topological spaces, in which closed sets and boundary sets coincide?

Let us bring to the attention of the reader that, in [24, Exercise 3B], it is asked to show that any closed subset of the plane $\mathbb{R}^2$ is the boundary of some set in $\mathbb{R}^2$ (note that therein boundary is called frontier). This is a good question, because the boundary of a set is always closed, but the converse is not necessarily true in every topological space (e.g., in a discrete space). However, we may ask what is so special about $\mathbb{R}^2$? Is this claim not true, for example, in $\mathbb{R}^n$? Therefore, we may first pose an appropriate question. What are the spaces X with the property that for any closed subset $F \subseteq X$ there is a subset $A \subseteq X$ with $F = b(A)$, where $b(A)$ denotes the boundary of A. Now, $b(A) = \overline{A} \setminus A^\circ$, where $\overline{A}$ is the closure of A and A° is the interior of A. Hence, since X is closed, we must have $X = b(Y) = \overline{Y} \setminus Y^\circ$ for some subset Y of X. This implies that $Y^\circ = \varnothing$ and $\overline{Y} = X$. So we have a clue, i.e., there must exist a dense subset with empty interior. This leads us to state a new gc-theorem which characterizes the closed sets, and hence the open sets, in any topological space X, a characterization which seems to have been overlooked in the literature.

Theorem 2.2. *Let X be a topological space. Then $F \subseteq X$ is a closed subset, if, and only if, for each dense subset $Y \subseteq X$ there exists a subset $F_Y \subseteq X$ such that $F = \overline{F_Y}$ with $F_Y^\circ = F^\circ \cap Y^\circ$.*

A proof of this theorem, which depends on the simple and well-known fact that $\overline{G \cap Y} = \overline{G}$, where Y (resp., G) is any dense (resp., open) set in X, is not hard and is left to the reader, see also [8, p. 28].

The following immediate corollary, which is also a gc-theorem, settles the above question in $\mathbb{R}^n$ (note that $\mathbb{Q}^n$ is a dense subset with empty interior in $\mathbb{R}^n$).

Corollary 2.3. *Let X be a topological space. Then every closed subset of X is the boundary of a subset of X if, and only if, X has a dense subset with empty interior.*

Let us conclude this article with two comments:

(1) Not only is the pq-style responsible for the overlooking of some of the converses related to useful non-gc-theorems in the literature, but also it is sometimes, equally responsible, for our inveterate tendency to overlook some obvious useful facts; see [12, Corollaries 1, 3, 4, Bisector Proposition] for some of these facts.
(2) We may also search the literature, especially the textbooks, to look for some interesting results which are non-gc-theorems. And then do our best to present them, if possible, in the form of gc-theorems. This would help to substantially reduce the number of non-gc-theorems in the literature, or at least in the textbooks, in the future. In particular, the mathematics education students who may follow the latter comment, could provide suitable materials for writing good dissertations.

Acknowledgements. The interest of the referee in our work and giving useful comments is appreciated.

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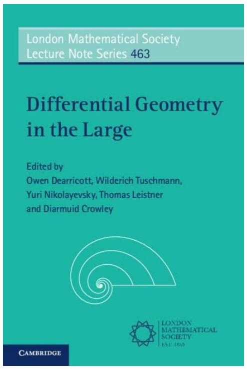
Book reviews

Differential Geometry in the Large by O. Dearricott, W. Tuschmann, Y. Nikolayevsky, T. Leistner and D. Crowley (eds.)

Reviewed by Bernd Ammann



In February 2019, an Australian–German workshop took place at the Mathematical Research Institute (MATRIX) in Creswick, Australia. The first week of the workshop, i.e., from 4 to 8 February, was an international conference with speakers from many countries. In the second week, from 11 to 15 February, the workshop was organized in three more specialized sessions, following the main topics "Geometric evolution equations and curvature flow," "Structures on manifolds and mathematical physics," and "Recent developments in non-negative sectional curvature."

The book under review is a collection of 17 contributions by speakers at this conference, grouped into the three main themes listed above. All parts were peer-reviewed. The intention of most of the contributions is to give an overview of new developments in the field. All contributions can be read independently, thus every reader is invited to choose those articles that best fit their interests; and this is what I will do as well in this review. My choice is based purely on my personal interests and experience, and no statement about the quality of the contributions is made.

