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New Challenges in the Interplay between Finance and Insurance

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Thorsten Schmidt, Freiburg

1 October – 6 October 2023

Abstract. The aim of this workshop was to convene experts for fostering the discussion and the development of innovative approaches in insurance and financial mathematics. New challenges like price instability, huge insurance claims and climate change are affecting the markets, while at the same time the possibility of using large volumes of data and continuously increasing computer power as well as recently developed mathematical methods offer new opportunities for modelling and risk assessment. Here we present an overview of these recent developments by providing the abstracts of the talks that were given during the week, together with a brief summary of the covered topics.

Mathematics Subject Classification (2020): 60, 62, 90, 91, 93.

Introduction by the Organizers

The last years have been a challenging period for financial and insurance markets. While stock markets experienced unexpected large price jumps, insurance and reinsurance companies suffered huge claims, but at the same time had the opportunity to use large volumes of data for their modelling, and the continuously increasing level of computer power gives rise to new approaches to make use of them. The impact of climate change poses a further challenge to both fields, and the consideration of sustainable investment policies and strategies becomes increasingly important.

This workshop brought together leading experts in all these fields to foster the discussion and the development of new and innovative approaches. In the following, we will provide the abstracts of the talks that were given during the week, and start with a brief summary of the covered topics.

Ralf Korn started with formulating stochastic control problems in the context of sustainable finance, and Peter Tankov gave an account of mean-field approaches for the decarbonization of financial markets. Emanuela Rosazza Gianin and Silvana Pesenti presented new results on consistency and robustness of dynamic risk measures, and on the application side for insurers. Filip Lindskog presented multiperiod approaches for the valuation of liabilities, and Michael Schmutz gave an update of the current view on risk measures from the regulatory perspective of Switzerland. Concerning challenges in life insurance, Peter Hieber talked about an approach to give policyholders more control in participating life contracts, Griselda Deelstra showed some new insights when combining financial and mortality risks, and Damir Filipovic presented a new flexible non-parametric data-driven approach to model long-term interest rates, which is an important challenge for life insurers facing long-tailed risks. Stéphane Loisel gave an account on how classical actuarial techniques may be used for the analysis of insurance risks prone to climate change, which was nicely complemented with a presentation of Valérie Chavez-Demoulin on techniques in the statistics of extremes when dealing with non-stationary situations like the one due to climate change. Johanna Ziegel and Pierre-Olivier Goffard then presented some recent advances on certain aspects of statistical methodology. There were several interesting contributions on model uncertainty in the context of optimal investment, with talks by Frank Riedel, Nicole Bäuerle, Mogens Steffensen and Katharina Oberpriller. Multivariate portfolio choice via quantiles was discussed by Carole Bernard. Christa Cuchiero showed how to use polynomial processes to model the capital distribution curves of financial markets, and, also along the lines of stochastic portfolio theory in the spirit of R. Fernholz, Josef Teichmann talked about ergodic robust maximization of asymptotic growth with stochastic factor processes. Extending classical mathematical finance concepts in other directions, Thilo Meyer-Brandis introduced cooperation in arbitrage theory, Irene Klein dealt with large financial markets and Cosimo Munari considered the case of frictions. Finally, Eckhard Platen gave an update of his alternative benchmark approach to financial modelling. On a conceptual side, Berenice Anne Neumann talked about Markovian randomized equilibria in general Dynkin games, Gudmund Pammer presented new results on stretched Brownian motion, Brandon Garcia Flores presented a new approach to use techniques from optimal transport for the identification of optimal reinsurance treaties, and Sigrid Källblad showed how to use optimal transport for adapted distance between the laws of SDEs. Furthermore, Monique Jeanblanc shaded new light on shrinked semimartingales, Anna Aksamit studied multi-action options under information delay, while Claudia Ceci and Alessandra Cretarola presented results on reinsurance using backward SDEs an dynamic contagion models. David Criens presented results on controlled mean field SPDEs, and Caroline Hillairet gave an account of recent advances in the study of Hawkes processes, which are relevant for instance in the insurance of cyber risk.

The week was very stimulating, with many scientific and social interactions of participants and seeds of new ideas and approaches, many of which will be pursued in the time to come.

Acknowledgement: The MFO and the workshop organizers would like to thank the Simons Foundation for supporting Eckhard Platen in the "Simons Visiting Professors" program at the MFO.

Workshop: New Challenges in the Interplay between Finance and Insurance

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Abstracts

Optimal Portfolios with Sustainable Assets – Aspects for Life Insurers

Ralf Korn
(joint work with A. Nurkanovic)

The talk has been based on [4]. With the task to transform our society to a more environmental-friendly and fair one, the interest in investing in sustainable assets has increased. Even more, potential customers have to be asked about their interest in sustainable investment before they enter a pension contract. Hence, the provider has to be prepared to offer suitable investment opportunities.

For various reasons, life insurers have already decided to invest in sustainable assets as part of their actuarial reserve fund. We therefore provide a new framework for optimal portfolio decisions of a life insurer and suggest new modeling approaches for the evolution of the demand for sustainable assets, for the hedging of the risk of sustainability rating changes and for the evolution of asset prices depending on their sustainability rating. While solving various portfolio problems under sustainability constraints explicitly and suggesting further research topics, we take a particular look at the role of the actuarial reserve fund and the annual declaration of its return.

We thus consider a portfolio optimization problem with asset price dynamics $B(t), S_i(t), i = 1, ..., d, t \in [0, T]$ (where $B(t)$ denotes the evolution of the money market account, $S(t)$ is the vector of stock price processes) and square integrable, progressively measurable portfolio processes $\pi(t), t \in [0, T]$. As new ingredients, our framework for sustainable investment contains

- the dynamics $D(t)$ of the cumulative demand of the customers for sustainable investments expressed in percent of their invested sum,
- the dynamics of sustainability ratings $R_i(t)$ of the different assets,
- and their possible influence on the dynamics of the asset prices.

The portfolio problem with a sustainability constraint has the form

$$\max_{\pi(.)\in A(x)} E\left(U\left(X^{\pi}(T)\right)\right) \tag{1}$$

$$\text{such that } R(t) \geq D(t)\ \forall t \in [0, T] \tag{2}$$

For the special choice of $U(x) = \ln(x)$ we can solve this problem in an explicit way and demonstrate various affects of the presence of the sustainability constraint. In particular, we highlight the special situation of a life insurer that is able to use its actuarial reserve fund as an asset

- with a sustainability rating and a constant rate of return for a full year,
- that can be rebuilt with respect to its sustainability rating over a one-year time span,
- and that can possibly be used as the basis for an insurance product against the threat of a sustainability rating downgrade.

As in current models the sustainability constraint leads to an optimal solution that is worse than an unconstrained optimal solution, a natural task is to provide a framework such that it will also be optimal to (mainly) include sustainable assets in the portfolio. Political decisions such as a special taxation on fossile resources based products or the promotion of sustainable production methods can lead to a different potential of future dividends of the corresponding companies and thus motivates the suggestion of new stock price models with a rating- or a demand-dependent drift that itself can depend on the sustainability rating. A possible form can be

$$dS(t) = S(t)\left[(b+\lambda(\hat{D}-D(t)))dt+\sigma dW(t)\right], \tag{3}$$

$$dD(t) = \delta\left(\hat{D}-D(t)\right)dt+\sigma\sqrt{D(t)(1-D(t))}dW_D(t) \tag{4}$$

with the two Brownian motions $W(t)$ and $W_D(t)$ possibly being correlated. I.e. we are using a Jacobi process (see [2] or [1] for its properties) for modeling the demand fluctuations over time. Considering a simple portfolio problem with a money market account and just this one stock, the optimal portfolio process can be shown to be given as

$$\pi(t) = \frac{1}{1-\gamma}\frac{b+\lambda(\hat{D}-D(t))-r}{\sigma^2} \tag{5}$$

for the case of $U(x) = x^\gamma/\gamma$ for $\gamma < 1, \gamma \neq 0$ if the two Brownian motions are independent. In the dependent case, we will obtain a further term that depends on $D(t)$. A proof for this and the explicit form of the optimal portfolio in this case is current work and will be presented soon.

Further model and conceptual challenges in the area of optimal investment with sustainable assets for life insurers can be found in [4].

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Decarbonization of financial markets: a mean-field game approach

Peter Tankov

(joint work with P. Lavigne)

Decarbonization of industry is an essential ingredient for a successful environmental transition, and the financial sector has a key role to play in meeting the financing needs of green companies and directing the funds away from brown, carbon intensive projects. The amount of assets invested in climate-aware funds increased more than two-fold in each year between 2018 and 2021, reaching USD 408 billion at the end of 2021, and several authors aimed to quantify the impact of these additional funding flows on the emission reductions in the real economy. Such impact can be achieved only if green-minded investors target a sufficiently large proportion of companies [1], and the environmental performance of each company depends on factors which are not directly controlled by investors, such as the general economic situation, financial health of the company, and future climate policies. The decarbonization of a financial market is therefore the result of interaction of a large number of companies, operating in an uncertain environment, and should be modeled as a dynamic stochastic game with a large number of players.

Here we develop a dynamic model for the decarbonization of a large financial market, arising from an equilibrium dynamics involving companies and investors, and built using the analytical framework of mean-field games. Mean-field games, introduced in [3] and [4] provide a rigorous way to pass to the limit of a continuum of agents in stochastic dynamic games with a large number of identical agents and symmetric interactions. In the limit, the representative agent interacts with the average density of the other agents (the mean field) rather than with each individual agent. This limiting argument simplifies the problem, leading to explicit solutions or efficient numerical methods for computing the equilibrium dynamics.

The key ingredient of our framework is the notion of mean-field financial market, which describes a large financial market with a continuum of small firms, where the performance of each firm is driven by idiosyncratic noise and a finite number of market-wide risk factors (common noise). We assume that the investors in this market are 'large' meaning that in every investor's portfolio the idiosyncratic risk of small firms is completely diversified, and the portfolio value depends only on market-wide risk factors. Consequently, and consistently with the classical finance theories, only market-wide risk factors are priced, and the stochastic discount factor depends only on the common noise and the 'mean-field'.

We then consider a mean-field market where shares of a continuum of carbon-emitting firms are traded. Each firm determines its dynamic stochastic emission schedule based on its own information and on the market-wide risk factors and market-wide decarbonization dynamics, rather than on the individual decisions of each other small firm, which it cannot observe. To fix its emission level, each firm optimizes a criterion depending on its financial and environmental performance. The financial performance is measured by the market value of the firm's shares and therefore depends on the stochastic discount factor, introducing an interaction between the firms. The environmental performance is measured by carbon emissions, which are penalized in the optimization functional of the firm. The strength of this emission penalty is stochastic, reflecting the uncertainty of climate transition risk. This "stochastic carbon penalty" is a key feature of our model, allowing us to analyze the impact of climate policy uncertainty on market decarbonization and asset prices in a diffusion setting. We show that higher uncertainty about future climate policies and transition risks creates incentive for all companies to emit more carbon and leads to higher share prices and higher spreads between share prices of carbon efficient and carbon intensive companies, confirming the findings of [2] in a more realistic setting with stochastic emission schedules.

The second key ingredient of our model is the interaction between two large investors (or two classes of investors), with different views about the future: while the regular investor uses the real-world measure, the green-minded investor uses an alternative measure, which may, for example, overweight the probability of some environmental policies, making the costs of climate transition more material. In the presence of such green-minded investors, all companies will reduce their emissions and pay lower dividends, leading to lower share prices. However, carbon intensive companies are affected much stronger than climate-friendly carbon efficient companies. This pressure on share prices, in turn, spurs the polluting companies to decrease their emissions.

We summarize the interaction channels and the structure of the game of the present article in figure 1 below. The interaction goes as follows:

- On the one hand, given a stochastic discount factor ξ, the firms choose optimal emissions ψ, driving their economic values V;
- On the other hand, the investors $i \in \{r, g\}$ optimize their wealth W_T^i depending on their greenness;
- All the players (the firms and the investors) are coupled through the terminal market clearing condition: the wealth of the investors equals the economic value of the firms.

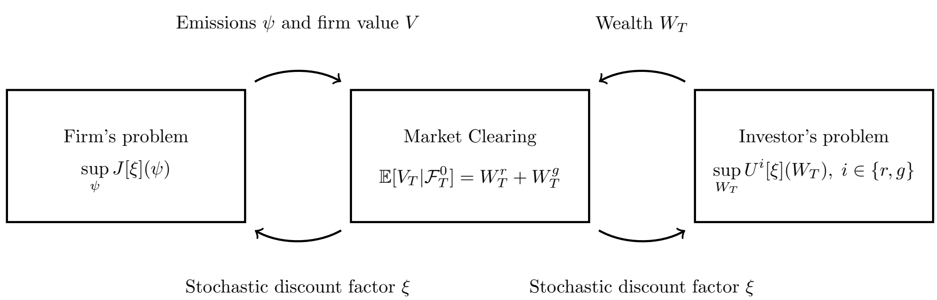

FIGURE 1. Structure of the game.

We rigorously prove the existence and uniqueness of the mean-field game Nash equilibrium for the contunuum of firms interacting through market prices of their shares, providing a robust solution to the stochastic "decarbonization game" in a competitive environment. The equilibrium is materialized by the equilibrium stochastic discount factor, which can be used to compute share prices and emission strategies for each firm. We then develop a convergent numerical algorithm to compute the equilibrium and use it to study the impact of climate transition risk and green investors on the market decarbonization dynamics and share prices.

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Fully-dynamic risk measures: horizon risk, time-consistency, and relations with BSDEs and BSVIEs

Emanuela Rosazza Gianin

(joint work with Giulia Di Nunno)

In a dynamic framework, we identify a new concept associated with the risk of assessing the financial exposure by a measure that is not adequate to the actual time horizon of the position. This will be called *horizon risk*. We clarify that *dynamic risk measures* are subject to horizon risk, so we propose to use the *fully-dynamic* version. To quantify horizon risk, we introduce *h-longevity* as an indicator. We investigate these notions together with other properties of risk measures as normalization, restriction property, and different formulations of time-consistency. We also consider these concepts for fully-dynamic risk measures generated by backward stochastic differential equations (BSDEs), backward stochastic Volterra integral equations (BSVIEs), and families of these. In particular, both for BSDEs and for BSVIEs, we show that h-longevity, restriction and the different formulations of time-consistency can be obtained under suitable conditions on the driver of the BSDE/BSVIE. Within this study, we provide new results for BSVIEs such as a converse comparison theorem and the dual representation of the associated risk measures.

Finally, inspired by the recent literature on cash-subadditive risk measures, we analyze - in full generality and in the framework of (families of) BSDEs - the case where cash-additivity of fully-dynamic risk measures is dropped. An example based on the generalized entropic risk measure (and the corresponding BSDE) will be also provided.

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Uncertainty Propagation and Dynamic Robust Risk Measures

Silvana M. Pesenti
(joint work with Marlon R. Moresco, Mélina Mailhot)

As uncertainty prevents perfect information from being attained, decision makers are confronted with the consequences of their risk assessments made under partial information. Incorporating model misspecification and Knightian uncertainty into dynamic decision making, thus robustifying one's decisions, has been studied in various fields, including economics [10, 15], mathematical finance [4, 12], and risk management [1]. Many circumstances require sequential decisions, where risk assessments are made over a finite time horizon and are based on the flow of information. Importantly, these decisions need to be time-consistent and account for the propagation of uncertainty. As uncertainty may change over time, we consider the dynamic risk of the entire processes rather than the total losses amount at terminal time. While the theory of time-consistent dynamic risk measures is growing [13, 6, 2, 9, 7, 3, 8], the time evolution of uncertainty is little explored.

In this work, we propose an axiomatic framework for quantifying uncertainty of discrete-time stochastic processes. Specifically, we introduce *dynamic uncertainty sets* consisting of a family of time-t uncertainty sets. Each time-t uncertainty set is a set of $\mathcal{F}_t$-measurable random variables summarising the uncertainty of the entire stochastic process at time t. The dynamic uncertainty sets may vary with each stochastic process, as the uncertainty of two processes may differ, even if they share the same law. That is, a time-t uncertainty set is a map $X_{t:T} \mapsto u_t(X_{t:T}) \subset L_t^\infty$ for any bounded discrete process X. This general framework includes, to the authors knowledge, all uncertainty sets encountered in the literature, from moment constraints, f-divergences, semi-norms, and the popular (adapted) Wasserstein distance.

Equipped with a dynamic risk measure represented by a family of one-step risk measures $\{\rho_t\}_{t\in\mathcal{T}}$ and a dynamic uncertainty set $\{u_t\}_{t\in\mathcal{T}}$, we define *dynamic robust risk measures* as sequences of conditional robust risk measures by taking the supremum of all risks in the uncertainty set. Mathematically, a time-t robust risk measure takes the form

$$R_{t:T}(X_{t+1:T}) = \operatorname{ess\,sup}\{\rho_t(Y) : Y \in u_{t+1}(X_{t+1:T})\},$$

for all discrete bounded process $X_{t+1:T}$ from time $t+1$ to T. In this procedure, the first step is to summarise the uncertainty and information of the process $X_{t+1:T}$ into a set of $(t+1)$-measurable random variables, the uncertainty set. The second step is to evaluate the risk of each of the candidate random variables and choose the largest.

This work proceeds by studying conditions on the dynamic uncertainty set that lead to well-known properties of dynamic robust risk measures such as convexity and coherence. To guarantee that the conditions are not overly strong, we seek not only sufficient conditions but also necessary ones. However, two different uncertainty sets can induce the same dynamic robust risk measure, and in fact, for each uncertainty set that satisfies a sufficient condition for a property of interest on the robust risk measure, one can find another uncertainty set that also satisfy it. Therefore, we introduce the *dynamic consolidated uncertainty set* $\{U_t\}_{t\in\mathcal{T}}$, which is the union of all uncertainty sets that agree on the dynamic robust risk measurement. We show that this consolidated uncertainty set also induces the same robust risk measure and can be written as

$$U_{t+1}(X_{t+1:T}) = \left\{Y \in L^\infty_{t+1} : \rho_t(Y) \le R^{\mathfrak{u}}_{t:T}(X_{t+1:T})\right\}.$$

Theorem 1 in the pre-print [11] connects the properties in the consolidated uncertainty set with the axioms of the dynamic robust risk measure.

Crucial to the dynamical framework are notions of time-consistencies, of which many have been introduced and studied in the literature. The most common is strong time-consistency, leading to a dynamic programming principle [6, 14, 5]. While the majority of works assume normalisation of the dynamic risk measures, in a robust setting, uncertainty does generally not lead to normalisation. Indeed, an important subject of debate is whether the value of zero – or more generally an $\mathcal{F}_{t-1}$-measurable random variable – contains uncertainty – at time t. We find that uncertainty sets induced by the f-divergence are normalised, while those generated by the Wasserstein distance or norms are not. Consequently, we introduce the new concept of non-normalised time-consistency to account for non-normalised uncertainty sets. We also work with weaker notions of time-consistency, such as rejection and weak time-consistency. We discuss time-consistency of the uncertainty sets, and show, in Theorem 2, that they are equivalent to the notions of time consistency in the robust risk measure. Figure 1 and Proposition 5 in the pre-print [11] summarise the relationship between the most common notions of time-consistencies.

One of the manuscript's key theorem generalises results from the seminal works of [6, 14]. Specifically, we show that a dynamic robust risk measure is strong or non-normalised time-consistent if and only if it admits a recursive representation of one-step robust risk measures. Furthermore, these one-step robust risk measures are characterised by dynamic uncertainty sets which possess the property of static. Static uncertainty sets arise in one-period settings and do not contain future information. Thus, we show that it is enough to consider the simpler subclass of static uncertainty sets when working with time-consistent dynamic robust risk measures. That is:

Theorem 4 (Recursive Relation). *A (normalised) dynamic robust risk measure R is non-normalised (strong) time-consistent if and only if there exists a static (and normalised) uncertainty set $\mathfrak{u}^{\varsigma} := \{u_t^{\varsigma}\}_{t\in\mathcal{T}}$ such that*

$$R_{t,T}\left(X_{t+1:T}\right) = R_t^{\mathfrak{u}^{\varsigma}}\left(Y_{t+1} + R_{t+1}^{\mathfrak{u}^{\varsigma}}\left(Y_{t+2} + R_{t+2}^{\mathfrak{u}^{\varsigma}}\left(Y_{t+3} + \ldots + R_{T-1}^{\mathfrak{u}^{\varsigma}}(Y_T)\ldots\right)\right)\right),$$

where $Y_t := X_t - R_t^{\mathfrak{u}^{\varsigma}}(0)$ for all $t \in \mathcal{T}$.

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Valuation of liability cash flows subject to capital requirements

Filip Lindskog
(joint work with N. Engler, H. Engsner, K. Lindensjö and J. Thøgersen)

I present two closely related approaches to valuation of liability cash flows motivated by current regulatory frameworks for the insurance industry.

In the first part I study market-consistent valuation of liability cash flows motivated by current regulatory frameworks for the insurance industry. The value assigned to an insurance liability is the consequence of (1) considering a hypothetical transfer of an insurance company's liabilities, and financial assets intended to hedge these liabilities, to an empty corporate entity, and (2) considering the circumstances under which a capital provider would want to achieve and maintain ownership of this corporate entity given limited liability for the owner and that capital requirements have to be met at any time for continued ownership. I focus on the consequences of the capital provider assessing the value of continued ownership in terms of a least favorable expectation of future dividends, meaning that I consider expectations with respect to probability measures in a set of equivalent martingale measures. I present natural conditions on the set of probability measures that imply that the value of a liability cash flow is given in terms of a solution to a backward recursion. This part of my talk is based on joint work with H. Engsner, K. Lindensjö and J. Thøgersen in [2] and [3].

The approach presented in the first part is attractive because it provides a general framework for market-consistent valuation of liability cash flows, taking repeated capital requirements and limit liability into account. However, it typically gives rise to computational challenges when accurate numerical estimates are required. The second part considers a specialized setting, yet sufficiently general for a wide range of applications, aiming for computational tractability.

This approach is motivated by computational challenges arising in multi-period valuation in insurance. Aggregate insurance liability cashflows typically correspond to stochastic payments several years into the future. However, insurance regulation requires that capital requirements are computed for a one-year horizon, by considering cashflows during the year and end-of-year liability values. This implies that liability values must be computed recursively, backwards in time, starting from the year of the most distant liability payments. Solving such backward recursions with paper and pen is rarely possible, and numerical solutions give rise to major computational challenges. The aim of the presented approach is to provide explicit and easily computable expressions for multi-period valuations that appear as limit objects for a sequence of multi-period models that converge in terms of conditional weak convergence. Such convergence appears naturally if one considers large insurance portfolios such that the liability cashflows, appropriately centered and scaled, converge weakly as the size of the portfolio tends to infinity. This part of my talk is based on joint work with N. Engler in [1].

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The risk margin and market risks

Michael Schmutz
(joint work with Christoph Möhr, Laurent Dudok de Wit)

The risk-based solvency frameworks Solvency II in Europe and the Swiss Solvency Test (SST) assess the capitalisation of insurance companies based on a risk evaluation over a one-year interval. The one-year horizon signifies that insurance companies, even in the case of long-term multi-year insurance contracts, basically only have to maintain capital at the regulatory required level of protection for one year. For the further settlement of the contracts, the risk margin (market value margin in the SST) should allow in later years to finance the capital necessary for the regulatory required level of protection or to raise this capital if required.

The risk margin thus plays a fundamental role in these frameworks. In practical implementations, it is often calculated via the sum of the multiplication of a cost of capital rate with the suitably discounted future expected capital requirements. The cost of capital rate represents the premium above the risk-free interest that an investor would demand from the insurance company for covering the corresponding risks. In a recent article [1], the risk margin and, in particular, the cost of capital rate are discussed in the context of an economic triangle of policyholders, shareholders, and regulator. The article uses well-established valuation procedures for illiquid balance sheet items and assumes that the insurance claims are nonhedgeable and independent of the financial market. In view of the in reality often present and sometimes substantial dependencies of insurance claims on financial market risks, we examine here somehow "the opposite". Namely, the dependency of the cost of capital rate on risks in traded financial assets. We focus here only on these risks and ignore further components, such as a potentially considerable illiquidity premium. Using substantial simplifications, we subsequently analytically discuss the fundamental influence of market risks on the cost of capital rate. Our aproach combines the practitioner's perspective with insights from Platen's benchmark approach to quantitative finance, cf. e.g. [5].

More concretely, we analyse the cost of capital from an investor's perspective. Let T denote the time at which all contracts have been settled and assume for simplicity that this date just falls at the end of a year. The capital realized at T is denoted by $\tilde{C}_T$ (i.e. value of assets − value of liabilities). The investor can exercise its limited liability put option if $\tilde{C}_T < 0$. Thus, the investor may price

$\tilde{C}_T^+$ at time $t = T - 1$ using "risk-neutral valuation", i.e.

$$C_t = \mathbb{E}_{\mathbb{Q}}\left(\frac{B_t \tilde{C}_T^+}{B_T}\Big|\mathcal{F}_t\right),$$

where $\mathbb{Q}$ is a suitable "risk-neutral" valuation measure, $(B_t)_{t\geq 0}$ represents the risk-free cash account, and $a^+ = \max(0, a)$ for $a \in \mathbb{R}$. Clearly, to prevent solvency problems, $C_t \geq SCR_t$ should hold for the regulatory required solvency capital SCR_t.

For simplicity, we consider from now on a Continuous Financial Market (CFM), cf. [5], with additional assumptions. The risky tradeables $((S_t^j)_{t\geq 0})_{j=1}^d$ therefore satisfy

$$dS_t^j = S_t^j a_t^j dt + S_t^j \sum_{k=1}^d b_t^{j,k} dW_t^k ,$$

with $((W_t^1, \ldots, W_t^d)_t)$ a d-dimensional standard Brownian Motion, $((a_t^j)_t)$ a suitable "drift", and $((b_t^{j,k})_t)$ a suitable volatility with respect to the k-th source of market risk. For simplicity, let $((b_t^{j,k})_t)_{j,k=1}^d$ be invertible for each t, with inverse matrix $((\bar{b}_t^{j,k})_t)$. For more detail, see e.g. [5, Chapter 10].

The *numéraire*-, or growth-optimal, portfolio $((S_t^*)_t)$, whose existence is assumed here, results from an "admissible trading strategy" in the chosen CFM and satisfies a number of interesting properties through which it can be defined differently but, under appropriate assumptions, equivalently. In particular, for any value process $((S_t^\delta)_t)$ of an "admissible trading strategy" with the same initial value as $((S_t^*)_t)$, the process $((\hat{S}_t^\delta)_t) = ((S_t^\delta / S_t^*)_t)$ is a *supermartingale*, i.e. $\hat{S}_s^\delta \geq \mathbb{E}_{\mathbf{P}}(\hat{S}_t^\delta | \mathcal{F}_s)$ for all $s \leq t$. For general background, see e.g. [5] or [4] for a kind of fundamental theorem that links the existence of this portfolio with an Absence of Arbitrage concept in an equivalent way. However, note that, according to the assumptions made on the existence of $\mathbb{Q}$, we are working within classical option pricing theory as it is e.g. also often used (in an extended form) for life insurance contracts. It turns out, see e.g. [5], that the numéraire portfolio S^* in our CFM can be represented by the following SDE:

$$dS_t^* = S_t^*(r_t + |\theta_t|^2)dt + S_t^* |\theta_t| dW_t ,$$

i.e. $S_t^* = \exp(\int_0^t (r_s + |\theta_s|^2) ds + \int_0^t |\theta_s| dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds)$, for $S_0^* = 1$, where (r_t) stands for the short-rate of the risk-free cash account, and (W_t), given by $dW_t = \frac{1}{|\theta_t|}\sum_{k=1}^d \theta_t^k dW_t^k$, is itself a (real-world, one-dimensional) standard Brownian motion by Lévy's characterization theorem. Here, $|\theta_t|$ stands for the "Total Market Price of Risk" given by $|\theta_t| = \sqrt{\sum_{k=1}^d (\theta_t^k)^2}$ and $\theta_t^k = \sum_{j=1}^d (a_t^j - r_t)\bar{b}_t^{k,j}$.

We assume in the above CFM that the density process (Z_t) is given by

$$Z_t = \frac{d\mathbb{Q}}{d\mathbf{P}}_{|\mathcal{F}_t} = \frac{B_t}{S_t^*},$$

and that it is a true **P**-martingale. The investor's view of the value C_t can then be reformulated via the generalized Bayes rule to give

$$\mathbb{E}_{\mathbf{P}}\left(\frac{S_t^*\tilde{C}_T^+}{S_T^*}\Big|\mathcal{F}_t\right)=\mathbb{E}_{\mathbf{P}}\left(\frac{\tilde{C}_T^+}{\exp\left(\int_t^T(r_s+|\theta_s|^2)ds+\int_t^T|\theta_s|dW_s-\frac{1}{2}\int_t^T|\theta_s|^2ds\right)}\Big|\mathcal{F}_t\right).$$

Thus, $|\theta_t|$ should have a link to the cost of capital rate. The cost of capital rate is typically taken as a constant rate of return above risk-free interest. For a Risk Return analysis on Multiple-Factor Beta Models, we refer e.g. to [3]. For a representation of the cost of capital rate based on the quotient of a conditional real-world and a risk-neutral expectation, we refer to [2]. The advantage of the following approach is that it leads to very explicit expressions with clear dependencies on parameters of the underlying financial market.

Unfortunately, $\tilde{C}_T^+$ is often too complicated for an analytical approach to the cost of capital rate. To gain insights into basic mechanisms, we assume, again very simplistically, that $\tilde{C}_T^+$ can be approximated by an Itô-process of the following form

$$d\tilde{C}_t^+=\mu_t\tilde{C}_t^+dt+\sigma_t\tilde{C}_t^+d\tilde{W}_t\,,$$

for suitable drift and volatility processes μ and σ, where $(\tilde{W}_t)$ also stands for a real-world standard Brownian motion and where for the covariation $[W,\tilde{W}]_t=\int_0^t\rho_s ds$ shall hold for a suitable process ρ. Itô -Calculus yields

$$d\left(\frac{\tilde{C}_t^+}{S_t^*}\right)=\frac{\tilde{C}_t^+}{S_t^*}(\mu_t-r_t-\sigma_t|\theta_t|\rho_t)dt+\frac{\tilde{C}_t^+}{S_t^*}\sigma_t d\tilde{W}_t-\frac{\tilde{C}_t^+}{S_t^*}|\theta_t|dW_t\,.$$

We use this to approximate $\frac{\tilde{C}_T^+}{S_T^*}$ *very roughly* from $t=T-1$ to T:

$$\frac{\tilde{C}_T^+}{S_T^*}\approx\frac{\tilde{C}_t^+}{S_t^*}(1+(\mu_t-(r_t+\sigma_t|\theta_t|\rho_t))\Delta t)+\frac{\tilde{C}_t^+}{S_t^*}\sigma_t\sqrt{\Delta t}\tilde{Z}-\frac{\tilde{C}_t^+}{S_t^*}|\theta_t|\sqrt{\Delta t}Z\,,$$

where $\Delta t=1$, $\tilde{Z}$ and Z are standard normally distributed random variables independent of $\mathcal{F}_t$, and all other terms are $\mathcal{F}_t$-measurable. With $1+x\approx\exp(x)$ one approximatively obtains

$$C_t=\mathbb{E}_{\mathbf{P}}\left(\frac{S_t^*\tilde{C}_T^+}{S_T^*}\Big|\mathcal{F}_t\right)\approx\frac{\mathbb{E}_{\mathbf{P}}\left(\tilde{C}_T^+|\mathcal{F}_t\right)}{1+r_t+\sigma_t|\theta_t|\rho_t}\,.$$

This provides a concrete link to classical Discounted Cash-Flow valuation methods from corporate finance, making $\sigma_t|\theta_t|\rho_t$ a concrete candidate for the cost of capital rate under the imposed assumptions. (This would then have to be supplemented by additional components such as an illiquidity premium.) The observation suggests, among other things, that market risks on a specific balance sheet can have a substantial impact on the cost of capital rate, with the covariation playing a major role, along with the volatility σ_t of the capital and, of course, the total market price of risk $|\theta_t|$. The concrete form of this representation paves the way for relating the cost of capital rate to concrete models for financial markets.

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Insurer's management discretion: Self-hedging participating life insurance

Peter Hieber
(joint work with Karim Barigou)

The performance of participating life insurance contracts depends on an underlying investment portfolio. For the policyholder, risks are limited as the insurance provider assures a minimum return. If the underlying portfolio performs well, the policyholder participates in its return.

The majority of scientific articles on participating life insurance assumes an exogenously given investment strategy for the underlying asset portfolio. This, however, strongly simplifies reality as the insurance provider has full control over the investment strategy of the underlying investment portfolio. He may adapt the portfolio's risk over time, for example contingent on the value of liabilities or asset-liability ratios. In this talk, we depart from the assumption of exogenously given investment strategies and consider more general endogenous investment strategies that adapt dynamically to market developments. The talk has three parts:

(1) Existing literature: We review approaches in the literature on endogenous strategies that are mostly based on the assumption of a complete financial market where all financial risks can be fully hedged. Examples include [2], [4], [3]. [3] transform the non-standard valuation problem into a fixed-point problem using the martingale method, which requires the evaluation of conditional expectations of highly path-dependent payoffs. They then use the Least-square Monte-Carlo (LSMC) approach to approximate such conditional expectations. [2] considers perfect hedging of a participating contract and derived a numerical method for the valuation. However, in both cases ([2], [3]), the focus is on the valuation problem and the determination of the optimal underlying hedging strategy remains an open research question.

(2) Solution in an incomplete market setting: As participating contracts invest for long time horizons, a more realistic assumption is that financial risks cannot be fully hedged. We discuss an objective function that minimizes the hedging risk and determine the corresponding optimal investment strategy. The financial model we consider is a Vasicek-Black-Scholes

model where interest rates are modelled stochastically by a Vasicek model. We consider a multi-period participating contract with an annual guarantee, a product that is very common in central Europe (Belgium, Germany, Switzerland). The implementation follows the neural network approach introduced in [1]. For special cases, we obtain closed form solutions for the optimal investment strategies that serve as a benachmark for our numerical results (see also [5]).

(3) Comparison to the complete market case: As a last step, we link our results to the complete market case and the results existing in the literature ([2], [4]). We specifically point at the resulting optimal hedging strategies. We stress the importance of endogenous investment strategies and their effect on the risk management of participating life insurance contracts. More specifically, we compare the solvency risks and contract values of participating life insurance contracts if investment strategies are (A) exogenously given and (B) chosen endogenously.

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Hybrid life insurance valuation based on a new standard deviation premium principle in a stochastic interest rate framework

Griselda Deelstra

(joint work with Oussama Belhouari, Pierre Devolder)

In this talk, we focus on the pricing of a class of hybrid life insurance products, which are dependent on both mortality and financial risks, and this in a stochastic interest rate framework.

Assuming a complete, arbitrage-free financial market, the valuation of future (purely) financial cash-flows can be based upon risk-neutral expectations and is related to the existence of hedging strategies. In insurance, the calculation of premiums is based on best estimate values and safety loadings, assuming that the law of large numbers can be applied by pooling independent contracts. Of course, in finance, markets appear in practice very often to be incomplete, whereas insurance risks are not always perfectly diversifiable (for instance by the presence of longevity risks or catastrophic risks).

Moreover, as hybrid life insurance contracts depend on both financial and insurance risks, defining a fair valuation of hybrid contracts requires a hybrid valuation principle combining the notions of financial and actuarial valuation. Different principles have been proposed in the literature in order to price these hybrid products (see, e.g., [5], [6], [1], [3], [2] and many others). In order to be consistent with the financial market, the concept of market-consistency is used in the literature, see e.g. [4] or [6]; whilst to be consistent with the actuarial market, the concept of actuarial-consistency has been introduced, see e.g. [2].

Focusing on the pricing of hybrid products in the presence of stochastic interest rates, we first conduct a profound study of the axioms that a valuation operator should verify in the presence of stochastic interest rates (see e.g. [1]) and we study both the market-consistency and actuarial-consistency properties. In particular, we present a generalized standard deviation premium principle in a stochastic interest rate framework, and discuss its integration in different valuation operators suggested in the literature, namely by [5], [6] and [3]. We illustrate our methods with a classical application in life insurance, namely a pure endowment with profit.

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Shrinking the term structure

Damir Filipović

(joint work with Markus Pelger and Ye Ye)

We develop a conditional factor model for the term structure of Treasury bonds, which unifies non-parametric curve estimation with cross-sectional asset pricing. Our factors are investable portfolios and estimated with cross-sectional ridge regressions. They correspond to the optimal non-parametric basis functions that span the discount curve and are based on economic first principles. Cash flows are covariances, which fully explain the factor exposure of coupon bonds. Empirically, we show that four factors explain the discount bond excess return curve and term structure premium, which depends on the market complexity measured by the time-varying importance of higher order factors. The fourth term structure factor capturing complex shapes of the term structure premium is a hedge for bad economic times and pays off during recessions.

Concretely, we denote by $d_t(x)$ the price at date t of a discount bond with time to maturity x [years]. The excess return over $t-1$ to t of this discount bond is then given by

$$r_t(x) = \frac{d_t(x)}{d_{t-1}(x+\Delta_t)} - \frac{1}{d_{t-1}(\Delta_t)},$$

where Δ_t denotes the time [in years] between business days $t-1$ and t. The goal of this project is to estimate and study the empirical properties of the unobserved discount bond excess return curve $r_t : [0,\infty) \to \mathbb{R}$. What is observed at any t are M_t coupon bond securities with prices $P_{t,i}$, cash flows $C_{t,ij}$ at cash flow dates $0 < x_1 < \cdots < x_N$, and their excess returns $R^{\mathrm{bond}}_{t,i} = \frac{P_{t,i}+C_{t-1,ii+1}}{P_{t-1,i}} - \frac{1}{d_{t-1}(\Delta_t)}$. By the absence of arbitrage, we know that a coupon bond is a portfolio of discount bonds. Formally, we obtain

$$R^{\mathrm{bond}}_t = \underbrace{Z_{t-1} r_t(\boldsymbol{x})}_{\text{fundamental returns}} + \underbrace{\epsilon_t}_{\text{return errors}}$$

where we define the normalized discounted cash flows $Z_{t-1,ij} := \frac{C_{t-1,ij+1} d_{t-1}(x_j+\Delta_t)}{P_{t-1,i}}$, and we denote by $f(\boldsymbol{x}) := (f(x_1),\dots,f(x_n))^\top$ the array of function values queried at $\boldsymbol{x} = (x_1,\dots,x_N)^\top$, for any function f.

We estimate r_t by solving the following regularized optimization problem

$$\min_{r_t \in \mathcal{H}_\alpha} \Bigg\{ \underbrace{\frac{1}{M_t} \left\| R^{\mathrm{bond}}_t - Z_{t-1} r_t(\boldsymbol{x}) \right\|_2^2}_{\text{return error}} + \underbrace{\lambda \left\| r_t \right\|^2_{\mathcal{H}_\alpha}}_{\text{smoothness}}, \Bigg\}. \tag{1}$$

We choose the regularization penalty by awarding smoothness of r_t. Smoothness of the return curve is motivated by economic principles, it puts limits to excessive returns of investments such as the butterfly trade $r_t(x-\Delta) - 2r_t(x) + r_t(x+\Delta) \approx r_t''(x)\Delta^2$. Our hypothesis space $\mathcal{H}_\alpha$ therefore consists of twice weakly differentiable functions satisfying the natural boundary conditions $r_t(0) = 0$ and $\lim_{x\to\infty} r_t'(x) = 0$, and finite weighted Sobolev type norm

$$\|r_t\|^2_{\mathcal{H}_\alpha} := \int_0^\infty r_t''(x)^2 \mathrm{e}^{\alpha x}\, dx.$$

We prove that $\mathcal{H}_\alpha$ is a reproducing kernel Hilbert space with kernel k given in closed form. Problem (1) is a kernel ridge regression with unique solution $\hat{r}_t$ in $\mathcal{H}_\alpha$, which is spanned by the N kernel basis functions $k(x_1,\cdot),\dots,k(x_N,\cdot)$. We orthonormalize the basis functions as follows. We show that the kernel matrix $\boldsymbol{K}_{ij} := k(x_i,x_j)$ is invertible, and thus admits spectral decomposition $\boldsymbol{K} = VSV^\top$, with eigenvectors $V = [v_1|\cdots|v_N]$, and strictly positive eigenvalues $s_1 \geq \cdots \geq$

$s_N > 0$. We obtain the orthonormal system of functions in $\mathcal{H}_\alpha$ given by $\boldsymbol{u} = (u_1, \ldots, u_N)^\top := S^{-1/2}V^\top k(\cdot, \boldsymbol{x})$.

After this transformation we obtain the following result.

Theorem 1 (Conditional Factor Model Representation)**.** *The unique solution $\hat{r}_t$ to* (1) *can be represented as factor model*

$$\hat{r}_t(\cdot) = \boldsymbol{u}(\cdot)^\top \hat{F}_t, \tag{2}$$

where the factors $\hat{F}_t$ are unique solution to the cross-sectional ridge regression

$$\min_{F_t \in \mathbb{R}^N} \left\{ \frac{1}{M_t} \left\| R_t^{\text{bond}} - \beta_{t-1}^{\text{bond}} F_t \right\|_2^2 + \lambda \left\| F_t \right\|_2^2 \right\},$$

where the conditional loadings $\beta_{t-1}^{\text{bond}}$ are given in terms of the normalized discounted cash flows (bond characteristics) Z_{t-1} by

$$\beta_{t-1}^{\text{bond}} := Z_{t-1} V S^{1/2}.$$

The factors $\hat{F}_t$ are given in closed form by

$$\hat{F}_t = \omega_{t-1} R_t^{\text{bond}},$$

which are the excess returns of traded bond portfolios with portfolio weights

$$\omega_{t-1} := \left({\beta_{t-1}^{\text{bond}}}^\top \beta_{t-1}^{\text{bond}} + \lambda M_t I_N \right)^{-1} {\beta_{t-1}^{\text{bond}}}^\top.$$

In summary, this is a flexible non-parametric data-driven approach, the smoothness penalty $\lambda > 0$ and maturity weight $\alpha > 0$ are selected empirically by cross-validation. We perform an extensive empirical analysis on a large sample of daily U.S. Treasury bond returns ranging from June 1961 to December 2020. In particular, we shrink the term structure and study low-dimensional approximations of the N-factor model (2), and empirically show that the first n factors describe the data accurately well, for $n = 4$. The paper is available at SSRN: `https://ssrn.com/abstract=4182649`, which contains an extensive list of references.

Climate change, insurance mathematics and optimal prevention

Stèphane Loisel

(joint work with H. Albrecher, C. Constantinescu, R. Gauchon, D. Kortschak, P. Ribereau, J.L. Rullière, J. Trufin)

In this blackboard talk, we start by describing the various impacts of climate change on the insurance industry. We present some theoretical results that demonstrate that quantitative risk management of uncertain and potentially worsening risks is completely different in presence of climate change. We also show the impact of the level of access to information of insurance risk managers on their ability to keep the insurance business safe enough. In presence of full uncertainty, without any possibility to adjust premium, we use our previous results obtained by Albrecher and Constantinescu to show that increasing capital requirements is not enough to make the ruin probability decrease to zero, and that there is a positive probability to be ruined anyway. Besides, the asymptotic rate of decay towards this positive probability level with respect to the initial capital is much slower than usually as well. In the opposite case, assuming that one is "magically" able to adjust instantaneously income premium rate to the worsened risk level, in the regular variation case, we use results obtained with Kortschak and Ribereau to study the effect of claim size distribution worsening. We consider two approaches: either the shape or the scale parameter changes over time. Comparing the two approaches, we note that when risks initially have infinite variance, a change in the scale parameter may have more impact than a change in the shape parameter. We also note that the company may cease its business due to climate change for several reasonns, including ruin, insolvency or mass lapse due to the rise of insurance premium to an unacceptable level. We then present recent works and works in progress to propose a risk management partial solution to this problem. We believe that one key ingredient is risk prevention. We briefly present some results of our recent works with Gauchon, Rullière and Trufin and explain the differences between our optimal prevention problem and classical optimal reinsurance problems. We highlight some results and explain in particular that the optimal prevention level does not depend on the initial surplus level in presence of one single kind of claims, while it depends on the initial surplus when there are two kinds of claims and when prevention only has some effect on one of them. We mention some work in process with Minier and Mamode Khan about prevention with INAR and BINAR processes. Following discussions during this Oberwolfach workshop, some concrete future collaborations have been started with Hansjoerg Albrecher on risk models in presence of climate change, as well as with Michael Schmutz on insurance regulation of long-term risk and short-term bias. In Oberwolfach discussions, we also planted the seed for other future collaborations, notably with Valérie Chavez-Demoulin on climate change risk for hailstorm risk management and with Caroline Hillairet on prevention and thinning.

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Extreme value theory in a changing world

Valérie Chavez-Demoulin
(joint work with Linda Mhalla)

The past few decades have seen extreme climate events affecting all regions of the world with catastrophic impacts on human society. Extreme value theory is the field of statistics dedicated to the study of events with low occurrence frequencies and large amplitudes. Such events are necessarily rare in relation to the bulk of a population, which makes them hard to model and difficult to predict. Classical methods of extreme value theory are based on the assumption that the data are independent and identically distributed (iid) or at least stationary and, in this case, the classical approaches rely on theoretical foundations that are well established and understood. In practice the iid or stationarity assumptions are generally violated, the nature of the series being non-stationary or depending on covariates. In this talk I have reviewed extreme value theory in the univariate and multivariate settings and under non-stationarity, attempting, in this case, to capture different sorts of dependence when estimating risk measures. Part of the work I presented contributes to the development of flexible frameworks for taking into account the effect of covariates on the (tail) dependence structure between two variables. In the context of multivariate extremes, we develop in [1] flexible, semi-parametric method for the estimation of non-stationary multivariate Pickands dependence functions. Related works in multivariate extremes, allowing extremal dependence structures that may vary with covariates are [2] and [3]. A new field of interest and very much linked to the understanding of effect of covariates is causality. The study of causality for extremes is in its infancy. Examples of related work are [4], who defined recursive max-linear models on directed acyclic graphs, [5], who define a causal tail coefficient capturing asymmetries in the extremal dependence between two random variables, [6], who use multivariate generalized Pareto distributions to study probabilities of necessary and sufficient causation as defined in the counterfactual theory of Pearl, and [7], who construct a causal inference method for tail quantities relying on Kolmogorov complexity of extreme conditional quantiles. [8] review the related basic probability schemes, inference techniques, and statistical hypotheses for extreme event attribution. In preparation, we are currently writing a Chapter about causality of extremes in a book entitled "Handbook on Statistics of Extremes".

Part of my presentation was related to a book entitled "Risk Revealed: Cautionary Tales, Understanding and Communication" I co-authored with Paul Embrechts and Marius Hofert, which will appear in 2024 in Cambridge University Press.

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Isotonic distributional regression

Johanna Ziegel

(joint work with Sebastian Arnold, Tilmann Gneiting, Alexander Henzi, Gian-Reto Kleger, Eva-Maria Walz)

Isotonic distributional regression (IDR) is a nonparametric distributional regression approach under a monotonicity constraint [9]. It has found application as a generic method for uncertainty quantification [12], in statistical postprocessing of weather forecasts [9, 11], and it is an integral part of distributional single index models [7, 2]. In this abstract, the construction and main properties of IDR are reviewed and it is explained how IDR can be generalized from empirical distributions to arbitrary distributions yielding isotonic conditional laws.

Assume that the covariate X takes values in a partially ordered space $(\mathcal{X}, \leq)$, and the outcome Y is real-valued. The main assumption of IDR is that when the covariate X increases, we expect an increase of the outcome Y. Mathematically, we assume that $x \leq x'$ for $x, x' \in \mathcal{X}$ implies $\mathcal{L}(Y \mid X = x) \preceq_{\text{st}} \mathcal{L}(Y \mid X = x')$, where $\mathcal{L}(Y \mid X = x)$ denotes the conditional distribution of Y given $X = x$ and $\preceq_{\text{st}}$ denotes the usual stochastic order.

For given data pairs $(x_i, y_i)_{i=1}^n$ with $(x_i, y_i) \in \mathcal{X} \times \mathbb{R}$, the IDR estimator is defined as the vector $\hat{\mathbf{F}} = (\hat{F}_i)_{i=1}^n = (\hat{F}_{Y|X=x_i})_{i=1}^n$ of cumulative distribution functions (cdfs) that satisfies

$$\hat{\mathbf{F}} = \arg\min_{(F_1,\dots,F_n)} \sum_{\ell=1}^{n} \mathrm{CRPS}(F_\ell, y_\ell), \tag{1}$$

where the minimum is taken over all vectors of cdfs $(F_1, \dots, F_n)$ that satisfy $F_i \preceq_{\text{st}} F_j$ whenever $x_i \leq x_j$. Here, the continuous ranked probability score (CRPS) is defined as

$$\mathrm{CRPS}(F, y) = \int_{\mathbb{R}} \left(F(z) - \mathbb{1}\{y \leq z\}\right)^2 \,\mathrm{d}z$$

for a cdf F and $y \in \mathbb{R}$.

The optimization problem at (1) has a unique solution that can be stated explicitly as a min-max formula. It turns out that for each $y \in \mathbb{R}$, $\hat{F}_1(y), \dots, \hat{F}_n(y)$ is the antitonic least-squares regression of the binary outcomes $\mathbb{1}\{y_1 \leq y\}, \dots, \mathbb{1}\{y_n \leq y\}$ [3]. Furthermore, the IDR solution is universal in the sense that the same solution arises when replacing the CRPS in (1) by any quantile- or threshold-weighted CRPS [6]. IDR can be efficiently computed using the pool adjacent violators (PAV) algorithm for each threshold $y \in \{y_1, \dots, y_n\}$ if there is a total order on the covariate space $\mathcal{X}$. For partial orders, the solution can be obtained as a quadratic programming problem for each $y \in \{y_1, \dots, y_n\}$. There is an R package and a Python implementation available [8]. The IDR solution is defined at observed covariate values only but predictions at new covariate values can be readily obtained by suitable interpolation techniques.

Statistical consistency results for IDR can be found in [5] for ordinal covariates, in [10] for real-valued covariates, and in [9] for vector-valued covariates. Furthermore, in [7, 2], the authors show that even if the partial is estimated from the data, consistency still holds.

Suppose that a vector $(X, Y) \in \mathcal{X} \times \mathbb{R}$ has distribution $(1/n) \sum_{i=1}^n \delta_{(x_i, y_i)}$. Then, IDR provides an approximation to the joint distribution of (X, Y) such that all conditional distributions of Y given X are ordered with respect to the stochastic order. It is a natural question to ask if such an isotonic approximation can be constructed starting with any distribution for (X, Y), where we assume that (X, Y) are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The answer is positiv as shown in [1], where the solution is termed the isotonic conditional law of Y given X. The isotonic conditional law of Y given X is constructed as the conditional law of Y given the σ-lattice generated by X.

More precisely, a σ-lattice $\mathcal{C} \subseteq \mathcal{F}$ is a system of sets that contains $\emptyset, \Omega$ and is closed under countable unions and countable intersections. A random variable Z is $\mathcal{C}$-measurable if $\{Z > a\} \in \mathcal{C}$ for all $a \in \mathbb{R}$. The conditional expectation $\mathbb{E}(Z \mid \mathcal{C})$ with respect to the σ-lattice $\mathcal{C}$ can be defined as the L^2-projection of Z onto the closed convex cone of $\mathcal{C}$-measurable random variables [4]. The conditional law $\mathcal{L}(Y \mid \mathcal{C})$ of Z with respect to $\mathcal{C}$ is then a Markov kernel from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\omega \mapsto \mathcal{L}(Z \mid \mathcal{C})(\omega, (a, \infty))$ is a version of $\mathbb{E}(\mathbb{1}\{Z > a\} \mid \mathcal{C})$ for any $a \in \mathbb{R}$. Furthermore, let $\mathcal{U}$ be the collection of all upper sets in $(\mathcal{X}, \leq)$. It is a σ-lattice, and a function $f : \mathcal{X} \to \mathbb{R}$ is increasing if and only f is $\mathcal{U}$-measurable, that is, $\{f > a\} \in \mathcal{U}$ for all $a \in \mathbb{R}$. Finally, for an ordered metric space $(\mathcal{X}, d, \leq)$, the σ-lattice generated by X is defined as

$$\mathcal{A}(X) = \{X^{-1}(B) \mid B \in \mathcal{B}(\mathcal{X}) \cap \mathcal{U}\}.$$

IDR is the isotonic conditional law of Y given X if the joint distribution of (X, Y) has finite support. Isotonic conditional laws can also be identified as CRPS minimizers in a suitable sense [1].

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Approximate Bayesian Computation for Insurance and Finance

Pierre-Olivier Goffard
(joint work with Patrick Laub)

Approximate Bayesian Computation (ABC) is a statistical learning technique to calibrate and select models by comparing observed data to simulated data. This technique bypasses the use of the likelihood and requires only the ability to generate synthetic data from the models of interest. We apply ABC to fit and compare insurance loss models using aggregated data. The talk is based on the work Goffard and Laub [3].

Over a fixed time period, an insurance company experiences a random number of claims called the *claim frequency*, and each claim requires the payment of a randomly sized compensation called the *claim severity*. The two could be associated in an equivalent way with a policyholder, a group of policyholders or even an entire nonlife insurance portfolio. The claim frequency is a counting random variable while the claim sizes are non-negative continuous random variables. Let us say that the claim frequency and the claim severity distributions are specified by the parameters θ_{freq} and θ_{sev} respectively, with $\theta = (\theta_{\text{freq}}; \theta_{\text{sev}})$. For each time $s = 1, \dots, t$ the number of claims n_s and the claim sizes $u_s := (u_{s,1}, u_{s,2}, \dots, u_{s,n_s})$ are distributed as

$$n_s \sim p_N(n|\theta_{\text{freq}}) \quad \text{and} \quad (u_s|n_s) \sim f_U(u|n, \theta_{\text{sev}}).$$

Fitting these distributions is key for claim management purposes. For instance, it allows one to estimate the expected cost of claims and set the premium rate accordingly. The mixed nature of claim data, with a discrete and a continuous component, has lead to two different claim modelling strategies. The first strategy is to handle the claim frequency and the claim severity separately, see for instance [1]. The second approach gathers the two constituents in a compound model for which data in aggregated form suffices. We take the later approach as we assume that the claim count and amounts $\{(n_1, u_1), \dots, (n_t, u_t)\}$ are unobservable. Instead, we only have access to some real-valued *summaries* of the claim data at each time, denoted by

$$x_s = \Psi(n_s, u_s), \quad s = 1, \dots, t. \tag{1}$$

Standard actuarial practice uses the aggregated claim sizes, defined as $\Psi(n, u) = \sum_{i=1}^{n} u_i$, and assumes that the claim frequency is Poisson distributed while the severities are governed by a gamma distribution, we refer to the works of [4].

A Bayesian approach to estimating θ would be to treat θ as a random variable and find (or approximate) the *posterior distribution* $\pi(\theta|x)$. Bayes' theorem tells us that

$$\pi(\theta|x) \propto p(x|\theta)\, \pi(\theta), \tag{2}$$

where $p(x|\theta)$ is the *likelihood* and $\pi(\theta)$ is the *prior distribution*. The prior represents our beliefs about θ before seeing any of the observations and is informed by our domain-specific expertise. The posterior distribution is a very valuable piece of information that gathers our knowledge over the parameters. A point estimate $\hat{\theta}$ may be derived by taking the mean or mode of the posterior. For an overview on Bayesian statistics, we refer to the book of [2].

The posterior distribution (2) rarely admits a closed-form expression, so it is approximated by an empirical distribution of samples from $\pi(\theta|x)$. Posterior samples are typically obtained using Markov Chain Monte Carlo (MCMC), yet a requirement for MCMC sampling is the ability to evaluate (at least up to a constant) the likelihood function $p(x|\theta)$. When considering the definition of x in (1), we can see that there is little hope of finding an expression for the likelihood function even in simple cases (e.g. when the claim sizes are **i.i.d.**). If the claim sizes are not **i.i.d.** or if the number of claims influences their amount, then the chance that a tractable likelihood for x exists is extremely low. Even when a simple expression for the likelihood exists, it can be prohibitively difficult to compute (such as in a big data regime), and so a likelihood-free approach can be beneficial.

We advertise here a likelihood-free estimation method known as *approximate Bayesian computation* (ABC). This technique has attracted a lot of attention recently due to its wide range of applicability and its intuitive underlying principle. One resorts to ABC when the model at hand is too complicated to write the likelihood function but still simple enough to generate artificial data. Given some observations x, the basic principle consists in iterating the following steps:

(1) generate a potential parameter from the prior distribution $\tilde{\theta} \sim \pi(\theta)$;
(2) simulate 'fake data' $\tilde{x}$ from the likelihood $(\tilde{x}|\tilde{\theta}) \sim p(x|\theta)$;
(3) if $\mathcal{D}(x, \tilde{x}) \leq \epsilon$, where $\epsilon > 0$ is small, then store $\tilde{\theta}$,

where $\mathcal{D}(\cdot,\cdot)$ denotes a distance measure and ϵ is an acceptance threshold. The algorithm provides us with a sample of θ's whose distribution is close to the posterior distribution $\pi(\theta|x)$.

The basic ABC algorithm outlined above is, arguably, the simplest method of all types of statistical inference in terms of conceptual difficulty. At the same time, this simple method is perhaps the most difficult form of inference in terms of computational cost. We must use a modified form of this basic regime to minimize (though not eliminate) the gigantic computational costs of ABC. ABC is a somewhat young field (like machine learning), and the methodology of ABC and the other likelihood-free algorithms are currently the subject of intense research. As such, there are many variations of ABC which are under investigation, and there is no ironclad consensus on which variation of the ABC algorithm is the best. For a comprehensive overview on ABC, we refer to the monograph of [7]; in finance and insurance, ABC has been considered in the context of operational risk management by [5] and for reserving purposes by [6].

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Sharing Model Uncertainty

Frank Riedel

(joint work with Chiaki Hara, Sujoy Mukerji, Jean-Marc Tallon)

Uncertainty, as opposed to risk, is a major concern in today's societies. Be it financial markets during the 2007-2009 crisis, policy makers when a new virus emerged, or farmers hit by climate change - in all situations, decision makers faced and face uncertainties that cannot be easily quantified probabilistically. It is therefore of crucial importance to understand whether and how economic institutions can deal with and possibly hedge against this uncertainty. In this paper, we study this question in the framework of identifiable environments in which (Knightian) uncertainty is resolved ex post, at least partially, when sufficient amounts of data have been collected, and agents exhibit smooth ambiguity-averse preferences ([17]), a setting that that has recently been axiomatized by [11]. Identifiability is a necessary condition for statistical learning to occur. We thus put ourselves in a framework where such perfect learning is possible, in principle. In experiments, we could think of an Ellsberg experiment in which the composition of urns is revealed after the experiment. In statistics, ergodic environments suffice for identifiability. In real life, perfect identification is not always achievable, of course. However, in the case of financial markets, e.g., the past volatility of a stock price is very well known ex post. A virus, to take another example, is understood much better aa couple of years after its first appearance. Even in climate change, it might be possible to state after a sufficiently long time that average temperature, e.g., has risen by at least one or two degrees. Our study thus sheds also light on the issue of learning under ambiguity, a notoriously difficult task so far.

In identifiable environments, agents can make their consumption plans contingent on *models*, thus allowing to make ex post insurance payments that depend on a certain probabilistic model being true. The farmer, to take up an example from above, can thus write an insurance contract on a temperature change of a certain amount being true after thirty years or so. This possibility allows to study uncertainty sharing in much more detail and to obtain more results than in general models in which uncertainty is not identifiable.

We are thus able to study models with aggregate uncertainty, in contrast to much of the literature on risk and uncertainty sharing that has focused on the simpler case of no aggregate uncertainty so far. We are able to identify the environments in which a representative agent of smooth ambiguity type exists. In such settings, we can compute quite explicitly the efficient uncertainty sharing rules and study how consumption shares vary with different uncertainty scenarios, depending on the respective individuals' risk and ambiguity aversion relative to society's risk and ambiguity aversion.

We investigate consequences of ambiguous model uncertainty on efficient allocations in an exchange economy, and departing from the literature, allow for ambiguous aggregate risk and heterogeneously ambiguity averse consumers. A model – a statistical view of the world, comprising of parameters and distinctive mechanisms– implies a specific probabilistic forecast about the states of the world. Furthermore, the parameters and mechanisms driving a model may be estimated and identified on the basis of objective data. However, at the point of decision-making, the data relevant to identifying the model is still missing. Hence, consumers are unsure what would be the appropriate probability measure to apply to evaluate consumption contingent on a state space Ω and keep in consideration a set $\mathcal{P}$ of alternative probabilistic laws. Importantly, because models are identified, the usual assumption that consumption plans are contingent on events in the state space now means that they can be made effectively contingent on models too.

We study the case where consumers in the economy are heterogeneously ambiguity averse with *smooth ambiguity* preferences [17]. Our primary focus lies in those economies that admit a *representative* consumer who is also of the smooth ambiguity type. This setting offers valuable and precise insights into efficient sharing rules and the characteristics of the representative consumer. Another advantage of the setting is that the insights obtained, initially assuming that $\mathcal{P}$ is *point-identified*, robustly extend to the case where models are only *set-identified*. When aggregate risk is unambiguous we show, quite generally, that ambiguity aversion makes no difference to the qualitative nature of efficient allocations: they are comonotone just as under expected utility. An economy with a smooth ambiguity averse representative consumer is characterized by consumers who exhibit linear risk tolerance with the same marginal risk tolerance. When aggregate risk is ambiguous, efficient sharing rules systematically deviate from the linearity that would arise under expected utility. The deviations –which make the slope and intercept of the linear rule model-contingent– arise to allow the more ambiguity averse consumers to have smoother expected utility across models.

Macro-finance models that study effects of ambiguity aversion consider single consumer economies with ambiguous aggregate risk. We show if we introduce heterogeneous ambiguity aversion the nature of the representative consumer can be very different from what is widely assumed in the literature. For instance, even if individual consumers have *constant* relative ambiguity aversion, the representative consumer is shown to have *decreasing* relative ambiguity aversion. Such a representative consumer makes for more compelling asset-pricing predictions than one based on homogeneous ambiguity aversion.

Related literature. Efficient risk-sharing in expected-utility economies was first studied by [4], further refined for the HARA class of utility functions by [25], [5] and [14] among others. Under ambiguity, [8] extended the comonotonicity result obtained under expected utility to Choquet expected utility with common capacity. [3], [22] and [12] further studied the case in which aggregate endowment is non-risky and preferences are more general than Choquet-expected-utility preferences (including, for the two latter references, the smooth ambiguity model). [23] and [9] characterized properties of efficient risk-sharing when the aggregate endowment is risky but not ambiguous. [2] extends some of these results to cases where agents have possibly heterogeneous, non-convex ambiguity sensitive preferences. [24] proves that, under HARA with common risk tolerance, a two-fund theorem holds for maxmin-expected-utility economies (and hence efficient allocations are comonotonic). To the best of our knowledge, no paper has studied risk-sharing with *ambiguous* aggregate endowments and *heterogeneous* ambiguity aversion.

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Optimal investment in ambiguous financial markets with learning

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(joint work with Antje Mahayni)

We investigate the effects of model ambiguity preferences on optimal investment decisions in a multi asset Black Scholes market. Since the seminal paper by [5], we know that decision makers may have a non-neutral attitude towards model ambiguity. As a result, preferences are decomposed into risk preferences (based on known probabilities) and preferences concerning the degree of uncertainty about the (unknown) model parameters and are evaluated separately. This is in particular relevant for portfolio optimization problems. [4] suggests that model ambiguity is at least as prominent as risk in making investment decisions.

There are different ways to incorporate model ambiguity in decision making. In our setting, model ambiguity refers to the drift uncertainty in the dynamics of asset prices and we apply the smooth ambiguity approach of [7] to deal with it. The risk in asset prices itself is evaluated by a utility function applied to the terminal wealth. Thus, the expected utility is itself a random variable (determined by the prior distribution of the drift parameters) which is evaluated by a second utility function (ambiguity function) capturing the model ambiguity. As in [1] we assume that both the risk aversion and ambiguity aversion of the investor are described by (CRRA) power functions. While [1] consider pre-commitment strategies, we take into account for the possibility that the investor is able to gradually learn about the drift by observing the asset price movements. Using duality results we are able to solve the problem analytically. To the best of our knowledge this has not yet been achieved before in our setting. Further, based on our theoretical results, we are able to shed light on the impact and consequences of ambiguity preferences.

The underlying financial market consists of d stocks and one riskless bond (for simplicity assumed to be identical to 1), defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with finite time horizon $T > 0$. The price process $S = (S_1(t), \ldots, S_d(t))_{t\in[0,T]}$ of the d stocks will for $i = 1, \ldots, d$ be given by

$$dS_i(t) = S_i(t)\Big[\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t)\Big] = S_i(t)\Big[\sum_{j=1}^{d} \sigma_{ij} dY_j(t)\Big], \tag{1}$$

where $W = (W_1(t), \ldots, W_d(t))^\top_{t\in[0,T]}$ is a d-dimensional Brownian motion, $\mu_i \in \mathbb{R}, \sigma_{ij} \in \mathbb{R}_+, i, j = 1, \ldots, d$ and $\sigma = (\sigma_{ij})$ is regular. We further set

$$Y(t) := W(t) + \Theta t, \quad \Theta^\top := \sigma^{-1}\mu, \quad \mu := (\mu_1, \ldots, \mu_d),$$

where Θ denotes the market price per unit of risk. We further assume that μ is not known and thus a random variable. This implies that the market price of risk Θ is also not known to the investor. However, she has a prior knowledge about Θ in form of a prior distribution $\mathbb{P}$ on $\mathbb{R}^d$.

Due to the self-financing condition, trading strategies $\pi = (\pi_1, \ldots, \pi_d)$ are d-dimensional stochastic processes, where $\pi_k(t)$ describes the amount invested in the k-th stock at time $t \in [0, T]$. Strategies π should be $\mathcal{F}^Y$-progressively measurable (which is the filtration generated by Y or equivalently S). This means that the agent is able to *learn* the right market price of risk. The associated wealth process denoted by $(X^\pi_t)_{t\in[0,T]}$ is given by

$$dX^\pi_t = \sum_{k=1}^{d} \pi_k(t) \frac{\mathrm{d}S_k(t)}{S_k(t)} = \pi(t)\sigma dY(t) \tag{2}$$

with initial capital $x_0 \in \mathbb{R}$. In what follows let $u(x) = \frac{1}{\alpha}x^\alpha, \alpha < 1, \alpha \neq 0$.

The investor aims to maximize her expected utility of terminal wealth. First we assume that the investor is ambiguity-neutral w.r.t. the unknown parameter and consider

$$V(x_0) = \sup_{\pi} \int \mathbb{E}_{\vartheta}[u(X_T^{\pi})]\mathbb{P}(d\vartheta) \tag{3}$$

where the supremum is taken over all $\mathcal{F}^Y$-adapted strategies π for which the stochastic integral and the expectations are defined and $X_T^{\pi} \geq 0$. We denote this set by $\mathcal{A}$. $\mathbb{E}_{\vartheta}$ is the conditional expectation, given $\Theta = \vartheta$. This problem is the well-known Bayesian adaptive portfolio problem. We summarize its solution in the following theorem ([6, 8]) (where $\|\cdot\|$ is the usual Euclidean norm):

Theorem 1. *The maximal expected utility attained in* (3) *is given by*

$$V(x_0) = \frac{x_0^{\alpha}}{\alpha}\left(\int_{\mathbb{R}^d}\left(\int \exp\left(z\cdot\vartheta - \frac{1}{2}\|\vartheta\|^2 T\right)\mathbb{P}(d\vartheta)\right)^{\gamma}\varphi_T(z)dz\right)^{1/\gamma}, \quad x_0 > 0 \tag{4}$$

where $\gamma = 1/(1-\alpha)$, φ_T is the density of the d-dimensional normal distribution $\mathcal{N}(0, TI)$ (I being the identity matrix). The optimal fractions invested in the stocks are also given by an explicit formula.

Now we are interested in an investor who takes model ambiguity into account, i.e. instead of (3) we consider for $v(x) = \frac{1}{\lambda}\, x^{\lambda}, \lambda < 1, \lambda \neq 0$ the problem ([1])

$$\sup_{\pi\in\mathcal{A}} v^{-1} \int v \circ u^{-1}\mathbb{E}_{\vartheta}[u(X_T^{\pi})]\mathbb{P}(d\vartheta) = \sup_{\pi\in\mathcal{A}} \left(\int \left(\mathbb{E}_{\vartheta}[(X_T^{\pi})^{\alpha}]\right)^{\lambda/\alpha}\mathbb{P}(d\vartheta)\right)^{1/\lambda} \tag{5}$$

This means that model ambiguity, represented by an uncertain market price of risk, is evaluated with a second utility function v which is here of the same form but with possibly different parameter. In case $\alpha > 0$ problem (5) is equivalent to

$$\sup_{\pi\in\mathcal{A}} \left(\mathbb{E}\left[\left(\mathbb{E}_{\Theta}[(X_T^{\pi})^{\alpha}]\right)^{\lambda/\alpha}\right]\right)^{\alpha/\lambda}, \tag{6}$$

Here we restrict to the case that $\lambda > \alpha > 0$ and define $\mathbf{p} := \lambda/\alpha > 1$. The economic interpretation is that the agent is ambiguity-loving (the ambiguity-averse case is similar). By using the $L^{\mathbf{p}}$ norm $\|\cdot\|_{\mathbf{p}}$ we can write problem (6) as

$$\sup_{\pi\in\mathcal{A}} \|\mathbb{E}_{\Theta}[(X_T^{\pi})^{\alpha}]\|_{\mathbf{p}} \tag{7}$$

where the norm is w.r.t. Θ. It is well-known that the $L^{\mathbf{p}}$ norm has the following dual representation for a r.v. $X \geq 0$, where $1/\mathbf{p} + 1/\mathbf{q} = 1$ (see e.g. [9]):

Lemma 1. *If $\mathbf{p} := \lambda/\alpha > 1$ we obtain for non-negative $X \in L^{\mathbf{p}}$*

$$\|X\|_{\mathbf{p}} = \sup\left\{\int X d\mathbb{Q} : \left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_{\mathbf{q}} \leq 1\right\}. \tag{8}$$

where on the right-hand side of (8) *the supremum is taken over all measures $\mathbb{Q}$ (not necessarily probability measures) which are absolutely continuous w.r.t. $\mathbb{P}$ and satisfy the constraint. Moreover, an optimal measure $\mathbb{Q}^*$ exists.*

In what follows define the set of measures $\mathfrak{Q}$ as the set of measures which satisfy the constraints in (8). This gives immediately rise to the following solution algorithm for our problem:

Theorem 2. *In the model of this subsection we have*

$$\sup_{\pi\in\mathcal{A}}\sup_{\mathbb{Q}\in\mathfrak{Q}}\int\mathbb{E}_\vartheta[(X_T^\pi)^\alpha]\mathbb{Q}(d\vartheta)=\sup_{\mathbb{Q}\in\mathfrak{Q}}\sup_{\pi\in\mathcal{A}}\int\mathbb{E}_\vartheta[(X_T^\pi)^\alpha]\mathbb{Q}(d\vartheta)=\int\mathbb{E}_\vartheta[(X_T^{\pi^*})^\alpha]\mathbb{Q}^*(d\vartheta).$$

After normalizing $\mathbb{Q}$, the inner optimization problem is exactly the Bayesian portfolio problem with distribution $\tilde{\mathbb{Q}} := \mathbb{Q}/\mathbb{Q}(\mathbb{R})$ for the unknown parameter. So solving (6) *boils down to solving the classical Bayesian portfolio problem first with value given in Theorem 1 and then in a second step finding the optimal prior distribution implied by $\mathbb{Q}^*$ which is obtained from the outer optimization problem. The optimal strategy π^* is then the one in Theorem 1 with $\mathbb{P}$ replaced by $\mathbb{Q}^*$.*

An approach like this may be generalized to situations where uncertainty and ambiguity are measured by other means (see e.g. [3]). The extended abstract is based on [2].

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Investment under Uncertain Preferences

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(joint work with S. Desmettre and J. Søe)

We consider classes of dynamic decision problems where an investor maximizes utility but faces random preferences. We consider three versions of the problem.

In one version, the investor optimizes expected utility where the expectation is taken with respect to both financial and preference uncertainty. That is based on Steffensen and Søe (2023). We formalize a consumption–investment–insurance problem with the distinction of a state-dependent relative risk aversion. The state dependence refers to the state of the finite state Markov chain that also formalizes insurable risks such as health and lifetime uncertainty. We derive and analyze the implicit solution to the problem and compare it with special cases in the literature.

Two other versions are based on certainty equivalents. We tackle the time-consistency issues arising from that formulation by applying the equilibrium theory approach.

In one version, the investor learns nothing about his preferences as time passes. That is based on Desmettre and Steffensen (2023). We provide the proper definitions and prove a rigorous verification theorem. We complete the calculations for the cases of power and exponential utility. For power utility, we illustrate in a numerical example, that the equilibrium stock proportion is independent of wealth, but decreasing in time, which we also supplement by a theoretical discussion. For exponential utility, the usual constant absolute risk aversion is replaced by its expectation.

The main results of Desmettre and Steffensen (2023) are gathered in the following verification theorem and corollary. Definitions and proofs can be found in Desmettre and Steffensen (2023). We model the parameter of a utility function γ as a real-valued random variable. Examples are the constant (known) relative and absolute risk aversions that are replaced by random variables. We form an optimization problem based on the idea to *maximize the certainty equivalent of terminal wealth w.r.t. a random risk aversion* in an equilibrium sense, i.e. we want to maximize the reward functional

$$J^\pi(t,x) := \int (u^\gamma)^{-1} \left(\mathbb{E}_{t,x}[u^\gamma(X^\pi(T))]\right) \, d\Gamma(\gamma)\,, \tag{1}$$

where Γ is the Cumulative Distribution Function (CDF) of γ, and we integrate over the support of the corresponding CDF. Moreover, we assume that the dependence of the utility function $u^\cdot$ on $\gamma \sim \Gamma$ is such that the integral in (1) is always well-defined. Note that now we decorate the utility function by subscript γ to highlight its dependence on risk aversion.

We now first formalize the equilibrium problem and then characterize its solution in a verification theorem. We introduce

$$y^{\pi,\gamma}(t,x) := \mathbb{E}_{t,x}\left[u^\gamma\left(X^\pi(T)\right)\right]\,, \tag{2}$$

such that the objective of the investor is to maximize the reward functional

$$J^\pi(t,x) := \int (u^\gamma)^{-1}\left(y^{\pi,\gamma}(t,x)\right) d\Gamma(\gamma) \tag{3}$$

in a given sense.

Theorem 1 (Verification Theorem)**.** *Assume that there exist functions $U \in C^{1,2}$, $Y^\gamma \in C^{1,2}$ for all γ, such that*

$$\begin{aligned} U_t(t,x) = \inf_\pi \Big\{ &-(r+\pi(\alpha-r))\,xU_x(t,x) - 0.5\pi^2x^2\sigma^2U_{xx}(t,x) + H_t(t,x) \\ &+ (r+\pi(\alpha-r))xH_x(t,x) + 0.5\pi^2x^2\sigma^2H_{xx}(t,x) \\ &- \int \iota^\gamma(Y^\gamma(t,x))(Y_t^\gamma(t,x) \\ &+ (r+\pi(\alpha-r)x)Y_x^\gamma(t,x) + 0.5\sigma^2\pi^2x^2Y_{xx}^\gamma(t,x))\,d\Gamma(\gamma)\Big\}\,, \end{aligned} \tag{4}$$

and

$$Y_t^\gamma(t,x) = -(r+\hat\pi(\alpha-r))xY_x^\gamma(t,x) - 0.5\sigma^2\hat\pi^2x^2Y_{xx}^\gamma(t,x)\,, \tag{5}$$

where $H(t,x) = \int (u^\gamma)^{-1}(Y^\gamma(t,x))d\Gamma(\gamma) \in C^{1,2}$ *and*

$$\begin{aligned}\hat\pi = \arg\inf_\pi \Big\{ & -(r+\pi(\alpha-r))\,xU_x(t,x) - 0.5\pi^2x^2\sigma^2U_{xx}(t,x) + H_t(t,x)\\ & + (r+\pi(\alpha-r))xH_x(t,x) + 0.5\pi^2x^2\sigma^2H_{xx}(t,x)\\ & - \int \iota^\gamma(Y^\gamma(t,x))(Y_t^\gamma(t,x)\\ & + (r+\pi(\alpha-r)x)Y_x^\gamma(t,x) + 0.5\sigma^2\pi^2x^2Y_{xx}^\gamma(t,x))\,d\Gamma(\gamma)\Big\}\,,\end{aligned} \tag{6}$$

with boundary conditions

$$U(T,x) = x, \text{ and } Y^\gamma(T,x) = u^\gamma(x) \text{ for all } \gamma\,. \tag{7}$$

Furthermore assume that U*,* H*, and* Y^γ *for all* γ*, belong to the space* $L^2(X^{\hat\pi})$*. Then* $\hat\pi$ *is an equilibrium control, and we have that*

$$V(t,x) = U(t,x)\,, \tag{8}$$

$$y^{\hat\pi,\gamma}(t,x) = Y^\gamma(t,x) \text{ for all } \gamma\,. \tag{9}$$

For the special form of H given by

$$H(t,x) = \int (u^\gamma)^{-1}(Y^\gamma(t,x))\,d\Gamma(\gamma)$$

we obtain as an immediate consequence:

Corollary 1. *From the pseudo HJB* (4) *we obtain by using*

$$\begin{aligned}H(t,x) &= \int (u^\gamma)^{-1}(Y^\gamma(t,x))\,d\Gamma(\gamma)\,,\\ H_t(t,x) &= \int \iota^\gamma(Y^\gamma(t,x))Y_t^\gamma(t,x)\,d\Gamma(\gamma)\,,\\ H_x(t,x) &= \int \iota^\gamma(Y^\gamma(t,x))Y_x^\gamma(t,x)\,d\Gamma(\gamma)\,,\\ H_{xx}(t,x) &= \int \iota^\gamma(Y^\gamma(t,x))Y_{xx}^\gamma(t,x)\,d\Gamma(\gamma) + \int (\iota^\gamma)'(Y^\gamma(t,x))(Y_x^\gamma(t,x))^2\,d\Gamma(\gamma)\,,\end{aligned}$$

the following form:

$$\begin{aligned}U_t(t,x) = \inf_\pi \Big\{ & -(r+\pi(\alpha-r))xU_x(t,x) - 0.5\sigma^2\pi^2x^2U_{xx}(t,x)\\ & + 0.5\sigma^2\pi^2x^2\int (\iota^\gamma)'(Y^\gamma(t,x))(Y_x^\gamma(t,x))^2\,d\Gamma(\gamma)\Big\}\,.\end{aligned} \tag{10}$$

In this formulation, the non-linearity arising within the time-inconsistent control problem is clearly visible, cf. [1, Section 16.2].

Risk aversion is an observed stochastic process in another version (work in progress, new results). That version can, e.g., be motivated by preferences that directly depend on the state of health. We introduce the notion of preferences concerning preference risk and find a case where the investor invests as if the (conditional) expected risk aversion were realized.

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Reduced-form framework under model uncertainty

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(joint work with Francesca Biagini, Andrea Mazzon)

The talk is based on [3],[4] and [5]. In this talk we introduce a reduced-form framework for multiple ordered default times under model uncertainty and study some applications in insurance and finance. To this purpose we define a sublinear conditional operator with respect to a family of probability measures possibly mutually singular to each other in presence of multiple ordered default times. In this way we extend the classical literature on credit risk in presence of multiple defaults, see for example [11], [12], [13] and [17] to the case of a setting where many different probability models can be taken into account.

Over the last years, several different approaches have been developed in order to establish such robust settings which are independent of the underlying probability distribution, see among others [1], [7], [8], [9], [10], [15], [16], [19], [20], [22], [23], [24] and [25]. However, the above results hold only on the canonical space endowed with the natural filtration. In credit risk and insurance modeling it is fundamental to model multiple random events occurring as a surprise, such as defaults in a network of financial institutions or the loss occurrences of a portfolio of policy holders. This requires to consider filtrations with a dependence structure. Such a problem is mentioned in [2] and solved for an initial enlarged filtration. In [6] they define a sublinear conditional operator with respect to a filtration which is progressively enlarged by one random time.

In this paper we extend the approach in [6] and define a sublinear conditional operator with respect to a filtration progressively enlarged by multiple ordered stopping times. Such an extension is connected to several additional technical challenges with respect to the construction in [6].

First, we cannot consider default times in all generality, but we need to focus on a family of ordered stopping times. In particular, we work in the setting of the top-down model for increasing default times introduced in [11], in order to model the loss of CDOs, as a generalization to the well known Cox model in [18]. More specifically, we start with a reference filtration $\mathbb{F}$ and define a family of ordered stopping times $\tau_1, ..., \tau_N$, in a similar way as done in [11]. We then progressively enlarge $\mathbb{F}$ with the filtrations $\mathbb{H}^n$ generated by $(\mathbf{1}_{\{\tau_n \leq t\}})_{t\geq 0}$, $n = 1, ..., N$, and define $\mathbb{G}^{(n)} := \mathbb{F} \vee \mathbb{H}^1 \vee ... \vee \mathbb{H}^n$, $n = 1, ..., N$. In our case, we construct $\tau_1 < ... < \tau_N$ in such a way that $\tilde{\tau}_n := \tau_n - \tau_{n-1}$ is independent of $\mathcal{H}_t^{n-1}$ for any $n = 2, ..., N, t \geq 0$ conditionally on $\mathcal{F}_\infty$. In particular, the intensities of the stopping times are driven by $\mathbb{F}$-adapted stochastic processes which may be used to model dependence structures driven by common risk factors and also contagion effects. We first address the problem of computing $\mathbb{G}^{(N)}$-conditional expectations of a given random variable under one given prior in terms of a sum of $\mathbb{F}$-conditional expectations depending on how many defaults have happened before time t. This is also a new contribution to the literature on ordered multiple default times in the classical case, i.e., in presence of only one probability measure. For an analogous result following the density approach for modeling successive default times, we refer to [12]. The main technical issue in our setting is to compute conditional expectations when a strictly positive number of defaults, but not *all* the N defaults, have happened. Already under a fixed prior the results for multiple ordered default times are not a trivial extension of the ones in a single default setting.

We then use this representation to define the sublinear conditional operator $\tilde{\mathcal{E}}^N$ under model uncertainty with respect to the progressively enlarged filtration $\mathbb{G}^{(N)}$. As in [6], our definition makes use of the sublinear conditional operator introduced by Nutz and van Handel in [21] with respect to $\mathbb{F}$. To this purpose we assume that $\mathbb{F}$ is given by the canonical filtration. In particular, we show that our construction is consistent with the ones in [21] in presence of no default and in [6] for $N = 1$, respectively. The main technical challenge is to prove a weak dynamic programming principle for the operator as done in [6] for the single default setting, as it requires to exchange the order of integration between the operator and expectations under a given prior. We then use the conditional sublinear operator to evaluate credit portfolio derivatives under model uncertainty. In particular, we focus on the valuation of the so called i-th to default contingent claims $\mathrm{CCT}^{(i)}$, for $i = 1, ..., N$. Moreover, we discuss if the valuation of such financial or insurance products with the sublinear conditional operator corresponds to a sensitive pricing rule. As done in [6] for the single default case, we can establish a relation between the sublinear conditional operator and the superhedging problem in a multiple default setting for a generic payment streams under given conditions. Furthermore, we show that the sublinear conditional operator can be used to price a contingent claim such that the extended market allows no arbitrage of the first kind under model uncertainty as in [7]. This result requires assumptions about the trading strategies which are, however, not restrictive in an insurance setting. By modeling the intensity processes as an affine process under uncertainty, introduced for example in [14] and [3], the valuation of several relevant payoffs can be numerically computed by solving non-linear PDEs.

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Multivariate Portfolio Choice via Quantiles

Carole Bernard
(joint work with Andrea Perchiazzo, Steven Vanduffel)

The talk was organized as follows. First, I recalled the quantile approach of [8] for an agent maximizing a one-dimensional objective function that is law-invariant and non-decreasing. The quantile approach builds on the concept of cost-efficiency originally proposed by [5, 6] and further discussed in [1]. Then I related the multivariate portfolio choice (see (1) below) to a risk sharing problem (see (3) hereafter) as studied e.g., by [3] in the context of a multivariate expected utility setting. We then show how the quantile approach used for univariate optimal portfolio choice can be also useful to solve the multivariate portfolio choice as in (1) below. To do so, we use the concept of multivariate cost-efficiency ([2]). Finally, two examples are fully solved: the optimization of a sum of expected CRRA utility functions and the infconvolution of the Range Value-at-Risk (RVaR). For this latter example, we make use of the explicit results of [7] and show that the portfolio problem that minimizes the sum of d RVaRs can be rewritten as a portfolio that maximizes a one-dimensional objective function, i.e., a distorted expectation. Furthermore, this problem has been explicitly solved in [4] and [9].

Specifically, we assume a frictionless and arbitrage-free financial market living on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a non-empty sample space, $\mathcal{F}$ is the σ-algebra generated by Ω and $\mathbb{P}$ denotes the probability measure on Ω. We consider a fixed investment horizon $T > 0$ without intermediate consumption in which a final payoff X_T received at time T has an initial price given as $\mathbb{E}\left[\xi_T X_T\right]$ where ξ_T is the pricing kernel, agreed by all agents, with positive density on $\mathbb{R}_+ \setminus \{0\}$. Let $V(\cdot)$ be a multivariate law-invariant objective function. We consider the problem

$$\sup_{(X_1, X_2, \ldots, X_d) \in \mathcal{A}} V\left(X_1, X_2, \ldots, X_d\right), \tag{1}$$

where $\mathcal{A} = \left\{(X_1, X_2, \ldots, X_d) \in \mathcal{K} \text{ s.t. } \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i\right] = w_0\right\}$, $\mathcal{K}$ is the set of random d-vector and $w_0 > 0$ denotes the total budget that must be allocated in d dimensions. The goal is to optimize a multivariate law-invariant objective function $V(\cdot)$ over a set of admissible $(X_1, \ldots, X_d) \in \mathcal{A}$ such that the total budget w_0 is allocated.

We assume that the objective function $V(\cdot)$ is law-invariant (that is, if two vectors $(X_1,\ldots,X_d)$ and $(Y_1,\ldots,Y_d)$ are equal in distribution, then $V(X_1,\ldots,X_d) = V(Y_1,\ldots,Y_d)$). Furthermore, we assume that $V(\cdot)$ is strictly increasing in at least one of the dimensions. Without loss of generality, we can thus assume that for any constant $a \in \mathbb{R}_+ \setminus \{0\}$, $V(X_1+a, X_2,\ldots,X_d) > V(X_1,X_2,\ldots,X_d)$. Finally, we assume that the general portfolio problem (see (1)) is well-posed in that there exists an optimal multivariate portfolio $(X_1^\star,\ldots,X_d^\star)$ leading to a maximum finite value for $V(X_1,\ldots,X_d)$.

To solve the general multivariate portfolio problem in (1), we first solve a multivariate risk sharing problem in the absence of a financial market that we then use to provide the solution to (1).

Let S be a random variable. Define the risk sharing of S as the following set of random vectors associated to S

$$A_d(S) := \left\{ (X_1, X_2, \ldots, X_d) \in \mathcal{K} : \sum_{i=1}^{d} X_i = S \right\}. \tag{2}$$

The optimal multivariate risk sharing associated to the total risk S solves

$$\sup_{(X_1,X_2,\ldots,X_d)\in A_{d(S)}} V(X_1,\ldots,X_d). \tag{3}$$

Denote by

$$(Y_1(S),\ldots,Y_d(S))$$

a solution to (3). In the context of the additive multivariate utility function, i.e., where $V(X_1,\ldots,X_d)$ is of the form $V(X_1,\ldots,X_d) = \sum_{i=1}^d U_i(X_i)$ in which U_i for $i = 1,\ldots,d$ are univariate exponential utility functions or univariate CRRA (Constant Relative Risk Aversion) utility functions, the multivariate risk sharing problem (3) can easily be solved explicitly. In the case of an objective function based on quantile risk measures (e.g., RVaR), a solution for the multivariate risk sharing problem is found in [7].

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Polynomial interacting particle systems and non-linear SPDEs for capital distribution curves

Christa Cuchiero
(joint work with Florian Huber)

The stability of the *capital distribution curves* over time, as shown in Figure 1, can be seen as a universal phenomenon in finance. By this we here mean a robust

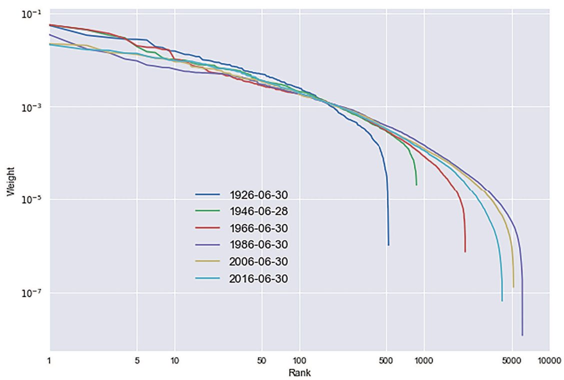

Figure 1. Capital distribution curves: 1926 - 2016, source [4]

empirical feature that holds universally across different markets, asset classes and in particular over time. Each of the above curves depicts the relative market capitalization in ranked order of the major US markets' stocks on a log-log scale from 1926 to 2016. The *relative market capitalization* or *market weight* is defined as the percentage of the market capitalization of a fixed company, i.e., the number of outstanding shares times the current price of one share, with respect to the capitalization of the whole market. The striking feature of these curves is their remarkably stable shape over the last century. Although the market weights of each company fluctuate stochastically the shape of the capital distribution curves differs (in first order) over the years only by the number of stocks present in the considered market. This fundamental observation was the starting point for R. Fernholz to develop *stochastic portfolio theory* about 20 years ago, see [1].

On the mathematical side of financial modeling we also encounter universal structures, such as the interplay of potentially infinitely many factors as well as mean field interactions and limits. Universal model classes that are able to capture these phenomena and appear throughout in mathematical finance, but also in other fields like population genetics and physics, are (infinite dimensional) affine and polynomial processes.

One goal of this work is to combine mathematical with financial universality and to model the capital distribution curves via polynomial processes, which have empirically proved to provide a very good fit to these curves.

More precisely, we extend volatility stabilized market models, a particular class of polynomial models introduced by Fernholz et al [2], by allowing for a common noise term such that the models remains polynomial. Indeed, we consider the following model for the N individual market capitalizations

$$dS_i(t) = \beta \sum_{j=1}^N S_j(t)dt + \sqrt{\alpha}\sqrt{S_i(t)\sum_{j=1}^N S_j(t)dW_t^i} + \sqrt{(N-\alpha)}S_i(t)dW_t^0,$$

where $\alpha \geq 0$, $\beta \geq \frac{\alpha}{2}$ and W^i for $i \in \{1,\dots,N\}$ are the idiosyncratic Brownian motions and W^0 the common one. The introduction of this common noise term permits to overcome the absence of correlation between the individual stocks in the original model of [2].

Inspired by M. Shkolnikov [5] who studied large volatility stabilized markets, we then analyze the limit as $N \to \infty$. To do so we rescale time, i.e. let time go slower as we add particles, and consider $X(t) := S(t/N)$

$$dX_i(t) = \frac{\beta}{N}\sum_{j=1}^N X_j(t)dt + \sqrt{\frac{\alpha}{N}}\sqrt{X_i(t)}\sqrt{\sum_{j=1}^N X_j(t)dW_t^i} + \sqrt{1-\frac{\alpha}{N}}X_i(t)dW_t^0.$$

Taking formal limits and denoting the typical particle in the limit by Y then yields

$$dY(t) = \beta\mathbb{E}[Y(t)|\sigma(W^0)]dt + \sqrt{\alpha Y(t)\mathbb{E}[Y(t)|\sigma(W^0)]}dW_t + Y(t)dW_t^0. \tag{1}$$

for some Brownian motion W independent of W^0, and where $\sigma(W^0)$ denotes the sigma-algebra generated by W^0. To make this rigorous we consider, as usual for McKean-Vlasov equations, the particles' empirical probability measure on path space, i.e.

$$\rho^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_i}$$

and its "mean-field limit" $(\rho_t^N)_{t\in[0,T]} \to (\rho_t)_{t\in[0,T]}$ in $C([0,T]; M_1(\mathbb{R}_+))$, where $T > 0$ denotes some finite time and $M_1(\mathbb{R}_+)$ probability measures over $\mathbb{R}_+$ with finite first moment, i.e.

$$M_1(\mathbb{R}_+) = \{\mu \in M(\mathbb{R}_+) | \int_{\mathbb{R}_+} x\mu(dx) =: \langle x, \mu\rangle < \infty\},$$

equipped with the Wasserstein-1 distance. Then, we show that the limit ρ, which is the unique solution of a degenerate, non-linear SPDE, corresponds to the conditional law of the typical particle Y, i.e. $\rho = \mathcal{L}(Y(\cdot)|\sigma(W^0))$. and $\langle \rho_t, \mathrm{id}_x\rangle = \mathbb{E}[Y(t)|\sigma(W^0)]$. Indeed, our two main results read as follows:

Theorem 1. *Under minor conditions on the initial values of the particle system, each convergent subsequence of* $(\rho^N_\cdot)_{N\in\mathbb{N}}$ *converges a.s. in* $C([0,T], M_1(\mathbb{R}_+))$ *to the unique probabilistically strong, analytically weak,* $M_1(\mathbb{R}_+)$*-valued solution* ρ *of the non-linear SPDE*

$$d\rho_t = (\frac{\alpha}{2}\langle \rho_t, \mathrm{id}_x\rangle \partial_x^2(x\rho_t) + \frac{1}{2}\partial_x^2(x^2\rho_t) - \beta\langle \rho_t, \mathrm{id}_x\rangle \partial_x \rho_t)dt - \partial_x(x\rho_t)dW_t^0. \tag{2}$$

Theorem 2. *Consider* (1) *with* $0 < Y(0) \in L^2(\Omega)$*, independent of* W^0*, and let* ρ *be the unique solution of* (2) *with* $\rho_0 = \mathcal{L}(Y(0))$.

- *Then, any solution to* (1) *satisfies* $\rho = \mathcal{L}(Y(\cdot)|\sigma(W^0))$ *as well as*
$$\mathbb{E}[Y(t)|\sigma(W^0)] = \langle \rho_t, \mathrm{id}_x\rangle = \langle \rho_0, \mathrm{id}_x\rangle \exp((\beta - \frac{1}{2})t + W_t^0) =: S(t).$$
- *The two-dimensional process* $(Y, \mathbb{E}[Y(t)|\sigma(W^0)]) =: (Y, S)$ *is a polynomial diffusion on* $\mathbb{R}^2_{++}$ *which is unique in law. Its dynamics are given by*
$$\begin{aligned} dY(t) &= \beta S(t)dt + \sqrt{\alpha}\sqrt{Y(t)S(t)}dW_t + Y(t)dW_t^0 \\ dS(t) &= \beta S(t)dt + S(t)dW_t^0, \quad S_0 = \langle \rho_0, \mathrm{id}_x\rangle. \end{aligned}$$

One of the mathematical subtleties of these results lies in the uniqueness proof which involves fine estimates with respect to weighted Sobolev norms. This uniqueness result then also allows us to conclude uniqueness in law of the polynomial process $(Y, \mathbb{E}\left[Y \mid \sigma(W^0)\right])$ which was open so far.

Let us remark that behind the intriguing polynomial property of $(Y, \mathbb{E}[Y(t)| \sigma(W^0)])$ is a generic structure. Indeed, consider (for simplicity) a one-dimensional conditional McKean-Vlasov SDE of the form

$$\begin{aligned} dZ_t &= b(Z_t, \mathbb{E}[Z_t^1|\sigma(W^0)], \dots, \mathbb{E}[Z_t^k|\sigma(W^0)])dt \\ &\quad + \sqrt{c(Z_t, \mathbb{E}[Z_t^1|\sigma(W^0)], \dots, \mathbb{E}[Z_t^k|\sigma(W^0)])}dW_t \\ &\quad + c^0(Z_t, \mathbb{E}[Z_t^1|\sigma(W^0)], \dots, \mathbb{E}[Z_t^k|\sigma(W^0)])dW_t^0, \quad 0 \le t \le T. \end{aligned}$$

Then, if c is quadratic in the first variable and b and c^0 are affine in the first variable, the conditional moments become a k-dimensional autonomous standard Itô-SDE driven by W^0. Provided that a (pathwise) unique solution exists for this SDE, its components then correspond to the conditional moments $\mathbb{E}[Z^i|\sigma(W^0)]$ for $i = 1, \dots, k$. From the theory of time-inhomogeneous polynomial processes (see [3]), one should then be able to deduce existence and uniqueness for a large class of conditional McKean-Vlasov SDEs beyond the standard conditions of Lipschitz continuity and uniform ellipticity. Proving this conjecture rigorously is subject of ongoing work.

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Ergodic robust maximization of asymptotic growth with stochastic factor processes

Josef Teichmann
(joint work with David Itkin, Martin Larsson, Benedikt Koch)

We consider a robust asymptotic growth problem under model uncertainty in the presence of stochastic factors. We fix two inputs representing the instantaneous covariance for the asset price process X, which depends on an additional stochastic factor process Y, as well as the invariant density of X together with Y. The stochastic factor process Y has continuous trajectories, but is not even required to be a semimartingale. Our setup allows for drift uncertainty in X and model uncertainty for the local dynamics of Y. There are several interpretation of Y: it could model stochastic covariance as it often happens in Finance, but it could also be a numerical model for uncertainty of the instantanous covariance function for X.

This work builds upon a recent paper of Kardaras & Robertson (AAP 2022), where the authors consider an analogous problem, however, without the additional stochastic factor process. Under suitable, quite weak assumptions we are able to characterize the robust optimal trading strategy and the robust optimal growth rate. The optimal strategy is shown to be functionally generated and, remarkably, does not depend on the factor process Y. We also construct a worst case model for the functionally generated strategy thereby fully solving the min-max problem.

Our result provides a comprehensive answer to a question proposed by Fernholz in 2002. We also show that the optimal strategy remains optimal even in the more restricted case where Y is a semimartingale and the joint covariation structure of X and Y is prescribed.

Our results are obtained using a combination of techniques from partial differential equations, calculus of variations, and generalized Dirichlet forms.

Collective Arbitrage and the Value of Cooperation

Thilo Meyer-Brandis
(joint work with Francesca Biagini, Alessandro Doldi, Jean-Pierre Fouque, Marco Frittelli)

The theory developed in this paper aims at expanding the classical Arbitrage Pricing Theory to a setting where N agents are investing in stochastic security markets and are allowed to cooperate through suitable exchanges. More precisely, we suppose that each agent is allowed to invest in a subset of the available assets

$(X^1, \dots, X^J)$, for a given $J \in \mathbb{N}$, and in a common riskless asset. Note that we do not exclude that such subset coincides with the full set $(X^1, \dots, X^J)$. The novel notions of Collective Arbitrage and Collective Super-replication, are based on the possibility that the N agents may additionally enter in a zero-sum risk exchange mechanism, where no money is injected or taken out of the overall system. Cooperation and the multi-dimensional aspect are the key features of Collective Arbitrage and Collective Super-replication. In this setting agents not only may invest in their respective market but may additionally cooperate to improve their positions by taking advantage of the risk exchanges. In the case of one single agent, the theory reduces to the classical one. There is an extensive literature in recent years on variations around the concept of one-agent No Arbitrage or No Free Lunch and we refer to the books Delbaen Schaermayer (2006) [5] and Föllmer and Schied (2014) [6], and references therein, for a detailed overview of the topic. Departing from this stream of literature, the main aim of this paper is to understand the consequences of the cooperation between several agents in relation to Arbitrage and Super-replication.

Before moving into the details of our new setup, we briefly summarize the classical one-agent situation. Let a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$, $\mathcal{T} = \{1, \dots, T\}$ be given, and denote by $X = (X^1, \dots, X^J)$ the J adapted stochastic processes representing the prices of J securities. The set of admissible trading strategies is denoted by $\mathcal{H}$ and let K be the set of time-T stochastic integral of $H \in \mathcal{H}$ with respect to X. The set K represents all the possible terminal time-T payoffs available in the market from admissible trading strategies and having zero initial cost.

An arbitrage opportunity is an admissible trading strategy $H \in \mathcal{H}$, having zero initial cost and producing a non negative final payoff $k \in K$, being strictly positive with positive probability. Equivalently, we have no arbitrage in this setting if the only non negative element in K is P-a.s. equal to 0, or more formally $K \cap L^0_+(\Omega, \mathcal{F}, P) = \{0\}$.

In this paper, we generalize the setting to multiple agents that might cooperate with each other. This leads to the new concepts of Collective Arbitrage and Collective Super-replication which we shortly describe in the following.

Collective Arbitrage. Since each agent $i = 1, \dots, N$ is allowed to invest only in a subset of the available assets $(X^1, \dots, X^J)$, we define, similarly to the notion of the set K, the market K_i of agent i, that is the space of all the possible time-T payoffs that agent i can obtain by using admissible trading strategies in his/her allowed investments and having zero initial cost.

Inspired by [4] we consider the set of all zero-sum risk exchanges

$$\mathcal{Y}_0 = \left\{ Y \in (L^0(\Omega, \mathcal{F}, P))^N \mid \sum_{i=1}^N Y^i = 0 \ P\text{-a.s.} \right\},$$

and the set $\mathcal{Y}$ of possible/allowed exchanges

$$\mathcal{Y} \subseteq \mathcal{Y}_0 \ \text{ such that } \ 0 \in \mathcal{Y}.$$

We stress that even if the overall sum is P-a.s. equal to 0, each components Y^i of $Y \in \mathcal{Y}$ is in general a random variable. If Y^i is positive on some event, agent i is receiving, on that event, from the collection of the other agents some (positive) amount of cash. So $Y \in \mathcal{Y}$ represents the amount that the agents may exchange among themselves with the requirement that the overall amount distributed is equal to zero.
A *Collective Arbitrage* is a vector $(k^1, \dots, k^N)$, where $k^i \in K_i$ for each i, and a vector $Y = (Y^1, \dots, Y^N) \in \mathcal{Y}$ such that

$$k^i + Y^i \geq 0, \quad P\text{-a.s. for all } i \in \{1, \dots, N\},$$

and

$$P(k^j + Y^j > 0) > 0 \text{ for at least one } j \in \{1, \dots, N\}.$$

One may immediately notice that if $N = 1$, then $Y \in \mathcal{Y}$ must be equal to 0 and thus a Collective Arbitrage reduces to a Classical Arbitrage.
However, for $N \geq 2$, in a Collective Arbitrage, agents are entangled by the vector of exchanges $Y \in \mathcal{Y}$: this additional possible cooperation may create a Collective Arbitrage even if there is No Arbitrage for each single agent.

We study the implications of the assumption of No Collective Arbitrage with respect to the set $\mathcal{Y}$, which we denote in short by $\mathbf{NCA}(\mathcal{Y})$. We also write $\mathbf{NA}_i$ for the No Arbitrage condition (in the classical sense) for agent i in market K_i and **NA** for the No Arbitrage condition (in the classical sense) in the global market K.

It is easy to verify that under very reasonable conditions the following implications hold

$$\mathbf{NA} \Rightarrow \mathbf{NCA}(\mathcal{Y}) \Rightarrow \mathbf{NA}_i \ \forall i \in \{1, \dots, N\},$$

but none of the reverse implication holds true in general. We show that the strongest condition **NA** is equivalent to $\mathbf{NCA}(\mathcal{Y})$ for the "largest" choice $\mathcal{Y} = \mathcal{Y}_0$, while the weakest condition, $\mathbf{NA}_i \ \forall i$, is equivalent to $\mathbf{NCA}(\mathcal{Y})$ for the "smallest" choice $\mathcal{Y} = \mathcal{Y}_0 \cap (L^0(\Omega, \mathcal{F}_0, P))^N$. The latter space actually consists of zero-sum deterministic vectors, when $\mathcal{F}_0$ is the trivial sigma algebra. However, for general sets $\mathcal{Y}$ the notions of $\mathbf{NCA}(\mathcal{Y})$ give rise to new concepts.

We analyse the conditions under which a new type of Fundamental Theorem of Asset Pricing holds, that we label Collective FTAP (CFTAP). Differently from the classical version, the CFTAP depends of course on the properties of the set of exchanges $\mathcal{Y}$, and so we provide several versions of such a theorem. On the technical side, in the classical case the **NA** condition implies that the set $(K - L^0_+(\Omega, \mathcal{F}, P))$ is closed in probability. This property is paramount to prove the FTAP and the dual representation of the super-replication price. Analogously, in our collective setting we need to show the closedness in probability of the analogue set denoted by $K^{\mathcal{Y}}$. We show such closure under some specific assumptions on the set $\mathcal{Y}$ and under the assumption of $\mathbf{NCA}(\mathcal{Y})$.

The key novel feature in the CFTAP is that equivalent martingale measures have to be replaced by vectors $(Q^1, \dots, Q^N)$ of equivalent martingale measures, one for each agent and theirs corresponding market, fulfilling in addition the polarity property

$$\sum_{i=1}^{N} E_{Q^i}[Y^i] \le 0\,, \quad \forall Y \in \mathcal{Y}. \tag{1}$$

We stress that the findings of this paper take particularly tractable, yet informative and meaningful forms in a finite probability space setup. Indeed, the fact that the agents are allowed to cooperate and the assumption of $\mathbf{NCA}(\mathcal{Y})$ has several consequences also in the pricing of contingent claims. This is particularly evident in the super-replication of N contingent claims.

Collective Super-replication. We consider the problem of N agents each super-replicating a contingent claim g^i, $i = 1, \dots, N$, which is a $\mathcal{F}$-measurable random variable. We set $g = (g^1, \dots, g^N)$. As an immediate extension of the classical super-replication price, we first introduce the overall super-replication price

$$\rho_+^N(g) := \inf\left\{\sum_{i=1}^{N} m^i \mid \exists k_i \in K_i, m \in \mathbb{R}^N \text{ s.t. } m^i + k^i \ge g^i \ \forall i\right\}.$$

If we use $\rho_{i,+}(g^i)$ for the classical super-replication of the single claim g^i, we may easily recognize that

$$\rho_+^N(g) = \sum_{i=1}^{N} \rho_{i,+}(g^i). \tag{2}$$

In the spirit of Systemic Risk Measures with random allocations in [2], we introduce the Collective super-replication of the N claims $g = (g^1, \dots, g^N)$ as

$$\rho_+^{\mathcal{Y}}(g) := \inf\left\{\sum_{i=1}^{N} m^i \mid \exists k_i \in K_i, m \in \mathbb{R}^N, Y \in \mathcal{Y} \text{ s.t. } m^i + k^i + Y^i \ge g^i \ \forall i\right\},$$

and show that under $\mathbf{NCA}(\mathcal{Y})$ the definition is well posed. The functional $\rho_+^{\mathcal{Y}}(g)$ and $\rho_+^N(g)$ both represent the minimal total amount needed to super-replicate simultaneously all claims $(g^1, ..., g^N)$. For the Collective super-replication price $\rho_+^{\mathcal{Y}}(g)$ we allow an additional exchange among the agents, as described by $\mathcal{Y}$. As $0 \in \mathcal{Y}$, we clearly have $\rho_+^{\mathcal{Y}} \le \rho_+^N$. Thus Collective super-replication is less expensive than classical super-replication: cooperation helps to reduce the cost of super-replication and $(\rho_+^N(g) - \rho_+^{\mathcal{Y}}(g)) \ge 0$ is the *value of cooperation* with respect to g.

Under the $\mathbf{NCA}(\mathcal{Y})$ assumption and using the closure of the set $K^{\mathcal{Y}}$, we prove the following version of the pricing-hedging duality

$$\rho_+^{\mathcal{Y}}(g) = \sup_{Q \in \mathcal{M}^{\mathcal{Y}}} \sum_{i=1}^{N} E_{Q^i}[g^i], \tag{3}$$

where $\mathcal{M}^{\mathcal{Y}}$ is the set of vectors of martingale measures satisfying the polarity condition (1). When problem (3) admits an optimum $\widehat{Q} = (\widehat{Q}^1, \dots, \widehat{Q}^N)$, which

clearly will depend on $\mathcal{Y}$, we derive the following formula
(4)

$$\rho_+^{\mathcal{Y}}(g) = \sum_{i=1}^{N} \inf \left\{ m \in \mathbb{R} \mid \exists k^i \in K_i, Y^i \text{ with } E_{\widehat{Q}^i}[Y^i] = 0 \text{ s.t. } m + k^i + Y^i \geq g^i \right\}.$$

Note that in (4), $(Y^1, \dots, Y^N)$ is not required to belong to $\mathcal{Y}$, but every Y^i must have zero cost under each component of the endogenously determined pricing vector $\widehat{Q}$. This is a strong fairness property associated to the value $\rho_+^{\mathcal{Y}}(g)$. Indeed, each term in the summation on the RHS of (4) is the *individual* super-replication price of the claim g^i under the assumption that the agent i is "pricing"using the pricing functional assigned by $\widehat{Q}^i$, so that both k^i and Y^i have zero value under $\widehat{Q}^i$. Thus the interpretation of $\rho_+^{\mathcal{Y}}(g)$ is twofold:

(i) $\rho_+^{\mathcal{Y}}(g)$ is the super-replication of the N claims $(g^1, \dots, g^N)$ when agents are allowed to exchange scenario dependent amounts under the condition that the overall exchanges $\sum_{i=1}^{N} Y^i$ is equal to 0;

(ii) $\rho_+^{\mathcal{Y}}(g)$ is the sum of the individual super-replication price of each claim g^i under the assumption that each agent is using the pricing measure $\widehat{Q}^i$.

This fairness aspect is discussed in the spirit of [3] and [1].

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Some thoughts on large financial markets under model uncertainty (discrete time)

Irene Klein

(joint work with Christa Cuchiero, Thorsten Schmidt)

All the ideas in the talk are based on joint work in progress with Christa Cuchiero and Thorsten Schmidt. Theorems 2 and 3 below currently are in the state of well-founded conjectures. The proofs still have to be made precise with all details.

We present some ideas for large financial markets in discrete time under model uncertainty. We consider a classical model of a large financial market (LFM) on a sequence of probability spaces as in Kabanov and Kramkov (1994) [3]. For each $n \geq 1$, the "small" market n in the sequence is defined as follows. Let $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t=0,1,\ldots,T_n})$ be a filtered measure space defined as in Bouchard and Nutz (2015) [1]. As there, let $\mathcal{P}^n$ be a convex set of probability measures on $(\Omega^n, \mathcal{F}^n)$. The risky assets are $d(n)$ Borel-measurable stocks $S^n_t = (S^{n,1}_t, ..., S^{n,d(n)}_t) : \Omega^n_t \to \mathbb{R}^{d(n)}$, where, for each $t = 0, 1, \ldots, T_n$ the set Ω^n_t is defined as in [1], i.e., the t-fold Cartesian product of a Polish space Ω^n_1 and Ω^n_0 is a singleton. Let $\mathcal{H}^n$ be the set of all predictable $\mathbb{R}^{d(n)}$-valued processes on $(\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t=0,1,\ldots,T_n})$. Then, a portfolio in market n with strategy $H^n \in \mathcal{H}^n$ is given by

$$X^n_t := (H^n \cdot S^n)_t = \sum_{k=1}^{d(n)} \sum_{u=1}^{t} H^{n,k}_u (S^{n,k}_u - S^{n,k}_{u-1}), \quad t = 1, \ldots T_n,$$

where $X^n_0 = 0$. Now we give the definition of a LFM under model uncertainty.

Definition 1. *A large financial market under model uncertainty is a sequence of small markets n as given above with $d(n)$ risky stocks in discrete time and time horizons $T_n < \infty$.*

As usual in the theory of large financial markets, we will assume that each small market n satisfies no arbitrage (NA) where we use the robust NA condition of [1]:

Definition 2. *The market n satisfies the condition NA($\mathcal{P}^n$) if for all $H^n \in \mathcal{H}^n$*

$$(H^n \cdot S^n)_{T_n} \geq 0 \quad \mathcal{P}^n\text{-q.s.} \quad \textit{implies} \quad (H^n \cdot S^n)_{T_n} = 0 \quad \mathcal{P}^n\text{-q.s.}$$

In the above definition q.s. stands for *quasi surely*. A property is said to hold $\mathcal{P}^n$-q.s. if it holds outside a polar set A' for $\mathcal{P}^n$, that is, a set A' such that $A' \subset A$ for some $A \in \mathcal{F}^n$ with $P^n(A) = 0$ for all $P^n \in \mathcal{P}^n$.

Let us recall the connection to martingale measures from [1]. On market n we define the following set $\mathcal{Q}^n$ of probability measures. (Note that, as $\mathcal{P}^n$ is a convex set by assumption, also $\mathcal{Q}^n$ is convex).

Definition 3.

$$\mathcal{Q}^n = \{Q^n \lll \mathcal{P}^n : Q^n \text{ is a martingale measure for } S^{n,k}, k = 1, \ldots, d(n)\},$$

where $Q^n \lll \mathcal{P}^n$ means that for $Q^n \in \mathcal{Q}^n$ there exists $P^n \in \mathcal{P}^n$ such that $Q^n \ll P^n$.

As a consequence of the NA assumption on each small market n the following existence of martingale measures hold:

Theorem 1 (FTAP (Bouchard, Nutz 2015))**.** *The following are equivalent:*

1. *NA($\mathcal{P}^n$) holds.*
2. *For all $P^n \in \mathcal{P}^n$ there exists $Q^n \in \mathcal{Q}^n$ such that $P^n \ll Q^n$.*
3. *$\mathcal{P}^n$ and $\mathcal{Q}^n$ have the same polar sets.*

We suggest now to define a notion of asymptotic arbitrage with model uncertainty on the large financial market. We will adapt here the concept of asymptotic arbitrage of first kind (AA1) as of [3]. Observe that this kind of asymptotic arbitrage is, if all Ω^n coincide, equivalent to the concept unbounded profit with bounded risk defined in Karatzas and Kardaras (2007) [5]. Note that this is a particularly important arbitrage property due to its connection to the growth optimal portfolio of Eckhard Platen.

Definition 4. *We say that the robust large financial market has an asymptotic arbitrage of first kind (AA1($\mathcal{P}^n$)) if the following holds: there exists a subsequence of markets n_k and a sequence of portfolios $X^k = (H^{n_k} \cdot S^{n_k})$ and a sequence of positive real numbers $\varepsilon_k \to 0$ such that*

(1) *for all $k \geq 1$ and all $t = 0, 1, \ldots, T(n_k)$, $X^k_t \geq -\varepsilon_k$ $\mathcal{P}^{n_k}$-q.s.*
(2) *there exists a sequence $(P^k)_{k\geq 1}$ with $P^k \in \mathcal{P}^{n_k}$ such that*

$$P^k(X^k_{T(n_k)} \geq \alpha) \geq \alpha,$$

for some $\alpha > 0$ and all $k \geq 1$.

We say that no asymptotic arbitrage of first kind *(NAA1($\mathcal{P}^n$)) is satisfied if the above does not exists.*

We can now suggest the following fundamental theorem of asset pricing under model uncertainty for large financial markets in discrete time. Observe that it looks very similar to Theorem 1 but now on the large financial market.

Theorem 2 (A (conjectured) FTAP under model uncertainty)**.** *NAA1($\mathcal{P}^n$) $\Leftrightarrow$ for each sequence $(P^n)_{n\geq 1}$ with $P^n \in \mathcal{P}^n$, for all n, there exists a sequence $(Q^n)_{n\geq 1}$ with $Q^n \in \mathcal{Q}^n$, for all n, such that $(P^n) \lhd (Q^n)$.*

Note that $(P^n) \lhd (Q^n)$ basically is the generalization of absolute continuity of measures to a sequences of measures and means that for each sequence $A^n \in \mathcal{F}^n$ with $Q^n(A^n) \to 0$ for $n \to \infty$ we have that $P^n(A^n) \to 0$ for $n \to \infty$.

Some ideas for the proof of Theorem 2: work in progress

Similarly as in [4] the idea is to find a generalized quantitative version of the Halmos-Savage Theorem. Here we suggest a version for convex sets of probability measures, see Theorem 3 below. With the help of this result it is quite standard to get Theorem 2 by using the superreplication of [1] which fits perfectly to the current setting. On the way we use the following characterization of NAA1 under model uncertainty which we can prove with all details.

Lemma 1. *NAA1($\mathcal{P}^n$) $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall n \geq 1$ and $\forall A^n \in \mathcal{F}^n$ such that $\exists P^n \in \mathcal{P}^n$ with $P^n(A^n) \geq \varepsilon$ there $\exists Q^n \in \mathcal{Q}^n$ with $Q^n(A^n) \geq \delta$.*

So, if our conjectured Theorem 3 below can be proved in the given form, the proof of Theorem 2 is done. Let us now formulate the conjectured Theorem 3, i.e., the quantitative Halmos–Savage–type result for convex sets of probability measures we are aiming at.

Theorem 3 (Conjecture: Quantitative Halmos-Savage Theorem for convex sets of probability measures)**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be a convex sets of probability measures on $(\Omega, \mathcal{F})$ such that $\mathcal{Q} \lll \mathcal{P}$. For fixed $\varepsilon > 0$ and $\delta > 0$ the following statement is true: Assume that for each $A \in \mathcal{F}$ such that there exists $P \in \mathcal{P}$ with $P(A) \geq \varepsilon$*

there exists $Q \in \mathcal{Q}$ such that $Q(A) \geq \delta$. Then for each $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that for each $A \in \mathcal{F}$ with $P(A) \geq 2\varepsilon$ we have that $Q(A) \geq \frac{\varepsilon\delta}{2}$.

Note that $\mathcal{Q} \lll \mathcal{P}$ in the statement of the theorem means that for every $Q \in \mathcal{Q}$ there exists $P \in \mathcal{P}$ such that $Q \ll P$.

Ideas for the Proof of Theorem 3. As a technical tool for the proof we will define a **convex** set $D^{\varepsilon,P}$: fix $P \in \mathcal{P}$ and $\varepsilon > 0$. Define

$$D^{\varepsilon,P} = \{h \in \bigcap_{P' \in \mathcal{P}} L^\infty(P') : 0 \leq h \leq 1 \quad \mathcal{P}-\text{q.s. and } E_P[h] \geq 2\varepsilon\}.$$

The assumption of Theorem 3 will lead to the following inequality:

$$\inf_{h \in D^{\varepsilon,P}} \sup_{Q \in \mathcal{Q}} E_Q[h] \geq \varepsilon\delta.$$

Now by finding appropriate dual locally convex topological vector spaces (E, E') and using a general Banach-Alaoglu-Bourbaki Theorem we think to be able to show that the convex set $D^{\varepsilon,P} \subset E'$ is $\sigma(E', E)$-compact. Then we aim at applying a Minmax Theorem as in Sion (1958) [6] to the given bilinear functional with a continuity property with respect to the chosen topology to get that:

$$\sup_{Q \in \mathcal{Q}} \inf_{h \in D^{\varepsilon,P}} E_Q[h] \geq \varepsilon\delta.$$

With this the statement of Theorem 3 follows. □

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Fundamental theorem of asset pricing with acceptable risk in markets with frictions

Cosimo Munari

We revisited the problem of market-consistent valuation of insurance liabilities from a financial economics perspective. The challenge is to define a range of prices at which an insurance company that has access to an outstanding financial market and is subject to a regulatory capital adequacy regime should be prepared to buy/sell a contract outside of the financial market. Our proposal was to call a price *market consistent with acceptable risk* (MCP) if there exists no portfolio of traded assets that can be bought/sold at a lower/higher price in the market and that super/sub-replicates the contract's payoff at an acceptable level of risk as prescribed by the regulatory solvency test. In the spirit of classical arbitrage pricing theory, the main goal was to provide a characterization of MCPs by way of special stochastic discount factors, called (strictly) consistent price deflators, that have to be chosen to respect market frictions as well as to be consistent with the regulator's solvency test. The presentation unfolded as follows:

- Formalization of the financial market and the capital adequacy test.
- Definition of MCPs.
- Primal characterization of MCPs based on super/sub-replication prices.
- Definition of (scalable) good deals as generalizations of arbitrage opportunities.
- Definition of (strictly) consistent price deflators as generalizations of stochastic discount factors.
- Extension of the fundamental theorem of asset pricing: The market is free of scalable good deals if and only if there exists a strictly consistent price deflator.
- Dual characterization of MCPs based on strictly consistent price deflators.
- Examples of price deflators that are strictly consistent with respect to Expected Shortfall and expectiles.

A number of future challenges was mentioned at the end, including at least:

- Extension to multi-period models.
- Extension to settings without a dominating probability.
- Characterization of optimal hedging portfolios/strategies with acceptable risk.
- Comparison with market-consistent valuation rules used in practice (best estimate of insurance liabilities plus risk margin).

We believe that the last point is especially pressing to bridge the gap between theory and practice and should ideally contribute to the ongoing discussion on the broad topic "valuation" in insurance regulation.

This work is related to the literature on good deal pricing. The goal there is to restrict the interval of arbitrage-free prices by discarding some "extreme" stochastic discount factors and the main problem is that of identifying, by way of an inverted fundamental theorem of asset pricing, the corresponding pricing bounds, the so-called good deal bounds. We refer, e.g., to:

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Apart from the general motivation, the key difference with our results is that the bulk of this literature focuses on frictionless markets and the only versions of the fundamental theorem of asset pricing involve, in our language, only consistent, instead of strictly consistent, price deflators. In particular, these versions cannot be used to characterize MCPs in dual terms.

Benchmark-Neutral Pricing for Entropy-Maximizing Dynamics

Eckhard Platen

The paper applies the benchmark approach to the modeling, pricing, and hedging of long-term contingent claims involving the growth optimal portfolio (GOP) of a large stock market. It employs the entropy-maximizing dynamics of the GOP of the stocks for modeling. Instead of risk-neutral or real-world pricing, the paper proposes the method of benchmark-neutral pricing, where it uses the GOP of the stocks as numéraire and the respective new benchmark-neutral pricing measure for taking conditional expectations. Under the entropy-maximizing dynamics of the GOP for stocks, the benchmark-neutral pricing measure turns out to be an equivalent probability measure. The risk-neutral pricing measure does not represent a probability measure. Consequently, benchmark-neutral pricing provides the minimal possible prices and hedges, whereas risk-neutral pricing becomes more expensive than necessary. The implementation of benchmark-neutral pricing and hedging is demonstrated. It is shown that the minimal possible prices, which benchmark-neutral pricing provides, can be significantly lower for long-term contingent claims than the respective risk-neutral ones.

The paper makes the following three key assumptions:
A1: The GOP exists.
A2: The normalized GOP forms a stationary scalar diffusion and its volatility is a function of its value.
A3: The market maximizes the relative entropy of the stationary density of the normalized GOP.

The first assumption is about the existence of the GOP and represents an intuitive and easily verifiable *no-arbitrage condition* because [4] have shown that the existence of the GOP is equivalent to their *No Unbounded Profit with Bounded*

Risk (NUPBR) condition. This no-arbitrage condition is weaker than the NFLVR condition of [1].

The maximization of the relative entropy is known to be equivalent to the minimization of the information rate; see [5]. Consequently, the resulting entropy-maximizing market dynamics does not leave any room for exploitable information and charaterizes the undisturbed market dynamics.

Conservation laws simplify in many areas the undisturbed dynamics of complex dynamical systems. According to [6], the maximization of a Lagrangian in the presence of Lie-group symmetries leads to the identification of conservation laws for the resulting model dynamics. The entropy-maximizing stationary dynamics of the normalized GOP turn out to have Lie-group symmetries and emerge as those of a time-transformed square root process, with conserved dimension four, and conserved logarithmic mean zero.

The modeling is performed on a filtered probability space $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, P)$, satisfying the usual conditions. We consider $d+1$, $d \in \{1, 2, ...\}$ adapted, nonnegative assets, denoted by $S_t^0, S_t^1, ..., S_t^d$, which we call the d *primary security accounts*, where all dividends or interests are reinvested. We interpret the d primary security accounts $S_t^1, ..., S_t^d$ as stocks, which are here denominated in units of the savings account $S_t^0 = 1$. Furthermore, we assume for the investment universe given by the d stocks that a continuous *growth optimal portfolio* (GOP) S_t^*, the *stock GOP*, exists. Every primary security account $\tilde{S}_t^j = \frac{S_t^j}{S_t^*}$, $j \in \{1, ..., d\}$, when denominated in the stock GOP, forms a right-continuous, integrable $(P, \underline{\mathcal{F}})$-local martingale. The stochastic differential equation (SDE) for the continuous stock GOP S_t^* is assumed to be of the form

$$\frac{dS_t^*}{S_t^*} = \lambda_t^* dt + \theta_t(\theta_t dt + dW_t)$$

for $t \in [0, \infty)$ with $S_0^* > 0$. We extend the above market formed by the d stocks by adding the savings account S_t^0 as an additional primary security account. In line with Theorem 7.1 in [3], the GOP S_t^{**} of the extended market satisfies the SDE

$$\frac{dS_t^{**}}{S_t^{**}} = \frac{\lambda_t^* + (\theta_t)^2}{\theta_t}(\frac{\lambda_t^* + (\theta_t)^2}{\theta_t}dt + dW_t)$$

for $t \in [0, \infty)$ and $S_0^{**} = 1$. For a replicable contingent claim $H_T \geq 0$ with maturity T the *real world pricing formula*

$$H_t = S_t^{**}\mathbf{E}^P(\frac{H_T}{S_T^{**}}|\mathcal{F}_t)$$

describes its unique fair price H_t at time $t \in [0, T]$, see [2]. Other pricing rules are possible but do never provide lower prices. The numéraire for real-world pricing is the GOP S_t^{**} of the extended market, which is, in reality, a highly leveraged portfolio and difficult to construct. Therefore, a change of numéraire is suggested that uses the strictly positive stock GOP S_t^* as numéraire. The Radon-Nikodym derivative

$$\Lambda_{S^*}(t) = \frac{dQ_{S^*}}{dP}|_{\mathcal{F}_t} = \frac{\frac{S_t^*}{S_t^{**}}}{\frac{S_0^*}{S_0^{**}}}$$

characterizes the respective *benchmark-neutral pricing measure* Q_{S^*}. For the entropy-maximizing dynamics, the Radon-Nikodym derivative $\Lambda_{S^*}(t)$ is shown to be a true $(P, \underline{\mathcal{F}})$-martingale and Q_{S^*} to be an equivalent probability measure. We call the new pricing method *benchmark-neutral pricing*, which uses the stock GOP S_t^* as numéraire and the benchmark-neutral pricing measure Q_{S^*} as pricing measure. One obtains directly the *benchmark-neutral pricing formula*

$$H_t = S_t^* \mathbf{E}^{Q_{S^*}}(\frac{H_T}{S_T^*}|\mathcal{F}_t)$$

for $t \in [0, T]$. The process $W^0 = \{W_t^0, t \in [0, \infty)\}$, satisfying the SDE

$$dW_t^0 = \sigma_{S^*}(t)dt + dW_t$$

for $t \in [0, \infty)$ with $W_0^0 = 0$, is under Q_{S^*} a Brownian motion. This result is of practical importance because it allows one to use the stock GOP as numéraire for pricing and hedging. Under benchmark-neutral pricing there is no need to estimate λ_t^* because this drift parameter becomes absorbed in the measure transformation.

Hedging under the benchmark-neutral pricing measure can be performed analogously as shown in [2] under the real world probability measure P, and can also be extended for non-replicable contingent claims.

When using a total return stock index as proxy for the stock GOP, it has been shown for zero-coupon bonds that long-term hedging over many decades can be accurately performed with very small hedge errors. These findings give access to new production methods for life insurance, pension, climate, and other long-term contracts that use the stock index as numéraire.

Since the risk-neutral pricing measure turns out to be not an equivalent probability measure under the entropy-maximizing dynamics, formally applied risk-neutral prices and hedges can become considerably more expensive than the minimal possible ones, which can be obtained via benchmark-neutral pricing and hedging.

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Markovian randomized equilibria for general Markovian Dynkin games in discrete time

Berenice Anne Neumann
(joint work with Sören Christensen, Kristoffer Lindensjö)

In discrete time Dynkin games each player $i \in \{1,2\}$ chooses a stopping time τ_i in order to maximize her expected reward

$$\mathbb{E}\left[F^i_{\tau_i}\mathbb{I}_{\{\tau_i<\tau_j\}} + G^i_{\tau_j}\mathbb{I}_{\{\tau_j<\tau_i\}} + H^i_{\tau_i}\mathbb{I}_{\{\tau_i=\tau_j\}}\right],$$

where $j = 3-i$ and F^i, G^i, H^i are integrable discrete time processes (with a suitable interpretation of H^i_n for $n=\infty$). In the case that $F^1 \leq H^1 \leq G^1$ and $F^2 \leq H^2 \leq G^2$ these games are well-understood. Under suitable integrability assumptions existence and characterization of Nash equilibria have been established [2, 3, 4]. However, the situation becomes more involved if we drop the assumption $F^1 \leq H^1 \leq G^1$ and $F^2 \leq H^2 \leq G^2$. First of all it is now necessary to consider mixed strategies [5]. Moreover, also using this class of strategies there are simple examples without a Nash equilibrium [6]. In general, only the existence of ϵ-equilibria can be established [7, 8].

In this talk we restricted our attention to discrete time Markovian Dynkin games. In this setting $(X_n)_{n\in\mathbb{N}}$ is a homogeneous Markov process with state space E and the reward of player i reads

$$\mathbb{E}_x\left[\alpha^{\tau_i} f_i(X_{\tau_i})\mathbb{I}_{\{\tau_i<\tau_j\}} + \alpha^{\tau_j} g_i(X_{\tau_j})\mathbb{I}_{\{\tau_j<\tau_i\}} + \alpha^{\tau_i} h_i(X_{\tau_i})\mathbb{I}_{\{\tau_i=\tau_j<\infty\}}\right],$$

where $j = 3-i$, α is the discount factor satisfying $0<\alpha<1$ and $f_i, g_i, h_i : E \to \mathbb{R}, i = 1,2$, are measurable functions satisfying an integrability assumption. In the talk we motivated that Markovian randomized stopping times are a natural class of randomized stopping times for these games. These Markovian randomized stopping times are stopping times, where at each time step n the player stops with a certain probability that only depends on the current state X_n of the underlying Markov process. Relying on this type of strategies we provide an explicit characterization and verification result of Wald-Bellman type. This result then allows us to construct equilibria in certain classes of zero-sum and symmetric games and to obtain necessary and sufficient conditions for the non-existence of pure strategy equilibria in zero-sum games. Moreover, we establish the existence of an equilibrium in Markovian randomized stopping times for general games whenever the state space of the underlying Markov chain is countable.

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Stretched Brownian motion: Analysis of a fixed-point scheme

Gudmund Pammer

(joint work with Beatrice Acciaio, Antonio Marini)

A central challenge in the theory of mathematical finance is the pricing of financial derivatives. In the classical theory this question is closely tied to the notion of martingale measures: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis and $S = (S_t)_{t\geq 0}$ be the $(\mathcal{F}_t)_{t\geq 0}$-adapted asset-price process. Under the no-arbitrage assumption, that is, we exclude the possibility of making profit without risk, the task of pricing a financial derivative Φ boils down to finding an equivalent martingale measure $\mathbb{Q}$. An equivalent martingale measure is simply a measure equivalent to $\mathbb{P}$ under which S is a martingale.

However, the true dynamics of the market, including the stochastic basis and the asset-price process, are unknown. Rather than directly specifying a model, we can extract information on the pricing measure $\mathbb{Q}$ from market data. The cornerstone of this approach is the famous observation by Breeden–Litzenberger [3], which culminates in the fitting problem (FP) in mathematical finance: The task is to find a stochastic basis supporting a martingale $S = (S_t)_{t\geq 0}$ that adheres to prescribed marginal constraints $S_t \sim \mu_t$ for $t \in I$ derived from market observations. Here $(\mu_t)_{t\in I}$ are one-dimensional marginals that are derived from market observations at a given time index set $I \subseteq \mathbb{R}_+$. Building on the Bass solution to the Skorokhod embedding problem and optimal transport, Backhoff, Beiglbock, Huesmann, and Kallblad [1] propose a solution to (FP) for the two-marginal problem, i.e., with constraints on two specific time-points $I = \{0, 1\}$. The stretched Brownian motion M^* is the unique-in-law optimizer of

$$\sup \{\mathbb{E}[M_1 \cdot B_1] : M \text{ solves (FP)}\},$$

where B is some Brownian motion. Notably rich in structure, this process is an Ito diffusion and a continuous, strong Markov martingale that emulates the behaviour of Brownian motion locally.

Following a similar approach, Conze and Henry-Labordere [2] recently introduced a novel alternative to the local volatility model. This model, rooted in an extension of the Bass construction, is efficiently computable through a fixed-point scheme. The goal is to find a fixed point of the map

$$\mathcal{A}\colon \text{CDF} \to \text{CDF}\colon F \mapsto F_{\mu_0} \circ \left(\gamma_1 * F_{\mu_1}^{-1}\left(\gamma_1 * F\right)\right),$$

where F is a cumulative distribution function (CDF), F_μ denotes the CDF of μ and γ_1 a normal distribution with variance 1. When α is a distribution whose CDF $\hat{F}$ is a fixed-point, then the process $\hat{M} = (\hat{M})_{t\geq 0}$, determined by

$$\hat{M}_t := \mathbb{E}[T(B_1)|B_t] = (\gamma_{1-t} * F_{\mu_1}^{-1}(\gamma_1 * \hat{F}))(B_t),$$

solves (FP).

In this work, we explore the intricate relationship between the fixed-point scheme and the stretched Brownian motion, revealing that in law $\hat{M} = M^*$. Furthermore, we give a precise criterion for the existence of a fixed-point and demonstrate its convergence. This study unveils that solving the fixed-point equation provides a highly efficient alternative to computing stretched Brownian motion. In particular, when μ_0 is concentrated on finitely many points, the fixed-point scheme exhibits linear convergence.

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On random reinsurance contracts and optimal transport

Brandon Garcia Flores

(joint work with Beatrice Acciaio, Hansjörg Albrecher)

Building upon the concept of random reinsurance treaties from [3] and [4], we establish a general framework for the study of optimal reinsurance problems. Traditionally, an optimal reinsurance problem consists in minimizing a risk measure $\mathcal{P}$ defined on a set of functions. The minimization is subject to the solution being in a set of constraints $\mathcal{S}$, which usually relates to demands set by either the cedent or the reinsurer. In this generality, one can hardly show the existence of any contract and is therefore restricted to deal with specific instances of the problem. The introduction of random reinsurance treaties is then reminiscent to the Monge-Kantorovich formulation of optimal transport (OT) which is used as a way of convexifying the problem, thus ensuring the existence of optimal solutions.

A random reinsurance treaty η is a probability measure in $\mathbb{R}^n \times \mathbb{R}^n$ supported in the set $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid 0 \leq y_i \leq x_i, i = 1, \ldots, n\}$ and such that $\pi_{1\#}\eta = \mu$, where μ is the distribution of the original claims. Here, $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the projection in the first coordinate and $\pi_{1\#}$ denotes the push-forward map induced by π_1. Denoting by X the original portfolio of claims, contracts of this kind can be simply seen as the joint distribution of X and the final risk exposure of the reinsurer, which now is not necessarily determined by X in a functional way. By means of standard OT methods, one can easily prove the following:

Theorem 1. *Let $\mathcal{M}$ denote the space of random reinsurance treaties endowed with the weak topology induced by bounded continuous functions. If $\mathcal{P} : \mathcal{M} \to \overline{\mathbb{R}}$ is lower semi-continuous and $\mathcal{S}$ is closed, then an optimal reinsurance contract η^* exists.*

While existence is guaranteed under relatively mild assumptions, one is then faced with the identification of optimal contracts. The rest of our work addresses this matter by using the idea of (local) linearization, a concept widely used in the are of optimization.

Assuming that the set of constraints is given as

$$\mathcal{S} = \{\eta \in \mathcal{M} \mid \mathcal{G}(\eta) \leq 0\}$$

for a lower semi-continuous function $\mathcal{G} = (g_1, \ldots, g_m) : \mathcal{M} \to \mathbb{R}^m$, one of the main results of our work is the following:

Theorem 2. *Let η^* be an optimal reinsurance contract and assume there exist continuous functions $p_{\eta^*} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ and $g_{\eta^*} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^m$ such that*

$$\lim_{t \to 0^+} \frac{\mathcal{P}((1-t)\eta^* + t\vartheta) - \mathcal{P}(\eta^*)}{t} = \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} p_{\eta^*}(x,y)(\vartheta - \eta^*)(dx, dy)$$

and

$$\lim_{t \to 0^+} \frac{\mathcal{G}((1-t)\eta^* + t\vartheta) - \mathcal{G}(\eta^*)}{t} = \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} g_{\eta^*}(x,y)(\vartheta - \eta^*)(dx, dy)$$

for every $\vartheta \in \mathcal{M}$. Moreover, assume that the partial minimization function,

$$m(x) = \inf_{y \in [0,x]} r p_{\eta^*}(x,y) + \lambda \cdot g_{\eta^*}(x,y)$$

is measurable for every $r \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}^m_+$. Then, there exist $r^* \in \mathbb{R}_+$ and $\lambda^* \in \mathbb{R}^m_+$ such that $\lambda^* \cdot \mathcal{G}(\eta^*) = 0$ and

$$\eta^* \Big(\{(x,y) \in \mathcal{A}_R \mid y \in \operatorname{argmin}_{t \in [0,x]} r^* p_{\eta^*}(x,t) + \lambda^* \cdot g_{\eta^*}(x,t)\} \Big) = 1.$$

If $\mathcal{G}$ is constant or there exists $\vartheta \in \mathcal{M}$ such that

$$\mathcal{G}(\eta^*) + \int_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} g_{\eta^*}(x,y)(\vartheta - \eta^*)(dx, dy) < 0,$$

then r^* can be taken to be equal to 1.

This theorem thus identifies the support of optimal reinsurance contracts relative to the functions p_{η^*} and g_{η^*}, and λ^*, all of which depend on η^*. However, in several common applications, p_{η^*} and g_{η^*} depend on the optimal contract through a (finite) set of parameters. Together with λ^*, one can then treat this set of parameters as variables and optimize over them, thus reducing the problem to a finite dimensional optimization problem, for which several techniques can be used.

One example that prominently falls into this category is when the risk measure is given by

$$\mathcal{P}(\eta) = f\left(\int_{\mathbb{R}_+^n} p_1(x,y)\,\eta(dx,dy), \dots, \int_{\mathbb{R}_+^n} p_\ell(x,y)\,\eta(dx,dy)\right)$$

subject to the constraints $\mathcal{G} = (g_1, \dots, g_m)$ given by

$$g_i(\eta) = h_i\left(\int_{\mathbb{R}_+^n} q_{i,1}(x,y)\,\eta(dx,dy), \dots, \int_{\mathbb{R}_+^n} q_{i,\ell_i}(x,y)\,\eta(dx,dy)\right)$$

where all the p_i's and $q_{i,j}$'s are continuous functions and f and the h_i's are differentiable. This type of risk measure includes, but is not limited to the cases where one would like to minimize the expectation, variance, skewness, coefficient of variation, etc. of the total retained amount subject on constraints depending on similar measures. Several of the optimal reinsurance problems that fall under this umbrella are treated in [1] and [5]. Adapting for non-continuities and differentiability, the techniques can be slightly generalized to deal with distortion risk measures, such as those dealt with in [2], which shows the generality of our approach.

Throughout the previous discussion, it was imperative that the set $\mathcal{S}$ was described by a finite set of inequalities. The final portion of our study then relaxes the requirement for $\mathcal{S}$ to be finitely representable by inequalities. Still inspired by the idea of local linearization, we make the following assumptions:

(1) If $\eta^* \in \mathcal{S}$ is an optimal reinsurance contract, then for every $\eta \in \mathcal{S}$ and $0 \le t \le 1$, we have

$$\mathcal{P}(\eta^*) \le \mathcal{P}((1-t)\eta^* + t\eta).$$

(2) For every $\eta \in \mathcal{S}$, $d\mathcal{P}(\eta;\cdot)$ exists for every direction in $\mathcal{S}-\eta$ and is given as an integral operator, i.e., there exists a measurable function $p_\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for every $\vartheta \in \mathcal{S}$,

$$d\mathcal{P}(\eta; \vartheta - \eta) = \int p_\eta(x,y)(\vartheta - \eta)(dx,dy)$$

These two assumptions jointly imply that

$$\int p_{\eta^*}(x,y)\,\eta^*(dx,dy) = \min_{\eta \in \mathcal{S}} \int p_{\eta^*}(x,y)\,\eta(dx,dy).$$

Letting q_{η^*} denote the function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $q_{\eta^*}(x,y) = \infty$ on the complement $\mathcal{A}_R$ and otherwise being equal to p_{η^*}, the previous equation can be stated as

$$\int q_{\eta^*}(x,y)\,\eta^*(dx,dy) = \min_{\nu \in \pi_2(\mathcal{S})} \mathcal{C}(\mu,\nu), \tag{1}$$

where

$$\mathcal{C}(\mu,\nu) = \min_{\eta \in \Pi(\mu,\nu) \cap \mathcal{S}} \int q_{\eta^*}(x,y)\,\eta(dx,dy), \tag{2}$$

and $\Pi(\mu, \nu)$ is the set of couplings between μ and ν. Equations (1) and (2) mean that the optimal contract satisfies a double minimization property, where the inner minimum is a constrained optimal transport problem. We conclude our work by showing how, by taking a point of view inspired by this OT approach, we are enabled to use tools from the area to provide novel solutions to old and new optimal reinsurance problems.

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Adapted Wasserstein distance between the laws of SDEs

Sigrid Källblad

(joint work with Julio Backhoff-Veraguas, Ben Robinson)

In applications where filtrations and the flow of information play a key role, the concepts of weak convergence and Wasserstein distances have proven to be insufficient for specifying convergence and distances between stochastic processes. For instance, neither usual stochastic optimisation problems (such as optimal stopping or utility maximisation) nor Doob–Meyer decompositions behave continuously with respect to these topologies. Over the last decades, several approaches have been proposed to overcome these shortcomings; we focus here on one such notion, namely the so-called adapted Wasserstein distance.

We refer to [1, 2, 3, 6] for more on the motivation and history of adapted distances and the closely related concepts of causal and bi-causal couplings.

Specifically, in this talk we study the adapted Wasserstein distance between the laws of solutions of one-dimensional Markovian SDEs when the space of continuous functions is equipped with the L^p-metric. We address this problem by embedding it into a class of bi-causal optimal transport problems featuring a specific type of cost function. Imposing fairly general conditions on the (Markovian) coefficients of the SDEs, we will discuss methods and results which can be summarised as follows:

(i) characterisation of the coupling attaining the infimum for a class of bi-causal optimal transport problems including the adapted Wasserstein distance;

(ii) a time-discretisation method allowing derivation of most continuous-time statements from their more elementary discrete-time counterparts;

(iii) a stability result for optimisers to some bi-causal optimal transport problems;
(iv) a result stating that the topology induced by the adapted Wasserstein distance coincides with several topologies (including the weak topology) when restricting to SDEs whose coefficients belong to an equicontinuous family;
(v) examples illustrating what to expect for path-dependent SDEs and in higher dimensions.

At a conceptual level, we connect two hitherto unrelated objects: the *synchronous coupling* of SDEs, which is the coupling arising when letting a single Wiener process drive two SDEs; and the *Knothe–Rosenblatt* rearrangement, which is a celebrated discrete-time adapted coupling that preserves the lexicographical order. In particular, we provide an optimality property for the Knothe–Rosenblatt rearrangement which extends earlier results of [4, 7]. We then make use of this result to argue that in a certain sense, the synchronous coupling is the continuous-time counterpart of the Knothe–Rosenblatt rearrangement.

Concerning the contributions (i) and (iv) above, similar statements have been made in the pioneering work of Bion-Nadal and Talay [5] for the problem of optimally controlling the correlation between SDEs with smooth coefficients. We here show that the bi-causal optimal transport problem, for general cost functions and between laws of possibly path-dependent SDEs, admits such a control reformulation. A posteriori, it is thus clear that (i) and (iv) were established for the adapted Wasserstein distance and smooth coefficients already in [5]. Our results in this direction can be understood as using probabilistic methods to generalise their findings to more general cost functions and SDEs.

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Shrinkage of semimartingales

Monique Jeanblanc
(joint work with Tomasz R. Bielecki, Jacek Jakubowski, Pavel V. Gapeev and Mariusz Niewkeglowski)

In this talk we study projections of semi-martingales on various filtrations, under specific assumptions. More precisely, $\mathbb{F}$ and $\mathbb{G}$ being two filtrations with $\mathbb{F} \subset \mathbb{G}$, and $Y^{\mathbb{G}}$ being a $\mathbb{G}$-semimartingale, we define the optional projection of $Y^{\mathbb{G}}$ as $Y_t = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t], \forall t \geq 0$ which is an $\mathbb{F}$-semimartingale under some conditions (see [7]) and we find some relationships between the decomposition of $Y^{\mathbb{G}}$ and Y.

1. A simple case

Let $\vartheta^{\mathbb{G}}$ be a $\mathbb{G}$-adapted bounded process. It is well known that

$$\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds|\mathcal{F}_t] = M_t + \int_0^t \vartheta_s ds$$

where M is an $\mathbb{F}$-martingale and $\vartheta_s = \mathbb{E}[\vartheta_s^{\mathbb{G}}|\mathcal{F}_s]$. (See, e.g., [5, lemma 8.3])

The goal is to identify M in terms of the $\vartheta^{\mathbb{G}}$ and one specific martingale which satisfy predictable representation property (PRP) on $\mathbb{F}$.

Assume for example that $\mathbb{F}$ is a Brownian filtration generated by W. In that case PRP holds, i.e., for any $\mathbb{F}$-martingale M there exists an $\mathbb{F}$-predictable process ψ such that $M_t = M_0 + \int_0^t \psi_s dW_s$.

For any $\mathbb{F}$-adapted bounded process φ one has, using tower property in the first equality

$$\begin{aligned}
\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] &= \mathbb{E}[\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \,|\mathcal{F}_t] \int_0^t \varphi_s dW_s] \\
&= \mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] + \mathbb{E}[M_t \int_0^t \varphi_s dW_s]
\end{aligned}$$

hence

$$\begin{aligned}
&\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] - \mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] \\
= \;& \mathbb{E}[M_t \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \psi_s \varphi_s ds]
\end{aligned}$$

To proceed, we need to apply integration by parts to the product of $\mathbb{G}$-semimartingales $\int_0^\cdot \vartheta_s^{\mathbb{G}} ds$ and $\int_0^\cdot \varphi_s dW_s$ (if $\int_0^\cdot \varphi_s dW_s$ is a $\mathbb{G}$-semimartingale!) which leads to

$$\mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) dW_s]$$

We now assume that there exists a $\mathbb{G}$-adapted process $\alpha^{\mathbb{G}}$ such that W is a $\mathbb{G}$-semimartingale with decomposition

$$W_t = W_t^{\mathbb{G}} + \int_0^t \alpha_s^{\mathbb{G}} ds$$

where $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion, then

$$\mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) dW_s] = \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds]$$

(See some conditions in [1, Ch 4 and 5]).

Using tower property in the second equality

$$\begin{aligned} \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} ds \int_0^t \varphi_s dW_s] &= \mathbb{E}[\int_0^t \vartheta_s^{\mathbb{G}} \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds] \\ &= \mathbb{E}[\int_0^t \vartheta_s \left(\int_0^s \varphi_u dW_u\right) ds] + \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds] \end{aligned}$$

we get (one has to check carefully that all local martingales that appear are true martingales) noting that

$$\mathbb{E}[\int_0^t \vartheta_s ds \int_0^t \varphi_s dW_s] = \mathbb{E}[\int_0^t \vartheta_s \left(\int_0^s \varphi_u dW_u\right) ds]$$

$$\mathbb{E}[\int_0^t \psi_s \varphi_s ds] = \mathbb{E}[\int_0^t \varphi_s \left(\int_0^s \vartheta_u^{\mathbb{G}} du\right) \alpha_s^{\mathbb{G}} ds]$$

and this being true for any φ, this yields

$$\psi_s = \mathbb{E}[\alpha_s \int_0^s \vartheta_u^{\mathbb{G}} du | \mathcal{F}_s] \,.$$

Remarks: If $\vartheta^{\mathbb{G}}$ is $\mathbb{F}$- adapted $M = 0$ and $\vartheta^{\mathbb{G}} = \vartheta$. This can be recover from

$$\psi_s = \mathbb{E}[\alpha_s^{\mathbb{G}} \int_0^s \vartheta_u^{\mathbb{G}} du | \mathcal{F}_s] = \int_0^s \vartheta_u du \, \mathbb{E}[\alpha_s^{\mathbb{G}} | \mathcal{F}_s] = 0$$

since $\mathbb{E}[\alpha_s^{\mathbb{G}} | \mathcal{F}_s] = 0$.

This can be easily extended to the case where $\mathbb{F}$ has a process (may be multi-dimensional or having jumps) which enjoy PRP for example if $\mathbb{F}$ is generated by a pair $(W, \widetilde{\mu})$ where W is a Brownian motion independent of a compensated marked point process $\widetilde{\mu}$.

2. Martingales

Let $\mathbb{F}$ be a filtration, M an $\mathbb{F}$-martingale (possibly multidimensional, or with jumps) enjoying PRP.

Let $\mathbb{G}$ be a filtration larger than $\mathbb{F}$ which enjoy PRP with respect to $M^{\mathbb{G}}$ where $M^{\mathbb{G}}$ is a (possibly multidimensional or with jumps) $\mathbb{G}$-martingale such that any $\mathbb{G}$-martingale $Y^{\mathbb{G}}$ has a decomposition as

$$Y_t^{\mathbb{G}} = \int_0^t \psi_s^{\mathbb{G}} dM_s^{\mathbb{G}}$$

We note with a superscript $\mathbb{G}$ processes that are $\mathbb{G}$-adapted as $Y^{\mathbb{G}}$.

Our goal is to find the decomposition of the $\mathbb{F}$-martingale $Y_t = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = \int_0^t \psi_s dM_s$.

The r.v. Y_t is characterized by

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s] = \mathbb{E}[Y_t \int_0^t \varphi_s dM_s]$$

for any $\varphi \in \mathbb{F}$.

In the one hand, using tower property

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s] = \mathbb{E}[Y_t \int_0^t \varphi_s dM_s] = \mathbb{E}[\int_0^t \psi_s \varphi_s d\langle M\rangle_s]\,.$$

To compute using integration by parts $\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s]$, we need to assume that M is a $\mathbb{G}$-semimartingale with decomposition

$$M_t = \int_0^t \beta_s^{\mathbb{G}} dM_s^{\mathbb{G}} + \int_0^t \alpha_s^{\mathbb{G}} d\langle M^{\mathbb{G}}\rangle_s\,.$$

This yields

$$\mathbb{E}[Y_t^{\mathbb{G}} \int_0^t \varphi_s dM_s] = \mathbb{E}[\int_0^t Y_s^{\mathbb{G}} \varphi_s dM_s] + \mathbb{E}[\int_0^t \Big(\int_0^s \varphi_u dM_u\Big) dY_s^{\mathbb{G}}] + \mathbb{E}[\langle Y^{\mathbb{G}}, \int_0^\cdot \varphi_s dM_s\rangle_t]$$

where in the first integral in the righthand side M is a $\mathbb{G}$-semimartingale as well as in the bracket and the second term is null. We compute the two remaining parts using that the local martingales are true martingales, this can proved by means of Burkolder Davis Gundy.

$$\mathbb{E}[\int_0^t Y_s^{\mathbb{G}} \varphi_s dM_s] = \mathbb{E}[\int_0^t Y_s^{\mathbb{G}} \varphi_s \alpha_s^{\mathbb{G}} d\langle M^{\mathbb{G}}\rangle_s]$$

and

$$\mathbb{E}[\langle Y^{\mathbb{G}}\,, \int_0^\cdot \varphi_s dM_s\rangle_t] = \mathbb{E}[\int_0^t \varphi_s \psi_s^{\mathbb{G}} \beta_s^{\mathbb{G}} d\langle M^{\mathbb{G}}\rangle_s]$$

$$\mathbb{E}[\int_0^t \psi_s \varphi_s d\langle M\rangle_s] = \mathbb{E}[\int_0^t \varphi_s \big(Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}} + \psi_s^{\mathbb{G}} \beta_s^{\mathbb{G}}\big) d\langle M^{\mathbb{G}}\rangle_s]$$

hence

$$\psi_s = \frac{\mathbb{E}[\big(Y_s^{\mathbb{G}} \alpha_s^{\mathbb{G}} + \psi_s^{\mathbb{G}} \beta_s^{\mathbb{G}}\big) d\langle M^{\mathbb{G}}\rangle_s|\mathcal{F}_s]}{d\langle M\rangle_s}$$

and, since $d\langle M\rangle = (\beta^{\mathbb{G}})^2 d\langle M^{\mathbb{G}}\rangle$

$$\psi_s = \mathbb{E}[\frac{Y_s^{\mathbb{G}}\alpha_s^{\mathbb{G}} + \psi_s^{\mathbb{G}}\beta_s^{\mathbb{G}}}{(\beta^{\mathbb{G}})_s^2}|\mathcal{F}_s]$$

See [3, 4] for details.

3. Semimartingales

It is well known, from [7], that if X is a $\mathbb{G}$-semimartingale and is $\mathbb{F}$-adapted where $\mathbb{F} \subset \mathbb{G}$, then X is an $\mathbb{F}$-semimartingale.

Note that if the $\mathbb{G}$-special semimartingale decomposes as $X = M + A$ and is $\mathbb{F}$-adapted, it may happen that M and A are not $\mathbb{F}$-adapted (see [7] or [2]). However, in our case X can be decomposed in both filtrations as (ℓ being a truncation function)

$$X_t = X_0 + X_t^{c,\mathbb{G}} + \int_0^t \int_E \ell(x)(\mu(dt,dx) - \nu^{\mathbb{G}}(dt,dx)) + B_t^{\mathbb{G}}(\ell) = M_t^{\mathbb{G}} + B_t^{\mathbb{G}}(\ell)$$

$$X_t = X_0 + X_t^{c,\mathbb{F}} + \int_0^t \int_E \ell(x)(\mu(dt,dx) - \nu^{\mathbb{F}}(dt,dx)) + B_t^{\mathbb{F}}(\ell) = M_t^{\mathbb{F}} + B_t^{\mathbb{F}}(\ell)$$

where B is a predictable process with finite variation. The process B is the first characteristic, the second characteristic is $\langle X\rangle$, the third characteristic is ν.

There exists a $\mathbb{G}$-predictable, locally integrable increasing process, say $A^{\mathbb{G}}$, predictable processes $b^{\mathbb{G}}$, $c^{\mathbb{G}}$ and a transition kernel K such that

$$B^{\mathbb{G}} = b^{\mathbb{G}} \cdot A^{\mathbb{G}}, \qquad C^{\mathbb{G}} = c^{\mathbb{G}} \cdot A^{\mathbb{G}}, \qquad \nu^{\mathbb{G}}(dt,dx) = K_t^{\mathbb{G}}(dx)dA_t^{\mathbb{G}}.$$

We assume that

$$A_t^{\mathbb{G}} = \int_0^t a_u^{\mathbb{G}} du,$$

where $a^{\mathbb{G}}$ is a $\mathbb{G}$ progressively measurable process. Then it can be shown (see [6]) that the $\mathbb{F}$- characteristic triple of X is given as

$$dB^{\mathbb{F}} = \int_0^{\cdot} {}^{o,\mathbb{F}}(b_s^{\mathbb{G}} a^{\mathbb{G}})_s ds, \quad C^{\mathbb{F}} = C^{\mathbb{G}}, \quad \nu^{\mathbb{F}}(dt,dx) = \left(K_t^{\mathbb{G}}(dx) a_t^{\mathbb{G}} dt\right)^{p,\mathbb{F}}$$

where ${}^{o,\mathbb{F}}Z$ is the $\mathbb{F}$-projection of Z and $U^{p,\mathbb{F}}$ is the dual predictable projection of U (see, e,g, [1]).

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Robust duality for multi-action options with information delay

Anna Aksamit
(joint work with Ivan Guo, Shidan Liu, Zhou Zhou)

We establish pricing–hedging duality under model uncertainty for multi-action options. Multi-action options form a class of contracts whose pay-off depends on the actions taken by a buyer of such contract. As an example we may consider American options, baskets of American options with constraints on execution times, or swing options.

We thus generalize the duality obtained in [2] to the case of exotic options that allow the buyer to choose some action from an action space, countable or uncountable, at each time step in the setup of [3]. Our ideas, however, go beyond that model and can be applied in various frameworks – including dominated setup.

We solve above problem by introducing an enlarged canonical space in order to reformulate the superhedging problem for such exotic options as a problem for European options. Then in a discrete time market with the presence of finitely many statically traded liquid options, we prove the pricing-hedging duality for such exotic options as well as the European pricing-hedging duality in the enlarged space. For the sake of simplicity we focus on the case without statically traded options in what follows.

Consider the discrete-time model introduced in [3]. Fix a time horizon $N \in \mathbb{N}$, and let $\mathbb{T} := \{0, 1, \dots, N\}$ be the time periods in this model. Let $\Omega_0 = \{\omega_0\}$ be a singleton and Ω_1 be a Polish space. For each $k \in \{1, \dots, N\}$, define $\Omega_k := \Omega_0 \times \Omega_1^k$ as the k-fold Cartesian product. For each k, define $\mathcal{G}_k := \mathcal{B}(\Omega_k)$ and let $\mathcal{F}_k$ be its universal completion. In particular, we see that $\mathcal{G}_0$ is trivial and denote $\Omega := \Omega_N$, $\mathcal{F} := \mathcal{F}_N$ and $\mathbb{F} = (\mathcal{F}_k)_k$.

Consider a market with $d \in \mathbb{N}$ financial assets that can be traded dynamically without transaction costs. We model the dynamically traded assets by an $\mathbb{R}^d$-valued process $S = (S_t)_{t\in\mathbb{T}}$ such that S_t is $\mathcal{G}_t$-measurable for $t \in \mathbb{T}$. For an $\mathbb{F}$-predictable, $\mathbb{R}^d$-valued process H, the terminal wealth of the hedging portfolio is given by $(H \circ S)_N = \sum_{j,k} H_k^j (S_k^j - S_{k-1}^j)$.

Model uncertainty is expressed via the family of possible models $\mathcal{P}$ which is constructed in the following manner. For a given $k \in \{0, \dots, N-1\}$ and $\omega \in \Omega_k$, we have a non-empty convex set $\mathcal{P}_{k,k+1}(\omega) \subseteq \mathfrak{P}(\Omega_1)$ of probability measures, representing the set of all possible models for the $(k+1)$-th period, given the state ω at time k. We assume that for each $k \in \{0, \dots, N\}$, $\mathrm{graph}(\mathcal{P}_{k,k+1}) \subseteq \Omega_k \times \mathcal{P}(\Omega_1)$ is analytic. We can then introduce the set $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$ of possible models for the multi-period market up to time N by

$$\mathcal{P} := \{\mathbb{P}_{0,1} \otimes \mathbb{P}_{1,2} \otimes \cdots \otimes \mathbb{P}_{N-1,N} : \mathbb{P}_{k,k+1}(\cdot) \in \mathcal{P}_{k,k+1}(\cdot)\}.$$

Let $\mathcal{A}$ be the space of actions at each time and introduce $\mathcal{C} := \mathcal{A}^{N+1}$ to be the collection of all possible plans, equipped with the Borel σ-algebra $\mathcal{B}(\mathcal{C})$ and a canonical filtration $(\mathcal{F}^c_k)_{0\le k\le N}$. In such set-up we are interested in the action dependent pay-off function $\Phi : \Omega \times \mathcal{C} \to \overline{\mathbb{R}}$, and its superhedging price given by

$$\pi(\Phi) := \inf\{x : \exists H \in \mathcal{H}, \text{ s.t., } x + (H(\cdot, c) \circ S)_N \ge \Phi(\cdot, c)\ \mathcal{P}\text{-q.s.}, \forall c \in \mathcal{C}\}$$

and define the set of dynamic trading strategies

$$\mathcal{H} := \Big\{H : \Omega \times \mathcal{C} \times \mathbb{T} \to \mathbb{R}^d \mid H(\cdot, \cdot, k+1) =: H_{k+1}(\cdot, \cdot) \text{ is } \mathcal{F}_k \otimes \mathcal{F}^c_k\text{-measurable}\Big\}.$$

Our main theorem states the duality result where dual representation of this superhedging price is established:

Theorem 1. *Suppose that the no arbitrage condition NA($\mathcal{P}$) holds, and let $\Phi : \Omega \times \mathcal{C} \to \overline{\mathbb{R}}$ be upper semianalytic. Then, one has*

$$\pi(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\chi\in\mathcal{D}} \mathbb{E}^{\mathbb{Q}}[\Phi_\chi].$$

In the above theorem set $\mathcal{D}$ consists of all feasible action plans $\chi : \Omega \times \mathbb{T} \to \mathcal{A}$ such that $\chi(\cdot, k)$ is $\mathcal{F}_k$-measurable for each k. Set $\mathcal{D}$ generalizes the set of stopping times to a multi-action set-up. Set $\mathcal{M}$ denotes the set of martingale measures for a process S on Ω, and is given by

$$\mathcal{M} = \big\{\mathbb{Q} \in \mathfrak{P}(\Omega) : \mathbb{Q} \lll \mathcal{P} \text{ and } \mathbb{E}^{\mathbb{Q}}[\Delta S_k \mid \mathcal{F}_{k-1}] = 0, \forall k = 1, \dots, N\big\}.$$

To prove Theorem 1, we apply the idea of space enlargement motivated by [2], which enables to view multi-action option as an European option on the space $\Omega \times \mathcal{C}$. Crucial argument is re-establishing dynamic programming principle based on Jankov-von Neumann analytic selection theorem. Since our framework allows for uncountable action space this argument becomes significantly more involved.

We complement our duality result with the study of the superhedging price of a multi-action option in the case of information delay. More precisely we cover the case where the seller of the option does not possess perfect information about the actions taken by the buyer, and is able to observe them with a delay. This framework takes into account this different type of uncertainty. The resulting duality for the superhedging price with information delay $\pi^{del}(\Phi)$ takes the following form:

$$\pi^{del}(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\chi\in\mathcal{D}^{ant}} \mathbb{E}^{\mathbb{Q}}[\Phi_\chi],$$

where, instead of previously appearing set of adapted feasible action plans $\mathcal{D}$, we have the set of the *anticipating* feasible action plans $\mathcal{D}^{ant}$. The dual side can be interpreted as the price which may be achieved by the buyer able to look into the future. *Looking into the future* feature is present here as information delay puts more constraints on the superhedging side.

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Optimal reinsurance via BSDEs in a partially observable model with jump clusters

Claudia Ceci

(joint work with Matteo Brachetta, Giorgia Callegaro, Carlo Sgarra)

Optimal reinsurance problems have attracted special attention during the past few years and they have been investigated in many different model settings. Insurance companies can hardly deal with all the different sources of risk in the real world, so they hedge against at least part of them, by re-insuring with other institutions. A reinsurance agreement allows the primary insurer to transfer part of the risk to another company and it is well known that this is an effective tool in risk management. Moreover, the subscription of such contracts is required by some financial regulators, see e.g. the Directive Solvency II in the European Union. Large part of the existing literature focuses mainly on classical reinsurance contracts such as the proportional and the excess-of-loss, which were extensively investigated under a variety of optimization criteria, e.g. ruin probability minimization, dividend optimization and expected utility maximization. Here we are interested in the latter approach (see Irgens and Paulsen [12], Mania and Santacroce [15], Brachetta and Ceci [3] and references therein). Some of the classical papers devoted to the subject assume a diffusive dynamics for the surplus process, while the more recent literature considers surplus processes including jumps.

The pioneering risk model with jumps in non-life insurance is the classical Cramér-Lundberg model, where the claims arrival process is a Poisson process with constant intensity. This assumption implies that the instantaneous probability that an accident occurs is always constant, which is in a way too restrictive in the real world, as already motivated by Grandell [10]. In recent years, many authors made a great effort to go beyond the classical model formulation. For example, Cox processes were employed to introduce a stochastic intensity for the claims arrival process, see e.g. Albrecher and Asmussen [1], Bjork and Grandell [2], Embrechts et al. [9]. Moreover, other authors introduced Hawkes processes in order to capture the self-exciting property of the insurance risk model in presence of catastrophic events. Hawkes processes were introduced by Hawkes [11] to describe geological phenomena with clustering features like earthquakes. Hawkes processes with general kernels are not Markov processes: they can eventually include long-range dependence, while Hawkes processes with exponential kernel exhibit the appealing property that the couple process-intensity is Markovian.

Dassios and Zhao [7] proposed a model which combines the two approaches by introducing a Cox process with shot noise intensity and a Hawkes process with exponential kernel for describing the claim arrival dynamics. Recently Cao, Landriault and Li [5] investigated the optimal reinsurance-investment problem in the model setting proposed by Dassios and Zhao [7] with a reward function of mean-variance type.

A different line of research related to the optimal-reinsurance investment problem focuses on the possibility that the insurer does not have access to all the information when choosing the reinsurance strategy. As a matter of fact, only the claims arrival and the corresponding disbursements are observable. In this case we need to solve a stochastic optimization problem under partial information. Liang and Bayraktar [14] were the first to introduce a partial information framework in optimal reinsurance problems. They consider the optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution are not known, but have to be inferred from the observations of claim arrivals. Ceci, Colaneri and Cretarola [6] derive risk-minimizing investment strategies when information available to investors is restricted and they provide optimal hedging strategies for unit-linked life insurance contracts. Jang, Kim and Lee [13] present a systematic comparison between optimal reinsurance strategies in complete and partial information framework and quantify the information value in a diffusion setting.

More recently, Brachetta and Ceci [4] investigate the optimal reinsurance problem under the criterion of maximizing the expected exponential utility of terminal wealth when the insurance company has restricted information on the loss process in a model with claim arrival intensity and claim sizes distribution affected by an unobservable environmental stochastic factor.

In the present paper we investigate the optimal reinsurance strategy for a risk model with jump clustering properties in a partial information setting. The risk model is similar to that proposed by Dassios and Zhao [7] and it includes two different jump processes driving the claims arrivals: one process with constant intensity describing the exogenous jumps and another with stochastic intensity representing the endogenous jumps, that exhibits self-exciting features. The externally-excited component represents catastrophic events, which generate claims clustering increasing the claim arrival intensity. The endogenous part allows us to capture the clustering effect due to self-exciting features. That is, when an accident occurs, it increases the likelihood of such events. The insurance company has only partial information at disposal, more precisely the insurer can only observe the cumulative claims process. The externally-excited component of the intensity is not observable and the insurer needs to estimate the stochastic intensity by solving a filtering problem. Our approach is substantially different from that of Cao et Al. [5] in several respects: firstly, we work in a partial information setting; secondly, the intensity of the self-excited claims arrival exhibits a slight more general dependence on the claims severity; finally, we maximize an exponential utility function instead of following a mean-variance criterion. In a partially observable framework, our goal is to characterize the value process and the optimal strategy. The optimal stochastic control problem in our case turns out to be infinite dimensional and the characterization of the optimal strategy cannot be performed by solving a Hamilton-Jacobi-Bellman equation, but via a BSDE approach.

A difficulty naturally arises when dealing with Hawkes processes: the intensity of the jumps is not bounded a priori, although a non-explosive condition holds. Hence we are not able to exploit some relevant bounds, which are usually required to prove a verification theorem and results on existence and uniqueness of the solution for the related BSDE. Nevertheless, we are going to show that the optimal stochastic control problem has a solution, which admits a characterization in terms of a unique solution to a suitable BSDE.

Our paper aims to contribute in different directions to the literature on optimal reinsurance problems: first, we provide a rigorous and formal construction of the dynamic contagion model. Second, we study the filtering problem associated to our model, providing a characterization of the filter process in terms of the Kushner-Stratonovich equation and the Zakai equation as well. To the best of our knowledge, this problem has not been addressed insofar in the existing literature. We refer to Dassios and Jang [8] for a similar problem without the self-exciting component. Third, we solve the optimal reinsurance problem under the expected utility criterion.

We remark that our study differs from Brachetta and Ceci [4] in many key aspects. The risk model is substantially different, requires a strong effort to be rigorously constructed and the study of a new filtering problem. What is more, a crucial assumption in Brachetta and Ceci [4] is the boundedness of the claims arrival intensity, which is not satisfied in our case, thus leading to additional technicalities in most of the proofs. This is what happens, for example, when one needs to prove existence and uniqueness of the solution of the BSDE. Moreover, we perform the optimization over a class of admissible contracts, instead of maximizing over the retention level. This feature allows us to cover a larger class of problems. Finally, we do not require the existence of an optimal control for the derivation of the BSDE, hence the general presentation turns out to be different.

The paper is organized as follows. In Section 1 we are going to introduce the risk model and to specify what information is available to the insurer. A rigorous mathematical construction is provided, based on a measure change approach, necessary to develop the following analysis in full details. In Section 2 the filtering problem is investigated in order to reduce the optimal stochastic control problem to a complete information setting. The stochastic differential equation satisfied by the filter is obtained, by exploiting both the Kushner-Stratonovich and the Zakai approaches. In Section 3 the optimal stochastic control problem is formulated, while in Section 4 a characterization of the value process associated with the optimal stochastic control problem is illustrated. Due to the infinite dimension of the filter, the approach based on the Hamilton-Jacobi-Bellman equation cannot be exploited, so the value process is characterized as the unique solution of a BSDE. In Section 5 the optimal reinsurance strategy is investigated under general assumptions and some relevant cases are discussed.

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Utility maximization for reinsurance policies in a dynamic contagion claim model

Alessandra Cretarola

(joint work with Claudia Ceci)

Optimal reinsurance and optimal investment problems for various risk models have gained a lot of interest in the actuarial literature in recent years. Thanks to the development of effective strategies, insurers can reduce potential claim risk (insurance risk) and optimize capital investments. Indeed, acquiring reinsurance serves as a safeguard for insurers against unfavorable claim experiences, while investing also enables insurers to diversify risks and potentially achieve higher returns on the cash flows within their insurance portfolio. Within the extensive body of literature devoted to risk theory, a classical task is to deal with optimal risk control and optimal asset allocation for an insurance company. Mainly in the case of classical reinsurance contracts such as the proportional and the excess-of-loss, different decision criteria have been adopted in the study of these problems e.g. ruin probability minimization, dividend optimization and expected utility maximization. Here, we focus on the latter approach (see Irgens and Paulsen [9], Mania and Santacroce [10], Brachetta and Ceci [4] and references therein). Earlier seminal papers on the topic adopt a diffusive dynamics for the surplus process, whereas more recent literature explores surplus processes that incorporate jumps.

The first risk model specification incorporating jumps in non-life insurance is represented by the classical Cramér-Lundberg model, in which the claims arrival process follows a Poisson process with a constant intensity. Since it is an assumption which is seriously violated in a large number of insurance contexts (e.g., climate risks), many researchers have suggested to employ a stochastic intensity for the claim arrival dynamics. For instance, clustering features due to exogenous (externally-excited) factors, such as earthquakes, flood, and hurricanes, might be captured using a Cox process, see e.g. Albrecher and Asmussen [1], Bjork and Grandell [2], Embrechts et al. [7]. Moreover, clustering effects due to endogenous (self-excited) factors, such as aggressive driving habits and poor health conditions, can be effectively described by a Hawkes process, see e.g. Hawkes [8]. Dassios and Zhao [6] introduced a dynamic contagion model by generalizing both the Cox process with shot noise intensity and the Hawkes process.

In recent years, Cao, Landriault and Li [5] analyzed the optimal reinsurance-investment problem for the compound dynamic contagion process introduced by Dassios and Zhao [6] via the time-consistent mean–variance criterion. Brachetta et al. [3] very recently investigated the optimal reinsurance strategy for a risk model with jump clustering features similar to that proposed by Dassios and Zhao [6] under partial information.

In this work, we study the optimal reinsurance problem via expected utility maximization in the risk model with jump clustering properties introduced in Brachetta et al. [3] under full information for general reinsurance contracts. Note that, the problem considered in Brachetta et al. [3] is the same but analyzed in a partial information setting. The study of the problem in the case of complete information is not addressed in the literature, and furthermore, it could allow for comparative analyses in a more tractable context than that of partial information. We discuss two different methodologies: the classical stochastic control approach based on the Hamilton-Jacobi-Bellman equation and a backward stochastic differential equation approach. It is important to stress that proving the existence of a classical solution to the Hamilton-Jacobi-Bellman equation corresponding to the optimal stochastic control problem under investigation is challenging due to its inherent complexity. This difficulty stems from the equation's nature as a partial integro-differential equation, compounded by an optimization component embedded within the associated integro-differential operator. This motivated the application of an alternative approach based on backward stochastic differential equations. It is worth noting that the resulting backward stochastic differential equation, whose unique solution characterizes the value process, differs from that studied in Brachetta et al. [3], due to the presence of an additional jump component.

The paper, still in progress, is organized as follows. Firstly, we introduce the mathematical framework including the dynamic contagion process. Then, we formally introduce the problem under investigation, which involves the controlled surplus process and the objective function. Afterwards, we discuss the Hamilton-Jacobi-Bellman approach in order to solve the resulting optimal stochastic control problem and represent the value process as the unique solution of a suitable backward stochastic differential equation. We also characterize the optimal strategy for a general reinsurance premium and provide more explicit results in some relevant cases. Currently, we are performing a comparison analysis, which should underline the risk due to the self-exciting component.

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Set-Valued Propagation of Chaos for Controlled Mean Field SPDEs

David Criens

The area of controlled McKean–Vlasov dynamics, also known as mean field control, has rapidly developed in the past years More recently, there is also increasing interest in infinite dimensional systems, see, e.g., [1, 6] for equations appearing in financial mathematics. We also refer to the recent paper [2], where the authors investigate controlled mean field stochastic PDEs (SPDEs) for which they establish well-posedness of the state equation, the dynamic programming principle and a Bellman equation.

Mean field dynamics are typically motivated by particle approximations (related to propagation of chaos). It is an important task to make the heuristic motivation rigorous. For finite dimensional frameworks, a suitable limit theory was developed in the seminal paper [8].

In this talk, we discuss recent results established in the paper [3] for an infinite dimensional variational SPDE framework as initiated by Pardoux [9] and Krylov–Rozovskii [7]. To reduce the technical level of the talk, we consider a specific interacting systems of controlled porous media equations of the form

$$dY^k_t = \Big[\Delta(|Y^k_t|^{q-2}Y^k_t) + \frac{1}{n}\sum_{i=1}^{n}(Y^k_t - Y^i_t) + \int c(f)\,\mathfrak{m}^k(t,df)\Big]dt + \sigma dW^k_t,$$
$$Y^k_0 = x,$$

with $q \geq 2$ and $k = 1, \dots, n$. Here, $\mathfrak{m}^1, \mathfrak{m}^2, \dots, \mathfrak{m}^n$ denote kernel that model the control variables, and $W^1, \dots, W^n$ are independent cylindrical Brownian motions. This corresponds to a relaxed control framework in the spirit of [4, 5].

Let $\mathcal{R}^n(x)$ be the set of joint empirical distributions of such particles together with their controls (latter are captured via $\mathfrak{m}^k(t, df)dt$ in a suitable space of Radon measures). The associated set of mean field limits is denoted by $\mathcal{R}^0(x)$. It consists of probability measures supported on the set of laws of $(Y, \mathfrak{m}(t, df)dt)$, where Y solves a controlled McKean–Vlasov equation of the form

$$dY_t = \Big[\Delta(|Y_t|^{q-2}Y_t) + (Y_t - E[Y_t]) + \int c(f)\,\mathfrak{m}(t,df)\Big]dt + \sigma dW_t \quad Y_0 = x.$$

For this setting, we discuss two types of results. Conceptually, the first one is probabilistic and deals with the convergence of the controlled particle systems, while the second one sheds light on the mean field limits from a stochastic optimal control perspective.

The probabilistic result states that the sets $\mathcal{R}^n(x)$ and $\mathcal{R}^0(x)$ are nonempty and compact (in a suitable Wasserstein space) and that

$$\mathcal{R}^n(x) \to \mathcal{R}^0(x)$$

in the Hausdorff metric topology. This result is considered as *set-valued propagation of chaos*. Indeed, when the sets $\mathcal{R}^n(x^n)$ and $\mathcal{R}^0(x^0)$ are singletons, we recover a classical formulation of the propagation of chaos property. To the best of our knowledge, the concept and formulation of set-valued propagation of chaos has not appeared in the literature before.

The optimal control result states that the value functions associated with $\mathcal{R}^n(x)$ and $\mathcal{R}^0(x)$ converge to each other (uniformly on compacts in their initial values x), i.e.,

$$\Big(x \mapsto \sup_{P \in \mathcal{R}^n(x)} E^P\big[\,\psi\,\big]\Big) \to \Big(x \mapsto \sup_{P \in \mathcal{R}^0(x)} E^P\big[\,\psi\,\big]\Big)$$

compactly, for any continuous input function ψ on the suitable Wasserstein space that is of certain growth. As a consequence, one also obtains limit theorems in the spirit of the seminal work [8]. Namely, it follows that all accumulation points of sequences of n-state nearly optimal controls maximize the mean field value function, and that any optimal mean field control can be approximated by a sequence of n-state nearly optimal controls.

The talk is concluded with the open problem to relax some weak monotonicity conditions from [3]. This problem appears to be challenging due to the non-local structure of McKean–Vlasov equations.

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Hawkes processes, Malliavin calculus and application to financial and actuarial derivatives

Caroline Hillairet

(joint work with Anthony Réveillac, Mathieu Rosenbaum)

In this talk, we are interested in the evaluation of financial or actuarial derivatives whose payoff depends on a cumulative loss (L_t)

$$L_t := \sum_{i=1}^{N_t} X_i, \quad t \in [0,T]$$

where $N := (N_t)_{t\in[0,T]}$ is a counting process (jumping at time $(\tau_i)_{i\in\mathbb{N}^*}$) that represents the claims arrival (frequency component) and the $(X_i)_{i\in\mathbb{N}^*}$ (iid random variables) are the claims sizes (severity component).

In the classical Cramer-Lundberg model, N is assumed to be a Poisson process, meaning that inter-arrivals $(\tau_i - \tau_{i-1})$ are assumed to be iid (with exponential distribution). Nevertheless, self-exciting and contagion effects have been highlighted such as for example in credit risk and in cyber risk, in favor of modeling the claims arrivals by a Hawkes process, that is adapted to model aftershocks of claims. A (linear) Hawkes process H is characterized by its stochastic intensity $\lambda(t)$ fully specified by the process H itself, namely

$$\lambda(t) := \lambda_0(t) + \int_{(0,t)} \Phi(t-s)dH_s = \lambda_0(t) + \sum_{\tau_n<t} \Phi(t-\tau_n) \quad t \in [0,T],$$

where Φ is the (deterministic) excitation kernel and λ_0 is the (deterministic) baseline intensity (hereafter taken as a constant μ). The main contribution is to derive an explicit closed form pricing formula for contracts with underlying a cumulative loss indexed by a Hawkes process.

From the probabilistic point of view, we consider a payoff of the form $K_T h(L_T)$ where (K_t) and (L_t) are two loss processes indexed by the same Hawkes. This quantity is at the core for determining the premium of a large class of insurance derivatives or risk management instruments : reinsurance contracts (such as Stop-Loss contracts), or credit derivatives (such as tranches of Collaterized Debt Obligations), or computation of the expected shortfall of contingent claims. It can be expressed as $\int_{(0,T]} Z_t dH_t F$ where Z is a predictable process and $F := h(L_T)$ is a functional of the Hawkes process. In the case where the counting process is a Poisson process (or a Cox process), Malliavin calculus enables one to transform this quantity. More precisely, if $H = N$ is an homogeneous Poisson process with intensity $\mu > 0$ (in other words the self-exciting kernel Φ is put to 0), the Malliavin integration by parts formula (Mecke formula, see [7]) allows us to derive that

$$\mathbb{E}\left[\int_{(0,T]} Z_t dN_t F\right] = \mu \int_0^T \mathbb{E}\left[Z_t F \circ \epsilon_t^+\right] dt, \tag{1}$$

where the notation $F \circ \epsilon_t^+$ denotes the functional on the Poisson space where a deterministic jump is added to the paths of N at time t. This expression turns out to be particularly interesting from an actuarial point of view since adding a jump at some time t corresponds to realising a stress test by adding artificially a claim at time t. Naturally, in case of a Poisson process, the additional jump at some time t only impacts the payoff of the contract by adding a new claim in the contract but it does not impact the dynamic of the counting process N.

We provide a generalization of Equation (1) in case the counting process is a Hawkes process H. The main ingredient consists in using a representation of a Hawkes process known as the "Poisson imbedding" (related to the "Thinning Algorithm", see [5]) in terms of a Poisson measure N on $[0,T] \times \mathbb{R}_+$ to which the Malliavin integration by parts formula can be applied.

$$\begin{cases} H_t = \int_{(0,t]} \int_{\mathbb{R}_+} \mathbf{1}_{\{\theta \le \lambda_s\}} N(ds, d\theta), \\ \lambda_t = \mu + \int_{(0,t)} \Phi(t-u) dH_u. \end{cases} \tag{2}$$

As the adjunction of a jump at a given time impacts the dynamic of the Hawkes process, we refer to the obtained expression more to an "expansion" rather than an "integration by parts formula" for the Hawkes process, as it involves what we name "shifted Hawkes processes" $H^{v_n,\dots,v_1}$ for which jumps at deterministic times $0 < v_n < \cdots < v_1$ are added to the process accordingly to the self-exciting kernel Φ. To illustrate this, a one shift Hawkes process at time v in $(0,T)$ can be expressed as follows

$$\begin{cases} H^v_t = \mathbf{1}_{[0,v)}(t)H_t + \mathbf{1}_{[v,T]}(t)\left(H^v_{v-} + 1 + \int_{(v,t]}\int_{\mathbb{R}_+} \mathbf{1}_{\{\theta \le \lambda^v_s\}} N(ds,d\theta)\right) \\ \lambda^v_t = \mathbf{1}_{(0,v]}(t)\lambda_t + \mathbf{1}_{(v,T]}(t)\left(\mu^{v,1}(t) + \int_{(v,t)} \Phi(t-u)dH^v_u\right), \\ \mu^{v,1}(t) := \mu + \int_{(0,v]} \Phi(t-u)dH^v_u = \mu + \int_{(0,v)} \Phi(t-u)dH_u + \Phi(t-v). \end{cases}$$

The main result is the following **expansion formula** (see [2]): Assuming Z a bounded $\mathbb{F}$-predictable process, F a bounded $\mathcal{F}_T$-measurable random variable and $\|\Phi\|_1 < 1$. Then

$$\mathbb{E}\left[F\int_{[0,T]} Z_t dH_t\right] = \mu\int_0^T \mathbb{E}\left[Z_v F^v\right]dv$$
$$+\mu\sum_{n=2}^{+\infty}\int_0^T\int_0^{v_1}\cdots\int_0^{v_{n-1}}\prod_{i=2}^n \Phi(v_{i-1}-v_i)\mathbb{E}\left[Z_{v_1}^{v_n,\dots,v_2}F^{v_n,\dots,v_1}\right]dv_n\cdots dv_1.$$

The first term $\mu\int_0^T \mathbb{E}\left[Z_v F^v\right]dv$ corresponds to the formula for a Poisson process (setting the self-exciting kernel Φ at zero). The sum in the second term can be interpreted as a correcting term due to the self-exciting property of the counting process H. The shifted processes $H^{v_n,\dots,v_1}$ appearing in the form of the premium are of the same complexity than the original Hawkes process H. However, they exhibit deterministic jumps at some times $v_1,\dots,v_n$ which are weighted by correlation factors of the form $\Phi(v_i - v_{i-1})$. We benefit from this formulation to derive a lower and an upper bound respectively for the quantity $\mathbb{E}[K_T h(L_T)]$: by controlling the different types of jumps of the shifted Hawkes process, one can perform bounds that are more accurate than those available so far.

As an extension (still assuming $\|\Phi\|_1 < 1$), we indicate how this methodology combining Poisson imbedding and Malliavin calculus, can be used to provide new results on Hawkes processes such as

- **Explicit "Pseudo-Chaotic" expansion** (see [3])

$$H_T = \sum_{k=1}^{+\infty}\int_{\mathbb{X}^k}\frac{1}{k!}c_k(x_1,\dots,x_k)N(dx_1)\cdots N(dx_k),$$

$$\begin{cases} c_1(x_1) = \mathbf{1}_{\{\theta_1\le\mu\}}, \\ c_k(x_1,\dots,x_k) = D^{k-1}_{(x_1,x_2,\dots,x_{k-1})}\mathbf{1}_{\{\theta_k\le\lambda_{t_k}\}} \end{cases}$$

  where $\mathbb{X} := [0,T]\times\mathbb{R}_+$; $x := (t,\theta)$; $dx = d\theta dt$ and D is the Malliavin derivative $(D_x F) := F\circ\epsilon_x^+ - F$.

- **Explicit correlation of a general Hawkes process** (see [4]). For $s \le t$

$$Cov(H_s,H_t) = \mu\int_0^s\left(1+\int_0^v \Psi(w)dw\right)\left(1+\int_v^s\Psi(y-v)dy\right)\left(1+\int_v^t\Psi(y-v)dy\right)dv,$$

where $\Psi := \sum_{n=1}^{+\infty} \Phi_n$ and Φ_n are the iterated convolution of the excitation kernel $\Phi_1 := \Phi, \quad \Phi_n(t) := \int_0^t \Phi(t-s)\Phi_{n-1}(s)ds, \quad t \in \mathbb{R}_+,\ n \in \mathbb{N}^*$.

- **Quantitative TCL** (see [1]). "Berry Esseen" bounds Central Limit Theorems for the compound Hawkes process $L_T := \sum_{i=1}^{H_T} X_i$ (with X_i iid and independent of H) using Malliavin-Stein method (as in Nourdin Pecatti [6])

$$d_W\left(\frac{L_T - m\int_0^T \lambda_s ds}{\sqrt{T}}, G\right) \leq \frac{C_{\Phi,\nu}}{\sqrt{T}}, \quad \forall T > 0, \quad G \sim \mathcal{N}(0,\sigma^2),$$

with $m = \mathbb{E}(X)$ and $\sigma^2 = \frac{\mu\mathbb{E}(X^2)}{1-||\Phi||_1}$.

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Arbeitsgemeinschaft: Cluster Algebras

Organized by
Roger Casals, Davis
Bernhard Keller, Paris
Lauren Williams, Cambridge MA

8 October – 13 October 2023

Abstract. Cluster algebras, invented by Sergey Fomin and Andrei Zelevinsky around the year 2000, are commutative algebras endowed with a rich combinatorial structure. Fomin–Zelevinsky's original motivations came from Lie theory but in the past two decades, cluster algebras have had strikingly fruitful interactions with a large array of other subjects including Poisson geometry, discrete dynamical systems, (higher) Teichmüller spaces, commutative and non-commutative algebraic geometry, representation theory, … . In this Arbeitsgemeinschaft, we have focused on 1) basic definitions and theorems, 2) cluster structures on algebraic varieties and 3) the recent connection between cluster algebras and symplectic topology, with its recent application to the construction of cluster structures on braid varieties.

Mathematics Subject Classification (2020): 13F60, 53D10, 57K33.

Introduction by the Organizers

The Arbeitsgemeinschaft Cluster Algebras, organised by Roger Casals, Bernhard Keller and Lauren Williams, attracted excellent researchers of various backgrounds from all over the world, including many graduate students and postdocs. It was organized with 48 on-site and 12 online participants. As usual for an Arbeitsgemeinschaft, the organisers had provided a detailed program and had distributed the talks to the participants. We had a total of 16 talks of one hour each with ample time for discussion and additional sessions for recaps, questions and answers, discussions and software demonstrations from eight to ten in the evenings. On Wednesday afternoon, we made an excursion to St. Roman and on Thursday evening, Andreas Thom moderated the discussion and vote on the next Arbeitsgemeinschaft in this series.

In this Arbeitsgemeinschaft, we focused on three main subjects:

A. the basic theory of cluster algebras (5 talks)
B. the most important classical examples of cluster structures on varieties (5 talks) and
C. the recent interaction between cluster algebras and symplectic topology and its application to the construction of cluster structures on braid varieties (6 talks).

The talks in part A were devoted to the definition and first examples of cluster algebras, the classification of the cluster-finite cluster algebras (parametrized by the finite root systems), the basic techniques for constructing cluster structures on (homogeneous) coordinate algebras of varieties with the example of the Grassmannian, additional notions and results on cluster combinatorics and the family of cluster algebras constructed from marked surfaces.

Part B started with a talk on more advanced techniques for constructing cluster structures on varieties followed by talks on the combinatorics of plabic graphs and the associated positroid cells, on webs and the cluster structure on the Grassmannian of 3-dimensional subspaces, on double Bruhat cells and generalizations and finally on Fock–Goncharov's cluster ensembles, which provide a more symmetric, geometric framework for the whole theory.

Part C focused on developments in symplectic geometry that have either used cluster algebras or been used to study them. In particular, this last series of lectures aimed at developing the intuitions and techniques from symplectic geometry (following Casals, Weng, Pascaleff–Tonkonog, Gao–Shen–Weng, ...) and the microlocal theory of sheaves (Kashiwara–Schapira, ...) to complement the more algebraic and combinatorial methods often used to study cluster algebras. On the one hand, these lectures explained new results in the study of Lagrangian surfaces, including the detection of infinitely many Lagrangian fillings, via techniques from cluster algebras (after Casals–Gao and Casals–Weng). On the other, the combinatorics of weaves were also presented from their original symplectic geometric viewpoint and then applied to prove new results in the study of cluster algebras. To wit, the lectures showed that the coordinate rings of braid varieties, which arise as certain moduli of Lagrangian fillings and generalize Richardson varieties, are indeed cluster algebras (after Casals–Gorsky–Gorsky–Simental and Casals–Gorsky–Gorsky–Le–Shen–Simental). For lack of time, we did not cover the alternative, more combinatorial construction of such cluster structures on braid varieties due to Galashin–Lam–Sherman-Bennett–Speyer.

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Arbeitsgemeinschaft: Cluster Algebras

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Abstracts

A1–Introduction to cluster algebras : Definition and first examples

Théo Pinet

The principal references for this note are the pioneering work of Fomin–Zelevinsky [1, 3], their book [4] and Keller's paper [5]. The main goal of the note is to introduce the notion of *cluster algebra* associated to a valued/ice quiver and to illustrate this notion on examples, with in particular the example of the homogeneous coordinate algebra of the Grassmannian of planes in $(n+3)$-dimensional space. Informally, the cluster algebra associated to a quiver Q with n vertices is a subalgebra of the field of rational functions $\mathbb{F} = \mathbb{Q}(x_1, \ldots, x_n)$ whose generators, the cluster variables, are grouped in clusters of size n and are constructed recursively, starting from the initial seed $(Q, (x_1, \ldots, x_n))$, using mutations. Let us now make this more precise.

Given a *good quiver* $Q = (Q_0, Q_1, s, t)$ (i.e. a finite directed graph with no loops or 2-cycles) and a vertex $k \in Q_0$, we define another good quiver $\mu_k(Q)$ from Q by

(1) adding an arrow $i \to j$ for all paths of the form $i \to k \to j$ in Q,
(2) inverting all arrows of the form $i \to k$ and $k \to j$ in Q, and
(3) removing all 2-cycles created from steps (1) and (2).

The good quiver $\mu_k(Q)$ is called *the mutation of* Q *at* k. Note that $\mu_k(\mu_k(Q)) = Q$. For example, mutating the *Markov quiver* Q_M below at vertex 1, gives us a quiver isomorphic to Q_M. We thus say that the *mutation class* of Q_M is $\{Q_M\}$.

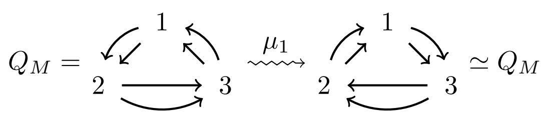

Figure 1. Example of quiver mutation with the Markov quiver Q_M.