Claude LeBrun's contribution starts with a nicely written introduction to one of his favourite subjects, namely, Einstein metrics on closed 4-dimensional manifolds. The article focusses on del Pezzo surfaces, which amounts to considering the cases $M = S^2 \times S^2$ and $M = \mathbb{C}P^2 \# m\overline{\mathbb{C}P^2}$, $m \in \{0, 1, \dots, 8\}$. It is already known that these manifolds carry Einstein metrics, but the associated moduli space is not fully determined: known Einstein metrics all satisfy $W^+(\omega, \omega) > 0$ for a non-zero self-dual harmonic 2-form ω, however it remains open whether further ones exist. The article excludes new types of Einstein metrics, namely those with $W^+(\omega, \omega) \geq 0$ where the zero set of ω is generic. Overall a pleasure to read.

Other articles summarize existing results in a nice way. This is, for example, the case for the contribution by Kröncke and Vertman, in which they review their progress about the Ricci flow on manifolds with conical singularities. This subject is interesting, as Ricci flows often run into singularities; traditionally one uses surgeries and one restarts the flow, but in view of some applications, e.g., in order to study moduli spaces of Riemannian metrics with some curvature properties, one would prefer that Ricci flows even survive the formation of such singularities in some weak sense, and then continue as Ricci flows with conical singularities.

The contribution by Ludewig summarizes joint work with Güneysu and Hanisch in which they construct rigorously a supersymmetric path integral. This article is motivated by mathematically very interesting and stimulating questions in quantum mechanics and statistical physics.

The chapter written by Gover and Waldron discusses a variant of the Yamabe equation, however the scaling function arises in a non-standard way. For a fixed conformal class $[\bar{g}]$ on a d-dimensional manifold one seeks metrics $g = \sigma^{-2}\bar{g}$ of scalar curvature $-d(d-1)$. This leads to a partial differential equation for σ. As long as σ is positive or negative, this defines a metric g. But the article also allows σ to become zero, and then the metric g develops an "end at infinity" at $\sigma^{-1}(0)$. For example, on the round sphere S^d there is a solution σ vanishing on the equatorial sphere S^{d-1} such that $(S^d \setminus S^{d-1}, g)$ is isometric to two copies of d-dimensional hyperbolic space. One interesting aspect of the article was for me that the zero set of σ satisfies nice conformally invariant equations, e.g., for $d = 3$ it is a stationary point of the Willmore energy, a so-called Willmore surface.

There are many more interesting contributions by renowned experts. I cannot comment on all of them, but let us also have a glance at the contribution by Stephan Klaus. I guess that many readers of the EMS Magazine have met him at a conference in Oberwolfach, Germany, without knowing his field of interest. Stephan Klaus is one of the key figures in this institute, and he contributes essentially to its smooth operation and effective organization. But he is also a strong mathematician, as you can see from his contribution about the simplicial Gauss–Bonnet–Chern theorem, the famous generalization of the Gauss–Bonnet theorem for closed surfaces. If you want to know about his research, his article gives great insight.

To sum-up, this collection of articles gives nice and self-contained introductions to the working fields of the authors. The reader should have a solid knowledge of Riemannian geometry, but more specialized expert knowledge is provided in the contributions. Some contributions also include interesting open problems. The book will be of interest for experts wishing to learn about recent developments, but also for young mathematicians who want to get an overview of current developments in the field.

O. Dearricott, W. Tuschmann, Y. Nikolayevsky, T. Leistner and D. Crowley (eds.), *Differential Geometry in the Large.* London Mathematical Society Lecture Note Series 463, Cambridge University Press, 2020, 398 pages, Paperback ISBN 978-1-108-81281-8, eBook ISBN 978-1-108-88413-6.

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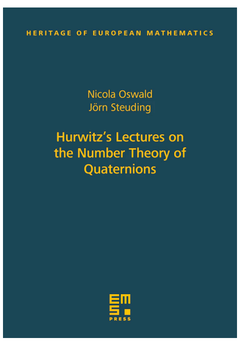
Hurwitz's Lectures on the Number Theory of Quaternions
by Nicola Oswald and Jörn Steuding

Reviewed by John Voight



Adolf Hurwitz published his pioneering work *Vorlesungen über die Zahlentheorie der Quaternionen* in 1919, just before his death. In this remarkable booklet, he sought to make the number-theoretic aspects of quaternions widely accessible and to encourage his (younger) colleagues to engage in further research.