Fix $n \in \mathbb{Z}_{\geq 0}$. A *seed* is a pair (Q, u) with Q a good quiver having n vertices and with $u = (u_1, \ldots, u_n) \in \mathbb{F}^n$ a sequence satisfying $\mathbb{F} = \mathbb{Q}(u_1, \ldots, u_n)$. Starting from a seed (Q, u) and a vertex $k \in Q_0$, the *mutated seed* $\mu_k(Q, u)$ *in direction* k is

$$\mu_k(Q, u) = (\mu_k(Q), u')$$

where $u' = (u_1, \ldots, u_{k-1}, u'_k, u_{k+1}, \ldots, u_n)$ with u'_k given by the *exchange relation*

$$u_k u'_k = \prod_{\substack{\alpha \in Q_1 \\ t(\alpha) = k}} u_{s(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ s(\alpha) = k}} u_{t(\alpha)}. \tag{1}$$

Fix now a good quiver Q with n vertices. A *cluster associated to* Q is a sequence $u' \in \mathbb{F}^n$ occuring in a seed (Q', u') that is linked to the *initial seed* $(Q, (x_1, \ldots, x_n))$ by a finite sequence of mutations. We call *cluster variables* the components of the clusters associated to Q and define the *cluster algebra $\mathcal{A}_Q$ corresponding to Q* as the subalgebra of $\mathbb{F}$ generated by all cluster variables. In other words,

$$\mathcal{A}_Q = \mathbb{Q}[\text{cluster variables associated to } Q] \subseteq \mathbb{F}.$$

Most cluster algebras, like the one associated to the Markov quiver, have infinitely many cluster variables. These algebras can thus be quite hard to describe explicitly. However, their complexity is somewhat limited by the theorem below, which is one of the most remarkable results proven during the early study of cluster algebras.

Theorem 1 (Laurent phenomenon, [1, 2])**.** *Fix $u' = (u'_1, \ldots, u'_n)$ a cluster of $\mathcal{A}_Q$. Then, the cluster variables of $\mathcal{A}_Q$ all live inside the ring $\mathbb{Z}[(u'_1)^{\pm 1}, \ldots, (u'_n)^{\pm 1}]$.*

In particular, the cluster algebra $\mathcal{A}_Q$ is contained in the algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{U}_Q$ where

$$\mathcal{U}_Q = \bigcap_{\substack{u'=(u'_1,\ldots,u'_n) \\ \text{cluster of } \mathcal{A}_Q}} \mathbb{Z}[(u'_1)^{\pm 1}, \ldots, (u'_n)^{\pm 1}]$$

is the *upper cluster algebra corresponding to Q*. Note nevertheless that $\mathcal{A}_Q \neq \mathcal{U}_Q$ in general since, for $Q = Q_M$ the Markov quiver, the Laurent polynomial

$$f(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}$$

belongs to $\mathcal{U}_Q$, but not to $\mathcal{A}_Q$ (see e.g. [4]).

Now, let us add *frozen nodes* $\{n+1, \ldots, m\}$ (with $m \geq n$) to our good quiver Q in order to obtain an *iced quiver of type* (n, m). These frozen vertices can connect to the original (i.e. unfrozen) vertices of our quiver Q in any way that do not create 2-cycles, but cannot be connected to another frozen vertex. Here is an example:

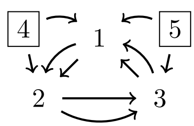

FIGURE 2. Example of iced quiver with 2 frozen nodes (indicated with a box) and an unfrozen part equal to the Markov quiver.

Given an iced quiver Q, we can define a cluster algebra $\mathcal{A}_Q$ exactly as above from the *initial seed* $(Q, (x_1, \ldots, x_m))$ by mutating at unfrozen vertices $\{1, \ldots, n\}$ (and at these vertices only). In this situation, the variables $x_{n+1}, \ldots, x_m$ belong to all clusters of $\mathcal{A}_Q$ and are called *coefficients* (instead of cluster variables). This slight generalization allows us to state the result below, again due to Fomin–Zelevinsky.

Theorem 2 ([2])**.** *Let X be a rational quasi-affine irreducible m-dimensional complex variety such that $\dim X = m$. Fix moreover Q an iced quiver of type (n, m). Suppose given functions φ_v and φ_{x_i} in the coordinate ring $\mathbb{C}[X]$ for all choices of cluster variables v of $\mathcal{A}_Q$ and all $n < i \leq m$. Suppose also that*

(i) *these functions altogether generate the coordinate ring $\mathbb{C}[X]$ and that*

(ii) *the map sending a cluster variable or a coefficient to the associated function sends exchange relations in $\mathcal{A}_Q$ to equalities in $\mathbb{C}[X]$.*

Then, the latter map extends to an algebra isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{A}_Q \simeq \mathbb{C}[X]$.

When the conditions in the above theorem are satisfied, we say that the coordinate ring $\mathbb{C}[X]$ *carries a cluster structure of type Q with initial seed* $\{\varphi_{x_i}\}_{i=1}^m$. For an example of such a situation, fix $m = n+3$ with $n \geq 1$ and denote by A the algebra of polynomial functions on the cone over the Grassmannian $\mathrm{Gr}_{2,m}(\mathbb{C})$ of planes in $\mathbb{C}^m$. Then, A is generated by the *Plücker coordinates* x_{ij} (with $1 \leq i < j \leq m$) which are subject to the *Plücker relations*

$$x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{i\ell}x_{jk} \tag{2}$$

whenever $1 \leq i < j < k < \ell \leq m$. Let now P be a m-gon with a fixed triangulation T. Then a well-known procedure (see e.g. [2, 4]) produces an iced quiver Q of type (n, m) from P and T. Here is an example with $m = 6$:

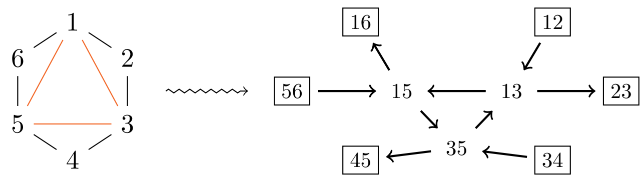

FIGURE 3. Iced quiver Q associated to hexagon P with triangulation T. Sides of P (diagonals of T) give frozen (resp. unfrozen) nodes, while arrows are obtained by turning in a counter-clockwise manner inside the triangles bounded by T (see e.g. [4]).

Theorem 3 ([2, 4])**.** *The algebra A carries a cluster structure with type the iced quiver Q above and with cluster variables (coefficients) the Plücker coordinates x_{ij} associated to diagonals (resp. sides) of P. Also, the clusters of A are the n-tuples of diagonals of P forming a triangulation and the exchange relations for the cluster algebra A (see* (1)*) are exactly the Plücker relations* (2)*.*

Let us at last finish this note by recalling that iced quivers of type (n, m) are in bijection with integral $m \times n$ matrices with skew-symmetric $n \times n$ top submatrix. Using this bijection, we can define the notion of *matrix mutation* which can in turn be generalized to the setting of integral $m \times n$ matrices having skew-symmetrizable $n \times n$ top submatrix. This then leads to *mutation for valued iced quivers* [4, 5].

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A2–Classification of cluster-finite cluster algebras

Kaveh Mousavand

This talk was a summary of the seminal work of Fomin and Zelevinsky [1] on the classification of those cluster algebras which admit only finitely many clusters. Such algebras are traditionally called of "finite type", and they are treated up to a suitable notion of isomorphism for cluster algebras. Before we recall the main ingredients and state the results, let us remark that there are other notions of finiteness in the study of cluster algebras (e.g. finite mutation type, or finitely generated cluster algebras, etc.) that are different from the problem considered in the talk. Also, we observe that in some textbooks, a commutative algebra is said to be of finite type if it is a quotient of a polynomial algebra in finitely many indeterminates. Unfortunately, this notion is different from the finiteness phenomenon treated in [1]. That is, there are examples of cluster algebras which are finite type as commutative algebras, but they admit infinitely many clusters (For instance, the coordinate algebras of maximal unipotent subgroups in [5], or any finitely generated cluster algebra with infinitely many clusters.). To avoid any confusion caused by the discrepancy in terminology, henceforth we adopt a less ambiguous term proposed by Benrhard Keller– one of the organizers of this Arbeitsgemeinschaft– and say that a cluster algebra is *cluster-finite* if it admits only finitely many clusters.

1. Notations, main ingredients and background

Here we only recall some standard terminology and notations that allow us to articulate the main problem and results. For detailed study of root systems, we refer to [4]. Moreover, all the required materials from cluster algebras that are used below can be found in [1].

Throughout, let Φ denote a finite irreducible crystallographic root system in the Euclidean space $\mathbb{R}^n$. It is known that, up to isometry and simultaneous rescaling of the vectors, Φ is uniquely determined by its Cartan matrix C_Φ, to which one can associate a unique Dynking graph. In particular, the Dynkin graphs of all finite irreducible crystallographic root systems are often denoted by A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, or G_2 (for details, see [2] and [4]). For Φ, and a fixed simple system Δ in Φ, by Φ^+ we denote the set of positive roots. Furthermore, the set of *almost positive roots* is defined as $\Phi_{\geq -1} := \Phi^+ \cup -\Delta$, where $-\Delta := \{-\alpha | \alpha \in \Delta\}$.

Now, we briefly recall the main ingredients of the most general construction of cluster algebras, as in [1]. Let $\mathbb{P}$ be a semifield, and by F denote the field of rational functions in n indeterminates with cooeficients in $\mathbb{ZP}$. This will be the ambient field containing the cluster algebra $\mathcal{A}$ of rank n, described below. Every *seed* in

F is a triple $\Sigma = (\underline{x}, \underline{p}, B)$, where $\underline{x}$ is called a *cluster*, consisting of n elements in F. These elements are known as the *cluster variables* and form a free generating set for a field extension over the field of fractions of $\mathbb{ZP}$ in F. Moreover, the coefficient $\underline{p} = (p_x^{\pm})_{x\in\underline{x}}$ is a 2-tuple of elements in $\mathbb{P}$ satisfying the normalization condition $p_x^+ \oplus p_x^- = 1$. Here, $\oplus$ denotes the auxiliary addition in the semifield $\mathbb{P}$. Finally, $B = (b_{xy})_{x,y\in\underline{x}}$ denotes a sign-skew-symmetric matrix whose rows and columns are indexed by the cluster variables in $\underline{x}$. Namely, for all $x, y \in \underline{x}$, either $b_{xy} = b_{yx} = 0$, or else $b_{xy}b_{yx} < 0$. Through the explicit formulas in section 1 of [1], one can mutate the seed $\Sigma = (\underline{x}, \underline{p}, B)$ in all n directions, that is, to simultaneously mutate the cluster $\underline{x}$, the coefficient $\underline{p}$, as well as the matrix B.

Starting from an initial seed Σ, perform all possible mutations on Σ, and then iterate this procedure at every output obtained in each step. This iteration may terminate after only finitely many steps, that is, we get no new seeds after a finite number of mutations, or else one can mutate and produce infinitely many different seeds. Let $\mathcal{S}$ denote the set of all seeds in F obtained via all possible iterations of mutations starting from Σ. By $\mathcal{X}$ and $\mathcal{P}$, respectively denote the set of all cluster variables and the set of all coefficients in the seeds belonging to $\mathcal{S}$. Let $\mathbb{Z}[\mathcal{P}]$ denote the subring of F generated by $\mathcal{P}$. Then, the normalized *cluster algebra* $\mathcal{A}$ is the $\mathbb{Z}[\mathcal{P}]$-subalgebra of F generated by $\mathcal{X}$. As shown in [1], $\mathcal{A}$ can be studied up to strong isomorphism of cluster algebras. More precisely, over a fixed semifield $\mathbb{P}$, if F and F' are two ambient fields as above, and $\mathcal{A} \subset F$ and $\mathcal{A}' \subset F'$ are two cluster algebras, then $\mathcal{A}$ and $\mathcal{A}'$ are *strongly isomorphic* if there exists a $\mathbb{Z}[\mathcal{P}]$-algebra isomorphism between F and F' which additionally transports any seed in F to F'. Such an isomorphism induces an algebra isomorphism between $\mathcal{A}$ and $\mathcal{A}'$ which preserves the cluster structure. We remark that, even over a fixed semifield, an arbitrary $\mathbb{Z}[\mathcal{P}]$-algebra isomorphism between two cluster algebras is not necessarily a strong isomorphism. In fact, there exist $\mathbb{Z}[\mathcal{P}]$-algebras which admit two different cluster structures that are not strongly isomorphic (for explicit examples, see [3]).

2. Main results

Before we state the first theorem, let us recall that for an arbitrary $n \times n$ integer square matrix $B = (b_{ij})$, the *Cartan counterpart* of B, which we denote by $C_B = (c_{ij})$, is defined by putting $c_{ij} := 2$, if $i = j$, and $c_{ij} := -|b_{ij}|$, otherwise. Observe that C_B is not necessarily a Cartan matrix, but it is a generalized Cartan matrix. Now, we are ready to state the first main result. Throughout, we use the notations and terminology introduced above.

Theorem 1. *Fomin-Zelevinsky [1]: Let $\Sigma = (\underline{x}, \underline{p}, B)$ be a seed in F such that $b_{xy}b_{xz} \geq 0$, for all x, y and z in $\underline{x}$. If the Cartan counterpart of B is the Cartan matrix C_Φ of a finite root system Φ, then $\mathcal{A}$ is cluster-finite. Conversely, up to strong isomorphism, every cluster-finite cluster algebra is of the above form, that is, it admits a seed with the aforementioned properties.*

By the preceding theorem, if the cluster algebra $\mathcal{A}$ of rank n is cluster-finite, a unique finite root system Φ in $\mathbb{R}^n$ is associated to $\mathcal{A}$. Consequently, $\mathcal{A}$ is called of type Φ, and has the corresponding Dynkin graph with n vertices. For a more detailed treatment of cluster-finite cluster algebras from this viewpoint, see [2].

The second main result is the following theorem which gives equivalent characterizations of cluster-finite cluster algebras, and further describes the connection between their cluster variables and certain roots in the corresponding root system.

Theorem 2. *Fomin-Zelevinsky [1]: For any cluster algebra $\mathcal{A}$, the following are equivalent:*

(1) *$\mathcal{A}$ is cluster-finite;*
(2) *$\mathcal{A}$ admits finitely many cluster variables, that is, $\mathcal{X}$ is a finite set;*
(3) *In every seed $\Sigma = (\underline{x}, \underline{p}, B)$ in $\mathcal{S}$, we have $|b_{xy}b_{yx}| \leq 3$, for all $x, y \in \underline{x}$.*

That being the case, let Φ be the root system of $\mathcal{A}$ and $\underline{x}_0 = (x_1, \cdots, x_n)$ the initial cluster. Then, there is a unique bijection between the almost positive roots in Φ and the cluster variables in $\mathcal{X}$, expressed in terms of $\underline{x}_0$. More precisely, if $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ is a simple system in Φ, for each $\alpha \in \Phi_{\geq -1}$, the corresponding cluster variable is $x[\alpha] = \frac{P_\alpha(\underline{x}_0)}{x^\alpha}$, where $P_\alpha(\underline{x}_0)$ is a polynomial over $\mathbb{ZP}$ in terms of cluster variables in $\underline{x}_0$ and has a non-zero constant term, and x^α is the monomial defined as $x^\alpha = x_1^{c_1} \cdots x_n^{c_n}$, where $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$. In particular, $x[-\alpha_i] = x_i$.

We end with some remarks on the above theorem and the more recent results on the cluster-finite cluster algebras obtained after their original treatment in [1].

First, observe that the implication (1) $\to$ (2) in the preceding theorem follows from the definition, but the converse is far from trivial. In particular, a finite set of cluster variables could *a priori* appear in infinitely many clusters that belong to different seeds in $\mathcal{S}$. However, the above theorem says this never happens. Second, note that part (3) gives an explicit condition in terms of entries of the matrices of each seed. However, we remark that one should verify this condition for all seeds in $\mathcal{S}$ to conclude that $\mathcal{A}$ is cluster-finite. In fact, there are cluster algebras which are not cluster-finite, but they admit a seed which satisfies condition (3). Third, with regard to the correspondence between the almost positive roots and the cluster variables of cluster-finite cluster algebras, we remark that an elegant construction is given by Keller [7], where one can begin from the initial cluster variables and through a concrete knitting algorithm recover the aforementioned bijection between the almost positive roots and all cluster variables. Finally, we note that some other conceptual characterizations of cluster-finite cluster algebras have been achieved after their first appearance in [1]. In particular, in [6] it is shown that a cluster algebra $\mathcal{A}$ is cluster-finite if and only if the set of cluster monomials forms an additive basis for $\mathcal{A}$.

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A3–The cluster structure of the Grassmannian coordinate algebra

Liana Heuberger

One of the first examples where a coordinate ring admits a cluster algebra structure in a non-trivial way arises in the case of the affine cone over the Grassmannian. During the proof of this result, we encounter a fundamental tool in cluster algebra theory: the celebrated *starfish lemma*. This talk showcases the power of the lemma by applying it to a familiar, yet nontrivial context.

The Grassmannian of a-subspaces of a b-dimensional $\mathbb{C}$-vector space is one of the first projective varieties one encounters in geometry beyond projective spaces themselves. Its homogeneous coordinate ring, also known as the *Plücker ring* has been extensively studied and is known to be generated by *Plücker coordinates*. Expressing this ring in terms of $\mathrm{SL}_a(\mathbb{C})$-invariant polynomials allows us to understand the Plücker coordinates as $a \times a$ minors of an $a \times b$ matrix.

There exist two known constructions of the cluster algebra structure of this ring, the first of which appeared in the work of Scott [1]. Scott chooses a seed whose cluster variables are themselves Plücker coordinates, and such that the one-step mutation at each variable yields a cluster variable which is again a Plücker coordinate. The combinatorial setup of this method, involving *alternating strand diagrams*, is less self-contained than that of the alternative construction of Fomin, Williams and Zelevinsky [2], whose proof we chose to present throughout this talk.

The seed chosen in [2] is formed of distinguished Plücker coordinates whose respective Young tableaux are *rectangles*. More precisely, one can associate a Plücker coordinate to any sub-rectangle of an $a \times (b-a)$ rectangle in a unique way, and we choose this set of coordinates as the seed of our cluster algebra. The frozen variables correspond to those coordinates with consecutive indices, while the remainder are cluster variables.

The proof involves a double inclusion: one has to prove that each mutation of this distinguished seed remains in the Plücker ring (as opposed to its fraction field), and conversely that every Plücker coordinate is generated by a subsequent mutation.

The first implication relies on the starfish lemma, which roughly guarantees that if one starts from a polynomial seed whose one-step mutations produce polynomial cluster variables, then the same holds for all subsequent mutations. For this distinguished seed, we no longer obtain Plücker coordinates after one-step mutations, yet we are still able to control the behaviour of the new cluster variables:

this achieved by combining well-known Plücker relations between the variables of the distinguished seed, and the exchange relations of the mutations. We show that the one-step cluster variables are indeed polynomial, thereby concluding the first half of the proof.

For the second implication, Fomin, Williams and Zelevinsky have an inductive approach via the *Muir embedding*. More specifically, one can embed rectangular quivers of smaller size inside a fixed rectangular quiver and use the inductive hypothesis to obtain some (but not all) Plücker coordinates. They then use *cyclic shifts*, shown to be mutations of the distinguished seed, to obtain the outstanding coordinates and the proof concludes.

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C1–Introduction to Lagrangian fillings

Yu Pan

Symplectic and contact geometry, rooted from classical mechanics, has experienced a rapid development in the last forty years. It mainly concerns manifolds with additional geometrical structures called symplectic and contact manifolds and special knots and surfaces in them called Legendrian knots and exact Lagrangian fillings.

A symplectic manifold is an even dimensional manifold with a non-degenerate closed 2-form. An example is the cotangent space $\mathbb{R}^4 = T^*\mathbb{R}^2, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Note that this symplectic manifold is also exact, i.e., $\omega = d\lambda$ (in the example $\lambda = -q_1 dp_1 - q_2 dp_2$). An odd dimensional counterpart is called contact manifold, which is an odd dimensional manifold with a contact structure given by the kernel of a 1-form α such that $\alpha \wedge d\alpha^n \neq 0$. An example of a 3-dimensional contact manifold is $\mathbb{R}^3_{std} = (\mathbb{R}^3, ker\ \alpha)$ where $\alpha = dz - ydx$. Darboux theorem shows that every symplectic (contact) manifold locally are the same. Therefore it is more interesting to explore the global geometrical (i.e., topological) properties of symplectic/contact manifolds.

For similar reason as the one for knots and surfaces being essential in low dimensional topology, it is also important to consider special knots and surfaces in contact and symplectic manifolds that cooperate well with the additional geometrical structures. These knots and surfaces are called Legendrian knots and exact Lagrangian surfaces.

In particular, a Legendrian knot $\Lambda \in \mathbb{R}^3_{std}$ in $(\mathbb{R}^3, ker\ \alpha)$ is a knot in $\mathbb{R}^3$ such that α vanishes on it. An important way to visualize it is through front projection $\Pi_F : \mathbb{R}^3 \to \mathbb{R}^2_{xz}$. Note that we do not loose information in the front projection of a Legendrian knot Λ since the y-coordinate can be recovered through $y = \frac{dz}{dx}$ (since the 1-form α vanishes on Λ). One can see the example of front projections of an unknot and a trefoil in Figure 1. As a generalization of the trefoil, the (-1)

closure of a positive braid β is sketched in the Figure 1 (c). This will be the main example of Legendrian links we will focus on in the latter C-lectures.

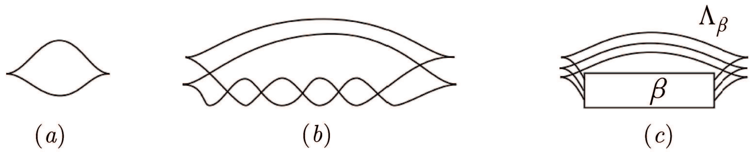

FIGURE 1. Front projections of unknot (a), trefoil (b) and (-1)-closure of positive braid β (c).

An exact Lagrangian filling L of $\Lambda \in \mathbb{R}^3_{std}$ in $(\mathbb{R}^4, \omega = d\lambda)$ is an embedded surface L bounded by Λ such that $\lambda|_{TL}$ is exact. The exact Lagrangian condition imposes strong rigidity on exact Lagrangian fillings. One evidence is that once a Legendrian knot has an exact Lagrangian filling, then the genus of the filling is fixed (differently compared with topological fillings, in which case the genus can increase freely), which is the 4-ball genus of the knot.

An essential question in symplectic geometry is that given a Legendrian knot in $\mathbb{R}^3_{std}$, how many exact Lagrangian fillings does it have in $\mathbb{R}^4$. Currently, the only known case is the maximum Thurston-Bennequin number (max-tb) unknot. By Eliashberg and Polterovich, the max-tb unknot has a unique exact Lagrangian filling. Note that the max tb condition is a necessary condition for a Legendrian to bound an exact Lagrangian filling. For the next easiest example, which is the Legendrian max-tb trefoil, which is also the (-1)-closure of a positive $(2,5)$ braid, we introduce a way to build exact Lagrangian fillings of it through concatenating elementary blocks together. The construction gives 5 exact Lagrangian fillings that are smoothly isotopic but are not Hamiltonian isotopic. This will match with the A_2 cluster structure will introduce in latter lectures for the positive $(2,3)$ braid.

As to other Legendrians, Casals and Gao in 2020 showed that (-1) closure of positive $(m, n+m)$ braids (which is a topological (m,n)-torus link), for $n \geq 3, m \geq 6$ or $(m,n) = (4.4), (4,5), (5,5)$, all have infinitely many exact Lagrangian fillings. This is essentially because of the fact that the positive (m,n) braid correspond to some cluster algebra of infinite type.

The goal of the C-lectures is to build connection of "the space of exact Lagrangian fillings of the Legendrian (-1)-closure of a positive braid β" with a cluster algebra so that we can use the cluster algebra structure to understand the geometrical space better. In particular, each exact Lagrangian filling has an $\mathbb{L}$-compressing disk system that corresponds to a quiver. The Lagrangian surgery operation that changes one exact Lagrangian filling to another corresponds to a mutation.

A4–More Cluster Combinatorics: g-vectors, c-vectors, F-polynomials

Merik Niemeyer

The goal of this talk was to deepen our understanding of cluster combinatorics by introducing c- and g-vectors, as well as F-polynomials. These come from a certain choice for the frozen part of the quiver, but contain enough information to reconstruct both the cluster variables and the y-variables of any cluster algebra associated to an ice quiver with the same mutable part. Moreover, we looked at some tropical dualities due to Nakanishi-Zelevinsky, which establish remarkable connections between c- and g-vectors. The talk largely followed Keller's survey paper [3].

1. Preparation

In the previous talks we have seen quivers and their corresponding exchange matrices, as well as ice quivers, which contain some frozen nodes, and can be described by extended exchange matrices. If the cluster variables of the initial seed are $x_1, ..., x_n$, and the frozen variables are $x_{n+1}, \dots, x_m$, every cluster variable will be a Laurent polynomial in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}[x_{n+1}, \dots, x_m]$. In order to phrase some of the results in the language of our reference material, let us slightly change perspective and set

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}} \in \mathrm{Trop}(x_{n+1}, \dots, x_m)\,,$$

for $1 \leq j \leq n$, where $\mathrm{Trop}(x_{n+1}, \dots, x_m)$ denotes a certain tropical semifield. These y-variables follow a 'tropical' mutation rule and capture how the frozen nodes are attached to the mutable nodes of the quiver. Therefore instead of keeping track of the extended exchange matrix and the cluster variables as we mutate, we can take the (principal part of the) exchange matrix, the cluster variables and the y-variables. This data constitutes a *seed*. Now, pick a vertex t_0 of the labeled n-regular tree $\mathbb{T}_n$, assign the initial seed to it, and then assign the seed mutated according to the edge labelling to the neighbouring vertices. Inductively, we obtain the *seed pattern*.

2. c-vectors, g-vectors and F-polynomials

2.1. **Definitions.** Let Q be a quiver (without frozen nodes), with nodes labelled $1, ..., n$. We first add frozen nodes in a particular way:

Definition 1. The *principal extension* Q_{pr} of Q is the quiver obtained from Q by adding nodes i' for $1 \leq i \leq n$ and arrows $i' \to i$.
The *cluster algebra with principal coefficients* associated to Q is the cluster algebra associated to Q_{pr}.

Let B be the exchange matrix of Q, then the extended exchange matrix of Q_{pr} is given by

$$\tilde{B} = \begin{pmatrix} B \\ \mathrm{Id}_n \end{pmatrix}.$$

This mutates according to the rules of matrix mutation, and thus we assign a matrix $\tilde{B}(t)$ to every vertex $t \in \mathbb{T}_n$, which has the form

$$\tilde{B}(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}.$$

Definition 2. The matrix $C(t)$ is the *matrix of c-vectors*, its columns are the *c-vectors* $c_j(t)$, $1 \leq j \leq n$.

Theorem 3. *Every c-vector is non-zero and its entries are either all non-negative or all non-positive.*

This appeared implicitly as a conjecture in [1] and was proved in full generality in [2].

Next, we define F-polynomials: Recall again, that every cluster variable $x_j(t)$, for $1 \leq j \leq n$ and $t \in \mathbb{T}_n$, is a Laurent polynomial in the initial cluster variables, with coefficients in $\mathbb{Z}[x'_1, ..., x'_n]$, where x'_i denotes the (frozen) variable associated to the node i'.

Definition 4. Let $1 \leq j \leq n$ and $t \in \mathbb{T}_n$. The *F-polynomial* $F_j(t) \in \mathbb{Z}[x'_1, ..., x'_n]$ is obtained by specializing $x_j(t)$ to $x_1 = ... = x_n = 1$.

In the original paper [1], Fomin and Zelevinsky prove that any F-polynomial is a ratio of two polynomials with positive integer coefficients, which implies that it can be evaluated in any semifield (we now know that every F-polynomial is in fact a polynomial with positive integer coefficients [2]). Moreover, they conjectured the following theorem, which is equivalent to the sign property of c-vectors given above.

Theorem 5. *Every F-polynomial has constant term 1.*

The final object we need to introduce are the g-vectors, which we obtain by endowing the Laurent ring $\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}, x'_1, ..., x'_n]$ with the following $\mathbb{Z}^n$-grading:

$$\deg(x_i) = e_i,$$
$$\deg(x'_i) = -Be_i,$$

for $1 \leq i \leq n$, where e_i denotes the i-th standard vector. Fomin and Zelevinsky proved that any cluster variable $x_j(t)$ is homogeneous with respect to this grading, allowing us to define:

Definition 6. Let $t \in \mathbb{T}_n, 1 \leq j \leq n$. The *$g$-vector* $g_j(t)$ is defined as

$$g_j(t) = \deg(x_j(t)).$$

The g-vectors are the columns of the *matrix of g-vectors*, denoted $G(t)$.

Again, we have a theorem which is equivalent to the two we gave previously:

Theorem 7. *The g-vectors are sign-coherent, meaning that for any $t \in \mathbb{T}_n$ every row of the matrix $G(t)$ is non-zero and either has only non-negative or only non-positive entries.*

As we had seen in previous talks, the cluster variables and entries of the exchange matrix are obtained recursively from the initial data via mutation. Consequently, one can deduce recursive formulas for all the above objects, and we gave an idea of how to do that.

2.2. **Separation formulas.** With all of this in place, we can reobtain both cluster and y-variables. These formulas are due to Fomin-Zelevinsky [1].

Theorem 8. *Let $t \in \mathbb{T}_n$, $\mathbb{P}$ any (coefficient) semifield, and $\mathcal{F} = \mathbb{Q}(\mathbb{P})(x_1, \ldots, x_n)$ the ambient field.*

(a) $y_j(t) = y_1^{c_{1j}(t)} \cdots y_n^{c_{nj}(t)} \prod_{i=1}^n F_i(t)|_{\mathbb{P}}(y_1, \ldots, y_n)^{b_{ij}(t)}$,
(b) $x_j(t) = x_1^{g_{1j}(t)} \cdots x_n^{g_{nj}(t)} \frac{F_j(t)|_{\mathcal{F}}(\hat{y_1}, \ldots, \hat{y_n})}{F_j(t)|_{\mathbb{P}}(y_1, \ldots, y_n)}$, *where* $\hat{y_j} = y_j \prod_{i=1}^n x_i^{b_{ij}}$.

Let us stress that this allows us to compute the cluster variables and coefficients for any cluster algebra just using the data obtained from the corresponding cluster algebra with principal coefficients.

3. Tropical dualities

Finally, we saw some tropical dualities, due to Nakanishi and Zelevinsky [4], which relate c- and g-vectors in various ways. To state these, we need to upgrade our notation slightly. We write $C(B, t_0, t)$ for the matrix of c-vectors obtained by starting with the exchange matrix B at $t_0 \in \mathbb{T}_n$ and mutating to t, and analogously for the matrix of g-vectors.

Theorem 9. *Let B be a skew-symmetrizable exchange matrix, $t_0, t \in \mathbb{T}_n$. Then:*

(a) $G(B, t_0, t)^T = C(-B^T, t_0, t)^{-1}$,
(b) $C(B, t_0, t) = C(-B(t), t, t_0)^{-1}$,
(c) $G(B, t_0, t) = G(-B(t), t, t_0)^{-1}$.

The c-vectors appearing in formula (a) belong to the *Langlands-dual* quiver which is obtained by replacing the exchange matrix B with $-B^T$.

In the last five minutes of the talk, we defined the notion of maximal green and reddening mutation sequences, notions which were further discussed in the evening session.

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A5–Cluster algebras from surfaces

Kayla Wright

Endowing mathematical objects with a cluster structure boomed after the axiomatization of cluster algebras by Fomin and Zelevnisky in the early 2000's. In this talk, we will explain how to endow a topological marked surface (S, M) with a cluster structure. Namely, we take a Riemannian orientable surface S with nonempty boundary and a finite set of marked points M on the boundary of S such that each boundary component contains at least one marked point. We triangulate (S, M) by drawing arcs between the marked points so that they are maximally non-crossing up to isotopy relative to the boundary. For example, if we take S to be a hexagon and M to be its 6 vertices, we triangulate S by drawing three non-crossing diagonals.

With this topological set up, we see beautiful bijections between arcs and cluster variables, triangulations and clusters, and skein relations and cluster mutation. This story can be further enhanced when incorporating the geometry of Teichmüller theory. Namely, if we look at the space of certain hyperbolic metrics on (S, M) and properly define lengths of geodesics on the surface, we are able to see a cluster structure on Teichmüller space, denoted $\mathcal{T}(S, M)$. More specifically, if we fix a metric in Teichmüller space and a choice of small circle around each marked point $m \in M$, we can define the length of a geodesic between marked points m, m' on (S, M) as the signed distance between the circles around m and m'. These small circles are called horocycles and the choice of horocycle at each marked point gives the data of decorated $\widetilde{\mathcal{T}(S, M)}$. We coordinatize this decorated version of Teichmüller space with Penner coordinates, also known as λ-lengths, which are an exponential version of the above defined length. These λ-lengths satisfy Ptolemy's Theorem which is the geometric version of the skein relations from the topological set-up. Altogether, this means that decorated Teichmüller space has a cluster structure, wheres cluster variables are in bijection with geodesics and cluster mutation is given by this hyperbolic version of Ptolemy's Theorem.

C2–Fronts and Lagrangian fillings of Legendrian links

Agniva Roy

The references for this talk are Section 4 of [1], and Sections 2 and 7 of [2].

1. Demazure weave fillings of positive braid closures

Definition 1 (Demazure Product)**.** Given a positive braid word β, the Demazure product of β, denoted $\delta(\beta)$, is the braid that corresponds to quotienting out the braid word using the relations $\sigma_i^2 = \sigma_i$, and also braid relations.

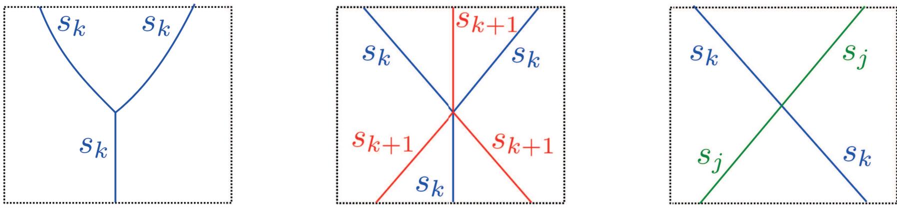

FIGURE 1. The figure is courtesy of the authors of [1].

Example. Given the word $\sigma_1^2\sigma_2^2$ representing a 3-stranded braid, the Demazure product is the braid $\sigma_1\sigma_2$.

In this section we describe an algebraic procedure that takes as input a positive braid β and outputs the braid corresponding to $\delta(\beta)$, the Demazure product of β. We will encode the braid purely by its crossings, as follows, and the three allowable moves will be *braid commutations*, *pinching a crossing* and a *braid move*, as shown in Figure 1.

A positive braid will be represented by encoding each Artin generator by a colour; thus an N-stranded braid with Demazure product w_0 will need $N-1$ colours. Then, the algorithm to build a Demazure weave proceeds by using commutations, braid moves and pinch moves to eliminate all powers of generators till we are left with just the Demazure product. Typically, we will use the moves to isolate the Demazure product on one side and then use pinch and braid moves successively to remove the powers of generators one by one.

The result of this procedure is called the **Demazure weave**. In Section 2, we will show how this algebraic procedure builds an exact Lagrangian filling for the (-1)-closure of the braid $\beta\delta(\beta)$.

Example. We give an example, see Figure 1, of the procedure using a 3-stranded braid $\beta = \sigma_1\sigma_2^2\sigma_1^2\sigma_2$. This example will not see any commuting relations being used. In this picture, we use blue to represent σ_1 and red for σ_2. The Demazure product of β is $\delta(\beta) = \sigma_1\sigma_2\sigma_1$.

We will interpret these diagrams as being properly embedded in a 2-disk, and call them N-graphs.

2. Legendrian surfaces from weaves

Given an N-graph G on D^2, one can construct an immersed surface in $\mathbb{R}\times D^2$, which is the front projection of a Legendrian surface $\Lambda(G)$ in $J^1(D^2)$ by weaving as follows. The objective is to create an immersed surface that projects to D^2, whose singularities are encoded by the N-graph:

- start with N sheets over D^2
- for every $(i, i+1)$-edge, introduce a line A_1^2-singularities between the corresponding two sheets

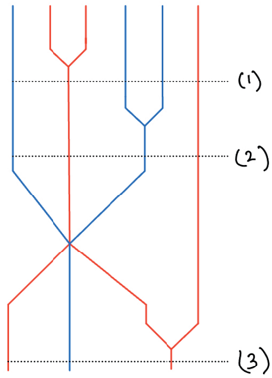

FIGURE 2. At level (1), we see the Demazure product on the left. At levels (2) and (3) respectively, we see the blue and red generators on the right being pinched so that at the end, we are left with $\delta(\beta)$.

- for every hexavalent vertex, introduce an A_1^3-singularity between the corresponding triple of consecutive sheets
- for every trivalent vertex, introduce a D_4^--singularity between the corresponding two sheets

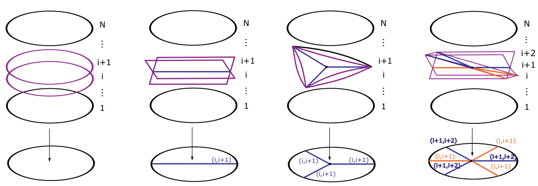

FIGURE 3. The weaving of singularities of fronts along the edges of the N-graph (courtesy of Roger Casals and Eric Zaslow). Gluing these local models according to the N-graph Γ yields the weave $\Lambda(\Gamma)$.

Some topological properties of the resulting surface: $\Lambda(G)$ is an N-fold branched cover over D^2 simply branched over the trivalent vertices of G.

(1) Euler characteristic $-\chi(\Lambda(G)) = N\chi(D^2) - v(G)$ where v is the number of trivalent vertices
(2) 1-cycles correspond to Y-trees. This is indicated in Figure 4.

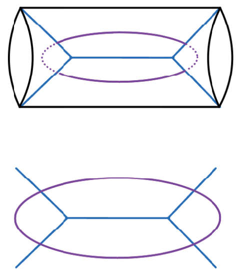

Figure 4. An I-tree corresponds to a cycle in the Legendrian surface, called an I-cycle.

Definition 2. An N-graph is called $free$ if the corresponding Legendrian weave has no Reeb chords.

The Demazure weaves built in Section 1 are free, hence the projection from $\mathbb{R}^5$ to $\mathbb{R}^4$ is embedded, as the only double points that could show up are due to Reeb chords. Also, by construction, the surfaces in $\mathbb{R}^4$ are exact Lagrangian, hence this procedure now produces an exact Lagrangian filling of the (-1)-closure of $\beta\delta(\beta)$ for any positive braid word β.

3. Quivers from Weaves

Associated to a Demazure weave, we can build a quiver that encodes the 1-cycles on the graph and their pairwise intersections. Further, there is a mutation operation one can do on the 1-cycles that show up in the weave to create another exact Lagrangian filling for the same braid, which may or may not be equivalent (up to Hamiltonian isotopy) to the previous one. We show how to do this in case of 2-weaves, i.e. weaves corresponding to 2-graphs, i.e. with only one colour. Firstly, given any 2-graph, encode all the I-trees as vertices on the quiver. Then, add arrows from every cycle to cycles that share a vertex with them, with arrows going from a cycle to one that is counter-clockwise of them.

Example. Consider the trefoil knot $T(2,3)$. It is the (-1)-closure of the braid σ_1^5, which we can consider to be $\beta\delta(\beta)$ for $\beta = \sigma^4$. We can see two I-cycles in the Demazure weave, and can build the A_2-quiver from them as shown in Figure 5. Mutating at an I-cycle corresponds to a local $I - H$ move.

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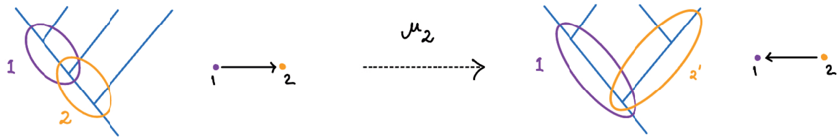

Figure 5. The Demazure weave and its mutation along the cycle denoted by 2, for the trefoil $T(2,3)$. These correspond to distinct exact Lagrangian fillings of the trefoil knot.

B1–Techniques for constructing cluster structures on varieties

Colin Krawchuk

In recent years, cluster structures have been discovered on the coordinate rings of many varieties, including open positroid varieties [8, 9, 4], double Bott-Samelson varieties [10] and braid varieties [5, 2]. The presence of a cluster structure has important implications for the geometry of an algebraic variety, including the existence of canonical linearly independent sets of regular functions.

It is therefore natural to ask how one might determine if a given variety inherits a cluster algebra structure. Identifying such a structure involves constructing an initial seed of regular functions, showing each cluster variable in the associated algebra is indeed a regular function, and showing that the cluster variables generate the coordinate ring of the variety. While there is no general method for this procedure, we recount several useful techniques that have been successfully applied to construct cluster structures on varieties.

One of the most useful criteria for showing that the cluster variables arising from a candidate seed are regular functions is the Starfish Lemma:

Lemma 1. [1, Starfish Lemma] Let $\mathcal{R} = \mathbb{C}[X]$ be the coordinate ring of an irreducible normal affine complex algebraic variety X. Let $(Q, \tilde{x})$ be seed of rank n in $\mathbb{C}(X)$ with $\tilde{x} = (x_1, \ldots, x_m)$ for $n \leq m$ whose variables lie in $\mathcal{R}$ such that

(1) the cluster variables in $\tilde{x}$ are pairwise coprime,
(2) for each cluster variable $x_k \in \tilde{x}$, the seed mutation μ_k replaces x_k with an element x'_k that lies in $\mathcal{R}$ and is coprime to x_k.

Then $\mathcal{A}(Q, \tilde{x}) \subset \mathcal{R}$.

The proof of the Starfish lemma relies on Hartogs' principle (showing that a function on X which is regular outside a subset of codimension 2 is regular everywhere). Under the conditions of the lemma, this property is satisfied not just for cluster variables but for elements of the upper cluster algebra of $\mathcal{A}(Q, \tilde{x})$.

To demonstrate the converse, that the cluster variables generate the coordinate ring of the variety, a frequent strategy is to first show that $\mathcal{A}(Q, \tilde{x})$ coincides with its upper cluster algebra. There are several reasons why this approach is beneficial. Often it is easier to show that regular functions on the variety are generated by elements of the upper cluster algebra than arbitrary cluster variables. Moreover, if we wish to apply the Starfish Lemma then this equality must hold in order for $\mathcal{A}(Q, \tilde{x})$ to be a cluster structure on $\mathbb{C}[X]$. On the other hand, cluster algebras that do not equal their upper cluster algebra are often unwieldy, and it can be challenging to show containment in these cases.

For these reasons, criteria for $\mathcal{A}(Q, \tilde{x})$ to be equal to its upper cluster algebra have been introduced by several authors. In [6] Muller introduced the class of locally acyclic cluster algebras, which admit a finite cover by certain simpler cluster algebras (called acyclic cluster localisations). A consequence of this definition is that any local property of acyclic cluster algebras is true of locally acyclic cluster algebras. In particular, we have the following useful result:

Theorem 2. [6] *If a cluster algebra is locally acyclic, then it coincides with its upper cluster algebra.*

Any locally acyclic cluster algebra $\mathcal{A}$ also inherits a covering of $Spec(\mathcal{A})$ by open subvarieties corresponding to cluster localisations. In [7] Muller and Speyer refined this idea by defining Louise cluster algebras that have the additional property that the cluster localisations associated to this covering satisfy a Mayer-Vietores decomposition of cluster algebras. As an application, they showed the following:

Theorem 3. [7] *Cluster algebras associated to Postnikov diagrams in the disk are Louise.*

Unfortunately, the definition of locally acyclic cluster algebras does not suggest a method to check whether a given cluster algebra possesses this property. However, if the quiver of a seed $(Q, \tilde{x})$ belongs to a class of quivers called Banff quivers, then the corresponding cluster algebra $\mathcal{A}(Q, \tilde{x})$ is locally acyclic [6]. Moreover, a recursive algorithm is given in [6] for checking if a quiver is indeed Banff. Similarly, the class of sink-recurrent quivers is defined in [5] and seeds with sink-recurrent quivers are shown to give rise to locally acyclic cluster algebras. Notably, this fact was used by the authors to prove that cluster algebras arising from 3D-Plabic graphs are locally acyclic.

A final strategy for showing that $\mathcal{A}$ coincides with its upper cluster algebra relies on quasi-homorphisms between cluster algebras in the sense of Fraser [3]. In particular, if the elements of a generating set for the upper cluster algebra belong to either $\mathcal{A}$ or a quasi-equivalent cluster algebra, then $\mathcal{A}$ coincides with its upper cluster algebra. This approach was taken in [2] where it is shown that cyclic rotations of braid words induce quasi-cluster transformations.

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B2–Combinatorics of plabic graphs

Peter Spacek

We introduced *plabic graphs*: *planar, bicolored graphs* properly embedded into the closed disk with b (uncolored) vertices on the boundary. (Loops and multiple edges are allowed.) We also defined *move-equivalence* of plabic graphs, i.e. two graphs are move-equivalent if they are related by the *square move* (exchanging colors on a square with alternatingly colored vertices), the *creative/destructive move* (inserting a colored vertex on an edge or removing a bivalent vertex), and finally the *(de)construction move* (merging two vertices of the same color connected by an edge or splitting a colored vertex into two connected by an edge).

We then discussed how to construct a quiver associated to a given plabic graph: a vertex for each face (if a face borders the boundary of the disk, the corresponding vertex is frozen), and an arrow between vertices for each edge of the plabic graph with a white vertex on the left and a black on the right (cancel out any 2-cycles arising from this). We noted that the square move leads to mutation of the associated quiver, as long as every two consecutive faces bordering the square are distinct.

Next, we related plabic graph to combinatorial objects that appeared before: we showed how to construct a plabic graph from a triangulation of a polygon and from (double) wiring diagrams. We quickly discussed how the quiver of a triangulation coincides with the quiver of the plabic graph arising from a triangulation, and mentioned that the same holds for (double) wiring diagrams.

We then defined *reduced plabic graphs*: namely, plabic graphs that are *not* move-equivalent to a plabic graph containing the “forbidden configurations”, namely the *hollow digon* (two vertices with two edges connecting them), and an internal leaf connected to a trivalent vertex of the other color that is not move-equivalent to a bivalent vertex. To obtain a more direct characterization, we introduced *trip permutations*: a *trip* is a path through the plabic graph following the “rules of the road”, turning to the right at black and to the left at white vertices; trips either start and end at a boundary vertex, or are *round trips* in the interior; the *trip permutation* (associated to a plabic graph G) is the permutation π_G of b elements that sends i to j if the trip in G starting at i ends at j.

We mentioned that move-equivalent plabic graphs have the same trip permutations, and that in reduced plabic graphs a fixed point i of the permutation implies that the component connected to the boundary vertex i is move-equivalent to a lollipop. This led to the definition of *decorated trip permutations* of reduced plabic graph: each fixed point i of the trip permutation of the reduced plabic graph is decorated with $\overline{i}$ or $\underline{i}$ if the lollipop attached to i is white resp. black. This allowed us to state the *fundamental theorem of reduced plabic graphs*: two reduced plabic graphs are move-equivalent if and only if their decorated trip permutations coincide. This in particular led to the observation that reduced plabic graphs are exactly those plabic graphs with a given trip permutation that have the minimal number of faces.

We continued by discussing the relation between reducedness and normalcy: a *normal plabic graph* is a bipartite plabic graph with trivalent white vertices and only black vertices connected to the boundary vertices. We say that a plabic graph has a *bad feature* if it contains either a round trip, a *essential self-intersection* (a trip that pass through the same edge twice), or a *bad double crossing* (two trips both crossing two given edges in the same order). We then stated the theorem that a normal plabic graph is reduced if and only if it contains *no* bad features. Afterwards, we sketched an algorithm that uses move-equivalences to turn a plabic graph into a normal plabic graph (or results into a non-reduced plabic graph), allowing the previous theorem to be applied to general plabic graphs. We also mentioned the existence of the *resonance property* to check reducedness.

Finally, we defined *source* and *target face labelings* of reduced plabic graphs: a face is labeled by the set of those i such that the trips starting (resp. ending) at i have the given face to the left of the trip. (This works due to the fact that trips in a reduced plabic graph bisect the disk.) We mentioned that the labels of the faces of a given reduced plabic graph all have the same cardinality. Finally, we defined the *positroid* associated to a reduced plabic graph given by the face labels of the boundary faces.

The main reference for this talk was Chapter 7 of [1]. The seminal reference for plabic graphs is [2].

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C3–Constructible sheaves on Legendrian knots

Yoon Jae Nho

Given a Legendrian knot $\Lambda \subset \mathbb{R}^3$, one can construct a D^--stack $\mathfrak{R}(\Lambda)$ which is a Legendrian isotopy invariant of Λ. If Λ is a positive braid knot, this stack can be identified with the open Bott-Samelson variety associated with β. One interpretation of $\mathfrak{R}(\Lambda)$ is that it is the "moduli" of exact Lagrangian fillings of Λ. Indeed, an exact Lagrangian filling L of Λ gives rise to an open toric chart

$(\mathbb{C}^*)^{b_1(L)}$ of $\mathfrak{R}(\Lambda)$, which can be verified by direct calculation in the case of free Legendrian weaves, using the machinery of [3].

Building $\mathfrak{R}(\Lambda)$ is a two-step process. First, one considers the category of constructible sheaves on $\mathbb{R}^2_{x,y}$ supported on Λ. These categories admit combinatorial descriptions, but they are not Legendrian isotopy invariants. Then, one can further restrict to sheaves with *singular support* on Λ. The theorem of GKS[4] then states that the category of such sheaves is indeed Legendrian isotopy invariant. Then, we can further restrict to "microlocal rank 1" sheaves with singular support on Λ with vanishing stalks at $y = -\infty$. The moduli of such sheaves then yield $\mathfrak{R}(\Lambda)$.

As a concrete example, in the case $\Lambda = \Lambda_\beta$ for the (-1)-closure of a positive braid-knot β with reduced word expression $\beta = s_{i_1}...s_{i_n}$, where s_i is the transposition of the ith strand with the $i+1$th strand, one can show that the moduli $\mathfrak{R}(\Lambda_\beta)$ is given by the moduli of tuples of complete flags $(F_1, ..., F_{n+1})$ with relative position conditions $F_j \sim_{s_{i_j}} F_{j+1}$, and $F_{n+1} = F_1$, which is indeed the open Bott-Samelson variety.

In this talk, we address the first part of the problem. Given a (regular cell refinement) of stratification induced by the front-projection of Λ on $\mathbb{R}^2_{x,y}$, we introduce the notion of *constructible sheaves*, i.e. sheaves whose restriction to each stratum are locally constant sheaves. Then, we compute constructible sheaves supported on the local model for the arc, the cusp and the crossing. We then use the local-to-global principle to express constructible sheaves supported on more general Legendrian knots as functors from the poset category induced by the stratification to the category of k-modules.

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B3–Webs and $Gr(3, n)$

Emine Yildirim

The goal of this talk to understand the cluster algebra structure in the homogeneous coordinate ring of Grassmannian $\mathbb{C}[\widehat{Gr(3,n)}]$ of 3-planes in $\mathbb{C}^n$ from Fomin-Pylyvaskyy perspective using Kuberberg's web basis. We mainly follow the following references: [4] and [1, Section 9.1]. We start the talk by explaining the definition of a tensor diagram. Then, we show how a tensor diagram encodes an element in the homogeneous coordinate ring of Grassmannian. Fomin and Pylyvaskyy show that if a tensor diagram is a planar tree, then the corresponding web invariant is a cluster or coefficient variable. We give a complete example of the cluster algebra structure in the case of $n = 6$. The cluster algebra for $Gr(3, 6)$ is of Dynkin type D_4; it has 22 cluster variables - six of which are frozen variables. Since Plücker coordinates are cluster variables, we have 20 Plücker cluster variables and two non-Plücker cluster variables in this case. We explicitly compute these two non-Plücker cluster variables using skein relations.

A tensor diagram is called a web if it is planar. Non-elliptic webs give rise to web invariants which form a linear basis in the ring of invariants. Let us now state some of Fomin-Pylyvaskyy's conjectures.

Conjectures.

(1) The set of cluster (and coefficient) variables coincide with the set of indecomposable arborizable web invariants.
(2) Two cluster variables lie in the same cluster if and only if they are compatible web invariants.
(3) If $n \geq 9$, there are infinitely many indecomposable non-arborizable web invariants.

Fomin-Pylyvaskyy [4] verify these conjectures in the finite type examples:

$Gr(2, n + 3)$	$Gr(3, 6)$	$Gr(3, 7)$	$Gr(3, 8)$
A_n	D_4	E_6	E_8

Note that this talk is a restrictive setting of Fomin-Pylyvaskyy paper - keep in mind the theorems and conjectures we mention in this abstract can be stated in a more general set up for $SL(V)$ invariant rings that is Fomin-Pylyvaskyy's main object in their paper [4]. Furthermore, C. Fraser [1] proves that for the cluster algebra in the homogeneous coordinate ring of Grassmannian $Gr(3, 9)$:

(1) Every cluster variable is an indecomposable arborizable web invariant.
(2) Every cluster monomial is a web invariant.
(3) There are infinitely many indecomposable non-arborizable web invariants.

These results are strong evidences for the validity of the conjectures. Finally, we would like to mention that webs may seem similar to dimers; [2] is a reference to see how they are related. Also, we refer curious audience to the paper [3] for further reading and a general view on this topic.

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C4–Microsupport and Legendrian fronts

Laurent Côté

Summary. Given Legendrian Λ in the cosphere bundle of $\mathbb{R}^2$, one can associate to it a variety $\mathcal{M}(\Lambda)$ whose properties carry useful information about Λ. This variety is defined as the moduli space of objects of the category of constructible sheaves *microsupported* along Λ. The purpose of this talk was to introduce the notions which enter into this construction.

1. Microsupport of sheaves

Conventions 1. Throughout this report, all sheaves are implicitly assumed to be constructible (with perfect stalks) and valued in the dg derived category of chain complexes over $\mathbb{C}$. All functors are implicitly derived. All stratifications are assumed to be Whitney. Finally, for consistency with some of the literature (e.g. [3]) we work throughout in the real analytic category.

Let M be a manifold. Fix a stratification $\mathcal{S}$ on M and a point $x \in M$. Let S_x be the stratum containing x. A function $f : Op(x) \to \mathbb{R}$ is said to be *stratified Morse* at $x \in M$ if either (a) $f|_{S_x}$ is non-critical at x or (b) $f|_{S_x}$ has a Morse critical points at x *and* $df_x(\tau) \neq 0$ for any $\tau \subset T_xM$ which is equal to a limit of tangent vectors of a larger stratum $Y > S_x$.

Construction-Definition 2. Given $\mathcal{F} \in sh(M)$, fix $x \in M$ and a function $f : Op(x) \to \mathbb{R}$ such that $f(x) = 0$. Fix $\epsilon, \delta > 0$ and set

$$M_{(x,f,\epsilon,\delta)}(\mathcal{F}) := cone\left(\mathcal{F}(B_\epsilon(x) \cap f^{-1}(-\infty, \delta)) \to \mathcal{F}(B_\epsilon(x) \cap f^{-1}(-\infty, -\delta))\right)$$

If f is stratified Morse at $x \in M$, then it can be shown that $M_{(x,f,\epsilon,\delta)}(\mathcal{F})$ stabilizes as $\epsilon, \delta \to 0$. In fact, the output only depends on $(x, df_x) \in T^*_xM$.

Definition 3. For $(x, \xi) \in S^*_xM$ and f stratified Morse at $x \in M$, we define the *Morse group* $M_{(x,\xi)}(\mathcal{F}) := M_{(x,f,\epsilon,\delta)}(\mathcal{F})$ for ϵ, δ small enough.

A covector $(x, \xi) \in S^*M$ is said to be *characteristic* if $M_{(x,\xi)}(\mathcal{F}) \neq 0$. Note that this notion depends on the stratification $\mathcal{S}$.

The characteristic covectors correspond precisely to the (co)directions along which the restriction map of $\mathcal{F}$ is non-trivial. This suggests that the set of characteristic co-vectors is a useful invariant of $\mathcal{F}$.

Definition 4 (Microsupport)**.** The *microsupport* (or *singular support*) of $\mathcal{F}$ is the set

$$SS(\mathcal{F}) := \overline{\{(x, \xi) \in S^*M \mid (x, \xi) \text{ is characteristic }\}}. \tag{1}$$

While the notion of a characteristic vector depends on the stratification, it can be shown that the microsupport does not depend on this choice. In fact, the microsupport can be defined without choosing a stratification and appealing to the theory of stratified Morse functions; see [2, Sec. 5.1]. However, the Morse-theoretic viewpoint is useful for intuition and computations.

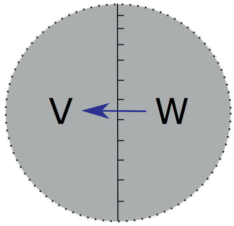

Figure 1. Here the vertical arrow is the front of a Legendrian arc $\Lambda \subset S^*\mathbb{R}^2$. The "hair" specifies a unique lift of the front.

2. The category $Sh_\Lambda(M)$

Let $\Lambda \subset S^*M$ be a Legendrian.

Definition 5. We let $sh_\Lambda(M) \subset sh(M)$ be the full subcategory on objects whose microsupport is contained in Λ.

To get a handle on this definition, let us suppose that $\pi(\Lambda) \subset M$ is a front. Then we can consider the category of sheaves $sh_{\mathcal{S}}(M)$ constructible with respect to any stratification $\mathcal{S}$ containing the front. According to the exit-path definition of a constructible sheaf, this is the same thing as a module over the exist path category. In other words, a constructible sheaf is the data of a stalk on each stratum and restriction maps from lower dimensional strata to higher dimensional strata.

The microsupport condition picks out a full subcategory $sh_\Lambda(M) \subset sh_{\mathcal{S}}(M)$ by forcing some of the restriction maps to be isomorphisms. This is illustrated in the following example.

Example. Suppose that Λ is a lift of the front drawn in Figure 2. Then the category of sheaves constructible with respect to the induced stratification is equivalent to the category of representations of the quiver $(\bullet \xleftarrow{\alpha} \bullet \xrightarrow{\beta} \bullet)$. However, for a constructible sheaf to lie in $sh_\Lambda(D^2)$, it must have the property that the restriction map corresponding to β is an isomorphism: indeed, the failure of this map to be an isomorphism would be witnessed by a point in the microsupport. But by definition of $sh_\Lambda(D^2)$, the microsupport of $\mathcal{F}$ in $S^{*,-}D^2$ is empty (the "hair" points in the $+$ direction). We conclude that the category $sh_\Lambda(D^2)$ is equivalent to the category of representations of the A_2 quiver.

The great virtue of the category $sh_\Lambda(M)$, as opposed to $sh_{\mathcal{S}}(M)$, is that it is an invariant of Λ. This is the content of the following theorem:

Theorem 6 (Fundamental theorem [1] (Guillermou–Kashiwara–Schapira))**.** *A Legendrian isotopy $\Lambda \rightsquigarrow \Lambda' \subset S^*M$ induces an equivalence of categories*

$$sh_\Lambda(M) \to sh_{\Lambda'}(M).$$

In general, $sh_\Lambda(M)$ can be very complicated. However, when $M = \mathbb{R}^2$, then the front projection of a Legendrian generically only has cusps and crossings. Hence

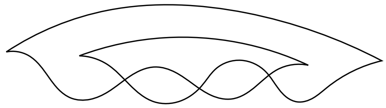

Figure 2. The front projection of a trefoil

the study of $sh_\Lambda(\mathbb{R}^2)$ can be reduced to local models. The simplest local model was computed in Example 2; the other two (cusp and crossing) were also computed in the talk. See [3, Sec. 3.3].

3. The moduli space of rank 1 objects

In order to access the category $sh_\Lambda(M)$, it is often useful to consider categorical invariants associated to it. The main class of invariants which were discussed in the talk are so-called "moduli spaces" of objects.

Definition 7. Suppose $\Lambda \subset S^*M$ is connected. The *microlocal rank* of $\mathcal{F} \in sh_\Lambda(M)$ is the rank of $M_{(x,\xi)}(\mathcal{F})$ for any $(x,\xi) \in \Lambda$.

Theorem 8 ([5] Toën–Vaquié)**.** *There exists a "derived stack" $\mathcal{M}^r(\Lambda)$ whose points are in bijection with isomorphism classes of objects of $sh_\Lambda(M)$ having microlocal rank r.*

This theorem is an abstract result valid for categories satisfying a certain finiteness assumption. Our standing assumption that constructible sheaves have perfect stalks is essential in order to appeal to it.

For many Legendrians which arise in practice, the output of this theorem (a priori a derived stack) is an ordinary variety which can be explicitly described.

Example. In the talk, we explicitly computed the moduli space of rank 1 objects where Λ is the (lift of the) front drawn in Figure 3. The answer is as follows. We first consider the moduli space

$$\widetilde{\mathcal{M}}^1(\Lambda) := \{(\ell_0, \dots, \ell_4) \in \mathrm{Mat}_{2\times 5}(\mathbb{C}) \mid \ell_i \in \mathrm{Mat}_{2\times 1}(\mathbb{C}), \ell_i \pitchfork \ell_{i+1}, i \in \mathbb{Z}/5\}$$

Then the moduli space of rank 1 objects is the quotient

$$\mathcal{M}^1(\Lambda) = \widetilde{\mathcal{M}}^1(\Lambda)/(GL_2(\mathbb{C}) \times \mathrm{Diag}_5(\mathbb{C})).$$

One can also consider a *framed* variant

$$\mathcal{M}^1_{fr}(\Lambda) = \widetilde{\mathcal{M}}^1(\Lambda)/GL_2(\mathbb{C}).$$

The main idea for performing such computations is to restrict ourselves to local models, for which (as explained above) the category of microlocal sheaves is fully understood. We refer to [3, Sec. 6] and [4, Sec. 3] for related computations.

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B4–Double Bruhat cells and generalisations

Matthew Pressland

1. Double Bruhat cells

One of the earliest results equipping the coordinate ring of an algebraic variety with a cluster algebra structure is due to Berenstein, Fomin and Zelevinsky [2], who achieve this for double Bruhat cells. Before describing their construction, we give the necessary set-up and definitions.

Fix a connected, simply connected, semisimple algebraic group G over $\mathbb{C}$, with opposite Borel subgroups B_+ and B_-. This determines a maximal torus $T = B_+ \cap B_- \cong (\mathbb{C}^\times)^n$, a Weyl group $W = \mathrm{Norm}_G(T)/T$, and a Dynkin diagram Δ. The Weyl group is generated by n simple reflections s_i, for $i \in \Delta_0$.

Each node $i \in \Delta_0$ determines a homomorphism $\varphi_i \colon \mathrm{SL}_2(\mathbb{C}) \to G$, taking upper triangular matrices into B_+ and lower triangular matrices into B_-. We may lift W to a subset (but not a subgroup) of G by identifying s_i with $\bar{s}_i = \varphi_i\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and a general element $w = s_{i_1} \cdots s_{i_\ell}$ with $\bar{s}_{i_1} \cdots \bar{s}_{i_\ell}$, where the expression for w as a product of simple reflections is chosen to be *reduced*, i.e. of minimal length $\ell = \ell(w)$. Viewing W as a subset of G in this way, we obtain a pair of *Bruhat decompositions*

$$G = \bigcup_{u \in W} B_+ u B_+ = \bigcup_{v \in W} B_- v B_-.$$

Definition 1. A *double Bruhat cell* is $G^u_v := B_+ u B_+ \cap B_- v B_-$, for $u, v \in W$.

To describe a cluster algebra structure on the coordinate ring $\mathbb{C}[G^u_v]$, we restrict for simplicity to the case that the Dynkin diagram Δ is simply-laced, that is, of type A, D or E. This will allow us to describe the initial seed via a quiver, rather than a more general valued quiver or skew-symmetrisable matrix. We will also deviate from the original presentation in [2], and using instead a description of this seed derived from work of Shen and Weng [7], which we will return to shortly.

Definition 2. Given $u, v \in W$, consider a trapezium with its upper edge cut into $\ell(u)$ segments, and lower edge cut into $\ell(v)$ segments. A *triangulation* of (u, v) is a choice of reduced expression for each of u and v, together with a decomposition

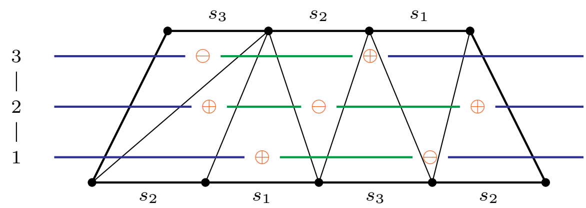
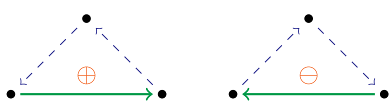


Figure 1. A triangulation of (u, v), with the associated string diagram overlaid, for $G = \mathrm{SL}_4(\mathbb{C})$, so $\Delta = \mathsf{A}_3$ (shown left), with $u = s_3s_2s_1$ and $v = s_2s_1s_3s_2$. Closed strings are shown in green, and open strings in blue.

of the trapezium into triangles such that exactly one edge of each triangle lies on the upper or lower edge of the trapezium. See Figure 1 for an example.

Given a triangulation of (u, v), we label the segments on the upper and lower edges via the chosen reduced expressions of u and v, reading from left to right. This labels exactly one edge of each triangle by a simple reflection, and hence induces a labelling of the triangles. A triangulation of (u, v) determines a string diagram in the following way. Draw $n = |\Delta_0|$ strands through the trapezium, indexed by the nodes of the Dynkin diagram. In a triangle labelled by s_i, cut strand i, and label the cut by $\oplus$ if the labelled edge of the triangle is on the bottom of the trapezium, and by $\ominus$ if the labelled edge is on the top. This process cuts the strands into *strings*, which can be either *closed* (incident with two cuts), or *open* (incident with at most one cut). Again, an example is shown in Figure 1.

Definition 3. The (ice) quiver $Q(t)$ of a triangulation t of (u, v) has as vertices the strings of the associated string diagram, with open strings frozen. At each cut, we see one of the following configurations in the quiver, depending on the sign.



Here the solid arrow connects the two strings from strand i meeting at the cut, and we draw a pair of dashed arrows as shown for each string passing through the triangle containing the cut and lying on a strand j with i and j joined by an edge of Δ. These dashed arrows are interpreted as 'half-arrows': in the final quiver, two half-arrows in the same direction add together to form a full (solid) arrow, while those in opposite directions cancel out. This process produces a natural collection of half-arrows between frozen vertices, but these play no role in defining the cluster algebra. See Figure 2 for the quiver associated to the triangulation in Figure 1.

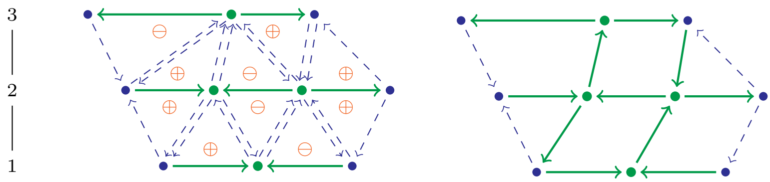


Figure 2. Constructing the quiver of the triangulation in Figure 1; the initial construction involving half-arrows (left), and the final quiver (right). Mutable vertices are green, and frozen vertices are blue.

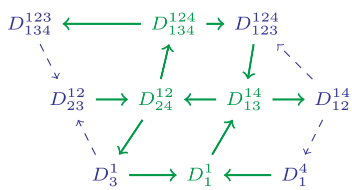
Theorem 4 (Berenstein–Fomin–Zelevinsky [2]). *Let $u, v \in W$, let t be a triangulation of (u, v), and let $\mathscr{A}(t)$ be the cluster algebra associated to $Q(t)$, with invertible frozen variables. Then there is an isomorphism*

$$\mathscr{A}(t) \xrightarrow{\sim} \mathbb{C}[G^u_v]$$

sending the initial cluster variables to generalised minors.

Strictly speaking, the original result from [2] gives an isomorphism with the upper cluster algebra $\mathscr{U}(t)$ associated to $Q(t)$. However, Muller and Speyer [6] show that this cluster algebra is locally acyclic, and hence $\mathscr{A}(t) = \mathscr{U}(t)$.

We do not give the general definition of generalised minors here, but note that in type A, where $G = \mathrm{SL}_{n+1}(\mathbb{C})$, they are ordinary matrix minors. There is an explicit combinatorial recipe for computing which minors are the images of the initial cluster variables under the isomorphism of Theorem 4. For our running example, the result is



where D^I_J denotes the minor on rows I and columns J.

2. Double Bott–Samelson varieties

Recall that the braid group $\mathrm{Br}(\Delta)$ is defined similarly to the Coxeter group of Δ (which is isomorphic to W), but excluding the relations $s_i^2 = e$. A *positive braid* is an element of $\mathrm{Br}(\Delta)$ expressible as a word in the letters s_i, $i \in \Delta_0$ (in contrast to a general braid, in which the letters s_i^{-1} may be necessary). Given $u, v \in \mathrm{Br}(\Delta)$, one can define a triangulation exactly as in Definition 2, replacing 'reduced expression' by 'positive braid word'. Given such a triangulation t, construct the associated string diagram as before, but viewing u and v as elements of $\mathrm{Br}(\widetilde{\Delta})$, for $\widetilde{\Delta}$ the associated affine diagram. Let $\widetilde{Q}(t)$ be the associated quiver, which differs from

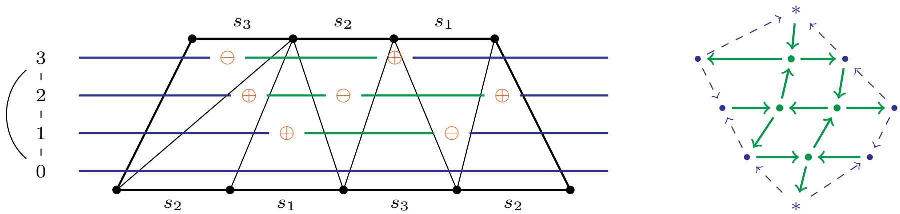


Figure 3. A triangulation of positive braid words (u, v), with the associated 'affine' string diagram overlaid (left), and a schematic of the associated quiver (right). To obtain the actual quiver, the two vertices labelled by $*$ should be identified.

$Q(t)$ by adding a single frozen vertex, corresponding to the single open string labelled by the extending vertex of $\widetilde{\Delta}$, and its incident arrows. An example is given in Figure 3; while this reuses the reduced expressions for elements of W from the previous example, we emphasise that the general construction applies to arbitrary positive braid words.

The cluster algebra $\widetilde{\mathscr{A}}(t)$ with invertible frozen variables associated to $\widetilde{Q}(t)$ also turns out to have a geometric interpretation.

Definition 5 (Shen–Weng [7])**.** Let $u = s_{i_1} \cdots s_{i_\ell}$ and $v = s_{j_1} \cdots s_{j_m}$ be positive braids. Then the *double Bott–Samelson variety* BS_v^u consists of tuples of flags $(x_0B_+, \ldots, x_\ell B_+, y_0B_-, \ldots, y_mB_-) \in G\backslash\big((G/B_+)^\ell \times (G/B_-)^m\big)$ (that is, each tuple is considered up to the left action of G on the product of flag varieties) subject to the conditions that

(1) $x_{k-1}^{-1}x_k \in B_+s_{i_k}B_+$ for $k = 1, \ldots, \ell$,
(2) $y_k^{-1}y_{k-1} \in B_-s_{j_k}B_-$ for $k = 1, \ldots, m$,
(3) $x_0^{-1}y_0 \in B_+B_-$ and $x_\ell^{-1}y_m \in B_+B_-$.

Letting $U_\pm$ denote the unipotent radicals of $B_\pm$, the *decorated Bott–Samelson variety* $\widehat{\mathrm{BS}}_v^u$ consists of those tuples

$$(x_0U_+, x_1B_+, \ldots, x_\ell B_+, y_0B_-, \ldots, y_{m-1}B_-, y_mU_-)$$

in $G\backslash\big(G/U_+ \times (G/B_+)^{\ell-1} \times (G/B_-)^{m-1} \times G/U_-\big)$ which map to points of BS_v^u under the natural projection.

Shen and Weng [7] show that both of these varieties depend, up to isomorphism, only on the positive braids u and v, and not on the choice of braid words.

Theorem 6 (Shen–Weng [7])**.** *Let u and v be positive braids and let t be a triangulation of (u, v). Then there is an isomorphism*

$$\widetilde{\mathscr{A}}(t) \xrightarrow{\sim} \mathbb{C}[\widehat{\mathrm{BS}}_v^u].$$

Remark 7. The ordinary Bott–Samelson variety BS_w associated to $w \in W$ was introduced [1] to provide a desingularisation of the Schubert variety $\overline{BwB}/B$. Double Bott–Samelson varieties are special cases of braid varieties, and so Theorem 6 is an important precursor to the general result that all such varieties carry cluster algebra structures [3, 4, 5].

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C5–Cluster algebras and symplectic topology: Microlocal holonomies and the Bott-Samelson case

Mikhail Gorsky

This talk concerns a point of view on cluster algebra structures on coordinate rings of certain affine algebraic varieties by means of symplectic geometry. Several families of varieties appearing in talks B1 – B4, such as open positroid varieties, double Bruhat cells, and double Bott-Samelson cells, can be described as moduli spaces of decorated microlocal rank-1 constructible sheaves on $\mathbb{R}^2$ supported on front projections of Legendrian links in $\mathbb{R}^3$ with the standard contact structure ξ_{st}. This perspective connects talks from series A and B with the framework of series C.

Consider a positive braid β with the Demazure product $w_0 \in S_n$. The Legendrian (-1)-closure of the braid represented by β is a Legendrian link Λ_β in $(\mathbb{R}^3, \xi_{st})$. If $\beta = \Delta\beta'$ for some positive braid β', where Δ is the half-twist, the link Λ_β is Legendrian isotopic to the rainbow closure of β', as considered in [7, Section 6.5]. With a Legendrian link Λ taken with a collection of marked points T one can naturally associate a moduli stack $\mathfrak{M}(\Lambda, T)$ of decorated microlocal rank-1 constructible sheaves on $\mathbb{R}^2$ whose support is contained in the front projection of Λ. It turns out that for (Λ_β, T) with T containing at least one marked point per link component, $\mathfrak{M}(\Lambda, T)$ is in fact a smooth affine algebraic variety: it can be realized, up to a torus factor, as a braid variety $X(\beta)$ in type A which will be discussed in more detail in talk C6. The smoothness of braid varieties follows from work of Escobar [3].

Shen and Weng in [8] and in a joint work with Gao [5] introduced several versions of double Bott-Samelson (BS) varieties associated with pairs of positive braid words. In particular, half-decorated double BS varieties for pairs (e, β')

were proved in [5] to be isomorphic to $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ for T having precisely one marked point per strand of $\Delta\beta'$. For decorated double BS varieties, a cluster $\mathcal{A}$-structure was defined in [8] in terms of generalized minors. This was done by extending standard approaches to cluster structures on double Bruhat cells via amalgamation techniques of Fock-Goncharov [4]. This algebra structure was translated to the symplectic framework in [5], where $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ was interpreted as the augmentation variety of the link $\Lambda_{\Delta\beta'}$. An undecorated variant of $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$, denoted by $\mathcal{M}_1(\Lambda_{\Delta\beta'}, T)$, is also isomorphic to a variant of a double BS variety, depending on the choice of T. The latter was proved in [8] to admit a cluster $\mathcal{X}$-structure, also known as a cluster Poisson structure, forming a cluster ensemble (as defined in talk B5) with $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ interpreted as a (half-)decorated double BS variety.

From the point of view of symplectic geometry, results of [5, 8] indicated the existence of cluster $\mathcal{A}$- and $\mathcal{X}$-structures on moduli spaces of microlocal rank-1 sheaves associated with Legendrian links, but the construction presented in these works was fairly unsatisfactory. In the talk, a "symplectic" construction of cluster $\mathcal{A}$-structures on $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ and of cluster $\mathcal{X}$-structures on $\mathcal{M}_1(\Lambda_{\Delta\beta'}, T)$ (the latter improving on earlier work [6]) was presented. This construction is due to Casals and Weng [1] who used technology of weaves introduced by Casals and Zaslow [2]. Weaves are certain coloured graphs representing Lagrangian fillings of Legendrian links, as explained in talk C2. The main result presented at the talk is the following.

Theorem 1. [1] *For a positive braid β' and a collection T of marked points on $\Lambda_{\Delta\beta'}$ with at least one point per component, the pair*

$$(\mathfrak{M}(\Lambda_{\Delta\beta'}, T), \mathcal{M}_1(\Lambda_{\Delta\beta'}, T))$$

forms a cluster ensemble, where the initial seeds of $(\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ and $\mathcal{M}_1(\Lambda_{\Delta\beta'}, T))$ are described in terms of an exact embedded Lagrangian filling L of $\Lambda_{\Delta\beta'}$ described via a certain explicit weave.

The construction and a sketch of the proof were presented. Cluster $\mathcal{A}$-variables are indexed by certain relative cycles $\eta \in H_1(L\backslash T, \Lambda\backslash T)$ and can be interpreted as so-called microlocal merodromies, which intuitively give parallel transport along η, while $\mathcal{X}$-variables are indexed by absolute cycles in $\gamma \in H_1(L)$ and can be interpreted as microlocal monodromies along γ. The language of weaves not only provided a symplectic interpretation of cluster algebra structures on the sheaf moduli spaces, but also allowed to simplify some of the proofs, compared to those in [8].

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C6–Cluster structures on braid varieties

Tonie Scroggin

Given a braid word β we may define an algebraic variety called a braide variety. In this talk we show that the coordinate ring of regular functions on any braid variety is a cluster algebra. By defining Lusztig cycles, intersections and functions on the Lusztig cycles we are able tio produce a quiver and cluster variables which constitutes the seed of the cluster algebra $\mathbb{C}[\chi(\beta)]$

Introduction to Cluster Ensembles and the Fock-Goncharov duality conjectures

Geoffrey Janssens

A geometric counterpart of Fomin-Zelevinsky's cluster algebras was introduced by Fock and Goncharov [2, 3] in which "seed tori" are glued together along cluster transformations, which are certain distinguished birational maps, to produce cluster varieties. These varieties come in pairs and form a so-called cluster ensemble.

In this talk we start by introducing the above concepts following the historical reference [3]. Secondely we explain the geometric gains of cluster ensembles. Hereby we emphasize the importance of some recent works, such as Gross-Hacking-Keel-Kontsevich [4, 5] and Argüz-Bousseau [1]. Finally, we give a brief introduction to Fock-Goncharov's duality conjectures. The recurrent example used during the talk is the one of (higher) Teichmüller theory. Indeed, cluster varieties have deep connections with several areas of mathematics, in particular in the study of the moduli space of local systems on topological surfaces [3].

1. Introduction to Cluster ensembles

In earlier talks cluster algebras associated to quivers with frozen vertices have been introduced and the translation to cluster algebras with coefficients was mentioned. For this talk we consider the general setting. In other words, let $(\mathbb{P}, \oplus, \cdot)$ be some semi-field and $(\mathbf{x}, \mathbf{y}, B)$ a labeled seed with B a skew-symmetrizable $n \times n$ matrix. In particular, $\mathbf{y} \in \mathbb{P}^n$ and the coordinates of $\mathbf{x} = (x_1, \dots, x_n)$ form a free generating set, over $\mathbb{Q}[\mathbb{P}]$, of a given field $\mathcal{F}$ of rational functions in n variables.

In order to associate an algebraic variety to every seed obtained from $(\mathbf{x}, \mathbf{y}, B)$ by mutation a coordinate-free point of view is more natural. More precisely, a seed can be viewed as the data $\vec{i} = (\Lambda, I, F, E, D)$ where

- Λ is a lattice of rank n (i.e. $\Lambda \cong \mathbb{Z}^n$) equipped with a skew-symmetric $\mathbb{Q}$-bilinear form $(\cdot,\cdot)$,
- I an index set with $F \subset I$ the frozen indices,
- $E = \{e_i\}$ is a basis of Λ and $D = (d_i)$ the multipliers. In particular, in the skew-symmetric case $d_i = 1$ for all i.

Forming the matrix $(\epsilon_{i,j} := d_j(e_i, e_j))_{i,j}$ recovers the transpose of the mutation matrix considered in the previous talks. However, with the above notion of a seed, mutation at some $k \in I \setminus F$ is defined as Y-seed mutation. The usual cluster mutation is found by considering the dual lattice $\Lambda^* = \hom(\Lambda, \mathbb{Z})$ with dual basis $\{e_i^*\}$. More precisely one needs $\Lambda^\circ = \mathrm{span}\{f_i := d_i^{-1} e_i\}$.

Now with the seed $\vec{i}$, via Λ and Λ°, one can naturally associate tori:

$$\mathcal{X}_{\vec{i}} = \operatorname{spec} k[\Lambda] = \hom(\Lambda, \mathbb{G}_m)$$

and similarly $\mathcal{A}_{\vec{i}} = \operatorname{spec} k[\Lambda^\circ]$. These tori are called the *seed $\mathcal{X}$-torus*, respectively *seed $\mathcal{A}$-torus.* If $\vec{i}' = \mu_k(\vec{i})$ is another seed obtained by mutating at $k \in I \setminus F$, then there are birational morphisms

$$\mu_k^{\mathcal{X}} : \mathcal{X}_{\vec{i}} \to \mathcal{X}_{\vec{i}'} \text{ and } \mu_k^{\mathcal{A}} : \mathcal{A}_{\vec{i}} \to \mathcal{A}_{\vec{i}'}$$

connecting the associated tori. It is usual to define these morphisms explicitly by pullback formulas at level of characters which mimic Y and $X-$cluster mutation. Using these maps one can glue all the tori in order to obtain a scheme structure on $\bigcup_{\vec{i}} \mathcal{X}_{\vec{i}}$ and also on $\bigcup_{\vec{i}} \mathcal{A}_{\vec{i}}$. For algebraic geometrical (complete) details we refer to [4, Proposition 2.4]. By doing so one obtains the tuple $(\mathcal{X}, \mathcal{A})$ called the *cluster ensemble* and which was introduced by Fock-Goncharov. As was pointed out, $\mathcal{A}$ is an honest variety, i.e. it is separated. However, in general $\mathcal{X}$ is not seperated.

Subsequently we explained that considering global regular functions on $\mathcal{A}$ one recovers the *upper cluster algebra* which by the Laurent phenomenon contains the cluster algebra. At level of the $\mathcal{X}$-variety the global regular functions yield the so-called *Poisson cluster algebra.* However the Laurent phenomenon doesn't hold in this case.

2. The geometric structure and duality phenomenons

The name Poisson cluster algebras refers to the fact that the $\mathcal{X}$-variety has a Poisson structure. More precisely, using the bilinear form $(\cdot,\cdot)$ one writes down an explicit Poisson structure on each torus $\mathcal{X}_{\vec{i}}$, which moreover is invariant under mutation. In particular it induces a Poisson structure on $\mathcal{X}$. On his turn the torus $\mathcal{A}_{\vec{i}}$ carries a mutation well-behaved closed 2-form Ω which induces a symplectic structure on $\mathcal{A}$.

These structures are connected to each other. To be more precise one needs to introduce a crucial map connecting the both varieties. To start one defines the skew-symmetrizable form $[e_i, e_j] = d_j(e_i, e_j)$ and subsequentely considers the map

$$\Lambda \to \Lambda^\circ : v \mapsto \sum_j [v, e_j] f_j.$$

Associated is the morphism of seed tori

$$\mathcal{A}_{\vec{i}} = \operatorname{spec} k[\Lambda^\circ] \to \mathcal{X}_{\vec{i}} = \operatorname{spec} k[\Lambda].$$

These maps behave well with mutaton and hence one obtains a morphism

$$p : \mathcal{A} \to \mathcal{X}$$

called the *assembly map* which, crucially, is monomial and positive. A reassuring fact now is that the symplectic structure on $p(\mathcal{A}_{\vec{i}})$ induced by Ω coincides with the symplectic structure given by the restriction of the Poisson structure on $\mathcal{X}_{\vec{i}}$.

The interplay however doesn't stop there and in the talk a glimpse was given of two deeper connections between the both varieties, both of a duality nature. For the first one needs an alternate description of cluster varieties by Gross-Hacking-Keel [4] using log Calabi-Yau varieties. In brief, they have shown that $\mathcal{X}$ is up to codimension 2 a blow-up of some concrete toric variety. In particular, besides the Fock-Goncharov dual variety $\mathcal{A}$ of $\mathcal{X}$, one can associate the mirror dual of the log Calabi-Yau variety (constructed in the framework of the Gross-Siebert program). It was recentely proven by Argüz-Bousseau [1] that the mirror to the $\mathcal{X}$ cluster variety is a degeneration of the $\mathcal{A}$ cluster variety and vice versa.

A second attractive conjectural duality between $\mathcal{X}$ and $\mathcal{A}$ is given by the Fock-Goncharov duality conjectures. During the talk we presented a short intuitive tropical path to the statement. This required to make the birational morphisms $\mu_k^{\mathcal{X}}$ and $\mu_k^{\mathcal{A}}$ explicit in terms of the coordinate functions z^{e_i}. With a slight abuse of notation, they are given by

$$\begin{aligned} \left(\mu_k^{\mathcal{X}}\right)^* &: z^v \mapsto z^v(1 + z^{e_k})^{-(v,e_k)} \\ \left(\mu_k^{\mathcal{A}}\right)^* &: z^\gamma \mapsto z^\gamma(1 + z^{(e_k,\cdot)})^{-\gamma(e_k)} \end{aligned}$$

where $v \in \Lambda$ and $\gamma \in \Lambda^\circ$. Thus, the gluing maps are substraction-free. A wonderful by-product of this is that one can take $\mathbb{P}$-points for any semi-field $\mathbb{P}$. Now choosing for $\mathbb{P}$ the tropical integers $\mathbb{Z}^{tr} = (\mathbb{Z}, +\max)$, a direct computation yields an intriguing phenomena. Namely, denoting $\mathcal{A}^{tr}$ for the $\mathbb{Z}^{tr}$-points of $\mathcal{A}$, the morphism $\mu_k^{\mathcal{A}^{tr}}$ is up to a change of e_k to $-e_k$ and of ϵ_{ij} to $-\epsilon_{ji}$ given by the same formula as $\mu_k^{\mathcal{X}}$ on Λ. In other words, $\mu_k^{\mathcal{X}}$ is the tropicalization of the Laglands dual $\mu_k^{\mathcal{A}^\vee(\mathbb{Z}^{tr})}$. The Fock-Goncharov duality conjecture states that the duality is far more reaching. For example the basis conjecture predicts that $\Gamma(\mathcal{X}^\vee, \mathcal{O}_{\mathcal{X}^\vee})$ has a basis indexed naturally by $\mathcal{A}^{tr}$ and vice-versa.

To finish the talk, we mentioned that in [4] the authors showed that the original Fock-Goncharov conjecture do not hold without certain positivity assumptions. Nevertheless, they suggest that some formal version of the conjecture should hold. In their seminal work Gross-Hacking-Keel-Kontsevich [5] proved the formal Fock-Goncharov conjecture, as well as the original Fock-Goncharov conjecture with the necessary positivity assumptions.

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Mini-Workshop: Poisson and Poisson-type algebras

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Abstract. The first historical encounter with Poisson-type algebras is with Hamiltonian mechanics. With the abstraction of many notions in Physics, Hamiltonian systems were geometrized into manifolds that model the set of all possible configurations of the system, and the cotangent bundle of this manifold describes its phase space, which is endowed with a Poisson structure. Poisson brackets led to other algebraic structures, and the notion of Poisson-type algebra arose, including transposed Poisson algebras, Novikov-Poisson algebras, or commutative pre-Lie algebras, for example. These types of algebras have long gained popularity in the scientific world and are not only of their own interest to study, but are also an important tool for researching other mathematical and physical objects.

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Introduction by the Organizers

Poisson algebras emerged naturally in the framework of Hamiltonian mechanics and the field developed rapidly following the advent of mathematical physics. Nowadays Poisson algebras play a central role in a wide range of areas in mathematics and physics, such as Poisson manifolds, algebraic geometry, operads, quantization theory, quantum groups, classical and quantum mechanics. A variety of related algebraic structures, the so-called Poisson-type algebras, gained popularity in recent years: these include Novikov-Poisson algebras, commutative pre-Lie algebras or the recently introduced transposed Poisson algebras, to name but a few. The purpose of this meeting was to bring together experts in various fields revolving around Poisson algebras, to discuss new approaches to open problems in the area and to initiate new research work. Discussions and talks were focused into the following directions:

- **Poisson-type structures:** Several talks considered the different Poisson-type structures existing in the literature. Guo's talk reported on the study of operads encoding algebraic structures with replicated copies of operations satisfying various compatibility conditions among these copies and explained the relations of the compatibility conditions with Koszul duality and Manin products. Admissible operads of various types have been discussed in the talk of Dzhumadil'daev. Burde presented various results and open conjectures concerning the existence of post-Lie algebra structures on a pair of Lie algebras over a fixed vectors space, emphasising the cases where either of the two Lie algebras is abelian, nilpotent, solvable, simple, semisimple, reductive, complete or perfect. The talk of Zusmanovich, considered the problem of whether an extension of the contact bracket (a natural generalisation of Poisson bracket) on the tensor product from the bracket on the factors is possible. Transposed Poisson algebras have been recently introduced as a dual notion of Poisson algebras, by exchanging the roles of the two multiplications in the Leibniz rule defining a Poisson algebra. The mini-workshop featured several talks which discussed the rich structure of transposed Poisson algebras. Bai presented a bialgebra theory for transposed Poisson algebras. Khrypchenko discussed transposed Poisson structures on Lie incidence algebras and Fernandez Ouaridi's talk focused on the simple transposed Poisson algebras.
- **Poisson algebras and superalgebras:** Another important part of the mini-workshop consisted of talks related to the study of certain specific classes of Poisson (super)algebras. Sierra reported on the study of the Poisson ideal structure of the symmetric algebras of the Virasoro algebra and the Witt algebra of algebraic vector fields on $\mathbb{C}^*$ and various other related Lie algebras. The talk of Yakimova highlighted the use of the symmetrisation map for obtaining various new explicit formulas for the generators of the Feigin-Frenkel center. Launois discussed algorithmic methods to study Poisson derivations of a semiclassical limit of a family of quantum second Weyl algebras. Agore presented certain universal objects for Poisson algebras and highlighted several applications of these constructions to the description of the automorphism group of a given Poisson algebra and to the classification of gradings by an abelian group. The talk of Siciliano gave an overview of the known results about solvable (truncated) symmetric Poisson algebras and their derived lengths as well as some open questions on these topics. The Gerstenhaber bracket on the Hochschild cohomology of a certain subalgebra of the Weyl algebra and its connection to the Virasoro Lie algebra have been highlighted by Lopes. Usefi's talk focused on the characterization of Lie superalgebras whose enveloping algebras satisfy a non-matrix polynomial identity.

Mini-Workshop: Poisson and Poisson-type algebras

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Abstracts

Universal constructions for Poisson algebras. Applications

Ana Agore
(joint work with Gigel Militaru)

We introduce and study some universal objects for Poisson algebras and highlight their main applications having as sourse of inspiration the previous work of Sweedler [15], Manin [11] and Tambara [13] for Hopf algebra (co)actions on associative algebras. From a categorical point of view, the existence of universal objects with a certain property, for a given category $\mathcal{C}$ can shed some light on the structure of the category $\mathcal{C}$ itself. In particular, the existence and description of universal objects (groups or "group like objects" such as Lie groups, algebraic groups, Hopf algebras, groupoids or quantum groupoids, etc.) which act or coact on a fixed object $\mathcal{O}$ in a certain category $\mathcal{C}$ has often various applications in many areas of mathematics. An elementary but illuminating example is the following: let $\mathcal{O}$ be a given object in a certain category $\mathcal{C}$ and consider the category $\mathrm{ActGr}_{\mathcal{O}}$ of all groups that *act* on $\mathcal{O}$, i.e. the objects in $\mathrm{ActGr}_{\mathcal{O}}$ are pairs (G, φ) consisting of a discrete group G and a morphism of groups $\varphi\colon G \to \mathrm{Aut}_{\mathcal{C}}(\mathcal{O})$, where $\mathrm{Aut}_{\mathcal{C}}(\mathcal{O})$ denotes the automorphisms group of the object $\mathcal{O}$ in $\mathcal{C}$. Then the category $\mathrm{ActGr}_{\mathcal{O}}$ has a final object, namely $\big(\mathrm{Aut}_{\mathcal{C}}(\mathcal{O}), \mathrm{Id}\big)$. Now, if we replace the discrete groups that act on the fixed object $\mathcal{O}$ in $\mathcal{C}$, by some other "groups like objects" from a certain more sophisticated category $\mathcal{D}$ (for instance, Lie groups, algebraic groups, Hopf algebras, etc.) which (co)act on $\mathcal{O}$ and if moreover we ask the (co)action to preserve the algebraic, differential or topological structures which might exist on $\mathcal{O}$, then things become very complicated. Indeed, the first obstacle we encounter is the fact that $\mathrm{Aut}_{\mathcal{C}}(\mathcal{O})$ might not be an object *inside* the category $\mathcal{D}$ anymore. However, even in this complicated situation, it is possible for the above result to remain valid but, however, the construction of the final object will be far more complicated. Furthermore, it is to expect that, if it exists, this final object will contain important information on the entire automorphisms group of the object $\mathcal{O}$. To the best of our knowledge, the first result in this direction was proved by Sweelder [15, Theorem 7.0.4] in the case where $\mathcal{C}$ is the category of associative algebras and $\mathcal{D}$ is the category of bialgebras: if A is a fixed associative algebra then the category of all bialgebras H that *act* on A (i.e. A is an H-module algebra) has a final object $\mathrm{M}(A)$, called by Sweedler the *universal measuring bialgebra* of A. The dual situation of *coactions* of bialgebras on a fixed algebra A, was first considered in the case when $\mathcal{C}$ is the category of graded algebras by Manin [11] for reasons related to non-commutative geometry, and in the general case by Tambara [13]. If A is an associative algebra, necessarily finite dimensional this time around, then the category of all bialgebras that coact on A (i.e. A is an H-comodule algebra) has an initial object $a(A)$. Furthermore, the usual automorphisms group $\mathrm{Aut}_{\mathrm{Alg}}(A)$ of A is indeed recovered as the group of all invertible group-like elements of the finite dual $a(A)^{\circ}$ [12, Theorem 2.1] and

$a(A)^{\rm o}$ is just Sweedler's final object in the category of all bialgebras that act on A [13, Remark 1.3]. The two results above remains valid if we take the category of Hopf algebras instead of bialgebras: in particular, the Hopf envelope of $a(A)$, denoted by $\underline{\rm aut}(A)$, is called in non-commutative geometry the *non-commutative symmetry group* of A [14] and its description is a very complicated matter. The existence and description of these universal (co)acting bialgebras/Hopf algebras has been considered recently in [1] in the context of Ω-algebras. The duality between Sweedler's and Manin-Tambara's objects has been extended to this general setting and necessary and sufficient conditions for the existence of the universal coacting bialgebras/Hopf algebras, which roughly explains the need for assuming finite-dimensionality in Manin-Tambara's constructions, are given. Furthermore, universal coacting objects for Poisson algebras have also been considered in [2] but from a different perspective, leading to entirely different constructions. We only point out that in [2], the universal coacting object considered is actually a Poisson Hopf algebra. For more background on the importance and the applications of universal bialgebras/Hopf algebras in various areas of mathematics, we refer to [3, 5, 6, 7, 9, 10].