Just over a hundred years later, Nicola Oswald and Jörn Steuding have translated, edited, and commented on Hurwitz's work in *Hurwitz's Lectures on the Number Theory of Quaternions* with largely the same goal in mind: to make these lectures broadly accessible. To further this goal, they also supplement their close reading with a concise and contextualized introduction to topics in algebraic number theory.

Hurwitz's lectures establish, essentially from scratch, the foundations of the arithmetic of quaternions. Within the notation and framework of his time, Hurwitz meticulously defines integer quaternions and establishes notions analogous to those of the usual integers, like greatest common divisors, ideals, and (noncommutative) prime factorization. An important distinguishing feature of the quaternions is their norm, the sum of four squares; the fact that it is multiplicative proves Euler's four-square identity. This treatment culminates in his elegant proof of Lagrange's theorem—that every non-negative integer can be represented as a sum of four integer squares—and a quaternionic solution to Euler's problem of finding 4×4 magic squares of squares.

The book is carefully organized to guide readers, spread over seven parts and an appendix. After a short introduction in Part I, the translation of and commentary on Hurwitz's lectures is provided in the main Part II. In Parts III–IV and Part VI, the authors provide additional historical context. Sandwiched in between, Part V provides a translation of Hurwitz's paper on the composition of quadratic forms, where he proves that an identity $(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = z_1^2 + \cdots + z_n^2$, where each z_k is a bilinear form in the variables x_i, y_j, exists if and only if $n = 1, 2, 4, 8$. Part VII concludes with a short epilogue, and a short appendix covers "Elementary Number Theory in a Nutshell."

Hurwitz's Lectures will be enjoyed by mathematicians with an interest in the historical development of algebraic number theory, as well as by students interested in these topics, because of the very few prerequisites. The book illuminates Hurwitz's contributions to the study of quaternions and provides a connection to modern theory, proving the enduring value of bridging classical inquiry with contemporary mathematical exploration—a fitting scholarly tribute, indeed.

Nicola Oswald and Jörn Steuding, *Hurwitz's Lectures on the Number Theory of Quaternions.* Heritage of European Mathematics, EMS Press, 2023, 311 pages, Hardcover ISBN 978-3-98547-011-2, eBook ISBN 978-3-98547-511-7.

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Report from the EMS Executive Committee online meeting, 17–20 November 2023

Richard Elwes

The EMS Executive Committee (EC) holds two major meetings in each (non-exceptional) year, plus regular shorter virtual meetings. For reasons of environmental and financial sustainability, one of the major meetings is now also held virtually. So, the EC plus guests met online between 17 and 20 November 2023.

1 Officers' reports

EMS President Jan Philip Solovej opened the meeting and reported on his recent activities, which included representing the EMS at several major events: a speech at the 2023 Abel Prize ceremony in Oslo; the inaugural Balkan Mathematics Conference in Pitești in July; the 29th Nordic Congress of Mathematicians in Aalborg in July; and ICIAM (International Congress on Industrial and Applied Mathematics) in Tokyo in August. He has also participated in the meetings of numerous EMS and EMS-affiliated committees. He reported on progress in several ongoing EMS activities (many of which will feature later in this report).

EMS Treasurer Samuli Siltanen then presented the EMS income statement as of 9 November 2023. Although the EMS finances remain healthy, it is also the case that the number of EMS supported activities has recently increased (for example the EMYA, see below). It is also true, of course, that EMS expenditure, like everything else, is affected by rising prices in the broader economy. Some savings are expected from recent moves, including changing the EMS Magazine to being online by default, with printed copies only for members who request it. However, these savings are not yet reflected in the balance sheets. The EC discussed several proposals to be put to the EMS Council in 2024 to further shore up the society's financial position.

EMS Secretary Jiří Rákosník delivered his report. Besides his many organisational activities for the EC, he has been busy in several other EMS and EMS Press activities, including in the maintenance of the EMS website, and creating the statutes for two major new prizes, the Simon Norton Prize for Mathematics Outreach and the Paul Lévy Prize in Probability Theory.

Vice-President Beatrice Pelloni reported on her role, including leading the evaluation of applications for EMS Topical Activity Groups (TAGs) and Strategic Activities (the call is currently paused), and for selecting new members of the EMS Young Academy (EMYA, see below).