The key object of our work, namely the *universal algebra* of two Poisson algebras P and Q, is a pair $\big(\mathcal{P}(P,\,Q),\,\eta\big)$ consisting of a commutative algebra $A := \mathcal{P}(P,\,Q)$ and a Poisson algebra homomorphism $\eta\colon Q \to P \otimes \mathcal{P}(P,\,Q)$ satisfying a certain universal property. If P is finite-dimensional, then the universal algebra $\mathcal{P}(P,\,Q)$ of P and Q exists and we provide an explicit construction of it. This result has two important consequences: for a fixed Poisson P-module U there exists a canonical functor $U \otimes -\colon {}_A\mathcal{M} \to {}_Q\mathcal{PM}$ from the category of usual A-modules (i.e. representations of the associative algebra A) to the category of Poisson Q-modules (i.e. Poisson representations of Q) and moreover, if U is finite-dimensional this functor has a left adjoint. Secondly, if V is an A-module, then there exists a canonical functor $- \otimes V\colon {}_P\mathcal{PM} \to {}_Q\mathcal{PM}$ connecting the categories of Poisson modules over P and Q and, furthermore, if V is finite-dimensional then the aforementioned functor has a left adjoint. These results provide answers, at the level of Poisson algebras, to the following general problem:

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two mathematical objects (not necessary in the same category), is it possible to construct "canonical functors" between the representation categories $\mathrm{Rep}(\mathcal{O}_1)$ *and* $\mathrm{Rep}(\mathcal{O}_2)$ *of the two objects?*

Three more applications of our constructions are considered. For a Poisson algebra P of dimension n, we denote $\mathcal{P}(P) := \mathcal{P}(P,\,P)$ and we construct $\mathcal{P}(P)$ as the quotient of the polynomial algebra $k[X_{ij}\,|\,i, j = 1, \cdots, n]$ through an ideal generated by $2n^3$ non-homogeneous polynomials of degree ≤ 2. $\mathcal{P}(P)$ has a canonical bialgebra structure and, moreover, $\mathcal{P}(P)$ is the *initial object* of the category $\mathrm{CoactBialg}_P$ of all commutative bialgebras coacting on P and, for this reason, we call it the *universal coacting bialgebra* of P. As in the case of Lie [4] or associative algebras [12], the universal bialgebra $\mathcal{P}(P)$ has two important applications, which provide the theoretical answer for Poisson algebras, of the following open questions:

(1) *Describe explicitly the automorphisms group of a given Poisson algebra* P;
(2) *Describe and classify all* G*-gradings on* P *for a given abelian group* G.

More precisely, we prove that there exists an isomorphism of groups between the group of all Poisson automorphisms of P and the group of all invertible group-like elements of the finite dual $\mathcal{P}(P)^{\circ}$. The second application is given is the following: for an abelian group G, all G-gradings on a finite dimensional Poisson algebra P are described and classified in terms of bialgebra homomorphisms $\mathcal{P}(P) \to k[G]$. By taking Takeuchi's commutative Hopf envelope of $\mathcal{P}(P)$, we obtain that the category CoactHopf_P of all commutative Hopf algebras coacting on P has an initial object $\mathcal{H}(P)$. It is reasonable to hope that $\mathcal{H}(P)$ will play the role of a non-commutative symmetry group of the Poisson algebra P. This expectation is based on the fact that the concept of Poisson H-comodule algebra which we are dealing with, is the algebraic counterpart of the action of an algebraic groups on an affine Poisson variety [8, Example 2.20].

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A bialgebra theory for transposed Poisson algebras

Chengming Bai
(joint work with Guilai Liu)

Transposed Poisson algebras are the dual notion of Poisson algebras by exchanging the roles of two binary operations in the Leibniz rule defining the Poisson algebras. The approach for Poisson bialgebras characterized by Manin triples with respect to the invariant bilinear forms on both the commutative associative algebras and the Lie algebras is not available for giving a bialgebra theory for transposed Poisson algebras. Alternatively, we consider Manin triples with respect to the commutative 2-cocycles on the Lie algebras instead. Explicitly, we first introduce the notion of anti-pre-Lie bialgebras as the equivalent structure of Manin triples of Lie algebras with respect to the commutative 2-cocycles since anti-pre-Lie algebras are regarded as the underlying algebraic structures of Lie algebras with nondegenerate commutative 2-cocycles. Then we introduce the notion of anti-pre-Lie-Poisson bialgebras, characterized by Manin triples of transposed Poisson algebras with respect to the bilinear forms which are invariant on the commutative associative algebras and commutative 2-cocycles on the Lie algebras, giving a bialgebra theory for transposed Poisson algebras. They are commutative and cocommutative infinitesimal bialgebras and anti-pre-Lie bialgebras satisfying certain compatible conditions. Finally the coboundary cases and the related structures such as analogues of the classical Yang-Baxter equation and $\mathcal{O}$-operators are studied.

Pre-Lie and Post-Lie Algebra Structures

Dietrich Burde
(joint work with Karel Dekimpe)

Post-Lie algebras and post-Lie algebra structures are an important generalization of left-symmetric algebras, also called pre-Lie algebras, and left-symmetric algebra structures on Lie algebras, which arise in many areas of algebra and geometry [1], such as left-invariant affine structures on Lie groups, affine crystallographic groups, simply transitive affine actions on Lie groups, convex homogeneous cones, faithful linear representations of Lie algebras, operad theory and several other areas.

In this talk we present several results and open conjectures concerning the existence of post-Lie algebra structures on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ over a fixed vector space V. In particular we are interested in cases, where either $\mathfrak{g}$ or $\mathfrak{n}$ has one of the following properties: it is abelian, nilpotent, solvable, simple, semisimple, reductive, complete or perfect. We may assume here that the algebras, if possible, do not belong to several classes simultaneously, to avoid an unnecessary overlap.

Over the last years we have obtained already several results on the existence of post-Lie algebra structures, see [2, 3, 4, 5]. The methods use the theory of Rota-Baxter operators, decomposition theory, cohomology theory and several other tools.

In a recent paper [6], we proved the following rigidity result.

Theorem. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras, where* $\mathfrak{g}$ *is semisimple and* $\mathfrak{n}$ *is arbitrary. Suppose that* $(\mathfrak{g}, \mathfrak{n})$ *admits a post-Lie algebra structure. Then* $\mathfrak{n}$ *is isomorphic to* $\mathfrak{g}$*.*

We will give a sketch of the proof. It uses several non-trivial results about decompositions of reductive Lie groups and Lie algebras by Onishchik. The result shows that the condition of $\mathfrak{g}$ being semisimple is very strong. A similar result for $\mathfrak{n}$ being semisimple does not hold.

Proposition. *Let* $\mathfrak{n}$ *be a semisimple Lie algebra. Then there exists a solvable non-nilpotent Lie algebra* $\mathfrak{g}$*, such that* $(\mathfrak{g}, \mathfrak{n})$ *is a pair of Lie algebras admitting a post-Lie algebra structure.*

Currently we are working on the generalization of the results for the semisimple case to the case of perfect Lie algebras. Here a Lie algebra L is called *perfect*, if $[L, L] = L$. A typical example of a perfect Lie algebra, which is not semisimple, is the Lie algebra $\mathfrak{sl}_n(\mathbb{C}) \ltimes V(n)$, where $V(n)$ is the natural irreducible representation of $\mathfrak{sl}_n(\mathbb{C})$. We have the following conjecture.

Conjecture. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras, where* $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \ltimes V(n)$ *and* $\mathfrak{n}$ *is nilpotent. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

This would be the first step to a more general conjecture, which is as follows.

Conjecture. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras, where* $\mathfrak{g}$ *is perfect non-semisimple, and* $\mathfrak{n}$ *is nilpotent. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

We have proved the second conjecture for the case, where $\mathfrak{g}$ is semisimple. However, the case of perfect Lie algebras is much more complicated.

A further question is about the case where $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is simple or semisimple. Using a classification of perfect Lie algebras of dimension $n \leq 8$ over $\mathbb{C}$, we proved the following result.

Proposition. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras, where* $\mathfrak{g}$ *is perfect non-semisimple, and* $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$*. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

We conjecture that the same conclusion holds for any semisimple Lie algebra $\mathfrak{n}$, and not only for $\mathfrak{sl}_3(\mathbb{C})$.

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Lie-Jordan elements and q-admissible operads

Askar Dzhumadil'daev

Notations: K be a field of characteristic $p \geq 0$; $\mathcal{M}ag = K[x_1, x_2, \ldots]$ free Magma, i.e., an algebra of non-commutative non-associative polynomials with generators $x_1, x_2, \ldots$; $\mathcal{M}ag(n)$ is multi-linear part of free magma of degree $n+1$; $\mathbb{T}_k$ and $\mathbb{T}_k^{np}$ are sets of planar and non-planar binary trees with $k+1$ leaves, $\mathbb{T} = \cup_{k\geq 0}\mathbb{T}_k$ and $\mathbb{T}^{np} = \cup_{k\geq 0}\mathbb{T}_k^{np}$; $\mathcal{T}, \mathcal{T}^{np}, \mathcal{T}_k, \mathcal{T}_k^{np}$ are linear spans of $\mathbb{T}, \mathbb{T}^{np}, \mathbb{T}_k, \mathbb{T}_k^{np}$; $\mathbb{T}_k^{\bar{q}}$ set of planar binary trees whose i-th inner vertex is colored by $q_i \in K$, $1 \leq i \leq k$, where $\mathbf{q} = (q_1, q_2, \ldots,)$; similar notations for $\mathbb{T}_k^{\mathbf{q}}, \mathbb{T}_k^{np,\mathbf{q}}, \mathbb{T}^{\mathbf{q}}, \mathbb{T}^{np,\mathbf{q}}$, etc. Then $\tau_q(\omega)(a,b) = \omega(a,b) + q\omega(b,a)$ corresponds to q-commutator of algebra (A, ω), We endow $\mathcal{T}$ by structure of algebra under bucket product $st = s \vee t$.

We introduce equivalency relation on non-planar trees: two such trees are equivalent if one can be obtained from the second one by permuting of branches. Take as representative of a non-planar tree a tree such that for any inner vertex its left sub-branch is no more than right-sub-branch. We identify an equivalency class with a representative and we can assume that $\mathbb{T}_n^{np} \in \mathbb{T}_n$.

Space of operations on free magma has a base generated by planar binary rooted trees. To construct elements of free magma one should label its leaves by elements of magma and inner vertices by multiplication of magma and correspond to each inner vertex products of its sons. Then an element obtained at a root is a product of leaf elements. For a tree, t denotes by $|t|$ its number of inner vertices. We obtain commutative monoid denoted G_1, under composition

$$\tau_q \tau_{q'} = (1 + q\,q')\tau_{\frac{q+q'}{1+qq'}}$$

It has unit τ_0 and the group of invertible elements is isomorphic to $\{q \in K | q^2 \neq 1\}$.

There are two kinds of extensions of coloring maps for any n. First way, all inner vertices are changed by q-commutator and $\tau_q : \mathbb{T}_n \to \mathbb{T}_n^{(q)}$ is defined in natural way. The second way, we numerate somehow inner vertices and each i-th inner vertex has its own color, say q_i, and in this case we have to consider $\tau_{\mathbf{q}} : \mathbb{T}_n \to \mathbb{T}_n^{\mathbf{q}}$, where $\mathbf{q} = (q_1, q_2, \ldots)$ is a sequence of colors. It is clear that the first case is a particular case of the second one: take $\mathbf{q} = (q, q, \ldots)$. Let

$$G_n = G_1 \times \cdots \times G_1 \cong K^n$$

be commutative monoid generated by coloring maps $\tau_{\mathbf{q}}$, where $\mathbf{q} = (q_1, q_2, \ldots)$.

Let

$$M_n = \{\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) | \alpha_i^2 = 1\}$$

and

$$M = \cup_{n\geq 1} M_n.$$

Say that $\boldsymbol{\alpha} \in M_n$ has (l_+, l_-)-*type* if numbers of components of $\boldsymbol{\alpha}$ are equal to $+1$ and -1 are l_+ and l_- respectively. Then $l_+ + l_- = n$. Let

$$M_n^{(l_+,l_-)} = \{\boldsymbol{\alpha} \in M_n \mid type(\boldsymbol{\alpha}) = (l_+, l_-)\}.$$

For $\boldsymbol{\alpha} \in M_n, t \in \mathbb{T}_n$ set

$$t_{\boldsymbol{\alpha}} = \tau_{\boldsymbol{\alpha}} t.$$

We call $t_{\boldsymbol{\alpha}}$ *Lie-Jordan element of type* $\boldsymbol{\alpha}$. Let $\mathcal{T}_n^{np,(l_+,l_-)}$ be subspace of $\mathcal{T}_n^{np}$ generated by trees $t_{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} \in M_n^{(l_+,l_-)}$. For $X \in \mathcal{T}_n$ say that X is *Lie-Jordan element of* $\pm$-*type* (l_+, l_-), or shortly homogeneous LJ-element, if X is a linear combination of elements constructed by trees $t \in \mathcal{T}_n^{np,(l_+,l_-)}$.

Well known that space of $(n+1)$-ary operations on $\mathcal{M}ag$ can be generated by planar rooted trees with n inner vertices and $\mathbb{T}_n$ can be selected as a base.

Theorem. *Set of Lie-Jordan elements $t_{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} \in M_n$ and $t \in \mathbb{T}_n^{np}$, forms base of the space of n-ary operations on $\mathcal{M}ag$. In particular, dim $\mathcal{M}ag(n) = 2^n(2n-1)!!$.*

Theorem. *The following conditions are equivalent*

- *X is Lie-Jordan element of type $\boldsymbol{\alpha} \in M_n$.*
- $\tau_{\boldsymbol{\alpha}} X = 2^n\, X$
- *$\tau_{\boldsymbol{\beta}} X = 0$, for any $\boldsymbol{\beta} \in M_n, \boldsymbol{\beta} \neq \boldsymbol{\alpha}$.*

In particular, the following conditions are equivalent

- *$X \in \mathcal{M}ag(n)$ is Lie element*
- *$\tau_{\boldsymbol{\alpha}} X = 2^n X$, where $\boldsymbol{\alpha} = (-1, -1, \ldots, -1) \in M_n$*
- *$\tau_{\boldsymbol{\beta}} X = 0$, for any $\boldsymbol{\beta} \neq (-1, -1, \ldots, -1).,\boldsymbol{\beta} \in M_n$*

and the following conditions are also equivalent

- *$X \in \mathcal{M}ag(n)$ is Jordan element*
- *$\tau_{\boldsymbol{\alpha}} X = 2^n X$, where $\boldsymbol{\alpha} = (1, 1, \ldots, 1) \in M_n$*
- *$\tau_{\boldsymbol{\beta}} X = 0$, for any $\boldsymbol{\beta} \neq (1, 1, \ldots, 1), \boldsymbol{\beta} \in M_n$*

Let $\mathcal{V} = \langle f_1, \ldots, f_k\rangle$ be an operad of algebras generated by polynomial identities $f_1 = 0, \ldots, f_k = 0$. Call an algebra $A = (A, \cdot)$ *q-admissible $\mathcal{V}$-algebra* and denote by $\mathcal{V}Adm^{(q)}$ class of such algebras, if A under q-commutator $A^{(q)} = (A, \cdot_q)$ became $\mathcal{V}$-algebra.

Theorem. *Let $q^2 \neq 1$. Then $\mathcal{V}Adm^{(q)}$ forms a variety of algebras, namely, variety generated by polynomial identities $f_1^{(-q)} = 0, \ldots, f_k^{(-q)} = 0$. As categories, varieties $\mathcal{V}$ and $\mathcal{V}Adm^{(q)}$ are isomorphic. Dimensions sequence of multi-linear parts of $d_{\mathcal{V},n} = dim\, \mathcal{V}(n)$ are not changed,*

$$d_{\mathcal{V},n} = d_{\mathcal{V}^{(q)},n}.$$

If $f_i, 1 \leq i \leq s$, are homogeneous Lie-Jordan polynomials, then

$$\mathcal{V}^{(q)} = \langle f_1, \ldots, f_s, f_{s+1}^{(-q)}, \ldots, f_k^{(-q)}\rangle.$$

If all $f_i, 1 \leq i \leq k$, *are homogeneous Lie-Jordan polynomials, then*

$$\mathcal{V}^{(q)} = \mathcal{V}.$$

An algebra with the following polynomial identities is called reverse-associative, anti-reverse-associative, and weak Leibniz, respectively:

$$a(bc) = (cb)a,$$

$$(a(bc) = -(cb)a,$$

$$[a, b]c = 2(a(bc) - b(ac)), \quad a[b, c] = 2((ab)c - (ac)b)$$

Applications of our results for these classes of algebras are given below.

Results on reverse-associative and anti-reverse-associative operads

Theorem. *Reverse-associative and anti-reverse-associative operads have the following properties.*

- **(a)** *Operads* $\mathcal{R}evas$ *and* $\mathcal{A}revas$ *are Koszul.*
- **(b)** *Any anti-reverse-associative algebra is associative-admissible and Lie-admissible.*
- **(c)** $\mathcal{R}evas^! = \mathcal{A}revas$.
- **(d)** $\mathcal{R}evas = \langle \{t_1, [t_2, t_3]\}, \quad [t_1, \{t_2, t_3\}]\rangle$.
- **(e)** $\mathcal{A}revas = \langle [t_1, [t_2, t_3]], \quad \{t_1, \{t_2, t_3\}\}\rangle$.
- **(f)** *Plus-colored trees generate a base of free reverse-associative operad. In particular,*

$$\mathcal{R}evas(n) = \mathcal{C}om^{+}(n) \oplus \mathcal{C}om^{-}(n), \qquad n > 1.$$

- **(g)** *Minus-colored trees generate a base of free anti-reverse-associative operad. In particular,*

$$Arevas(n) = Com^{\pm}(n) \oplus Com^{\mp}(n), \qquad n > 1,$$

- **(h)** $dim\ \mathcal{A}revas(n) = dim\ \mathcal{R}evas(n) = 2(2n-1)!!, \qquad n > 1,$

$$dim\ \mathcal{A}revas(1) = dim\ \mathcal{R}evas(1) = 1.$$

- **(i)** $\mathcal{A}revas = \mathcal{C}omNil_2 \star \mathcal{A}comNil_2$, *where*

$$\mathcal{C}omNil_2 = \langle t_1t_2 - t_2t_1, (t_1t_2)t_3\rangle,$$

$$\mathcal{A}comNil_2 = \langle t_1t_2 + t_2t_1, (t_1t_2)t_3\rangle.$$

- **(j)** *Multipliaction table in free reverse-associative algebra* $F_{revas}(X)$ *generated by elements* $X = \{x_1, x_2, \ldots, x_n\}$ *can be defined by*

$$x_ix_j = x_i \bullet x_j + x_i \circ x_j,$$

$$x_iu = x_i \bullet u_+ + x_i \circ u_-,$$

$$ux_j = u_+ \bullet x_j + u_- \circ x_j,$$

$$uv = u_+ \bullet v_+ + u_- \circ v_-,$$

where $u, v \in F_{revas}(X)^2$, $1 \leq i, j \leq n$.

(k) *Multipliaction table in free anti-reverse-associative algebra $F_{arevas}(X)$ generated by elements $X = \{x_1, x_2, \dots, x_n\}$ can be constructed as in reverse-associative case,*

$$x_i x_j = x_i \bullet x_j + x_i \circ x_j,$$

$$x_i u = x_i \bullet u_- + x_i \circ u_+,$$

$$u x_j = u_- \bullet x_j + u_+ \circ x_j,$$

$$uv = u_- \bullet v_- + u_+ \circ v_+,$$

for any $u, v \in F_{arevas}(X)^2$, $1 \leq i, j \leq n$.

Results on associative-admissible operad

Recall that Non-Anti-Commutative Lie operad $\mathrm{Lie}^\flat$ is generated by Jacobi identity $jac = 0$, reverse-associative identity

$$revas(t_1, t_2, t_3) = t_1(t_2 t_3) - (t_3 t_2) t_1 = 0,$$

and the identity

$$t_1(t_2 t_3 + t_3 t_2) = 0.$$

Left-Leibniz and right-Leibniz operads are defined by identities

$$llei = 0 \text{ and } rlei = 0$$

respectively, where

$$llei = (t_1 t_2) t_3 - t_1(t_2 t_3) + t_2(t_1 t_3),$$

$$rlei = t_1(t_2 t_3) - (t_1 t_2) t_3 + (t_1 t_3) t_2.$$

So, two-sided Leibniz operad $\mathcal{L}ei$ is defined by left- and right-Leibniz identities

$$\mathcal{L}ei = \langle llei, rlei \rangle.$$

Theorem. *Associative-admissible operad has the following properties.*

(a) $\mathcal{L}ie^\flat = \mathcal{L}ei$.
(b) *$\dim \mathcal{L}ie^\flat(n) = (n-1)!$, if $n \neq 2$ and $= 2$, if $n = 2$.*
(c) *Operads $\mathcal{A}sAdm$ and $\mathcal{L}ie^\flat$ are Koszul.*
(d) $\mathcal{A}sAdm^! = \mathcal{L}ie^\flat$.
(e) $\mathcal{A}sAdm = \mathcal{A}sCom \star \mathcal{A}com$.
(f) *Dimensions of multi-linear parts of associative-admissible operad $d_n = \dim \mathcal{A}sAdm(n)$ can be found by the following recurrence relations*

$$d_n = \sum_{k=1}^{n-1} k! F_{k+2} B_{n-1,k}(d_1, d_2, \dots, d_{n-k}), \quad n > 1,$$

$$d_1 = 1,$$

where F_n are Fibonacci numbers and $B_{n,k}(x_1, \dots, x_{n-k+1})$ are Bell polynomials.

Theorem. *Let p be prime. Dimensions $d_n = dim\ \mathcal{A}s\mathcal{A}dm(n)$ of multi-linear part of associative-admissible operad satisfy the following congruences*

$$d_{p-1} \equiv \begin{cases} 1(mod\, p), & if\ p \neq 3, \\ -1 & if\ p = 3, \end{cases}$$

$$d_p \equiv \begin{cases} 1(mod\, p), & if\ p \neq 2, \\ 0 & if\ p = 2, \end{cases}$$

$$d_{p+1} \equiv 2(mod\, p),$$

$$d_{p+2} \equiv 10(mod\, p).$$

Results on weak Leibniz operad

Let us define left-weak Leibniz and right-weak-Leibniz polynomials by

$$lwlei = [t_1, t_2]t_3 - 2t_1(t_2t_3) + 2t_2(t_1t_3),$$

$$rwlei = t_1[t_2, t_3] - 2(t_1t_2)t_3 + 2(t_1t_3)t_2.$$

Let $\mathcal{L}wlei = \langle lwlei\rangle$ and $\mathcal{R}wlei = \langle rwlei\rangle$ are Left-weak-Leibniz and Right-weak-Leibniz operads, So, two-sided weak-Leibniz operad $\mathcal{W}lei$ is defined by

$$\mathcal{W}lei = \langle lwlei, rwlei\rangle.$$

Let I be some finite set of integers, $\varepsilon_i \in K$, for any $i \in I$. Let $L(I, \varepsilon)$ be infinite-dimensional algebra with base $\{e_i | i \in \mathbf{Z}\}$ and multiplication

$$e_i \cdot e_j = (i - j)e_{i+j} + \sum_{k \in I} \varepsilon_k e_{i+j+k}.$$

Theorem.

- $\mathrm{Wlei}^! = \mathrm{Wlei}$.
- *Weak Leibniz operad is not Koszul.*
- *Any weak Leibniz algebra is associative-admissible. Any weak Leibniz algebra is two-sided Alia, if $p \neq 3$. In particular, any weak Leibniz algebra is Lie-admissible, if $p \neq 3$.*
- *There exist weak Leibniz algebras that are not Leibniz.*
- *The algebra $L(I, \varepsilon)$ is a simple weak Leibniz algebra for any I and $\varepsilon_i \in K, i \in I$.*

Results on Associative-admissible and Lie-admissible operad

Let $\mathcal{A}sLieAdm$ be operad representing associative-admissible and Lie-admissible algebras, i.e. algebras with the following identities

$$\{t_1,\{t_2,t_3\}\} = \{\{t_1,t_2\},t_3\},$$

$$[[t_1,t_2],t_3] + [[t_2,t_3],t_1] + [[t_1,t_2],t_3] = 0.$$

Let $\mathcal{A}sCom$ be operad for associative and commutative algebras,

$$t_1(t_2t_3) = (t_1t_2)t_3, \quad t_1t_2 = t_2t_1.$$

Let

$$aslia' = rwlei - lwlei,$$

$$aslia(t_1,t_2,t_3) = aslia'(t_1,t_2,t_3) + 2\langle t_3,t_1,t_2\rangle = [t_1,[t_2,t_3]] - 2(\langle t_2,t_1,t_3\rangle - \langle t_3,t_1,t_2\rangle).$$

In other words,

$$aslia' = aslia'(t_1,t_2,t_3) = [t_1,[t_2,t_3]] + 2(arevas(t_2,t_3,t_1) - arevas(t_3,t_2,t_1)) =$$

$$[t_1,[t_2,t_3]] + 2(t_2(t_3t_1) - t_3(t_2t_1) - (t_1t_2)t_3 + (t_1t_3)t_2).$$

Theorem. *The operad $\mathcal{A}sLieAdm$ has the following properties.*

- *The operad $\mathcal{A}sLieAdm$ is Koszul.*
- $\mathcal{A}sLieAdm = \langle aslia\rangle$, *if* $p \neq 3$.
- $\mathcal{A}sLieAdm = \mathcal{A}sCom \star \mathcal{L}ie$.
- $\mathcal{A}sLieAdm^! = \langle revas, lwlei \text{ or } rwlei\rangle$.
- $d^!_n = dim\ \mathcal{A}sLieAdm^!(n) = (n-1)! + 1$.
- *Poincare series* $f^!_{\mathcal{A}sLieAdm}(x) = \sum_{i\geq 1} d^!_i \frac{x^i}{i!} = -1 + e^x - x - \ln(1-x)$
- $d_n = dim\ \mathcal{A}sLieAdm(n) = \sum_{k=1}^{n-1}(-1)^k\lambda_k B_{n-1,k}(d_1,d_2,\dots,d_{n-k})$, *where*

$$\lambda_k = \sum_{s=1}^{k}(-1)^s s! B_{k,s}(1!+1, 2!+1, \dots, i!+1, \dots, (k-s+1)!+1).$$

Simple transposed Poisson algebras and Jordan superalgebras

Amir Fernández Ouaridi

Transposed Poisson algebras (TPAs, for short) were introduced as a dual class of the Poisson algebras in the sense that the roles of the two multiplications in the Leibniz rule are swapped [1]. Precisely, we have the identity

$$2x \circ \{y,z\} = \{x \circ y, z\} + \{y, x \circ z\}.$$

This identity can be realized as the left multiplication of the associative commutative algebra is a $\frac{1}{2}$-derivation of the Lie algebra. These derivations of Lie algebras are well-studied (for example, see [4]). The interest on this class has increased rapidly in the last four years (see [2] and the references therein). Some known facts about TPAs include the closure undertaking tensor products, the Koszul self-duality as an operad or the correspondence with weak Leibniz algebras by depolarization. TPAs coincide with commutative Gelfand-Dorfman algebras [1, 6]. In this talk, we will discuss about simple transposed Poisson algebra. For a further read on the topic of simple TPAs, we refer to the paper of the author [3].

Recall that an ideal of a Poisson-type algebra is a proper subspace such that it is simultaneously an ideal of both multiplications. Kantor [5] introduced an invertible way to construct a Jordan superalgebra from a Poisson algebra (the Kantor double), this construction preserves the simplicity in both directions, so a classification of the complex simple finite-dimensional Poisson algebras was obtained from the classification of simple Jordan superalgebras.

Our first approach to the problem of classifying simple TPAs took us to the study of the Kantor double of a TPA. It turns out that, as in the Poisson case, the Kantor double of a TPA is a Jordan superalgebra. Hence, TPAs are Jordan brackets. This motivates the following open question.

Question. *Characterize the subclass of Jordan brackets that arise from TPAs. Are these Jordan algebras special?*

The construction of simple TPA from simple Jordan superalgebras is partially possible. Indeed, we proved that a TPA is simple with perfect associative part if and only if its Kantor double is simple. Although we can not construct all the simple TPAs, a straightforward check of the inverse Kantor double of a complex simple finite-dimensional Jordan superalgebras shows that none of them produce non-trivial TPAs. In other words, there are no complex simple finite-dimensional TPAs with perfect associative part.

This result was improved, thanks to the next key result: over an arbitrary field and for any dimension, a TPA is simple if and only if its Lie bracket is simple. It is worth to point out that this result is also valid for the super case. The main idea to prove this fact is the introduction of the notion of a transposed quasi-ideal (see [3] for details).

As a consequence of the cited result, any complex simple finite-dimensional TPA is trivial. This is thanks to a result of Filippov, who showed that every simple complex finite-dimensional Lie algebra has trivial $\frac{1}{2}$-derivations [4]. However, there are simple finite-dimensional TPAs over fields of characteristic $p > 2$. This motivates the following question.

Question. *Classify the finite-dimensional TPAs.*

Another consequence is that any TPA with simple associative part has either abelian Lie part or simple Lie part. An example of a TPA with both multiplications being simple was presented by A. Dzhumadildayev on the field of formal series during the mini-workshop. This motives the following question:

Question. *Is there any finite-dimensional simple TPA with both multiplications being simple?*

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Compatible structures of operads by polarization, Koszul duality, and Manin products

Li Guo

(joint work with Xing Gao, Huhu Zhang)

Traditionally, a compatible structure is usually referring to a linearly compatible structure, where a vector space is equipped with two identical copies of operations in a given algebraic structure (Lie algebra, for example) so that the sums of these two copies of operations still yield the same algebraic structure. Together with several other algebraic compatible structures, they have been widely studied in mathematics and mathematical physics.

The first instance of linearly compatible structures appeared in the pioneering work [9] of Magri on bi-Hamiltonian systems, in which a Poisson algebra has two linearly compatible Poisson (Lie) brackets. Such a structure was abstracted to the notion of a bi-Hamiltonian algebra and was studied in the context of operads and Koszul duality [3]. Compatible Lie algebras have been studied in connection with integrable systems, classical Yang-Baxter equation, loop algebras and elliptic theta functions [5, 6, 7, 10]. In [2], quantum bi-Hamiltonian systems were built on linearly compatible associative algebras [11, 12].

Other algebraic structures with multiple copies of operations related by various compatibility conditions have appeared in recent studies in broad areas.

For example, a multiple pre-Lie algebra emerged in the remarkable work of Bruned, Hairer and Zambotti [1, 4] on algebraic renormalization of regularity structures. Matching Rota-Baxter algebras appeared in the algebraic study of Voterra integral equations [8, 15]. These structures can be broadly grouped into linearly compatible, matching, and totally compatible structures.

General studied of such structures using operads have been carried out with various restrictions [13, 14]. This talk presents some recent progress aiming at giving a unified approach to these various structures that can be applied to an arbitrary operad. We first introduce the notion of polarization in operads, leading to the notion of linearly compatible operads. Refining the polarization by the process of taking foliation, we obtain a general notion of matching type operads including those that have appeared. When we make all matching compatibilities of a given operad equal, we obtain the totally compatible operad of this operad.

For unary/binary quadratic operads, we prove that the linear compatibility and the total compatibility are in Koszul dual to each other, and there is a Koszul self-duality among the matching compatibilities. For binary quadratic operads, these three compatible operads can also be obtained by Manin products. For a finitely generated binary quadratic Koszul operad, we prove that the three types of compatible operads are also Koszul.

Back to the Poisson algebra origin of this study, natural questions arise, such as

(1) Study the transpose bi-Hamiltonian algebra?
(2) What should be the algebraic structure when the Poisson bracket in a Poisson algebra is replaced by any of the compatible Lie algebras, with the bi-Hamiltonian system or a bi-Hamiltonian algebra as a special case?
(3) The same question can be asked for the transposed Poisson algebra.

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Transposed Poisson structures on Lie incidence algebras

Mykola Khrypchenko
(joint work with Ivan Kaygorodov)

A *transposed Poisson algebra* [1] is a triple $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ consisting of a vector space $\mathfrak{L}$ with two bilinear operations $\cdot$ and $[\cdot, \cdot]$, such that

(1) $(\mathfrak{L}, \cdot)$ is a commutative associative algebra;
(2) $(\mathfrak{L}, [\cdot, \cdot])$ is a Lie algebra;
(3) the *"transposed" Leibniz law* holds: $2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y]$ for all $x, y, z \in \mathfrak{L}$.

A *transposed Poisson algebra structure* on a Lie algebra $(\mathfrak{L}, [\cdot, \cdot])$ is a (commutative associative) multiplication $\cdot$ on $\mathfrak{L}$ such that $(\mathfrak{L}, \cdot, [\cdot, \cdot])$ is a transposed Poisson algebra.

A transposed Poisson structure $\cdot$ on $(\mathfrak{L}, [\cdot, \cdot])$ is said to be *of Poisson type* if it is at the same time a usual Poisson structure on $\mathfrak{L}$. It was proved in [1] that this happens if and only if

$$x \cdot [y, z] = [x \cdot y, z] = 0$$

for all $x, y, z \in \mathfrak{L}$. Another class of transposed Poisson structures that can be defined on any Lie algebra $(\mathfrak{L}, [\cdot, \cdot])$ is as follows. Fix $c \in Z([\mathfrak{L}, \mathfrak{L}])$ and consider the following *mutation* of the product $[\cdot, \cdot]$:

$$a \cdot_c b = [[a, c], b] = [a, [c, b]].$$

Then $\cdot_c$ is a transposed Poisson structure on $\mathfrak{L}$ called *mutational*.

Given two binary operations $\cdot_1$ and $\cdot_2$ on a vector space V, their *sum* $*$ is defined by

$$a * b = a \cdot_1 b + a \cdot_2 b.$$

We say that $\cdot_1$ and $\cdot_2$ are *orthogonal*, if

$$V \cdot_1 V \subseteq \operatorname{Ann}(V, \cdot_2) \text{ and } V \cdot_2 V \subseteq \operatorname{Ann}(V, \cdot_1).$$

In this case $*$ defined above is called the *orthogonal* sum of $\cdot_1$ and $\cdot_2$.

Clearly, the sum $*$ of two transposed Poisson structures $\cdot_1$ and $\cdot_2$ on $(\mathfrak{L}, [\cdot, \cdot])$ is commutative and satisfies the transposed Leibniz law. If $\cdot_1$ and $\cdot_2$ are orthogonal, then $*$ is also associative, so we get the following.

Proposition. *The orthogonal sum of two transposed Poisson structures on a Lie algebra $\mathfrak{L}$ is a transposed Poisson structure on $\mathfrak{L}$.*

Observe that any mutational transposed Poisson structure on a Lie algebra $\mathfrak{L}$ is orthogonal to any transposed Poisson structure of Poisson type on $\mathfrak{L}$.

Corollary. *The (orthogonal) sum of a mutational transposed Poisson structure on $\mathfrak{L}$ and a transposed Poisson structure of Poisson type on $\mathfrak{L}$ is a transposed Poisson structure on $\mathfrak{L}$.*

Let X be a finite poset and K a field. Recall that the *incidence algebra* $I(X,K)$ of X over K (see [3]) is the associative K-algebra with basis $\{e_{xy} \mid x \le y\}$ and multiplication is given by

$$e_{xy}e_{uv} = \begin{cases} e_{xv}, & y = u, \\ 0, & y \ne u, \end{cases}$$

for all $x \le y$ and $u \le v$ in X. Given $f \in I(X,K)$, we write $f = \sum_{x \le y} f(x,y)e_{xy}$, where $f(x,y) \in K$. Let us denote $e_x := e_{xx}$, and for arbitrary $Y \subseteq X$ put $e_Y := \sum_{y \in Y} e_y$. Then e_Y is an idempotent and $e_Y e_Z = e_{Y \cap Z}$, in particular, $e_x e_y = 0$ for $x \ne y$. Notice that $\delta := e_X$ is the identity element of $I(X,K)$.

We consider $I(X,K)$ as a Lie algebra under the commutator product $[f,g] = fg - gf$. If X is connected, then one can easily prove that

$$Z(I(X,K)) = \langle \delta \rangle \text{ and } [I(X,K), I(X,K)] = \langle e_{xy} \mid x < y \rangle.$$

Moreover,

$$Z([I(X,K), I(X,K)]) = \langle e_{xy} \mid \mathrm{Min}(X) \ni x < y \in \mathrm{Max}(X) \rangle.$$

Diagonal elements of $I(X,K)$ are $f \in I(X,K)$ with $f(x,y) = 0$ for $x \ne y$. They form a commutative subalgebra $D(X,K)$ of $I(X,K)$ with basis $\{e_x \mid x \in X\}$. As a vector space,

$$I(X,K) = D(X,K) \oplus [I(X,K), I(X,K)].$$

Thus, each $f \in I(X,K)$ is uniquely written as $f = f_D + f_J$ with $f_D \in D(X,K)$ and $f_J \in [I(X,K), I(X,K)]$. Observe that $Z(I(X,K)) \subseteq D(X,K)$.

In this talk, we describe transposed Poisson structures on $(I(X,K), [\cdot,\cdot])$, where X is a finite connected poset and K is a field of characteristic zero. It is obvious that any transposed Poisson structure of Poisson type on $I(X,K)$ is of the form

$$e_x \cdot e_y = \mu(x,y)\delta$$

for some $\mu : X^2 \to K$ with $\mu(x,y) = \mu(y,x)$, where the associativity of the product is equivalent to

$$\mu(x,y) \sum_{v \in X} \mu(z,v) = \mu(y,z) \sum_{v \in X} \mu(x,v).$$

Observe that we write only non-trivial products here.

Each $\nu \in Z([I(X,K), I(X,K)])$ defines the mutational structure whose non-trivial products are:

$$e_x \cdot e_y = e_y \cdot e_x = \begin{cases} \nu(x,y)e_{xy}, & \mathrm{Min}(X) \ni x < y \in \mathrm{Max}(X), \\ -\sum_{x < v \in \mathrm{Max}(X)} \nu(x,v)e_{xv}, & x = y \in \mathrm{Min}(X), \\ -\sum_{\mathrm{Min}(X) \ni u < x} \nu(u,x)e_{ux}, & x = y \in \mathrm{Max}(X). \end{cases}$$

The definition of the third structure requires some preparation. We say that a pair (x,y) of elements of X is *extreme*, if $x < y$ is a maximal chain in X and there is no cycle in X containing x and y. Denote $X_e^2 = \{(x,y) \in X^2 \mid (x,y) \text{ is extreme}\}$. We set $\mathrm{sgn}_{u_0}(x,y) := 1$, if there is a path starting at u_0 and ending at (x,y).

Otherwise there is a path starting at u_0 and ending at (y, x), in which case set $\operatorname{sgn}_{u_0}(x, y) := -1$. Given $(x, y) \in X_e^2$ and $u, v \in X$, we say that u and v are *on the same side* with respect to (x, y), if there is a path from u to v that does not have (x, y) and (y, x) as edges. Otherwise u and v are said to be *on the opposite sides* with respect to (x, y). Fix $u_0 \in X$. For any $(x, y) \in X_e^2$ denote

$$V_{xy} = \{v \in X \mid u_0 \text{ and } v \text{ are on the opposite sides with respect to } (x, y)\}.$$

An arbitrary $\lambda : X_e^2 \to K$ determines the λ-*structure* on $I(X, K)$ as follows:

$$e_x \cdot e_{xy} = e_{xy} \cdot e_x = -e_{xy} \cdot e_y = -e_y \cdot e_{xy} = \lambda(x, y)e_{xy}, (x, y) \in X_e^2,$$

$$e_x \cdot e_y = e_y \cdot e_x = \begin{cases} \operatorname{sgn}_{u_0}(x, y)\lambda(x, y)e_{V_{xy}}, & (x, y) \in X_e^2, \\ -\sum_{(x,v)\in X_e^2} \operatorname{sgn}_{u_0}(x, v)\lambda(x, v)e_{V_{xv}}, & x = y \in \operatorname{Min}(X), \\ -\sum_{(u,x)\in X_e^2} \operatorname{sgn}_{u_0}(u, x)\lambda(u, x)e_{V_{ux}}, & x = y \in \operatorname{Max}(X). \end{cases}$$

Lemma. *Any λ-structure $\cdot$ is a transposed Poisson structure on $I(X, K)$ orthogonal to any structure of Poisson type.*

Lemma *The sum of any mutational structure and any λ-structure is a transposed Poisson structure on $I(X, K)$.*

The following theorem is the main result of our talk [2].

Theorem. *A binary operation $\cdot$ on $I(X, K)$ is a transposed Poisson algebra structure on $I(X, K)$ if and only if $\cdot$ is the sum of a structure of Poisson type, a mutational structure and a λ-structure.*

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Poisson derivations of a semiclassical limit of a family of quantum second Weyl algebras

Stéphane Launois

(joint work with Isaac Oppong)

In [1], we studied deformations $A_{\alpha,\beta}$ of the second Weyl algebra and computed their derivations. In this talk, we identify the semiclassical limits $\mathcal{A}_{\alpha,\beta}$ of these deformations and compute their Poisson derivations. Our results show that the first Hochschild cohomology group $\mathrm{HH}^1(A_{\alpha,\beta})$ is isomorphic to the first Poisson cohomology group $\mathrm{HP}^1(\mathcal{A}_{\alpha,\beta})$.

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Gerstenhaber algebra structure on Hochschild cohomology

Samuel A. Lopes

(joint work with Andrea Solotar)

The Hochschild cohomology $\mathsf{HH}^\bullet(\mathsf{A})$ of an associative algebra A encodes many non-trivial properties and features of the algebra, including crucial information about its deformations. In [2], Gerstenhaber constructed two operations on $\mathsf{HH}^\bullet(\mathsf{A})$: the cup product and a (graded) Lie bracket. Together, they form what is now called a Gerstenhaber algebra structure on $\mathsf{HH}^\bullet(\mathsf{A})$. In general, a Gerstenhaber algebra is just a graded Poisson algebra of degree -1. Another example is the exterior algebra $\Lambda^\bullet\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ or, more generally, $\Lambda^\bullet_R L$, for a Lie–Rinehart algebra (R, L).

The Lie bracket on $\mathsf{HH}^\bullet(\mathsf{A})$ is easily defined on the bar resolution, but in general it is quite difficult to compute from a minimal resolution of A. Nevertheless, the graded Lie algebra structure of $\mathsf{HH}^\bullet(\mathsf{A})$ can be quite interesting; in particular, $\mathsf{HH}^\bullet(\mathsf{A})$ is a Lie module for the Lie algebra $\mathsf{HH}^1(\mathsf{A})$ of outer derivations of A.

We will compute the Gerstenhaber bracket on $\mathsf{HH}^\bullet(\mathsf{A})$ in case $\mathsf{A} = \mathsf{A}_h$ is the subalgebra of the Weyl algebra $\mathsf{A}_1 = \mathbb{F}\{x, y\}/\langle [y, x] = 1\rangle$ generated by x and $h(x)y$, for an arbitrary polynomial $h(x)$ over a field $\mathbb{F}$ of characteristic 0. In this case, $\mathsf{HH}^1(\mathsf{A}_h)$ is related to the Virasoro Lie algebra and we will show that the Lie module $\mathsf{HH}^\bullet(\mathsf{A}_h)$ is related to the intermediate series modules for the Virasoro Lie algebra. This is the result of joint work with Andrea Solotar [4].

Some questions: A Gerstenhaber algebra is a Batalin–Vilkovisky algebra (BV algebra, for short) if the Lie bracket coming from the Gerstenhaber algebra is induced by a degree -1 operator Δ with $\Delta^2 = 0$. Thus,

$$[a, b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b) + a\Delta(1)b.$$

BV structures appeared in mathematical physics in connection with the quantization of gauge theories but it is interesting in general to determine when a Gerstenhaber algebra is a BV algebra.

(1) In particular, when is $\mathsf{HH}^\bullet(\mathsf{A})$ a BV algebra?
(2) The former question has a positive answer in case A is a twisted Calabi–Yau algebra with a semisimple Nakayama automorphism [3]. The algebras A_h were shown in [4] to be twisted Calabi–Yau, although the Nakayama automorphism is not in general semisimple. Are there BV structures in $\mathsf{HH}^\bullet(\mathsf{A}_h)$ when the Nakayama automorphism of A_h is not semisimple?

(3) One can look at Poisson analogues of the above setting via semiclassical limit and try to answer similar questions (see also [1]).

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Solvability of symmetric Poisson algebras

Salvatore Siciliano

Let P be a Poisson algebra over a field $\mathbb{F}$. We recall that P is said to satisfy a nontrivial Poisson identity (or that P is a Poisson PI algebra) if there exists a nonzero element in the free Poisson algebra of countable rank which vanishes under any substitution in P (see e.g. [2]). A basic theory of Poisson PI algebras was carried out by Farkas [2, 3], and further developments on this theory were next considered by several authors. In particular, in [6], Mishchenko, Petrogradsky, and Regev developed the theory of so called codimension growth in characteristic zero, and proved that the tensor product of Poisson PI algebras is a Poisson PI algebra.

Now, let L be a Lie algebra over $\mathbb{F}$ and $\{U_n \mid n \geq 0\}$ the canonical filtration of its universal enveloping algebra $U(L)$. Set $U_{-1} = 0$ and consider the symmetric algebra $S(L) = \mathrm{gr}(U(L)) = \oplus_{n=0}^{\infty} U_n/U_{n-1}$, which we identify with the polynomial ring $\mathbb{F}[x_1, x_2, \ldots]$, where $x_1, x_2, \ldots$ is an $\mathbb{F}$-basis of L (cf [1, §2.3]). By linearity and the Leibniz rule, the Lie bracket $[\cdot,\cdot]$ of L can be uniquely extended to a Poisson bracket $\{\cdot,\cdot\}$ of $S(L)$ so that this algebra becomes a Poisson algebra, called the *symmetric Poisson algebra* of L. Moreover, when $\mathbb{F}$ has characteristic $p > 0$, the Poisson bracket of $S(L)$ naturally induces the structure of a Poisson algebra on the factor algebra $\mathbf{s}(L) = S(L)/I$, where I is the ideal generated by the elements x^p with $x \in L$. We will refer to $\mathbf{s}(L)$ as the *truncated symmetric Poisson algebra* of L.

Poisson identities of symmetric Poisson algebras of Lie algebras were first studied by Kostant [4], Shestakov [8], and Farkas [2, 3]. In particular, in [3] Farkas proved that, in characteristic zero, $S(L)$ satisfies a nontrivial Poisson identity if and only if L contains an abelian subalgebra of finite codimension. Some years later, in [5], Giambruno and Petrogradsky extended Farkas' result to arbitrary characteristic and, moreover, established when the truncated symmetric Poisson algebra of a restricted Lie algebra satisfies a nontrivial multilinear Poisson identity.

More recently, in [7], Monteiro Alves and Petrogradsky investigated the Lie identities of $S(L)$ and $\mathbf{s}(L)$. In particular, they determined necessary and sufficient conditions on L such that $S(L)$ and $\mathbf{s}(L)$ are Lie nilpotent, studied the Lie nilpotence class of $\mathbf{s}(L)$ and, in characteristic $p \neq 2$, established when $S(L)$ and $\mathbf{s}(L)$ are solvable. On the other hand, the harder problem of the solvability of $S(L)$ and $\mathbf{s}(L)$ in characteristic 2 remained unsettled and a related conjecture formulated. Afterwards, in [11] a corrected version of that conjecture was proved, thereby completing the classification. Further developments of these topics have been also carried out in [9, 10].

The aim of this talk is to present an overview of the known results about solvable (truncated) symmetric Poisson algebras and their derived lengths. We first recall some theorems about the Lie structure of ordinary and restricted enveloping algebras, which originally motivated the present subject. Next, we summarize results on the existence of nontrivial Poisson identities in symmetric and truncated symmetric Poisson algebras. Finally, we consider Lie nilpotence and solvability of these Poisson algebras and discuss some results concerning the derived lengths and the Lie nilpotence classes.

Some open questions on these topics are the following:

Problem 1. *Let L be a Lie algebra over a field of characteristic $p > 0$ such that $\mathbf{s}(L)$ is Lie nilpotent. It is shown in* [7] *that the Lie nilpotence class and the strong Lie nilpotency class of $\mathbf{s}(L)$ are the same, provided $p \geq 5$. Is this true also in characteristics $p = 2, 3$?*

Problem 2. *Let L be a Lie algebra over a field of characteristic $p > 2$ such that $\mathbf{s}(L)$ is solvable. Do the derived length and the strong derived length of $\mathbf{s}(L)$ coincide?*

Note that the derived lengths of a truncated symmetric Poisson algebra can be actually different in characteristic 2 (see [9, Remark 4.4]).

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The Poisson spectrum of the symmetric algebra of the Virasoro algebra

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(joint work with Alexey Petukhov)

Let G be a connected algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, and consider the coadjoint action of G on $\mathfrak{g}^*$. This is a beautiful classical topic, with profound connections to areas from geometric representation theory to combinatorics to physics. Algebraic geometry tells us that coadjoint orbits in $\mathfrak{g}^*$ correspond to G-invariant radical ideals in the symmetric algebra $S(\mathfrak{g})$.

As is well known, $S(\mathfrak{g})$ is a Poisson algebra under the Kostant-Kirillov bracket:

$$\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial e_i} \frac{\partial f}{\partial e_j} [e_i, e_j]$$

where $\{e_i\}$ is a basis of $\mathfrak{g}$. A basic fact is that I is G-invariant if and only if I is Poisson.

Thus to compute the closure of the coadjoint orbit of $\chi \in \mathfrak{g}^*$, let $\mathfrak{m}_\chi$ be the kernel of the evaluation morphism

$$\mathrm{ev}_\chi : \mathrm{S}(\mathfrak{g}) \to \mathbb{C},$$

and let $P(\chi)$ be the *Poisson core* of $\mathfrak{m}_\chi$: the maximal Poisson ideal contained in $\mathfrak{m}_\chi$. By definition, an ideal of the form $P(\chi)$ is called *Poisson primitive*; by a slight abuse of notation, we refer to $P(\chi)$ as the *Poisson core of* χ. The closure of the coadjoint orbit of χ is defined by $P(\chi)$:

$$\overline{G \cdot \chi} = V(P(\chi)) := \{\nu \in \mathfrak{g}^* \mid ev_\nu(P(\chi)) = 0\}, \tag{1}$$

and so $\chi, \nu \in \mathfrak{g}^*$ are in the same G-orbit if and only if $P(\chi) = P(\nu)$. In the case of algebraic Lie algebras over $\mathbb{C}$ or $\mathbb{R}$, coadjoint orbits are symplectic leaves for the respective Poisson structure.

We investigate how this theory extends to the *Witt algebra* $W = \mathbb{C}[t, t^{-1}]\partial_t$ of algebraic vector fields on $\mathbb{C}^\times$, and to its central extension the *Virasoro algebra* $Vir = \mathbb{C}[t, t^{-1}]\partial_t \oplus \mathbb{C}z$, with Lie bracket given by

$$[f\partial_t, g\partial_t] = (fg' - f'g)\partial_t + Res_0(f'g'' - g'f'')z, \quad z \text{ is central.}$$

(We also consider some important Lie subalgebras of W.) These infinite-dimensional Lie algebras, of fundamental importance in representation theory and in physics, have no adjoint group [3], but one can still study the Poisson cores of maximal ideals, and more generally the Poisson ideal structure of $S(W)$ and $S(Vir)$. Motivated by (1), we will say that functions $\chi, \nu \in Vir^*$ or in W^* are in the same *pseudo-orbit* if $P(\chi) = P(\nu)$. These (coadjoint) pseudo-orbits can be considered as algebraic symplectic leaves in Vir^* or W^*.

Taking the discussion above as our guide, we focus on prime Poisson ideals and Poisson primitive ideals of $S(Vir)$ and $S(W)$. Important questions here, which for brevity we ask here only for Vir, include:

- Given $\chi \in Vir^*$, can we compute the Poisson core $P(\chi)$ and the pseudo-orbit of χ? When is $P(\chi)$ nontrivial?
- How can we understand prime Poisson ideals of $S(Vir)$? Can we parameterise them in a reasonable fashion, ideally in a way which gives us further information about the ideal? How does one distinguish Poisson primitive ideals from other prime Poisson ideals?
- It is known, see [4, Corollary 5.1], that $S(Vir)$ satisfies the ascending chain condition on prime Poisson ideals. The augmentation ideal of $S(Vir)$, that is, the ideal generated by $Vir \subset S(Vir)$, is clearly a maximal Poisson ideal. What are the others? Conversely, does any nontrivial prime Poisson ideal have finite height?
- Do prime Poisson ideals induce any reasonable algebraic geometry on the uncountable-dimensional vector space Vir^*?

We answer all of these questions, almost completely working out the structure of the Poisson spectra of $S(Vir)$ and $S(W)$.

Let us begin by discussing the idea of algebraic geometry on Vir^*. A priori, this seems intractable as Vir^* is an uncountable-dimensional affine space; little interesting can be said about $S(\mathfrak{a})$ where $\mathfrak{a}$ is a countable-dimensional *abelian* Lie algebra. However, Vir and W are extremely noncommutative and so Poisson ideals in their symmetric algebras are very large: in particular, by a result of Iyudu and the second author [1, Theorem 1.3], if I is a nontrivial Poisson ideal of $S(W)$ (respectively, a non-centrally generated Poisson ideal of $S(Vir)$), then $S(W)/I$ (respectively, $S(Vir)/I$) has polynomial growth. This suggests that a Poisson primitive ideal, and more generally a prime Poisson ideal, might correspond to a finite-dimensional algebraic subvariety of Vir^*, which we could investigate using tools from affine algebraic geometry. We will see that this is indeed the case.

From the discussion above, it is important to characterise which functions $\chi \in Vir^*$ have nontrivial Poisson cores. Strikingly, we show that such χ must vanish on the central element z. Further, the induced function $\overline{\chi} \in W^*$ is given by evaluating *local* behaviour on a proper (that is, finite) subscheme of $\mathbb{C}^\times$. We have:

Theorem. *Let $\chi \in Vir^*$. The following are equivalent:*

(1) *The Poisson core of χ is nontrivial: that is, $P(\chi) \supsetneqq (z - \chi(z))$.*
(2) *$\chi(z) = 0$ and the induced function $\overline{\chi} \in W^*$ is a linear combination of functions of the form*

$$f\partial_t \mapsto \alpha_0 f(x) + \ldots + \alpha_n f^{(n)}(x)$$

where $x \in \mathbb{C}^\times$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{C}$.
(3) *The isotropy subalgebra Vir^χ of χ has finite codimension in Vir.*

We call functions $\chi \in Vir^*$ satisfying the equivalent conditions of Theorem *local functions* as by condition (2) they are defined by local data.

Motivated by condition (3) of Theorem , we investigate subalgebras of Vir of finite codimension. We prove:

Theorem. *Let $\mathfrak{k} \subseteq Vir$ be a subalgebra of finite codimension. Then there is $f \in \mathbb{C}[t, t^{-1}] \setminus \{0\}$ so that $\mathfrak{k} \supseteq \mathbb{C}z + f\mathbb{C}[t, t^{-1}]\partial_t$. In particular, any finite codimension subalgebra of Vir contains z.*

As an immediate corollary of Theorem, we show:

Corollary. *If $0 \neq \zeta \in \mathbb{C}$, then $S(Vir)/(z - \zeta)$ is* Poisson simple*: it has no nontrivial Poisson ideals.*

We then study the pseudo-orbits of local functions on Vir, W, and related Lie algebras; we describe our results here for Vir only. If $\chi \in Vir^*$ is local, then by combining Theorem and [1, Theorem 1.3] $S(Vir)/P(\chi)$ has polynomial growth and we thus expect the pseudo-orbit of χ to be finite-dimensional. We show that pseudo-orbits of local functions in Vir^* are in fact orbits of a finite-dimensional solvable algebraic (Lie) group acting on an affine variety which maps injectively to Vir^*, and we describe these orbits explicitly. This allows us to completely determine the pseudo-orbit of an arbitrary local function in Vir^* and thus also determine the Poisson primitive ideals of $S(Vir)$. We also classify maximal Poisson ideals in $S(Vir)$: they are the augmentation ideal, the ideals $(z - \zeta)$ for $\zeta \in \mathbb{C}^\times$, and the defining ideals of all but one of the two-dimensional pseudo-orbits.

Through this analysis, we obtain a nice combinatorial description of pseudo-orbits in W^*: pseudo-orbits of local functions on W, and thus Poisson primitive ideals of $S(W)$, correspond to a choice of a partition λ and a point in an open subvariety of a finite-dimensional affine space $\mathbb{A}^k$, where k can be calculated from λ. We further expand this correspondence to obtain a parameterisation of all prime Poisson ideals of $S(W)$ and $S(Vir)$. We also study the related Lie algebra $W_{\geq -1} = \mathbb{C}[t]\partial_t$, and prove that Poisson primitive and prime Poisson ideals of $S(W_{\geq -1})$ are induced by restriction from $S(W)$.

Our understanding of prime Poisson ideals allows us to determine exactly which prime Poisson ideals of $S(Vir)$ obey the *Poisson Dixmier-Moeglin equivalence*, which generalises the characterisation of primitive ideals in enveloping algebras of finite-dimensional Lie algebras due to Dixmier and Moeglin. The central question is when a Poisson primitive ideal of $S(Vir)$ is *Poisson locally closed*: that is, locally closed in the Zariski topology on Poisson primitive ideals. (If $\dim \mathfrak{g} < \infty$ then a prime Poisson ideal of $S(\mathfrak{g})$ is Poisson primitive if and only if it is Poisson locally closed [2, Theorem 2].) We show that (z) is the only Poisson primitive ideal of $S(Vir)$ which is not Poisson locally closed. We further prove that $S(W)$ has no nonzero prime Poisson ideals of finite height.

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Non-matrix polynomial identities on enveloping algebras

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(joint work with David Riley, Jeff Bergen)

A variety of associative algebras over a field $\mathbb{F}$ is called non-matrix if it does not contain $M_2(\mathbb{F})$, the algebra of 2×2 matrices over $\mathbb{F}$. A polynomial identity (PI) is called non-matrix if $M_2(\mathbb{F})$ does not satisfy this identity. Latyshev in his attempt to solve the Specht problem proved that any non-matrix variety generated by a finitely generated algebra over a field of characteristic zero is finitely based [9]. The complete solution of the Specht problem in the case of characteristic zero is given by Kemer [7].

Although several counterexamples are found for the Specht problem in the positive characteristic [1], the development in this area has lead to some interesting results. Kemer has investigated the relation between PI-algebras and nil algebras. Amitsur [2] had already proved that the Jacobson radical of a relatively-free algebra of countable rank is nil. Restricting to non-matrix varieties, Kemer [6] proved that the Jacobson radical of a relatively-free algebra of a non-matrix variety over a field of positive characteristic is nil of bounded index. These varieties have been further studied in [5, 6] and recently generalized for alternative and Jordan algebras in [14].

Enveloping algebras satisfying polynomial identities were first considered by Latyshev [10] by proving that the universal enveloping algebra of a Lie algebra L over a field of characteristic zero satisfies a PI if and only if L is abelian. Latyshev's result was extended to positive characteristic by Bahturin [4]. Passman [11] and Petrogradsky [13] considered the analogous problem for restricted Lie algebras.

Let $A = A_0 \oplus A_1$ be a vector space decomposition of a non-associative algebra over a field $\mathbb{F}$ of characteristic not 2. We say that this is a $\mathbb{Z}_2$-grading of A if $A_iA_j \subseteq A_{i+j}$, for every $i, j \in \mathbb{Z}_2$ with the understanding that the addition $i+j$ is mod 2. The components A_0 and A_1 are called even and odd parts of A, respectively. Note that A_0 is a subalgebra of A. One can associate a Lie super-bracket to A by defining $(x, y) = xy - (-1)^{ij}yx$ for every $x \in A_i$ and $y \in A_j$. If A is associative, then for any $x \in A_i$, $y \in A_j$ and $z \in A$ the following identities hold:

(1) $(x, y) = -(-1)^{ij}(y, x)$,
(2) $(x,(y, z)) = ((x, y), z) + (-1)^{ij}(y,(x, z))$.

The above identities are the defining relations of Lie superalgebras. Furthermore, A can be viewed as a Lie algebra by the usual Lie bracket $[u, v] = uv - vu$. The bracket of a Lie superalgebra $L = L_0 \oplus L_1$ is denoted by $(,)$. We denote the enveloping algebra of L by $U(L)$. Lie superalgebras whose enveloping algebras satisfy a PI were characterized by Bahturin [3] and Petrogradsky [12]. In this talk we characterize Lie superalgebras whose enveloping algebras satisfy a non-matrix PI. Our first main result is as follows.

Theorem. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field of characteristic $p > 2$. The following conditions are equivalent:*

(1) *$U(L)$ satisfies a non-matrix PI.*
(2) *$U(L)$ satisfies a PI, L_0 is abelian, and there exists a subspace M of L_1 of codimension at most 1 such that $(L_0, L_1) \subseteq M$ and $(M, L_1) = 0$.*
(3) *The commutator ideal of $U(L)$ is nil of bounded index.*
(4) *$U(L)$ satisfies a PI of the form $([x, y]z)^{p^m} = 0$, for some m.*

The equivalence of (1) and (4) is well known to hold for all algebras: it follows easily from standard PI-theory. The deeper fact that (1) and (3) are equivalent follows from the structure theory of PI algebras. We emphasize that the term Lie solvable is used with respect to the usual Lie bracket $[\,,]$.

Theorem *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field of characteristic not 2. Then $U(L)$ is Lie solvable if and only if (L, L) is finite-dimensional, L_0 is abelian, and there exists a subspace M of L_1 of codimension at most 1 such that $(L_0, L_1) \subseteq M$ and $(M, L_1) = 0$.*

Kemer [8] proved that an algebra R over a field of characteristic zero satisfies a non-matrix PI if and only if R is Lie solvable. The following is now easily deduced from Theorem.

Corollary. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra over a field of characteristic zero. The following conditions are equivalent:*

(1) *$U(L)$ satisfies a non-matrix PI.*
(2) *$U(L)$ is Lie solvable.*
(3) *(L, L) is finite-dimensional, L_0 is abelian, and there exists a subspace M of L_1 of codimension at most 1 such that $(L_0, L_1) \subseteq M$ and $(M, L_1) = 0$.*

The adjoint representation of L is given by ad $x : L \to L$, ad $x(y) = (y, x)$, for all $x, y \in L$. The notion of restricted Lie superalgebras can be easily formulated as follows.

Definition. *A Lie superalgebra $L = L_0 \oplus L_1$ over a field $\mathbb{F}$ of characteristic $p \geq 3$ is called* restricted, if there is a pth power map $L_0 \to L_0$, denoted as $^{[p]}$, satisfying

(a) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$, for all $x \in L_0$ and $\alpha \in \mathbb{F}$,
(b) $(y, x^{[p]}) = (y,_p x)$, for all $x \in L_0$ and $y \in L$,
(c) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, for all $x, y \in L_0$ where is_i is the coefficient of λ^{i-1} in $(\text{ad }(\lambda x + y))^{p-1}(x)$.

In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module by the adjoint action of the even subalgebra. For example, every $\mathbb{Z}_2$-graded associative algebra inherits a restricted Lie superalgebra structure.

Let L be a restricted Lie superalgebra over a field $\mathbb{F}$ of characteristic $p \geq 3$. We denote the enveloping algebra of L by $u(L)$. Restricted Lie superalgebras whose enveloping algebras satisfy a polynomial identity have been characterized by Petrogradsky [12].

Theorem. *Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra over a perfect field and denote by M the subspace spanned by all $y \in L_1$ such that (y, y) is p-nilpotent. The following statements are equivalent:*

(1) *$u(L)$ satisfies a non-matrix PI.*
(2) *The commutator ideal of $u(L)$ is nil of bounded index.*
(3) *$u(L)$ satisfies a PI, (L_0, L_0) is p-nilpotent, $\dim L_1/M \leq 1$, (M, L_1) is p-nilpotent, and $(L_1, L_0) \subseteq M$.*

We show that (3) implies (2) over any field. However, given that $u(L)$ satisfies a non-matrix PI, the restriction on the field is necessary to be able to show that $\dim L_1/M \leq 1$. We show that over a non-perfect field there exists a restricted Lie superalgebra $L = L_0 \oplus L_1$ such that $\dim L_1 = 2$, $u(L)$ is Lie solvable and yet (y, y) is not p-nilpotent, for every $y \in L_1$. This is in complete contrast with Theorem .

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Symmetrisation and the Feigin–Frenkel centre

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Let $\mathfrak{g}$ be a complex reductive Lie algebra. The Feigin–Frenkel centre $\mathfrak{z}(\widehat{\mathfrak{g}}) \subset \mathcal{U}(t^{-1}\mathfrak{g}[t^{-1}])$ is a remarkable commutative subalgebra. Its structure is described by a theorem of Feigin and Frenkel (1992), if $\ell = \operatorname{rk}\mathfrak{g}$ and $\tau = -\partial_t$, then $\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[\tau^k(S_i) \mid k \geqslant 0, 1 \leqslant i \leqslant \ell]$, where the generators $\tau^k(S_i)$ are algebraically independent.

The classical counterpart of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the Poisson-commutative subalgebra of $\mathfrak{g}[t]$-invariants in $\mathcal{S}(\mathfrak{g}[t,t^{-1}])/(\mathfrak{g}[t]) \cong \mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])$, which is a polynomial ring with infinitely many generators according to a direct generalisation of a Raïs–Tauvel theorem (1992). Unlike the finite-dimensional case, no natural isomorphism between $\mathcal{S}(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{g}[t]}$ and $\mathfrak{z}(\widehat{\mathfrak{g}})$ is known. Explicit formulas for the elements S_i appeared first in type A [1, 2] and were extended to all classical types in [3]. In [4], it is shown that for all classical Lie algebras, the symmetrisation map ϖ can produce generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$. Note that ϖ is a homomorphism of $\mathfrak{g}[t^{-1}]$-modules and it behaves well with respect to taking various limits.

One of the tools in [4] is a certain map $\mathsf{m}\colon \mathcal{S}^k(\mathfrak{g}) \to \Lambda^2\mathfrak{g} \otimes \mathcal{S}^{k-3}(\mathfrak{g})$. Let $F[-1] \in \mathcal{S}^k(\mathfrak{g}t^{-1})$ be obtained from $F \in \mathcal{S}^k(\mathfrak{g})^{\mathfrak{g}}$ by the canonical isomorphism $\mathfrak{g}t^{-1} \cong \mathfrak{g}$. Then $\varpi(F[-1]) \in \mathfrak{z}(\widehat{\mathfrak{g}})$ if and only if $\mathsf{m}(F) = 0$. More generally, if $H \in \mathcal{S}^k(\mathfrak{g})^{\mathfrak{g}}$ is such that

$$\mathsf{m}^d(H) = \mathsf{m}(\mathsf{m}^{d-1}(H)) \in \mathcal{S}(\mathfrak{g}) \quad \text{for all} \quad 1 \leq d < k/2,$$

then there is a way to produce an element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ corresponding to H.

For each classical $\mathfrak{g}$, there is a generating set $\{H_i \mid 1 \leqslant i \leqslant \ell\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\mathsf{m}(H_k) \in \mathbb{C}H_j$ with $j < k$ for each k. In types A and C, we are using the coefficients Δ_k of the characteristic polynomial, for $\mathfrak{g} = \mathfrak{so}_n$, we work with coefficients Φ_{2k} of $\det(I_n - q(F_{ij}))^{-1}$.

In type A_{n-1}, $\mathsf{m}(\Delta_k) = \frac{(n-k+2)(n-k+1)}{k(k-1)}\Delta_{k-2}$; in type C_n, we have

$$\mathsf{m}(\Delta_{2k}) = \frac{(2n-2k+3)(2n-2k+2)}{2k(2k-1)}\Delta_{2k-2};$$

and finally for $\mathfrak{g} = \mathfrak{so}_n$, we have $\mathsf{m}(\Phi_{2k}) = \frac{(n+2k-3)(n+2k-2)}{2k(2k-1)}\Phi_{2k-2}$. This leads to the following sets of *Segal–Sugawara vectors* $\{S_i \mid 1 \leq i \leq \ell\}$ [4]:

$$\{S_{k-1} = \varpi(\Delta_k[-1]) + \sum_{1 \leq r < (k-1)/2} \binom{n-k+2r}{2r} \varpi(\tau^{2r}\Delta_{k-2r}[-1]){\cdot}1 \mid 2 \leq k \leq n\}$$

in type A_{n-1};

$$\{S_k = \varpi(\Delta_{2k}[-1]) + \sum_{1 \leq r < k} \binom{2n-2k+2r+1}{2r} \varpi(\tau^{2r}\Delta_{2k-2r}[-1]){\cdot}1 \mid 1 \leq k \leq n\}$$

in type C_n;

$$\{S_k = \varpi(\Phi_{2k}[-1]) + \sum_{1\le r<k} \binom{n+2k-2}{2r} \varpi(\tau^{2r}\Phi_{2k-2r}[-1]){\cdot}1 \mid 1 \le k < \ell\}$$

for $\mathfrak{so}_n$ with $n = 2\ell - 1$ with the addition of $S_\ell = \varpi(\mathrm{Pf}[-1])$ for $\mathfrak{so}_n$ with $n = 2\ell$.

The advantage of our method is that it reduces questions about elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ to questions on the structure of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ in a type-free way. For example, it is possible to deal with type G_2 by hand [4]. It is quite probable, that other exceptional types can be handled on a computer. Conjecturally, each exceptional Lie algebra possesses a set $\{H_k \mid 1 \le k \le \ell\}$ of generating symmetric invariants such that for each k there is i with $\mathsf{m}(H_k) \in \mathbb{C}H_i$.

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Contact brackets and other structures on the tensor product

Pasha Zusmanovich

The purpose of this report is once more to call attention to an elementary and, in some cases, very effective technique of computing various kinds of structures on tensor products. Such problems often can be reduced to the simultaneous evaluation of kernels of several tensor product maps, i.e., maps of the form $S \otimes T$, where S and T are linear operators on the respective spaces of linear maps. Using the fact that

$$Ker(S \otimes T) = Ker(S) \otimes * + * \otimes Ker(S), \tag{1}$$

the question reduces to evaluation of the intersection of several linear spaces having the form as on the right-hand side of (1), for various operators S and T. The intersection of two such spaces satisfies the distributivity, and so can be handled effectively, due to the following elementary linear algebraic lemma:

Lemma ([4, Lemma 1.1]). *Let U_1, U_2 be subspaces of a vector space U, and V_1, V_2 be subspaces of a vector space V. Then*

$$(U_1\otimes V+U\otimes V_1)\cap(U_2\otimes V+U\otimes V_2) = (U_1\cap U_2)\otimes V+U_1\otimes V_2+U_2\otimes V_1+U\otimes(V_1\cap V_2).$$

This technique was used for the first time in [4] to derive some formulas for the low degree cohomology of current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra, and A is an associative commutative algebra. The paper [5] contains further results about such cohomology, as well as about Poisson and

Hom-Lie structures on current and related Lie algebras. The last our result in this direction is in [6], which answers a recent question from [2] about extension of contact bracket on the tensor product from the bracket on the factors.

Recall that the contact bracket on a commutative associative algebra A with unit is a bilinear map $[\,\cdot\,,\cdot\,] : A \times A \to A$ such that

$$[ab, c] = [a, c]b + [b, c]a + [c, 1]ab$$

for any $a, b, c \in A$. Contact brackets are an obvious generalization of Poisson brackets, the latter being contact brackets satisfying $[A, 1] = 0$. It was asked in [2] whether, given contact brackets on two algebras A and B, is it always possible to extend them to the tensor product $A \otimes B$? In [6], using some general formulas for the space of contact brackets on some particular classes of algebras, a procedure was devised for constructing examples showing that such extension is not always possible.

This linear algebraic method is sometimes very effective, but its applicability is severely limited by the fact that no statement similar to Lemma is true for intersection of three or more spaces. The proper contexts of Lemma might be criteria for distributivity of a set of subspaces of a vector space (for an exposition, see, for example, [3, Chap. 1, §7]) and, more speculatively, the "four subspaces problem" of Gelfand–Ponomarev, [1].

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Mini-Workshop: Felix Klein's Foreign Students: Opening Up the Way for Transnational Mathematics

Organized by
Danuta Ciesielska, Warsaw
Renate Tobies, Jena

15 October – 20 October 2023

Abstract. Extending existing analyses of the topic, the workshop aimed to investigate the influence of Felix Klein on the development of mathematics (especially number theory, algebra, geometry, analysis, applications of mathematics in scientific and technical fields as well as in mathematics education) in countries other than Germany. The goal of the workshop was to take a look at mathematicians of foreign origin who studied with Klein that have received little attention so far (including Czech, Greek, Hungarian, Japanese, Polish, Russian, and Ukrainian mathematicians) and uncover how Klein guided them through his lectures and seminars. The protocols of the lectures held in Klein's seminars (from 1872 to 1912 in Göttingen, Erlangen, and Leipzig), which are a unique and so far largely unexplored source, were the basis for the workshop.

Mathematics Subject Classification (2020): Primary: 01A55, 01A60, 01A65, 01A70; secondary: 01A72, 01A73, 97A30.

Introduction by the Organizers

The mini-workshop *Felix Klein's Foreign Students: Opening Up the Way for Transnational Mathematics*, organised by Danuta Ciesielska (Warsaw) and Renate Tobies (Jena), aimed to investigate (extending existing analyses of the topic) the influence of Felix Klein on the development of mathematics (especially number theory, algebra, geometry, analysis, and applications of mathematics in scientific and technical fields as well as in mathematics education) in countries other than Germany.

The idea for this international collaborative project came from Danuta Ciesielska, who for several years has been researching (together with two Polish colleagues) how Polish mathematicians studied in Göttingen with Klein and Hilbert, resulting in a recently published Polish monograph [1].

The mini-workshop *Felix Klein's Foreign Students: Opening Up the Way for Transnational Mathematics* joined together 17 researchers from different countries, familiar not only with their own mathematical traditions, but also with the development of national identities as well as political and cultural histories of the various regions.

Previous research showed that Klein did not have to "court" students from abroad throughout his career. Rather, they were sent to him from Scandinavian countries, from Italy, France, Great Britain, America, the Netherlands, Russia, Switzerland, Austria-Hungary, Greece, etc. We now have a good overview of Klein's first international students and we also have a good analysis of all the women who studied under Klein (from 1893).

The mini-workshop aimed to examine the causes of Klein's international success. Before the workshop, we had arrived at the following hypotheses, partially based on ([3]):

(1) Klein deliberately aimed to found a *mathematical school* as early as 1872. In a letter to Gaston Darboux, Klein spoke of recreating a "school of geometrical production" as he had come to know it under Alfred Clebsch, who had just died. This was later to be considered as "a style of mathematical life that promised colossal successes for the future" ([2]).
(2) This goal required that Klein readily share his own ideas and seek to advance them through *cooperative work*, but now, unlike Clebsch, on an international level – increasingly incorporating new methods into his practice.
(3) Klein's early efforts to become acquainted with various mathematical schools at home and abroad led to good personal contacts with mathematicians of numerous countries, who recommended their own students to Klein. Even when Klein was still in Erlangen in the early 1870s, Sophus Lie recommended Scandinavian students to go to Klein because they *would be encouraged* there (which would not happen if they went to Berlin).

In order to test these hypotheses, it was necessary to look deeper into the sources, especially into the protocol books containing handwritten records of the talks given at Klein's seminars from 1872 to 1912. These 29 volumes are available online:

- https://www.uni-math.gwdg.de/aufzeichnungen/klein-scans/klein/
- https://page.mi.fu-berlin.de/moritz/klein/

In the case of (b) an attempt has been made to identify the complete names of the presenters. There are numerous errors, however, especially with foreign persons. Therefore, we also want to correct the sources with the help of our experts in the future. In addition to Klein himself, the speakers in the seminars were his students or distinguished visitors, many of them foreigners.

During the workshop the state of current knowledge about students with different languages, ethnicities and traditions at Klein's various career stations, especially in Göttingen, were discussed. Participants investigated the similarities and differences between them, while trying to identify all foreign participants in Klein's seminars and analyzing their contributions. Because of that, it was possible to achieve a better understanding of the socio-geographic profiles of the students coming to Klein, their own professional development and their subsequent impact on mathematics and mathematical life in their homelands.

The work on the records of the foreign participants in Klein's seminars allowed for addressing more detailed questions about these participants, including the following:

- Why did they want to work with Klein?
- What was their mathematical preparation before they arrived?
- What topics did Klein assign to them for their own seminar presentation?
- How did Klein further encourage them to work on these or related topics?
- Did this encouragement lead them to creating results of their own in the field?
- Were their results published, e.g. in the *Mathematische Annalen* (which was edited by Klein)?
- Were they later involved in other projects of Klein (e.g. *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*)?

The mini-workshop also addressed some questions of a general character, in particular these:

- To what extent did Klein influence (directly or indirectly) persons who later achieved outstanding results in individual mathematical fields?
- In what ways did former students of Klein impact the organisation of mathematical life in their homelands (university education, publishing, mathematical societies)?

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Mini-Workshop: Felix Klein's Foreign Students: Opening Up the Way for Transnational Mathematics

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Abstracts

Founding a School for Mathematical Production – processed and open issues

Renate Tobies

The starting point of the presentation was Klein's network of people. The breadth of the network is explained above all by Klein's vision of creating a school of mathematical production. Connected to this was the empowerment and the urge to lead young talents to their own creative results. The prominent Berlin mathematician Leopold Kronecker believed that mathematicians did not need to form a (scientific) school and that collaborative work would hinder progress in the field. Klein, in contrast, aspired to reproduce Alfred Clebsch's model and thus to create "a school of geometric production". Klein followed Clebsch's program of "uniting people from different fields of work". He created connections between different areas of mathematics (geometry, algebra, function theory, number theory), and made connections between mathematics and its neighboring disciplines. His general way of working was based on his approach to mathematical research, which required cooperation.

Klein tested out as many colleagues and students as possible for their potential as collaborators, among them foreign colleagues and students. This presentation provided an overview of Klein's foreign students, and named their contexts of work as well as interesting open questions. In order to conduct a detailed and thorough analysis of Klein's impact on later developments, this introductory lecture also aimed to explain basic sources, including errors contained therein (online available audience lists of Klein's lecture courses, participants in the research seminars, the Appendix to Klein's Collected Mathematical Papers, vol. 3; the Poggendorff Bio-Bibliographical Hand Dictionary, etc.).

We also gave an overview about when and why foreign students studied with Klein. The success of these studies was discussed by means of examples. On the one hand, consideration was given to the insights gained with the revised Klein biography ([1]), and the recently published book on Klein and Georg Pick ([2]). On the other hand, we also looked into further seminars and minutes of Klein's seminars in order to classify some foreign students about whom we still know too little (students from Hungary, Russia, etc.).

The list of more than 300 people (including two female mathematicians from St. Petersburg) who donated money for the portrait of Felix Klein painted by Max Liebermann was shown. This reveals connections between Klein and other mathematicians as far away as India, Australia and Japan. Some of the examples were used to show what other sources can be consulted to explain the respective person and their mathematical results.

Finally, we highlighted the special role of Klein's interdisciplinary research seminars, which he was able to establish in Göttingen. Klein succeeded in getting

Hilbert appointed on 1 April 1895 and immediately involved him in the leadership of his own seminar, in which the focus was on approximation analysis. In the following semesters, Klein held further joint seminars with Hilbert on number theory, function theory, and mechanics. With the appointment of other younger colleagues, also for applied mathematics, physics, astronomy, statistics, Klein also involved them in the management of his seminars. Examples show how far-reaching Klein's international impact was in these areas (some lesser-known examples: William F. Baker (2015) developed a design tool for plane trusses using an extended Airy stress function based on Klein and Wieghardt (1905); Timoshenko's beam theory; Kármán's vortex street; the Painlevé-Klein problem in the theory of friction). The name *Technomathematik* is just over twenty years old; it was created in Kaiserslautern in 1979 by Helmut Neunzert (*1936) for a new study program intended to merge mathematics and technology. Neunzert recently confirmed that he was inspired to do so by Felix Klein's combination of pure and applied mathematics.

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Felix Klein vs. Henri Poincaré, 1881–82: On the Birth of the Theory of Automorphic Functions

David E. Rowe

Between June 1881 and September 1882 Klein und Poincaré exchanged letters that led to their competition to find and prove general uniformization theorems in complex analysis. These letters were first published in 1923 by Mittag Leffler in Volume 39 of Acta Mathematica (see [7], pp. 111–133). In that same year, Klein also published them with commentary in [5], pp. 577–621; see further [6], pp. 334–381. As a famous topic in the careers of both men, one can find many discussions about it, not only in the standard biographies [2] and [9], but also in more technical historical studies such as [8] and [1]. The present account mainly aims to add some further contextualization by drawing on the four letters Brunel sent to Poincaré during June and July 1881 when he was studying under Klein in Leipzig.

Georges Brunel (1856–1900) entered the École normale in 1877. After graduation in 1880, he spent the academic year 1880–81 in Leipzig working under Felix Klein. In his first letter to Poincaré, Brunel introduced himself as a "comrade", i.e. fellow normalien of Paul Appell and Émile Picard, though both were older than he. In 1884 Brunel obtained the chair for pure mathematics in Bordeaux. His predecessor was Jules Hoüel, a leading authority on non-Euclidean geometry, having in 1867 translated works by Lobachevsky and Bolyai.

Brunel had already spent a good deal of time in Leipzig before he wrote to Poincaré, and this stay abroad had not been easy for him. Still, he felt a deep urge to serve his country, while behaving properly as a guest in a foreign land. As he explained to Poincaré, he hoped to learn what German mathematicians had to teach the French. This view was promoted by Charles Hermite, who urged his pupils to follow the new currents of research pursued on the other side of the Rhine.

Klein's seminar during the winter semester 1880/81 dealt with various topics in geometry and complex function theory. Adolf Hurwitz and Walther Dyck, two of his most important German students, both attended. Along with Brunel, others came from foreign countries: Giuseppe Veronese, then an assistant under Luigi Cremona in Rome, and Washington Irving Stringham, a student of J.J. Sylvester at Johns Hopkins University. Shortly before Christmas 1880, Brunel spoke about Riemann's approach to the genus of surfaces and its role in algebraic curve theory. Klein assigned him this topic as well as some relevant literature with which to prepare his talk.

In January 1881, Brunel spoke for the second time on a related topic: Riemann's theory of Abelian functions and Enrico Betti's generalization to higher dimensions. Brunel also discussed the pioneering topological studies undertaken by the Göttingen physicist and mathematician Johann Benedict Listing. This reflects Klein's longstanding fascination with this older tradition. In fact, Betti and Riemann first met in Göttingen, though their friendship grew far closer during Riemann's final years when he spent much of his time in Pisa and elsewhere in northern Italy.

During the summer semester of 1881, Klein's lectures moved deeply into Riemann's theory of functions of a single complex variable. Nearly all the talks in his seminar dealt with topics closely tied to this course. The one striking exception was Brunel's presentation of Cantor's new theory of point sets. This included a proof that the algebraic numbers constitute a countable subset within the uncountable infinity of real numbers. Besides works by Cantor, Brunel also discussed papers by several other German authors, including J. Lüroth, E. Netto, and E Jürgens. He also treated Liouville's classical method for constructing transcendental numbers which, following Cantor's theorem, form an uncountably infinite set. This was thus a second new field of research that had not yet made inroads into France, showing that Brunel was well prepared to act as an early envoy for recent mathematics in Germany. To a certain extent, he actually took up this role. On returning to France, Brunel published a review of Klein's booklet *Ueber Riemanns Theorie der Algebraischen Functionen und ihrer Integrale* (republished in [5], pp. 479–573).

Already in his second letter to Poincaré, written on 19 June 1881, Klein informed him that Brunel had been studying in Leipzig. He advised him further that Brunel would be able to give Poincaré details about Klein's research program. In his seminar, Klein vented his anger over the fact that Poincaré had ignored the published literature:

> . . . [Klein] complained that the "young French" didn't know what had been published in Germany; he said that in France one probably didn't know that the Mathematische Annalen existed (to which I [Brunel] could only have replied that, in Berlin itself, this Journal was considered to be problematic), that one didn't read Crelle's Journal (where did you read Fuchs's work?), etc., etc.

Klein was especially angry that Poincaré had honored Lazarus Fuchs, whose work had helped inspire his own:

> I protest against the name Fuchsian functions. The fundamental idea belongs to Riemann, and the credit for applying Riemann's idea belongs to Schwarz. . . . Later, I myself worked in this direction and . . . I presented some results which are the basis of Mr. Poincaré's work. As for Mr. Fuchs, who once wanted to deal with similar questions, he only succeeded in this: in showing us that he understood absolutely nothing about them.

In March 1882, after Klein published a note in Mathematische Annalen rejecting the names Poincaré had introduced, Mittag Leffler wrote him: "But certainly Mr. Klein is right in saying you are wrong to call your functions Fuchsian and Kleinian functions. They must be named Poincaré functions. It's the only name that's fair and reasonable."

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Mathematicians Connected with the Czech Lands as Klein's Students and Collaborators

Martina Bečvářová

From the second half of the 19th century, the most talented and outstanding German and Czech mathematicians from the University in Prague, later from the German University in Prague or German Technical University in Prague went abroad thanks to government scholarships or other funds. Czech mathematicians travelled mainly to Italy, France or Germany because of many political and cultural reasons, German mathematicians travelled mainly to Germany. Both of them, they studied in the most prestigious mathematical centres of the period, at Berlin, Göttingen, Hamburg, Leipzig, Munich, Paris, Strasbourg, Milano, and Rome.

What were their main goals and professional interests to travel abroad? They tried to expand the horizons of their mathematical knowledge and establish contacts with the best experts from the famous European mathematical centres. They were interested in new, modern and promising mathematical topics that were missing or not given enough attention at the University in Prague. They wanted to be more involved in the latest mathematical trends and methods and to get in touch with the newest mathematical ideas. They wanted to publish their first scientific works in world-renowned and respected journals and their first monographs in internationally known publishing houses. They aspired to obtain doctorates at leading European universities and, after returning home, to achieve a better career, i.e. to habilitate and later become full professors at schools in their homeland. They also wanted to recognize the most advanced education methods and get them to universities and polytechnics in the Czech lands.

What were their typical activities during their study stay abroad? This depended on at what stage in their career they got the opportunity to study abroad. Regular students visited some special lectures or seminars. Graduates of basic studies participated in more advanced seminars as passive visitors or active lecturers. They prepared their dissertations and broadened their horizons for the doctoral process. Both used libraries where new monographs, journals or thesis were. They wrote their first articles and discussed their topics, ideas or first scientific results. They participated in the life of mathematical communities (professional as well as social) and it was very important part of their stays. Many of them became later a member of the Deutsche Mathematiker-Vereinigung and promoted German mathematics and culture.

Based on archival sources deposited in the Czech Republic, Germany and Italy, we discovered only 12 mathematicians connected with the Czech lands who studied or worked under the influence of Felix Klein (1849–1925) at the Polytechnic in Munich in the 1870s, at the University in Leipzig in the 1880s and at the University in Göttingen in the early 20th century as his regular students, or prepared their doctoral thesis, or passed their doctoral procedures and took active participation in his special mathematical seminars, published their first mathematical results thanks to his helps or inspirations or collaborated with him during all their lives as good mathematicians and personal friends.

Only one of them was a Czech mathematician [Ludvík Kraus (1857–1884)], the others were German mathematicians [Anton Puchta (1851–1903), Karl Bobek (1855–1899), Seligmann Kantor (1857–1903), Georg Alexander Pick (1859–1942), Wilhelm Weiss (1859–1904), Emil Waelsch (1863–1927), Joseph Grünwald (1876–1911), Georg Hamel (1877–1954), Ernst Fanta (1878–1939), Lothar Schrutka (1881–1945) and Paul Georg Funk (1886–1969)]. Nine of them connected a greater or lesser part of their lives, pedagogical or professional activities with Prague universities (the Czech University, the German University, the German Technical University; Puchta, Kantor, Kraus, Bobek, Pick, Waelsch, Weiss, Grünwald and Funk). Four of them connected part of their lives with the German Technical University in Brno (Waelsch, Hamel, Fanta and Schrutka). Five of them went abroad after a shorter or longer career in the Czech lands (Puchta, Hamel, Fanta, Schrutka and Funk), one completely resigned from his academic career after a short period of his pedagogical activities (Kantor). For detailed information on their personal life, academic career, teaching and other activities see [1, 2, 3].

Three of them were for one academic year Klein's regular students (Hamel, Schrutka and Funk). Five of them started their doctoral procedure with Klein's help or inspirations at the University in Prague, Leipzig or Erlangen (Puchta, Kantor, Bobek, Waelsch and Weiss). Only Kantor was unsuccessful, and that for formal and not for professional reasons. Three of them completed a study stay with Klein before starting their regular habilitation procedures at their Alma Mater in Prague or Vienna (Kraus, Grünwald and Fanta). Only one of them after a successful habilitation at the German University in Prague completed a study stay at Klein as his equal colleague (Pick).

Especially important moments for the professional growth of young mathematicians from the Czech lands were their participation in Klein's special mathematical seminars, where he himself, his guests, doctoral students or the best university students presented the latest results of their research or reported on newly published articles. The essential source on the history of Klein's seminars are the so-called *Sämtliche Protokolle 1872–1912*, i.e. Klein's seminar protocol-books which are online available [6]. Twenty-nine books provide interesting information on Klein's seminars from the summer semester 1872 until the summer semester 1912. The names of the lecturers, the titles and abstracts of the lectures written by the lecturers themselves (1 or more pages) and the lists of participants can be found there. Thanks to these records, we know that ten mathematicians connected with the Czech lands lectured at Klein's seminars from 1876 until 1912, one took part in the seminars without any lecture (Kantor). Only Schrutka apparently did not participate in Klein's seminars. Our participants presented 30 lectures in the German language. Their topics were analysis (10), geometry (8), algebra and theory of numbers (6), mechanics (3), instruments (2) and other (1).

Name	Place	Time	Nr.	Topic
Puchta Anton	Munich	WS 1876/1877 – SS 1877/1878	9	Geo, Al, An
Kraus Ludvík	Munich	WS 1878/1879 – WS 1879/1880	7	Geo, Al, An
Bobek Karl	Leipzig	WS 1881/1882 – SS 1881/1882	2	An
Pick Georg Alexandr	Leipzig	WS 1883/1884 – SS 1883/1884	5	An
Waelsch Emil	Leipzig	SS 1884/1885	1	Al, Geo
Weiss Wilhelm	Leipzig	SS 1884/1885	1	Geo
Grünwald Josef	Göttingen	WS 1899/1900	1	Mech
Hamel Georg	Göttingen	SS 1899/1900 – WS 1900/1901	2	Instr, Mech
Fanta Ernst	Göttingen	WS 1901/1902	1	Instr, Mech
Funk Georg Paul	Göttingen	WS 1911/1912	1	An

Geo – geometry, Al – algebra, An – analysis, Mech – mechanics,
Instr – instruments

The mathematicians from the Czech lands as others wrote abstracts of their lectures (usually from 1 to 5 pages in 1870s, from 2 to 15 pages in 1880s, from 4 to 23 pages later). Pick's abstracts were long, perfect, inspiring and beautifully written with full references. Hamel's abstracts were the same, with interesting pictures, but almost unreadable. For more information on the seminars, see [4, 6].

Mathematicians from the Czech lands thanks to Klein's help published some of their results in the journal Mathematische Annalen which covered a wide spectrum of mathematics and was published from 1869 until 1919 by Teubner (in Leipzig), since 1920 by Springer (in Berlin). Klein was its redactor from 1876 until 1924 and influenced its content and focus. Six German mathematicians from the Czech lands published their first articles in this journal with Klein's support; they were devoted to number mathematical branches. The articles are online available [7].

Name	Nr.	Time	Notes
Kantor Seligmann	3	1879–1882	geometry, configurations
Bobek Karl	2	1884,1887	elliptic functions, geometry of curves
Pick Georg Alexandr	16	1883–1915	transformations, algebraic geometry, number theory, special functions, functional spaces
Weiss Wilhelm	1	1887	geometry
Hamel Georg	10	1903–1935	special functions, geometry, ordinary differential equations
Schrutka Lothar	2	1912, 1941	number theory

It should be emphasized that Klein had a significant influence on Pick, who brought modern mathematics at the German University in Prague and, thanks to his almost half-century-long pedagogical work at that school, raised a new generation of Prague German mathematicians (as for example K. Löwner, H. Löwig, A. Winternitz, W. Fröhlich, O. Varga). For more information, see [1, 4, 5]. Pick's, Hamel's, Kantor's and Funk's results are widely known, contemporary recognized and still cited. The other mathematicians who studied with Klein had only a local influence in the Czech lands (teaching, creating textbooks, training technicians, etc.) for many reasons (political, religious, economic, health, personal etc.).

Mathematicians from the Czech lands were in corresponding touch with German mathematicians, i.e. with Klein too. Their correspondence is a great resource for better understanding the depth of their cooperation, Klein's professional influences, personal assistances and helps. The correspondence of mathematicians from the Czech lands has been partly preserved in Germany. Unfortunately, we do not know anything about the correspondence of mathematicians from Germany. For many historical reasons, it was not preserved in the Czech archives after the World War II.

Sender	Addressee	Nr.	Time	Notes
Puchta A.	Klein F.	7	1878–1886	a
Puchta A.	Cantor M.	4	1882–1883	b
Kraus L.	Klein F.	1	1879	a
Pick G. A.	Klein F.	131	1884–1898	a
Pick G. A.	Hilbert D.	2	1885, 1900	c
Pick G. A.	Hurwitz A.	1	1893	d
Pick G. A.	Schwarzschild K.	1	1914	e
Pick G. A.	Gordan P.	1	1892	f
Pick G. A.	von Kraus C.	12	1908–1913	e
Kantor S.	Klein F.	1	1884	a
Bobek K.	Klein F.	2	1880, 1884	a
Waelsch E.	Klein F.	3	1889–1890	a
Hamel G.	Klein F.	3	1908–1922	a
Küpper K.	Klein F.	3	1889	a

a – Nachlass Felix Klein, Göttingen, b – Nachlass Moritz Cantor, Heidelberg, c – Nachlass David Hilbert, Göttingen, d – Mathematiker-Archiv, Göttingen, e – Nachlass Karl Schwarzschild, Göttingen, f – Universitätsbibliothek, Erlangen-Nürnberg – Bayerische Staatsbibliothek, Munich.

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Mellen Woodman Haskell in Göttingen and Leipzig

Henning Heller

Mellen Woodman Haskell (1863–1948) was among the first American students of Felix Klein. The Harvard graduate and later professor at Berkeley stayed in Leipzig and Göttingen from October 1885 until June 1889. Unfortunately, no first-hand sources remain from Haskell's formative years. Using civil, school, and university archives, this presentation sheds some light on Haskell's family background and education, his study years in Leipzig and Göttingen, his participation at Klein's seminars, and the circumstances of his PhD examination.

1. Family background and early life

Haskell was born on 17 March 1863 in Salem, Massachusetts, as the first child of the clergyman Augustus Mellen Haskell (1832–1893) from the village of Poland, Maine, and Catherine Woodman (1827–1866) from the neighboring town of New Gloucester [1]. The young parents lived in Salem from the year of their marriage, 1861, until the birth of their second son, Augustus Storey Haskell (1866–1949) [2]. In that year, the family moved to Manchester, New Hampshire, where Augustus found a new post, but tragically, Catherine died only months later of tuberculosis. Augustus remarried one year later with Anna Johnson (1826–1909) from Salem. In 1870, the family settled to West Roxbury, today a suburb of Boston, which remained their long-lasting home. In 1873, Mellen entered the Roxbury Latin School at the age of 10, three years younger than his peers. He even skipped a year and was admitted to Harvard College at the incredibly young age of 15 years, but decided to repeat his last year at school [3]. Haskell entered Harvard College in 1879 and studied mathematics at Harvard University from 1883 to 1885. He was awarded the Parker Fellowship to pursue a PhD degree in Germany.

2. Studying in Leipzig and Göttingen

Haskell matriculated in Leipzig in winter 1885/86 to study under the supervision Felix Klein, but did not take any courses there (he probably learned German instead) [4]. When Klein moved to Göttingen for the next summer term, Haskell came with him. In Göttingen, Klein delivered an advanced mathematical lecture series that followed his own research interests [5]. During the span of seven semesters, Klein lectured on algebra (S86–W86/87), elliptic modular functions

(S86), hyperelliptic functions (S87–W87/88), and abelian functions (S88–S89). In all but the last semester, also a seminar was held. Haskell was by far the most active student of this lecture cycle as he visited (in one case: probably) all lecture courses and seminars [6]. Additionally, he was co-responsible for the production of lecture notes in three lectures courses. During these years, a number of mathematicians, which were later influential for the setup of the mathematical research community in the United States, arrived in Göttingen. These include the German postgraduates Oskar Bolza and Heinrich Maschke, the American postgraduate F.N. Cole, and the American PhD students H.D. Thompson, W.F. Osgood, H.S. White, H.W. Tyler, and Maxime Bôcher [7]. Haskell was perhaps the only peer who personally met *all* of these figures, but as no personal communication remains, it is hard to estimate his status and influence.

3. Presentations

Haskell held four presentations in Klein's seminars, all of them during his first three semesters in Göttingen [8]. (He later participated at the seminars without presenting himself.) His first presentation concerned the explicit calculation of a degree-4 resolvent of the *octahedral equation*. This resolvent stems from a subgroup relation $S_3 \subset S_4$, while Klein in his *Lectures on the Icosahedron* [9] only considered a chain of resolvents stemming from the subnormal series $S_4 \rhd A_4 \rhd C_4 \rhd C_2 \rhd \{1\}$. Haskell's presentation thus filled a small gap in Klein's book. Although mathematically unspectacular, Haskell's consideration can be understood as a preparation to Klein's resolvents of his *icosahedral equation*, which likewise stem from non-normal subgroup relations ($D_5 \subset A_5$ and $A_4 \subset A_5$, respectively). In this sense Haskell did his share to "complete" the heuristics of Klein's geometrical approach to solving equations. Haskell's other three presentations concerned a recent publication of Lazarus Fuchs, and two "work-in-progress" presentations on his PhD thesis.

4. PhD examination

The content of Haskell's PhD dissertation – *On the multiple covering of the plane belonging to the curve* $\lambda^3\mu + \mu^3\nu + \nu^3\lambda = 0$ *in the projective sense* – was already considered in [11], so I focus on some new biographical insights [12]. In his review, Klein emphasized the difficulty of Haskell's subject, and praised that Haskell overcame them "with extraordinary diligence in a thoroughly satisfactory manner". He also noted "a number of individual investigations that claim independent importance". Haskell was thus accepted for his oral examination in mathematics and physics (as he had to choose a second subject), which took place on 6 June 1888. Unfortunately, Haskell did not pass the physics examination by Woldemar Voigt. Haskell was allowed to repeat the examination 6 months later, but decided to take his time and only repeated on 18 June 1889. He passed this second examination (again by Klein and Voigt), and two weeks later returned to the United States. From the steamboat "Aller", he thanked Klein for his support, and concluded about his study time in Göttingen:

> I have finally learned what it means to *work*. It is a hard thing to learn, but I hope it sticks with me. [13]

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Klein *ohne* Klein: Studentship at a distance in Chicago (1893-1910)

Nicolas Michel

Some ten years into his tenure as the first head of the mathematics department at the University of Chicago, E. H. Moore paid the following tribute to Felix Klein:[1]

> Certainly in the domain of mathematics, German scholars in general and yourself in particular have played, **by way of example and counsel and direct and indirect inspiration**, quite the leading role in the development of creative mathematics in this country, and on behalf of my colleagues here I wish to express our most grateful recognition and appreciation of our profound debt.

Despite only visiting the United States twice in his life, Klein seemingly exerted a lasting and multi-faceted influence on the shaping of American mathematics at the turn of the 20th century. But what were the modalities of this 'direct and indirect' influence? In their landmark account of the development of the American

[1] Letter from Moore to Klein, dated March 23rd 1904, Klein Nachlass X, NSUB, Göttingen, cited in [2, p. 324]. Emphasis mine.

mathematical community, Parshall and Rowe stress the importance of Klein's participation to the 1893 Chicago World Fair and his ensuing public lectures in Evanston, and map out the ways in which American students picked up on research themes *en vogue* in Germany and made them their own. In this paper, I elected to focus on another dimension of this 'influence', namely the fostering of a certain collegial and scientific culture at the University of Chicago modelled after and adapted from that which Klein had developed in Göttingen.

Indeed, we now know that an account of mathematics in Klein's Göttingen cannot be complete without an assessment of its culture of oral and informal conversations (whether at a professor's personal home or during *Spaziergängen*), of its seminar life full of intensity and competition, as well as of its mathematical library where visitors, students, and faculties would frequently meet and interact [1]. This paper, along similar lines, seeks to provide elements of a 'thick description' of the oral culture at the Mathematics Department in Chicago and to trace its inception back to Klein's Göttingen.

In fact, a first step in that direction had already been taken during a previous Oberwolfach meeting, centered around the figure of Oswald Veblen–a towering figure in the history of American mathematics who initially studied in the Moore-led department at Chicago. In this meeting, three other scholars and I collectively transcribed and analyzed a notebook written by Veblen as he studied in Chicago, and more specifically as he attended Moore's 1901 seminar on the foundations of geometry [3]. One outcome of this project was to highlight the rich interplay between seminar life, research activities, and the fostering of graduate students to which this notebook bore witness. Both Moore and Veblen would make key contributions to axiomatics and geometry in the wake of this seminar; contributions in which they both highlight this seminar as a site of collaborative and productive learning. What's more, we find the same pattern whereby seminars and dialogues between students and professors lead to important contributions by both groups amongst other students of Moore's and (at a later stage) of Veblen himself, who reproduced this social organization of mathematical life.

To better understand the origins of this form of mathematical life, one must look at another one of the three professors who initially constituted the Mathematics Department at the University of Chicago upon its creation in 1893: namely, Oskar Bolza. In his 1936 autobiography *Aus meinem Leben*, Bolza describes his own experience as a student in Germany looking to obtain a doctorate and to go into a mathematical career of his own. Upon meeting with leading mathematicians in Berlin, where experts in his subjects of choice resided, he was faced with scholars who had no interest nor desire to engage meaningfully with a young scholar-to-be. Neither Kronecker nor Weierstrass helped him design an appropriate research question for a dissertation, nor did they advise him as to how to work on said question: they would simply wait for him to bring a manuscript of sufficient quality to them, and then examine him.

Discouraged by this experience, Bolza then travelled to Göttingen, initially to work with Schwarz. In so doing, he encountered Klein and discovered an entirely different model for scholarly conduct. Klein would not only propose research questions and provide mathematical advice; he would also foster constant dialogue with and amongst students, meet weekly with them, and help insert them into German academia. Not all aspects of this practice were enjoyed by Bolza, however: after spending two semesters in 1886-1887 within this intense and dynamic community, he despaired to keep up with Klein's masterful weaving of so many mathematical concepts and intuitions and lost faith in his own ability as a researcher.

Bolza's research output would only rarely interact with the central themes of Klein's own mathematical product. Yet, he regarded the latter as the person who influenced him the most after Weierstrass. I contend that this influence mostly lies in the shaping of this model for scholarly conduct. Evidence to this idea can be adduced by considering the archives of the Mathematics Department under Moore's and his successor's (G. A. Bliss) leadership; sources which now are preserved at the Hanna Holborn Gray Special Collections Research Center in Chicago.

One such set of sources is a collection of letters sent to Bliss by dozens of former students of Bolza's, at Chicago. These letters, written in celebration of the 50th anniversary of Bolza's doctorate, all point to a mathematician's lasting commitment to many of the epistemic virtues which characterized Klein's tenure at Göttingen. Recollections of Bolza's availability and of the regular conversations he held with students at his own house and of his active stance towards the fostering of graduate students and the selection of appropriate research questions, for instance, feature heavily in these letters. Interestingly enough, however, the elitism that dominated mathematical conversations in Göttingen seems mostly absent from these students' experience–perhaps a remnant of Bolza's own anxieties regarding the overwhelming effects of genius.

Moreover, these letters provide a rich description of Bolza's much-appreciated teaching practices, including his constant reliance on historical exposition to motivate the study of a given subject-matter, to distinguish between ancient and modern (i.e., fit for graduate research) approaches to said subject, and to constitute a canon of classical texts whose knowledge he expected of students. Further evidence to flesh out this description can be found in the many notebooks written by students of Bolza's seminars, also preserved at the University of Chicago, as well as in the publications of Bolza and his best students (include Bliss himself). All those pedagogical traits can be profitably compared with the rich portrait of Klein as a teacher and a historian of mathematics which Renate Tobies has built over the years [4].

Together, these historical elements point to another sense in which Chicago mathematicians can be regarded as Klein's students *at a distance*, through the intermediary of Bolza's reproduction of a Göttingen-inspired figure of the mathematician as scholar, as teacher, and as researcher.

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Klein's influence in Britain

June Barrow-Green

In 1873 Klein made the first of several visits to Britain travelling in both Scotland and England. On the invitation of Henry Smith, the leading English number theorist and geometer, he attended the annual British Association for the Advancement of Science meeting which that year was held in Bradford, with Smith as President of the Mathematics & Physics section. It was at this meeting that Klein first met Arthur Cayley, William Clifford and James Clerk Maxwell, as well as the Irish astronomer Robert Stawell Ball. Clifford and Maxwell would die in 1879 but Klein's association with Cayley was enduring. Both Smith and Cayley sent postgraduate students to Klein including Arthur Buchheim, Arthur Berry, and Grace Chisholm. Another student who travelled from Britain to Göttingen during Klein's tenure was Horatio Carslaw who went there to study with Sommerfeld. Among the other British mathematicians who one way or another encountered Klein in their youth were the Cambridge educated mathematicians Henry Frederick Baker, Augustus Love and Edmund Taylor Whittaker.

Klein's reflections on his 1873 visit, as revealed in letters to Sophus Lie, give valuable insights into his interactions with the British mathematical community.[1] For example, not only did Klein find Sylvester to be more brilliant than Cayley, but he found "everyone in London" to be "generally of the same opinion" [1, p.148]. On that first visit, as well as on others, Klein travelled with the Scottish mathematician (later orientalist and Old Testament scholar) William Robertson Smith whom he had originally met in Bonn in 1867 (when Smith was studying with Plücker), and with whom he had cemented a friendship when they were both in Göttingen in 1869 [2].

Klein's first British student was Arthur Buchheim (1859–1888) who in 1881 made three presentations on Abelian integrals in Klein's seminar in Leipzig. Buchheim had been a student of Henry Smith's at Oxford and Smith thought extremely highly of him. However, when Buchheim returned from Germany, rather than continue at Oxford he made his career as a schoolmaster. In his obituary of Buchheim, Sylvester attributed Buchheim's refusal to apply for a vacant Fellowship at Oxford, despite being "strongly pressed by the authorities to do so", to Buchheim's sojourn with Klein having put him too much out of "the style of ordinary English University Examinations" [3]. Thus, it would appear that Buchheim did not think

[1]Several extracts from this correspondence are reproduced in [1].

he would pass the New College Fellowship examination while Smith thought that he would.

Klein's contacts with Cayley, and later with Andrew Russell Forsyth, who 1895 succeeded Cayley in the Sadleirian chair at Cambridge and whom Klein visited often, resulted in several postgraduate students going to Göttingen. Arthur Berry (1862–1929), who later worked in elliptic functions and differential equations, and became known for his *History of Astronomy* (1898), had been Senior Wrangler[2] in 1885, and went to study with Klein in 1887, giving a seminar on 'Differential Invariants'. Later Berry himself would encourage female students, such as Hilda Hudson (1881–1965) and Lorna Swain (1891–1936), to continue their studies in Germany.

The fulsome remarks in praise of Klein by Henry Frederick Baker (1866–1956) in the preface of his book *Abel's Theorem and the Allied Theory of Theta Functions* (1897), appear to be the only surviving evidence of Baker's meetings with Klein. Nevertheless, they make clear Klein's influence on Baker's mathematical thought. In addition, Baker published several papers in *Mathematische Annalen* (in English) in the 1890s, his choice of publication presumably deriving from his time in Göttingen. It also seems likely that the idea for Baker's geometry seminar (known colloquially as Baker's 'tea party') in Cambridge, which began in 1914 (with Baker's accession to the Lowndean chair) and which was the first, and for a long time the only, seminar in Britain, can be traced back to Göttingen [4].

Both Augustus Love (1863–1940) and Edmund Taylor Whittaker (1873–1956) spent time with Klein when he visited England, and they both wrote articles for Klein's *Encyklopädie*. In addition, in 1887 Love was commissioned by Klein to write an article on English work on vortex motion [5], and later Klein arranged for the translation of Love's text on elasticity into German [6]. Klein also arranged for the translation into German of Horace Lamb's text on hydrodynamics [7], both translations appearing in 1907.

With regard to translations, Klein's influence extended in the opposite direction too, with the translation of his own works into English. As well as his well-known *Lectures on the Ikosahedron and the Solutions of Equations of the Fifth Degree* (1888) which was enthusiastically reviewed by Cayley, there was his *On Riemann's Theory of Algebraic functions and their Integrals* (1893), the translation of which was done by Frances Hardcastle (1866–1941), a Cambridge student who completed the work while she was in the United States at Bryn Mawr, and published it at her own expense.

A number of British female students attended Klein's seminars in the 1890s, the most notable of whom was Grace Chisholm who studied for her PhD under Klein and in 1895 achieved the distinction of being the first woman anywhere to be awarded a traditional PhD in mathematics [8], [9].[3] Isabel Maddison, who was

[2]The Senior Wrangler is the top student in the Cambridge Mathematica Tripos.

[3]Chisholm was the subject of Elisabeth Mühlhausen's talk [10].

a contemporary of Chisholm's at Girton College in Cambridge,[4] subsequently went to Bryn Mawr to work under the direction of Charlotte Scott (a student of Cayley's) before studying with Klein and Hilbert in Göttingen during 1893/4. She was followed there by Ada Johnson (who had surpasssed all the men in Part II of the Cambridge Mathematical Tripos in 1894), in 1895/1896. (Rather curiously, all attempts to ascertain Johnson's area of research have so far failed. She returned to Cambridge where she was an Associate at Newnham College until 1908 but then drops out of sight.) A later visitor to Göttingen was Lorna Swain who, encouraged by Berry, went there just before the outbreak of the First World War but had to return to England in haste when war was declared. Swain would go back to Göttingen in the late 1920s to work with Ludwig Prandtl.

Another British traveller to Göttingen was Charles Tweedie (1868–1925) who in 1891 studied under both Klein and Schwarz [12]. Tweedie was unusual in that he had not been an undergraduate at Cambridge but at Edinburgh where one of his teachers was P. G. Tait whom Klein had met on his first visit to Britain. Tweedie made his career in Edinburgh and became best known for his work in history of mathematics, notably biographies of Colin Maclaurin and James Stirling, having earlier published papers in geometry and combinatorics.

That Klein was very well known in Britain is evident from the recognition he received. He was made a Fellow of the Royal Society of London in 1885 and in 1912 was awarded the Copley Medal, the Society's most prestigious award. He is the only foreign mathematician to have been awarded the London Mathematical Society's most important prize, the De Morgan medal (1893), and in 1897 the University of Cambridge awarded him an honorary doctorate.

From the above it is evident that Klein had many direct connections to members of the British mathematical community. Further research is required to establish more precisely the extent to which Klein can be said to have had an influence on them, both with respect to their mathematical development and with respect to their careers.

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From Naples to Pavia, passing from Göttingen. The scientific trajectory of Ernesto Pascal and his relationship with Felix Klein

Maria Giulia Lugaresi

The Italian mathematician Ernesto Pascal (1865-1940), born in Naples in 1865, completed his primary and secondary education in his hometown. Attracted by the mathematical teachings of Nicola Trudi, Emanuele Fergola, Achille Sannia and, most of all, Giuseppe Battaglini, he graduated in mathematics at the University of Naples in 1887. Soon after his degree, Pascal obtained a training scholarship for the academic year 1887-88 at the University of Pisa, where he had the opportunity to attend the lessons of Enrico Betti, Ulisse Dini, Luigi Bianchi and the young Vito Volterra. In the next academic year Pascal, encouraged by Eugenio Beltrami – who at that time was Professor at the University of Pavia – decided to go to the University of Göttingen to improve his studies.

From November 1888 to August 1889 Pascal was in Göttingen where he could meet and study with Hermann Amandus Schwarz and, most of all, Felix Klein, who contributed to orient Pascal's research towards Sigma abelian functions. Soon after his return to Italy, Pascal was appointed "Extraordinary Professor"(1890-95) and then "Full Professor"(1895-1907) at the University of Pavia, after the death of Felice Casorati. In 1907 Pascal was called at the University of Naples, where he remained until his retirement in 1935. He kept the chair of Higher Analysis, in 1910 he moved to the chair of Complementary Algebra and maintained for assignment the chair of Higher Analysis. In the same year he became editor in chief of the journal "Giornale di Matematiche di Battaglini", replacing Alfredo Capelli. Pascal died in Naples in 1940.

The main episodes of Pascal's academic life and his scientific trajectory of research can be better understood through the reading of the letters he wrote during all his professional life. As of 1889, Pascal was in correspondence with Klein. The Göttingen State and University Library preserved eleven manuscripts (ten letters and one draft) that Pascal sent to Klein between October 1889 and August 1913

([12]). The correspondence, even if composed only by eleven manuscripts, offers useful pieces to enrich Pascal's academic life. The correspondence began soon after Pascal's return to Italy. The letters proved Pascal's positive memory of his German experience. He recalled with enthusiasm, but also with nostalgy his stimulating meetings with Klein. The Italian mathematician took part in Klein's course in Summer Semester 1889, that were devoted to the theory of Abelian functions.

During his stay in Göttingen Pascal met and could work with many German and foreign mathematicians who came to Göttingen to study under Klein. In the letters to Klein Pascal referred to some of these mathematicians with whom he remained in touch after his return to Italy. Among the mathematicians who were in Göttingen in the same period he quoted Heinrich Burkhardt, Henry White, Mellen Woodman Haskell. Klein presented the research of his students in the sessions of the Göttingen Academy of Science. The first results of Pascal's studies about Abelian sigma functions appeared in two short articles, presented by Klein, and were published in the volume of 1889 of the "Nachrichten von der K. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität": *Zur Theorie der ungeraden Abel'schen Sigmafunctionen* (pp. 416-423); *Zur Theorie der geraden sigma-Funktionen* (pp. 547-553). The two articles were republished in a longer version in the volume 18 of the Annali of Brioschi in 1889 ([3]; [4]).

Between 1889 and 1895 Pascal's research dealt with Abelian, hyperelliptic and elliptic functions. This wide field of research was developed by Pascal taking inspiration from Klein's lectures and publications and gave birth to seven articles that appeared in the volumes 17-19 of the Annali of Brioschi ([3]; [4];[5];[6];[7];[8];[9]). Other influences of Klein can be found in Pascal's works about sigma elliptical functions, that were published in 1895 ([10]).

The correspondence with Klein was interrupted between March 1895 and February 1901. In this period Pascal's research continued, but his publications were mainly oriented towards handbooks for university teaching. Many monographs, prepared for his university courses, appeared between 1895 and 1897, first in a litographic version and then they were printed in paperback size by the editor Hoepli in Milan: *Esercizi e note critiche di calcolo infinitesimale* (1895); *Teoria delle funzioni ellittiche* (1896); *I determinanti: teoria ed applicazione con tutte le più recenti ricerche* (1897); *Calcolo delle variazioni e calcolo delle differenze finite* (1897); *Repertorio di matematiche superiori* (1897-1900, 2 volumes). Some years later these books were translated into German by the mathematicians Hermann Leitzmann and Adolf Schepp. These translations contributed to spread Pascal's works outside Italy. The main mathematical handbooks of Pascal had also a Polish translation thanks to the editorial work of the Polish mathematician Samuel Dickstein.

The *Repertorio* constituted an excellent contribution to a significant assessment of nineteenth-century mathematical production. It responded to the way in which studies were organised in Germany, providing an overall vision of a single discipline (analysis or geometry), in opposition to the extremely sectorial approach of Germany. Themes related to the development and the teaching of mathematics were particularly important for Pascal. In Naples he gave a great stimulus to the teaching of mathematics thanks to the creation of mathematical seminars and laboratories, the so-called "mathematical cabinets" (*gabinetti scientifici*). Pascal treasured his German experience when he decided to realise in Naples these scientific places and he talked about them in a letter to Klein (1913, August 14th).

In my paper I will give an overview of the content of Pascal's letters preserved in Klein's archive in Göttingen in order to reconstruct the development of Pascal's research following suit Klein. The letters represented also a proof of Pascal's devotion and respect for Klein. The Italian mathematician strongly supported two scientific and celebrating ventures in Naples: a prize for summarising Klein's results about hyperelliptic and Abelian functions and the appointment of Klein as a foreign member of the Royal Academy of Sciences of Naples.

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Wilhelm Wirtinger (1865–1945) and his publications on Abelian functions, in particular theta functions

Peter Ullrich

Even though Wilhelm Wirtinger himself saw a strong cultural, in particular scientific, connection between the German-speaking parts of the Austro-Hungarian Empire and the Deutsche Reich, which had been formed between 1867 and 1871 under Prussian leadership, he can be counted among Klein's "foreign students":

In 1866 Austria and Prussia fought the Seven Weeks' War – which also had the consequence that Göttingen became a Prussian university –.

Wilhelm Wirtinger was born on July 19, 1865 in Ybbs at the Danube where his father was chief physician at a predecessor of the Vienna psychatric clinic. At school Wilhelm was almost exclusively interested in mathematics and physics. Already as schoolboy he studied original works of Isaac Newton (1642/43–1727), Leonhard Euler (1707–1783), Carl Neumann (1832–1925) and others including texts on Abelian functions. During his studies of mathematics and physics at the University of Vienna from 1884 until 1887 he seems to have been mainly under the influence of Emil Weyr (1848–1894): Even before his doctorate, he published two papers [9], [10] which belong to synthetic geometry as the latter's research area. Also the topic of his doctoral thesis, on a cubic involution in the plane, comes from this part of mathematics.

After having taken his doctorate on December 23, 1887, Wirtinger received a scholarship from the Todesco foundation which enabled him to spend the winter semester 1888/89 in Berlin where he attended lectures with Lazarus Fuchs (1833–1902), Leopold Kronecker (1823–1891), and Karl Weierstraß (1815–1897). Even though he obviously was quite familiar with the standards of rigor as introduced by Weierstraß, it was the following summer semester 1889 in Göttingen that had the most decisive influence on his scientific career: He attended the last part of Felix Klein's (1849–1925) lecture courses on Abelian functions and then delved into research on this topic: Starting already in Göttingen and continuing after his return to Vienna, he published several notes [11], [12], [13], [14] on Kummer surfaces of genus 3 and the Abelian functions and theta series associated to them. Even more, as one learns from his letter to Klein dated December 28, 1892, he worked on the theory of general theta series already at that time. Klein had designed this topic for the prize question of the Beneke foundation for 1895 which was administered by the Philosphical Faculty at the University of Göttingen. On the basis of his book [16], which is connected to his articles [15], [17], Wirtinger received the prize.

This immediately helped his academic career: After his return to Vienna, Wirtinger had completed his habilitation at the University of Vienna in 1890 and had received the position of assistant to Emanuel Czuber (1851–1925) at the Polytechnic of Vienna. Following the announcement of his winning the prize of the Beneke foundation in 1895, he was appointed extraordinary professor at the University of Innsbruck. One year later, there he was promoted to an ordinary professorship. In 1903 he followed a call as an ordinary professor at the University of Vienna.

There he became the supervisor or at least a reviewer for almost all important mathematicians who received their doctorates at the University of Vienna between 1905 and 1930. Johann Radon (1887–1956), who had been one of these students, has called him "der größte Mathematiker Österreichs" (= "Austria's greatest mathematician") [6]. Wirtinger retired in 1935 and died on January 16, 1945, also in Ybbs at the Danube. (For further information on Wirtinger's life see [1], [2], [5], [8].)

Wirtinger not only published original research on Abelian functions, but also invested a lot of time and energy in publication projects that Klein had initiated: He was one of the co-editors of the three-part volume 2 on analysis of the "Encyklopädie der mathematischen Wissenschaften", wrote its article on algebraic functions and their integrals [18], provided templates for its article on elliptic functions, which appeared in 1913, and co-authored, together with Adolf Krazer (1858–1926), its article on Abelian functions and general theta functions [3]. Additionally, together with Max Noether (1844–1921), he edited the supplements to Riemann's collected works [4].

Within the period between 1901 and 1920, when these text were published, lies Wirtinger's "unfruchtbares Jahrzehnt" (= "barren decade") between 1909 and 1919 when he published no original results of research at all. One reason for this could be problems with the publications for the "Encyklopädie" project. But he also suffered severe blows of fate within his family: His eldest son died in an accident in 1912, his youngest son died in action during the First World War in 1915. And, naturally, the circumstances of this war and its consequences will have reduced his ability to conduct scientific research.

In a handwritten autobiography from 1939, Wirtinger states that it was the task of writing an article "Klein und die Mathematik der letzten fünfzig Jahre" (= "Klein and the mathematics of the last fifty years") [19] on the occasion of Klein's seventieth birthday in 1919 that brought him back to mathematical productivity. However, it is remarkable that from this time onwards, Abelian functions and theta series were no longer the focus of his research, cf. e. g., [7].

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Klein's Göttingen seminars on hydrodynamics (1903-1904, 1907-1908, 1908) and the development of the notion of applied mathematics until Richard von Mises (1883-1953) in the 1920s

Reinhard Siegmund-Schultze

Our most important source about Felix Klein's seminars–which stretched over about 40 years and took place in several universities (Göttingen, Leipzig, Munich)–is Renate Tobies' biography of Klein of 2019/20 [1]. Of the three seminars on hydrodynamics between 1903 and 1908, the one in the winter semester 1907/08 has been analysed in detail by Michael Eckert [2]. The same author has devoted an English monograph to the history of the turbulence problem which pervades much of the three hydrodynamics seminars [3]. Eckert and Tobies show that the seminar talks cannot be completely understood without considering the Göttingen context in general, in particular Klein's lectures on hydrodynamics and handwritten notes, both kept at the Manuscript Division (Handschriftenabteilung) in Göttingen.

In my talk I stressed the need to look at the talks which were given parallel to the seminars at the Göttingen Mathematical Society (Mathematische Gesellschaft) as well. This Society was also basically run by Klein from 1892. Of the talks one finds abstracts of varying length published in the Jahresbericht of the DMV. At the Gesellschaft, Klein usually reported about his seminars. In addition, talks by established mathematicians, physicists and engineers visiting or resident in Göttingen were presented there, including talks on applied topics. The hydrodynamics seminar talks were predominantly given by students. They treated basic topics such as Boussinesq's theory of fluids [4], the separation of the stream from the wall (Theodor von Kármán) and of vortices from the stream (Hiemenz, Koch, Fuhrmann), and Heinrich Blasius' discussion of turbulence [5]. Blasius was one of the first students of Ludwig Prandtl. Blasius' discussion of the difference between onsetting and fully developed turbulence in his seminars in January and February 1908 was taken over in 1916 by Prandtl. Other groundbreaking new ideas such as the Hungarian Győző Zemplén's theory of shock waves (1904), Prandtl's boundary layer (1904), and the Hungarian von Kármán's vortex street (1911) were reserved for talks at the Mathematische Gesellschaft and for parallel publications. Carl Runge and Prandtl, who from 1904 were directors of Göttingen's Institute for Applied Mathematics and Mechanics [6], preferred the venue of the Mathematische Gesellschaft for their presentations. They were absent as speakers at the three hydrodynamics seminars, although—according to Klein—were co-organizers and probably present at the seminars in 1907 and 1908.

A third effort, which completes the picture of applied mathematics and mechanics research done in Göttingen at the time, is Klein's Encyclopaedia of Mathematical Sciences. Here the Austrian Richard von Mises (1883-1953), who, at the time of the hydrodynamics seminars, was assistant to Georg Hamel at the German Technical University of Brünn, came into play. He had an intense correspondence with Klein and his assistant Conrad Müller between 1907 and 1912 which accompanied his article in the Encyclopaedia "Dynamical Problems of Mechanical Engineering" [7]. The correspondence shows Klein's high expectations for the young engineer and mathematician von Mises, who never presented a talk at the seminars and would later in the 1920s become the director of the institute for applied mathematics in Berlin. The correspondence also shows Klein's keen interest even in specific, technical applications such as Otto Schlick's patent (1894) for a gyroscope to prevent ship lurching. On this patent the future theoretical physicist Paul Ehrenfest had reported in Klein's seminar in 1902. Klein persuaded von Mises to include another invention with a similar purpose (Frahm's water tanks) in his article.

Klein passed on von Mises' critical remarks about another Encyclopaedia article to the author von Kármán (Strength of materials in mechanical engineering). In this episode, as in some others, the spirit of collaboration and competition becomes palpable which was typical of the Göttingen atmosphere both in pure and applied mathematics. In my talk I quoted some remarks from the autobiographical memoirs (1936) of Hans Lorenz (1865-1940) which were critical of Klein's alleged dictatorial ways of running the seminars [8]. Lorenz, who was originally considered by many as being able to imbue engineering and technical physics with a new level of mathematical sophistication, had been appointed by Klein in 1900. But disappointed by his lack of willingness to cooperate, Klein managed to replace him by Prandtl in 1904. Von Mises' scathing review of the insufficient mathematical treatment in Lorenz's turbine theory of 1906 may have endeared the 24 year old von Mises to Klein. However, the later development of both Lorenz and von Mises shows a certain frustration with some Göttingen tendencies at domination and towards "nostrification" (make them ours) of results obtained outside the Göttingen environment, a feeling which the two scientific engineers shared.

Nevertheless, von Mises remained a staunch admirer of Klein and devoted an article to him on his 75th birthday in his new *Journal for Applied Mathematics and Mechanics* (*ZAMM*) in 1924. In a public dispute (1927) with Richard Courant, Klein's successor as an organizer in Göttingen, von Mises claimed that the realization of Klein's efforts was being more loyally pursued in Berlin than in Göttingen [9].

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On the impact of Felix Klein on his students and their successors. Austro–Polish stories

Danuta Ciesielska

In the period from 1874 to 1912, more than 50 Poles studied mathematics with Klein in Munich, Leipzig and Göttingen. Between 1885 and 1911, 40 Poles or Polish students from Russia attended Klein's lectures, courses and seminars, including 14 men and women who gave lectures in Klein's seminars [11]. In this report I will focus on some of these people, works and future results.

The very first Poles, in total 12, studied with Klein in Munich. In Leipzig only two Poles attended Klein's courses and in Göttingen at least 43 Polish young students or scholars participated in Klein's lectures or seminars, 14 of them presented 19 talks during Klein's seminars [11]. Future physicist Józef Wierusz-Kowalski (1866–1927, Joseph von Kowalski, Josef de Kowalski) was the first who presented talk [11].

Many Poles who were former Klein's students did not work at universities until 1918 but some were professors at those universities that operated in Austria with Polish as a language of instruction: the universities in Kraków and Lwów, and the Lwów Technical School. Some became professors of University of Warsaw (opened by German Governor in 1915), Warsaw Polytechnic (1915) and Polish Free University in Warsaw (Wolna Wszechnica Polska). When Poland regained independence in 1918, the universities in Poznań and in Vilnius completed the list. Here I focuse on those Klein's students who worked in Kraków or Lwów before 1918, in land ruled by Austria.

The only Pole to appear in Klein's *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, vol. 2 (Berlin 1927) is Kazimierz Żorawski (1866–1953). Żorawski, from Źórawski noble family, studied in Warsaw and next with Sophus Lie in Leipzig (1889/90; 1890), where he was inspired to write his doctoral thesis (1891). From the winter semester of 1890/91 to the winter semester of 1891/92 he studied with Klein in Göttingen. He presented a paper on the Grünwald–Letnikov derivatives of fractional order in Klein's seminar in June 1891 [11]. In the summer semester of 1891 seminar took place between 14 May and 8 July 1891. In the meetings participated 12 people, and the talks were presented by Ernst Ritter (PhD 1891 in Göttingen under Klein, moved to the USA), Friedrich Schilling (PhD 1893 in Göttingen under Klein, Achen, Karlsruhe, Dantzing, president DMV), James Harrington Boyd (PhD 1892 in Princeton, professor at Macalester College, reader at Chicago), Eduard Burr von Vlek (PhD 1893 in Göttingen under Klein, professor at the University of Wisconsin Madison, now a chair of mathematics here holds his name) and Kazimierz Żorawski. The notes from this meetings are in volume **10**, 120–179.

In his contribution Żorawski presented basic information about the derivative of fractional order. The very first attempt to derivatives of fractional order are due to Euler and Liouville. The formal definitions were proposed by German mathematician from Prague Anton Karl Grünwald (1838–1920) and by Russian mathematician Aleksey Vasilyevich Letnikov (1837–1888). Grünwald in 1867 posed a definition [10]. Letnikov in his master thesis in 1868 proposed a similar definition [3] which was later elaborated by Pavel Alekseevich Nekrasov (1853–1924). Żorawski in his talk presented short history of an investigation and recall definitions, among them Grünwald's: $[D^p f(x)]_u^x = \frac{1}{\Gamma(-p)} \int_u^x \frac{f(z)}{(x-z)^p} dz$, next he focused on interesting examples. Later Żorawski published only one paper on the similar topic. It was an article in Polish about derivatives of infinitely large order [14]. Nevertheless, he was very active as a scholar and in academic policy. His scientific results include applications of Lie groups in geometry and the theory of differential equations and mechanics. Some of the results of Żorawski's research were obtained again later, published and attributed to other mathematicians. Żorawski's works were highly appreciated by Lie, Élie Cartan, and Klein. He published more than 60 scientific papers in German, French, Polish and Czech, he also contributed in history and wrote textbooks.

Żorawski was a dean of faculty, rector, a director in Ministry of Education in Poland but in my opinion the most interesting is his role as a Chair of Kretkowski's Fund. A very rich nobleman and a mathematician Władysław Kretkowski (1840–1910) donated in his last will the huge fortune for abroad studies in mathematics (1911–1919), for extra lectures and seminars, library, and a chair of application of mathematics in Kraków. Thanks to this donations 11 young mathematicians went abroad for undergraduated studies, and three of them studied in Göttingen: Franciszek Włodarski (1889–1944), Władysław Ślebodziński (1884–1972), Stanisław Ruziewicz (1889–1941), for more information see [1].

In the winter semester 11 people took part in the meetings of seminars, some of the previous participants had left, but some started participation. Among them were Poles: Stanisław Kępiński from Kraków and Bolesław Młodziejowski from Moscow. Kępiński spent two years in Göttingen (1891–1893) and participated in Klein's seminar with a lecture (see [11], vol. **11**). He started studies at the Jagiellonian University where he obtained doctorate; his thesis concerned partial differential equations. In Göttingen Kępiński studied with: SS 1891 – Schwarz, Klein, Burkhardt, WS 1891/1892 – Klein, Burkhardt and SS 1891/1892 – Klein, Weber, and Burkhardt. He wrote extensive reports from this studies (Archive of the Jagiellonian University), fragments of them were published in [1] and [2]. In 1894 he obtained *vienam legendi* from the Jagiellonian University and worked in Kraków for a year. Next he continued his academic and scientific career at Lwów Polytechnic, he was elected a rector and member of Galizian (province od Austria) parliament in 1903. Four Kępiński's letter to Klein are the only letters sent by Polish scientists to him [5].

The second part of contribution deals with the influence of Klein and Lie research on the group of symmetries on the second generation of his former students. The concept of the abstract topological group is connected with Hilbert's fifth problem. In 1900 in Paris Hilbert asked if it was possible to have "Lie's Begriff der kontinuierlichen Transformationsgruppe ohne die Annahme der Differentzbarkierait der di Gruppe deffnireden Functionen" [Lie's concept of the continuous transformation group without the assumption of the differenceability of the functions defined in the group] (see: [4], p. 269). It started a new concept of group in Euclidean space but without transformation. That idea finally lead to the formal definition of topological group. In 1925 Franciszek Leja (1885–1979) and Otto Schreier (1901–1929) independently presented it in their reports [7], [12] and two years later in papers [8], [9], [13]. This amazing, almost mystic, coincidence is one of dozens in the history of mathematical research, but here there is no doubt that both heroes was influenced by the ancestor, including Felix Klein. A former student of University of Lwów, Leja worked on his dissertation on invariants of partial differential equations under mentioned before Żorawski. Leja's later research on the application of Lie groups to differential equations led him to a definition of the abstract topological group as group and topological space at the same time, such that the group operation: product and inverse map are continuous. An Austrian Otto Schreier was a doctoral student of Philipp Furtwängler (1869–1940), who was Klein's doctoral student in Göttingen. Schreier was working on abstract algebra and this led him to the definition of abstract topological group; moreover, he proved that it must be abelian.

Figure 1. Members of Mathematical Society in Göttingen, 1895. SUB Göttingen, 2 Sammlung Voit: Gruppenbild 4. Sitting (from left): Eduard Götting^{p}, Ernst Schering, Henrich Weber, Woldemar Voigt, Eduard Riecke and Wilhelm Schur. Standing (from left): Ernst Harald Schütz^{p}, Ernst Ritter^{t}, Henrich Burkhardt^{t}, Ludwig Harald Schütz^{p}, Ignatz Robert Schütz, Julius von Braun, Georg Bohlmann, Erich Prümm^{p}, Rudolf Schmidt^{p}, Wilhelm Felgentraeger, Otto Blumenthal^{t}, Stanisław Tołłoczko^{t}, Sophus Marxsen^{t}, Adolf Jost, Maximilian August Toepler, Teophil Friesendorf^{t}, Dychhoff, Wilhelm Lorey^{t}. p – studied with Klein, t – presented talk at Klein's seminar.

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Göttingen, far away: Felix Klein's influence on mathematics in Japan

Harald Kümmerle

While Göttingen figures prominently in the history of mathematics in Japan as a destination of study abroad, it is difficult to clearly identify Japanese students of Klein. This is importantly because the founder of mathematics as a discipline in Japan, Fujisawa Rikitaro (1861-1933), had laid a strong emphasis on pure mathematics. Investigating the cases of Yoshiye Takuji (1874-1974) and Kuroda Minoru (1878-1922), both of whom studied in Göttingen, gives insight into the mathematical "establishment" in early 20th century Japan. In contrast, two of the most prolific recipients of Klein's contributions to the movement for the reform of mathematical education, Hayashi Tsuruichi (1873-1935) and Ogura Kinnosuke (1885-1962), had very untypical biographies and never studied in Göttingen (nor Germany more generally). Despite this, they were based at Tohoku Imperial University (in Sendai), where the whole department of science officially give itself the motto to become "Japan's Göttingen". When investigating their activities in mathematical research and education as well as the hurdlest, they faced, Hayashi and Ogura appear as two exponents of the "anti-establishment". Ironically but not coincidentally, the movement for the reform of mathematical education in Japan

– together with other ambitions of Felix Klein – achieved success only when militarism was taking hold during the 1930s. Thus, the talk provides an additional perspective on the complex relation of mathematics and modernity.

Getting to Göttingen: Support for women's mathematical research at Bryn Mawr and Girton

Brigitte Stenhouse

The end of the nineteenth century saw the emergence of mathematical research as a formal part of university life, legitimised by the awarding of doctoral degrees. The seminars of Felix Klein epitomised the ideals of a certain style of university research that became highly respected (and imitated) by a broad mathematical community. The seminars fulfilled many functions, from connecting mathematicians on an international scale, enabling collaboration, supporting students in the pursuit of a PhD, and substantially directing research through the posing and answering of questions [1]. Although participation in university life was highly gendered, this story of rigorous higher education and *careers* in mathematical research is no less true for women as for men.

In this talk we explore the communities and structures built to enable women to pursue research in mathematics. We focus especially on the anglophone women who came from Bryn Mawr College in Philadelphia, USA and Girton College in Cambridge, UK to participate in Klein's Göttingen seminar.

Girton College was founded in 1869 as the first residential college for the degree-level education of women. The founder had a clear vision of what she wanted to achieve with the college: "Miss [Emily] Davies was resolved that women students should submit to the same tests, in order that they might share the same opportunities, as men" [2, pp. 16–17]. This occasionally put Davies in conflict with the founders of Newnham College, also in Cambridge, who were happy to provide separate lectures and exams specifically for women, rather than fighting to integrate women into the established Cambridge systems (though it should be noted that this was partly in response to a low opinion of the Cambridge exams!). High school education for girls was still at a low standard in the late nineteenth century, and the early Girton students had to almost start from scratch learning the mathematics, latin, and other subjects necessary to pass the intermediary exams — such as the 'Little-Go' — before moving on to the prestigious Tripos examinations taken in their final year. In order to sit the exams, special permission was required from the examiners, each and every year, with no guarantee it would be granted. Nevertheless, the tenacity of the teachers and their students paid off in 1905 when Trinity College, Dublin offered retrospective reciprocal degrees to any woman who had previously satisfied all the necessary degree requirements. The hundreds of women who travelled by steamboat to Ireland to claim their degree (forty years before Cambridge would award degrees to women on an equal footing with men) were thus dubbed the 'Steamboat Ladies'.

One of the earliest Girton students to sit the Mathematical Tripos was Charlotte Angas Scott, who was placed as 8th Wrangler in the First Class in 1880. (All undergraduates who sat the exam were ranked in order, and those achieving a first class were named Wranglers with the top student named Senior Wrangler. Women were not officially included in the ranking.) Frances Hardcastle was bracketed 53rd in Part I in 1891, and then achieved a second class in Part II in 1892. In 1892, two students from Girton sat Part I of the Mathematical Tripos and attained the equivalent ranking of Wrangler, namely Isabel Maddison and Grace Chisholm (later Chisholm Young). Grace Chisholm subsequently sat Part II in 1892, being ranked in the 3rd class. All of these women were subsequently involved with Klein's seminar, as were many of the men who they were taught by at Girton, such as Arthur Cayley, Arthur Berry, and Andrew Russell Forsythe [3].

Meanwhile, in the USA, Brywn Mawr College was founded in 1879 with classes beginning in 1885 [4]. Similarly to Girton, Brywn Mawr had a strong focus on providing an education for women to rival that received by their male contemporaries. Indeed, Bryn Mawr was the first women's college in the USA to have a graduate department, whilst many of the others focused on offering high-school-level tuition to compensate for the inadequate extant provision for women. The college was substantially shaped by the first Dean and later President, Martha Carey Thomas, who had been refused a PhD in linguistics from Göttingen in 1882, owing to her sex. Charlotte Angas Scott was appointed as Associate Professor in Mathematics at Bryn Mawr in 1885, and would supervise eight PhD students during her tenure which lasted until her retirement in 1924. One of those PhD students was Isabel Maddison, who subsequently worked at Bryn Mawr in various roles until 1926, including as Assistant to the President (Thomas) and Reader in Mathematics. Three other PhD students who passed through Göttingen include Emilie Norton-Martin, Virginia Ragsdale, and Helen Elizabeth Schaeffer [5].

A key motivation for many higher education institutions was to train women into qualified teachers who could then provide a higher quality education for girls. This is reflected in the career trajectories of the alumni of Girton and Bryn Mawr, at least for those who remained unmarried and thus in paid employment [6]. However, through these institutions women were simultaneously building infrastructure to support themselves in pursuing alternative careers in mathematical research [7].

Research Fellowships were offered to graduate students at Bryn Mawr specifically to enable a period of study and research to be undertaken in Europe. Maddison, Norton-Martin, and Schaeffer all made use of the Garret European Fellowship in order to study with Klein in Göttingen during their doctoral studies; Ragsdale was supported by the Bryn Mawr European Fellowship. The importance of funding research activities was directly recognised and championed by Girton alumni, who collectively funded a Research Studentship worth £100 a year for two years. By the 1930s the fellowship infrastructure at Girton had considerably expanded to include multiple opportunities aimed directly at researchers in the physical sciences, including mathematics. Money was also used to recognise and celebrate achievements of alumni, which would assist in reputation building as a mathematician. Maddison and Chisholm-Young were both awarded the Gamble Prize from Girton, for published articles in 1895 and 1915 respectively. Hardcastle similarly won the Gamble Prize in 1897 for an article on point groups, and she was subsequently invited to write a report on this topic for the British Association for the Advancement of Science; her work in this area was supported by a Pfeiffer Studentship, again from Girton.

Both Bryn Mawr and Girton provided employment opportunities for women, but it is ambiguous as to whether this helped or hindered research activity. Teaching loads could be heavy, leaving little time for other pursuits, and contracts were often precarious, only running year by year. It is pertinent to consider the case of Grace Chisholm-Young, perhaps the most research active of all women who attended Klein's seminar, and yet she never held an official academic position, instead collaborating with her husband who had easier access to university jobs as a Cambridge-educated man [8]. Nevertheless, women's colleges provided gainful employment and access to an academic community that was not always easily found elsewhere; learned societies in Europe and North America only gradually accepted women as full members from the end of the nineteenth century onwards.

In conclusion, whilst there are many specificities to studying the trajectory of women's research careers at the end of the nineteenth century, doing so allows us to reflect more broadly on the changing landscape of academic research. We can witness the emerging means of publishing and gaining recognition for work done; the beginnings of research taking on a complementary role to teaching within university spaces; the need for robust high school education to prepare students for university; the importance of scholarships and research fellowships; and the high value of building networks and social organisation which thus directs and enables the development of new ideas in mathematics.

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Felix Klein's first female doctoral student Grace Emily Chisholm Young (1868–1944). A livelong connection concerning mathematical research and much more

Elisabeth Mühlhausen

In October 1893 Grace Emily Chisholm arrived in Göttingen to study mathematics, physics and astronomy. Having successfully completed her studies at Girton College in Cambridge her lecturer Andrew R. Forsyth had suggested that she should continue there because he knew about his colleague Felix Klein's commitment to women's studies.

With enthusiasm and energy she followed the lectures, she joined discussions with her new academic acquaintances. During this time she enjoyed the invitations of Klein and his family. She also improved her German language skills and gave her first lecture at Klein's seminar early in 1894 about spherical trigonometry. This subject was suggested by Klein and the basis for her dissertation on "Algebraisch-gruppentheoretische Untersuchungen zur sphärischen Trigonometrie". She obtained her Ph.D. degree "magna cum laude" in April 1895 and was the first woman in Prussia to do so.

On the famous photography of the Göttingen Mathematics Club of 1902 you can see her keeping eye contact with Klein in the centre who is surrounded by his younger colleagues. Meanwhile Grace had married the mathematician William Henry Young (1863–1942). On their first mathematical journey to Italy, another suggestion of Klein's of course, they got to know the mathematicians who belonged to his network. In winter 1898 they studied for some months at the University of Turin with professor Corrado Segre (1863–1924) who held the chair of higher geometry and was a regular correspondent of Klein. In the field of algebraic geometry the Youngs wanted to get on the latest research level. One year later their results were published in Turin, in Italian naturally.

Their successful joint work started 1900 when Klein advised them to read the report on set theory of Arthur Schönflies (1853–1926). The next 25 years they concentrated their research on this evolving field. One year before Klein had adviced Schönflies to write an article on the state of set theory for the Encyclopedia and soon thereafter Schönflies published his much longer report "Die Entwicklung der Lehre von den Punktmannigfaltikeiten" in the journal of the Deutsche Mathematiker-Vereinigung.

It was the starting point for their contribution to set theory and its applications. Between 1900 and 1905 they published their joint work results in about 20 papers and concluded a contract with Cambridge University Press to publish a textbook in set theory that introduced this new field to the UK. "The Theory of Sets of Points" was published already 1906.

They also worked in related areas like measure theory and integration, Fourier series, and the foundation of differential calculus.

Until 1929 they published 214 papers which are listed in the bibliography by Ivor Grattan-Guinness. Most of them appeared under William Young's name, 13 were jointly authored and 18 have Grace Chisholm Young as the sole author.

For me as a former teacher of mathematics and biology it was particularly interesting that Grace Chisholm Young also found time to write children's books. She published in 1905 "Bimbo" and in 1907 a successor called "Bimbo and the Frogs" in which she describes the life of a family that is strongly reminiscent of the Young family. At that time they already had four children: Frances *1897, called Bimbo since their time in Italy, Cecily *1900, called Rosebud, Janet*1901, called Lenchen and the just born Helen *1903, in the books mentioned as Dortchen in the cradle. Every family member is part of a turbulent family life and in between the focus is on a boy, Bimbo, who asks questions that are answered with illustrated biological descriptions of the development of plants and animals especially frogs.

In 1905 Grace Chisholm Young published another book, her first joint publication with her husband with the title: "Beginner's book of Geometry". It describes the basic features of geometry in terms of folding paper. It is obvious that she took up Klein's ideas how to teach mathematics in a visual way.

The book is entertaining not only because of many illustrations but a child needs perseverance and parental help to get through the complicated folding instructions. Fortunately every 3D model can be folded without using any glue. The child-friendly structure of the book from the simple to the difficult is well done.

In 1908 there appeared a German translation "Der kleine Geometer" by Felix Bernstein (1878–1956) who at that time he had just published his dissertation concerning set-theory. Felix Klein was glad to see it and was very fond of it. In his "Elementarmathematik vom höheren Standpunkte aus", published 1908, it is described as a new original way to introduce the child to geometric understanding by starting with all kinds of three-dimensional models.

Apparently Felix Klein sensed the mathematical creativity and power Grace Chisholm Young possessed. His trust in her talent was certainly a strengthening factor for her mathematical confidence. Under his influence she changed from a student to a research mathematician. I think, it was one of Felix Klein's special talents to recognise and support talented mathematicians regardless of gender.

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From St. Petersburg to Göttingen. About two female Felix Klein undergraduate students

Joanna Zwierzyńska

The second half of the 19th century was a pivot period for women's higher education. They were finally allowed to study at universities in several European countries such as Switzerland or France. In the Russian Empire, they were still not formally admitted to universities, but institutions aimed exclusively at them gradually emerged, making higher education possible. The largest and best-known such institution in the Russian Empire was the Bestuzhev Courses – a four-year course founded in 1878 in St. Petersburg that enabled women deprived of access to universities to obtain higher education.

The idea of women's higher education was promoted in the society of the Russian Empire by female activists leading the women's movement in this country: Nadezhda V. Stasova, Maria V. Trubnikova, and Anna P. Filosofova. They gained support from wide circles of democratic intelligentsia, among others professors of St. Petersburg University, to create such an institution [5]. Higher Women's Courses were named Bestuzhev Courses after Konstantin Nikolayevich Bestuzhev-Ryumin (1829–1897), professor of history and their first director.

Bestuzhev Courses opened their doors to women of all social classes. To allow women with worse financial situation to study, Filosofova, Stasova, and other female activists created the Society for Providing Means of Support for the Higher Women's Courses, which organised book sales, lotteries, lectures, and concerts so they could award scholarships to deserving women [4].

The Higher Women's Courses had two faculties: one in history and philology and the second one in mathematics and natural sciences. They stood out for their excellent teaching staff as well as their facilities: they had excellently equipped classrooms, and in 1880, the first chemical laboratory for women was provided. In 1903, a mathematical reading room was opened due to the students' request [8].

The first women graduated from Bestuzhev Courses in 1882, four years after the school was opened. Graduates of the courses were pioneers in almost every scientific field in the Russian Empire and beyond - suffice it to say that among them were for example the first woman to become an employee of the Pulkovo Observatory, Russia's first female petrographer and palaeontologist, founder of the empire's first women's accounting courses, the first female climatologist in Russia or the first female university professor in Romania.

Among the graduates of the Bestuzhev Courses were Helena Bortkiewicz (Helene von Bortkewitsch) and Aleksandra Stebnicka (Alexandrine von Stebnitzky), both of Polish origin and of noble birth. Both came to Göttingen attracted by the opportunity to study under Felix Klein's tutelage. Klein was known as a firm believer in the equal abilities of men and women, and he accordingly believed that they should have access to the same educational opportunities [6].

Helena Bortkiewicz (Helene von Bortkewitsch) was born on 3.08.1870. Her father was Józef Bortkiewicz, a Polish nobleman who served in the Russian army with the rank of colonel, lecturer in artillery and mathematics at the military academy, and author of textbooks on mathematics, economics, and bookkeeping. Her mother was Helena (Helene) Bortkiewicz, née Rokicki (von Rokicka). Helena (daughter) was the sister of the statistician and economist Władysław (Ladislaus) Bortkiewicz (von Bortkewitsch) (1869–1931).

Aleksandra (Alexandra, Alexandrine) Stebnicka (von Stebnitzky) was born 23.04.1870 in Tbilisi. She was a daughter of the Polish engineer, general Hieronim Stebnicki (1832–1897), a cartographer, geodesist, geophisicist and a corresponding member of the Academy of Sciences in St. Petersburg, who worked in the orbervatory in Pulkovo, and his wife Praskowia. Alexandra's sister Olga was mother of Pyotr Kapitsa (Peter Kapitza), a famous phisicist and Noble laureate.

Both Helena Bortkiewicz and Aleksandra Stebnicka came to Göttingen in the academic year 1894/1895 (Władysław Bortkiewicz already had a PhD degree obtained in 1893 in Göttingen, but in 1895 he worked in Strasbourg). Because of their gender, they were not allowed to enter the matriculation book, but they could attend lectures and seminars of professors who gave their consent. They not only participated in Klein's courses but also presented papers in his seminar. Both enrolled in Klein's lectures on differential (SS 1895) and integral (WS 1896/1897 – only Bortkewitsch) calculus, number theory (WS 1895/1896), theory of the top (WS 1895/1896) and the mathematical theory of the gyroscope. Both participated in Klein's seminar (SS 1895, WS 1895/1896, and only Bortkewitsch in WS 1896/1897) Each of them gave two talks at Klein's seminar:

- 28 May 1895, Tuesday, Helene von Bortkewitsch, *Differenzenrechnung.*
- 31 May 1895, Friday, Alexandra von Stebnitzky, *Summationsrechnung.*
- 27 November 1895, Wednesday, Alexandra von Stebnitzky, *Ueber die ganzen Zahlen im Körper(i) und ihre Zerlegungssätze.*
- 11 December 1895, Wednesday, Helene von Bortkewitsch, *Grundlegung der Idealtheorie.*

An analysis of the notes shows that Aleksandra Stebnicka and Helena Bortkiewicz were well-prepared to deliver a fairly advanced lecture. Unfortunately – neither of them was given the opportunity to pursue a real scientific career. With their immense talents, excellent education and ample opportunities, they did not realise a career in science.

Helena Bortkiewicz, after her stay in Göttingen came back to Russia. As Wolfgang Karl Härdle and Annette B. Vogt realised, ,,[she] published papers in Russian journals, but the situation was not comfortable for her since the only widely accepted professions for women were as a physician or a teacher in a girls school" [2]. She worked for three years as a teacher of mathematics and languages. In February 1917 she became a staff member in a St. Petersburg bank [2].

After the October Revolution, Helena Bortkiewicz moved to Berlin, where she lived from 1919 in her brother's apartment in Berlin-Halensee After his death in 1931, she suffered severe financial problems [2]. She died 29.10.1939.

Aleksandra Stebnicka after studies came back to St. Petersburg and tried to work scientifically. She was primarily an astronomer, not a mathematician, so it is not a surprise that she became an astronomer in Pulkovo near St. Petersburg, where she independently conducted astronomical observations, sponsored by the Imperial Academy of Sciences in St. Petersburg. She died 28.04.1928.

We can only speculate how would Helena Bortkiewicz's and Aleksandra Steblicka's fate look like if they did not meet the restrictions because of their gender. It is, however, interesting to compare their story with that of their yearmate Teofil

Friesendorff, also of Polish origin, also a member of the St. Petersburg Mathematical Society and also a student in Göttingen in 1895, in particular, presenting papers at a Klein seminar.

Teofil was immortalised in 1895 in a photograph of the Mathematical Society in Göttingen, in which it is in vain to find any women, particularly his female Polish colleagues. The same year, nine women studying mathematics in Göttingen, including Helena and Aleksandra, formed a club, which can be interpreted as the formation of the first women's network of mathematicians [7]. Still, we do not have any photograph of their society.

Two years after participation in Klein's seminar, in 1897, Friesendorff attended the First International Congress of Mathematicians in Zurich, and a few years later, became a professor at the Electrotechnical Institute in St. Petersburg. Of course, we can only speculate whether he was the most talented or hardworking of their three – but I have not found any evidence to support this statement.

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Julio Rey Pastor, precursor of the modernization of mathematics in Spain and Argentina under the inspiration of Felix Klein's Erlangen Program

Grodecz Alfredo Ramírez Ogando

1. Historical Mathematical Context: Argentina in the 19th century

After gaining its independence in 1820, Argentina found itself in the need to develop knowledge in both engineering and science. There are several attempts to modernize mathematics by sending students to Europe or bringing scientists from Europe. Two examples from the mid-19th century were shown. The first was Santiago Cáceres who after studying theology and philosophy at the University of Cordoba, in 1853 the university sent him to study mathematics, physics and astronomy with Weber and Gauss. When Cáceres returned, he was supposed to occupy the chair of exact sciences. This did not happen due to bureaucratic problems within the University of Cordoba.[12] The second example was Valentin Balbín (1851-1901), he went to Oxford in 1872 to studied mathematics. In 1884 he returned to the University of Buenos Aires to the chair of Higher Mathematics. In 1889 Balbín founded the "Revista de Matemáticas Elementales" and in 1892 to 1896 Balbín was rector of the Unvercity of Bueno Aires [12].

2. Historical Mathematical Context: Spain in the 19th century and beginning of the 20th century

José Echegaray y Eizaguirre (1832-1916) mathematician and writer, Nobel Prize for Literature in 1904, gave in 1866 an acceptance speech to the "Real Academia de las Ciencias Exactas, Físicas y Naturales" entitled: "Historia de las matemáticas puras en nuestra España". In it he seeks to show the level of regression of Spain in the area of mathematics and proposed to promote basic sciences. The speech was highly exaggerated to provoke a reaction in the Spanish scientific community and the Spanish government [1].

There did not seem to have been a reaction on the part of the political or university authorities, but rather a counter-reaction, since in 1875 academic freedom was limited in all the universities of Spain and the study plans were drawn up for all the universities from the university of Madrid. In response to this limitation, the academic community began to found institutes outside the universities, such as the "Instituto de enseñanza libre". In its journal it has contributions from Rusell, Tolstoy, Montessori, Darwin among other personalities of the scientific, cultural and pedagogical world [8].

Another institution founded to promote scientific and cultural exchange was the "Junta de Ampliación de Enseñanza e Investigaciones Científicas" (JAE) founded in 1907. The JAE was in charge of coordinating the sending of Spanish students to other countries. At a later stage, it was in charge of creating autonomous laboratories independent of the universities. This is important, since the JAE is an effort of Spanish academics to have more contact with scientists outside Spain and thus improve the level of science in Spain [3].

3. Zoel García de Galdeano y Yanguas

In spite of this control in the periphery, some academic freedom was exercised, as it happened at the University of Zaragoza in the Faculty of Sciences in the mathematics course. This was the case of Zoel García de Galdeano y Yanguas (1846-1924), who was professor of infinitesimal calculus in Zaragoza from 1889 to 1918 and taught courses in set theory, non-Euclidean geometry, algebraic geometry and in 1907 he gave a short exposition of Felix Klein's Erlangen program. With them he sought to bring to his students the avant-garde mathematics that was being carried out in Germany, France, England and Italy [1]. García de Galdeano had contact with several German mathematicians such as George Cantor. In 1899

García de Galdeano wrote a book on geometry which he dedicated to Felix Klein, to whom he sent the book, as can be seen in a letter in the archives of the University of Göttingen [2],[9],[6].

The interest in this international contact was shown by his participation in the International Congresses of Mathematicians (1897, 1900, 1904, 1908, 1912, 1920). In addition, at the Rome congress of 1908, he was appointed Spanish delegate to the "International Commission on Mathematics Education" (ICME), where Felix Klein was the president [1].

One tool García de Galdeano used to promote mathematical development among his students was his personal library, known by his students as "La biblioteca de Don Zoel". This library had more than three thousand books with works by Klein, Riemann, Cauchy, Darboux, Cantor, Weierstrass, Mittag-Leffler, Lie, Poincaré and many others [7].

4. Julio Rey Pastor

Julio Rey Pastor (1888-1962), was born in la Rioja not so far from Zaragoza, was a student of García de Galdeano from 1904 to 1908 and did his doctorate at the University of Madrid, since at that time it was only possible to do a doctorate in mathematics in Madrid or Barcelona [5]. As a student, he published several articles in the "Revista trimestral de matemáticas" and in the "Anales de la Facultad de Ciencias de Zaragoza" [11]. He obtained his doctorate degree under the advice of Eduardo Torroja, with the thesis in synthetic geometry entitled: "Correspondencia de figuras elementales: Con aplicación al estudio de las figuras que engendran". He went to the Universtity of Oviedo for the chair of Mathematical Analysis at the University of Oviedo in 1911 where he gave a provocative speech in a tone similar to Echegaray's, denying the existence of Spanish contributions to science [8]. In the same year in October he obtains a scholarship to study at the University of Berlin mathematical analysis and advance geometry. In Berlin he studied with Schwarz, who was his tutor, and took courses with Frobenius, Knopp, Schotty and Schur [12]. During his stay in Germany he had the opportunity to conduct research at the university of Munich for a historical work on Spanish mathematicians of the 16th century [12].

In 1913 Rey Pastor obtains the chair of Mathematical Analysis at the University of Madrid and in the summer of he went to Gottingen.This was a 14-month stay. He attended courses taught by Carathéodory, Courant and Hilbert. He also took part in seminars on number theory, led by Landau and on function theory led by Herzglotz and Koebe.He had to suspend his stay in Göttingen earlier than planned because of the war and before returning to Spain he went to Italy where he visited Italian mathematicians [12].

As a result of his stays in Germany he publishes two very different articles in geometry. The first one in 1912: "Geometric Theory of Polarity" more an article on syntetic Geometry. The second in 1916: "Fundamentals of Higher Projective Geometry." The second is on algebraic geometry and in 1917, Rey Pastor worked on problems related to the uniformization theorem of Riemann surfaces.

What is notorious in some sources is that they say that in Göttingen he took lectures or was a student of Felix Klein. However, in the Göttingen records of the students who took courses with Klein, the name of Rey Pastor is not found. It is possible that he may have met him or taken a course with him however this claim cannot be proven.

In 1915 Rey Pastor gaves a series of six lectures on contemporary problems of mathematics at the Ateneo de Madrid. The lectures were printed under the title "Introducción a la Matemática Superio" and Hermann Weyl made a review of this work in which he asked Rey Pastor: How did you manage to develop in six lectures the essential ideas of contemporary mathematics? I do not cease to admire as I read each line [12].

Rey Pastor implements at the university of Madrid with the help of JAE structures for mathematical research in the tradition of Felix Klein's seminar at the university of Göttingen. In 1916 he founds the "Seminario de Investigación Matemática" [10].

Rey Pastor was invited in 1917 by the "Institución Cultura Española" to give a series of lectures at the university of Buenos Aires. This institution was created by Spanish immigrants to Argentina and this institution had a chair within the University of Buenos Aires. He gave the lectures of the "Introducción a la Matemática Superior" and his stay in Argentina lasts five months, more than what was planned [12], [10].

In 1921 Rey Pastor obtained a chair at the University of Buenos Aires, without losing his chair at the University of Madrid where he went to give courses for a few months every year until his retirement in 1962 with some interruptions [12].

In Buenos Aires the entrance in the university of Rey Pastor brings the modernization of mathematics by founding, in the tradition of Felix Klein, mathematical research seminars and their respective journals [12].

5. Conclusions

No evidence was found that Julio Rey Pastor was a student of Felix Klein. However, both in Spain and in Argentina he applied Klein's program and the traditions learned in Göttingen with the research seminars. It should be mentioned that Rey Pastor already possessed this mathematical academic culture before his stays in Germany through his teacher Zoel García de Galdeano. In this case, the work done in Spain by García de Galdeano for the intruduction of the Erlangen Program and the modernization of mathematics was transcendent and Rey Pastor was the one who acted as an amplifier for the change of mathematical culture within the Klein tradition, both in Spain and in Argentina.

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Adding Another Dimension – Felix Klein's Influence from the Perspective of David Hilbert, Adolf Hurwitz, and Hermann Minkowski

Jule Hänel

Felix Klein's mathematical network was without doubt extensive. Part of this network as his student was Adolf Hurwitz and as his later colleagues David Hilbert and Hermann Minkowski, from whose perspective Klein's influence on the mathematical community between 1885 and 1919 is investigated here. The foundation for this is the extensive correspondence between the three mathematicians, in which Klein is one of the most frequently referenced personalities. In particular, comments appearing on international students that are listed in the Klein protocols [4] documenting his seminars in Göttingen are examined and evaluated.

Their early mathematical careers led the three mathematicians to Königsberg: Adolf Hurwitz (1859-1919) was a Jewish mathematician who studied in Munich, Leipzig, and Berlin, where he completed his habilitation under Schwarz and Weierstrass before becoming an extraordinary professor in Königsberg in 1884. Hermann Minkowski (1964-1909), also of Jewish origin, studied in Königsberg and Berlin and spent his career between 1887-1896 as a lecturer and extraordinary professor in Bonn and Königsberg. Finally, David Hilbert (1862-1943), who was born in Königsberg, was a student in Berlin and Paris, extraordinary and later ordinary professor in Königsberg until 1895. Klein comments on their interaction in [1]: "*And fortunately around 1885, for almost a decade, a trio of young researchers came together [...] in Königsberg, who put this tendency into practice in a new way [...]*" (p. 327f), where the last part refers to the combination of the mathematical research areas of invariant theory, equation theory, function theory, geometry, and number theory (compare [1], p. 327).

Having formed a friendship and established the habit of *mathematical walks* during their common period in Königsberg 1884-1892, many personal and mathematical matters arise in the correspondence between Hilbert, Hurwitz, and Minkowski. Between 1885 and 1919, 324 letters and postcards of the correspondence have been found so far. Letters to Minkowski have been lost, whereas the letters from Minkowski to David Hilbert have already been published in [2]. Most of the letters between Hilbert and Hurwitz can be found in the archive of the university of Göttingen, signatures Cod Ms D Hilbert 160 and Cod Ms Math Arch 76, a few can be found in the archiv of the ETH Zürich library under the signature HS 583:51,52. The publication of the whole correspondence in chronological form is joint work in progress with Nicola Oswald, Jörn Steuding, and Klaus Volkert.

Felix Klein is mentioned in over 90 letters/postcards in the correspondence. While Adolf Hurwitz was a student under Klein in Leipzig and Munich and spent most of his mathematical career in Zürich from 1892 until he died in 1919, David Hilbert and Hermann Minkowski were colleagues of his in Göttingen. Klein succeeded in hiring David Hilbert in 1895, marking the beginning of their common time there until Klein died in 1925. Minkowski came to Göttingen in 1902 but died early in 1909. The three had a changing and at times close relationship with Klein.

International students of Klein, i.e. mathematicians who attended Klein's seminars and are listed in the seminar protocols, appear at various points of the correspondence. Examples are Charles Jaccottet from Switzerland, who earned his doctorate under Felix Klein in 1895 and was recommended to him by Hurwitz from Zürich, Luigi Bianchi from Italy, Anne Lucy Bosworth Focke from America, Annie Louise MacKinnon Fitch from Canada, and Giuseppe Veronese from Italy. The most frequently mentioned international student is the Danish mathematician Charlotte Wedell (Baronesse Wedell-Wedellsborg), who was one out of four women to participate in the first ICM meeting in Zürich in 1897. In the Klein protocols [4], information can be found on when the students participated in the seminars and what talks they gave; the participation of the female mathematicians is documented nicely in [3], Table 1.1.

There is however much more to gain about the perception of Felix Klein's international influence from the correspondence between Hilbert, Hurwitz, and Minkowski. For example, around 1893, many letters treat Chicago's World Fair and how Klein calls the three mathematicians for articles and reports from his stay in America. His Evanston Lectures and other international and national responsibilities of Klein are mentioned as well as (joint) seminars in Göttingen. Of course, many social aspects describing the relationship between the three and Felix Klein appear throughout the entire correspondence, changing tenor in phases. In this talk, impressions of this correspondence will shed light on how Klein and his influence are perceived by David Hilbert, Adolf Hurwitz, and Hermann Minkowski.

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Greek traces regarding F. Klein's activity

Christine Phili

In Göttingen "a seat of an international Congress of Mathematicians permanently in session"as Carathéodory characterized it, some Greek mathematicians had the privilege to attend Klein's lectures.

Although Cyparissos Stephanos (1857-1917) never attended his lectures during the preparation of his Thèse d'Etat, under Hermite's supervision, started his correspondence, with Felix Klein, who for all his life had a number of students came from abroad.

The intensive mathematical activity of Stephanos, as well as the development of international contacts from 1880 to 1890 appeared in his eight letters written in French to Felix Klein, which exists in the Archives of the University of Göttingen. From his post of archivist of the French mathematical Society, he contributed to organize regularly the exchanges between French and German scientific societies.

Through Klein's disciple Walther von Dyck, Stephanos in his unpublished letter of the 29 September 1883 indicated his vivid interest to publish in Acta the French translation of Erlanger Programm.

However in spite of Poincaré's and Klein's efforts as appeared in their letters to Mittag-Leffler on the 14th of August 1883 and on the cod. F. Klein 21th June 1885, the Swedish mathematician never accepted this proposition.

In 1908 at the International Congress of mathematicians in Rome, Stephanos thanks to Klein's support was designed to be the Greek representative to the International Commission of Mathematical Instruction. From this post he tried to reform the teaching of Mathematics in Greece.

Athanassios Karagiannides (1868-?) by a fellowship from Greek government continued his post doctoral studies in Göttingen; where he attended Klein's lectures during the academic year 1890-1891. In Klein's Archives exists two letters of this Greek student. The first one, written on the 23th of December 1891 constitutes a formal letter of thanks but in his second letter one the Greek mathematician developed his ideas in order to prove that "any polynom became a polygon".

In Karagiannides's libel, *The non Euclidean geometry from the antiquity until today. A historic-critical study* (Berlin, 1893), in which refused the existence of the non-Euclidean geometry attacked Klein's contribution "So, arbitrary and façon de parler, Mr. Klein named the non Euclidean geometry with three names, hyperbolic, elliptic and parabolic geometry".

Nicolaos Hatzidakis (1872-1942) continued his post doctoral studies firstly in Paris and in 1898-1899 in Göttingen, where attended the lectures of Klein, Hilbert and Schönfliess. In 1899 Hatzidakis presented his paper on differential geometry in Klein's seminar.

Unfortunately as the Archives of Hatzidakis' family seems to be lost we could find Klein's letter of introduction in Hatzidakis' edited curriculum vitae. After Hatzidakis' election in the University of Athens in 1904, the next year introduced the institution of seminars in Greece. When in 1918 the Greek Mathematical Society was founded, Nicolaos Hatzidakis was unanimously elected his first president and one of his first target was to establish Realschule in Greece.

Constantine Carathéodory (1873-1950), who lead the most profound and permanent relationship with Klein, a successor of his chair, he maintained his esteem and respect during all his life.

In this paper, we focus on two letters, written in Smyrna, where Carathéodory after Venizelos' invitation accepted to organize a new University. In his project appeared Klein's influence mainly in the department of Ethnology.

In his first letter written on the 11th of March 1921 after expressing his thanks for the consignment of the first volume of his Gesammelte Abhandlungen, informed his mentor for his activity in Smyrna.

In his second letter, on the 2th February 1922 written from in the magnificent house of his ancestors in Istanbul, aware of the gravity of the situation and feeling that the catastrophe will be a matter of time, unveiled the truth stressing that "we are leaving the country".

Unfortunately Toynbee's statement regarding this chimeric institution came true. The Ionian University never opened its gates as the tragic events of September 1922 swept.

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Mini-Workshop: Nonlinear Approximation of High-dimensional Functions in Scientific Computing

Organized by
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15 October – 20 October 2023

Abstract. Approximation techniques for high dimensional PDEs are crucial for contemporary scientific computing tasks and gained momentum in recent years due to the renewed interest in neural networks. It seems that especially nonlinear parametrizations will play an essential role in efficient and tractable approximations of high dimensional problems. We held a mini-workshop on the relation and possible synergy of neural networks and tensor product approximation. To reliably evaluate the prospect of different numerical experiments, the traditional talks were accompanied by live coding sessions.

Mathematics Subject Classification (2020): 15A69, 68T07, 35Q93, 65F55, 41A46.

Introduction by the Organizers

The workshop *Nonlinear Approximation of High-dimensional Functions in Scientific Computing*, organised by Mathias Oster (RWTH Aachen), Janina Schütte (WIAS Berlin) and Philipp Trunschke (École Centrale de Nantes) was attended by 17 people (16 on-site and 1 online) with affiliations for example in Germany, the US, the UK, France, Italy and the Netherlands. The program consisted of 16 talks (50 minutes) and three coding sessions (90 minutes), allowing for extended discussions throughout the workshop. Conversations with all participants lead to a positive conclusion. The workshop was a success fostering new collaborations, strengthening standing connections and providing the space to learn about other attendees research in the talks, while also having time to discuss new ideas during breaks and the coding sessions.

Topic. Numerous state-of-the-art applications in engineering and physics rely on the efficient solution of high-dimensional *partial differential equations* (PDEs) with controllable precision and reliable error bounds. But classical methods like finite differences, finite elements and finite volumes are limited to low dimensions due to an exponential growth in complexity. To circumvent this curse of dimensionality, new approximation methods such as sparse approximations, tensor product approximations and neural networks have been developed.

This mini-workshop explored the benefits and limitations of contemporary methods for neural network and tensor network approximations of high-dimensional functions and used the generated insights to discuss possible new and improved tools. Here, the coding session allowed the participants to explore some new ideas on-site, as for example using a combination of (global) linear transformation and tensor trains to reduce the ranks, exploring the implicit bias observed for linear networks as well as synthesising tensor trains with neural networks by using functional tensor trains whose basis functions are parametrised by neural networks. The following topics have been discussed in the workshop.

- **Theory-to-practice gap** The theory-to-practice gap describes two orthogonal phenomena in machine learning. On the one hand, it is often observed that neural networks outperform their theoretical expressivity bounds when the required accuracy is moderate. In particular, many proofs for approximation rates of neural networks show that certain network architectures are able to model classical approximation schemes. It is thus natural to ask when the trained networks can perform better than these classical algorithms and manifest the first interpretation of the theory-to-practice gap. On the other hand, the theory-to-practice gap describes the practical difficulty of estimating neural networks from point evaluations. Theoretical constructions demonstrate that the required sample size may suffer from the curse of dimensionality and practical experiments substantiate that even the approximation of "simple" functions, like the square $x \mapsto x^2$, is difficult to high-accuracy. This obviously depends on the distribution of the data and may be alleviated by model- and problem-dependent importance sampling schemes. However, theoretical results in this direction are currently sparse and first advances for the special case of tensor networks have been discussed in the workshop. As of now, it remains unclear if the theory-to-practice gap for general neural networks can be bridged or if it is a fundamental limitation of the model class akin to the concept of the "condition number" in numerical linear algebra.
- **Neural Operators** For neural operator techniques as Deep-O-Net or Fourier-Neural-Operators it is often claimed that they can approximate mappings from one functional space to another functional space with "discretisation invariant" schemes. These invariance claims have have been discussed and some counter examples have been presented. This also leads to interesting tasks of correct sampling of functional spaces ("Besov priors").

- **Mean-Field Limit** Two mean field generalisations of deep learning, based on *neural ordinary differential equations* (neural ODEs) have been discussed. The first approach considered the learning problem in the mean-field limit of the data. In this setting, the learning problem can be interpreted as an optimal control problem in Wasserstein space, where the initial data distributions is transported by means of a neural network (the control). Another approach, presented the infinite width and depth limit of neural networks as neural ODEs with Barron functions as vector fields and formulated an corresponding abstract optimal control problem with measure-valued controls.
- **Optimisation** The abundance of local minima in learning tensor networks and neural networks leads to an influence of the chosen optimisation scheme on the resulting generalisation performance. Of particular interest in this context is the implicit regularisation in the context of overparameterisation (more parameters than training data), i.e., which networks are favoured by such algorithms. This implicit bias was discussed for neural networks with linear and non-linear activation functions. In the optimisation of tensor methods optimal sampling strategies and active learning have been of interest.
- **Synthesising Techniques** Finally, part of the workshop was concerned with combining tensor decomposition methods with more classical approaches, such as sparse approximation schemes, for solving time-space discretisations of parabolic PDEs and model order reduction techniques for optimal controls of the Navier–Stokes equation.

As expected, these complex open problems were not solved in one week. Nevertheless, discussions in all considered areas were productive and new ideas and collaborations were found.

Mini-Workshop: Nonlinear Approximation of High-dimensional Functions in Scientific Computing

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Abstracts

Semi-global Optimal Control Problems and their Applications to Machine Learning

Mathias Oster
(joint work with Angela Kunoth, Reinhold Schneider)

Learning a function $f : \mathbb{R}^d \to \mathbb{R}$ by deep neural networks with activation function σ in for example the L^2 norm can be interpreted as an abstract optimal control problems with measure-valued controls $\mu(t)$ of the form

$$\min_{\mu(\cdot)} \mathcal{J}(\mu(\cdot)), \quad \mathcal{J}(\mu(\cdot)) = \int_{\mathbb{R}^d} \|f(x) - \int a\sigma(A\ z(T,x) + b)\ d\mu(t;a,A,b)\|^2 dx$$
$$s.t. \frac{d}{dt} z(t,x) = \int a\sigma(Az(t,x) + b)\ d\mu(t;a,A,b), \quad z(0,x) = x$$

and provides an interesting mathematical framework to analyse the expressivity and optimization of deep neural networks from a continuous point of view. This control problem can be seen as an infinitely deep neural network with distinguished last layer. Here we exploit the ideas of Barron spaces as continuous interpretation of infinitely wide shallow networks and neural odes as infinitely deep residual network architectures. This continuous interpretation might allow one to deduce new adaptive algorithms for neural network that change the depth and width of the neural network during the training process.

First, we show the existence of minimizers to the optimal control problem by using Prokhorov's theorem on tight measures and some regularity assumptions on the activation function and classical compactness and continuity arguments.
Secondly, we analyse analyse the gradient flows corresponding to optimizing the map $\mu(\cdot) \to \mathcal{J}(\mu(\cdot))$ in the space of probability measures. To that end, we introduce a fibered Wasserstein metric on probability measures with bounded second moment and fixed first marginal and define the notion of absolute continuous curves. Furthermore, we define a notion of Wasserstein gradient and exemplify it on the example of a potential functional $\mathcal{E}(\mu) = \int V(u) d\mu(u)$ for some twice continuously differentiable function V. By using the equivalence of absolute continuous curves and solutions to the continuity equation we can state the gradient flow equations for the optimal control problem and we sketch the proof of existence of gradient flows based on the so-called generalized minimizing movement.

Lastly, we propose a first naïve algorithm to deal with flexible architectures and provide some very first examples.

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Approximation of high-dimensional functions with tensors and neural networks

Ivan Oseledets

Approximation of multivariate functions is a notoriously difficult task. In this talk, I discussed two different approaches: tensor decompositions and neural networks/operators.

The idea behind tensor decompositions is based on the separation of variables. Several tensor formats exist that utilize this idea: the simple canonical decomposition, which has well-known problems with stability if used as a general approximation tool, and SVD-based tensor formats such as tensor train and Hierarchical Tucker (H-Tucker). Using those formats, one can often approximate functions with high precision. Moreover, for a special class of functions written in the so-called inverse Polish notation, we can constructively represent tensors with optimal ranks. Some applications include computation of the matrix permanent and cooperative games, for more details see [3]. The idea of quantized tensor train (QTT) uses the procedure of tensorization. For example, given a function $f(x) = \sin(x)$ we can create a vector $v = 2^d$ of length d of values of this function on a uniform grid and reshape it into a $2 \times 2 \times \ldots \times 2$ d-dimensional tensor. For this example, the QTT-ranks will be equal to 2, giving logarithmic complexity. Moreover, one can show that for a certain class of functions QTT-representation gives the approximation of a function with complexity $\mathcal{O}(\log^{\alpha} \varepsilon)$, where ε is the approximation accuracy [4, 5].

However, it is also clear that there are important cases when tensor approximation fails, for example, for function with diagonal singularities like

$$f = e^{-x^2/2} e^{-y^2/2} e^{-|x-y|}.$$

A big alternative are neural networks, which are universal function approximators. However, the converger of the error with respect to the number of parameters is not well understood. A promising class of functions seems to be Deep-ReLU networks, especially due to the results of Yarotsky [7]. It can be shown, for example, that a function $f(x) = x^2$ can be well-approximated using DeepReLU network and the error decays exponentially with the depth. Based on this result, one can show that polynomials can be well-approximated and large classes of functions. In [6] we showed that even for the simplest one-dimensional example it is not possible to recover such a good Deep ReLU representation: instead of 10^{-6} we get $10^{-2} - 10^{-3}$ error of approximation at its best. The reason for that is the loss function is very "narrow" in this particular point. The current understanding of the situation is that deep feedforward networks can be very unstable in training, and we need to look for alternatives.

A promising direction is the approximation not of the solutions, but of the mappings using so-called *neural operators*. Neural operator is a parametrized mapping from a function (element of a Banach space) to another function (Banach space), and they are quickly gaining popularity. Popular approaches include DeepONet [8] and Fourier Neural Operator (FNO). All of them still can not be considered as real operators, and they do not improve with better discretizations, as standard methods. However, in many cases they provide an extremely fast surrogate model.

Among open problem for training neural operators, I want to highlight the following one. A standard approach is to construct a dataset of input-output pairs. The input pair (for example, coefficient in the diffusion equation) is sampled from a certain probability distribution over functions. But this distribution is taken empirically, like random mixture of Gaussians or random trigonometric polynomials. However, it is not clear why these functions are used for training, and what is the motivation for using such kind of functions. The research question, that needs an answer is what the optimal (or quasioptimal) way of sampling input data for different kinds of problems, where neural operators are used? Understanding and the solution of the problem may be the key for the generalization of such neural operators and their wider usage.

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Optimal Sampling for Approximate Gradient Descent

Philipp Trunschke
(joint work with Robert Gruhlke, Charles Miranda, Anthony Nouy)

We consider the problem of minimising a *loss functional*

$$\text{minimise}_{v\in\mathcal{M}}\,\mathcal{L}(v), \qquad \mathcal{L}(v) := \int \ell(v;x)\,\mathrm{d}\rho(x)$$

over a possibly nonlinear *model class* $\mathcal{M} \subseteq \mathcal{H}$ in a Hilbert space $\mathcal{H}$. When computing the integral is infeasible, a common approach is to replace the exact loss $\mathcal{L}$ with a Monte Carlo estimate before employing a standard gradient descent scheme. This results in the well-known *stochastic gradient descent* (SGD) method. However, using an estimated loss instead of the true loss can result in a "generalisation error". Rigorous bounds for this error usually require compactness of $\mathcal{M}$ and Lipschitz continuity of $\mathcal{L}$ while providing a very slow decay with increasing sample size. This slow decay is unfavourable in settings where high accuracy is required or sample creation is costly.

To address this issue, we propose a new approach that performs successive corrections on local linearisations of $\mathcal{M}$. To be specific, we suppose that in every step $t \in \mathbb{N}$ there exists a linear space $\mathcal{T}_t$ that approximates $\mathcal{M}$ locally around the current iterate u_t. Given the gradient $g_t := \nabla\mathcal{L}(u_t)$ and an estimator P_t^n of the $\mathcal{H}$-orthogonal projector P_t onto $\mathcal{T}_t$, we then perform a linear update $\bar{u}_{t+1} := u_t - s_t P_t^n g_t$ in direction of the (empirically) projected negative gradient $-P_t^n g_t$. This yields the intermediate iterate $\bar{u}_{t+1}$. Since the $\bar{u}_{t+1}$ is not guaranteed to lie in the original model class $\mathcal{M}$, we perform a recompression step $u_{t+1} := R_t(\bar{u}_{t+1})$, where $R_t : \mathcal{H} \to \mathcal{M}$ takes the linear update $\bar{u}_{t+1}$ back to the model class $\mathcal{M}$ with a controllable error in the loss $\mathcal{L}$. The proposed algorithm can thus be presented in the two equations

$$\begin{aligned}\bar{u}_{t+1} &:= u_t - s_t P_t^n g_t, \qquad g_t := \nabla\mathcal{L}(u_t),\\ u_{t+1} &:= R_t(\bar{u}_{t+1}).\end{aligned}$$

We show that under certain assumptions on the loss $\mathcal{L}$ and the sequences of projectors P_t^n, step sizes s_t and recompressions R_t, the resulting optimisation scheme converges almost surely to a stationary point of the true loss. The corresponding rates of convergence are displayed in Table 1. The proposed algorithm exhibits the same convergence rates as classical *gradient descent* (GD) in the best case but can never perform worse than SGD. We pay particular attention to the estimation of the projectors P_t^n, which must be carried out using optimally weighted samples in order to achieve the presented rates.

	GD	Best-case	Worst-case	SGD
L-smoothness	$\mathcal{O}(t^{-1})$	$\mathcal{O}(t^{-1+\varepsilon})$	$\mathcal{O}(t^{-1/2+\varepsilon})$	$\mathcal{O}(t^{-1/2+\varepsilon})$
strong convexity	$\mathcal{O}(a^t)$	$\mathcal{O}(a^t)$	$\mathcal{O}(t^{1-2\varepsilon})$	$\mathcal{O}(t^{1-2\varepsilon})$

Table 1. Almost sure convergence rates for different algorithms with $\varepsilon \in (0, \frac{1}{2})$ and $a \in (0, 1)$ depending on the chosen step size.

Tensor train approximation of deep transport maps

Sergey Dolgov
(joint work with Tiangang Cui, Robert Scheichl, Olivier Zahm and workshop participants)

A challenging example of high-dimensional functions is joint probability density (or distribution) functions of multiple random variables. Sampling and computation of expectations of high-dimensional random variables is one of the fundamental challenges in stochastic computation. We develop a deep transport map that is suitable for sampling concentrated distributions defined by an unnormalised density function [1]. We approximate the target distribution as the pushforward of an easy reference distribution under a composition of inverse Rosenblatt transformations of coordinates. Each transformation is formed by a tensor-train (TT) decomposition of a bridging density, which is a simplified version of the target density. This composition of maps moving along a sequence of bridging densities alleviates the difficulty of approximating the concentrated target density directly. In contrast to neural network layers, each Rosenblatt map is fully defined by its bridging density, and can be computed independently of next layers by fast TT cross algorithms. We propose two bridging strategies suitable for wide use: tempering of the target density with a sequence of increasing powers [1], and smoothing of an indicator function with a sequence of sigmoids of increasing scales [3]. The latter strategy opens the door to efficient computation of rare event probabilities in Bayesian inference problems. Numerical experiments on problems constrained by differential equations show little to no increase in the computational complexity with the event probability going to zero, and allow to compute hitherto unattainable estimates of rare event probabilities for complex, high-dimensional posterior densities.

One drawback of the TT decomposition though is its sensitivity to the order of variables. Probability density functions with locally correlated variables exhibit typically low TT ranks [4], whereas if the same variables are permuted in such a way that strongly correlated variables are far apart in the random vector, the TT ranks may increase up to an exponential factor. Permutation (or even better, rotation) of variables may significantly expand the applicability of TT-driven approximation methods to higher dimensions and more complicated functions. In principle, this is the problem that is tackled by the Rosenblatt map. However, if the initial dimension is very high, it may still be daunting to compute a TT approximation, even for simple bridging densities. In this case it may be useful to identify unimportant variables (e.g. those in which the function is almost constant), and truncate them altogether. If the function to be approximated is a posterior density function of exponential family, the eigenvalue decomposition of the information matrix computed from the gradient of the log-likelihood can be used to inform the permutation or rotation of variables [2]. This allowed us to solve a Bayesian inverse problem constrained by an elasticity PDE with a thousand of random variables.

Both techniques outlined above require a function to be of a probability density form to compute the Rosenblatt map or the information matrix. Efficient tensor methods for very high dimensional functions which are neither positive nor easily differentiable are still lacking. During the workshop, we have come up with an idea of learning a matrix of linear change of variables simultaneously with a low-rank TT decomposition from data such as random samples of the function. Preliminary experiments with simple functions demonstrated that a nearly optimal rotation of variables is achievable using a moderate amount of function evaluations. However, further research is needed to make this technique useful for higher dimensions and concentrated functions, sampling of which is difficult.

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Curse-of-dimensionality-free deep-learning approaches to deterministic control problems

Lars Grüne

(joint work with Dante Kalise, Luca Saluzzi, and Mario Sperl)

It is known that deep neural networks have the ability to represent certain classes of high-dimensional functions without being affected by the curse of dimensionality. One of these classes are the so-called Barron functions. However, the usual way to check that a function falls into this class is by checking suitable smoothness properties, which cannot be expected to hold for the functions to be approximated in typical deterministic control problems.

Another prominent function class for which the curse of dimensionality can be avoided, the so-called compositional functions, have recently been shown to be a promising system class for problems involving deterministic dynamical systems

[2, 4]. In this talk, we have explained the ability of the simplest functions in this class, the so-called separable functions, to approximate control Lyapunov functions and optimal value functions.

For control Lyapunov functions, the requirement of separability is closely linked to the kind of Lyapunov functions that can be obtained from nonlinear small-gain theory, which is used for this purpose in a control context e.g. in [1]. While this approach is in principle constructive, it suffers from the fact that the construction of the resulting control Lyapunov functions is quite complicated. Here neural networks can provide a remedy, because the theory is only used for designing the architecture of the network, while the actual separable structure is learned in the training process of the network [3]. More precisely, small-gain theory ensures the *existence* of a control Lyapunov function V of the separable form

$$V(x) = \sum_{i=1}^{s} V_i(z_i), \quad \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix} = Tx,$$

where the low-dimensional subvectors z_i are obtained from the original high-dimensional state vector x by some coordinate transformation T, but the *computation* of T and of the individual V_i is left to the training process of the neural network.

For optimal value functions, separability is in general a too demanding property, as exploiting the interaction between different subsystems is usually a prerequisite for achieving optimality. However, when the subsystems are connected via a graph, it seems reasonable to expect that subsystems that are far away (in terms of the graph distance) only interact with each other very weakly. This heuristic expectation can be made rigorous in the framework of decaying sensitivity [5] and exploited for a curse-of-dimensionality-free approximation of optimal value functions V via *overlapping separable* functions

$$V(x) = \sum_{i=1}^{s} W_i(z_i), \quad z_i = \begin{pmatrix} x_{j_1} \\ \vdots \\ x_{j_k} \end{pmatrix},$$

where each component x_j of the state vector may occur in several of the subvectors z_i but the number k of components appearing in each z_i is bounded independent of the overall dimension. Under an exponential sensitivity assumption, first rigorous error estimates for such an overlapping separable approximation were obtained in [6].

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A statistical Tensor Train - POD approach for feedback boundary optimal control in fluid dynamics

Luca Saluzzi

(joint work with Sergey Dolgov and Dante Kalise)

Consider the optimal control problem

$$\begin{cases} \inf\limits_{u\in\mathcal{U}} J(u(\cdot,x)) := \int_0^{+\infty} y(s)^\top Q y(s) + u^\top(s) R u(s)\, ds\,, \\ \text{subject to } \dot{y}(s) = f(y(s)) + B(y(s))u(s), \quad s \in (0,+\infty), \end{cases} \tag{1}$$

where $y(0) = x$ and $\mathcal{U} = L^\infty([0,+\infty);U)$ is the set of admissible controls. For a given initial condition $x \in \mathbb{R}^d$, we define the value function associate to the optimal control problem (1) as

$$V(x) = \inf_{u\in\mathcal{U}} J(u(\cdot,x))$$

which, by standard dynamic programming arguments, satisfies the following Hamilton-Jacobi-Bellman PDE

$$\min_{u\in U} \left\{ (f(x) + B(x)u)^\top \nabla V(x) + x^\top Q x + u^\top R u \right\} = 0, \quad x \in \mathbb{R}^d. \tag{2}$$

The HJB PDE (2) is a challenging first-order fully nonlinear PDE cast over $\mathbb{R}^d$, where d can be arbitrarily large, and thus intractable through conventional grid-based methods. However, in the unconstrained case, *i.e.* $U = \mathbb{R}^m$, the minimizer of the l.h.s. of eq. (2) can be computed explicitly as

$$u^*(x) = -\frac{1}{2} R^{-1} B(x)^\top \nabla V(x). \tag{3}$$

In this context we propose to approximate the value function together with its gradient in a *data-driven* approach, learning a surrogate model for the value function via adaptive sampling of the solution of the HJB (2). The synthetic data are generated via the so-called State-Dependent Riccati Equation (SDRE), an extension of the Riccati solution to nonlinear dynamics. By writing the dynamics in semilinear form

$$\dot{y} = A(y(t))y(t) + B(y(t))u(t), \tag{4}$$

equation (2) can be approximated as

$$A^\top(x)\Pi(x) + \Pi(x)A(x) - \Pi(x)B(x)R^{-1}B(x)^\top\Pi(x) + Q = 0\,, \tag{5}$$

which is obtained by applying the ansatz $V(x) = x^\top \Pi(x) x$ with a gradient approximation $\nabla V(x) \approx 2\Pi(x)x$. At this point, similarly to [1], the value function is represented in Functional Tensor Train (FTT) format

(6)

$$V(x) \approx \tilde{V}(x) := \sum_{\alpha_0=1}^{r_0} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_d=1}^{r_d} G^{(1)}_{(\alpha_0,\alpha_1)}(x_1) \cdots G^{(k)}_{(\alpha_{k-1},\alpha_k)}(x_k) \cdots G^{(d)}_{(\alpha_{d-1},\alpha_d)}(x_d),$$

with

$$G^{(k)}_{(\alpha_{k-1},\alpha_k)}(x_k) = \sum_{i=1}^{n_k} \Phi_k^{(i)}(x_k) H^{(k)}_{(\alpha_{k-1},i,\alpha_k)},$$

where $\{\Phi_k^{(i)}(x_k)\}_{i=1}^{n_k}$ are prescribed basis functions and $\{r_k\}_{k=1}^{d}$ are called TT ranks.

Given certain sample points $\{x_i\}_{i=1}^N$ and the dataset $\{ V(x_i), \nabla V(x_i)\}_{i=1}^N$ computed by SDRE, we are interested in determining the coefficient tensors

$\{H^{(1)}, \ldots, H^{(d)}\}$ which characterize the FTT representation $\tilde{V}(x)$ introduced in (6), solving the regression problem

$$\min_{H^{(1)},\ldots,H^{(d)}} \sum_{i=1}^{N} |\tilde{V}(x_i) - V(x_i)|^2 + \lambda \|\nabla \tilde{V}(x_i) - \nabla V(x_i)\|^2,$$

which is approximated by an alternating direction strategy and a TT *cross interpolation* technique [2, 5]. The TT Cross enables to adapt the sampling sets to minimize the conditioning of the interpolation problem, avoiding the evaluation of the function on the whole tensorial grid. The methodology has been successfully applied to the optimal control of a multi-agent system, where the TT ranks of the approximation of the value function presented a constant behaviour varying the dimension of the system, yielding an effective mitigation of the curse of dimensionality. However, the dimension of the value function is still that of the state space, leading to a very large number of unknowns in the approximation ansatz and training data. A possible way to tackle this problem is given by the application of Model Order Reduction (MOR) techniques. One of the most famous MOR method is the Proper Orthogonal Decomposition (POD), which synthesizes a set of snapshots capturing the behaviour of the system and looks for basis functions that capture the major variations in the data. In contrast to existing techniques, we propose a Statistical Proper Orthogonal Decomposition (SPOD) which takes into account controlled trajectories treating boundary conditions and initial condition as random variables. The corresponding reduced basis is chosen to minimize the empirical risk for the controlled solution, avoiding any linearisation of the dynamical system. Once computed the basis and projected the system, the reduced dynamics can be employed for either a fast online computation of the optimal control or an efficient synthesis of a dataset for the construction of a TT surrogate model. The methodology has been tested on the vorticity stabilization of the 2D Navier-Stokes equations, whose discretization employs several thousands of degrees of freedom.

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A Mean-Field Optimal Control Approach to the Training of NeurODEs & AutoencODEs

Cristina Cipriani

(joint work with Benoît Bonnet, Massimo Fornasier, Hui Huang, Alessandro Scagliotti and Tobias Wöhrer)

In recent years, neural networks have emerged as a significant tool in artificial intelligence. However, there exists a pressing need for a robust mathematical framework to systematically analyze their intricate characteristics. A key theoretical advancement involves interpreting deep neural networks with residual connections (or shortcut connections) as dynamical systems, as outlined in the works [1] and [2]. The information flow from input to output in a network with an infinite number of layers can be expressed in the continuum limit as:

$$\dot{X}(t) = \mathcal{F}(X(t), \theta(t)),$$

This leads to nonlinear *neural ODEs* (NeurODEs), where time takes the role of the continuous-depth variable. This perspective allows the interpretation of neural network learning problems as continuous-time control problems, which provides access to the extensive literature of mathematical control theory, potentially enhancing the overall explainability of learning algorithms. Relevant works in this direction include [3] and [4].