Vice-President Jorge Buescu reported on his role, including involvement in the preparation of the Simon Norton Prize for Mathematics Outreach. He was also the EMS delegate to the European Open Science Cloud (EOSC) General Assembly in Brussels in May and is a member of the EOSC "Upskilling Task Force" (replacing former EMS VP Betül Tanbay).

2 Membership and website

Elvira Hyvönen from the EMS Secretariat, in attendance as a guest, took the committee through the latest membership statistics. The EC approved 130 new applications for individual membership. No new applications for corporate membership, or class change, have been received. The EC discussed possible strategies to attract more members, as well as for increasing donations to the Committee for Developing Countries. Increased collaboration with EMS Press will form an important part of the approach to advertising in future.

3 Scientific meetings and society activities

The 9th European Congress of Mathematicians (ECM) will be held in Sevilla, 15–19 July 2024.[1] The chair of the organising committee Juan González-Meneses delivered a report on the preparations, which are progressing well. To date, 27 satellite events have been announced (22 in Spain and 5 in Portugal) and all 10 plenary speakers and 32 invited speakers accepted their invitations. The EMS prizes will be presented at the congress; they have recently expanded to include the new Paul Lévy Prize in Probability Theory with a cash prize of 20,000 euros donated by BNP Paribas.

The 10th ECM will be held in 2028. Three bids to organise the meeting were received, with one subsequently being withdrawn. The EC agreed to make visits to the sites of the two remaining bids

[1] www.ecm2024sevilla.com

and invite both bidders to the next EC meeting. The final decision on the location will be taken by the EMS Council in July 2024.

The first Balkan Mathematical Conference (BMC) was held in on 1 July 2023 in Pitești (Romania). Preparations for BMC II in summer 2025 were discussed. The 29th Nordic Congress of Mathematicians was held very successfully in Aalborg (Denmark), 3–7 July 2024. The next edition is expected in 2028.

The EC approved funding for 10 EMS summer schools over 2024, and 9 other scientific meetings, plus invitations to two EMS Distinguished Speakers.

Vice-President Beatrice Pelloni reported on the activities of EMYA, the EMS Young Academy launched in spring 2023. The EC was delighted to welcome María Ángeles García Ferrero, the new representative of EMYA to the EC, to continue this discussion. EMYA's input will be important for the future of many EMS activities. For instance, EMYA will meet to discuss zbMATH Open, and will hold a discussion panel on sustainability at the next ECM.

The EC discussed plans for a new online system for handling grant applications, and to appoint a new community manager jointly with EMS Press who would take over some of the duties currently carried out by the secretary and other members of the EC, the publicity officer, EMS office, and EMS Press staff. The EC also considered several technical modifications to society procedures to be proposed to the council in July 2024 for approval.

4 Standing committees and projects

The EC considered reports from the chairs of the ten standing committees which carry out so much of the EMS's work. Some new members were approved to replace those whose terms are ending. The committees are:

- Applications and Interdisciplinary Relations (CAIR), which has focussed on promoting grants of the European Research Council to the mathematical community.
- Developing Countries, which has been running the EMS Simons for Africa programme, and recently approved 12 travel grants.
- Education, which is highly active in several areas related to mathematical education at high school level and above.
- European Research Centres in the Mathematical Sciences (ERCOM).[2] Two new member institutes have recently been accepted: the Mathematical Institute of the Serbian Academy of Sciences and Arts (Belgrade) and SwissMAP (Les Diablerets, Switzerland).
- Ethics, which considers allegations of malpractice in mathematics. This committee intends to revise the EMS Code of Practice and other documents, including creating a new document regarding teaching mathematical ethics. The EC discussed material that it would be appropriate to include in these.

[2] www.ercom.org

- European Solidarity, which considers travel and funding requests for mathematicians from economically less favoured European countries.
- Meetings, which handles applications for EMS summer schools and other scientific events, and makes recommendations to the EC.
- Publications and Electronic Dissemination (PED), whose members intend to organise a mini-symposium at 9ECM.
- Mathematics Outreach and Engagement, which has finalised the procedures for the new Simon Norton Prize for Mathematics Outreach to be presented in years 2024, 2026 and 2028. The next deadline for nominations is 30 April 2024. It also runs an outreach listings page.[3]
- Women in Mathematics (WiM), which, in cooperation with EMS Press, has been monitoring the participation of women in editorial boards of journals published by EMS Press. The situation is improving step by step. Following a successful 2023 EMS WiM Day where Ana Caraiani and Isabelle Gallagher delivered online lectures, the committee is organising a sequel on 17 May 2024 within the broader "12 May" initiative in honour of Maryam Mirzakhani.