Our work in [5] focuses on the mean-field formulation of the control problem, specifically addressing the scenario of an infinitely large dataset. We examine the evolution of the distribution μ_0 of initial data through the network as a partial differential equation, subsequently considering the corresponding mean-field optimal control problem. In [5], we establish first-order optimality conditions through a mean-field Pontryagin Maximum Principle, derived as a consequence of an abstract Lagrange multiplier rule in the Banach space of Radon measures.

However, it is crucial to note that NeurODEs encounter limitations when modeling neural networks with discrepancies in dimensionality between consecutive layers. Skip connections with identity mappings necessitate a "rectangular" network shape, where the width of layers is uniform. To address this limitation

and enhance the network's capacity, we introduce a novel design of the vector field driving the dynamics in [6]. This continuous-time model accommodates various width-varying neural networks and builds upon insights from our previous work [5]. Furthermore, in [6] we extend our framework to encompass the low-Tikhonov regularization regime. For the continuous-time version of Autoencoders (AutoencODEs), we propose a novel discrete architecture and an alternative training method based on the Pontryagin Maximum Principle. To demonstrate the effectiveness of our approach, we present informative numerical examples offering valuable insights into the resulting algorithm.
Finally, we leverage the well-established theory of optimal control to address the lack of robustness in neural networks against data manipulation, commonly known as adversarial attacks. These attacks involve small changes of the inputs, which lead to significant modifications in the model outputs. In [7], we interpret the adversarially robust learning problem arising in machine learning as a minimax control problem

$$\min_u \mathbb{E}_{(x^0,y)\sim\mu}\left[\max_{\|\alpha\|\leq\epsilon} \mathrm{Loss}(\theta, x^0+\alpha, y)\right],$$

where the initial data and labels (x^0, y) are drawn from an underlying data distribution μ, and $\mathrm{Loss}(u, x^0, y)$ quantifies the prediction accuracy. We derive the Pontryagin Maximum Principle for this problem using separation of Boltyanski approximating cones, as presented in [8], and develop a numerical method to address the robust learning problem, which is used for low-dimensional examples.

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Spectral approximation of Lyapunov operator equations with applications in non-linear feedback control

Bernhard Höveler
(joint work with Tobias Breiten)

Let a (non-linear) dynamical system be given as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= f(x(t)), \qquad \text{for } t \in (0,\infty) \\ x(0) &= z \end{cases}$$

for some $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and let us define the Lyapunov function v to a given cost $g\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}_+$ as follows

$$v(z) := \int_0^\infty g(\Phi^t(z))\,\mathrm{d}t \qquad \text{for } z \in \Omega$$

where the flow $\Phi^t(z)$ is defined as the mapping from the initial value z to the state $x(t)$ with $x(0) = z$ at time t, i.e $\Phi^t(z) := x(t)$. Computing such a function is a challenging task both from the numerical as from the analytical side. One of the main numerical challenges arises, when n is large and therefore the system is high dimensional.
One of the main results of this talk is that we can define a weak-* continuous semigroup

$$\begin{aligned} S^*(t)\colon \quad X^* &\to X^* \\ \phi &\mapsto \phi \circ \Phi^t \end{aligned}$$

and that there exists a preadjoint $S(t)$. Here X and X^* are some specially weighted $\mathrm{L}^p(\Omega)$ spaces. The weighting assures the exponential decay under some assumptions. It is shown that – if the cost function g admits the decomposition $g(x) = \sum_{i=1}^\infty c_i(x)^2$ – the Lyapunov function v can be written as

$$v(x) = \sum_{i=1}^\infty p_i(x)^2$$

where p_i are the eigenfunctions of the symmetric bilinear form

$$\langle \phi, \psi \rangle_P = \int_0^\infty \langle C\phi, C\psi \rangle_{\ell_2} \ \mathrm{d}t \quad \text{with} \quad C\phi := \Big(\langle \phi, c_i \rangle_{X,X^*} \Big)_{i \in \mathbb{N}}.$$

Furthermore, it can be shown that the error to a finite rank approximation decays with a rate that is depending on the regularity of the c_i and f. Lastly, the generator A of the semigroup S can be used to show that P is the solution to an operator Lyapunov equation of the form

$$\langle A\phi, \psi \rangle_P + \langle \phi, A\psi \rangle_P + \langle C\phi, C\psi \rangle_{\ell_2} = 0 \qquad \text{for all } \phi, \psi \in \mathcal{D}(A) \subseteq X$$

which can be exploited for a numerical method. The proposed scheme relies on a low rank approximation and a splitting integrator to solve a corresponding time

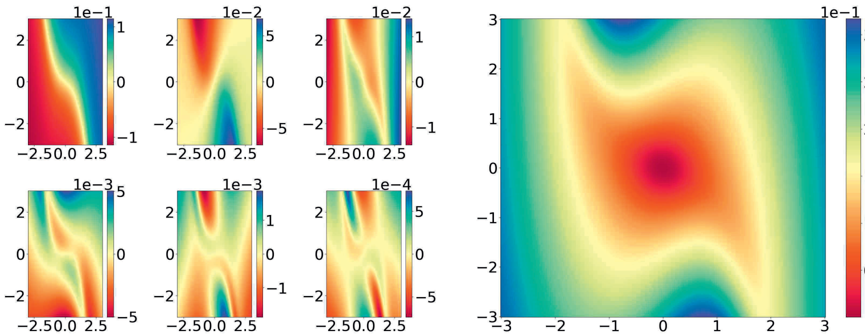

FIGURE 1. First six eigenfunctions (left) and the Lyapunov function (right) of a modified van der pool oscillator.

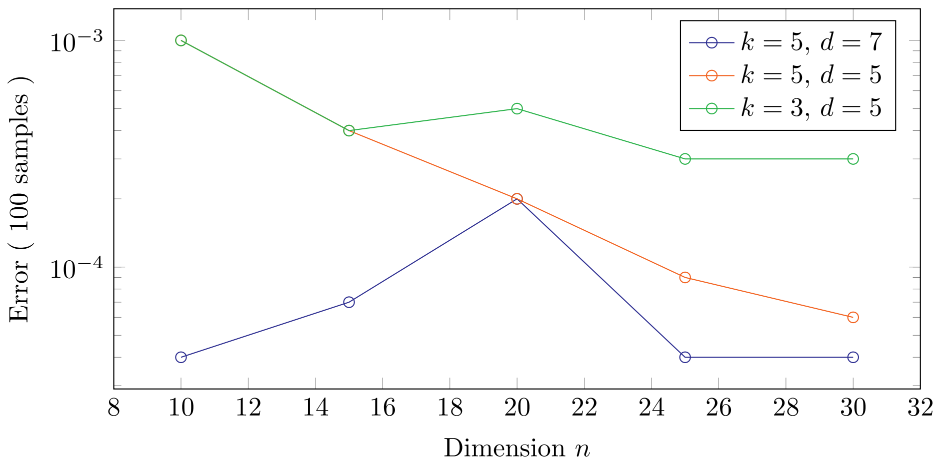

FIGURE 2. Maximum error of the proposed scheme applied to the discretized Allen Cahn model.

dependent problem. To overcome the curse of dimensionality tensor trains (TT) are used. This leads to an approximation of the Lyapunov function of the form

$$v_h(x_1, \dots, x_n) := \operatorname{Re} \sum_{j,j'}^{k} \prod_{i=1}^{n} G_i^{(j)}(x_i) \tilde{M}_{j,j'} \prod_{i=1}^{n} G_i^{(j')}(x_i)$$

where $G_i^{(j)} : [-1, 1] \to \mathbb{R}^{r_i^{(j)} \times r_{i+1}^{(j)}}$ are matrix valued functions for $j = 1, \dots, k$ and $i = 1, \dots, n$ while $\tilde{M} \in \mathbb{C}^{k \times k}$.

However, in contrast to a neural network the TT-approximation depends on the chosen basis and from an analytical standpoint it is not immediately clear what a good choice of basis might be. A possible mitigation might be to optimize over the choice of basis as well, which leads to an optimization over the Stiefel manifold.

Another area of interest is the inclusion of control. Ongoing research suggests that a non-linear operator equation similar to the Riccati equation is suitable.

$$\langle A\phi, \psi\rangle_P + \langle \phi, A\psi\rangle_P - \frac{1}{2}\sum_{k=1}^{\infty}\left(\langle M_k\phi, B_k\psi\rangle_P + \langle B_k\phi, M_k\psi\rangle_P\right) + \langle C\phi, C\psi\rangle_{\ell_2} = 0$$

Where:

$$B_k^*\phi := p_k b^\top \nabla\phi \quad \text{and} \quad M_k\phi := b^\top \nabla p_k \phi$$

However, the non-linear nature makes the analysis of this equation much more difficult.

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Functional SDE approximation inspired by a deep operator network architecture

Martin Eigel

(joint work with Charles Miranda)

We are concerned with the efficient generation of solution trajectories of SDEs by training a specific neural network (NN) architecture called SDEONet. This architecture is inspired by recent development in the area of operator learning, where operators in infinite dimensional spaces are represented with NNs. In particular, we refer to the analysis on deep operator networks (DeepONets) in [1]. These are composed of two NNs, a branch and a trunk network, representing learned basis coefficients (branch) of a linear combination of a learned reduced basis (trunk), respectively. To transfer this functional framework to the task of solving SDEs, we make use of the representability of any process $X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ in terms of a Wiener chaos expansion

$$X_t = \sum_{k\geq 0}\sum_{|\alpha|=k} x_\alpha(t) \underbrace{\prod_{i=1}^{\infty} H_{\alpha_i}\left(\int_0^T e_i(s)\,\mathrm{d}W_s\right)}_{\Psi_\alpha}, \tag{1}$$

with univariate Hermite polynomials H_n of degree n and a basis $(e_i)_{i\geq 1}$ of $L^2([0,T])$, which we choose to be the Haar basis. The coefficients x can be obtained by projection onto the Wiener chaos but also follow the dynamics of an

ODE [2]

$$\text{(2)} \qquad \frac{\mathrm{d}x_\alpha}{\mathrm{d}t}(t) = \mu(t, X_t)_\alpha + \sum_{j=1}^{\infty} \sqrt{\alpha_j} e_j(t) \sigma(t, X_t)_{\alpha^-(j)},$$

$$\text{(3)} \qquad x_\alpha(0) = 1_{\alpha=0} x_0.$$

Our SDEONet architecture is a mapping from Brownian increments to the realization of the respective SDE trajectory as depicted in Figure 1 with input G consisting of integrals of e_i as in (1). It can hence be seen as an alternative approach to the standard Euler-Maruyama simulation scheme.

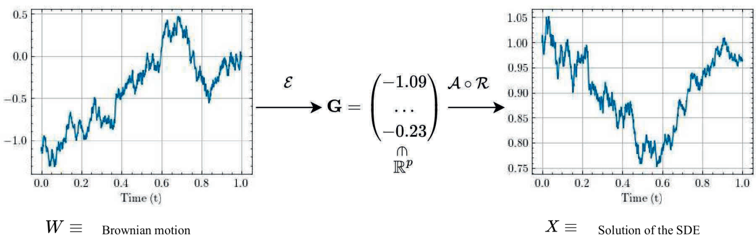

Figure 1. Sketch of SDE trajectory generation by the SDEONet architecture.

We consider the continuous stochastic process $(X_t)_{t\in[0,T]}$ that satisfies the stochastic differential equation (SDE)

$$\text{(4)} \qquad \mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \text{ with } X_0 = x_0,$$

and $(W_t)_{t\in[0,T]}$ a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.

Figure 2 illustrates the representation of the functional mapping (of the stochastic process operator $\mathcal{G}$) by the SDEONet architecture. First, the encoder maps the Brownian increments W to $(G_i)_{i=0}^{m-1}$ with $G_i := \int_0^T e_i(t)\,\mathrm{d}W_t$. Second, the approximator maps $(G_i)_{i=0}^{m-1}$ to approximate polynomial chaos $\Psi_{k_j^*}$. These two operations constitute the branch net. The trunk net approximates the coefficient functions $x_{k_j^*}(t)$. The reconstructor uses branch and trunk to approximate the trajectory $(X_t^{m,p^*})_{t\in[0,T]}$.

For the convergence and complexity analysis, we consider a decomposition of the error E into (Wiener chaos) truncation [2], NN (Hermite) polynomial approximation [4] and reconstruction (with approximate ODE coefficients) [3],

$$E := \left(\int_0^T \mathbb{E}[|X_t - \widetilde{X}_t^{m,p}|^2]\,\mathrm{d}t \right)^{1/2} \leq E_{\text{Trunc}} + E_{\text{Approx}} + E_{\text{Recon}}.$$

For all three terms, convergence rates and NN complexity bounds can be derived.

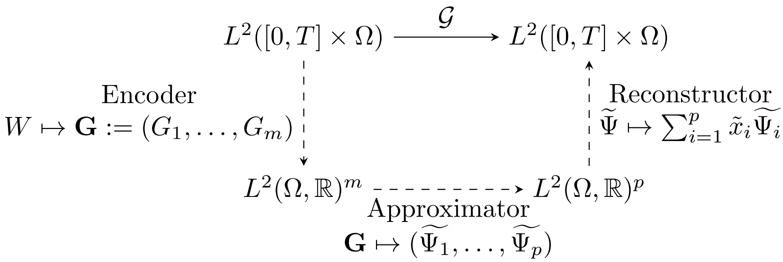

FIGURE 2. Diagram of the SDEONet operator mapping.

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Approximating Langevin Monte Carlo with ResNet-like Neural Network architectures

CHARLES MIRANDA

(joint work with Martin Eigel, Janina Schütte, David Sommer)

Deep Neural Networks (DNNs) have demonstrated their success in solving complex numerical problems, such as image classification, regression, kernel learning and solving partial differential equations (PDEs). Therefore, significant attention is given to establishing theoretical guarantees on the expressive abilities of DNNs. Deep neural networks (DNNs) have overcome the curse of dimensionality, especially when it comes to approximating Kolmogorov partial differential equations (PDEs) [1]. The latter demands a polynomial growth of parameters with the increase in dimension and expected precision, yet DNNs offer a potent workaround, thus presenting a remarkable achievement. Our study aims to sample from smooth log-concave probability distributions $\mathrm{d}\mu_\infty(x) \propto \exp(-V(x))\mathrm{d}x$. The primary objective is to create a deep neural network (DNN) with the ability to generate samples from the target distribution. The DNN's performance is evaluated based on the 2-Wasserstein distance, using input samples from a simple reference distribution such as the standard normal distribution. Our investigation is focused on the approximation of the Langevin Monte Carlo (LMC) algorithm, which is the

Euler-Maruyama discretisation of the stochastic differential equation

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t$$

through ResNet-like neural network structures

$$\begin{aligned} x_0 &:= X_0 \\ x_k &:= x_{k-1} + \phi_k(x_{k-1}) + \xi_k,\ x \in 1, \dots, K \end{aligned}$$

where ϕ_k are fully connected neural networks and ξ_k are i.i.d. standard normal random variable. Notably, we pay special attention to error analysis in the context of the 2-Wasserstein distance. The suggested approach emulates LMC by connecting feed-forward neural networks as above. The approximation of the drift term with epsilon accuracy occurs in an appropriate L^2 space established by the current law of the process. Namely, our analysis is done on the quantity

$$\| - \nabla V - \phi_{k+1}\|_{L^2_{\nu_k}(\mathbb{R}^d;\mathbb{R}^d)}$$

where ν_k is the law of x_k.

We demonstrate that if the above quantity is smaller than ε

$$\mathcal{W}_2(\mu_\infty, \nu_K) \leq (1 - mh)^K \mathcal{W}_2(\mu_\infty, \mu_0) + \frac{7\sqrt{2}}{6}\frac{M}{m}\sqrt{hd} + \frac{1-(1-mh)^K}{m}\varepsilon$$

where m is the strong-convexity parameter of V, M the Lipschitz constant of ∇V and $h \leq 2/(m+M)$ the step size in the LMC algorithm. By exploiting the properties of the initial distribution, which is the standard Normal distribution, and of the ResNet-like architecture, we are able to show that the measures ν_k are sub-Gaussian. We show that the ε-accuracy can be achieved for all ϕ_k such that the number of parameters for the ResNet-like architecture is bounded by $K(N(d, r, \varepsilon/\sqrt{d}, m, M) + 2d^2 + 2)$, where $N(d, r, \varepsilon/\sqrt{d}, m, M)$ is the number of parameters for a single fully connected neural network ϕ to satisfy

$$\| - \nabla V - \phi\|_{L^\infty(B_r(0);\mathbb{R}^d)} \leq \frac{\varepsilon}{\sqrt{2d}}$$

where

$$r \in \mathcal{O}\left(d^{7/4}\varepsilon^{-1}(d^{9/4}\varepsilon^{-1})^{3(1.5^K-1)}\right)$$

Unfortunately, the aforementioned result indicates that the radius of the ball must increase exponentially in the number of steps. We conjecture that due to the strong convexity of V and the Lipschitz continuity of ∇V, there exists a neural network capable of approximating $-\nabla V$ with a linear error growth. The experiments also indicate that the proposed architecture can sample from μ_∞ with the same convergence rate even if the potential V is no longer strongly convex, such as in a Gaussian mixture.

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The implicit bias phenomenon in deep learning

Holger Rauhut

It is common in deep learning to use many more parameters than training examples. Despite traditional statistical wisdom, which would predict overfitting, the learned neural networks usually generalize well to new unseen data [17]. In this overparameterized setting many networks exist that interpolate the data exactly. They all lead to global minimizers of the empirical loss function, which sums up the losses of a neural network over the training data. In this situation, the employed optimization algorithm (usually variants of gradient descent or stochastic gradient descent), including hyperparameters such as initialization, step sizes etc., significantly influences the computed minimizer. This phenomenon is called **implicit bias** of the learning algorithm. It is puzzling that the implicit bias of (stochastic) gradient descent and variations is often towards solutions that generalize well. Although there is a growing research literature available, see e.g. [1, 2, 4, 3, 5, 9, 10, 11, 13, 15, 16, 17], many aspects of this phenomenon are not well understood yet.

One working hypothesis is that (stochastic) gradient descent with suitable initialization favors networks of low complexity, i.e., networks that can be represented with much fewer parameters than the number of trainable network weights. Low complexity may be understood in a broad sense here and it may be a challenge to determine suitable low complexity models for concrete types of data and network models. Examples may be sparse representations [3, 5, 7] as well as low rank matrix [1, 4] and tensor representations [14].

In order to gain theoretical understanding of the implicit bias phenomenon, it is useful to study simpler optimization problems that share two characteristics with the overparameterized deep learning scenario:

- many (infinite number of) global minimizers;
- a factorization/compositional structure.

In [3, 7] the problem of minimizing the function

$$L(x) = \frac{1}{2}\|Ax - y\|_2^2$$

is considered where $A \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^m$ with $m < n$. In this case, L has infinitely many global minimizer. In fact, if A has full rank, they form the affine subspace of solutions x to $Ax = y$. In order to induce a factorization structure we set

$$x = w^{(1)} \odot w^{(2)} \odot \cdots \odot w^{(N)},$$

where $(w^{(1)} \odot w^{(2)})_j = w_j^{(1)} w_j^{(2)}$ is the Hadamard product. This structure can be interpreted as a linear diagonal neural network. Plugging into the function L, we define

$$L^N(w^{(1)}, \ldots, w^{(N)}) = L(w^{(1)} \odot w^{(2)} \odot \cdots \odot w^{(N)}) \tag{1}$$

$$= \frac{1}{2}\|A(w^{(1)} \odot w^{(2)} \odot \cdots \odot w^{(N)}) - y\|_2^2. \tag{2}$$

Initializing identically with $w^{(\ell)}(0) = \alpha \mathbf{1}$, $\ell = 1, \ldots, N$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$, we consider the gradient flow

$$\frac{d}{dt} w^{(\ell)}(t) = -\nabla_{w^{(\ell)}} L^N(w^{(1)}(t), \ldots, w^{(N)}(t)), \quad \ell = 1, \ldots, L.$$

We are interested in the convergence behavior and implicit bias of the product flow

$$v(t) = \prod_{\ell=1}^{N} w^{(\ell)}(t).$$

For identical initialization (as assumed) the vectors $w^{(\ell)}(t)$ remain identical, $w^{(1)}(t) = \cdots = w^{(N)}(t) = w(t)$, so that $v(t) = w^{\odot N}(t)$, where $w(t)$ is the gradient flow for

$$\widetilde{L}^N(w) = \frac{1}{2} \|A w^{\odot N} - y\|_2^2.$$

Theorem. Assume that $S_+ = \{z \geq 0 : Az = y\}$ is nonempty, and let $N \geq 3$. Then $v_\infty = \lim_{t \to \infty} v(t) = w^{\odot N}(t)$ exists and $v_\infty \in S$. Let $Q = \min_{z \in S_+} \|z\|_1$ and $\beta = \|v(0)\|_1 = \alpha \sqrt{n}$. If $\beta < Q$ then

$$\|v_\infty\|_1 - Q \leq N \left(\frac{\beta}{Q} \right)^{1 - \frac{2}{N}} Q.$$

Since ℓ_1-minimization promotes sparse solutions, see e.g. [8], this result basically states that the implicit bias of gradient flow is towards sparse solutions if the initialization scale α is small enough compared to the ℓ_1-norm of the ℓ_1-minimizer.

This result can be extended to the recovery of vectors with not necessarily non-negative coefficients by using a difference of two factorizations, i.e., $v = w_1^{\odot N} - w_2^{\odot N}$, see [3] for details. Furthermore, by splitting $w = ru$, where r is a scalar und u is a vector on the unit sphere, and considering the gradient flow for both r and w with different learning rates – also referred to as weight normalization – gives similar results [5] as stated in the theorem above, however, allowing for larger initialization scale α, which leads to faster convergence.

In order to make a step closer to realistic neural networks, deep linear fully connected networks of the form $V = W^{(N)} \cdots W^{(1)}$ are considered in several works [1, 4, 6, 11, 15]. The current results suggest implicit towards low rank solutions, but a theorem similar to the one stated above is not yet available.

Of course, the next step will be to extend to nonlinear networks. Preliminary results for two-layer networks are available, see e.g. [12], but in general the understanding of the implicit bias phenomenon in deep learning is widely open.

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The Role of Statistical Theory in Understanding Deep Learning

Sophie Langer
(joint work with Alina Braun, Gabriel Clara, Michael Kohler, Johannes+Schmidt-Hieber, Harro Walk)

In recent years, there has been a surge of interest across different research areas to improve the theoretical understanding of deep learning (see, e.g., [1] and [8]). A particular promising approach is the statistical one, which interprets deep learning as a nonlinear or nonparametric generalization of existing statistical models. For instance, a simple fully connected neural network is equivalent to a recursive generalized linear model with a hierarchical structure. Given this connection, many papers in recent years derived convergence rates of neural networks in a nonparametric regression or classification setting (see, e.g., [12], [3], [10]) . Nevertheless, phenomena like overparameterization seem to challenge the statistical principle of bias-variance trade-off (see [15]). Therefore, deep learning cannot only be explained by existing techniques of mathematical statistics but also requires a radical overthinking. In this talk, we will delve into the dual aspects of the role statistics plays in comprehending deep learning: its significance and its limitations, emphasizing the need to bridge with other research domains. Our discussion centeres around three distinct topics:

Empirical risk minimizers vs. estimators learned by gradient descent. The statistical performance of deep neural networks is often analyzed within a non-parametric regression framework. The objective here is to construct an estimator m_n for the true regression function m such that

$$\mathbf{E} \int |m_n(x) - m(x)|^2 P_X(dx) \tag{1}$$

is *small* with a particular interest in the behavior of the bound as the number of data points n increases - the rate of convergence. Previous studies (see, e.g., [12], [3], [10]) adopted the empirical risk minimizer

$$m_n \in \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} |f(X_i) - Y_i|^2,$$

based on a specific class of neural networks. For this kind of estimators rate of convergence results were derived under different assumptions on m, which all have in common that the rate, i.e., the bound on (1), no longer depends on the input dimension d of the problem but on a lower dimension d^* and thus promises fast convergence even in high-dimensional spaces. While these results show interesting approximation and generalisation results for neural networks, they are subject to a fundamental problem: they sidestep the optimization process of neural networks by assuming an empirical risk minimizer, limiting the holistic understanding of the procedure. To adress this gap, we showed in a simplified setting (see [4]), i.e., for regression functions with suitable decaying Fourier transform (similar to the so-called Barron class in [2]) and for shallow neural networks with sigmoidal activation function a rate of convergence of $n^{-1/2}$. While these results offer hope for a statistical analysis that considers training, they also underscore the indispensability of integrating optimization considerations, especially for deeper network structures and less restrictive assumptions on the regression function.

Understanding dropout in a linear model. Overparameterized neural networks have gained significant attention in recent years due to their remarkable ability to achieve high accuracy on complex tasks. However, these networks are prone to overfitting, where they memorize the training data rather than learning the underlying patterns. To address this issue, researchers have developed various regularization schemes. In addition to explicit regularization techniques such as ℓ_2- or ℓ_2-penalization, algorithmic regularization approaches have been employed. Among them, dropout has emerged as a technique that randomly drops neurons during training, and it has demonstrated its effectiveness in various applications (see [13]). However, despite its empirical success, a comprehensive theoretical understanding of how dropout achieves regularization is still somewhat limited.

In the case of a linear model, it was shown that under an averaged form of dropout the least squares minimizer performs a weighted variant of ℓ_2-penalization. In turn, the heuristic "dropout performs ℓ_2-penalization" has even made it in popular textbooks (see [6] and [7]). We challenge this relation by investigating the statistical behavior of iterates generated by gradient descent with dropout (see [5]). In particular, non-asymptotic convergence rates for the expectation and covariance matrices of the iterates are derived. While in expectation the connection between dropout and ℓ_2-penalization can be verified, we show sub-optimality of the asymptotic variance compared to the estimator resulting from direct minimization of averaged dropout. To us, this result highlights once again, that simplification in analyzing deep learning can also lead to wrong conclusions.

Statistical analysis of image classification. The availability of massive image databases resulted in the development of scalable machine learning methods such as convolutional neural network (CNNs) filtering and processing these data. While the very recent theoretical work on CNNs focuses on standard nonparametric denoising problems, the variability in image classification datasets does, however, not originate from additive noise but from variation of the shape and other characteristics of the same object across different images. To address this problem, we consider a simple supervised classification problem for object detection on grayscale images (see [11]). While from the function estimation point of view, every pixel is a variable and large images lead to high-dimensional function recovery tasks suffering from the curse of dimensionality, increasing the number of pixels in our image deformation model enhances the image resolution and makes the object classification problem easier. We propose and theoretically analyze two different procedures. The first method estimates the image deformation by support alignment. Under a minimal separation condition, it is shown that perfect classification is possible. The second method fits a CNN to the data. We derive a rate for the misclassification error depending on the sample size and the number of pixels. Both classifiers are empirically compared on images generated from the MNIST handwritten digit database. The obtained results corroborate the theoretical findings. To us, the introduced image deformation model offers a new way of analyzing image classification theoretically with rates of convergence that are in line with practical observations. Furthermore, it highlights the necessity of critically questioning and revising existing statistical models.

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Optimal sampling and tensor learning

Anthony Nouy
(joint work with Robert Gruhlke, Bertrand Michel, Charles Miranda, Philipp Trunschke)

We consider the approximation of functions in L^2 from point evaluations, using linear or nonlinear approximation tools. For linear approximation, recent results show that weighted least-squares projections allow to obtain quasi-optimal approximations with near to optimal sampling budget [1, 2]. This can be achieved by drawing i.i.d. samples from suitable distributions (depending on the linear approximation tool) and subsampling methods. In a first part of this talk, we review different strategies based on i.i.d. sampling and present alternative strategies based on repulsive point processes that allow to achieve the same task with a reduced sampling complexity. In a second part, we show how these methods can be used to approximate functions with nonlinear approximation tools, in an active learning setting, by coupling iterative algorithms and optimal sampling methods for the projection onto successive linear spaces. We particularly focus on the approximation using tree tensor networks, an approximation tool with high expressive power [3, 4] and with an architecture allowing for an efficient implementation of optimal sampling procedures within coordinate descent algorithms.

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Low-rank tensor solvers for high-dimensional parabolic PDEs

Markus Bachmayr

(joint work with Henrik Eisenmann, Manfred Faldum, Emil Kieri, André Uschmajew)

In this talk, we consider two different approaches for numerically solving second-order parabolic initial value problems on high-dimensional product domains using low-rank tensor approximations. A typical model problem takes the form

$$\partial_t u - \nabla \cdot M \nabla u = f \quad \text{in } (0,T) \times \Omega = \Omega_1 \times \cdots \times \Omega_d, \tag{1}$$

subject to the initial condition $u(0,\cdot) = u_0$ in Ω and the boundary condition $u = 0$ on $(0,T) \times \partial\Omega$. As the following results show, using methods based on low-rank approximations of solutions this problem can be treated also for large d.

The two types of low-rank approximations that we consider are conceptually quite different, one based on dynamical low-rank approximation, the other on an adaptive solver for a space-time variational formulation. In both cases, we assume a Gelfand triplet $V \subset H \subset V'$, where in the case of (1), $V = H^1_0(\Omega)$ and $H = L_2(\Omega)$. In the first approach based on dynamical low-rank approximation, one obtains approximate dynamics under the additional constraint that for all times $t \in [0,T)$, one has $u(t) \in \mathcal{M}$, where $\mathcal{M}$ is a manifold of low-rank tensors such as $\mathcal{M} = \left\{ \sum_{i=1}^r \phi^1_k \otimes \phi^2_k : \ \phi^1_k \in L_2(\Omega_1), \phi^2_k \in L_2(\Omega_2) \right\} \subset H$ in the case $d = 2$.

The Dirac-Frenkel variational principle then yields an accordingly projected problem, which as shown in [3] can also be formulated in a weak formulation of (1): Given $f \in L_2(0,T;V')$ and $u_0 \in \mathcal{M} \cap H$, find $u \in W(0,T;V,V') = \{u \in L_2(0,T;V) \colon u' \in L_2(0,T;V')\}$ such that for almost all $t \in [0,T]$,

$$\begin{aligned} &u(t) \in \mathcal{M}, \\ &\langle u'(t) + A(t)u(t), v\rangle = \langle f(t), v\rangle \quad \text{for all } v \in T_{u(t)}\mathcal{M} \cap V, \\ &u(0) = u_0, \end{aligned} \tag{2}$$

where $T_{u(t)}\mathcal{M}$ denotes the tangent space at $u(t)$ and where $A(t)\colon V \to V$ is the elliptic part of the operator, assumed to be Lipschitz continuous with respect to t. Under natural conditions on $\mathcal{M}$ and the additional regularity requirements $f \in L_2(0,T;H)$ and $u_0 \in \mathcal{M} \cap V$, in addition to a splitting of $A(t) = A_1(t) + A_2(t)$ where $A_1(t)$ maps $\mathcal{M}$ to the respective tangent space and $A_2(t)$ satisfies a suitable boundedness condition as a mapping from $\mathcal{M} \cap V$ to H, in [3] we obtain existence and uniqueness of solutions $u \in W(0,T^*;V,H) \cap L_\infty(0,T^*;V)$ whenever u_0 has positive distance from the boundary of $\mathcal{M}$. Here either $T = T^*$ or $u(t)$ approaches the boundary of $\mathcal{M}$ as $t \to T^*$. In [4], this result is shown to be applicable to manifolds $\mathcal{M}$ of tensor trains and hierarchical tensors in H, and thus to problems with large d. We also show the resulting approximation to be stable with respect to perturbations of the problem data and that spatial semidiscretizations converge under natural assumptions.

Numerical solvers with favorable properties are available for the reduced problems on $\mathcal{M}$ defined by (2). However, with this approach in general one cannot ensure that the solutions of (2) are close to the unconstrained evolution given by $u'(t) + A(t)u(t) = f(t)$; as a simple example, one obtains a systematic error when $u_0 \perp_H f(0)$. Ensuring that such effects are avoided is difficult in practice. Such issues do not arise in the second approach that we consider.

This alternative construction of a low-rank solver for parabolic problems such as (1) is based on a space-time variational formulation. In the basic case of the heat equation, it reads: with $\mathcal{X} = W(0,T;V,V')$ and $\mathcal{Y} = L_2(0,T;V) \times H$, find $u \in \mathcal{X}$ such that for all $(v,w) \in \mathcal{Y}$,

$$\text{(3)} \quad \int_0^T \langle \partial_t u, v\rangle_{V',V} + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \gamma_0 u\, w\,\mathrm{d}x = \int_0^T \int_\Omega f\, v\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0\, w\,\mathrm{d}x,$$

where $\gamma_0 u \in H$ denotes the inital trace of u. Here we restrict ourselves to the model case $\Omega = (0,1)^d$ for simplicity. Similarly to [6], our approximations of u are based on basis functions $\{\theta_\mu\}_{\mu\in\mathcal{I}}$ on $(0,T)$ with the Riesz basis properties $\|\mathbf{v}\| \eqsim \bigl\| \sum_{\mu\in\mathcal{I}} \mathbf{v}_\mu \frac{\theta_\mu}{\|\theta_\mu\|_S} \bigr\|_S$ for all $\mathbf{v} \in \ell_2(\mathcal{I})$ and $S \in \{L_2(0,T), H^1(0,T)\}$ and $\{\psi_\nu\}_{\nu\in\mathcal{J}}$ on $(0,1)$ such that $\|\mathbf{v}\|_{\ell_2(\mathcal{J})} \eqsim \bigl\| \sum_{\hat\nu\in\mathcal{J}} \mathbf{v}_{\hat\nu} \frac{\psi_{\hat\nu}}{\|\psi_{\hat\nu}\|_S} \bigr\|_S$ for all $\mathbf{v} \in \ell_2(\mathcal{J})$ and $S \in \{H^1_0(0,1), L_2(0,1), H^{-1}(0,1)\}$. A concrete example of suitable such basis functions is provided by spline (multi-)wavelets.

A novel aspect in the method that we obtain in [5] is that we combine a sparse expansion in time with adaptive low-rank approximations in the spatial variables. Specifically, we compute approximations of u in the form

$$\text{(4)} \quad u(t,x_1,\ldots,x_d) \approx \sum_{\mu\in\Lambda_{\mathrm{t}}\subset\mathcal{I}} \theta_\mu(t) \sum_{(\nu_1,\ldots,\nu_d)\in\Lambda_\mu} \mathbf{u}_{\mu,\nu_1,\ldots,\nu_d}\, d^{\mathcal{X}}_{\mu,\nu} \psi_{\nu_1}(x_1)\cdots\psi_{\nu_d}(x_d)$$

with finite $\Lambda_{\mathrm{t}} \subset \mathcal{I}$ and $\Lambda_\mu = \Lambda^1_\mu \times\cdots\times \Lambda^d_\mu \subset \mathcal{J}\times\cdots\times\mathcal{J}$ that are potentially different for each μ. Here the coefficient tensors $\mathbf{u}_\mu = (\mathbf{u}_{\mu,\nu_1,\ldots,\nu_d})_{\nu\in\Lambda_\mu}$ are represented in hierarchical tensor format separately for each μ.

Based on a generalization of the strategy with a single hierarchical tensor representation of the approximate solution developed in [2] (see also [1]) for elliptic problems, an adaptive solver operating on the Riesz basis representation of the problem is obtained in [5] that refines the index sets Λ_{t} and Λ_μ, $\mu \in \Lambda_{\mathrm{t}}$, while at the same time computing approximate coefficient tensors $\mathbf{u}_\mu$ with adaptively adjusted ranks. A central role is played by suitable low-rank approximations of the scaling factors $d^{\mathcal{X}}_{\mu,\nu}$ in (4) that yield the appropriate normalization to a Riesz basis of $\mathcal{X}$. These can be chosen as

$$d^{\mathcal{X}}_{\mu,\nu} = \frac{\|\psi_{\nu_1} \otimes \cdots \otimes \psi_{\nu_d}\|_{H^1}}{\|\psi_{\nu_1} \otimes \cdots \otimes \psi_{\nu_d}\|^2_{H^1} + \|\theta_\mu\|_{H^1}}.$$

For each fixed μ and $a_\mu = \|\theta_\mu\|_{H^1}$, low-rank approximations by exponential sums of these expressions are obtained by applying quadrature to the integral representations

$$\frac{\sqrt{s}}{s + a_\mu} = \int_0^\infty \frac{1}{\sqrt{\pi y}} \bigl(1 - 2\sqrt{a_\mu y} F(\sqrt{a_\mu y})\bigr) \exp\bigl(-ys\bigr)\, \mathrm{d}y, \quad s > 0,$$

where F is the Dawson function, and by setting $s = \|\psi_{\nu_1} \otimes \cdots \otimes \psi_{\nu_d}\|^2_{H^1} = \sum_{i=1}^d \|\psi_{\nu_i}\|^2_{H^1}$. This yields approximate low-rank diagonal preconditioning for (3).

The resulting method can always be guaranteed to converge in $\mathcal{X}$-norm to the exact solution u of (3). Under benchmark approximability assumptions on the problem data and on u, it is also shown to yield approximations with optimality properties analogous to those obtained for the elliptic case in [2], especially on the arising tensor ranks. In particular, the curse of dimension can be avoided both concerning the complexity of approximations and the required number of operations in their computation. This is confirmed by the numerical tests in [5], where the total computational costs are observed to grow polynomially in d in the case of the heat equation as in (3).

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Parametric PDE-induced Neural Networks and Network Training by Hierarchical Tensors

Thong Le
(joint work with Martin Eigel, Lars Grasedyck, Janina Enrica Schütte)

In our research, we investigate the potential of integrating low-rank tensor decompositions in neural network training. Our approach involves discretizing the loss function

$$\mathcal{L}_\Phi : \mathbb{R}^d \mapsto \mathbb{R}, \quad W \mapsto \mathcal{L}_\Phi(W).$$

with a grid of size n^d and afterwards finding the position of the minimum absolute entry which corresponds to the weights of the neural network. Calculating all entries is not possible because of the curse of dimensionality so we make use of the Hierarchical Tucker format to circumvent the curse of dimensionality. This not only enhances the networks' ability to optimize but could also facilitate more effective weight initialization, potentially leading to better network training. There are two different approaches one could choose:

- First idea is to create a fine grid in order to better approximate the minimum loss value but this would lead to higher n,
- Second idea is to use a grid refinement strategy to adaptively approach the minimum loss value which could be done with small n.

In this workshop we focused on the latter idea.

Furthermore we propose an idea to construct Feedforward Neural Networks using hierarchical domain decompositions of the parameter field of the parametric PDE which in our case is a cookie-shaped domain.

Our focus throughout the workshop is the Darcy partial differential equation as the model problem within a cookie-shaped parameter domain. Using this model problem we provide numerical results.

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Convolutional neural networks for parametric PDEs

Janina Schütte
(joint work with Martin Eigel)

Deep learning has emerged as a flexible tool, extending its reach beyond famous applications, such as in natural language processing and image recognition, into the realm of solving parametric partial differential equations (pPDEs).

The significance of solving pPDEs lies in their crucial role across diverse fields such as physics, engineering, finance, and environmental science. Understanding the impact of varying parameters on a system is essential for predicting outcomes and making informed decisions.

Deep learning offers a novel approach to tackle the complexity of pPDEs. By training neural networks on appropriate data sets, the models learn intricate relationships between parameters and the corresponding system behavior. This expedites the solution process and therefore provides a chance to observe different states of the system under the influence of many different parameters.

There exist well developed mathematical concepts to solve PDEs specifically finite element (FE) and finite volume methods. There are works incorporating these methods into the setting of parametric PDEs, such as the Adaptive Stochastic Galerkin FEM [2] or the Variational Monte Carlo method [3], which are based on a polynomial chaos expansion and tensor approximation. A method based on convolutional neural networks (CNNs) was proposed in [1].

Parametric Darcy problem. The introduced methods are sample based and can be applied to data generated with a large class of linear and nonlinear pPDEs. In the analysis, the focus lies on the *parametric Darcy problem*, or stationary diffusion equation, which we also use as a benchmark problem in the numerical experiments. We formulate it in the following way. Let $D \subset \mathbb{R}^d$ be a spatial domain and $\Gamma \subset \mathbb{R}^{\mathbb{N}}$ a possibly countable infinite parameter space. Let $f : D \to \mathbb{R}$. We approximate the map $u : \Gamma \times D \to \mathbb{R}$, which satisfies

$$\begin{cases} \nabla_x \cdot (\kappa(y,x)\nabla_x u(y,x)) = f(x) & \text{for } x \in D \text{ and} \\ u(x) = 0 & \text{for } x \in \partial D \end{cases} \tag{1}$$

for the parameter field $\kappa : \Gamma \times D \to \mathbb{R}$ and where the derivatives are applied to the variable x.
The dependence of the parameter field κ on the parameter vector y can be characterized in different ways. For instance, for the *cookie problem*, the parameter field is defined for $D = [0,1]^2$ and $\Gamma = [0,1]^p$. Let $y \in \Gamma$ with $y_k \sim U[0,1]$ for $k = 1, \dots, p$ and define

$$\kappa(x,y) = a_0 + \sum_{k=1}^{p} y_k \chi_{D_k}(x),$$

where D_k are disks with fixed centers and radii and $a_0 > 0$ is constant. A visualization of the cookie parameters and the corresponding solutions can be seen in figure 1 in the top and bottom row, respectively.

CNN approximating an adaptive finite element method. To solve this problem a CNN architecture is proposed, which maps the coefficients of a FE discretization of $\kappa(y, \cdots)$ to those of $u(y,\cdot)$. For a FE space V_h we denote the interpolation of $\kappa(y,\cdot)$ into V_h by κ_h and the Galerkin projection of the solution of problem (1) $u(y,\cdot)$ onto V_h by $h_h(y,\cdot)$. Well suited $P1$ finite element spaces V_h

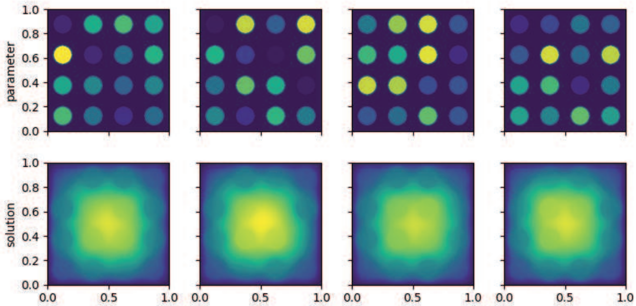

FIGURE 1. Realizations of parameter fields for the cookie problem and the corresponding solutions to the parametric Darcy problem

are build to control the discretization error

$$\mathcal{E} = \|u(y,\cdot) - u_h(y,\cdot)\|_{H_0^1(D)}$$

for any $y \in \Gamma$. The space is build in an adaptive manner by starting with a coarse FE space and repeating:

$$\begin{aligned} &\text{Solve on current space} \\ \rightarrow &\text{Estimate } \mathcal{E} \text{ locally} \\ \rightarrow &\text{Mark large error regions} \\ \rightarrow &\text{Refine marked regions} \end{aligned}$$

A CNN architecture is derived, which can approximate every step of the above iteration. There exists a constant $C > 0$ such that for any $\varepsilon > 0$ and V_h the final space of the described algorithm after $K \in \mathbb{N}$ steps with maximally $L \in \mathbb{N}$ refinement steps in every region, there exists a CNN $\Psi : \mathbb{R}^{2\times \dim V_L} \to \mathbb{R}^{\sum_{\ell=1}^{L} \dim V_\ell}$ such that the number of parameters is bounded by $CLK \log(\varepsilon^{-1})$ and

$$\|u(y,\cdot) - \mathcal{F}(\Psi(\kappa_L(y), f_L))\|_{H^1(D)} \le \|u(y,\cdot) - u_h(y,\cdot)\|_{H^1(D)} + \varepsilon,$$

where $\mathcal{F}$ maps the coefficients of the CNN output to the corresponding FE function.

Approximation of corrections. As the derived CNN can approximate steps of an adaptive finite element method, individual parts of the network can be trained separately. A first part of the network can approximate the solution on a coarse grid, while the following parts of the network approximate correcitons of the solution on finer grids, as depicted in figure 2. The training of only few parameters at a time yields an advantage, when optimizing the network. Furthermore, the influence of later corrections quickly decreases, which gives a need for good approximations in the first steps and requires less accuracy in later corrections. This can be translated into smaller networks for later corrections or only few fine grid training samples.

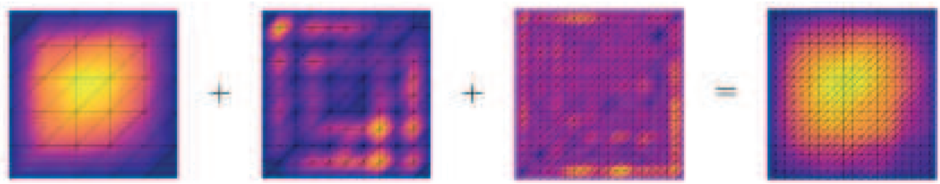

Figure 2. Visualization of the multilevel decomposition

Coclusions and outlook. Convolutional neural networks are an efficient tool to solve parametric partial differential equations. Theoretically small approximation errors can be achieved with network sizes growing only logarithmically with the the inverse of the required error bound. Numerically, the multilevel decomposition of the data allows for efficient training of small networks and with few expensive and many cheap data points. Solving a parametric PDE for a given parameter with the trained neural network only takes one forward pass through network, which can be evaluated quickly.
Applying this network to different applications, such as the inverse problem mapping the solution to the parameter, is of great interest.

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Mini-Workshop: Mathematics of Many-body Fermionic Systems

Organized by
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Abstract. Fermionic quantum systems are well described by the linear many-body Schrödinger equation. For interacting systems the full Schrödinger theory is extremely complicated and theoretical as well numerical investigations are not feasible. In practice, macroscopic properties of large systems can therefore only be accessed by means of approximate theories. The intention of this workshop was to showcase the most recent advances in the mathematical study of many-body interacting fermionic systems and to stimulate discussions among different research groups.

Mathematics Subject Classification (2020): 35P05, 35Q20, 35Q40, 35Q55, 35Q60, 35Q83, 81Q10, 82B10, 82C10.

Introduction by the Organizers

The mini-workshop *Mathematics of Many-body fermionic systems*, organized by Nikolai Leopold (Basel), Phan Thành Nam (Munich) and Chiara Saffirio (Basel) gathered sixteen participants, including the organizers. The group covered a broad range of expertise and maintained a well-rounded balance in terms of both age and gender. The main goals of the workshop were to showcase the most recent mathematical techniques in many-body interacting fermionic systems and to foster the interaction between different research groups. The newest results in the field were presented in thirteen one-hour lectures. Several free slots as well as an open problem discussion session provided opportunities for in-depth scientific discussions on cutting-edge methodologies and potential future research directions.

The talks were centered on the following core themes: ground state energies of many-body fermionic systems, effective dynamics for quantum systems and quantum systems in interaction with radiation fields.

The majority of the contributions dealt with ground state energies of many-fermion systems. Christian Hainzl opened the workshop with a presentation concerning the correlation energy of the electron gas in the mean-field regime. Related developments were reported by Martin Christiansen about spectral estimates for fermionic n-body operators and reduced density matrices, by Emanuela Giacomelli about the low density Fermi gas in three dimensions towards the Huang-Yang conjecture, and by Blazej Ruba concerning the bosonization for strongly interacting Fermi gases. The topic was in addition addressed by Volker Bach who discussed unitary renormalization group flows for fermion systems, Mathieu Lewin who gave an overview on mathematical results in density functional theory, and Charlotte Dietze who presented semiclassical estimates for Schrödinger operators with Neumann boundary conditions on Hölder domains.

The time evolution of quantum systems was treated by Peter Pickl with his presentation about effective evolution equations for tracer particles in interaction with either bosonic or fermionic gases. François Golse gave insights into the random batch method in the context of large N limit (uniform in $\hbar$) of the Wigner transform of the single-particle reduced density matrix associated with an N-body quantum system. Jani Lukkarinen's talk was concerned with the propagation of chaos via cumulant hierarchies in two example models: the discrete nonlinear Schrödinger evolution and the stochastic Kac model.

Systems with radiation fields have been considered by Tadahiro Miyao who presented a unified mathematical framework to describe the magnetic properties of ground states in many-electron systems, and Simone Rademacher who discussed the Landau-Pekar conjecture on the effective mass problem for the classical polaron. The workshop ended with the talk of Manfred Salmhofer reviewing results on the Hubbard model and the Fermi liquids, based on renormalization group techniques.

Wednesday morning was devoted to a collaborative discussion session aiming to maximize the interaction between the participants. The attendees were split into four subgroups, each dedicated to exploring a given topic for an hour. Subsequently, the findings were shared in a large plenary session, sparking further discussions. This format, recommended to the organizers by Mathieu Lewin, proved highly successful with many topics continuing to be explored during the traditional afternoon hike. The subjects listed below were the main themes of the discussion.

Correlation estimates: The discussion revolved around the study of the energy in terms of reduced density matrices, and in particular around Coulson's challenge related to the reconstruction of the N-particle states originated from a two-body density matrix. At present it is believed that in practical applications the so-called P-Q-G-T1-T2 conditions on two-body density matrices suffice for the reconstruction of the N-body states up to a very high precision. In 2013, Volker Bach, Hans

Konrad Knörr and Edmund Menge showed that conditions P-Q-G imply the validity of the Hartree-Fock approximation, thus particularly explaining the success of earlier numerical tests by Eric Cancès, Mathieu Lewin and Gabriel Stoltz. On the other hand, the T1 and T2 conditions are obtained by suitable 3-body density inequalities, and they seem to be hidden in recent developments in the correlation energy. Potential links between the P-Q-G-T1-T2 conditions and the random phase approximation were suggested, leading to interesting open problems to be investigated in the upcoming years.

Effective dynamics: The second group focused on the derivation of effective evolution equations for many particle systems. Two key open problems were identified. Firstly, there is an interest in deriving effective equations for longer time scales than those thus far explored. Secondly, a highly desirable goal is to establish the derivation of the Vlasov–Poisson equation from the classical dynamics of many particles with Coulomb interaction. The plenary discussion revolved around the latter challenge, specifically addressing the fact that Sylvia Serfaty and Mitia Duerinckx have successfully derived the Vlasov–Poisson equation with Coulomb potential in the monokinetic case. The discourse then centered on exploring whether the assumptions on the solutions of the pressureless Euler–Poisson equation (linked to monokinetic solutions of the Vlasov–Poisson equation) can be relaxed in the monokinetic derivation. Additionally, it was addressed how a derivation beyond the monokinetic scenario could be accomplished.

Kinetic equations: The group directed its attention towards the derivation of the quantum Boltzmann equation from the many-body Schrödinger equation. The weakly interacting and dilute regime were identified for derivations of the quantum Boltzmann equation with cubic collision operator. Recent findings about the derivation of the quantum Boltzmann equation by Thomas Chen, Michael Hott and Esteban Cárdenas as well as concerning the derivation of the wave kinetic equation by Tristan Buckmaster, Yu Deng, Pierre Germain, Zaher Hani and Jalal Shatah were highlighted. The main focus of the discussion then shifted to the technical aspects of the derivation. On the one hand it was investigated how Gronwall-type estimates could be optimized to be more useful in the kinetic regime. In this context it was discussed if the introduction of randomness could be helpful. On the other hand attention was directed towards finding suitable macroscopic observables for the derivation such as cumulants.

Semiclassical limits (including systems with radiation fields): Natural connections between semiclassical analysis and density functional theory were mentioned, including several problems on semiclassical estimates. In particular, challenging questions on asymptotic behaviors of large Coulomb systems were promoted. Concerning systems with radiation fields it was discussed in which way Maxwell's equations emerge from the quantized electromagnetic field with large photon number. Existing results were pointed out and a derivation of Maxwell's equations from non-relativistic quantum electrodynamics in a many-fermion limit as open problem identified. Additionally, the question if it is possible to define and analyze a microscopic model of a laser was raised. First results in this direction by Jean-Bernard Bru and Walter de Siqueira Pedra were pointed out.

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Abstracts

Correlation energy of the electron gas in the mean-field regime

Christian Hainzl
(joint work with M. R. Christiansen and P. T. Nam)

In [5] we prove a rigorous upper bound on the correlation energy of interacting fermions in the mean-field regime for a wide class of interaction potentials. Our result covers the Coulomb potential, and in this case we obtain the analogue of the Gell-Mann–Brueckner formula [6] $c_1\rho\log(\rho) + c_2\rho$ in the high density limit. We do this by refining the analysis of our bosonization method in [3] to deal with singular potentials, and to capture the exchange contribution which is absent in the purely bosonic picture.

In a forthcoming paper we will actually also prove the corresponding lower bound. Before stating the Theorem we give a precise definition of the model.

We consider N (spinless) electrons in the unit torus $\Omega = [0, 2\pi]^3$ (periodic b.c.) where

$$N = |B_F| = |B(0, k_F) \cap \mathbb{Z}^3|, \quad k_F \sim N^{1/3},$$

B_F denoting the Fermi ball and k_{B_F} the Fermi momentum. The N-body Hamiltonian on $L^2_a(\Omega^N)$ has the form

$$H_N = \sum_{i=1}^N (-\Delta_{x_i}) + \frac{1}{k_F} \sum_{1\le i<j\le N} V(x_i - x_j)$$

with mean-field periodic Coulomb potential

$$\frac{1}{k_F}V(x) = \frac{1}{k_F(2\pi)^3} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \hat{V}_k e^{ik\cdot x}, \quad \hat{V}_k = \frac{4\pi}{|k|^2}.$$

The main theorem about recovering the Gell-Mann-Brueckner formula for the correlation energy reads as follows, where E_{HF} is the Hartree-Fock energy.

Theorem.

$$E_N = E_{\mathrm{HF}} + E_{\mathrm{corr,bo}} + E_{\mathrm{corr,ex}} + o(k_F)_{k_F\to\infty}$$

with bosonic correlation contribution,

$$E_{\mathrm{corr,bo}} = \frac{1}{\pi} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \int_0^\infty F\left(\frac{k_F^{-1}\hat{V}_k}{(2\pi)^3} \sum_{p\in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \right) dt \sim k_F \log(k_F)$$

and exchange correlation contribution

$$E_{\mathrm{corr,ex}} = \frac{1}{4(2\pi)^6} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{p,q\in L_k} \frac{k_F^{-2}\hat{V}_k\hat{V}_{p+q-k}}{\lambda_{k,p} + \lambda_{k,q}} \sim k_F$$

$$F(x) = \log(1+x) - x, \quad \lambda_{k,p} = \tfrac{1}{2}(|p|^2 - |p-k|^2) > 0, \quad p \in L_k = (B_F + k)\setminus B_F.$$

The main idea of the proof can be summerized as follows. Starting from the Hamiltonian in second quantization, one can approximate the main contribution of the Hamiltonian by the following pseudo-quadratic Hamiltonian

$$H_{\mathrm{eff}} \approx \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \sum_{p\in L_k} 2\lambda_{k,p} b^*_{k,p} b_{k,p} + \sum_{k,p,q} \frac{\hat{V}_k k_F^{-1}}{2(2\pi)^3} (2b^*_{k,p} b_{k,q} + b_{k,p} b_{-k,-q} + b^*_{-k,-q} b^*_{k,p}),$$

where the operators $b_{k,p}$ describe a pair of Fermions,

$$b_{k,p} = a^*_{p-k} a_p, \quad p \in L_k = (B_F + k)\setminus B_F,$$

where a^*_{p-k} annihilates a hole in the Fermi sea and a_p annihilates a particle outside the Fermi sea. These $b_{k,p}$'s behave approximately like bosons. Following Sawada [8, 9] we diagonalize the Hamiltonian as if these operators were bosons and obtain the stated result. Since we track the non-bosoniscity of the b-operators exactly we also recover the exchange contribution in contrast to Sawada. Using a different approach, more precisely patching the Fermi sea, a similar result for smooth potentials was obtained earlier, see [1, 2]. In a perturbative form a similar result was obtained in [7]. In a similar way one can also track the elementary excitations. Plugging in the Coulomb potential into the final formula, one obtains the so called plasmon spectrum [4].

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Spectral Estimates for Fermionic n-Body Operators

Martin Ravn Christiansen

Fermionic n-Body Operators. Let $\mathfrak{h}$ be a Hilbert space and let $\Psi \in \bigwedge^N \mathfrak{h}$ be a normalized N-particle state. Then the n-body operator associated to Ψ, $\gamma_n^\Psi : \bigotimes^n \mathfrak{h} \to \bigotimes^n \mathfrak{h}$, is defined with respect to elementary tensors by

$$\left\langle (\varphi_1 \otimes \cdots \otimes \varphi_n), \gamma_n^\Psi (\psi_1 \otimes \cdots \otimes \psi_n) \right\rangle = \langle \Psi, c^*(\psi_1) \cdots c^*(\psi_n) c(\varphi_n) \cdots c(\varphi_1) \Psi \rangle .$$

Here $c^*(\cdot)$ and $c(\cdot)$ denote creation and annihilation operators, which obey the canonical anticommutation relations (CAR)

$$\{c(\varphi), c^*(\psi)\} = \langle \varphi, \psi \rangle , \quad \{c(\varphi), c(\psi)\} = 0 = \{c^*(\varphi), c^*(\psi)\} .$$

γ_n^Ψ is a positive self-adjoint operator on $\bigotimes^n \mathfrak{h}$, and if $(u_k)_k$ is an orthonormal basis for $\mathfrak{h}$ then its action can be recast as

$$\left\langle \Phi, \gamma_n^\Psi \Phi \right\rangle = \left\| \sum_{k_1,\ldots,k_n} \overline{\Phi_{k_1,\ldots,k_n}} c_{k_n} \cdots c_{k_1} \Psi \right\|^2$$

where $\Phi_{k_1,\ldots,k_n} = \langle u_{k_1} \otimes \cdots \otimes u_{k_n}, \Phi \rangle$ for $\Phi \in \bigotimes^n \mathfrak{h}$ and $c_k = c(u_k)$ for $k \in \mathbb{N}$.

The n-body operator γ_n^Ψ is trace-class with $\mathrm{tr}(\gamma_n^\Psi) = \frac{N!}{(N-n)!}$. This trivially implies that also $\|\gamma_n^\Psi\|_{\mathrm{op}} \leq \frac{N!}{(N-n)!} \sim N^n$, which is optimal in the bosonic case. For fermions this is untrue however, as e.g.

$$\left\langle \varphi, \gamma_1^\Psi \varphi \right\rangle = \langle \Psi, c^*(\varphi) c(\varphi) \Psi \rangle \leq \langle \Psi, \{c^*(\varphi), c(\varphi)\} \Psi \rangle = \|\varphi\|^2$$

by the CAR, which shows that $\left\|\gamma_1^\Psi\right\|_{\mathrm{op}} \leq 1$.

In terms of the basis $(u_k)_k$, this can be expressed as $\| \sum_k \alpha_k c_k \|_{\mathrm{op}}^2 \leq \sum_k |\alpha_k|^2$ for any coefficients $(\alpha_k)_k$. This implies an improvement on the bound for $\|\gamma_n^\Psi\|_{\mathrm{op}}$ for any n, since

$$\begin{aligned}\sqrt{\langle \Phi, \gamma_n^\Psi \Phi \rangle} &\leq \sum_{k_1,\ldots,k_{n-1}} \left\| \left(\sum_{k_n} \overline{\Phi_{k_1,\ldots,k_n}} c_{k_n} \right) c_{k_{n-1}} \cdots c_{k_1} \Psi \right\| \\ &\leq \sqrt{\sum_{k_1,\ldots,k_n} |\Phi_{k_1,\ldots,k_n}|^2} \sqrt{\sum_{k_1,\ldots,k_{n-1}} \left\| c_{k_{n-1}} \cdots c_{k_1} \Psi \right\|^2} = \frac{N!}{(N-n+1)!} \|\Phi\|\end{aligned}$$

implies that $\|\gamma_n^\Psi\|_{\mathrm{op}} \leq \frac{N!}{(N-n+1)!} \sim N^{n-1}$.

Yang's Estimates. For $n = 2$ this simply reads $\|\gamma_2^\Psi\|_{\mathrm{op}} \leq N$, which was first proved by Yang in [1], who also showed it to be optimal (for even N). Based on his analysis of the optimizers, he conjectured - and later proved - the following:

Theorem. (Yang, [1, 2]) For any normalized $\Psi \in \bigwedge^N \mathfrak{h}$ it holds that for all $n \in \mathbb{N}$

$$\|\gamma_n^\Psi\|_{\mathrm{op}} \leq C_n N^{\lfloor \frac{n}{2} \rfloor}$$

for constants $C_n > 0$ depending only on n.

This bound follows from two main points. The first is that if we define $\Lambda_n^N = \sup_{\Psi\neq 0} \|\gamma_n^\Psi\|_{\mathrm{op}}\|\Psi\|_{\mathrm{op}}^{-2}$ - i.e. the quantity we wish to control - it is seen that

$$\begin{aligned}\sqrt{\langle \Phi, \gamma_n^\Psi \Phi\rangle} &\leq \sum_{k_1}\left\|\Big(\sum_{k_2,\ldots,k_n}\overline{\Phi_{k_1,\ldots,k_n}}c_{k_n}\cdots c_{k_2}\Big)c_{k_1}\Psi\right\| \\ &\leq \sqrt{\Lambda_{n-1}^{N-1}}\sum_{k_1}\sqrt{\sum_{k_2,\ldots,k_n}\left|\Phi_{k_1,\ldots,k_n}\right|^2}\,\|c_{k_1}\Psi\| \\ &\leq \sqrt{\Lambda_{n-1}^{N-1}}\sqrt{\sum_{k_1,\ldots,k_n}\left|\Phi_{k_1,\ldots,k_n}\right|^2}\sqrt{\sum_{k_1}\|c_{k_1}\Psi\|^2} = \sqrt{N\Lambda_{n-1}^{N-1}}\,\|\Phi\|\end{aligned}$$

which implies the recursive estimate $\Lambda_n^N \leq N\Lambda_{n-1}^{N-1}$.

The second point is that an argument of Bell [3] implies that $\Lambda_n^N \lesssim C_n'\Lambda_{n-1}^N$ for odd n - combining these two estimates then yields Yang's estimate $\Lambda_n^N \leq C_n N^{\lfloor \frac{n}{2}\rfloor}$ by induction.

To illustrate Bell's argument, consider $n=3$: Then as for $n=1$

$$\left\langle \Phi, \gamma_3^\Psi \Phi\right\rangle \leq \left\langle \Psi, \left\{\Big(\sum_{k,l,m}\overline{\Phi_{k,l,m}}c_m c_l c_k\Big)^*, \Big(\sum_{k,l,m}\overline{\Phi_{k,l,m}}c_m c_l c_k\Big)\right\}\Psi\right\rangle$$

and since 3 is odd, the anticommutator reduces to a sum of terms containing at most 4 creation/annihilation operators, rather than 6. Indeed, assuming without loss of generality that the coefficients $\Phi_{k,l,m}$ are antisymmetric, this anticommutator is

$$9\sum_k\left|\sum_{l,m}\overline{\Phi_{k,l,m}}c_m c_l\right|^2 - 18\sum_{k,l}\left|\sum_m\overline{\Phi_{k,l,m}}c_m\right|^2 + 6\sum_{k,l}\left|\Phi_{k,l,m}\right|^2$$

which implies that $\Lambda_3^N \leq 9\Lambda_2^N + 6$.

Hilbert-Schmidt Estimates for γ_2^Ψ and $\gamma_2^{\Psi,T}$. The argument of Bell was recently used to obtain Hilbert-Schmidt estimates on 2-body operators and their truncated versions $\gamma_2^{\Psi,T} = \gamma_2^\Psi - (1-\mathrm{Ex})\,(\gamma_1^\Psi\otimes\gamma_1^\Psi)$.

First let us note that by the identity $\|\gamma_2^\Psi\|_{\mathrm{tr}} = N(N-1)$ and Yang's optimal estimate $\|\gamma_2^\Psi\|_{\mathrm{op}} \leq N$, it easily follows that $\|\gamma_2^\Psi\|_{\mathrm{HS}} \leq N^{\frac{3}{2}}$. This can however be improved significantly:

Theorem. ([4]) For any normalized $\Psi \in \bigwedge^N \mathfrak{h}$ it holds that

$$\|\gamma_2^\Psi\|_{\mathrm{HS}} \leq \sqrt{5}N, \quad \|\gamma_2^{\Psi,T}\|_{\mathrm{HS}} \leq \sqrt{5N\,\mathrm{tr}(\gamma_1^\Psi - (\gamma_1^\Psi)^2)}.$$

Note that the bound $\|\gamma_2^\Psi\|_{\mathrm{HS}} \leq \sqrt{5}N$ is of the same order with respect to N as Yang's bound $\|\gamma_2^\Psi\|_{\mathrm{op}} \leq N$ - informally speaking this implies that although γ_2^Ψ can have eigenvalues of order N, it can not have "too many" large eigenvalues. Furthermore, for Slater states Ψ it holds that $\|\gamma_2^\Psi\|_{\mathrm{HS}} = \sqrt{2}N$, so this order is optimal.

The first estimate follows by noting that

$$\operatorname{tr}(A\gamma_2^\Psi) = -\sum_n \left\langle \sum_{k,l,m} \overline{A_{k,l,m,n}} c_m^* c_l c_k \Psi, c_n \Psi \right\rangle$$

for any Hilbert-Schmidt operator A, whence

$$\left|\operatorname{tr}(A\gamma_2^\Psi)\right| \leq \sqrt{N \sum_n \langle \Psi, T_n^* T_n \Psi\rangle} \leq \sqrt{N \sum_n \langle \Psi, \{T_n^*, T_n\}\, \Psi\rangle}$$

for $T_n = \sum_{k,l,m} \overline{A_{k,l,m,n}} c_m^* c_l c_k$. Since this is again a sum of terms with 3 fermionic operators, the anticommutator simplifies significantly, with the consequence that (assuming without loss of generality an antisymmetry condition on $A_{k,l,m,n}$)

$$\sum_n \{T_n^*, T_n\} \leq \sum_{m,n} \left| \sum_{k,l} \overline{A_{k,l,m,n}} c_l c_k \right|^2 + 4 \sum_{k,n} \left| \sum_{l,m} A_{k,l,m,n} c_l^* c_m \right|^2,$$

and since not only $\| \sum_k \alpha_k c_k \|_{\mathrm{op}}^2 \leq \sum_k |\alpha_k|^2$ but also $\| \sum_k \alpha_k c_k^* \|_{\mathrm{op}}^2 \leq \sum_k |\alpha_k|^2$ this can be bounded as

$$\sum_n \{T_n^*, T_n\} \leq 5 \left(\sum_{k,l,m,n} |A_{k,l,m,n}|^2 \right) \left(\sum_k |c_k|^2 \right) = 5N \|A\|_{\mathrm{HS}}^2$$

for the claim. The estimate on $\|\gamma_2^{\Psi,T}\|_{\mathrm{HS}}$ follows by a similar argument after noting the identity

$$\begin{aligned}\left\langle (\varphi_1 \otimes \varphi_2), \gamma_2^{\Psi,T} (\psi_1 \otimes \psi_2) \right\rangle &= \left\langle \Psi, c(\gamma_1^\Psi \varphi_2) c^*(\psi_1) c^*(\psi_2) c(\varphi_1) \Psi \right\rangle \\ &\quad - \left\langle \Psi, c^*(\psi_1) c^*(\psi_2) c(\varphi_1) c((1-\gamma_1^\Psi)\varphi_2) \Psi \right\rangle.\end{aligned}$$

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Unitary Flows for Fermion Systems

Volker Bach
(joint work with Jakob Geisler, Konstantin Merz)

1. Unitary Flows on Fermion Operators

1.1. **Fermion Systems and Fermi Gases.** Here we present a mathematical study of fermion systems. Although we ultimately aim at treating atoms and molecules, we focus on Fermi gases here. For $d, L \in \mathbb{N}$, the configuration space of the system is the d-dimensional torus $\Lambda := \mathbb{R}^d / L\mathbb{Z}^d$ of sidelength $L \gg 1$, the corresponding momentum space is $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^d$. States of the system are represented by vectors in fermion Fock space $\mathfrak{F} = \mathfrak{F}_f(\mathfrak{h})$, where $\mathfrak{h} = L^2(\Lambda)$ is the Hilbert space of a single fermion. The system's dynamics is generated by the second-quantized Hamiltonian

$$\widetilde{\mathbb{H}}_{g,\nu} \;=\; \sum_{k \in \Lambda^*} (k^2 - \nu)\, \widehat{a}_k^*\, \widehat{a}_k \;+\; \frac{g}{2} \sum_{q,k,k' \in \Lambda^*} \frac{\hat{v}_q}{|\Lambda|}\, \widehat{a}_{k+q}^*\, \widehat{a}_{k'}^*\, \widehat{a}_{k'+q}\, \widehat{a}_k\,, \tag{1}$$

where $\nu > 0$ is the chemical potential, $g \geq 0$ is the coupling constant, $\hat{v} : \Lambda^* \to \mathbb{R}_0^+$ is the restriction to $\Lambda \subseteq \mathbb{R}^d$ of the Fourier transform $\mathfrak{F}[V] \in \mathcal{S}(\mathbb{R}^d)$ of a pair potential $V \in \mathcal{S}(\mathbb{R}^d)$, both assumed to be nonnegative, smooth functions of rapid decrease, for simplicity.

The Fermi gas under consideration is characterized by the spectral properties of $\widetilde{\mathbb{H}}_{g,\nu}$. These have been an object of research for almost a century, just like quantum mechanics itself. The discovery of High-T_c superconductivity brought these models into the focus of mathematical physics some 35 years ago. Monographs that provide an overview are [10, 7, 9]

1.2. **Hartree–Fock Theory and Bogoliubov Transformations.** One of the most important approximations to the ground state energy of a many-fermion system is the Hartree–Fock approximation which is defined by restricting the Rayleigh–Ritz variational principle to Slater determinants,

$$E_{\mathrm{HF}}(g,\nu) \;:= \inf\Big\{ \big\langle f_1 \wedge \cdots \wedge f_N \big|\, \widetilde{\mathbb{H}}_{g,\nu}(f_1 \wedge \cdots \wedge f_N) \big\rangle \;\Big|\; N \in \mathbb{N}_0\,,\;\; \langle f_i | f_j \rangle = \delta_{i,j} \Big\}. \tag{2}$$

In [6, 2] it was shown that the Hartree–Fock energy $E_{\mathrm{HF}}(g,\nu)$ coincides with the smallest energy expectation value of wave functions, which are Bogoliubov transforms $\mathbb{U}\Omega$ of the vacuum vector Ω,

$$E_{\mathrm{HF}}(g,\nu) \;=\; \inf\Big\{ \big\langle \Omega \big|\, \mathbb{U}^*\, \widetilde{\mathbb{H}}_{g,\nu}\, \mathbb{U}\Omega \big\rangle \;\Big|\; \mathbb{U} \in \mathrm{Bog}(\mathfrak{F}) \Big\}. \tag{3}$$

Here, $\mathrm{Bog}(\mathfrak{F}) \subseteq \mathcal{U}(\mathfrak{F})$ denote the Bogoliubov transforms on $\mathfrak{F}$, i.e., all unitary operators that act linearly on creation and annihilation operators. If we impose translation invariance of $|\mathbb{U}\Omega\rangle\langle\mathbb{U}\Omega|$, then the best choice for $\mathbb{U}$ is

$$h_k^* \;:=\; \mathbb{U}_\mu \, \widehat{a}_k^* \, \mathbb{U}_\mu^* \;:=\; \widehat{a}_k^* \,, \qquad k^2 \geq \mu \,, \tag{4}$$

$$\ell_k^* \;:=\; \mathbb{U}_\mu \, \widehat{a}_k^* \, \mathbb{U}_\mu^* \;:=\; \widehat{a}_k \,, \qquad k^2 < \mu \,, \tag{5}$$

$$\mathbb{U}_\mu \Omega \;:=\; \Big(\prod_{k^2<\mu} \widehat{a}_k^* \Big) \Omega \,, \tag{6}$$

where $\mu \equiv \mu(g) = \nu + \mathcal{O}(g)$ is chosen as to minimize $\langle \Omega | \mathbb{U}_\mu^* \widetilde{\mathbb{H}}_{g,\nu} \mathbb{U}_\mu \Omega \rangle$. Note that translation invariance of $|\mathbb{U}\Omega\rangle\langle\mathbb{U}\Omega|$ for the minimizing Bogoliubov transformation $\mathbb{U}$ is a plausible assumption and actually violated sometimes; BCS theory builds up on this assumption. For more details see [1] and references therein.

After conjugation with $\mathbb{U}_\mu$, the Hamiltonian reads

$$\mathbb{H}_g \;:=\; \mathbb{U}_\mu^* \, \widetilde{\mathbb{H}}_{g,\nu} \, \mathbb{U}_\mu \;=\; E_{\text{tHF}}(g,\nu) \,+\, \mathrm{d}\Gamma(\omega) \,+\, g\, \mathbb{Q} \,, \tag{7}$$

where $E_{\text{tHF}}(g,\nu)$ is the Hartree–Fock energy restricted to translation-invariant states, $\omega_k = |k^2 - \mu|$, and $\mathbb{Q}$ is purely quartic in h_k^*, ℓ_k^*, h_k, and ℓ_k.

1.3. **Flow Equations for Fermion Systems in Standard Representation.** We report on joint work in progress with *J. Geisler* and *K. Merz*. A suggestive formulation of the renormalization group (RG) is given by a family $\big(\mathbb{W}(t)\big)_{t\geq 0}$ of unitarily equivalent operators determined by the evolution equation

$$\forall\, t > 0 : \quad \dot{\mathbb{W}}(t) \;=\; i\big[\mathbb{G}(t)\,,\, \mathbb{W}(t)\big]\,, \qquad \mathbb{W}(0) = \mathbb{H}_g \,, \tag{8}$$

where $\mathbb{G}(t) = \mathbb{G}^*(t)$ is chosen as to eliminate ("diagonalize away") the undesired terms in $\mathbb{H}_g$. A concrete implementation of this idea is the Brockett-Wegner flow [5, 11, 3, 4]. A main difficulty for setting up the flow (8) is to find an appropriate Banach space on which it possesses basic properties such as (local and global) existence in the flow parameter $t \geq 0$.

A natural idea is to write $\mathbb{W}(t) = \mathbb{Q}[\underline{w}^{(t)}] \in \mathcal{B}[\mathfrak{F}]$ and $\mathbb{G}(t) = \mathbb{Q}[\underline{g}^{(t)}] \in \mathcal{B}[\mathfrak{F}]$ as images of symbols $\underline{w}^{(t)}, \underline{g}^{(t)} \in \mathcal{W}$ under a linear quantization map $\mathbb{Q} : \mathcal{W} \to \mathcal{B}[\mathfrak{F}]$, where $\mathcal{W}$ is a suitable Banach space of coefficients. Here and henceforth, we assume, for simplicity, Λ^* to be finite and, hence, the one-fermion space $\mathfrak{h} = \ell^2(\Lambda^*)$ and also the fermion Fock space $\mathfrak{F} = \mathfrak{F}[\mathfrak{h}]$ to be finite-dimensional, so that all operators are bounded. The Banach space $\mathcal{W} = \bigoplus_{m,n\geq 0} \mathcal{W}_{m,n}$ contains collections $\underline{w}^{(t)} = \big(\underline{w}_{m,n}^{(t)}\big)_{m,n\geq 0}$ and $\underline{g}^{(t)} = \big(\underline{g}_{m,n}^{(t)}\big)_{m,n\geq 0}$ of antisymmetric functions $\underline{w}_{m,n}^{(t)}, \underline{g}_{m,n}^{(t)} : (\Lambda^*)^m \times (\Lambda^*)^n \to \mathbb{C}$. Given $\underline{w} = \big(\underline{w}_{m,n}\big)_{m,n\geq 0} \in \mathcal{W}$, its quantization $\mathbb{Q}[\underline{w}]$ is defined as

$$\mathbb{Q}[\underline{w}] \;:=\; \sum_{m,n=0}^{\infty} \; \sum_{x_1^m \in (\Lambda^*)^m} \; \sum_{y_1^n \in (\Lambda^*)^n} w_{m,n}(x_1^m | y_1^n)\, a^*(x_1^m)\, a(y_1^n) \,, \tag{9}$$

where $x_1^m = (x_1, \ldots, x_m)$ with $x_1 < \ldots < x_m$ and $y_1^y = (y_1, \ldots, y_n)$ with $y_1 < \ldots < x_n$, for some fixed total order on Λ^*. Moreover, $a^*(x_1^m) = a_{x_1}^* \cdots a_{x_m}^*$ and

$a(y_1^n) = a_{y_n} \cdots a_{y_1}$. Now observe that, for $\underline{v} = \big(\underline{v}_{m,n}\big)_{m,n\geq 0}, \underline{w} = \big(\underline{w}_{m,n}\big)_{m,n\geq 0} \in \mathcal{W}$, we have

$$\mathbb{Q}[\underline{v}]\,\mathbb{Q}[\underline{w}] \;=\; \mathbb{Q}[\underline{v} * \underline{w}]\,, \tag{10}$$

with the convolution product defined by

$$\begin{aligned}(\underline{v} * \underline{w})_{M,N}(x_1^M | y_1^N) \;:=& \\ \sum_{r=0}^{\infty}\sum_{m=0}^{M}\sum_{n=0}^{N}&(-1)^{mn+r}\, r!\binom{m+r}{r}\binom{n+r}{r} \\ &\mathcal{A}_{M,N}\Bigg[\sum_{z_1^r\in(\Lambda^*)^r} v_{M-m,n+r}(x_{m+1}^M | z_1^r, y_1^n)\, w_{m+r,N-n}(z_1^r, x_1^m | y_{n+1}^N)\Bigg]\,,\end{aligned} \tag{11}$$

and $\mathcal{A}_{M,N}$ being the antisymmetrization operator. While the parametrization (9) seems natural, the No-Go theorem of Geisler [8] shows that, under some mild assumption on its form, no choice of norm $\|\cdot\|_{\mathcal{W}}$ on $\mathcal{W}$ will

- make the convolution product (11) submultiplicative, i.e.,

$$\forall\, \underline{v}, \underline{w} \in \mathcal{W}: \qquad \|\underline{v} * \underline{w}\|_{\mathcal{W}} \;\leq\; \|\underline{v}\|_{\mathcal{W}}\, \|\underline{w}\|_{\mathcal{W}}\,, \tag{12}$$

- and at the same time control the operator norm, i.e.,

$$\forall\, \underline{w} \in \mathcal{W}: \qquad \|\mathbb{Q}[\underline{w}]\|_{\mathcal{B}[\mathfrak{F}]} \;\leq\; \|\underline{w}\|_{\mathcal{W}}\,. \tag{13}$$

1.4. **New Representation of Fermion Operators.** We continue to report on joint work in progress with *J. Geisler* and *K. Merz*. Given the negative result of [8] that (12) and (13) plus some further natural assumptions lead to a contradiction, we propose to change the parametrization of operators on fermion Fock space altogether. We replace $\mathcal{W}$ by a different Banach space $\widehat{\mathcal{W}}$ of interaction coupling functions $\underline{\hat{w}} : \mathfrak{P}(\Lambda^*)^3 \to \mathbb{C}$ of the form $\underline{\hat{w}} = \big(\hat{w}_{I,J,K}\big)_{I\dot\cup J\dot\cup K\subseteq\Lambda^*}$, where $\mathfrak{P}(\Lambda^*)$ is the collection of subsets (power set) of Λ^*, and $\dot\cup$ denotes disjoint union. The quantization $\widehat{\mathbb{Q}} : \widehat{\mathcal{W}} \to \mathcal{B}[\mathfrak{F}]$ is defined by

$$\widehat{\mathbb{Q}}[\underline{\hat{w}}] \;:=\; \sum_{I\dot\cup J\dot\cup K} \hat{w}_{I,J,K}\, a_I^*\, n_K\, a_J\,, \tag{14}$$

where the summation is defined as

$$\sum_{I\dot\cup J\dot\cup K} F(I,J,K) \;:=\; \sum_{K\subseteq\Lambda^*}\;\sum_{J\subseteq\Lambda^*\setminus K}\;\sum_{I\subseteq\Lambda^*\setminus(K\cup J)} F(I,J,K)\,, \tag{15}$$

$a_\emptyset^* := a_\emptyset := n_\emptyset := 1$, and

$$a_A^* \;:=\; a_{\alpha_1^*}\cdots a_{\alpha_n}^*\,, \quad a_A \;:=\; a_{\alpha_n^*}\cdots a_{\alpha_1}^*\,, \quad n_A \;:=\; a_A^*\, a_A\,, \tag{16}$$

for $A = \{\alpha_1, \cdots, \alpha_n\} \subseteq \Lambda^*$ with $\alpha_1 < \cdots < \alpha_n$. We can show that this quantization possesses the following properties.

- For any $\underline{\hat{v}} = \big(\hat{v}_{I,J,K}\big)_{I\dot\cup J\dot\cup K\subseteq\Lambda^*}, \underline{\hat{w}} = \big(\hat{w}_{I,J,K}\big)_{I\dot\cup J\dot\cup K\subseteq\Lambda^*} \in \widehat{\mathcal{W}}$, the product of their quantizations

$$\widehat{\mathbb{Q}}[\underline{\hat{v}}]\,\mathbb{Q}[\underline{\hat{w}}] \;=\; \widehat{\mathbb{Q}}[\underline{\hat{v}} * \underline{\hat{w}}]\,, \tag{17}$$

induces a convolution product on $\widehat{\mathcal{W}}$ given for disjoint $I, J, K \subseteq \Lambda^*$ by

$$(18) \quad (\underline{\hat{v}} * \underline{\hat{w}})_{I,J,K} = \sum_{I'\dot{\cup}J'\dot{\cup}K'} \sum_{I''\dot{\cup}J''\dot{\cup}K''} \sum_{G\subseteq J'\dot{\cup}I''} S^{I,J,K}_{I',J',K';\, I'',J'',K'';\, G} \;\hat{v}_{I',J',K'} \;\hat{w}_{I'',J'',K''},$$

where $S^{I,J,K}_{I',J',K';\, I'',J'',K'';\, G} \in \{-1, 1\}$ is an explicit function.

- For $\xi, \eta \geq 1$ with $\xi^2 \geq 1+\eta$ define a norm on $\widehat{\mathcal{W}}$ for $\underline{\hat{w}} = \big(\hat{w}_{I,J,K}\big)_{I\dot{\cup}J\dot{\cup}K\subseteq\Lambda^*} \in \widehat{\mathcal{W}}$ by

$$(19) \quad \big\|\underline{\hat{w}}\big\|_{\xi,\eta} := \sum_{I\dot{\cup}J\dot{\cup}K} \xi^{|I|+|J|}\,\eta^{|K|}\,|\hat{w}_{I,J,K}|\,.$$

  Then

$$(20) \quad \big\|\underline{\hat{v}} * \underline{\hat{w}}\big\|_{\xi,\eta} \leq \big\|\underline{\hat{v}}\big\|_{\xi,\eta}\,\big\|\underline{\hat{w}}\big\|_{\xi,\eta}\,.$$

- For $\underline{\hat{w}} \in \widehat{\mathcal{W}}$, the operator norm of $\widehat{\mathbb{Q}}[\underline{\hat{w}}]$ is bounded by the norm of $\underline{\hat{w}}$,

$$(21) \quad \big\|\widehat{\mathbb{Q}}[\underline{\hat{w}}]\big\|_{\mathcal{B}[\mathfrak{F}]} \leq \big\|\underline{\hat{w}}\big\|_{\xi,\eta}\,.$$

Our current activity aims at implementing the diagonalizing flow (8) with $\mathbb{W}(t)$ and $\mathbb{G}(t)$ given by $\widehat{\mathbb{Q}}[\underline{\hat{w}}(t)]$ and $\widehat{\mathbb{Q}}[\underline{\hat{g}}(t)]$, respectively, for suitable $\underline{\hat{w}}(t), \underline{\hat{g}}(t) \in \widehat{\mathcal{W}}$.

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Bosonization for strongly interacting Fermi gases

Blazej Ruba

(joint work with S. Fournais and J. P. Solovej)

We study a gas consisting of $N \gg 1$ spinless fermions interacting through a two-body potential v modulated by the factor $N^{-\alpha}$, where α is a numerical parameter. The gas is described by the Hamiltonian

$$H_N = \sum_{i=1}^{N}(-\Delta_i) + N^{-\alpha} \sum_{1 \le i < j \le N} v(x_i - x_j) \tag{1}$$

on the Hilbert space of functions in $L^2((\mathbb{R}/\mathbb{Z})^{3N})$ antisymmetric with respect to permutations of the N copies of $(\mathbb{R}/\mathbb{Z})^3$. We are interested mostly in the ground state energy E_N of H_N.

Assuming that the Fourier series of v has non-negative coefficients $\widehat{v}(k)$ satisfying $\sum_k |k|\widehat{v}(k) < \infty$, we have elementary bounds

$$E_N^{(0)} \le E_N \le E_N^{(0)} + cN^{\frac{2}{3}-\alpha} \sum_k |k|\widehat{v}(k) + o(N^{\frac{2}{3}-\alpha}), \tag{2}$$

where $c > 0$ is an explicit constant and

$$E_N^{(0)} = \min_{\substack{p_1,\dots,p_N \in 2\pi\mathbb{Z}^3 \\ \text{distinct}}} \sum_{i=1}^{N} |p_i|^2 + \frac{N^{-\alpha}}{2} \int v - \frac{N^{1-\alpha}}{2} v(0). \tag{3}$$

One may ask whether one of the bounds in (2) is sharp up to $o(N^{\frac{2}{3}-\alpha})$. The answer is, at least for regular enough v: for $\alpha > \frac{1}{3}$ the upper bound is sharp, for $\alpha = \frac{1}{3}$ neither is sharp, and for $\alpha < \frac{1}{3}$ the lower bound is sharp. The last statement is our main new result.

The choice $\alpha = \frac{1}{3}$, often called the mean field scaling, has been studied extensively. In [1] Hamiltonians in the mean field scaling with small and very regular v were studied. It was explained how in such models one can use second order perturbation theory rigorously. If we take $\alpha > \frac{1}{3}$, which corresponds to interactions weaker than in the mean field scaling, the perturbative expansion is an even better approximation. In particular the reasoning in [1] shows that the upper bound in (2) is sharp up to our desired accuracy. The next term in E_N, given by second order of the perturbative expansion, is of order $N^{1-2\alpha} \ll N^{\frac{2}{3}-\alpha}$.

The understanding of the ground state energy in the mean field scaling was further improved in two series of works, [2, 3, 4] and [5, 6], where two approaches to approximate bosonization were developed. In both treatments one introduces operators which, in some sense, satisfy approximate canonical commutation rules and derives an effective Hamiltonian quadratic in the approximate bosons. Then that Hamiltonian is diagonalized using a Bogoliubov transformation. It is unclear whether the last step of this procedure can be justified also for $\alpha < \frac{1}{3}$, because then the generator of the Bogoliubov transformation is so large that it is difficult to control errors of the bosonic approximation.

In order to avoid the difficulties of working with a large Bogoliubov transformation, we do a variational calculation with the class of all vectors which can be obtained from the ground state of a non-interacting gas by acting with a polynomial in approximate bosons. More precisely, we construct a linear map from the Fock space of exact bosons to the fermionic Hilbert space and show that it is approximately isometric and approximately intertwines between H_N and the effective Hamiltonian. The quality of the approximation gets better as N increases, but it deteriorates very rapidly with the number of bosons in the state. Considering general states with $O(1)$ bosons and optimizing over the state after taking the limit $N \to \infty$ we obtain our result:

$$E_N = E_N^{(0)} + o(N^{\frac{2}{3}-\alpha}). \tag{4}$$

If one performs the Bogoliubov calculation non-rigorously, i.e. without controlling the error terms, one arrives at

$$E_N \approx E_N^{(0)} + \frac{N^{\frac{1-\alpha}{2}}}{\sqrt{2}} \sum_k |k| \sqrt{\widehat{v}(k)}. \qquad \textbf{(conjectural!)} \tag{5}$$

It seems that in order to justify this formula using the bosonization method one would have to understand how to control errors in calculations with states involving many bosons.

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Estimation of propagation of chaos via cumulant hierarchies in two example models

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(joint work with Aleksis Vuoksenmaa)

Propagation and generation of "chaos" is an important ingredient in rigorous control of applicability of kinetic theory, in general. Chaos can here be understood as sufficient statistical independence of random variables related to the "kinetic" observables of the system. Cumulant hierarchy of these random variables thus often gives a way of controlling the evolution and the degree of such independence, i.e., the amount of chaos in the system.

Motivated by recent successes of "direct" perturbation expansion results, such as those in [1, 2], we propose a way to combine such techniques into a simpler method to rigorously control the evolution of the cumulant hierarchy in two, qualitatively different, example cases for which kinetic theory is believed to be applicable: the discrete nonlinear Schrödinger evolution (DNLS) with suitable random, spatially homogeneous initial data, and the stochastic Kac model. In both cases, we set up suitable random variables and propose methods to control the evolution of their cumulant hierarchies. In this abstract, we focus only on the latter results.

The stochastic Kac model is a toy model introduced by Mark Kac in 1956 [3] for deriving a Boltzmann equation. It consists of N particles, where only velocities v_i, $i = 1, 2, \ldots, N$, of the particles are tracked, and collisions between particles take place stochastically. The collisions are determined by a Poisson clock whose rate is scaled to match the time-scale of the kinetic evolution. Once the clock rings, the labels of the two colliding particles are picked randomly and, for the chosen pair, their velocities are mixed randomly in such a way that the total energy is always preserved in a collision.

Assuming, for simplicity, that the initial distribution has energy density one, it has been proven that the distribution of the system approaches uniform distribution on the corresponding constant energy surface. However, this convergence can be quite slow, taking order N time units for typical initial data. Also, already in his original work, Kac proved a version of propagation of chaos for this system: if the initial data is approximately of a product form, then it will remain approximately in a product form for later times and the single velocity marginal can be well approximated by the solution to a corresponding Boltzmann-type evolution equation. Summary of the related results and literature may be found from [4, 5].

In our work in progress, we have been able to improve these results in two ways: (1) We have fairly accurate estimates for finite cumulants which become very small (consistent with approximate independence) already at times which are order one. (2) Since our initial data is less restricted, we are also able to conclude *generation of chaos* for these cumulants.

A more precise summary, whose proof and detailed assumptions can be found from our upcoming work, is given in the following Theorems. The results concern cumulants of the energies of the particles, i.e., the random variables $e_i := v_i^2$ and, for simplicity, we only consider the one-dimensional case, $v_i \in \mathbb{R}$.

Theorem 1 (preliminary for non-repeated cumulants). *Assume that the initial distribution is exchangeable, i.e., label permutation invariant. Suppose there is $B \geq 0$ such that the initial non-repeating energy cumulants $M_n^N(0)$ satisfy a bound $|M_n^N(0)| \leq B(n!)^2$. Then there is a constant A which only depends on B, such that the time-evolved non-repeating cumulants $M_n^N(t)$ satisfy the following bound for $n \geq 3$ and $t \geq 0$:*

$$|M_n^N(t)| \leq A^n (n!)^2 \left(\frac{1}{(N-1)^{n-1}} + \mathrm{e}^{-\frac{n}{4}t} \right).$$

This indeed proves generation of chaos for all finite order non-repeating cumulants whenever $\ln N \gg \ln n$. The first term in the bound is uniformly $O(N^{-(n-1)})$ hence goes to zero as $N \to \infty$. This is consistent with the above mentioned convergence to a stationary distribution since the variables e_i are mildly correlated under the uniform distribution on the energy surface.

Theorem 2 (preliminary for general cumulants). *Assume that the initial distribution is exchangeable, i.e., label permutation invariant. Suppose there is $B \geq 0$ such that the initial energy cumulants $\kappa_0^{n,N}(e_s)$ satisfy a bound $|\kappa_0^{n,N}(e_s)| \leq B^{n^2}(n!)^2$. Then there is a constant A, which only depends on B, and $N_0(n) \in \mathbb{N}$, such that for every $N \geq N_0(n)$ the time-evolved cumulants $\kappa_t^{m,N}(e_s)$ satisfy the following bound for any $t \geq 0$ and any sequence s of m labels, $m \in \{3, 4, \ldots, n\}$:*

$$|\kappa_t^{m,N}(e_s)| \leq A^{m^2}(m!)^2 \left(\frac{1}{(N-1)^{len(s)-1}} + \mathrm{e}^{-\frac{1}{4}t} \right)$$

In the above, s is a sequence of m labels from $\{1, 2, \ldots, N\}$, and $\mathrm{len}(s)$ is the number of different labels in this sequence. The earlier bound for non-repeating cumulants is $\mathrm{e}^{-\frac{\mathrm{len}(s)}{4}t}$ while here we are only able to prove $\mathrm{e}^{-\frac{1}{4}t}$. The proof is based on using an order on the structure of certain partition classifiers to control the linear part of the evolution and then iteratively propagating the upper bound.

In addition to the above generation of chaos bounds, it would be of interest to also look at relaxation for fixed N: Could we control (exponential) convergence of cumulants to their values in the uniform distribution on the sphere, as $t \to \infty$ for a fixed N? Would it be possible to control the accuracy of kinetic theory predictions such as improving the earlier estimates for the accuracy of the solution of the Boltzmann equation?

How much of these techniques can be used for cumulant hierarchies of other models, such as the nonlinear Schrödinger equation, is still under investigation.

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The low density Fermi gas in three dimensions

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(joint work with Marco Falconi, Christian Hainzl, Marcello Porta)

In 1957 Huang-Yang (HY) conjectured a formula for the asymptotic expansion of the ground state energy density of the Fermi gas at low density and in the infinite volume limit (see [1]), the rigorous validation of which is still an open problem. Here we present some recent results aimed at paving the way for rigorously proving the HY formula. We consider N interacting fermions with spin $\sigma = \{\uparrow, \downarrow\}$ in a box $\Lambda_L := [-L/2, L/2]^3$, with periodic boundary conditions. The Hamiltonian of the system is

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{i<j=1}^{N} V(x_i - x_j), \tag{1}$$

and it acts on $L^2_a(\Lambda_L^{N_\uparrow}) \otimes L^2_a(\Lambda_L^{N_\downarrow})$, where $L^2_a(\Lambda_L^{N_\sigma})$ is the antisymmetric tensor product of N_σ copies of $L^2(\Lambda_L)$ with N_σ denoting the number of particles with spin $\sigma = \{\uparrow, \downarrow\}$ ($N = N_\uparrow + N_\downarrow$). Correspondingly, we set $\rho_\sigma := N_\sigma/L^3$ ($\rho = \rho_\uparrow + \rho_\downarrow$). In the following, we will assume that our system is dilute, i.e., $\rho_\sigma \ll 1$. The interaction potential V is such that

$$V(x-y) = \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} \hat{V}_\infty(p) e^{ip\cdot(x-y)}, \qquad \hat{V}_\infty(p) = \int_{\mathbb{R}^3} dx\, V_\infty(x) e^{-ip\cdot x}, \tag{2}$$

where V_∞ is supposed to be non negative, radial, smooth and compactly supported.

We are interested in the thermodynamic limit, meaning that $N_\sigma, L \to \infty$ keeping ρ_σ fixed. In this setting, it is well know [2] that, in units such that $\hbar = 1$ and putting the masses of the particles equal to $1/2$, the ground state energy per unit volume can be approximated as

$$e(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + o(\rho^2) \qquad \text{as} \quad \rho \to 0. \tag{3}$$

The first term in the above expansion is purely kinetic (the kinetic energy of the free Fermi gas), and the fact that this contribution is proportional to $\rho^{5/3}$ is a consequence of the fermionic nature of the wave function. The effect of the interaction appears at the next order, via the parameter a, which is the scattering length of the interaction potential. In particular the contribution $\mathcal{O}(\rho^2)$ in (3) corresponds to the leading order in the asymptotic expansion for the correlation energy, which is defined as the difference between the ground state energy and that of the free Fermi gas. In [1] Huang-Yang conjectured a refined version of the asymptotics in (3) in the case where $\rho = \rho_\uparrow/2 + \rho_\downarrow/2$:

$$(4)\qquad e(\rho) = \frac{3}{5}(3\pi^2)^{\frac{2}{3}}\rho^{\frac{5}{3}} + 2\pi a\rho^2 + \frac{4(11-2\log 2)}{35\pi^2}\left(\frac{3}{4\pi}\right)^{\frac{4}{3}} a^2\rho^{\frac{7}{3}} + o\left(\rho^{\frac{7}{3}}\right).$$

as $\rho \to 0$. In 2021, the same asymptotics as the one in (3) have been re-derived in [3]. Differently than in [2], in [3] more restrictions are put in the interaction potential, but better error estimates are obtained. However, the main difference between [2] and [3] is the approach taken. In particular, the main novelty in [3] is the use of Bogoliubov theory applied to pairs of fermions (particle-hole pairs) that behave approximately as bosonic particles. Developing further this approach, in [4] refined asymptotics estimates are obtained, as stated below.

Theorem 1. *Let V, V_∞ as in (2) with V_∞ non negative, radial, smooth and compactly supported. There exists $L_0 > 0$ such that for $L \geq L_0$, it holds*

$$(5)\qquad e_L(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a\rho_\uparrow\rho_\downarrow + r_L(\rho_\uparrow, \rho_\downarrow),$$

where a is the scattering length of the interaction potential V_∞ and

$$-C\rho^{2+\frac{1}{5}} \leq r_L(\rho_\uparrow, \rho_\downarrow) \leq C\rho^{\frac{7}{3}}.$$

Note that the upper bound in Theorem 1 is optimal, in the sense that it agrees with the HY formula in (4). We also mention that very recently the ground state energy of the dilute spin-polarized Fermi gas was studied in [6] and with similar techniques an almost optimal upper bound for the ground state energy of a dilute spin 1/2 Fermi gas was derived (via cluster expansion) in [5].

The general strategy of the proof of Theorem 1 is based on the use of almost bosonic operators to describe the low energy excitations around the Fermi ball. These operators are defined as

$$(6)\qquad b^*_{p,\sigma} = \sum_{\substack{k\in\mathcal{B}_F^\sigma\\ k+p\notin\mathcal{B}_F^\sigma}} a^*_{k+p,\sigma}a^*_{k,\sigma}, \qquad b_{p,\sigma} = \sum_{\substack{k\in\mathcal{B}_F^\sigma\\ k+p\notin\mathcal{B}_F^\sigma}} a_{k,\sigma}a_{k+p,\sigma}$$

where a^*, a are the fermionic creation/annihilation operators. In (6), $\mathcal{B}_F^\sigma$ denotes the Fermi ball, i.e., $\mathcal{B}_F^\sigma := \{k \in (2\pi/L)\mathbb{Z}^3 \,|\, |k| \leq k_F^\sigma\}$, and k_F^σ is the Fermi momentum which, for fixed densities and in the limit $L \to \infty$, can be written as $k_F^\sigma = (6\pi^2)^{1/3}\rho_\sigma^{1/3} + o(1)$. The reason why we refer to the operators in (6) as *almost-bosonic operators* is that, when acting on states with few particles, they approximately behave as bosonic creation/annihilation operators (i.e., they almost satisfy the canonical commutation relations). Once these operators are introduced, the main idea is to express the relevant contributions to the correlation energy in terms of b, b^* and to diagonalise this effective energy via an (almost-bosonic)

Bogoliubov transformation, which is explicitly written as

$$T = \exp\left\{ \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} \hat{\varphi}(p) \hat{b}_{p,\uparrow} \hat{b}_{-p,\downarrow} - \text{h.c.} \right\}. \tag{7}$$

Note that, the choice of $\hat{\varphi}$ is responsible for getting the right dependence on the scattering length in the constant term we want to extract, i.e., $8\pi a \rho_\uparrow \rho_\downarrow$. In other words, $\hat{\varphi}$ is related to the scattering equation. Finally, we emphasise that since we are working directly in the thermodynamic limit, both in [3, 4] we need to introduce some localizations in order to obtain decay estimates that are not true in the original setting. More specifically, we need to use a regularised version of the almost bosonic creation/annihilation operators. In [4][1] this corresponds to discarding some momenta inside the Fermi ball, i.e., all the $k \in (2\pi/L)\mathbb{Z}^3$ in $k_F + \rho^{2/3} < |k| < k_F$ and some others outside[2], i.e., all the momenta in the annulus $k_F < |k| < 2k_F$ or $|k| > \rho^{-\beta}$. In our approach, however, we not only need to regularise the almost bosonic operators but we also need to localise $\hat{\varphi}$: it turns out that it is convenient to do this localisation in configuration space. The way we do it is different in [3] and [4]. We conclude by comparing these two different approaches. In [3] we take[3] $\varphi \equiv \varphi_\gamma$, where φ_γ is the periodization of the solution of the Neumann problem in a ball $B \equiv B_{\rho^{-\gamma}}(0) \subset \mathbb{R}^3$ centered at zero and with radius $\rho^{-\gamma}$. More precisely, φ_γ is the periodization of $\varphi_{\gamma,\infty}$ which is the solution of

$$-2\Delta(1-\varphi_{\gamma,\infty}) + V_\infty(1-\varphi_{\gamma,\infty}) = \lambda_\gamma(1-\varphi_{\gamma,\infty}), \quad \varphi_{\gamma,\infty} = 2\nabla\varphi_{\gamma,\infty} = 0 \text{ on } \partial B,$$

where $|\lambda_\gamma| \leq C\rho^{3\gamma}$. In [4], instead, $\varphi(x)$ is taken to be the periodization of a localized version of the solution of the zero energy scattering equation in $\mathbb{R}^3$, which reads as

$$\varphi_\infty(x) := \varphi_0(x)\chi(x/\rho^{-1/3}), \quad 2\Delta\varphi_0 V(1-\varphi_0) = 0, \quad \varphi_0(x) \to 0 \text{ as } |x| \to \infty,$$

where χ is a smooth cut-off function in $\mathbb{R}^3$, which varies smoothly between 0 and 1 in the annulus $\rho^{-1/3} \leq |x| \leq 2\rho^{-1/3}$. As a consequence of our localization, $\varphi_\infty = \varphi_0$ in the support of the interaction potential V_∞ and it is such that φ_∞

$$2\Delta\varphi_\infty + V_\infty(1-\varphi_\infty) \sim -2a\left[\frac{2\rho^{\frac{1}{3}}}{|x|^2} + \frac{\rho^{\frac{2}{3}}}{|x|}\right]\chi(|x| \sim \rho^{-\frac{1}{3}}),$$

where a is the scattering length of the interaction potential. This different way of doing the localization allows us to better estimate many error terms recovering the optimal estimate in the proof of the upper bound in Theorem 1.

[1] In [3] a different choice of the cut-off is used.
[2] The smoothness of the interaction potential is needed to justify the ultraviolet cut-off.
[3] In [3], we take $\gamma = 2/9$ for the upper bound and $\gamma = 1/3$ in the lower bound.

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A tracer particle interacting with a cold quantum gas

Peter Pickl
(joint work with Viet Hoang, Maximilian Jeblick, Jonas Lampart, David Mitroskas, Sören Petrat)

The computation of effective equations in many body systems is an interesting area of research and was the topic of the Oberwolfach Min-Workshop where this presentation was held. In the talk I will present recent findings on the dynamics of a so called tracer particle entering a cold quantum gas. Both cases, gases made of Fermions and of Bosons, will be considered. The question is of physical interest, since the influence of the gas on the dynamics of the tracer can be used to gain information of the underlying interactions of the system. In the Bosonic case it has been shown recently, that – assuming constant density of the gas and a respective scaling of the interaction of the gas particles with the tracer – the system is effectively described by the Bogoliubov-Fröhlich Hamiltonian: the interaction of the tracer will be of leading order influenced by the Bogoliubov excitations in the gas [1] and excite itself further particles of a similar number as the Bogoliubov excitations. In the Fermionic case the rigidity of the Fermi-ball plays an important role for the dynamics of the tracer. It suppresses the effective interaction significantly and leads to free evolution of the tracer for relatively strong couplings [2] respectively an effective interaction between tracers when shooting more than one tracer particle into the gas [3]. At the end of the talk, new, so far unpublished findings for the Fermi gas will be presented. Together with Viet Hoang we could show that, increasing the tracer-gas coupling, the leading order dynamics will be given by a tracer coupled to a phonon field. The phonons describe the pair-excitations in the gas and behave like Bosons.

Possible extensions and open questions are the generalization of these findings to systems of large volumes. Further it should, at least in the case of an absence of interaction within the gas, be possible to extend the time scales for which one can proof validity of the effective descriptions to times polynomial in the density rather than logarithmic. On large time scales, new physical phenomena should become visible.

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Some mathematical results in Density Functional Theory

Mathieu Lewin

(joint work with Elliott H. Lieb and Robert Seiringer)

I review some mathematical results in Density Functional Theory (DFT) following [4]. Consider N electrons in $\mathbb{R}^3$ and assume that they have the one-particle density ρ (a non-negative function such that $\int_{\mathbb{R}^3} \rho(x)\,dx = N$). *Lieb's functional* [5] provides the lowest possible energy of these electrons at the given ρ:

$$F(\rho) := \inf_{\rho_\Gamma=\rho} \operatorname{tr}\left(H^N(0)\Gamma\right). \tag{1}$$

The infimum is over all N-particle mixed states Γ on $\bigwedge_1^N L^2(\mathbb{R}^3,\mathbb{C}^2)$ having density $\rho_\Gamma = \rho$. Here

$$H^N(V) := \sum_{j=1}^N (-\Delta_{x_j} + V(x_j)) + \sum_{1\le j<k\le N} \frac{1}{|x_j - x_k|}$$

is the usual Coulomb N-particle Hamiltonian in an external potential V — in (1) we took $V \equiv 0$. The main interest of $F(\rho)$ is that the ground state energy in any external potential V can be expressed as

$$\min \sigma(H^N(V)) = \inf_{\substack{\rho \\ \int_{\mathbb{R}^3}\rho=N}} \left\{ F(\rho) + \int_{\mathbb{R}^3} \rho(x)V(x)\,dx \right\}. \tag{2}$$

This minimization problem is settled in the physical space $\mathbb{R}^3$ and not in the N-particle space $\mathbb{R}^{3N}$. If we knew how to compute $F(\rho)$, this would dramatically decrease the computational cost of the ground state energy. Unfortunately, $F(\rho)$ is a highly nonlinear and nonlocal unknown functional which is defined in terms of N-particle states. The cost of computing $F(\rho)$ is probably at least as high as solving Schrödinger's equation. The purpose of (orbital-free) DFT is to provide simple but efficient approximations of $F(\rho)$.

The most famous is the *Local Density Approximation* (LDA), where it is assumed that ρ is locally flat and the local energy is taken to be that of an infinite gas of constant density, per unit volume:

$$F(\rho) \approx F_{\rm LDA}(\rho) := \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}dx\,dy + \int_{\mathbb{R}^3} f\big(\rho(x)\big)\,dx,$$

where $f:\mathbb{R}_+\to\mathbb{R}$ is the energy per unit volume of the infinite uniform electron gas. The LDA was rigorously justified for the first time in [3], for the grand-canonical version of $F(\rho)$. It is an open problem to justify the LDA for the canonical functional $F(\rho)$.

Another regime is the *large density limit*, where the kinetic energy dominates and the system becomes non-interacting. This limit can be stated as

$$\lim_{\lambda\to\infty}\frac{F(\rho_\lambda)}{\lambda^2} = T(\rho), \qquad \rho_\lambda(x) = \lambda^3\rho(\lambda x).$$

The kinetic energy can be expressed in terms of the one-particle density matrix γ (a trace-class operator on $L^2(\mathbb{R}^3)$) as

$$T(\rho) = \inf_{\substack{\Gamma\\ \rho_\Gamma=\rho}} \operatorname{tr}\Big(\sum_{j=1}^N(-\Delta)_{x_j}\Big)\Gamma = \inf_{\substack{0\le\gamma=\gamma^*\le 1\\ \rho_\gamma=\rho}}\operatorname{tr}(-\Delta)\gamma. \tag{3}$$

Let us now discuss some known bounds on $T(\rho)$ and work in any dimension $d\geq 1$. The Hoffman-Ostenhof inequality state that

$$T(\rho)\geq \int_{\mathbb{R}^d}|\nabla\sqrt{\rho}(x)|^2\,dx.$$

This implies that $\sqrt{\rho}\in H^1(\mathbb{R}^d)$ is a necessary condition for $T(\rho)$ to be finite. It was proved in [5] that this is also a sufficient condition. For the proof one takes as trial state the Slater determinant $\Psi = (N!)^{-1/2}\det(\phi_j(x_k))$ with $\phi_j = \sqrt{\rho/N}e^{i\theta_j(x)}$. The phases θ_j are chosen so that the ϕ_j are orthonormal, with $\int_{\mathbb{R}^d}\rho|\nabla\theta_j|^2<\infty$. It is very hard to construct such phases and get good bounds. In [5] Lieb got

$$T(\rho)\leq\left(\frac{\pi^2}{3}N^2+CN\right)\int_{\mathbb{R}^d}|\nabla\sqrt{\rho}(x)|^2\,dx.$$

This blows up quite fast with N. The growth was later improved to the optimal rate $CN^{2/d}$ in [1]. In fact, an extensive bound cannot only involve gradients. It should at least also include the semi-classical approximation of $T(\rho)$,

$$T(\rho)\approx c_{\rm TF}\int_{\mathbb{R}^d}\rho(x)^{1+\frac{2}{d}}\,dx, \qquad c_{\rm TF} = \frac{\pi^2 d}{(d+2)q^{2/d}}\left(\frac{d}{|\mathbb{S}^{d-1}|}\right)^{\frac{2}{d}}$$

where $c_{\rm TF}$ is the semi-classical (a.k.a. Thomas-Fermi) constant. Here q is the number of spin states, which is $q=2$ for electrons. Choosing the phases θ_j appropriately, March and Young [6] had already obtained in dimension $d=1$

$$T(\rho)\leq\frac{\pi^2}{12}\int_{\mathbb{R}}\rho(x)^3\,dx+\int_{\mathbb{R}}\Big|\big(\sqrt{\rho}\big)'(x)\Big|^2\,dx. \tag{4}$$

The first constant is just $c_{\rm TF}$ in $d = 1$, which led them to conjecture the following inequality in any dimension

$$T(\rho) \overset{?}{\leq} c_{\rm TF} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2}{d}}\, dx + C \int_{\mathbb{R}^d} |\nabla\sqrt{\rho}(x)|^2\, dx, \tag{5}$$

This conjecture is still open in dimension $d \geq 2$, but recent works came arbitrarily close to the result in the following sense.

Theorem 1 (Semi-classical estimates on the kinetic energy functional)**.** *Let $d \geq 1$. There exists a constant $C = C(d)$ such that*

$$\begin{aligned} c_{\rm TF} e^{-\varepsilon} \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}} - \frac{C}{\varepsilon}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2 &\leq T(\rho) \\ &\leq c_{\rm TF}(1+\varepsilon)\int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}} - \frac{C(1+\varepsilon)}{\varepsilon}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2 \end{aligned} \tag{6}$$

for any $\varepsilon > 0$ and any $\rho \geq 0$ with $\sqrt{\rho} \in H^1(\mathbb{R}^d)$.

The lower bound was first proved in [7] but with the coefficient $C/\varepsilon^{3+4/d}$ in front of the second term. The upper bound was shown in [3], using the simple trial state

$$\gamma = \int_0^\infty \sqrt{\eta\left(\frac{t}{\rho(x)}\right)}\, \mathbb{1}\left(-\Delta \leq \frac{d+2}{d} c_{\rm TF} t^{\frac{2}{d}}\right) \sqrt{\eta\left(\frac{t}{\rho(x)}\right)}\, \frac{dt}{t},$$

where $\eta \in C_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$ is so that $\int_0^\infty \eta(t)\, dt = 1$ and $\int_0^\infty t^{-1}\eta(t)\, dt \leq 1$. Here the two functions $\sqrt{\eta(t/\rho(x))}$ are interpreted as multiplication operators. The main idea is to locate the places where $\rho(x) \approx t$ using the cut-off function η and to place a free Fermi gas of density t there. Concentrating η about 1 at scale ε one obtains the upper bound in (6). This idea was recently pursued in [8] to also provide the stated lower bound. In fact, taking ε large enough and using the Hofmann-Ostenhof inequality provides a very simple proof of the Lieb-Thirring inequality. If we scale a density ρ as $\rho(\hbar x)$ and take $\varepsilon = \hbar$ in(6), we find

$$T(\rho_\hbar) = c_{\rm TF}\hbar^{-d} \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}} + O(\hbar^{-d+1}).$$

An interesting open problem is to justify the next order, which is predicted to be

$$\frac{d-2}{3d}\hbar^{-d+2} \int_{\mathbb{R}^d} |\nabla\sqrt{\rho}(x)|^2\, dx.$$

The negativity of the coefficient in $d = 1$ is related to the non-optimality of the semi-classical constant for the Lieb-Thirring inequality in dimension $d = 1$ [2].

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Spectral estimates for Schrödinger operators with Neumann boundary conditions on Hölder domains

Charlotte Dietze

Netrusov and Safarov proved Weyl's law

$$N\left(-\Delta_\Omega^N-\lambda\right)=\frac{|B_1^d(0)|}{(2\pi)^d}|\Omega|\lambda^{\frac{d}{2}}+o\left(\lambda^{\frac{d}{2}}\right)\text{ as }\lambda\to\infty, \tag{1}$$

for γ-Hölder domains Ω with Neumann boundary conditions for all Hölder exponents $\gamma\in\left(\frac{d-1}{d},1\right)$ [1, Corollary 1.6]. They also showed that Weyl's law fails for all $\gamma\in\left(0,\frac{d-1}{d}\right]$. More precisely, for those γ, there exists a γ-Hölder domain Ω such that (1) is not true [1, Theorem 1.10].

We consider Weyl's law for Schrödinger operators on Hölder domains Ω

$$N\left(-\Delta_\Omega^N+\lambda V\right)=(2\pi)^{-d}\left|B_1^d(0)\right|\lambda^{\frac{d}{2}}\int_\Omega|V|^{\frac{d}{2}}+o\left(\lambda^{\frac{d}{2}}\right)\text{ as }\lambda\to\infty, \tag{2}$$

where $V:\Omega\to(-\infty,0]$. In view of [1], one might expect that (2) holds for all $\gamma\in\left(\frac{d-1}{d},1\right)$ and $V\in L^{d/2}(\Omega)$. For every $\gamma\in\left(\frac{d-1}{d},1\right)$ we give an explicit example for a γ-Hölder domain Ω, and $V\in L^{d/2}(\Omega)$, where (2) fails [2, Theorem 1.1]. However, if we assume more integrability on V, namely that it is in some weighted L^p-space for some $p=p(d,\gamma)>d/2$, we prove (2) [2, Theorem 1.3].

The proof of (2) relies on a Cwikel-Lieb-Rozenblum-type bound for the number of negative eigenvalues of the Schrödinger operator $-\Delta_\Omega^N+V$ [2, Theorem 1.2]. In the proof of this Cwikel-Lieb-Rozenblum-type bound, we use a new covering theorem, and a new Poincaré-Sobolev inequality for suitably chosen small rectangles intersected with Ω.

Further details and explanations can also be found in [3].

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The Random Batch Method for Quantum Dynamics

François Golse

(joint work with Shi Jin & Thierry Paul)

Consider the quantum Hamiltonian for a system of N identical particles

$$\mathcal{H}_N := \sum_{m=1}^{N} -\tfrac{1}{2}\hbar^2\Delta_{x_m} + \tfrac{1}{N-1}\sum_{1\le l<n\le N} V(x_l - x_n)\ ,$$

where V is an even, real-valued function. The cost of computing the interaction potential is $\frac{1}{2}N(N-1)$ evaluations of V and additions.

The Random Batch Method (RBM) is, at each time step (1) to replace the total interaction of each particle with the $N-1$ other particles by the interaction with $p \ll N$ other particles chosen at random multiplied by $(N-1)/p$ (with a computing cost $Np \ll \frac{1}{2}N(N-1)$ operations), and (2) to reshuffle the particles at each time step (with a computating cost $O(N)$ by Durstenfeld's algorithm [3]).

1. Formulation of the RBM: Case $p = 2$

Let $N \ge 2$ be an even integer. Let $\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots$ be a sequence of random mutually independent elements of $\mathfrak{S}_N$ distributed uniformly. Given $\Delta t > 0$, define

$$\mathbf{T}_t(l,n) := \begin{cases} 1 \text{ if } \{l,n\} = \{\sigma_{[t/\Delta t]+1}(2k-1), \sigma_{[t/\Delta t]+1}(2k)\} \text{ for some } k = 1, \ldots \frac{N}{2}\ , \\ 0 \text{ otherwise.} \end{cases}$$

and

$$\tilde{\mathcal{H}}_N(t) := \sum_{m=1}^{N} -\tfrac{1}{2}\hbar^2\Delta_{x_m} + \sum_{1\le l<n\le N} \mathbf{T}_t(l,n) V(x_l - x_n)\ .$$

The RBM dynamics is defined by the Cauchy problem

$$i\hbar\partial_t \tilde{R}_N(t) = [\tilde{\mathcal{H}}_N(t), \tilde{R}_N(t)]\ , \qquad \tilde{R}_N(0) = R_N^{in}\ .$$

In the sequel, we seek to compare

$$R_N(t) := e^{-it\mathcal{H}_N/\hbar} R_N^{in} e^{it\mathcal{H}_N/\hbar}\ , \quad \text{and} \quad \tilde{R}_N(t)\ .$$

2. Convergence of the RBM

The convergence of the RBM is couched in terms of the Wigner function. Any trace-class operator S on $\mathfrak{H}=L^2(\mathbf{R}^d)$ is defined by an integral kernel $s\equiv s(x,y)$ such that (see Lemma 2.1 in [1])

$$z\mapsto[x\mapsto s(x+z,x-z)]\in C_b(\mathbf{R}^d_z;L^1(\mathbf{R}^d_x))\,.$$

Its Wigner function is the Fourier transform (in $\mathcal{S}'(\mathbf{R}^d\times\mathbf{R}^d)$)

$$W_\hbar[S](x,\xi):=\tfrac{1}{(2\pi)^d}\int_{\mathbf{R}^d}s(x+\tfrac12\hbar y,x-\tfrac12\hbar y)e^{-i\xi\cdot y}dy\,.$$

We shall also need the dual norm

$$|||f|||_{-m}:=\sup\left\{\left|\int_{\mathbf{R}^{2d}}f(z)\overline{a(z)}dz\right|\,:\,a\in C_c^\infty(\mathbf{R}^{2d})\text{ and }\max_{0<|\alpha|\le m}\|\partial^\alpha a\|_{L^\infty}\le 1\right\}.$$

Finally, the notion of symmetrized 1-particle marginal $\tilde R_{N:1}(t)$ of $\tilde R_N(t)$ is defined as follows: for all $A\in\mathcal{L}(\mathfrak{H})$,

$$\operatorname{tr}_{\mathfrak{H}}(\tilde R_{N:1}(t)A):=\tfrac1N\sum_{k=1}^N\operatorname{tr}_{\mathfrak{H}_N}(\tilde R_N(t)J_kA)$$

where $\mathfrak{H}_N=\mathfrak{H}^{\otimes N}=L^2(\mathbf{R}^{dN})$ and

$$J_kA:=I_{\mathfrak{H}}^{\otimes(k-1)}\otimes A\otimes I_{\mathfrak{H}}^{\otimes(N-k)}\,.$$

Theorem. [4] *Assume that V is an even real-valued function on $\mathbf{R}^d$ with Fourier transform $\hat V\in L^2(\mathbf{R}^d;(1+|\omega|^2)d\omega)$, while $(R_N^{in})^*=R_N^{in}\ge 0$ satisfies $\operatorname{tr}_{\mathfrak{H}_N}R_N^{in}=1$. Then, for all $\Delta t,\hbar\in(0,1)$, all even $N\ge 2$ and all $t\ge 0$,*

$$\begin{aligned}&|||W_\hbar[\mathbf{E}\tilde R_{N:1}(t)-R_{N:1}(t)]|||_{-[d/2]-3}\\ &\le 5\Delta t\cdot\gamma_dL(V)(1+(1+2\sqrt d)L(V)t)e^{6t\max(1,\sqrt d L(V))}\,,\end{aligned}$$

where $L(V)=(2\pi)^{-d}\|(d+|\omega|^2)\hat V\|_{L^1}$.

This result proves the convergence of 1-particle observables for the RBM as the reshuffling time step $\Delta t\to 0$, uniformly in the particle number N and in the Planck constant $\hbar$. We have treated only the case $p=2$; larger values of p can be handled in essentially the same manner — in fact, one expects that the larger p, the better the RB approximation will be.

3. Metrizing the Set of Density Operators

Let $\mathcal{D}(\mathfrak{H})=\{R=R^*\in\mathcal{L}(\mathfrak{H})\,:\,R\ge 0\text{ and }\operatorname{tr}_{\mathfrak{H}}(R)=1\}$ the set of density operators on $\mathfrak{H}$. For $R,S\in\mathcal{D}(\mathfrak{H})$, set

$$d_\hbar(R,S):=\sup_{A\in\mathcal{L}(\mathfrak{H})}\{|\operatorname{tr}_{\mathfrak{H}}((R-S)A)|\,:\,\sup_{1\le j,k\le d}SC_{j,k}(A)\le 5\hbar^2\}\,,$$

where

$$\begin{aligned}SC_{j,k}(A):=&\hbar\|[x^j,A]\|+\hbar\|[-i\partial_{x^j},A]\|+\|[x^k,[x^j,A]]\|\\ &+\|[-i\hbar\partial_{x^k},[x^j,A]]\|+\|[-i\hbar\partial_{x^k},[-i\hbar\partial_{x^j},A]]\|\,.\end{aligned}$$

Proposition. [4]
(1) The functional $d_\hbar : \mathcal{D}(\mathfrak{H}) \times \mathcal{D}(\mathfrak{H}) \to [0, +\infty]$ *is an extended metric.*
(2) There exists $\gamma_d > 0$ *depending only in the space dimension* d *such that*

$$|||W_\hbar[R - S]|||_{-[d/2]-3} \leq \gamma_d d_\hbar(R, S)\,, \qquad R, S \in \mathcal{D}(\mathfrak{H})\,.$$

The proof of this proposition is based on the duality formula

$$\int_{\mathbf{R}^{2d}} W_\hbar[T](x,\xi)\overline{a(x,\xi)}dxd\xi = \mathrm{tr}_{\mathfrak{H}}(TA^*)$$

where A is the Weyl operator with symbol a, and on the Calderón-Vaillancourt theorem [2].

The definition of $d_\hbar$ is reminiscent of the Kantorovich-Rubinstein duality formula (Theorem 1.14 in [6]) for the Monge-Kantorovich distance $\mathcal{MK}$ between Borel probability measures on $\mathbf{R}^n$ with finite first order moment

$$\mathcal{MK}(\mu,\nu) := \sup_{\mathrm{Lip}(\phi)\leq 1} \left| \int_{\mathbf{R}^n} \phi(z)\mu(dz) - \int_{\mathbf{R}^n} \phi(z)\nu(dz) \right|$$

The quantum analogue of this metric for $R, S \in \mathcal{D}(\mathfrak{H})$ is

$$MK_\hbar(R,S) := \sup_{A\in\mathcal{L}(\mathfrak{H})} \{|\mathrm{tr}_{\mathfrak{H}}((R-S)A)| \,:\, \sup_{1\leq j\leq d} \widetilde{SC}_j(A) \leq \hbar\}\,,$$

where

$$\widetilde{SC}_j(A) = \max(\|[x^j, A]\|, \|[-i\hbar\partial_{x^j}, A]\|)\,.$$

The analogy comes from the correspondence principle, which says that

$$\tfrac{i}{\hbar}[x^j, \cdot] \to \{x^j, \cdot\} = -\partial_{\xi^j}\,, \quad \tfrac{i}{\hbar}[-i\hbar\partial_{x^j}, \cdot] \to \{\xi^j, \cdot\} = \partial_{x^j}\,,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined on pairs of C^1 functions on phase space, while ξ^j is the j-th component of the classical momentum variable ξ, which is conjugate to the jth position coordinate x^j.

The proof of the theorem of convergence of the RBM is based on proving that

$$d_\hbar(\mathbf{E}\tilde{R}_{N:1}(t), R_{N:1}(t)) \leq 5\Delta t L(V)(1 + (1 + 2\sqrt{d})L(V)t)e^{6t\max(1,\sqrt{d}L(V))}$$

by a duality argument. In the course of the proof, one needs to control commutators with the interaction potential V in terms of $d_\hbar$. This is done with the following lemma.

Lemma. *Let* $f \equiv f(x)$ *such that* $\hat{f}$ *and* $\widehat{\partial_x f}$ *belong to* $L^1(\mathbf{R}^d)$*. Then*

$$\|[f, T]\| \leq \max_{1\leq j\leq d} \|[x^j, T]\| \cdot \tfrac{1}{(2\pi)^d} \sum_{j=1}^{d} \|\widehat{\partial_{x^j} f}\|_{L^1}\,.$$

For a quick proof of this inequality, see formula (55) in [5].

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Magnetic properties of ground states in many-electron systems

Tadahiro Miyao

(joint work with K. Nishimata, H. Tominaga)

In crystals, electrons exhibit the following fundamental properties: (i) Fermi statistics, (ii) spin, (iii) Coulomb repulsion, and (iv) itinerancy. Explaining ferromagnetism in metals solely based on these properties remains a significant goal in condensed matter physics.

In 1963, Gutzwiller, Kanamori, and Hubbard proposed a simple model to describe electrons on a crystal lattice and analyzed the magnetic properties of the ground state [1, 2, 3]. This model, known today as the Hubbard model, is one of the simplest models that captures the four fundamental properties mentioned earlier.

Subsequently, various studies, including numerical calculations, have been conducted on this model in theoretical physics. However, a precise explanation of the quantum origin of metallic ferromagnetism remains incomplete.

In this talk, I will first explain fundamental issues in the rigorous analysis of metallic ferromagnetism. Next, I will elaborate on the basic theorems in this field, namely the Marshall–Lieb–Mattis theorem [5, 6], Lieb's theorem [4], and their stability [9, 10]. After revealing the similar structures inherent in these three theorems, I will consider the following problem: constructing a unified mathematical theory that can describe them all. In this talk, I will formulate this theory using the standard form of von Neumann algebras. As a result, I will establish the existence of a set of Hamiltonians $\mathcal{C}_{\rm MLM}$ possessing the following properties [11, 13]:

(1) All ground states of Hamiltonians in $\mathcal{C}_{\rm MLM}$ exhibit properties akin to those stated in the Marshall–Lieb–Mattis theorem for the ground states of Heisenberg models.
(2) $\mathcal{C}_{\rm MLM}$ contains a countably infinite number of elements.

We refer to $\mathcal{C}_{\rm MLM}$ as the Marshall–Lieb–Mattis (MLM) stability class. The MLM stability class enables the explanation of the magnetic properties of ground states in half-filling many-electron systems on bipartite lattices. For instance, it is demonstrated that Hamiltonians describing systems where many electrons interact with phonons or photons belong to $\mathcal{C}_{\mathrm{MLM}}$. Consequently, the magnetic properties of the ground states of these Hamiltonians satisfy the aforementioned property (1), see [9, 10, 11]. This implies the stability of magnetic properties in the ground states under electron-lattice interactions.

In addition to the MLM stability class, several stability classes can be constructed, such as the Nagaoka–Thouless stability class [11, 13, 14]. These stability classes are determined by factors such as the filling factor, crystal structure, and Coulomb interaction, each corresponding to different scenarios in many-electron systems. Each stability class describes the stability of magnetic states in many-electron systems in different situations.

A modified version of the MLM stability class, known as the deformed MLM stability class, encompasses models such as the Kondo lattice model and the periodic Anderson model. Moreover, it includes Hamiltonians derived by introducing electron-phonon interactions to these models [12, 15]. Consequently, the stability of magnetic properties in the ground states of the Kondo lattice model and periodic Anderson model, influenced by these interactions, can be expounded through the attributes of this deformed MLM stability class.

In this talk, due to time constraints, I will elucidate findings pertaining to finite lattice systems. For discussions on the infinite volume limit using conditional expectations between von Neumann algebras, please refer to [13].

In addition to the rigorous results mentioned here, the flat-band ferromagnetism theory proposed by Mielke and Tasaki is considered to describe more realistic many-electron systems [7]. A methodology for constructing stability classes describing flat band ferromagnetism is becoming evident, and I am presently engaged in the detailed investigation of this phenomenon.

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The effective mass problem for the classcial polaron

Simone Rademacher
(joint work with Dario Feliciangeli and Robert Seiringer)

The polaron is a quasi-particle that models an electron in a charged crystal. While moving through the crystal, the electron interacts with its self-induced polarization field that is mathematically either described by a quantum field (Fröhlich model) or by a classical field (Landau-Pekar equations). Here we consider the classical field description: For that we consider a pair $(\psi, \varphi) \in H^1(\mathbb{R}^3) \times L^2_\varepsilon(\mathbb{R}^3)$ where ψ denotes the L^2-normalized wave function of the electron and φ the polarization field that is an element of the weighted L^2-space

$$L^2_\varepsilon(\mathbb{R}^3) := \{ f : \mathbb{R}^3 \to \mathbb{C} | \int \varepsilon(k) |f(k)|^2 dk < \infty \} \tag{1}$$

for a positive function $\varepsilon : \mathbb{R}^3 \to \mathbb{R}_+$. The dynamics of the classical polaron is given by the solution (ψ_t, φ_t) to the Landau-Pekar (LP) equations

$$i\partial_t \psi_t = h_{\sqrt{\alpha}\varphi_t} \psi_t, \quad i\varepsilon^{-1}(k)\, \partial_t \varphi_t(k) = \varphi_t(k) + \sqrt{\alpha} \sigma_{\psi_t}(k) \tag{2}$$

where $\alpha > 0$ denotes the coupling constant and

$$h_\varphi := -\tfrac{\Delta}{2m} + 2\text{Re} \int v(k) \varphi(k) e^{ik\cdot x} dk, \quad \sigma_\psi(k) = (2\pi)^{3/2} \frac{v(k)}{\varepsilon(k)} \widehat{|\psi|^2}(k) \tag{3}$$

for some $v : \mathbb{R}^3 \to \mathbb{R}$. Landau and Pekar [5, 6, 9] first described the classical polaron in the strong coupling regime, $\alpha \to \infty$, by (2) with the choice

$$v(k) = |k|^{-1}, \quad \varepsilon = 1 \tag{4}$$

for the form factor resp. the dispersion of the underlying medium.

The dynamics of the polaron is closely related to the polaron's effective mass: While interacting with the self-induced polarization field, the electron slows down; and thus the polaron's effective mass increases. Landau and Pekar [6] formulated a quantitative conjecture on the effective mass of the polaron in the strong coupling regime, whose mathematical verification is an outstanding open problem. For the effective mass problem of the quantum (Fröhlich) polaron there is recent progress for lower [1, 2, 10] and upper bounds [3] improving earlier results [11].

Here we address the effective mass problem for the classical polaron given by (2) as originally studied by Landau and Pekar. The heuristic arguments of Landau and Pekar are based on traveling wave solutions to (2) that are given for $\mathrm{v} \in \mathbb{R}^3$ by initial states $(\psi_{\mathrm{v}}, \varphi_{\mathrm{v}})$ such that

$$(\psi_t(x), \varphi_t(k)) = (e^{i\omega_{\mathrm{v}} t}\psi_{\mathrm{v}}(x - \mathrm{v}t), e^{ik\cdot \mathrm{v}}\varphi_{\mathrm{v}}(k)) \tag{5}$$

solves the Landau-Pekar equations (2) for some phase factor $\omega_{\mathrm{v}} \in \mathbb{R}$. Due to a vanishing speed of sound for the choice (4) we, however, conjecture that there are no traveling wave solutions for the classical polaron with the choice (4).

Considering a regularized polaron model with non-vanishing speed of sound, i.e. considering the dispersion ε of the underlying medium such that

$$\mathrm{v}_c := \inf_k \varepsilon(k)/|k| > 0 \tag{6}$$

for $\mathrm{v} \leq \mathrm{v}_c$, I prove [8] that there exist traveling wave solutions of the form (5):

Theorem 1. [8] *Let* ε *satisfy* (6) *and* $v/\varepsilon^{1/2} \in L^2_{(|k|+1)^4}(\mathbb{R}^3)$, $v/\varepsilon^{1/2}(k) \geq |k|^{-1/4}$. *For* $|\mathrm{v}| \leq \mathrm{v}_c$ *there exist traveling wave solutions of the form* (5)*.*

In this case, the heuristic arguments of Landau and Pekar can be made rigorous and the effective mass of the classical regularized polaron can be defined through an energy-velocity expansion of sub-sonic traveling waves, i.e.

$$(\psi_{\mathrm{v}}, \varphi_{\mathrm{v}}) \quad \text{satisfying (4) with} \quad |\mathrm{v}| \leq \mathrm{v}_c, \;\; \omega_{\mathrm{v}} \geq -e_\alpha + \mathrm{v}^2/4 \tag{7}$$

that have low energy, i.e. such that

$$\mathcal{G}(\psi_{\mathrm{v}}, \varphi_{\mathrm{v}}) = \langle \psi_{\mathrm{v}}, h_{\varphi_{\mathrm{v}}}\psi_{\mathrm{v}}\rangle + \|\varphi\|^2_{L^2_\varepsilon} < e_\alpha + \kappa \tag{8}$$

for sufficiently small $\kappa > 0$ and where $e_\alpha = \inf_{\psi,\varphi} \mathcal{G}(\psi, \varphi)$. I prove the following expansion for states of the set

$$\mathcal{I}_{\mathrm{v}} := \{(\psi_{\mathrm{v}}, \varphi_{\mathrm{v}}) \in H^1(\mathbb{R}^3) \times L^2_\varepsilon(\mathbb{R}^3) \mid (7), (8) \text{ hold } \} \,. \tag{9}$$

Theorem 2. [8] *Let* ε *satisfy* (6) *and* $v/\varepsilon^{1/2} \in L^2_{(|k|+1)^4}(\mathbb{R}^3)$, $v/\varepsilon^{1/2} \geq |k|^{-1/4}$. *Then for all* $\alpha \geq \alpha_0$*, and* $\alpha \ll 1$*, it is*

$$E_{\mathrm{v}} := \inf\nolimits_{\mathcal{I}_{\mathrm{v}}} \mathcal{G}(\psi, \varphi) = e_\alpha + \frac{m_{\text{eff}}\mathrm{v}^2}{2} + O(\alpha \mathrm{v}^3) \,. \tag{10}$$

The constant m_{eff} is explicitely given and, in particular, satisfies in the strong coupling limit

$$\lim_{\alpha\to\infty} \alpha^{-1} m_{\text{eff}} = \lim_{\alpha\to\infty} \alpha^{-1} \lim_{\mathrm{v}\to 0} \frac{E_{\mathrm{v}} - e_\alpha}{\mathrm{v}^2/2} = \frac{2(2\pi)^3}{3}\|kv\varepsilon^{-3/2}\|_2^2 \tag{11}$$

that agrees with findings for the effective mass the regularized (quantum) Fröhlich model [7]. Moreover, I prove in [8] that alternatively the effective mass can be defined based on an energy-momentum expansion of low-energy states with fixed total momentum.

However, for the Landau-Pekar equations with the choice (4) for the dispersion relation resp. the form factor, as originally considered by Landau and Pekar, non of these two approaches work (the reason is related to the vanishing speed of sound in this case). In [4] we therefore provide a novel approach for the definition of the effective mass based on an energy-velocity expansion for solutions to the LP equations with that gives a first verification the conjecture of Landau and Pekar for the classical polaron.

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Hubbard Models, Fermi Liquids, and Renormalization

Manfred Salmhofer

I review some ideas and results in mathematical condensed-matter physics, specifically about many-electron models for metals, magnets, and superconductors, along the lines of [1].

A prototypical such model is the *Hubbard model*, a quantum many-body system on a square or cubic lattice, introduced independently by Gutzwiller, Hubbard, and Kanamori [4] in the late 1950s. The particles obey Fermi statistics, their kinetic term is the discrete Laplacian, and their interaction is an on-site repulsion. More general Hubbard-type models involve different lattices, general short-range hopping amplitudes in the kinetic term, and also more general short-range interactions that may also be attractive. Since the late 1980s these models have received enormous attention as microscopic models for high-temperature superconductors [2, 3]. Because the high-T_c materials have a layered structure, where hopping amplitudes between layers are at least one order of magnitude smaller than those within a layer, the Hubbard model on a two-dimensional lattice is of particular interest.

For fermions on a finite lattice, the Fock space is finite-dimensional. Thus the Hamiltonian H is bounded below, the ground state is well-defined, and so is the thermal state of quantum statistical mechanics in finite volume, as a positive linear functional on the fermionic C^* algebra, given by the appropriately normalized trace with $e^{-\beta(H-\mu N)}$, where $\beta > 0$ is the inverse temperature (the grand canonical ensemble, with the number operator N and the chemical potential μ fixing the average density). The interest lies in finding and proving statements that hold in the thermodynamic limit where the volume becomes infinite, or that are uniform in volume at large volumes.

In spite of the simplicity of the Hamiltonian, many of the properties of the ground state and the thermal state remain controversial. Some remarkable rigorous results, in particular about magnetism in these models, are reviewed in [4, 5].

In the physical application, the interaction is very often strong, i.e. the typical two-body interaction energy is much larger than the band width defined by the kinetic term. But it is already very nontrivial to treat the weakly coupled case, which arises from noninteracting Fermi gases at positive particle density when a weak, short-range interaction is included. The positive density implies that the Fermi gas has an extended Fermi surface, in particular the spectrum of the kinetic energy operator is gapless in the infinite-volume limit.

This property is essential for much of the phenomenological importance of these models – the presence of a Fermi surface is the basis for metallic behaviour – but it also presents an essential mathematical difficulty, in that naive perturbation theory diverges at zero temperature and gives a wrong temperature dependence at small positive temperatures. This necessitates renormalization to give a rigorous treatment of interaction effects.

It is a fundamental question whether the low-lying excitations of the weakly interacting system, i.e. the states energetically just above the ground state, have the character of fermionic quasiparticles, which very loosely speaking means that the states in Hilbert space correspond to wave packets with a small damping, which satisfy Fermi statistics. Landau's *Fermi liquid (FL) theory* asserts that a rather general class of fermion systems (which also includes ones with strong interactions) has this property [6, 7]. From the mathematical point of view, FL theory remained largely conjectural for some time, partly because a precise definition of a Fermi liquid is not straightforward, partly because of the difficulty of the problem. In one dimension, the exact solution of the Luttinger model shows that FL theory is not valid. This was proven to extend to Hubbard-type models in one dimension in [8, 9]. FL theory was very successful in many three-dimensional fermion systems, but its limitations became obvious in the high-T_c materials, which exhibit striking deviations from the predictions of FL theory, such as a linear rise of electrical resistivity as a function of temperature above the critical temperature for superconductivity.

A mathematically precise condition for a weakly interacting Fermi system to be a Fermi liquid at positive temperature was formulated in [10], as follows. The quantum-field theoretical fermionic two-point function of a Fermi gas has Fourier transform $\hat{C}(\omega, k) = (i\omega - e(k))^{-1}$, where ω is an odd multiple of $\frac{\pi}{\beta}$, $e(k) = \varepsilon(k) - \mu$, and $k \mapsto \varepsilon(k)$ is the Fourier transform of the hopping amplitude. The level set $S = \{k : e(k) = 0\}$, where $\hat{C}$ becomes large, and singular in the zero-temperature limit $\beta \to \infty$, is called Fermi surface in three dimensions, and Fermi curve in two dimensions (for brevity, always referred to as the Fermi surface in the following). The system with an interaction with coupling strength λ is a Fermi liquid at sufficently low temperatures (say, $\beta > 1$) if there is $C > 0$, independent of the volume, such that (a) renormalized perturbation theory converges on the set $\mathcal{R}$ of all pairs (λ, β) satisfying $|\lambda| \log \beta < C$ and (b) on $\mathcal{R}$, the fermionic two-point function has Fourier transform $\hat{G}(\omega, k) = (i\omega - e(k) - \Sigma(\lambda, \omega, k))^{-1}$ and the *fermionic self-energy* Σ is a C^2 function of (ω, k), with sup norms of the second derivatives bounded uniformly on $\mathcal{R}$.

This condition is fine enough to separate FL from the one-dimensional Luttinger liquids: in one dimension, (a) holds but (b) fails, since already the first derivative of Σ diverges on the zero set of e (which in one dimension is a set of two points). This second-order divergence is the first indication for the anomalous decay exponents of the full solution in one dimension. In two spatial dimensions, the detailed calculation of the order-λ^2 contribution to the self-energy [14] shows that (b) is the best one can hope to get, and in the limit $\beta \to \infty$, the second derivative blows up.

The deeper motivation for condition (a) is that, because of the Kohn-Luttinger effect [11], one should not expect a Fermi system that satisfies $e(-k) = e(k)$ to be a FL at zero temperature. Specifically, for $\varepsilon(k) = k^2$ the ground state will be superconducting for any $\mu > 0$, i.e. it has off-diagonal long range order that spontaneously breaks the $U(1)$ particle number symmetry of the action. In other words, the restriction to the set $\mathcal{R}$ places the temperature β^{-1} above the critical temperature for the superconducting transition.

When analyzing the Fermi system, it becomes clear that the validity or failure of FL theory in this sense is intimately tied to the geometry of the Fermi surface S: if the Fermi surface is regular, i.e. $\nabla e(k) \neq 0$ for all $k \in S$, and if it obeys a relatively weak non-nesting condition, then the first derivatives with respect to momentum and frequency are bounded [12, 13]. (The singularity in one dimension arises because there is no curvature.) If the interior of S is strictly convex and if S is regular and positively curved, (b) holds. These properties were proven to all orders in λ in all dimensions $d \geq 2$ in [13, 14, 15]. In subsequent work, the above FL condition was proven to hold for models with positively curved Fermi surfaces in two dimensions in [17, 18, 19, 20, 21]. Conversely, it was proven not to hold in the half-filled case, where the Fermi surface is perfectly nested [22]. It was also shown that even in the absence of nesting, the presence of Van Hove singularities, i.e. points $k \in S$ where $\nabla e(k) = 0$, leads to singularities in the fermionic self-energy Σ that violate (b) [23, 24]. The regularity of Σ as a function of ω and of k is different, unlike in the one-dimensional Luttinger model [24].

Fermi systems with $e(-k) \neq e(k)$ (defined by precise conditions) were shown to be Fermi liquids in a more general sense by Feldman, Knörrer, and Trubowitz [25].

Remarkably, their proof holds in the limit of zero temperature, $\beta \to \infty$, i.e. in [25] the region $\mathcal{R}$ plays no role, and the only condition on the coupling is $|\lambda| < const.$ The intuitive reason is that the asymmetry of the band function in k removes the Cooper pairing instability, but the full proof requires more, namely rather subtle bounds on particle-hole contributions to the effective interaction. The result of [25] includes the proof that the fermionic occupation density has a discontinuity at the Fermi surface (which is never true at any positive temperature).

In all the above proofs, mathematical renormalization group methods [26, 27, 28] were used; the same methods also serve to prove an inversion theorem [16] that justifies renormalization. A variant of these methods have also been successful in theoretical physics studies. Their application to the Hubbard model in the parameter range interesting for the high-T_c materials explains the phase diagram of these systems and sheds light on the interplay of antiferromagnetic and superconducting correlations in the Hubbard model [24, 29, 30, 31, 1].

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Mini-Workshop: Standard Subspaces in Quantum Field Theory and Representation Theory

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29 October – 3 November 2023

Abstract. Real standard subspaces of complex Hilbert spaces are long known to provide the right language for Tomita-Takesaki modular theory of von Neumann algebras. In recent years they have also become an object of prominent interest in mathematical quantum field theory (QFT) and unitary representation theory of Lie groups. This workshop brought together mathematicians and physicists working with standard subspaces, particularly in QFT (construction of QFT models, characterization of entropy, information-theoretic aspects), nets of standard subspaces on causal homogeneous spaces and aspects of reflection positivity and euclidean models related to standard subspaces and modular theory.

Mathematics Subject Classification (2020): 22Exx, 46Lxx, 47Bxx.

Introduction by the Organizers

Standard subspaces originate from the theory of von Neumann algebras, where they encode the modular data of a von Neumann algebra w.r.t. a cyclic and separating (standard) vector. They can however also be defined independently of von Neumann algebras in the simple setting of a complex Hilbert space; a standard subspace is then a closed real linear subspace which contains no complex line and has dense complex linear span. In recent years it has become increasingly clear that this point of view allows for a rich and still unfolding theory that is of interest in its own right and has fascinating applications in various fields.

The mini-workshop *Standard Subspaces in Quantum Field Theory and Representation Theory* was a meeting designed to bring together researchers working with standard subspaces from different perspectives, with an emphasis on people in quantum field theory and representation theory of Lie groups.

In quantum field theory (QFT), standard subspaces serve as a means to encode localization regions in a spacetime manifold and are thus a basic aspect of any model QFT. Typical questions involve the modular data of standard subspaces belonging to particular localization regions (for massive QFT, the modular group for a double cone is still unknown), the usage of standard subspaces in the formulation of examples of interacting QFTs, the role played by standard subspaces to define an intrinsic notion of entropy, or the interplay of standard subspaces with KMS-condition and reflection positivity, which appears in reconstruction theorems for Euclidean field theories.

In the presence of a spacetime symmetry Lie group, one considers nets of standard subspaces transforming under a unitary representation of this group, which immediately explains the close link to Lie group representations. Here typical questions concern the interplay between the geometric configurations, such as wedge regions for modular flows on causal homogeneous spaces and the types of unitary group representations that can host corresponding nets of standard subspaces. Another important aspect is to detect natural finite-dimensional spaces of distribution vectors for the representations, specified by a suitable KMS condition, that are invariant under large subgroups and from which well-behaved nets of real subspaces can be constructed by a smearing process.

The mini-workshop format has turned out to be the perfect choice for discussing these questions in an efficient and productive manner. The areas of expertise of the 17 participants were close enough to allow for easy discussions, and at the same time far enough apart for learning new results, points of view and ideas from each other. For the younger participants the event also offered the highly appreciated opportunity to get to know more colleagues, discuss and present their projects, and grow their scientific networks.

Thanks to the mini-workshop format, we could also successfully implement some informal discussion sessions in addition to more typical seminar talks. In these sessions, participants presented ideas, observations and questions in an unfinished format which led to long and intense discussions between many people.

An example of such a discussion was a session on inclusions of standard subspaces. Here the main question is how to decide whether an inclusion $K \subset H$ is irreducible, and the discussion related this to questions in von Neumann algebras (split property, modular nuclearity), entropy (the boundedness of the cutting projection decides about the existence of irreducible extensions), symmetric inner functions and the distribution of their zeros (closely connected to $\dim(K' \cap H)$ in particular examples), and more.

Another group discussion was centered around positive energy representations of gauge groups. As these groups are infinite-dimensional, the highly developed finite-dimensional structure theory does not apply to these groups, but their positive energy representations appear naturally in physical models, such as Conformal Field Theory (CFT), where positive energy representations of loops groups are crucial in the construction for models, such as the $U(1)$-current and its derivatives. The extension of the elaborate geometric side of standard subspaces for finite-dimensional groups to important classes of infinite-dimensional ones is an important problem for future research.

Some further group discussions concerned the, by far not fully understood, aspect of reflection positivity and the existence of euclidean models. In this context it is not clear how the modular objects, such as modular operator and conjugation, corresponding to standard subspaces, should be represented on the euclidean side. Natural candidates involve unitary representations of the non-connected group $O_2(\mathbb{R})$, satisfying suitable positivity conditions.

The following abstracts provide an excellent picture of the current state of the art and the diverse research directions concerning various aspects of standard subspaces and their applications. Topics that appeared in several presentations were aspects of entropy (Longo, Cadamuro), deformations of second quantization processes (Lechner, Correa da Silva), reflection positivity and euclidean models (Adamo, Tanimoto), and connections between nets of standard subspaces and unitary representations (Morinelli, Ólafsson, Beltiţă, Neeb).

Acknowledgement: The organizers thank the director Prof. Dr. Gerhard Huisken, and the Oberwolfach staff for offering an outstanding environment for this workshop and support in all phases of the planning.

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Mini-Workshop: Standard Subspaces in Quantum Field Theory and Representation Theory

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Abstracts

Standard subspaces in Representation Theory

Vincenzo Morinelli
(joint work with K.-H. Neeb, G. Ólafsson)

A model in Algebraic Quantum Field Theory (AQFT) is specified by a map associating to any open region of the spacetime, its von Neumann algebra of local observables acting on a fixed complex Hilbert space $\mathcal{H}$ (the state space), satisfying fundamental quantum and relativistic assumptions as Isotony, Locality, Poincaré covariance, positivity of the energy, cyclicity of the vacuum vector for local algebras [Haa96]. One can take as an example an AQFT on Minkowski spacetime. Here, the Rindler wedge and its Poincaré transforms are fundamental localization regions called wedges. They are determined by the one-parameter group of boost symmetries (properly parametrized) that fix them as a subset of the Minkowski spacetime. The algebraic canonical construction of the free field provided by Brunetti–Guido–Longo (BGL) builds on the the wedge-boost identification, the Bisognano-Wichmann (BW) property and the PCT Theorem, cf. [BGL02]. In particular, given a particle, namely an irreducible representation of the proper Poincaré group U that is unitary on the connected component of $\mathbf{1}$ and antiunitary on the connected component of $-\mathbf{1}$, it is possibile to canonically determine the states in the Hilbert space $\mathcal{H}_U$ supporting U localized in any wedge, having as a fundamental input the unitary representation of the one-parameter group of boosts associated to the wedge and the antiunitary operator implementing the wedge reflection. For instance, consider the wedge region $W_R = \{x \in \mathbb{R}^{1+d} : |x_0| < x_1\}$, the real standard subspace[1] $\mathsf{H}(W_R) \subset \mathcal{H}_U$ of states localized in a wedge region W_R is uniquely determined as follows: let

$$\Delta^{it}_{\mathsf{H}(W_R)} := U(\Lambda_{W_R}(-2\pi t)), \qquad \text{(BW) property}$$

$$J_{\mathsf{H}(W_R)} := U(r_{W_R}), \qquad \text{(PCT) Theorem}$$

where we have that $\Lambda_{W_R}(t)x = (\cosh(t)x_0 + \sinh(t)x_1, \sinh(t)x_0 + \cosh(t)x_1, \mathbf{x})$, $j_R x = (-x_0, -x_1, \mathbf{x})$, for $x \in \mathbb{R}^{1+d}$, $\mathbf{x} \in \mathbb{R}^{d-1}$, then $\mathsf{H}(W_R) = \ker(1 - J_{\mathsf{H}(W_R)}\Delta^{\frac{1}{2}}_{\mathsf{H}(W_R)})$. Note that $S_{\mathsf{H}(\mathsf{W_R})} = J_{\mathsf{H}(\mathsf{W_R})}\Delta^{\frac{1}{2}}_{\mathsf{H}(\mathsf{W_R})}$ is the *Tomita operator* of the standard subspace $\mathsf{H}(W_R)$ (for the Tomita theory of standard subspaces we refer to [Lon08]). Then for every open region $O = \bigcap_{W \supset O} W$ with W wedge region, one can define the set of states localized in O by intersection $\mathsf{H}(O) = \bigcap_{W \supset O} \mathsf{H}(W)$. The free field net of von Neumann algebras is then constructed via second quantization, see [BGL02, LRT78].

In this presentation, we will provide an overview on the analysis developed in the last years together with K.-H. Neeb and G. Ólafsson where we generalize this one-particle picture from a geometrical perspective. The core of this analysis relies on the understanding of a deep connection between the geometry of standard

[1] a real closed subspace $\mathsf{H} \subset \mathcal{H}$ is standard if $\overline{\mathsf{H} + i\mathsf{H}} = \mathcal{H}$ and $\mathsf{H} \cap i\mathsf{H} = \{0\}$

subspaces, given by the Tomita modular operator and modular conjugation, and the geometry of specific elements in the Lie algebra of a Lie group G called Euler elements and their representation theory. This approach provides feedbacks for representation theory and for the algebraic approach to Quantum Field Theory without restrictions to second quantization models.

Let G be a connected Lie group and let h be an Euler element in its Lie algebra $\mathfrak{g}$, namely $\mathrm{ad}\, h$ is diagonalizable and $\mathrm{Spec}(\mathrm{ad}\, h) = \{-1, 0, 1\}$, then $\tau_h = \exp(i\pi\, h)$ generates an involution on $\mathfrak{g}$. Simple Lie algebras containing Euler elements are classified, cf. [MN21, Kan00]. Assume that τ_h integrates to an involution on G and let G_{τ_h} be the semidirect product group generated by G and the involution τ_h. A G-equivariant set of wedges is defined by

$$\mathcal{G}_E = \{W = (x, \tau_x) \in \mathfrak{g} \times \tau_h G : x \text{ is and Euler element}\}$$

where the G-action is defined by $g.W = (\mathrm{Ad}\, g(x), g\tau_x g^{-1})$. Once a cone C in the Lie algebra $\mathfrak{g}$ is given, then wedge inclusions can be defined. Furthermore, the causal complement of a abstract wedge is given by $W' = (-x, \tau_x)$. In particular we have defined on a abstract level a local poset of abstract wedge regions, for the general picture see [MN21].

Let $W = (x_W, \sigma_{x_W}) \in \mathcal{G}_E$, given an (anti-)unitary representation of G_{τ_h} on a Hilbert space $\mathcal{H}$, then $U(\exp(-2\pi t\, x_W))$ and $U(\sigma_{x_W})$ identify the standard subspace $\mathsf{H}(W) \subset \mathcal{H}$ by the Tomita theory. Due to the general theory of standard subspaces we can define a generalized framework for one particle nets of standard subspaces that strictly extends the set of one-particle models from AQFT that can be constructed through the BGL-construction, cf. [MN21].

Causal homogeneous spaces $M = G/H$ play the role of the spacetime in AQFT models. On these spaces concrete wedge regions, as positive subsets for the Euler element flow, can be defined, see [NÓ22, NÓ23, MNÓ23a, MNÓ23b]. If G is centerfree and the wedge subset is connected in M, then there is a correspondence between abstract wedges (Euler couples) and wedge subsets of a causal homogeneous space [MN23]. Given a unitary representation U of G, one can associate real subspaces on general open regions by using the language of distribution vectors [FNÓ23, NÓ23, NØÓ21, NÓ21]. Bosonic second quantization associates to a one-particle net an isotonous, G-covariant net of von Neumann algebras acting on the Fock space [MN21].

We are in the position of defining an axiomatic framework for nets of von Neumann algebra on abstract wedges as well as on open regions of a causal symmetric spaces. One can deduce properties of wedge symmetries and wedge von Neumann algebras for this generalized AQFT from local properties of the net.

The following results are contained in [MN23]. Firstly, given a one-parameter subgroup $\lambda(t)$ of a connected Lie group G, a unitary representation U of G with discrete kernel on an Hilbert space $\mathcal{H}$ and a standard subspace inclusion $\mathsf{K} \subset \mathsf{H} \subset \mathcal{H}$ such that $\Delta_{\mathsf{H}}^{it} = U(\lambda(-2\pi t))$ (BW property) and $U(g)\mathsf{K} \subset \mathsf{H}$ when g is in an open neighbourhood of the identity (regularity property), then $\lambda(t)$ is generated by an Euler element. So, *Euler elements appear naturally in this framework as a consequence of the (BW) and the regularity properties.*

Let G, U and λ as before. Assume that $\lambda(t)$ is generated by an anti-elliptic element of $\mathfrak{g}$ and consider a von Neumann algebra inclusion $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with a common unique G-fixed cyclic and separating vector in $\mathcal{H}$ (vacuum vector). If the Bisognano-Wichmann property holds, namely $\Delta^{it}_{\mathcal{A},\Omega} = U(\lambda(-2\pi t))$ where $\Delta^{it}_{\mathcal{A},\Omega}$ is the modular group of $\mathcal{A}$ with respect to Ω, and an analogue regularity property holds for the von Neumann algebra inclusion $\mathcal{N} \subset \mathcal{M}$ with respect to the adjoint G-action, then the algebra $\mathcal{M}$ is a type III_1 factor with respect to Connes' classification.

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Signal communications and modular theory

Roberto Longo

We propose a conceptual frame to interpret the prolate differential operator

$$W = \frac{d}{dx}(1-x^2)\frac{d}{dx} - x^2\,,$$

which appears in Communication Theory, as an entropy operator; indeed, we write its expectation values as a sum of terms, each subject to an entropy reading by an embedding suggested by Quantum Field Theory.

This adds meaning to the classical work by Slepian et al. on the problem of simultaneously concentrating a function and its Fourier transform, in particular to the "lucky accident" that the truncated Fourier transform $\mathbf{F}_B$

$$\mathbf{F}_B = E_B \mathbf{F} E_B$$

commutes with the prolate operator; here, $\mathbf{F}$ is the unitary Fourier transform on $L^2(\mathbb{R})$ and E_B the orthogonal projection onto $L^2(B)$, with $B = (-1, 1)$ the unit ball.

The key is the notion of entropy $S(\Phi\|H)$ of a vector Φ of a complex Hilbert $\mathfrak{H}$ space with respect to a real linear subspace H, recently introduced by the author, and extended with collaborators, by means of the Tomita-Takesaki modular theory of von Neumann algebras; if H is a factorial standard subspace of $\mathfrak{H}$ with modular operator Δ_H, we have

$$S(\Phi\|H) = \Re(\Phi, iP_H i \log\Delta_H \Phi) = \Re(\Phi, \mathbf{E}_H \Phi)\,.$$

Here, P_H is the cutting projection

$$P_H : \Phi + \Phi' \mapsto \Phi\,, \quad \Phi \in H,\ \Phi' \in H'\,,$$

with H' the symplectic complement of H. P_H can be explicitly expressed in terms of the modular data.

$\mathbf{E}_H = i \log\Delta_H$ is a real-linear, selfadjoint, positive operator, that we regard as an entropy operator inasmuch as its expectation values are entropy quantities.

We consider a generalization of the prolate operator to the higher dimensional case and show that it admits a natural extension commuting with the truncated Fourier transform; this partly generalizes the one-dimensional result by Connes to the effect that there exists a natural selfadjoint extension to the full line commuting with the truncated Fourier transform.

We consider the entropy operator $\mathbf{E}_H$ when $\mathfrak{H}$ is the one-particle Hilbert space of a free, massless, scalar Boson, H is the local subspace associated with the unit ball B in $\mathbb{R}^d$, and $\Phi \in \mathfrak{H}$ is a wave packet. Then $S(\Phi\|H)$ is the information contained by Φ in B.

In this case, on Cauchy data in $L^2(B) \oplus L^2(B)$, $\mathbf{E}_H$ has (up to constants) two components: $-L$, with $L = \nabla(1-r^2)\nabla$ the Legendre operator, and M, the multiplication operator by $(1-r^2)$.

We infer that the prolate operator is an entropy operator, thus a natural a priori candidate to commute with $\mathbf{F}_B$.

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The massive modular Hamiltonian for a double cone

Daniela Cadamuro

Since it has been set up in the 1970's due to works by Tomita that became public with lectures by Takesaki [Tak70], as well as by Araki [Ara76], Tomita-Takesaki modular theory has been one of the most important developments in the theory of operator algebra, as well as in quantum theory. However, in relevant examples from quantum (field) theory, obtaining an "explicit" form of the modular generator $\log\Delta$ has been the strenous work of many researchers along the time. At least in the following situations, a model-independent answer is known:

- If $\mathcal{M}$ is the algebra of all observables and Ω represents a thermal equilibrium state (KMS condition), then $\log\Delta$ is the generator of time translations (up to a factor) [HHW67].
- If $\mathcal{M} = \mathcal{A}(\mathcal{W})$ is the algebra associated with a spacelike wedge region $\mathcal{W}$ in quantum field theory, and Ω is the Minkowski vacuum, then $\log\Delta$ is the generator of boosts along the wedge [BW75].

But what about the algebra of a double cone, $\mathcal{M} = \mathcal{A}(\mathcal{O})$, in a quantum field theory? To answer this question we consider the example of a real scalar free field ϕ of mass $m > 0$. We consider the Fock vacuum as our cyclic and separating vector. The local algebras, as well as the modular operator [EO73], are determined by second quantization, so that we only need to consider the modular operator at one-particle level, which is defined as follows.

On the (complex) one-particle Hilbert space $\mathcal{H}_1$ of the theory, we consider a (closed real) local subspace $\mathcal{L}_1(\mathcal{O}) \subset \mathcal{H}_1$, which is "standard" and "factorial" ($\overline{\mathcal{L}_1 + i\mathcal{L}_1} = \mathcal{H}_1$, $\mathcal{L}_1 \cap \mathcal{L}_1' = \mathcal{L}_1 \cap i\mathcal{L}_1 = \{0\}$), where "prime" denotes the symplectic complement. We define the one-particle Tomita operator on $\mathcal{H}_1$ as

$$T_1 : f + ig \mapsto f - ig, \quad f, g \in \mathcal{L}_1(\mathcal{O}), \tag{1}$$

the polar decomposition of its closure is $T_1 = J_1\Delta_1^{1/2}$. (We shall drop the index "1" from now on.)

We can rewrite the one-particle modular generator as follows. Let P be the real-linear projector onto $\mathcal{L} \subset \mathcal{H}$ with kernel $\mathcal{L}' \subset \mathcal{H}$. Then, on a certain domain, we write

$$P = (1 + T)(1 - \Delta)^{-1}. \tag{2}$$

A computation then shows that

$$\log\Delta = -2\,\mathrm{arcoth}(P - iPi - 1). \tag{3}$$

This determines $\log\Delta$ from P, and hence from $\mathcal{L}$ [FG89]. We now write this formula in a different manner, by writing $\mathcal{H}$ in time-0 formalism in configuration space. Here, $\mathcal{H}$ is parametrized by time-0 initial data of field and field momentum $f = (f_+, f_-)$. The scalar product and the complex structure, with $A = -\nabla^2 + m^2$, are given by

$$\mathrm{Re}\langle f, g\rangle_{\mathcal{H}} = \left\langle f, \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{-1/2} \end{pmatrix} g \right\rangle_2, \quad i_A = \begin{pmatrix} 0 & A^{-1/2} \\ -A^{1/2} & 0 \end{pmatrix}. \tag{4}$$

The local subspaces are defined as follows. Let $\mathcal{B}$ be the base of $\mathcal{O}$ at time 0, then we define

$$\mathcal{L} = \overline{\mathcal{C}_0^\infty(\mathcal{B}) \oplus \mathcal{C}_0^\infty(\mathcal{B})}, \quad P = \chi \oplus \chi, \tag{5}$$

where χ multiplies with the characteristic function of $\mathcal{B}$. Inserting this in formula (3), we have

$$\log\Delta = i_A \begin{pmatrix} 0 & M_- \\ -M_+ & 0 \end{pmatrix}, \tag{6}$$

where

$$M_\pm = 2A^{\pm\frac14}\,\mathrm{arcoth}(B) A^{\pm\frac14}, \quad B = \overline{A^{1/4}\chi A^{-1/4}} + \overline{A^{-1/4}\chi A^{1/4}} - 1. \tag{7}$$

Hence, Δ is determined from χ and A. However, "explicitly" finding the spectral decomposition of B as a selfadjoint operator on $L^2(\mathbb{R}^s)$ is very difficult. There are however known examples:

- If $\mathcal{O}$ is the wedge in x_1-direction, then M_- multiplies with $2\pi x_1$, indepedent of m.
- If $\mathcal{O}$ is a double cone of radius r and $m = 0$, then M_- multiplies with $\pi(r^2 - \|\boldsymbol{x}\|^2)$ [HL82].

Now the questions we would like to answer in the case of double cones and $m > 0$ are the following: Is M_- mass indepedent? Is M_- a multiplication operator? Since answering these questions anaytically is very difficult, we do it numerically, namely we evaluate B and M_- numerically to check this hypothesis.

Using numerical approximation means approximating A^s and χ with finite-dimensional matrices. For that, we need to choose an orthonormal basis and finite dimensional in one summand of $\mathcal{H}$, and we need to approximate $A^{\pm 1/4}$ and χ with a matrix in this basis. Then, we can apply numerical eigendecomposition in order to evaluate the arcoth, and therefore approximate the operator B. We do this with no rigorous estimates on the approximation. Explicitly, A, χ acts on $L^2_{\mathbb{R}}(\mathbb{R})$ by $A = -\partial_x^2 + m^2$, and χ is determined by the region considered: $\chi(x) = \Theta(x)$ for a wedge, or $\chi(x) = \Theta(1+x)\Theta(1-x)$ for the standard double cone.

As our basis functions, we choose suitable piecewise linear functions [BCM23], and the discretization is first done for $A^{-1/4}$ which is bounded and has a known convolution kernel; we then obtain $A^{1/4}$ by numerical matrix inversion. We can then approximate (the integral kernel of) M_- using the formula (7); this is done by

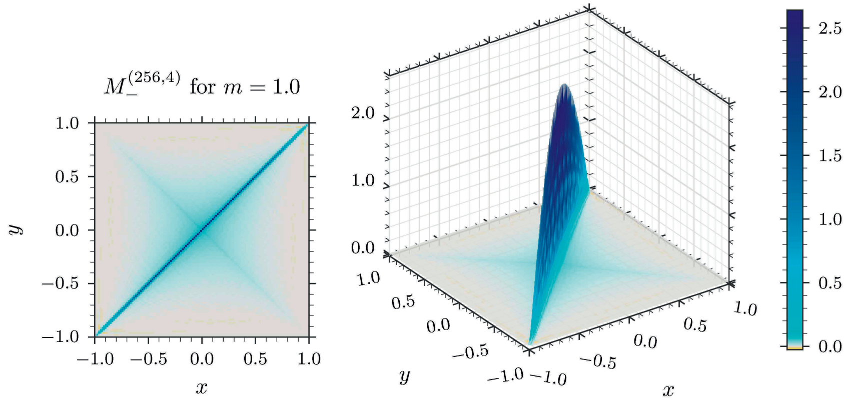

Figure 1

functional calculus of matrices, and the computation turns out to require extended floating point precision of 400–600 decimal digits. We expect convergence against the undiscretized result in the weak sense, i.e., if $M^{(N,b)}$ denotes the integral kernel at a number N of basis elements covering the interval $[-b, b]$,

$$\iint g(x)M_-^{(N,b)}(x,y)h(y)dx\,dy \xrightarrow[N,b\to\infty]{} \iint g(x)M_-(x,y)h(y)dx\,dy.$$

We choose $g = h$ to be a Gaussian located near a point μ, then we vary this point μ.

The results in the wedge case turn out to be compatible with known results. In the case of a double cone, we find that the discretized kernel M_- is concentrated predominantly on the diagonal, see Figure 1. There appear to be some contributions along the antidiagonal, but it is unclear whether this is due to numerical errors or whether there is really a subdominant non-diagonal contribution. An explicit expression for the curves displayed in Figure 1 and Figure 2 is not known. The smeared version of the discretized kernel M_-, see Figure 2, shows that the kernel is mass-dependent. In particular, the black parabola corresponds to the case $m = 0$ and therefore to the quadratic result of Hislop-Longo, while the two straight black lines (piecewise linear) for large mass correspond to the result of a left and a right wedge. Indeed, large masses correspond to small correlation lengths, and hence a heuristic explanation for the approximate “double wedge” structure may be that at one end of the interval, the contribution from the other end of the interval is very small, so that the modular operator for the interval approximately behaves like the one for a half-line.

A similar analysis can be done for a double cone in the 3+1-dimensional field using its spherical symmetry. It turns out in this case that the modular operator also depends on angular momentum [BCM23].

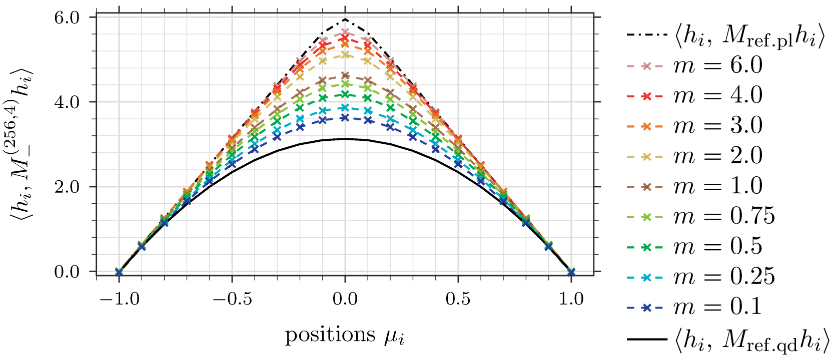

FIGURE 2

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Modular Generators Implementing Conformal Flows in $1 + 1$ de Sitter Space

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(joint work with Urs Achim Wiedemann)

We construct *modular Hamiltonians*, which satisfy the Virasoro algebra relations. They give rise to one-parameter groups of unitary operators in the Fock representation for the free massless field, which implement the geometric flows associated to the conformal Killing vector fields on the 1+1-dimensional de Sitter space **dS**. Previous results by Longo and Kawahigashi [KL05] and Longo, Martinetti, and Rehren [LMR10] on chiral quantum fields suggest that the modular Hamiltonians on de Sitter space should be given by *Connes' spatial derivatives* for pairs of *product states*, build up from *rescaled vacuum states.* We show that this is indeed the case, and that we can provide explicit expressions for Connes' spatial derivatives in terms of *higher ladder operators* on Fock space.

To be more specific, we first compute the *conformal Killing vector fields* on the two-dimensional de Sitter space $\mathbf{dS} \doteq \{x \in \mathbb{R}^{1+2} \mid x_0^2 - x_1^2 - x_2^2 = -r^2\}$. We show that these vector fields can be analytically continued to conformal Killing vector fields defined on $\mathbb{S}^2 \setminus \{(ir, 0, 0), (-ir, 0, 0)\}$, where $\mathbb{S}^2 \doteq \{(ix_0, x_1, x_2) \in (i\mathbb{R}) \times \mathbb{R}^2 \mid x_0^2 + x_1^2 + x_2^2 = r^2\} \subset \mathbf{dS}_{\mathbb{C}}$ is the Euclidean sphere embedded in the complexified de Sitter space. Only six vector fields allow analytic continuations to all of $\mathbb{S}^2$, as this is the maximum number of conformal Killing vector fields allowed on $\mathbb{S}^2$. The other (analytically continued) vectors fields diverge at the poles $(\pm i, 0, 0)$ of $\mathbb{S}^2$. We solve the flows equations for the conformal Killing vector fields and the analytically continued conformal Killing vector fields. Inspecting Killing vector fields for higher $k > 2$, we find that also there, analytic continuation of the flow to $t \to i\pi/k$ yields a discrete space-time transformation that amounts to mirroring any point $a \in \mathbf{dS}$ at the source or sink of the corresponding vector field that lies the closest to a.

Next, we provide [BJM23] a new realisation of the representations $D_1^{\pm}$ first studied by Bargmann in his classification of the unitary irreducible representations of $SO_0(1, 2)$. While Bargmann used functions supported on the forward light cone in $\mathbb{R}^{1+2}$, our representation space consists of functions supported on the Cauchy surface $\mathcal{C} = \{x \in \mathbf{dS} \mid x_0 = 0\}$, sometimes called the *time-zero circle*. The scalar product[1]

$$\langle h_1, h_2 \rangle = -\tfrac{1}{2} \int_{\mathcal{C}} \mathrm{d}\psi \, \overline{h_1(\psi)} \int_{\mathcal{C}} \mathrm{d}\psi' \, \ln\bigl(2 - 2\cos(\psi - \psi')\bigr) h_2(\psi) \, . \tag{1}$$

and the generators of the rotations and the two Lorentz boosts, denoted by k_0, l_1 and l_2, have a particular simple form in our formulation: the corresponding unitary groups are

$$D_1^{\pm}(R_0(\alpha)) = \mathrm{e}^{i\alpha k_0^{\pm}} \, , \qquad k_0^{\pm} = -i \tfrac{\mathrm{d}}{\mathrm{d}\psi} \, .$$

and

$$D_1^{\pm}(\Lambda_1(t)) = \mathrm{e}^{it\nu r \cos_\psi} \, , \quad (\cos_\psi h)(\psi) \doteq \cos\psi \cdot h(\psi) \, , \quad h \in \mathfrak{h}^{\pm} \, .$$

The unitary group of the second boost is $D_1^{\pm}(\Lambda_2(t)) = \mathrm{e}^{it\nu r \sin_\psi}$.

In the sequel, we associate to any (generalized) function h in the one-particle Hilbert space $\mathfrak{h}$, a distribution $\widehat{h}$ on the de Sitter space, with support on the Cauchy surface $\mathcal{C}$. With the help of the fundamental solution E, we construct solutions $\mathbf{\Phi}_{\widehat{h}} \doteq E * \widehat{h}$ of the wave equation with *Cauchy data*

$$\phi_{\widehat{h}} = -(\nu r)^{-1} \Im h \quad \text{and} \quad \pi_{\widehat{h}} = \Re h \, , \qquad h \in \mathfrak{h} \, .$$

We verify that the generators k_i and l_i, $i = 0, 1, 2$, implement the geometric flows $\Phi_t^{\mathfrak{X}_i}$ and $\Phi_t^{\mathfrak{Y}_i}$ associated to the (conformal) Killing vector fields $\mathfrak{X}_i$ and $\mathfrak{Y}_i$, $i =$

[1] The logarithmic singularity of the kernel is integrable and hence ultra-violet divergencies are mild, while infra-red divergencies are absent due to the fact that $\mathcal{C}$ is compact.

0, 1, 2. For instance,

$$\Re\, \mathrm{e}^{itl_i} h = \big(\tfrac{\partial}{\partial \eta} E * (\widehat{h} \circ \Phi_t^{\mathfrak{X}_i})\big)(0, \,.\,) \;,$$
$$-(\nu r)^{-1} \Im\, \mathrm{e}^{itl_i} h = -\big(E * (\widehat{h} \circ \Phi_t^{\mathfrak{X}_i})\big)(0, \,.\,) \;, \qquad i = 1, 2 \;.$$

This geometric picture allows us to look out for Cauchy data supported in an interval $I \subset \mathcal{C}$. The lack of a zero mode requires care when defining the standard subspaces $\mathfrak{h}_I$ associated to an interval I on the Cauchy surface: the Cauchy data $(f', g) \in \mathfrak{h}_I$ gives rise to a solution of the wave equation which has support in the space-time points that can be connected with light rays to points in the closed interval $\overline{I}$, *once an overall constant solution is subtracted*. As expected, $\mathfrak{h}_I = \mathfrak{h}_I^+ \oplus \mathfrak{h}_I^-$ can be decomposed into right-movers and left-movers without destroying the localisation properties, as left and right movers are uncorrelated. The map $I \mapsto \mathfrak{h}_I$ has a number of desirable properties: isotony, preservation of intersections, additivity, locality, Haag duality, anti-locality and the one-particle Reeh-Schlieder property all hold. Rescaling is a delicate operation on the one-particle Hilbert space $\mathfrak{h}$ due to the fact that the Cauchy surface is compact. The *higher conformal flows* defined maps on the set of solutions of the wave function, and therefore also on the Cauchy data. The positive and negative frequencies remain separated.

The next step is to build up the Tomita operator associated to a wedge $W \subset \mathbf{dS}$ from the boost leaving the wedge invariant, and the reflection at the edge of the wedge. Since both building blocks arise from group theory, the Tomita operator associated to a wedge is given intrinsically by the representation theory of the space-time symmetry group. We show that modular localisation, invented by to Brunetti, Guido and Longo [BGL02], assigns $\mathbb{R}$-linear subspaces $\mathfrak{h}(\mathcal{O})$ of $\mathfrak{h}$ to causally complete space time regions $\mathcal{O} \subset \mathbf{dS}$, in agreement with our physical expectations, as it yields the localisation of Cauchy data for solutions of the wave equation because

$$\mathfrak{h}(\mathcal{O}_I) := \bigcap_{W \subset \mathcal{O}_I} \mathfrak{h}(W) = \mathfrak{h}_I$$

for all double cones $\mathcal{O}_I$ with base $I \subset \mathcal{C}$.

The generators l_0, l_1, l_2 and k_1, k_2 are modular Hamiltonians, *i.e.*, they are the generators of modular groups associated to $\mathbb{R}$-linear subspaces $\mathfrak{h}(X)$ of $\mathfrak{h}$. The localisation region $X \subset \mathbf{dS}$ turns out to be the region where the associated Killing vector field (select one among $\mathfrak{Y}_0, \mathfrak{X}_1, \mathfrak{X}_2$ and $\mathfrak{Y}_1, \mathfrak{Y}_2$) is time-like and future directed. As the vector field $\mathfrak{X}_0$ is nowhere time-like, the angular momentum operator k_0 is not a modular Hamiltonian. Beside the five modular Hamiltonians already mentioned, there exits many more such Hamiltonians which arise by applying $SO_0(2, 2)$ transformations to the space-time regions already considered. A particular interesting case arises by shrinking a wedge to a double cone.

In the sequel, we introduce Fock space and we define smeared-out field operators

$$\varphi(h) \doteq \tfrac{1}{\sqrt{2}}\big(a^*(h) + a(h)\big) \;, \qquad h \in \mathfrak{h} \;,$$

satisfying the canonical commutation relations $\big[\varphi(f), \varphi(g)\big] = i \Im \langle f, g \rangle_{\mathfrak{h}}$, $f, g \in \mathfrak{h}$. The *canonical momenta* are defined by setting $\pi(g) \doteq \varphi(i\nu r g)$, $\nu r g \in \mathfrak{h}$, g real valued. Next, we provide unitary operators on Fock space which implement the higher conformal flows. The generators

$$X_1^{(n)} \doteq \tfrac{1}{2}\big(K_n^+ + K_{-n}^+ + K_n^- + K_{-n}^-\big) ,$$
$$Y_1^{(n)} \doteq \tfrac{1}{2}\big(K_n^+ + K_{-n}^+ - K_n^- - K_{-n}^-\big) ,$$

can be expressed in terms of *higher ladder operators* K_n^+, K_{-n}^+, K_n^- and K_{-n}^- satisfying the *Virasoro algebra* relations

$$\big[K_n^\pm, K_m^\pm\big] = (n-m)K_{n+m}^\pm + \tfrac{1}{12}\, n(n^2-1)\delta_{n,-m}\, 1 ,$$

while $[K_n^\pm, K_m^\mp] = 0$ for all $n, m \in \mathbb{Z}$. The generators satisfy

$$i\big[X_1^{(n)}, \nabla\varphi(f) + \pi(g)\big] = \pi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} f\big) + \nabla\varphi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} g\big) ,$$
$$i\big[Y_1^{(n)}, \nabla\varphi(f) + \pi(g)\big] = \nabla\varphi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} f\big) + \pi\big(\cos_{n\psi} \tfrac{\mathrm{d}}{\mathrm{d}\psi} g\big) .$$

Similar formulas hold for $X_2^{(n)}$ and $Y_2^{(n)}$, $n \in \mathbb{N}$. In the the sequel we show that these operators are the generators of modular groups for product states consisting of rescaled vacuum state.

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Standard Subspaces and Distribution Vectors

Gestur Ólafsson

(joint work with J. Frahm, V. Morinelli, K.-H. Neeb and I. Sitiraju)

Construction of fields of standard subspaces using antiunitary representations and the geometry of causal symmetric spaces has recently been studied in a series of articles including [BN23, FNÓ23, MN21, MNÓ23a, MNÓ23b, NØÓ21, NÓ23a, NÓ23b, ÓS23]. We give here a short overview of the main ideas.

In the following G will always denote a connected semisimple Lie group with finite center, $\tau : G \to G$ a nontrivial involution and $(G^\tau)_e \subseteq H \subset G^\tau$, where $G^\tau = \{x \in G \mid \tau(x) = x\}$. We denote by G_{τ_h} the semidirect product of G and $\{1, \tau_h\}$. The derivative of τ induces a Lie algebra homomorphism $\tau_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ which leads to the ± 1-eigenspace decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{q}$ is isomorphic as a H-space to the tangent space of M at $x_0 = eH$. We assume that the symmetric space $M = G/H$ is noncompactly causal, i.e., there exists an H-invariant closed pointed and generating cone $C \subset \mathfrak{q}$ such that the elements of C^o are hyperbolic.

We always choose a Cartan involution that commutes with τ and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian. Denote by Δ the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$ and let $\Omega_{\mathfrak{a}} = \{x \in \mathfrak{a} \mid (\forall \alpha \in \Delta)\, |\alpha(x)| < \pi/2\}$. We then let $A_{i\Omega} = \exp i\Omega$ and define $\Xi_{G/K} = \Xi = GA_{i\Omega_{\mathfrak{a}}}x_0 \subset G_{\mathbb{C}}/K_{\mathbb{C}}$, the crown of the Riemannian symmetric space G/K. We also define $\Xi_G = GA_{i\Omega}K_{\mathbb{C}} = q_{G/K}^{-1}(\Xi_G)$, the crown domain in $G_{\mathbb{C}}$. Here $q_{G/K} : G \to G/K$ denotes the canonical projection $x \mapsto xK$. Ξ is open in $G_{\mathbb{C}}/K_{\mathbb{C}}$ and Ξ_G is open in $G_{\mathbb{C}}$. Both sets are by definition G-invariant. Denote by $\partial_d \Xi$ the distinguished boundary of Ξ (see [GK02]).

Theorem 1 (Gindikin-Krötz, [GK02]). *Assume that $G \subset G_{\mathbb{C}}$, $G_{\mathbb{C}}$ simply connected. Let $M = Gz_0K_{\mathbb{C}}$ be an open G-orbit in $\partial_d\Xi$. If M is a symmetric space then M is ncc and $z_0^{-1}H_{\mathbb{C}}z_0 = K_{\mathbb{C}}$, $H = G^{z_0}$. Furthermore, up to covering, every ncc space G/H can be realized in this way.*

Example 2 (The de Sitter space). For $v = (v_0, \mathbf{v}), w = (w_0, \mathbf{w}) \in \mathbb{R}^{1+d}$ let $[v, w] = v_0w_0 - \langle \mathbf{v}, \mathbf{w}\rangle$. Let $G = \mathrm{SO}_{1,d}(\mathbb{R}) \supset H = \mathrm{SO}_{1,d-1}(\mathbb{R})$. We have $G(ie_0) = G/K = \mathbb{H}^d = \{iv \in i\mathbb{R}^{d+1} \mid [v, v] = 1\}$ and $Ge_1 = G/H = \mathrm{dS}^d$. Let $h : (x_0, x_1, \tilde{x}) \mapsto (x_1, x_0, 0, \cdots, 0)$. Then h is an Euler element and $\mathfrak{a} = \mathbb{R}h$. Furthermore, $\Xi_{\mathbb{H}^d} = (\mathbb{R}^{d+1} + iV_+) \cap \mathrm{dS}^d$ where V_+ is the forward light cone. We have $\exp(-ith)(ie_0) = i(\cos t)e_0 + (\sin t)e_1 \to e_1$ as $t \to \pi/2$ and dS^d is the only open G-orbit in the distinguished boundary of Ξ.

Let $(U, \mathcal{H})$ be an anti-unitary representation of G_{τ_h}. Recall that a real subspace $\mathtt{V} \subset \mathcal{H}$ is *standard* if $\mathtt{V}$ is closed, $\mathtt{V} \cap i\mathtt{V} = \{0\}$ and $\mathtt{V} + i\mathtt{V} \subset \mathcal{H}$ is dense. Denote by $\mathcal{H}^\infty$ the space of smooth vectors and by $\mathcal{H}^{-\infty}$ the conjugate linear dual of $\mathcal{H}$. For a finite dimensional H-invariant subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ and $\mathcal{O} \subseteq M = G/H$ open define

$$\mathtt{H}_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}}\{U^{-\infty}(\phi)\mathtt{E} \mid \phi \in C_c^\infty(q_M^{-1}(\mathcal{O}), \mathbb{R})\}} \subset \mathcal{H}. \tag{1}$$

$\mathtt{H}_{\mathtt{E}}$ defines a net of real subspaces which is clearly isotone and covariant, but locality, Reeh-Schlieder and Bisonanao-Wichmann do not hold in general.

Let h be an Euler element such that $z_0 = \exp(\frac{\pi i}{2}h)$, z_0 as in Theorem 1. Denote by $\alpha_t(m) = \exp(th)m$, $m \in M$, the modular flow and $X_h^M(m) = d/dt|_{t=0}\, \alpha_t(m)$. The positivity domain is $W_M^+(h) = \{m \in M \mid X_h^M(m) \in V_+(m)\}$ (see [MNÓ23a, NÓ23a, NÓ23b]).

Let $(U, \mathcal{H})$ be a irreducible antiunitary representation of G_{τ_h}. Denote by $\mathcal{H}^{-\infty,8[\mathfrak{h}]}$ the space of $\mathfrak{h}$-finite distribution vectors and by $\mathcal{H}^{[K]}$ the space of K-finite vectors. If $G \subset G_{\mathbb{C}}$ is linear then for $\xi \in \mathcal{H}^{[K]}$ the orbit map $g \mapsto U_g\xi$ extends to a holomorphic function on Ξ_G ([KS04]). Define

$$\beta^{\pm}(\xi) = \lim_{t \to \pm\pi/2} U(\exp -ith)\xi \in \mathcal{H}^{[\mathfrak{h}]}$$

whenever the limit exists.

Theorem 3 (FNÓ23). *Assume that G is linear or locally isomorphic to $\mathrm{SO}_{1,2}(\mathbb{R})$. Assume that $M = G/H$ is ncc. Let $(U, \mathcal{H})$ be an irreducible antunitary representation of G_{τ_h}, $\mathcal{E} \subset \mathcal{H}^{[K]}$ finite dimensional subspace invariant under K and $J = U_{\tau_h}$. Let $\mathtt{E} = \mathcal{E}^J$. Then the following holds:*
1) $\beta^{\pm} : \mathcal{H}^{[K]} \to \mathcal{H}^{-\infty,[\mathfrak{h}]}$ exists and both maps are injective.
2) Let $\mathtt{E}_H = \beta^+(\mathtt{E}) \subseteq \mathcal{H}^{-\infty}$. Then the net $\mathtt{H}^M_{\mathtt{E}_H}$ defined in (1) on M is isotone, covariant and has the Reeh-Schlieder and Bisognano-Wichmann property, where $W = W^+_M(h)_{eH}$ is the connected component of the positivity domain of h on M, containing the base point.

The holomorphic extension of the orbit map $g \mapsto U_g\xi$ leads to a sesquiholomorphic G-invariant positive definite kernel on $\Xi \times \Xi$, $\Phi^U_\xi(z, w) = \Phi_w(z) = \langle U_w\xi, U_z\xi\rangle$ (the inner product is linear in the second factor). The GNS construction then leads to a realization of $(U, \mathcal{H})$ in spaces of holomorphic functions on Ξ_G. If $\xi \in \mathcal{H}^K$ then Φ^U_ξ lives on $\Xi_{G/K}$ and $\phi^U_\xi = \Phi_{eK}$ is a spherical function if $\|\xi\| = 1$. Keeping one of the variable, say z, in Ξ we have $y \to \Phi(z, y)$ extends to an analytic function on M and hence a well defined distribution M. The question is then again if $\lim_{z\to eH} \Phi(z, \cdot)$ exists in $\mathcal{D}'(G/H)$. For this we discuss as an example the case of dS^d. Let $U = U_\lambda$ be a unitary spherical representation of $\mathrm{SO}_{1,d}(\mathbb{R})_e$ with spectral parameter λ. In this case we have, with $\rho = (n-1)/2$:

$$\Psi(z, w) = {}_2F_1\left(\rho + \lambda, \rho - \lambda; \frac{n}{2}; \frac{1 - [z, \bar{w}]}{2}\right), \quad \lambda \in i\mathbb{R}_+ \cup (0, \rho].$$

The hypergeometric function ${}_2F_1(z)$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$. For $x \in \mathrm{dS}^d$ let $\Gamma(x) = \{z \in \mathrm{dS}^d \mid [z - x, z - x] > 0\} = \Gamma^+(x) \cup \Gamma^-(x)$ where the $\pm$ indicate $\pm(z - x)_0 > 0$. We have

Lemma 4. 1) $\{[z, \bar{w}] \mid z, w \in \Xi\} = \mathbb{C} \setminus (-\infty, -1]$.
2) $[\mathrm{dS}^d, \Xi] \cap \mathbb{R} = (-1, 1)$.
3) *If $x \in \mathrm{dS}^d$ then $[x, y] \leq -1$ if and only if $x + y \in \Gamma(x)$.*

(1) Shows that $\Psi(z, w)$ is well defined on the crown. (2) shows that $\Psi(z, y)$ is well defined for $z \in \Xi$ and $y \in \mathrm{dS}^d$. Finally, it was shown in [FNÓ23, ÓS23] that the distributional limit exists and defines a H-invariant distribution on dS^d.

There is also the *conjugate* crown, the crown of $G(-ie_0)$, so we have two natural crowns $\Xi^{\pm}$ and two kernels $\Psi^{\pm}$ and hence two distributions $\eta^{\pm}_\lambda$. The explicit formulas above show that $\eta^+ - \eta^-$ is supported on the closed forward and backward light cone (see [ÓS23, Cor. 6.9])

Similar calculations work for all K-types for principal series representations of $\widetilde{\mathrm{S}L}_2(\mathbb{R})$ expressed in terms of hypergeometric functions.

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Arrow of time and quantum physics

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(joint work with Klaus Fredenhagen)

The arrow of time is a subject of ongoing debate ever since this term was coined by Eddington almost a hundred years ago. In brief, this topic can be described as follows: the time parameter that enters into the fundamental equations of physics can be reversed, which in principle seems to allow physical systems to move backwards in time. On the other hand, there is overwhelming evidence that this does not happen. The standard resolution of this apparent clash between theory and reality is based on the argument that such time reversed processes are exceedingly unlikely (Second Law). Therefore, they were and will never be observed.

In this contribution a complementary view is presented [BF23]. It is based on the hypothesis that time translations form a semi-group acting on all systems, there is no inverse and hence no return to the past. Information about the past is encoded in material bodies which accompany us, such as books or other devices and media, not least our brains. We can extract from them information about past events, observations, experiments, data taken, and theories developped on their basis. The informations obtained in this way can be described in common language, including mathematics. In order to check their truth value, one has to repeat past experiments. But this can only happen in the future. Thus the past may be regarded as factual, whereas the future is indeterminate.

While this hypothesis seems consistent with reality, it raises some questions concerning the *theoretical* treatment of time. These questions and the proposed answers are outlined in this abstract.

1. Is the hypothesis of an intrinsic arrow of time compatible with the successful theoretical treatment of time as a group? In order to answer this question one considers a unital C*-algebra $\mathfrak{A}(\mathcal{V}_o)$, describing local observables in a given future-directed lightcone $\mathcal{V}_o$ in Minkowski space $\mathcal{M}$. On this algebra acts the abelian semi-group of time translations $V_+ := \{\tau = t(1, \boldsymbol{v}) : t \geq 0\,,\ |\boldsymbol{v}| < 1\}$ by endomorphisms α, *viz.* $\alpha_\tau(\mathfrak{A}(\mathcal{V}_o)) = \mathfrak{A}(\mathcal{V}_o + \tau)$. With this input one can identify vacuum states in the region $\mathcal{V}_o$. The following result can then be established.

Proposition 1. *In the GNS-representation induced by a vacuum state*

(i) *there exists a continuous unitary representation U_0 of the semi-group V_+ whose adjoint action implements the time translations on the observables.*
(ii) *there exists an extension of U_0 to a continuous unitary representation U of the group of spacetime translations $\mathbb{R}^d$ on $\mathcal{M}$. It satisfies the relativistic spectrum condition. Its adjoint action on the represented observables in $\mathcal{V}_o$ defines a net of observables in all of Minkowski space $\mathcal{M}$.*

This proposition shows that the hypothesis of a fundamental arrow of time is compatible with the familiar theoretical assumption according to which the group of spacetime translations $\mathbb{R}^d$ acts on the observables. In this way a theoretical picture of the past is obtained that is consistent with the theory for future observations.