5 Publishing and publicity

EMS Publicity Officer Richard Elwes, in attendance as a guest, reported on his activities. The EMS social media channels continue to grow healthily, although overall Twitter (now called "X") is deteriorating as a platform. The EMS is currently available on:

- Twitter,[4]
- Facebook,[5]
- LinkedIn,[6]
- Mastodon,[7]
- and YouTube.[8]

The managing director of EMS Press, André Gaul, also attending as a guest, presented an update from EMS Press. It has been very busy, with 8 books published this year and 2 more coming soon. The Subscribe to Open (S2O) publishing model for journals continues to operate very well, with 21 titles fully open access under S2O in 2023, alongside a further three Diamond open access publications, and prospects for 2024 also looking promising. EMS Press is a forward-looking publishing house as regards technology and is also an invaluable source of support to the EMS in this regard.

[3] www.popmath.eu
[4] www.twitter.com/euromathsoc
[5] www.facebook.com/euromathsoc
[6] www.linkedin.com/company/european-mathematical-society
[7] www.mathstodon.xyz/@EuroMathSoc
[8] www.youtube.com/c/EuropeanMathematicalSociety

The editor-in-chief of the EMS Magazine, Fernando da Costa, another guest at the meeting, reported on the Magazine, including the fact that his term will end in June 2024, along with those of three other editors. The EC agreed that the Magazine is in excellent shape and discussed possible replacement editors.

The quarterly EMS Digest is currently paused. It is expected to restart in a new format in 2024.

The editor-in-chief of zbMATH Open, Klaus Hulek, also attending as a guest, reported to the committee. There has been significant increase in the number of users since zbMATH became Open Access. Particularly encouraging is the increase of users from developing countries. The president congratulated the zbMATH Open team for their excellent progress, and on behalf of the EMS expressed deep gratitude to outgoing editor-in-chief Klaus Hulek for all his work.

6 Funding, political, and scientific organisations

The committee discussed its relationship with several external organisations, including the European Research Council (ERC), the European Open Science Cloud (EOSC), the International Mathematical Union (IMU), the Bernoulli Society, and the International Council for Industrial and Applied Mathematics (ICIAM). The president and other officers represented the EMS at ICIAM 2023 Tokyo in August.

The president closed the meeting with his thanks to all the EC members and guests. The next major EC meeting was scheduled for March 2024 in Helsinki, with regular short online meetings continuing in the meantime.

Richard Elwes is currently the EMS publicity officer (until July 2024) and a senior lecturer at University of Leeds (UK). As well as teaching and researching mathematics, he is involved in mathematical outreach and is the author of five popular mathematics books.

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European Mathematical Society

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- Option to join the EMS as reviewer of zbMATH Open.

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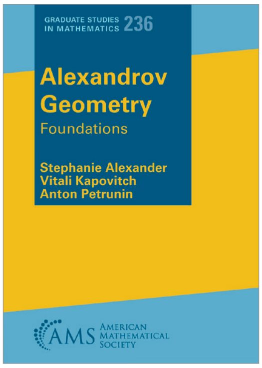

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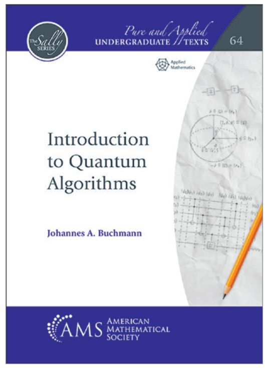

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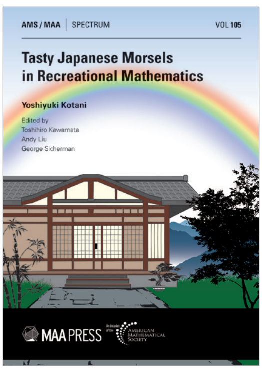

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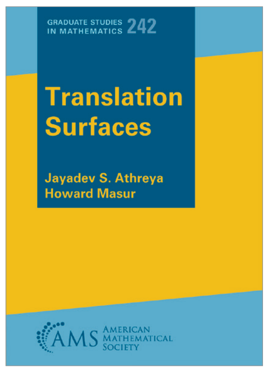

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