2. Are there uncertainties in the theoretical description of the past and how do they manifest themselves? It turns out that the unitary representation U_0 of the semigroup V_+ to a representation U of the group $\mathbb{R}^d$ is in general not unique. The pertinent information is encoded in the largest projection Z in the weak closure of the algebra of observables in $\mathcal{V}_o$ that annihilates the vacuum state. It commutes with all translations U. Whenever $Z < (1 - P_0)$, where P_0 is the projection onto the vacuum state, the extension U is not unique. As a consequence, the theoretical description of the past is ambiguous. This feature becomes manifest in the structure of the energy-momentum spectrum.

Proposition 2. *If the past is ambiguous, i.e. the extensions U of the time translation U_0 are not unique, their spectrum consists of the closed cone $\overline{V}_+$ in energy-momentum space. So there exist excitations of arbitrarily small mass. Conversely, the existence of massless excitations implies that the extensions U are not unique.*

Since there are massless excitations in reality, the photons, complete information about the past (the wave function of the universe) is a theoretical fiction.

3. How big is the loss of information on the properties of states over time that arises from these ambiguities? An answer is given by noticing that the uncertainties concerning the past are due to states Φ in the kernel of the projection Z, involving massless excitations. The corresponding information in a lightcone $\mathcal{V}_o + \tau$ can be quantified by a convenient measure of information $I_\tau(\Phi)$, which is related to the concept of relative entropy introduced by H. Araki. It was invented by R. Longo, who used the theory of standard subspaces for its definition. Making use of this notion, the following result obtains.

Proposition 3. *Let Φ be a state in the kernel of Z. Then*

(i) $I_\tau(\Phi) \in [0, \infty]$ *and there is a dense set of states Φ for which this information is finite.*
(ii) $I_\tau(\Phi) \leq I_\sigma(\Phi)$ *if $\tau - \sigma \in V_+$.*

The information contained in the stationary vacuum state is equal to 0.

Thus the information contained in the states decreases in the course of time.

4. Does the arrow of time enforce the quantum features of operations that are described by common language (classical terms), extracted from past information? The preceding results suggest that a statistical description of future experiments is unavoidable in view of the lacking information about the past. It turns out that the arrow of time adds to it the non-commutative features of quantum physics.

We outline here the simple case of a classical, non-interacting, smooth field $x \mapsto \phi(x)$ in Minkowski space $\mathcal{M}$ with relativistic Lagrangian density

$$x \mapsto L(x)[\phi] = (1/2)(\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2) \, .$$

One considers operations affecting the field which are induced by perturbations of the Lagrangian. In the present simple case these are given by

$$L(x)[\phi] \to L(x)[\phi] + c(x) + f(x)\phi(x) \, ,$$

where c, f are real test functions with compact support. Their spacetime integrals describe functionals $\phi \mapsto F[\phi]$ on the field. The support of the functionals F in $\mathcal{M}$ is identified with the support of f. Constant functionals have empty support.

The effect of these pertubations on ϕ, encoded in the functionals F, is described by symbols $S_L(F)$. They define a dynamical group $\mathcal{G}_L$. It is the free group generated by these symbols, modulo the following defining relations:

(1a) $S_L(F)S_L(G) = S_L(F+G)$ if $\operatorname{supp} F$ lies above and $\operatorname{supp} G$ lies beneath some Cauchy surface. Here the arrow of time enters, the swapped product is not fixed and depends on the dynamics. Only if the supports of F and G are spacelike separated, the product is commutative according to this definition.
(1b) $S_L(c) = e^{ic} 1 \, , \quad c \in \mathbb{R}$.
(2) $S_L(F) = S_L(F^{\phi_0} + \delta L(\phi_0))$, where $F^{\phi_0}[\phi] := F[\phi + \phi_0]$ for given external smooth field ϕ_0 with compact support; $\delta L(\phi_0)$ is the corresponding variation of the action, determined by L. In this relation the Lagrangian and hence the dynamics enters. If $F = 0$, the underlying field is unaffected.

Proceeding from the group $\mathcal{G}_L$ to the enveloping dynamical C*-algebra $\mathcal{A}_L$, one arrives at the following result.

Proposition 4. *Let L be the non-interacting Lagrangian, given above. The algebra $\mathcal{A}_L$ coincides with the Weyl algebra of a local quantum field, satisfying the Klein-Gordon equation. It is generated by the exponentials of the field, integrated with real test functions f.*

For Lagrangians L describing interacting fields, the corresponding algebras $\mathcal{A}_L$ comply with all Haag-Kastler axioms of local quantum physics. These results obtain without imposing any quantization rules. It is the arrow of time which, together with the dynamics, leads to the quantization of the classical input. In view of these results, it seems worthwhile to take a fresh look at the foundations of quantum physics, based on this new paradigm.

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Nets of standard subspaces on homogeneous spaces

Karl-Hermann Neeb

(joint work with Vincenzo Morinelli)

We discuss some recent results concerning nets of real subspaces indexed by open subsets $\mathcal{O}$ of a homogeneous space $M = G/H$ of a Lie group G. We assume that M carries a G-invariant causal structure, i.e., a field of pointed generating closed convex cones $C_m \subseteq T_m(M)$ that is invariant under the G-action. Typical examples are time-oriented Lorentzian manifolds on which G acts by time-orientation preserving symmetries or conformal maps. Natural properties of such nets are closely related to those of nets of von Neumann algebras.

For a unitary representation $(U, \mathcal{H})$ of a connected a Lie group G and a homogeneous space $M = G/H$, we consider families $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq M}$ of closed real subspaces of $\mathcal{H}$, indexed by open subsets $\mathcal{O} \subseteq M$ with the following properties:

- (Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$
- (Cov) **Covariance:** $U(g)\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.
- (RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.
- (BW) **Bisognano–Wichmann property:** There exists an open subset $W \subseteq M$ (called a *wedge region*) and $h \in \mathfrak{g}$, such that $\exp(\mathbb{R}h)W \subseteq W$ and $\mathsf{H}(W)$ is standard with modular group $\Delta_{\mathsf{H}(W)}^{-it/2\pi} = U(\exp th)$, $t \in \mathbb{R}$.

Given a unitary representation $(U, \mathcal{H})$, we would like to understand, and possibly classify, all nets with these properties. To this end, the first problem is to understand which Lie algebra elements h and which regions $W \subseteq M$ occur in (BW). The following theorem shows that we may restrict our attention to the case where h is an *Euler element*, i.e., $\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h)$ for $\mathfrak{g}_\lambda(h) = \ker(\lambda\mathbf{1} - \mathrm{ad}h)$.

Theorem 1. (Euler Element Theorem, [MN23]) *Let G be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Let $(U, \mathcal{H})$ be a unitary representation of G with discrete kernel. Suppose that $\mathtt{V}$ is a standard subspace and $N \subseteq G$ an identity neighborhood such that*

(a) *$U(\exp(th)) = \Delta_{\mathtt{V}}^{-it/2\pi}$ for $t \in \mathbb{R}$, i.e., $\Delta_{\mathtt{V}} = e^{2\pi i\,\partial U(h)}$, and*
(b) *$\mathtt{V}_N := \bigcap_{g\in N} U(g)\mathtt{V}$ is cyclic, i.e., $\mathtt{V}_N + i\mathtt{V}_N$ is dense in $\mathcal{H}$.*

Then h is an Euler element and the conjugation $J_{\mathtt{V}}$ satisfies

$$J_{\mathtt{V}}U(\exp x)J_{\mathtt{V}} = U(\exp \tau_h(x)) \quad \textit{for} \quad \tau_h = e^{\pi i \mathrm{ad} h}, x \in \mathfrak{g}. \tag{1}$$

If (Iso), (Cov), (RS) and (BW) are satisfied, then the preceding theorem applies with $\mathtt{V} = \mathsf{H}(W)$. For any relatively compact open subset $\mathcal{O} \subseteq W$ we find a symmetric e-neighborhood with $N.\mathcal{O} \subseteq W$, and then $\mathsf{H}(\mathcal{O}) \subseteq \mathtt{V}_N$ by (Iso), (Cov) and (BW). Hence (RS) implies that $\mathtt{V}_N$ is cyclic. Accordingly, we may assume in the following that h is an Euler element, and by (1), that U extends to a representation of the extended group $G_{\tau_h} = G \rtimes \{e, \tau_h\}$ with $U(\tau_h) = J_{\mathsf{H}(W)}$.

So we may now start with an (anti-)unitary representation $(U, \mathcal{H})$ of G_{τ_h} with discrete kernel, and consider the standard subspace $\mathtt{V} := \mathtt{V}(h, U)$, specified by

$$J_{\mathtt{V}} = U(\tau_h) \quad \text{and} \quad \Delta_{\mathtt{V}}^{-it/2\pi} = U(\exp th), t \in \mathbb{R}.$$

For any homogeneous space $M = G/H$ and an open $\exp(\mathbb{R}h)$-invariant subset $W \subseteq M$, we may then consider the nets

$$\mathsf{H}^{\max}(\mathcal{O}) := \bigcap_{g \in G, \mathcal{O} \subseteq gW} U(g)\mathtt{V} \quad \text{and} \quad \mathsf{H}^{\min}(\mathcal{O}) := \overline{\sum_{g \in G, gW \subseteq \mathcal{O}} U(g)\mathtt{V}}. \tag{2}$$

Both nets are easily seen to be isotone and covariant. It is also rather easy to verify that they satisfy (BW) in the sense that $\mathsf{H}^{\max}(W) = \mathsf{H}^{\min}(W) = \mathtt{V}$ if and only if we have the following inclusion of subsemigroups of G:

$$S_W := \{g \in G \colon gW \subseteq W\} \subseteq S_{\mathtt{V}} := \{g \in G \colon g\mathtt{V} \subseteq \mathtt{V}\}. \tag{3}$$

We refer to [MN23] for details. An interesting consequence is that the existence of a net H on open subsets of M, satisfying (Iso), (Cov) and (BW), implies (3) and that

$$\mathsf{H}^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}^{\max}(\mathcal{O})$$

for all open subsets of $\mathcal{O} \subseteq M$.

As this point, the next step consists in a better understanding of condition (3). Here W and the semigroup S_W are the most intricate points, but the semigroup $S_{\mathtt{V}}$ has a rather explicit description ([Ne22, Thms. 2.16, 3.4]):

$$S_{\mathtt{V}} = G_{\mathtt{V}} \exp(C_+ + C_-) = \exp(C_+) G_{\mathtt{V}} \exp(C_-), \tag{4}$$

where $G_{\mathtt{V}} = \{g \in G \colon \mathrm{Ad}(g)h = h, \tau_h(g)g^{-1} \in \ker(U)\}$, and

$$C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h), \quad C_U := \{x \in \mathfrak{g} : -i\partial U(x) \geq 0\}.$$

In this sense $S_{\mathtt{V}}$ can be obtained from h, $\ker U$, and the positive cone C_U of the representation U.

Natural choices of wedge regions $W \subseteq M$ are the connected components of the *positivity domain*

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\}$$

of the so-called *modular vector field* $X_h^M(m) := \frac{d}{dt}\big|_{t=0} \exp(th).m$. These "wedge regions" have been studied for compactly and non-compactly causal symmetric spaces in [NÓ23b] and [NÓ23a, MNÓ23], respectively. In many situations $W_M^+(h)$ is connected, and then one has good information on S_W (cf. [MN23, Prop. 2.9]):

$$\mathbf{L}(S_W) := \{x \in \mathfrak{g}\colon \exp(\mathbb{R}_+ x) \subseteq S_W\} = \mathfrak{g}_0(h) + (C_W \cap \mathfrak{g}_1(h)) - (C_W \cap \mathfrak{g}_{-1}(h)),$$

where $C_W := \{y \in \mathfrak{g}\colon (\forall m \in W)\ X_y^M(m) \in C_m\}$ contains the invariant cone

$$C_M := \{y \in \mathfrak{g}\colon (\forall m \in M)\ X_y^M(m) \in C_m\}.$$

In all examples for which we have explicit information on these cones, we have $C_M \cap \mathfrak{g}_{\pm 1}(h) = C_W \cap \mathfrak{g}_{\pm 1}(h)$. Note that C_M can be considered as the "positive cone" of the G-action on the causal manifold M, so that the semigroups S_{V} and S_W are closely related to the cones C_U and C_M in a similar fashion.

Example 2. For Minkowski space $M = \mathbb{R}^{1,d}$, $G = \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})^\uparrow$, the Poincaré group, the Lie algebra $\mathfrak{g}$ contains only one adjoint orbit of Euler elements, represented by the boost generator $h.x = (x_1, x_0, 0, \ldots, 0)$. Then

$$W = \{x \in \mathbb{R}^{1,d}\colon x_1 > |x_0|\}$$

is the Rindler wedge, $S_W = \overline{W} \rtimes (\mathrm{SO}_{1,1}(\mathbb{R})^\uparrow \times \mathrm{SO}_{d-1}(\mathbb{R}))$, and for an antiunirary representation of $G_{\tau_h} = \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})$, the compatibility condition $S_W \subseteq S_{\mathsf{V}}$ is equivalent to the positive energy condition that

$$V_+ = \{x \in \mathbb{R}^{1,d}\colon x_0 > \mathbf{x}^2, x_0 > 0\} \subseteq C_U.$$

If the semigroup S_W is a connected group, then the compatibility condition $S_W \subseteq S_{\mathsf{V}}$ imposes no essential restriction on the representation, such as positive spectrum conditions. In this context, a central result of [MN23], based on the irreducible case that is dealt with in [FNÓ23], is:

Theorem 3. *For every connected reductive linear Lie group G and any Euler element $h \in \mathfrak{g}$, there exists a causal symmetric space $M = G/H$ such that for* **all** *conneced components $W \subseteq W_M^+(h)$ and* **all** *(anti-)unitary representations $(U, \mathcal{H})$ of G_{τ_h}, the net $\mathsf{H}^{\max}$ satisfies* (Iso), (Cov), (BW) *and* (RS).

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Quasi-free isomorphisms of second quantization von Neumann algebras and modular theory

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(joint work with Roberto Conti)

1. Motivations

We present the work [CM], whose main motivation is to try to understand a classical result by Eckmann and Fröhlich on the local quasi-equivalence of vacua of different masses of the Klein-Gordon field [EF74] (proven with methods form constructive QFT) in terms of modular theory. The main tool employed towards this end is the quasi-equivalence criterion of Araki and Yamagami [AY82].

2. Abstract result

Our setting is the following. Let H be a complex Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and let $e^H = \bigoplus_{n=0}^{+\infty} H^{\otimes sn}$ be the associated symmetric Fock space, in which the coherent vectors $e^x := \bigoplus_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} x^{\otimes n}$, $x \in H$, form a total set. We also consider on e^H the Weyl unitaries $W(x), x \in H$, defined by their action on $\Omega := e^0$ (vacuum vector) and by the canonical commutation relations (CCR):

$$W(x)\Omega := e^{-\frac{1}{4}\|x\|^2} e^{ix/\sqrt{2}}, \qquad x \in H,$$
$$W(x)W(y) = e^{-\frac{i}{2}\Im\langle x,y\rangle} W(x+y), \qquad x, y \in H.$$

For any standard subspace K of H (i.e., a real subspace such that $\overline{K + iK} = H$ and $K \cap iK = \{0\}$), the von Neumann algebra

$$A(K) = \{W(h) \mid h \in K\}''$$

on e^H, is called the second quantization algebra of K. Moreover, K defines a closed, densely defined conjugate linear operator

$$s : K + iK \to K + iK, \quad s(h + ik) = h - ik, \qquad h, k \in K.$$

and if $s = j\delta^{1/2}$ is the polar decomposition, j and δ are the modular conjugation and the modular operator of K. Their second quantizations $J = \Gamma(j)$, $\Delta = \Gamma(\delta)$ are respectively the modular conjugation and the modular operator of $A(K)$ with respect to Ω [EO73, Lon].

A *Bogolubov transformation* between standard subspaces $K_1, K_2 \subset H$ is a real linear bijection $Q : K_1 \to K_2$ preserving the symplectic form, i.e., $\Im\langle Qh, Qk\rangle = \Im\langle h, k\rangle$, $h, k \in K_1$. Given such a map, the C*-algebras generated by the Weyl operators $W(k)$ and $W(Qk)$, $k \in K_1$, are isomorphic, and it is then natural to ask under which condition this isomorphism extends to an isomorphism between the respective von Neumann algebras $\phi : A(K_1) \to A(K_2)$. If this is the case, ϕ is called the *quasi-free isomorphism* induced by Q.

The problem of the existence of the quasi-free isomorphism is equivalent to the problem of the quasi-equivalence of the states ω and ω_Q on the C*-algebra generated by $W(k)$, $k \in K_1$, defined by

$$\omega(W(k)) = e^{-\frac{1}{4}\|k\|^2}, \quad \omega_Q(W(k)) = e^{-\frac{1}{4}\|Qk\|^2}, \qquad k \in K_1.$$

The relevance of the modular structures of K_1, K_2 for this problem can be understood from the fact that they relate the symplectic structures and the real Hilbert space ones of K_1, K_2. Indeed, if

$$R_j := i\frac{\delta_j - 1}{\delta_j + 1}, \qquad j = 1, 2,$$

is the *polariser* of K_j, there holds $\Im\langle h, k\rangle = \Re\langle h, R_j k\rangle$, $h, k \in K_j$.

Applying then the very general quasi-equivalence criterion of [AY82], one obtains the following result, in which $Q^\dagger : K_2 \to K_1$ is the adjoint of Q w.r.t. the real scalar products of K_1, K_2, and $Q^\dagger Q$ is extended to $K_1 + iK_1$ by complex linearity.

Theorem 1. *The Bogolubov transformation $Q : K_1 \to K_2$ induces a quasi-free isomorphism if and only if:*

(i) *Q is bounded (w.r.t. the norm of H);*
(ii) *$(1 + iR_1)^{1/2} - (Q^\dagger Q + iR_1)^{1/2}$ is Hilbert-Schmidt on $K_1 + iK_1$, endowed with the graph scalar product of s_1.*

A sufficient condition for (ii) is that $1 - Q^\dagger Q$ is of trace class on K_1, while a necessary condition is that $1 - Q^\dagger Q$ is Hilbert-Schmidt on K_1. Moreover, (ii) is also equivalent to the fact that the operators

$$(*) \qquad 1 - Q^\dagger Q, \quad \frac{1}{\sqrt{1+\delta_1}} - Q^{-1}\frac{1}{\sqrt{1+\delta_2}}Q$$

are both Hilbert-Schmidt on $K_1 + iK_1$. In the case in which δ_1, δ_2 are bounded, using powerful results of [BS67], it is possible to show that the fact that $1 - Q^\dagger Q$ is Hilbert-Schmidt on K_1 is equivalent to (ii).

3. Applications to QFT

The one particle space of the Klein-Gordon field in d spatial dimensions is $H = L^2(\mathbb{R}^d)$. On it, the operator $\omega_m := (-\Delta + m^2)^{1/2}$ is defined by functional calculus. For $d = 1$ and $I \subset \mathbb{R}$ an open interval, the space

$$K_m(I) := \left\{\omega_m^{-1/2} f + i\omega_m^{1/2} g \, : \, f, g \in C_c^\infty(I, \mathbb{R}), \, \textstyle\int_I f = 0 = \int_I g\right\}^-, \qquad m \geq 0,$$

is a standard subspace of H for $m > 0$, and of

$$H_0 := \left\{\omega_0^{-1/2} f + i\omega_0^{1/2} g \, : \, f, g \in C_c^\infty(\mathbb{R}, \mathbb{R}), \, \textstyle\int_{\mathbb{R}} f = 0 = \int_{\mathbb{R}} g\right\}^-$$

for $m = 0$. The restriction to zero average functions is needed to avoid the infrared divergence of the scalar field in $d = 1$. The map

$$Q : K_m(I) \to K_0(I), \quad \omega_m^{-1/2} f + i\omega_m^{1/2} g \mapsto \omega_0^{-1/2} f + i\omega_0^{1/2} g,$$

is a Bogolubov transformation, and it can be shown that $1 - Q^\dagger Q$ is of trace class [CM20], so, by the above results, it induces a quasi-free isomorphism of the respective second quantization von Neumann algebras. Equivalently, the restriction of the massive and massless vacua to the nets generated by the derivative of the time zero field and momentum are locally quasi-equivalent.

For $d = 2, 3$ and $B \subset \mathbb{R}^d$ the unit ball, the real subspace

$$K_m(B) := \mathcal{L}_-(B) + i\mathcal{L}_+(B), \qquad \mathcal{L}_\pm(B) := \overline{\omega_m^{\pm 1/2} C_c^\infty(B, \mathbb{R})}, \qquad m \geq 0.$$

is a standard subspace of H for all $m \geq 0$, and, given $m > 0$,

$$Q : K_m(B) \to K_0(B), \quad \omega_m^{-1/2} f + i\omega_m^{1/2} g \mapsto \omega_0^{-1/2} f + i\omega_0^{1/2} g,$$

is a Bogolubov transformation, for which one can compute

$$Q^\dagger Q = \left(E_- \frac{\omega_m}{\omega_0} E_- + iE_+ \frac{\omega_0}{\omega_m} E_+ i \right) \Big|_{K_m(B)},$$

with $E_\pm : H \to \mathcal{L}_\pm(B)$ the real orthogonal projections. Contrary to the $d = 1$ case, now $1 - Q^\dagger Q$ is most likely not of trace class. However, the above formula can be used to estimate the integral kernel of $E_{K_m(B)}(1 - Q^\dagger Q)E_{K_m(B)}$ (with $E_{K_m(B)}$ the real orthogonal projection onto $K_m(B)$), and obtain the following partial result towards the existence of the quasi-free isomorphism induced by Q.

Theorem 2. *The operator $1 - Q^\dagger Q$ is Hilbert-Schmidt on $K_m(B) + iK_m(B)$ (and then on $K_m(B)$), and its Hilbert-Schmidt norm vanishes for $m \to 0^+$.*

Unfortunately, proving the Hilbert-Schmidt property of the second operator in $(*)$ seems to require a much more detailed knowledge of the massive modular operator than is presently available.

As a byproduct of the above result, if $\delta_{m,B}$ is the modular operator of $K_m(B)$, then the Hilbert-Schmidt norm (on $K_m(B) + iK_m(B)$) of $(\lambda - \delta_{m,B})^{-1} - Q^{-1}(\lambda - \delta_{0,B})^{-1}Q$, $\lambda \in \mathbb{C} \setminus [0, +\infty)$, vanishes for $m \to 0^+$, i.e., the resolvents of the local modular operators depend continuously on the mass.

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Twisted Araki-Woods Algebras: structure and inclusions

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(joint work with Gandalf Lechner)

Fock spaces and second quantization are central concepts in algebraic quantum theory and exist in various forms. From the physics perspective, the better known examples are the Boltzmann-Fock space $\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, the Bose-Fock space $\mathcal{F}_F(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n \mathcal{H}^{\otimes n}$, and the Fermi-Fock space $\mathcal{F}_{-F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{A}_n \mathcal{H}^{\otimes n}$, where $\mathcal{S}_n$ and $\mathcal{A}_n$ are the symmetrization and anti-symmetrization operators, which are used in the description of interaction-free Bosonic and Fermionic models [BR97]. The use of symmetrization and anti-symmetrization maps on the n-particle components capture the fact that bosons satisfy CCR and fermions CAR, respectively, and more general commutation relations such as the q-deformed commutation relations require the introduction of twisted Fock spaces [BS91], whose construction holds in much more generality than only q-deformed commutation relations and are relevant in studying representations of Wick algebras [JSW95]. Analogous spaces, called S-symmetric Fock spaces, are also relevant in integrable models in quantum field theory when a prescribed two-particle scattering matrix S is given [Lec23], [AL17].

Twisted Fock Spaces and Twisted Araki-Woods Algebras. Following [BS91] and [JSW95], given a separable Hilbert $\mathcal{H}$ and an operator $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ with $\|T\| \leq 1$ we define, for $n \in \mathbb{N}$, the operators $T_j, R_{T,n}, P_{T,n} \in \mathcal{B}(\mathcal{H}^{\otimes n})$, $1 \leq j \leq n-1$, by $T_j = 1^{\otimes(j-1)} \otimes T \otimes 1^{\otimes(n-j-1)}$, $R_{T,n} := 1 + T_1 + T_1T_2 + \ldots + T_1 \cdots T_{n-1}$, $P_{T,1} := 1$, $P_{T,n+1} := (1 \otimes P_{T,n}) R_{T,n+1}$. In case $P_{T,n}$ is positive for all $n \in \mathbb{N}$ we say that T is a twist and denote the set of all twists $\mathcal{T}_{\geq}$.

In case $T \in \mathcal{T}_{\geq}$, we define $\mathcal{H}_{T,n}$ as the closure of the quotient $\mathcal{H}^{\otimes n}/\ker(P_{T,n})$ with respect to the inner product $\langle [\psi_n], [\phi_n] \rangle_{T,n} := \langle \psi_n, P_{T,n}\phi_n \rangle$, where the square-brackets denote equivalence classes and $\psi_n, \phi_n \in \mathcal{H}^{\otimes n}$. Finally, the twisted Fock space is $\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_{T,n}$ provided with the natural inner product $\langle \cdot, \cdot \rangle_T$. It is worth mentioning that, as the afore-used notation suggests, the case $T = 0$, $T = F$, and $T = -F$, where F is the tensor flip, correspond respectively to the Boltzmann-, Bose-, and Fermi-Fock spaces, but there are many more operators that are twists. In fact, it is known that if $T = T^*$ and T satisfies one of the following conditions (*i*) $\|T\| \leq \frac{1}{2}$; (*ii*) $T \geq 0$; (*iii*) $\|T\| \leq 1$ and the Yang-Baxter equation holds, *i.e.* $T_1T_2T_1 = T_2T_1T_2$, then $T \in \mathcal{T}_{\geq}$.

The recursive formula defining $P_{T,n}$ makes, for each $\xi \in \mathcal{H}$, the twisted left creation operator $a^{\star}_{L,T}(\xi) : \mathcal{H}^{\otimes n}/\ker P_{T,n} \to \mathcal{H}^{\otimes(n+1)}/\ker P_{T,n+1}$ given by the formula $a^{\star}_{L,T}(\xi)[\Psi_n] := [\xi \otimes \Psi_n]$ a well-defined operator which naturally extends to densely define operator on $\mathcal{F}_T(\mathcal{H})$ denoted by the same symbol. Its adjoint with respect to $\langle\cdot,\cdot\rangle_T$ can be calculated and turns out to be the twisted left annihilation operator $a_{L,T}(\xi)[\Psi_n] = [a_L(\xi)R_{T,n}\Psi_n]$, where $a_L(\xi)$ is the usual (untwisted) annihilation operator.

As usual, we can define the essentially self-adjoint field operators $\phi_{L,T}(\xi) = a^\star_{L,T}(\xi) + a_{L,T}(\xi)$ and define, following [CdSL23], for a standard subspace $H \subset \mathcal{H}$, the left twisted Araki-Woods algebras

Definition 1. Given a closed real subspace $H \subset \mathcal{H}$ and a twist $T \in \mathcal{T}_\geq$, we define the (left) T-twisted Araki-Woods von Neumann algebra

$$\mathcal{L}_T(H) := \{\exp(i\phi_{L,T}(h)) \,:\, h \in H\}'' \subset \mathcal{B}(\mathcal{F}_T(\mathcal{H})).$$

It is easy to prove that H being cyclic in $\mathcal{H}$ implies the Fock vacuum Ω to be cyclic for $\mathcal{L}_T(H)$. The natural question to be asked is under which conditions Ω is separating for $\mathcal{L}_T(H)$ and what is the modular data of this pair.

Twisted Araki-Woods Algebras and Standard Vectors. Under the assumption that Ω is separating, we have two modular data to consider: The one originating from H, denoted by J_H, and Δ_H (see [Lon08]); and the one originating from the pair $(\mathcal{L}_T(H), \Omega)$, denoted by J and Δ. In order to have $\Delta_{|_{\mathcal{H}\cap\mathrm{Dom}(\Delta)}} = \Delta_H$, we introduce the concept of compatibility:

Definition 2. Let $H \subset \mathcal{H}$ be a standard subspace. The twists *compatible with* H are the elements of

$$\mathcal{T}_\geq(H) := \{T \in \mathcal{T}_\geq : [\Delta_H^{it} \otimes \Delta_H^{it}, T] = 0 \text{ for all } t \in \mathbb{R}\}.$$

Under the assumption of compatibility and Ω being separating for $\mathcal{L}_T(H)$, one can explore the KMS condition to prove two conditions about the twist:

(i) T is braided, *i.e.* T satisfies the Yang-Baxter equation $T_1T_2T_1 = T_2T_1T_2$;

(ii) T is *crossing-symmetric*, *i.e.* for all $\psi_i \in \mathcal{H}$, $1 \leq i \leq 4$

$$T(t) := \langle\psi_1 \otimes \psi_2, (\Delta_H^{it} \otimes 1)T(1 \otimes \Delta_H^{-it})\psi_3 \otimes \psi_4\rangle$$

must have a continuous and bounded extension to the strip in the complex plane with $0 \leq \mathrm{Im}(t) \leq \frac{1}{2}$ and analytic in its interior satisfying the boundary condition

$$T\left(t + \frac{i}{2}\right) := \langle\psi_2 \otimes J_H\psi_4, (1 \otimes \Delta_H^{-it})T(\Delta_H^{it} \otimes 1)J_H\psi_1 \otimes \psi_3\rangle.$$

On the other hand, in case T satisfies the Yang-Baxter equation and crossing-symmetry, the analogous construction for right twisted operators is possible and it is easy to see that the twisted right Araki-Woods algebra satisfies $\mathcal{R}_T(H') \subset \mathcal{L}_T(H)'$ where H' is the symplectic complement of the standard subspace H. These results can be collect in the following theorem which is one of the main results on [CdSL23]:

Theorem 3. *Let $H \subset \mathcal{H}$ be a standard subspace and $T \in \mathcal{T}_\geq(H)$ a compatible twist. The following are equivalent:*

(1) Ω is separating for $\mathcal{L}_T(H)$;

(2) T is braided and crossing symmetric w.r.t. H.

Inclusions of Twisted Araki-Woods Algebras. From the quantum field theory perspective, one is interested in a net of von Neumann algebras indexed by the open regions of a manifold. Among several other physically motivated conditions, we mention: (i) *isotony*, meaning that if two space-time regions $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$; and *causality*, meaning that if $\mathcal{O}_1$ is space-like separated from $\mathcal{O}_2$, then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$. In the standard subspace language, it justifies considering the relative commutant of the inclusion $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$, where $K \subset H$ are standard subspaces, namely, $\mathcal{C}_T(K,H) := \mathcal{L}_T(H) \cap \mathcal{L}_T(K)'$.

Two situations are studied in [CdSL23] and [CdSL], one showing that the relative commutant can be very big (a type III von Neumann algebra) and the other showing that the relative commutant may consists only of multiples of the identity.

Theorem 4. *Let $K \subset H$ be an inclusion of standard subspaces and let $T \in \mathcal{T}_{\geq}(H)$ be a braided crossing-symmetric twist w.r.t to H with norm $\|T\| < 1$.*

(1) If L^2-nuclearity holds on the standard subspace level, i.e. $\|\Delta_H^{1/4}\Delta_K^{-1/4}\|_1 < 1$, *where $\|\cdot\|_1$ is the trace norm on $\mathcal{H}$, and T is also compatible with K, then $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ satisfies L^2-nuclearity and is quasi-split. If, in addition, $\mathcal{L}_T(H)$ is of type III, also the relative commutant $\mathcal{C}_T(K,H)$ is of type III.*

(2) If $\Delta_H^{\frac{1}{4}} E_K$ is not compact, where E_K is the real orthogonal projection onto K, then $\mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathbb{C}1$.

The assumptions on item (1) on the above theorem are, in general, too strong and item (2) shows that physical models with $\|T\| < 1$ are usually non-local. Understanding what happens in the situation when $\Delta_H^{\frac{1}{4}} E_K$ is compact, but L^2-nuclearity doesn't hold, and when $\|T\| = 1$ are still under investigation.

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Holomorphic extension in a locally convex setting and standard subspaces

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(joint work with Karl-Hermann Neeb)

In the framework of one-parameter operator groups on locally convex spaces, we discussed holomorphic extensions with respect to the parameter, from the real line to suitable horizontal strips in the complex plane. In the special case of one-parameter unitary groups $e^{itH} = (e^H)^{\mathrm{i}t}$ on Hilbert spaces, we recover the complex powers $e^{\mathrm{i}zH} = (e^H)^{\mathrm{i}z}$ of the positive operator defined as the exponential of the infinitesimal generator. This Hilbert space setting is however too special for the applications to certain constructions of nets of standard subspaces in the framework of Lie group representations, as they appear in Algebraic Quantum Field Theory in connection with the Kubo–Martin–Schwinger (KMS) boundary conditions. The constructions of this type are our main motivation. They require one-parameter operator groups on spaces of distribution vectors of unitary representations of Lie groups as presented below in some more detail.

A general KMS boundary condition. We assume the following setting:

- $\mathcal{S}_\pi := \mathbb{R} + \mathrm{i}(0,\pi) \subset \mathbb{R} + \mathrm{i}[0,\pi] =: \overline{\mathcal{S}}_\pi \subset \mathbb{C}$
- $\mathcal{Y}$ a complex Hausdorff locally convex space.
- for every subset $\Gamma \subseteq \mathbb{C}$ we denote by $\mathcal{O}_\partial(\Gamma, \mathcal{Y})$ the set of all continuous functions $f\colon \Gamma \to \mathcal{Y}$ that are weakly holomorphic on the interior of Γ
- $(U_t)_{t\in\mathbb{R}}$ is a 1-parameter subgroup of $\mathrm{GL}(\mathcal{Y})$
- $J\colon \mathcal{Y} \to \mathcal{Y}$ is an anti-linear continuous map
- the following compatibility condition is satisfied: $(\forall t \in \mathbb{R}) \quad JU_t = U_t J$

Then $v \in \mathcal{Y}$ is said to satisfy the *KMS condition* ($v \in \mathcal{Y}_{\mathrm{KMS}}$) if there exists a function $f \in \mathcal{O}_\partial(\overline{\mathcal{S}}_\pi, \mathcal{Y})$, satisfying the boundary condition

$$(\forall t \in \mathbb{R}) \quad f(t) = U_t v,\ f(t + \mathrm{i}\pi) = JU_t v \quad (= Jf(t)).$$

A construction of standard subspaces in a representation theoretic setting. We now assume the following:

- G is a finite-dimensional real Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp_G\colon \mathfrak{g} \to G$.
- $U\colon G \to \mathcal{U}(\mathcal{H}),\ g \mapsto U_g$ is a unitary representation of G with continuous orbit maps $U^\xi(g) = U_g\xi$.
- $\mathcal{H}^\infty := \{\xi \in \mathcal{H} : U^\xi \in \mathcal{C}^\infty(G, \mathcal{H})\}$ is endowed with its unique Fréchet topology for which the inclusion map $\mathcal{H}^\infty \hookrightarrow \mathcal{H}$ is continuous.
- $\mathrm{d}U\colon \mathfrak{g} \to \mathcal{L}(\mathcal{H}^\infty)$, $\mathrm{d}U(x)v := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} U_{\exp_G(tx)}v$;
- The space $\mathcal{H}^{-\infty}$ of continuous antilinear functionals on $\mathcal{H}^\infty$ is endowed with its weak-$*$-topology and we write

  $$\langle \cdot, \cdot \rangle\colon \mathcal{H}^\infty \times \mathcal{H}^{-\infty} \to \mathbb{C}$$

  for the antiduality pairing that coincides on $\mathcal{H}^\infty \times \mathcal{H}$ with the scalar product of $\mathcal{H}$ (antilinear in the first variable).

- $U^{\pm\infty}: G \to \mathrm{GL}(\mathcal{H}^{\pm\infty})$ are the representations naturally associated to the unitary representation $U: G \to \mathcal{U}(\mathcal{H})$.
- We also define for $h \in \mathfrak{g}$ and $t \in \mathbb{R}$

$$U_{h,t} := U(\exp_G(th)),\ U^{\pm\infty}_{h,t} := U^{\pm\infty}(\exp_G(th)).$$

- $J: \mathcal{H} \to \mathcal{H}$ is a conjugate-linear surjective isometry satisfying $JU_{h,t} = U_{h,t}J$ for every $t \in \mathbb{R}$, and moreover $J\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$, hence we also have its corresponding operators $J^{\pm\infty}: \mathcal{H}^{\pm\infty} \to \mathcal{H}^{\pm\infty}$.

We then obtain a standard subspace of $\mathcal{H}$ defined by

$$\mathtt{V} := \{v \in \mathcal{D}(\Delta^{1/2}) : \Delta^{1/2}v = Jv\} \quad \text{for} \quad \Delta := e^{2\pi i \mathrm{d}U(h)}$$

(cf. [NÓ17, §3.1]). Moreover, [NØÓ21, Prop. 2.1] implies $\mathtt{V} = \mathcal{H}_{\mathrm{KMS}}$.

The position of the standard subspace within the space of distribution vectors. Our main results can now be stated as follows:

- $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is the (weak-$*$-)closure of $\mathtt{V}$ in $\mathcal{H}^{-\infty}$ ([BN23, Thms. 6.2 and 6.5]);
- $\mathcal{H}^{-\infty}_{\mathrm{KMS}} \cap \mathcal{H} = \mathtt{V}$ ([BN23, Thm. 6.4]).
- $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is the annihilator of $J\mathtt{V} \cap \mathcal{H}^\infty$ with respect to the imaginary part of the pairing ([BN23, Cor. 6.8]).

Here we define the space $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ via the KMS boundary condition with respect to the 1-parameter group $(U^{-\infty}_{h,t})_{t\in\mathbb{R}}$ in $\mathrm{GL}(\mathcal{H}^{-\infty})$ and the continuous antilinear map $J^{-\infty}: \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$.

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Inclusions of Standard Subspaces

Gandalf Lechner

(joint work with Ricardo Correa da Silva)

Standard subspaces naturally appear in the context of von Neumann algebras, where any von Neumann algebra in standard form gives rise to a standard subspace encoding its modular data, and in quantum field theory, where standard subspaces encode localization regions. From this perspective, standard subspaces appear as auxiliary objects. There is however growing evidence that standard subspaces are interesting objects in their own right – for example, they lead to an independent notion of entropy [CLR19], can naturally be constructed on the basis of suitable Lie group representations [MN21], and lie at the basis of the recently introduced twisted Araki-Woods algebras [CdSL23].

In these and other applications, one is typically not interested in a single standard subspace (the set of all standard subspaces H of a complex Hilbert space $\mathcal{H}$ can easily be classified, see [Lon08, Cor. 2.1.5]), but rather in families of standard subspaces and their intersection, inclusion and covariance properties. The topic of this talk was therefore to initiate an abstract discussion of inclusions

$$K \subset H \subset \mathcal{H}$$

of standard subspaces, without reference to von Neumann algebras or group representations. This can be seen as an analogue of the study of inclusions of von Neumann algebras, or more specifically subfactors.

We review some known results about inclusions of standard subspaces and then reported on joint work in progress with R. Correa da Silva [CdSL].

Inclusions and irreducible inclusions. Given a standard subspace K, can we embed it properly into a larger standard subspace H, or can we properly embed a smaller standard subspace into K? This question is answered in the following lemma:

Lemma 1. [FG00] *Let $K \subset \mathcal{H}$ be a standard subspace. Then the following are equivalent:*

(1) There exists a standard subspace $H \subset \mathcal{H}$ such that $K \subsetneq H$.
(2) There exists a standard subspace $H \subset \mathcal{H}$ such that $H \subsetneq K$.
(3) The modular operator Δ_K is unbounded.

Guided by the comparison with subfactor theory, we are particularly interested in understanding *irreducible* inclusions, which by definition are inclusions $K \subset H$ with $K' \cap H = \{0\}$. Here K' denotes the symplectic complement of K. Clearly, this requires in particular $K' \cap K = \{0\}$, i.e. K must be a factorial subspace (a factor, for short). Recall that a factor has a well-defined cutting projection $P_K : K + K' \to K$, $k + k' \mapsto k$ [CLR19].

The basic result in this regard is a reformulated version of a proposition from [FG00].

Proposition 2. *Let $K \subset \mathcal{H}$ be a standard subspace. Then the following are equivalent:*

(1) There exists a standard subspace $H \subset \mathcal{H}$ such that $K \subsetneq H$ is irreducible.
(2) There exists a standard subspace $H \subset \mathcal{H}$ such that $H \subsetneq K$ is irreducible.
(3) The modular operator Δ_K is unbounded, K is a factor, and the cutting projection P_K of K is unbounded.

This proposition states that irreducible inclusions of standard subspaces exist in abundance. A central question is then how to detect whether a given inclusion is irreducible, or how to detect whether the relative symplectic complement $K' \cap H$ is cyclic (hence standard).

Detecting irreducibility. Let K, H be a pair of standard subspaces. Then [BGL02, Prop. 4.1]

$$K' \cap H + i(K' \cap H) = \{v \in \mathrm{dom}(S_K^* S_H) \,:\, S_K^* S_H v = v\}.$$

This characterization is however often difficult to use as it leads to intricate domain questions. The same holds true for other characterizations that we derived for $K' \cap H$ in terms of polarizers and projections [CdSL].

Comparing with the von Neumann algebraic situation, two notions that are helpful tools in the understanding of relative commutants are the split property [DL84] and modular nuclearity [BDL90]. We give standard subspace formulations for both of them and investigate their consequences in [CdSL]. Here we focus on the nuclearity aspects.

Definition 3. An inclusion $K \subset H$ of standard subspaces is said to satisfy modular nuclearity if the real linear operator $\Delta_H^{1/4} E_K$, where $E_K : \mathcal{H} \to K$ is the real orthogonal projection onto K, is trace class.

Making use of [LRT78, BDL90, LS16], we then prove:

Theorem 4. [CdSL] *Let $K \subset H$ be an inclusion of factor standard subspaces satisfying modular nuclearity. Then* $\dim(K' \cap H) = \infty$.

A class of examples. As a concrete class of examples, we consider the irreducible one-dimensional standard pair, namely the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, \frac{dp}{p})$ and the standard subspace $H \subset \mathcal{H}$ given by the data (see [LL14, Sect. 4] for this and other equivalent formulations)

$$(\Delta_H^{it}\psi)(p) = \psi(e^{-2\pi t}p), \qquad (J_H\psi)(p) = \overline{\psi(p)}.$$

The one-parameter group of unitaries $(U(x)\psi)(p) = e^{ipx}\psi(p)$ acts half-sidedly by endomorphisms of H, namely $U(x)H \subset H$, $x \geq 0$. It is known that the semigroup of all unitaries $V \in \mathcal{U}(\mathcal{H})$ that commute with $U(x)$, $x \in \mathbb{R}$, and satisfy $VH \subset H$, are precisely the unitaries of the form $V = \varphi(P)$, where P is the generator of U and φ an inner function on the upper half plane satisfying the symmetry condition $\varphi(-p) = \overline{\varphi(p)}$, $p > 0$ [LW10, Thm. 2.3].

We are therefore presented with the family of concrete inclusions $\varphi(P)H \subset H$. In the talk it was explained that the modular nuclearity condition fails except for quite specifically chosen inner functions φ. Nonetheless it is possible to understand and sometimes explicitly compute the relative symplectic complement $\varphi(P)H' \cap H$, which can be $\{0\}$, finite-dimensional, infinite-dimensional, or cyclic depending on φ. In particular, there are interesting relations relating the number of zeros of the inner function φ and the dimension of $\varphi(P)H' \cap H$.

The structures found in this class of examples are currently being investigated alongside more general methods for analyzing relative symplectic complements of standard subspaces [CdSL].

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Unitarity and reflection positivity in two-dimensional conformal field theory

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(joint work with Maria Stella Adamo, Yuto Moriwaki)

Two-dimensional conformal field theories (2d CFTs) have been studied extensively in various setting, from algebraic to analytic. One of the algebraic settings is Vertex Operator Algebras (VOAs), which formalize chiral components of a 2d CFT in terms of formal series in z. A VOA that corresponds to a quantum field theory must satisfy a condition called unitarity [DL14].

It is known [CKLW18] that one can construct Wightman fields on S^1 from quasi-primary fields in a unitary VOA satisfying so-called polynomial energy bounds. As S^1 can be seen as the one-point compactification of $\mathbb{R}$, one can see these Wightman field as Wightman fields on one of the lightrays in $\mathbb{R}^{1+1}$. On the other hand, there are Osterwalder-Schrader axioms (OS axioms) [OS73, OS75] that can accomodate many interacting QFTs, mostly the massive ones. From the Schwinger functions satisfying the OS axioms, one can reconstruct Wightman fields. As the

VOA formalism considers the Euclidean setting, it should be possible to construct Schwinger functions satisfying the OS axioms, at least for a nice class of 2d CFTs.

The OS axioms in the two-dimensional Euclidean space concern the Schwinger functions $\{S_n(z_1, \cdots, z_n)\}$, where S_n is a distribution on a subset of $\mathbb{R}^{2n}$ excluding the coinciding points, $z_j \neq z_k$ for $j \neq k$. Among the OS axioms, we consider only reflection positivity, which assures the Hilbert space structure in the resulting Wightman field theory. Up to a conformal transformation [FFK89], this means that, for a finite sequence of test functions $\{f_n(z_1, \cdots, z_n)\}$ supported in the region $|z_1| < |z_2| < \cdots < |z_n|$, one should have

$$0 \leq \sum_{j,k} \int \overline{f_j(\bar{z}_j^{-1}, \cdots, \bar{z}_1^{-1})} f_j(z_{j+1}, \cdots, z_{j+k}) S_{j+k}(z_1, \cdots, z_{j+k}) \tag{1}$$
$$|J(z_1) \cdots J(z_n)| dx_1 dy_1 \cdots dx_{j+k} dy_{j+k},$$

where $z_j = x_j + iy_j \in \mathbb{R}^2$ and $J(z)dxdy$ is a measure on $\mathbb{R}^2$ invariant under the reflection $z \mapsto \bar{z}^{-1}$ when including the scaling factor coming from the conformal transformations for fields (we define $S_n(z_1, \cdots, z_n) = \langle \Omega, \phi(z_1) \cdots \phi(z_n)\Omega \rangle$, see below).

Let $V, Y(\cdot, z)$ be a unitary VOA and $v \in V$ be a quasi-primary vector. For a given v, we denote $\underline{\phi}(z) = Y(v, z) = \sum_n \underline{\phi_n} z^{-n-d}$, where $d \in \mathbb{N}$ is the conformal dimension of v and $\underline{\phi_n} \in \mathrm{End}(V)$. Unitarity means that V is equipped with a positive-definite inner product $\langle \cdot, \cdot \rangle$ and V is generated by quasi-primary fields satisfying $(\phi_n)^* = \phi_{-n}$ [CKLW18]. We put $\phi(z) = \sum_n \phi_n z^{-n}$ and define

$$\begin{aligned} S_n(z_1, \cdots, z_n) &= \langle \Omega, \phi(z_1) \cdots \phi(z_n)\Omega \rangle \\ &= \sum_{k_1, \cdots, k_n} \langle \Omega, \phi_{k_1} \cdots \phi_{k_n} \Omega \rangle z_1^{-k_1} \cdots z_n^{-k_n}, \end{aligned} \tag{2}$$

where $\Omega \in V$ is the vacuum. This is, at this point, a formal series in $z_1, \cdots, z_n$.

We assume polynomial energy bounds for V [CKLW18]. This means that $\|\phi_n \Psi\| \leq C(|n| + 1)^s \|(L_0 + \mathrm{id})^p \Psi\|$ for some $C, s, p > 0$. Then one can show that the series (2) converges for $|z_1| < \cdots < |z_n|$. Using again polynomial energy bounds, it is also possible that S_n defines a distribution as required in the OS axioms.

If we consider $z \in S^1$, we have the relation $(\phi(z)^*) = \phi(z)$. By analytically continuing this equation, we have (weakly) $\phi(z)^* = \phi(\bar{z}^{-1})$. As for reflection positivity, by the positive definiteness of the scalar product, we have $\langle \Psi, \Psi \rangle \geq 0$, where

$$\Psi = \sum_j \int f_j(z_1, \cdots, z_j) \varphi(a_1, z_1) \cdots \varphi(a_j, z_j)\Omega \, |J(z_1) \cdots J(z_j)| dx_1 dy_1 \cdots dx_j dy_j$$

is a vector in the completion $\overline{V}$ of V. One can show that $\langle \Psi, \Psi \rangle$ is equal to the right-hand side of (1), therefore, reflection positivity holds under unitarity and polynomial energy bounds. Other OS axioms can be checked as well [Mor22], and also the linear growth condition [OS75] from polynomial energy bounds. Altogether, quasi-primary fields in unitary VOA can generate Schwinger functions, from which one can construct Wightman fields as is done in Constructive QFT [GJ87].

We hope to extend this to full vertex operator algebras [Mor20], as we already have Wightman fields for a class of full CFT [AGT]. We hope to find Hilbert space structure for the Euclidean fields. This Euclidean construction could be useful when one tries to perturb CFTs to obtain massive models [JT].

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Reflection Positivity for finite dimensional Lie groups

Maria Stella Adamo

(joint work with Karl-Hermann Neeb, Jonas Schober)

Reflection positivity appears as one of Osterwalder–Schrader axioms, used to study a large class of quantum field theories (QFTs) [OS73, OS75]. Such axioms are used in Constructive QFT, for example, to construct interacting or massive QFTs, see, e.g., [GJ87]. Osterwalder–Schrader axioms are given for a *Euclidean* field theory, providing tools to reconstruct Wightman fields for a Minkowskian (Lorentzian) quantum field theory by analytic continuation of Euclidean Schwinger functions.

As a consequence of a similar duality between the Euclidean motion Lie group and the Poincaré Lie group on the Minkowski space, see [LM75], one can investigate instead reflection positivity for unitary Lie group representations $\mathcal{U}$ on Hilbert spaces $\mathcal{H}$ equipped with a θ-positive closed subspace $\mathcal{H}_+ \subseteq \mathcal{H}$, where θ is an involution on $\mathcal{H}$.

The represented Lie groups G are paired with a subsemigroup S and an involution τ such that $\tau(\mathrm{S})^{-1} = \mathrm{S}$. Note that S and τ play the role of $\mathcal{H}_+$ and θ respectively in the group theoretic context. A unitary representation $\mathcal{U} : \mathrm{G} \to \mathrm{U}(\mathcal{H})$ is said to be reflection positive if $\mathcal{U}$ is a representation of (G, S) on $(\mathcal{H}, \mathcal{H}_+)$, i.e., $\mathcal{U}(\mathrm{S})\mathcal{H}_+ \subseteq \mathcal{H}_+$, and $\mathcal{U}$ and θ verify a compatibility condition between θ and τ of the form $\theta\mathcal{U}(g)\theta = \mathcal{U}(\tau(g))$ for all $g \in \mathrm{G}$, for further reading, see [NO18]. When the other conditions are satisfied, the compatibility condition is usually difficult to establish.

From a quadruple $(\mathcal{U}, \mathcal{H}, \mathcal{H}_+, \theta)$ as before, in a canonical way, one obtains a new $*$-representation $\widehat{\mathcal{U}}$ of $(\mathrm{S}, \sharp)$ on $\widehat{\mathcal{H}}$, where $s^\sharp := \tau(s)^{-1}$ is the involution induced by τ in S. $\widehat{\mathcal{H}}$ indicates the completion of the quotient of $\mathcal{H}_+$ by the null vectors with respect to the norm induced by θ. This construction of $\widehat{\mathcal{U}}$ from $\mathcal{U}$ involves the so-called Osterwalder-Schrader transform and so we regard $(\mathcal{U}, \mathcal{H}, \mathcal{H}_+, \theta)$ as a Euclidean realization of $(\widehat{\mathcal{U}}, \widehat{\mathcal{H}})$.

One of the simplest example of $(\mathrm{G}, \mathrm{S}, \tau)$, yet rich in information, is given by the real line $\mathbb{R}$ with its subsemigroup of the positive half line $\mathbb{R}_+$ and $\tau = -\mathrm{id}_{\mathbb{R}}$. Analogously, one can consider the triple $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$. In [ANS22] we consider only regular representations $\mathcal{U}$, namely those for which $\cap_{g\in\mathrm{G}}\mathcal{U}(g)\mathcal{H}_+ = \{0\}$ and $\overline{\cup_{g\in\mathrm{G}}\mathcal{U}(g)\mathcal{H}_+} = \mathcal{H}$. For these representations $\mathcal{U}$, every $\mathcal{U}(\mathrm{G})$-invariant subspace in $\mathcal{H}_+$ is trivial and the only $\mathcal{U}(\mathrm{G})$-invariant subspace that contains $\mathcal{H}_+$ is $\mathcal{H}$ itself.

For the real line $\mathbb{R}$, a regular representation $\mathcal{U}$ is a 1-parameter group, that by the spectral form of the Lax–Phillips Representation Theorem, is realized by multiplication on $\mathrm{L}^2(\mathbb{R}, \mathcal{M})$, and its positive subspace corresponds to the Hardy space $\mathrm{H}^2(\mathbb{C}_+, \mathcal{M})$ of the upper half-plane $\mathbb{C}_+$ with values in some higher dimensional multiplicity space $\mathcal{M}$ [LP81, NO18]. Furthermore, in [ANS22] we assume that the multiplicity space is one-dimensional. However, the Lax–Phillips Theorem doesn't recover the involution, and thus doesn't give information on the compatibility condition between θ and τ. Thus, we investigate the issue of classifying the involution θ which verify the compatibility condition and thus produce reflection positive representations.

Under these assumptions, θ is of the form $\theta = \varphi R$, where $\varphi \in L^\infty(\mathbb{R})$ takes values on the unit circle and, for $x \in \mathbb{R}$, $Rf(x) := f(-x)$ is a reflection on the real line. By using a similar characterization of Hankel operators given for the unit disk $\mathbb{D}$, we show that θ defines a Hankel operator by $\mathrm{H}_\theta := P_+\theta P_+^*$. Therefore, we obtain a 1-1 correspondence between positive Hankel operators and unitary reflection positive representations, see [ANS22, Example 1.7 (a)], cf. [Nik02, Nik19, Par88, RR94].

Hankel operators can be characterized through Carleson measures. By Nehari's Theorem [ANS22, Nik02], such a measure has a symbol $h \in \mathrm{L}^\infty(\mathbb{R})$ with values in $\mathbb{S}^1$, and thus a kernel, that allows us to define a new weighted space $\mathrm{L}^2(\mathbb{R}, \nu) \cong \mathrm{L}^2(\mathbb{R})$ through a *-isometric isomorphism that preserves the Hardy space (the positive part) and produces a reflection positive representation $(\mathcal{U}, \mathrm{L}^2(\mathbb{R}, \nu), \mathrm{H}^2(\mathbb{C}_+, \nu), \theta_h)$ [ANS22, Theorem 4.5]. Recently, in [Sch23], if the positive Hankel operator on $\mathrm{H}^2(\mathbb{C}_+)$ is contractive, then there exists a involution θ_h such that $(\mathrm{L}^2(\mathbb{R}), \mathrm{H}^2(\mathbb{C}_+),$

θ_h) is a reflection positive Hilbert space, without modification of the measure. Accordingly, by using the Wold decomposition as a normal form for regular unitary operators $\mathcal{U}$ on $\mathbb{Z}$, we obtain similar results for the triple $(\mathrm{L}^2(\mathbb{S}^1), \mathrm{H}^2(\mathbb{D}), \mathcal{U})$, where $\mathcal{U}$ acts as a multiplication operator by z.

For reflection positive representations on $\mathbb{Z}$ and $\mathbb{R}$ respectively, the positive part of the Hilbert space is realized as a Hardy space H^2 on $\mathbb{D}$ and on $\mathbb{C}_+$ respectively. Such domains are biholomorphically equivalent to the β-strip $\mathbb{S}_\beta$ of all $z \in \mathbb{C}$ such that $\operatorname{Im} z \in (0, \beta)$. Nonetheless, the β-strip exhibits different geometrical features compared to $\mathbb{D}$ and $\mathbb{C}_+$, e.g., the biholomorphism to the upper half-plane $\mathbb{C}_+$ is given by the exponential map, which is not a Möbius transformation, and the boundary of $\mathbb{S}_\beta$ has two connected components, whereas the boundaries of $\mathbb{D}$ and $\mathbb{C}_+$ are both connected. Such a domain naturally appears when one studies reflection positivity for the circle group $(\mathbb{T}_\beta, \mathbb{T}_{\beta,+}, \tau_\beta)$ for $\beta > 0$, where $\mathbb{T}_\beta := \mathbb{R}/\beta\mathbb{Z}$, $\mathbb{T}_{\beta,+}$ is the half-circle and $\tau_\beta(z) := -\mathrm{id}_{\mathbb{T}_\beta}$. Compared to the previous cases of $\mathbb{R}$ and $\mathbb{Z}$, $\mathbb{T}_{\beta,+}$ is *not* a semigroup.

We will start our investigation of reflection positivity for the circle group $\mathbb{T}_\beta$ by looking at reflection positive functions φ, since they provide a way to produce reflection positive representations through GNS-like construction by using the positive definite kernels induced by φ [NO15, NO18]. In the special case of reflection positive functions on the real line $\mathbb{R}$ which verify the β-KMS condition, they constitute a source of standard subspaces. Indeed, in [NO19] such functions on $\mathbb{R}$ are shown to be of the form $\varphi_{\mathbb{R}}(t) := \langle v, \Delta^{-it/\beta} v\rangle$ for $t \in \mathbb{R}$, where v belongs to a standard subspace $\mathrm{V} \subseteq \mathcal{H}$ and (Δ, J) denotes its modular pair.

For the circle group $\mathbb{T}_\beta$, general reflection positive functions $\varphi_{\mathbb{T}_\beta}$ admit an integral representation with respect to a finite Borel measure μ on $\mathbb{R}_+$ [KL81, NO15]. This allows to extend $\varphi_{\mathbb{T}_\beta}$ to a continuous function on $\overline{\mathbb{S}_\beta}$, which is holomorphic in $\mathbb{S}_\beta$ and to obtain by restriction to the lower boundary of $\mathbb{S}_\beta$ a reflection positive function on $\mathbb{R}$ which verify the β-KMS condition [NO15, NO19]. Using the integral representation of $\varphi_{\mathbb{T}_\beta}$, such a restriction is given by the Fourier transform of a finite positive Borel measure ν on $\mathbb{R}$ that verifies β-reflection, i.e., $d\nu(-p) = e^{-\beta p} d\nu(p)$.

On the other hand, in [ANS], finite positive Borel measures ν on $\mathbb{R}$ for which β-reflection holds are in 1-1 correspondence with finite positive Borel measures μ on $\mathbb{R}_+$. To define $\varphi_{\mathbb{T}_\beta}$, we consider the restriction on $[0, \beta]i$ of the Fourier transform of the finite positive Borel measure ν on $\mathbb{R}$, which satisfies the β-reflection condition. Therefore, we can directly show that reflection positive functions $\varphi_{\mathbb{T}_\beta}$ are in 1-1 correspondence with reflection positive functions $\varphi_{\mathbb{R}}$ which verify the β-KMS-condition.

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Maximal Quantum f−Divergences in von Neumann Algebras

Alessio Ranallo

(joint work with Stefan Hollands)

Inspired by recent advances in the study of the capacity of quantum channels between finite-dimensional factors by means of *Geometric Rényi Divergences* [FF21], we study the notion of Maximal Quantum f−Divergences in the setting of von Neumann algebras and Algebraic Quantum Field Theory.

Divergences are used to distinguish between couples of probability measures (and quantum states). Araki's notion of relative entropy is an example of divergence. In Quantum Information Theory an account of *quasi-entropies* is given in [OP93] and a more systematic account of various types of Quantum f−Divergences can be found in [Hia19].

Consider two probability measures $p = \{p_i\}_i, q = \{q_i\}_i \in \mathcal{P}(X)$ on a finite set X, such that $p \ll q$. The relative entropy between these two reads

$$S(p\|q) = -\sum_i p_i \log\left(\frac{q_i}{p_i}\right) = -\sum_i p_i \log\left((R_p^q)_i\right)\ ,$$

where R_p^q denotes the *Radon-Nikodym* derivative of q w.r.t. p. Relative entropy can be generalized to the quantum setting in a number of ways. The Araki's notion of relative entropy comes from the Umegaki's one, where the Radon-Nikodym derivative is replaced by the *relative modular operator*. Indeed, let $\mathcal{M}$ be a von Neumann algebra in standard form acting on the Hilbert space $\mathfrak{H}$. Let $\psi, \varphi \in \mathcal{M}_{*,+}$ be two normal, bounded, and positive functionals with (standard) vector representatives $\Psi, \Phi \in \mathfrak{H}$. For the sake of simplicity, suppose that both ψ and φ are faithful, then

$$S(\varphi\|\psi) = -\langle\Phi\,, \log\left(\Delta_{\psi,\varphi}\right)\Phi\rangle$$

is the formula for Araki's relative entropy. It is then easy to show that in finite dimensions Araki's notion reduces to the Umegaki's original definition

$$S(\varphi\|\psi) = -\mathrm{Tr}\left(\rho_\varphi\left(\log\left(\rho_\psi\right) - \log\left(\rho_\varphi\right)\right)\right)\,,$$

where ρ_φ, resp. ρ_ψ, denotes the matrix representing φ, resp. ψ. However, in finite dimensions, the choice of the operator $\rho_\psi^{1/2}\rho_\varphi^{-1}\rho_\psi^{1/2}$ induce another "entropy-like" quantity

$$S_{\mathrm{BS}}(\varphi\|\psi) := -\mathrm{Tr}\left(\rho_\varphi \log\left(\rho_\psi^{1/2}\rho_\varphi^{-1}\rho_\psi^{1/2}\right)\right)\,.$$

Here, BS is for Belavkin-Staszewski [BS82], where this entropy was introduced. Note that $S_{\mathrm{BS}}(\varphi\|\psi) = S(\varphi\|\psi)$ whenever ρ_φ and ρ_ψ commutes. Araki's relative entropy is an example of *standard divergence*, while the Belavkin-Staszewski notion is an example of *maximal divergence*, see [Hia19] for a more systematic treatise.

We prove that a Kosaki-type formula holds for the Belavkin-Staszewski divergence.

$$S_{\mathrm{BS}}(\varphi\|\psi) = \sup\sup\left\{\varphi(1)\log n - \int_{1/n}^{\infty}\left[\varphi(x_t x_t^*) + \frac{1}{t}\psi(y_t y_t^*)\right]\frac{dt}{t}\right\},$$

where the first sup is taken over $n \in \mathbb{N}$, while the second is over finite range step functions $x_{(\cdot)} : (\frac{1}{n}, \infty) \to \mathcal{M}$ such that $x_t = 1$ for sufficiently small t, such that $x_t = 0$ for sufficiently large t, and where $y_t := 1 - x_t$.

Given two normal linear maps $\alpha\,, \beta : \mathcal{N} \to \mathcal{M}$ that are completely positive and unital, normal *channels* for short, we are able to provide a notion of Belavkin-Staszewski divergence of α w.r.t. to β. The definition generalizes the one for matrix algebras introduced (for generalized divergences of which S_{BS} is an instance of) in [LKDW18], where one takes the sup over all states induced from states on the enlarged system obtained from coupling our initial system, e.g. $\mathcal{M}$ above, with an ancillary system $\mathcal{A}$ (arbitrary), and then precomposing with the dilated channels $\alpha \otimes \mathrm{id}_{\mathcal{A}}$, $\beta \otimes \mathrm{id}_{\mathcal{A}}$:

$$S_{\mathrm{BS}}(\alpha\|\beta) = \sup_{\substack{\mathcal{A} \\ \psi \in (\mathcal{M}\otimes\mathcal{A})_{*,+,1}}} S_{\mathrm{BS}}(\psi\circ(\alpha\otimes\mathrm{id}_{\mathcal{A}})\|\psi\circ(\beta\otimes\mathrm{id}_{\mathcal{A}}))\,.$$

The motivation behind this definition comes from the fact that refined information about the action of channels can be obtained through entanglement. In the case of von Neumann algebras of general type, we provide a generalization of this definition based on the notion of bimodules between two von Neumann algebras.

After presenting some results on channel divergences, we discuss briefly some open questions of relevance to the workshop.

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Localization of positive energy representations for gauge groups on conformally compactified Minkowski space

Bas Janssens

(joint work with Karl-Hermann Neeb)

For a gauge theory associated to a principal K-bundle $P \to M$, the relevant group $\mathcal{G}$ of gauge transformations depends rather sensitively on the boundary conditions at infinity. It contains the group $\mathrm{Gau}_c(P) = \Gamma_c(M, \mathrm{Ad}(P))$ of compactly supported vertical automorphisms[1] of P (the 'local' gauge transformations), but it is usually larger. For instance, if $P = M \times K$ is the trivial bundle, one would expect $\mathcal{G}$ to contain the group K of constant gauge transformations, which are certainly not compactly supported.

On the other hand, if one requires that $\mathcal{G}$ preserves boundary conditions for the (classical) fields at infinity, then the relevant group $\mathcal{G}$ of gauge transformations may be significantly smaller than the group $\mathrm{Gau}(P) = \Gamma(M, \mathrm{Ad}(P))$ of *all* vertical automorphisms. Any input on the following question would be most welcome:

[1] My convention here is that $\mathrm{Ad}(P) = (P \times K)/K$ is the bundle of Lie groups over M associated to P by the conjugation, and $\mathrm{ad}(P) = (P \times \mathfrak{k})/K$ is the bundle of Lie algebras associated to P by the adjoint action.

Question

What, in the gauge theory and geometric setting of your choice, would be examples of relevant groups $\mathrm{Gau}_c(P) \subset \mathcal{G} \subset \mathrm{Gau}(P)$ *of gauge transformations?*

Different gauge theories and space-time geometries will probably lead to different answers. For instance, if the boundary conditions are in terms of a fall-off rate $1/r^k$ for the curvature $F \in \Omega^2(M, \mathfrak{k})$ of the principal connection, then the requirement on the infinitesimal gauge transformations $\xi \in \Gamma(\mathrm{ad}(P))$ will be that the fall-off rate for $\delta F = [\xi, F]$ does not exceed $1/r^k$ for any field F with this property. If K is abelian, then this condition is vacuous. If K is semisimple however, then the above condition is fulfilled only if $\xi \sim 1$ is bounded. (See [Ash83] for a more refined version of this type of argument, taking into account the 'peeling-off' behaviour of F.)

Let me say a few words about the background of the above question, and sketch the implications of one possible answer. I will be rather brief because the details have appeared elsewhere [JN23].

Together with Karl-Hermann Neeb, we have proven a localization theorem for certain projective unitary representations of the compactly supported gauge group $\mathrm{Gau}_c(P)$. This is in an equivariant setting, with a Lie group H of 'space–time symmetries' whose action on M lifts to an action on P by bundle automorphisms. In the Lie algebra $\mathfrak{h}$ of H we specify a distinguished cone $\mathcal{C}$ of 'timelike generators'. If, for example, $M = \mathbb{R}^d$ is Minkowski space and $H = \mathbb{R}^d \rtimes \mathrm{SO}(d-1, 1)$ is the Poincaré group, then it is natural to choose $\mathcal{C} = \{p \in \mathbb{R}^d\,;\, \eta(p,p) \leq 0\}$ to be the forward light cone.

We are interested in *positive energy representations*; projective unitary representations of $\mathrm{Gau}_c(P)$ that extend to the semidirect product of $\mathrm{Gau}_c(P)$ with the group H of space-time symmetries in such a way that every timelike generator $p \in \mathcal{C}$ gives rise to a Hamilton operator $H(p)$ with spectrum bounded from below.

This positive energy condition is surprisingly restrictive. If the structure group K of the principal fibre bundle $P \to M$ is compact, semisimple and 1-connected, one can prove the following result.

Theorem 1 (Localization theorem)**.** *Suppose that the action of* $\mathcal{C}$ *on* M *has no fixed points. Then for every positive energy representation* $(\overline{\rho}, \mathcal{H})$ *of the identity component* $\Gamma_c(M, \mathrm{Ad}(P))_0$*, there exists a 1-dimensional,* H*-equivariantly embedded submanifold* $S \subseteq M$ *and a positive energy representation* $\overline{\rho}_S$ *of* $\Gamma_c(S, \mathrm{Ad}(P))$ *such that the following diagram commutes,*

$$\begin{array}{ccc} \Gamma_c(M, \mathrm{Ad}(P))_0 & \xrightarrow{\;\overline{\rho}\;} & \mathrm{PU}(\mathcal{H}) \\ \big\downarrow{\scriptstyle r_S} & \nearrow{\scriptstyle \overline{\rho}_S} & \\ \Gamma_c(S, \mathrm{Ad}(P)), & & \end{array}$$

where the vertical arrow denotes restriction to S*.*

Loosely speaking: if there are no fixed points for the action of the space-time symmetry group H, then positive energy representations come from 1-dimensional H-orbits.

One way to fix boundary conditions on Minkowski space is to require the gauge fields (weighted by an appropriate conformal factor) to extend smoothly to the *conformal compactification* $M = S^1 \times S^{d-1}$. In this setting, the relevant gauge group is $\mathcal{G} := \Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$. This is larger than the group $\Gamma_c(\mathbb{R}^d, \mathrm{Ad}(P))$ of 'local' gauge transformation (because compactly supported gauge transformations extend trivially to infinity), but it is strictly smaller than the group $\Gamma(\mathbb{R}^d, \mathrm{Ad}(P))$ of *all* vertical automorphisms (which does not require any limiting behaviour at infinity). This is one example of a choice of boundary conditions for which

$$\Gamma_c(\mathbb{R}^d, \mathrm{Ad}(P)) \subseteq \mathcal{G} \subseteq \Gamma(\mathbb{R}^d, \mathrm{Ad}(P)).$$

If we take H to be the connected Poincaré group, then our theorem does not immediately apply. The reason for this is that although the action of H on null infinity is fixed point free, spacelike infinity ι_0 (which is identified with timelike infinity $\iota_\pm$ in the compactification) is a fixed point.

Let us start by taking M to be the noncompact manifold $M = S^1 \times S^1/\{\iota_0\}$, the conformal compactification of $\mathbb{R}^2$ with spatial infinity removed. The action of the Poincaré group then has three orbits: Minkowski space $\mathbb{R}^2$, and the two one-dimensional components $\mathcal{I}_{L/R} \simeq \mathbb{R}$ of null infinity (corresponding to left and right moving modes). So the only 1-dimensional Poincaré-invariant orbits are $\mathcal{I}_{L/R}$! In this setting, the localization theorem implies that every positive energy representation of $\Gamma_c(S^1 \times S^1/\{\iota_0\}, P)$ is determined entirely by two positive energy representations of the pointed loop group $\Gamma_c(S^1/\{\iota_0\}, \mathrm{Ad}(P))$, one for $\mathcal{I}_L \simeq S^1/\{\iota_0\}$ and one for $\mathcal{I}_R \simeq S^1/\{\iota_0\}$.

Although we cannot directly apply the above form of the localization theorem to the full compactification $S^1 \times S^1$, a more refined analysis reveals that for every positive energy representation $(\overline{\rho}, \mathcal{H})$ of $\Gamma(S^1 \times S^1, \mathrm{Ad}(P))$, the projective unitary operator $\overline{\rho}(g) \in \mathrm{PU}(\mathcal{H})$ associated to a gauge transformation $g \in \Gamma(S^1 \times S^1, \mathrm{Ad}(P))$ can only depend on the *values* of g at null infinity $\mathcal{I}_{L/R}$, and on the *2-jets* at spatial infinity ι_0.

For Minkowski space $\mathbb{R}^d$ with $d > 2$, the conformal compactification $S^1 \times S^{d-1}$ has one orbit of dimension d (the open dense subset $\mathbb{R}^d$), one orbit of dimension $d-1$ (null infinity), and a single fixed point (spacelike infinity ι_0, which is again identified with past and future timelike infinity $\iota_\pm$ in the compactification). If we again apply the localization theorem to the noncompact manifold $S^1 \times S^{d-1}/\{\iota_0\}$, we now find that every positive energy representation of $\Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$ is *trivial* on $\Gamma_c(S^1 \times S^{d-1}/\{\iota_0\}, \mathrm{Ad}(P))$. In other words: for every positive energy representation $(\overline{\rho}, \mathcal{H})$ of $\Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$, the projective unitary transformation $\overline{\rho}(g) \in \mathrm{PU}(\mathcal{H})$ assigned to a gauge transformation g depends only on the *germ* of g around spacelike infinity ι_0.

In fact, a more refined analysis shows that $\overline{\rho}(g)$ depends only on the *1-jet* of g at ι_0. Taking into account the Poincaré group as well, this reduces the relevant symmetry group from $(\mathrm{SO}_0(d-1,1) \rtimes \mathbb{R}^d) \ltimes \Gamma(S^1 \times S^{d-1}, \mathrm{Ad}(P))$ to the *finite* *dimensional* Lie group $(\mathrm{SO}_0(d-1,1) \rtimes \mathbb{R}^d) \ltimes J^1_{\iota_0}\mathrm{Ad}(P)$. Now the group of 1-jets $J^1_{\iota_0}\mathrm{Ad}(P)$ is isomorphic to $K \ltimes \mathfrak{k} \otimes \mathbb{R}^d$, where the first term captures the values of the gauge transformation at ι_0 and the (abelian) second term captures the first derivatives of the gauge transformation at ι_0. Putting it all together, we end up with a semidirect product

$$(\mathrm{SO}_0(d-1,1) \times K) \ltimes (\mathbb{R}^d \oplus \mathfrak{k} \otimes \mathbb{R}^d)$$

of a semisimple Lie group $G := \mathrm{SO}_0(d-1,1) \times K$ with the abelian Lie group $V := \mathbb{R}^d \oplus \mathfrak{k} \otimes \mathbb{R}^d$ (considered as a vector space with addition). The projective unitary representations of this group can be found using Mackey's imprimitivity theorem; they are given by a G-orbit in V, together with a projective unitary representation of the little group $L_\nu \subseteq G$, the stabiliser of a point ν in the orbit.

For $d = 4$, $K = \mathrm{SU}(3)$ and $\nu = p \oplus X \otimes p$ with $\eta(p,p) = 0$, one obtains representations that are induced from the little group $L_\nu = E(2) \times \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$, where $E(2) \subseteq \mathrm{SO}(3,1)$ is the group of two-dimensional euclidean motions. It is tempting to speculate that these representations might be connected to symmetry breaking phases.

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On separable states

Ko Sanders

1. Introduction

Entanglement is the phenomenon in quantum physics where measurements in spacelike separated regions give rise to correlations that cannot be explained by classical physics. Although this defining feature of quantum physics is rather counter-intuitive, it is not at all rare. Physical systems naturally entangle themselves with their environment at no cost to the experimenter. On the contrary, to prevent this decoherence is difficult and expensive in terms of effort and energy.

The omnipresence of entanglement is reflected in the structure of relativistic quantum field theory (QFT). The Reeh-Schlieder Theorem [RS61] states that the vacuum vector $\Omega_0 \in \mathcal{H}$ of any Wightman QFT in Minkowski space is cyclic for all local algebras of observables. This entails that the vacuum is entangled between any two spacelike separated open regions A and B of spacetime, cf. Corollary 1 in Section 5.1 of [HS18]. This property is shared by many other states, cf. [San09, Wit18]. Indeed, $\mathcal{H}$ contains a dense G_δ of vectors with the Reeh-Schlieder property [DM71].

Even though entanglement is the rule in QFT, rather than the exception, there are at least two good reasons to have a closer look at separable states, i.e. states which are not entangled between A and B. Firstly, most of our physical concepts are classical and hence arise in a context where all states are separable. Secondly, to quantify the amount of entanglement between A and B in a given state ω, one uses an entanglement measure, which compares ω to the nearest separable state. Here, the word "nearest" can be made mathematically precise in various ways, leading to a range of entanglement measures, cf. [HS18] and references cited therein.

E.g., the entanglement entropy in vacuum typically falls off when the separation between A and B increases. This suggests an explanation as to why the physical world looks so classical on large scales: a smaller entanglement entropy should make it harder to exploit any entanglement present in the system and make it visible. Unfortunately, I am not aware of any results that quantify the word "harder" in terms of the energy (density) needed in relativistic QFT.

It is known that there exist normal separable states under quite general circumstances, namely when a QFT satisfies the split property, cf. [BDF87] (see also [Buc74]). However, normality is a rather weak condition on quantum states and one might like to ascertain further physical properties, e.g. that separable states can share the symmetry of a system and/or have a finite energy (density), etc. Furthermore, it would be interesting to know how much energy needs to be expended to create and/or maintain a separable state. In this talk, based on [San23], I present a result that gives partial answers to these questions in a toy model system. The proof of this result required novel methods involving test functions of positive type, which I will also discuss.

2. An existence theorem for separable states

To formulate the main result of [San23], let us fix an inertial coordinate frame in Minkowski space and write $x = (x_0, \mathbf{x})$. $\omega_2(x, x')$ denotes the two-point distribution of a state.

Theorem. Consider a free scalar QFT of mass $m > 0$ in 4-dimensional Minkoswki space. Given any $R > 0$ there exists a quasi-free, Hadamard, stationary, homogeneous, isotropic state ω, s.t.

(i) $\omega_2(x, x') = 0$ if $(x, x') \in S = \{\|\mathbf{x} - \mathbf{x}'\| > R + |x_0 - x'_0|\}$,

(ii) $\omega(T^{\mathrm{ren}}_{00}(x)) \leq 10^{31} m^4 \frac{e^{-\frac{1}{4}mR}}{(mR)^8}$.

Item (i) ensures that ω is a product state between A and B, as soon as these regions are separated by a distance $\geq R$. By a standard spacetime deformation argument one can also establish the existence of separable states for massless fields and in curved spacetimes with topology $\mathbb{R}^4$.

To find ω, we will compare ω_2 to the vacuum two-point distribution ω_2^0, i.e.

$$\omega_2(x - x') = \omega_2^0(x - x') + w(x - x')\,,$$

where we exploited the translation invariance. w must satisfy $(\Box + m^2)w = 0$ with initial data $w_0(\mathbf{x}) = w|_{x_0=0}(\mathbf{x})$ and $w_1(\mathbf{x}) = \partial_0 w|_{x_0=0} \equiv 0$ such that

(1) w_0 is real-valued, smooth and rotation invariant.
(2) w_0 is of positive type, i.e. $\widehat{w_0} \geq 0$.
(3) $w_0(\mathbf{x}) = -\omega_2^0(0,\mathbf{x})$ if $\|\mathbf{x}\| > R$.
(4) $\omega(T_{00}^{\mathrm{ren}}(x)) = (-\Delta + m^2)w(0) \leq 10^{31} m^4 \frac{e^{-\frac{1}{4}mR}}{(mR)^8}$.

The strategy to find w (and hence ω_2) is to modify the initial data of ω_2^0, taking

$$w_0(\mathbf{x}) = -\chi_\infty(\|\mathbf{x}\|)\omega_2^0(0,\mathbf{x}) + f(\mathbf{x})\,.$$

Here χ_∞ is a smooth, rotation invariant function that vanishes near $\mathbf{x} = 0$ and equals 1 when $\|\mathbf{x}\| \geq R$, removing all unwanted correlations. $f \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$ is supported in the ball of radius R and is needed to achieve the positive type of w_0, i.e. $\hat{f} \geq \mathcal{F}\left[\chi_\infty \omega_2^0|_{x_0=0}\right]$ (the Fourier transform), which leads to the study of test functions f of positive type and *lower* bounds on $\hat{f}$. To my knowledge such bounds had not been considered before, except asymptotically for $|k| \to \infty$ [FF15].

3. Test functions of positive type

A standard construction of test functions starts with the characteristic function χ of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and takes repeated convolutions (cf. [Hor90]). Given $a = \sum_{n=1}^\infty a_n < \infty$ with $a_1 \geq a_2 \geq \cdots > 0$, one obtains a test function $f \in C_0^\infty([-a,a], \mathbb{R})$ by taking the limit

$$f := \chi\left(\frac{\cdot}{a_1}\right) * \chi\left(\frac{\cdot}{a_2}\right) * \cdots\,.$$

This construction leads to good control on f, e.g. on $\|\partial_x^n f\|_\infty$ for all $n \geq 0$ and on upper bounds on $|\hat{f}|$. Because $\widehat{\chi * \chi} = \hat{\chi}^2 \geq 0$ we can even get $\hat{f} \geq 0$. However, we have no good control over lower bounds on $\hat{f}$. Indeed, $\hat{f}$ will have zeroes. To remedy this, one can modify the construction and replace χ by $\eta = \frac{3}{2}(\chi * \chi)^2$ with

$$\frac{1}{1 + \frac{7}{40}k^2} \leq \hat{\eta}(k) \leq \frac{1}{1 + \frac{1}{20}k^2}\,.$$

For $f \in C_0^\infty(\mathbb{R}, \mathbb{R})$, $\hat{f}$ falls off faster than any polynomial and $|\hat{f}(k)| \leq e^{-|k|}$ iff $f \equiv 0$. More precisely, using the construction with χ Ingham [Ing34] showed:

Theorem. Given $l > 0$ and $\epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ decreasing, there exists $f \in C_0^\infty([-l,l], \mathbb{R})$ such that $|\hat{f}(k)| \leq e^{-k\epsilon(|k|)}$ iff

$$\int_1^\infty \frac{\epsilon(k)}{k}\, dk < \infty\,. \tag{1}$$

In analogy, [San23] proves a lower bound using the construction with η:

Theorem. Given $l > 0$, $\epsilon : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ decreasing and $\gamma \in (0,1)$ such that (1) holds and $\lim\limits_{k\to\infty} k^\gamma \epsilon(k) = \infty$, there exists a non-negative, even $g \in C_0^\infty([-l,l], \mathbb{R})$ such that $\int g(x)dx = 1$ and $|\hat{g}(k)| \geq e^{-k\epsilon(|k|)}$.

Analogous results hold in higher dimensions. Examples include test functions of arbitrarily small support that dominate Gevrey type functions. The constructions involved provide enough detailed control over test functions of positive type to prove the main theorem in Section 2, but the estimate on the energy density is not sharp. It would be interesting to see if the methods introduced here can be developed further to yield sharper results.

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Mini-Workshop: Positivity and Inequalities in Convex and Complex Geometry

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Abstract. The workshop convened researchers from algebraic geometry, convex geometry, and complex geometry to explore themes arising from the Alexandrov-Fenchel and Brunn-Minkowski inequalities. It featured three introductory talks delving into the basics of Lorentzian polynomials, valuations in convex geometry, and plurisubharmonic functions, that served as a foundation for the subsequent research talks. As anticipated, significant overlap emerged among the varied perspectives within these three areas, evident in the presentations and ensuing discussions.

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Introduction by the Organizers

The workshop was organized by Andreas Bernig (Goethe-Universität Frankfurt), Julius Ross (University of Illinois at Chicago) and Thomas Wannerer (Friedrich-Schiller-Universität Jena). It was held over 5 days and included five introductory talks over three topics, and 13 research talks.

The mini-workshop revolved around a recent theme that has connected many seemingly different areas of mathematics, the so-called "Kähler package" that contains Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann bilinear relations. Originally coming from Kähler and algebraic geometry, it is now understood that this also appears in algebra, combinatorics and convex geometry. For example, each of the following admits a version of the Kähler package: McMullen's algebra generated by the Minkowski summands of a simple convex polytope, the combinatorial intersection cohomology of a convex polytope, the Chow ring of a matroid, and the ring of algebraic cycles modulo homological equivalence on a smooth projective variety via Grothendieck's standard conjectures on algebraic cycles. A powerful idea in the groundbreaking work of the Fields medalist June Huh and his collaborators is that the existence of a log-concave sequence is strong evidence for a Kähler package in the background. The celebrated Alexandrov–Fenchel inequality of convex geometry is an important example of a log-concave sequence, and therefore it is no surprise that this inequality can be deduced from at least three different incarnations of the Kähler package.

The aim of the mini-workshop was to bring together researchers interested in different aspects of the Kähler package, with an emphasis on aspects that relate most closely to complex and convex geometry.

The first introductory talk was titled *Plurisubharmonic functions and complex Brunn–Minkowski theory* and was given by Bo Berndtsson. He introduced the class of plurisubharmonic functions, sketched Bedford–Taylor theory, and discussed the complex version of Prekopa's theorem. The second introductory talk, over two hours, was titled *Valuations and convex geometry* and was given by Semyon Alesker. He introduced the algebraic structures on the space of valuations on convex bodies that are fundamental in the (partly conjectured) Kähler package for valuations. The final introductory talk, over two hours, was titled *Lorentzian polynomials* and given by Hendrik Süß. He introduced the concept of Lorentzian polynomials according to the work by Brändén and Huh and then explained various operations that map Lorentzian polynomials to Lorentzian polynomials. Thanks to the introductory talks, a common background knowledge was established at the beginning of the week on which later talks could rely.

The research talks covered various related topics such as the Alexandrov–Fenchel inequality and related inequalities for mixed volumes, the theory of valuations on convex bodies and on manifolds, the Hodge–Riemann bilinear relations on Kähler manifolds, Weighted Ehrhart theory, Gamma-positivity, Superforms and Hodge–Riemann classes coming from ample vector bundles.

The stimulating atmosphere of the mini-workshop led to many fruitful discussion that strengthened the links between different, but in fact closely related, areas of mathematics.

Mini-Workshop: Positivity and Inequalities in Convex and Complex Geometry

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Abstracts

Plurisubharmonic functions and complex Brunn-Minkowski theory

Bo Berndtsson

This was an introductory lecture. I first defined the notion of plurisubharmonic function. A function ϕ, defined in an open subset of $\mathbb{C}^n$ is plurisubharmonic if it is upper semicontinuos and its restriction to any complex line is subharmonic as a function of one complex variable. If the function is smooth, this is equivalent to saying that its complex Hessian

$$(\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k})$$

is positively semidefinite everywhere. With a plurisubharmonic function one can associate the *positive* differential form – or current –

$$i\partial\bar\partial\phi := i\sum\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}dz_j\wedge d\bar{z}_k.$$

If ϕ is smooth, this is a differential form of bidegree $(1,1)$ (meaning that it contains one barred differential and one unbarred); in general it is a current (meaning that the coefficients should be interpreted as distributions). That this form is positive means, in the smooth case, that the coeffcient matrix is positive semidefinite everywhere. In the general case it means that for any vector $(\lambda_1, ...\lambda_n)$

$$\sum\frac{\partial^2\phi}{\partial z_j\partial\bar{z}_k}\lambda_j\bar\lambda_k,$$

is a positive distribution, i.e. a positive measure.

If our funtion $\phi(z)=\phi(x+iy)=\phi(x)$ depends only on the real part of z, then ϕ is plurisubharmonic if and only if it is convex, and its complex Hessian coincides with the real Hessian, modulo a factor $1/4$. Moreover, one checks that

$$i\partial\bar\partial\phi = (1/2)\sum\phi_{jk}dx_j\wedge dy_k$$

then. So, this expression has a meaning as a current for any convex function, not necessarily smooth.

It was discovered by E. Bedford and B.A. Taylor, that the $(1,1)$ currents associated to plurisubharmonic functions can be multiplied, provided that the functions are locally bounded, so that e.g.

$$i\partial\bar\partial\phi\wedge i\partial\bar\partial\psi$$

is a well defined current. This is remarkable since distributions, or even measures, cannot be multiplied in general, but it turns out that the cancellation from the wedge product of differential forms works in our favour. When ϕ depends only on the real part of z, this gives in particular that

$$(\sum\phi_{jk}dx_j\wedge dy_k)^n/n!$$

is a well defined measure, the Monge-Ampère measure of ϕ. It can be shown that this coincides with Alexandrov's definition of Monge-Ampère measure.

The second topic of the lecture further developed the analogy between convexity and plurisubharmonicity. The analog for convex functions of the Brunn-Minkowski inequality for convex sets is Prékopa's theorem. Prékopa's theorem says that if $\phi(t,x)$ is a convex on $\mathbb{R}^{n+1}$, then

$$\tilde{\phi}(t) := -\log \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx$$

is again convex. One well known proof of this, by Brascamp and Lieb, uses a certain Poincaré-type inequality: If u is a function on $\mathbb{R}^n$ sucht that

$$\int u e^{-\phi} dx = 0,$$

then

$$\int u^2 e^{-\phi} \leq \int |du|^2_{(\phi^{jk})} e^{-\phi}.$$

Here $|du|^2_{(\phi^{jk})} := \sum u_j u_k \phi^{jk}$ is the norm of du measured with the inverse of the Hessian of ϕ.

The Brascamp-Lieb inequality can be seen as the real variable counterpart of Hormander's L^2-estimates for the $\bar{\partial}$-equation. A natural question is then if there is a complex version of Prékopa's theorem? In one sense the answer is no: A counterexample by Kiselman shows that the function

$$\tilde{\phi}(\tau) := -\log \int_{\mathbb{C}^n} e^{-\phi(\tau,z)} dm(z)$$

is in general not subharmonic for plurisubharmonic $\phi(\tau, z)$. It does, however, hold under various extra assumptions, most notably if ϕ is S^1-invariant in z, for fixed τ:

$$\phi(\tau, e^{i\theta} z) = \phi(\tau, z).$$

One way to see this is via the Bergman kernel, here defined as

$$B_\tau(z) := \sup_h \frac{|h(z)|^2}{\int |h|^2 e^{-\phi(\tau,z)} dm(z)},$$

with the supremum taken over all holomorphic functions. The real variable analog of this would be to take supremum over all constant functions, leading to the function $\tilde{\phi}(\tau)$ (as $\log B_\tau$) introduced above. The first complex Prékopa theorem says that in general

$$\log B_\tau(z)$$

is plurisubharmonic in (τ, z). In the special case of S^1-invariance it is easily seen that

$$B_\tau(0) = 1/\int_{\mathbb{C}^n} e^{-\phi(\tau,z)} dm(z),$$

giving a more concrete Prekopa theorem in this case.

The complex Prékopa (or Brunn-Minkowski) theory is, however, considerably richer than this. One way to explain the general picture is to start from the observation that the Bergman kernel is the squared norm of the evaluation functional

$$h \to h(z)$$

on the Hilbert space

$$A^2_\tau := \{h \in H(\mathbb{C}^n), \int |h|^2 e^{-\phi(\tau,z)} dm(z) < \infty\}.$$

It turns out that one can replace the evaluation functional by any other family of functionals μ_τ, that depend holomorphically on τ in the sense that

$$\tau \to \mu_\tau(h)$$

is holomorphic for holomorphic h. This means, intuitively, that the bundle of Hilbert spaces $\tau \to A^2_\tau$ has positive curvature.

This is finally the most general statement along these lines , in the setting of Euclidean space. The complex case is, however, more naturally studied in the setting of complex manifolds. We then replace $\mathbb{C}^{n+1}$ by a complex manifold X, and the projection from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ by a surjective holomorphic map to another manifold. It turns out that one can develop a similar theory in this setting, under the crucial assumption that X be Kahler.

Valuations and convex geometry

Semyon Alesker

(1) I gave two introductory talks on translation invariant valuations on convex sets focusing mostly on the structures on the space of smooth translation invariant valuations (product, convolution, Fourier type transform), their relations to the recent Kotrbatý's conjectures on mixed hard Lefschetz (mHL) and mixed Hodge-Riemann (mHR) type results, to McMullen's polytope algebra, and to toric varieties. Below we briefly summarize main relevant definitions and theorems.

(2) Let V be a finite dimensional real vector space, $n = \dim V$. Let $\mathcal{K}(V)$ denote the family of all convex compact non-empty subsets of V. Its elements are also called convex bodies.

Definition 1. A valuation is a functional $\phi\colon \mathcal{K}(V) \to \mathbb{C}$ which is finitely additive:

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$.

Definition 2. A valuations ϕ is called translation invariant if

$$\phi(K + v) = \phi(K) \text{ for any } K \in \mathcal{K}(V), v \in V.$$

(3) Let us denote by $Val(V)$ the set of all continuous (in the Hausdorff metric) translation invariant valuations. It is a vector space over $\mathbb{C}$. Being equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$ it becomes a Banach space.

Definition 3. A valuation ϕ is called α-homogeneous if

$$\phi(\lambda K) = \lambda^\alpha \phi(K) \text{ for any } K \in \mathcal{K}(V), \lambda > 0.$$

Let $Val_\alpha(V) \subset Val(V)$ denote the subset of α-homogeneous valuations. Clearly it is a closed linear subspace. The following structural result is very important in the theory.

Theorem 4 (P. McMullen [6], 1977)**.** *One has the decomposition*

$$Val(V) = \oplus_{i=0}^n Val_i(V).$$

It is known that:
(1) $Val_0(V) = \mathbb{C} \cdot \chi$. This is trivial.
(2) $Val_n(V) = \mathbb{C} \cdot vol$. This is not trivial and due to Hadwiger.
The space $Val(V)$ has an important distinguished dense subspace $Val^\infty(V)$ of so called smooth valuations which carries rich algebraic structures. The definition was given in the lectures.

(4) The space $Val^\infty(V)$ carries a canonical multiplicative structure. Let us fix a positive Lebesgue measure vol_V on V.

Theorem 5 (Alesker [2], 2004)**.** *(1) $Val^\infty(V)$ has a canonical (thus $GL(V)$-equivariant) continuous (in the Garding topology) product $Val^\infty \times Val^\infty \to Val^\infty$ which is uniquely characterized by the following property: Let $\phi(K) = vol_V(K + A), \psi(K) = vol_V(K + B)$. Then*

$$(\phi \cdot \psi)(K) = vol_{V^2}(\Delta(K) + (A \times B)),$$

where $vol_{V^2} := vol_V \times vol_V$ is the product measure, and $\Delta: V \to V \times V$ is the diagonal imbedding, i.e. $\Delta(x) = (x, x)$.

(2) Equipped with this product $Val^\infty(V)$ is an associative commutative algebra with a unit ($= \chi$).

(3) $Val^\infty(V)$ is a graded: $Val_i^\infty \cdot Val_j^\infty \subset Val_{i+j}^\infty$.

(4) Poincaré duality: the bilinear map

$$Val_i^\infty \times Val_{n-i}^\infty \to Val_n = \mathbb{C} \cdot vol$$

is a perfect pairing, i.e. for any $0 \neq \phi \in Val_i^\infty$ there exists $\psi \in Val_{n-j}^\infty$ such that $\phi \cdot \psi \neq 0$.

Furthermore $Val^\infty(V)$ satisfies a version of the hard Lefschetz theorem which is a combination of results of Alesker [1, 3] and Bernig-Bröcker [4].

(5) Another important structure is the Bernig-Fu convolution.

Theorem 6 (Bernig-Fu [5],2006)**.** *(1) $Val^\infty(V)$ has a continuous (in the Garding topology) convolution $Val^\infty \times Val^\infty \to Val^\infty$ commuting with the* *group of linear volume preserving transformaitons which is uniquely characterized by the following property: Let* $\phi(K) = vol_V(K+A), \psi(K) = vol_V(K+B)$. *Then*

$$(\phi * \psi)(K) = vol_V(K + A + B).$$

(2) Equipped with this convolution $Val^\infty(V)$ *is an associative commutative algebra with a unit (=* vol_V*).*

(3) $Val^\infty_{n-i} * Val^\infty_{n-j} \subset Val^\infty_{n-i-j}$.

Poincaré duality and hard Lefschetz theorem (for intrinsic volumes) are also satisfied by convolution.

(6) Alesker [1, 3] has constructed an isomorphism of topological algebras, called Fourier type transform,

$$\mathbb{F}\colon (Val^\infty(V), \cdot)\tilde{\to}(Val^\infty(V^*), *)$$

commuting with the group of linear volume preserving transformations.

(7) The Kotrbatý's conjectures are formulated in terms of convolution and they are mixed hard Lefschetz and mixed Hodge-Riemann type results for valuations. I explained a heuristic argument in favor (imho) of the conjectures. It is based on the connection of valuations to the McMullen's polytope algebra established by Bernig and Faifman. Then I indicated a relation to toric varieties.

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Gamma-positivity for symmetric edge polytopes

Martina Juhnke-Kubitzke

Symmetric edge polytopes are a class of lattice polytopes that has seen a surge of interest in recent years for their intrinsic combinatorial and geometric properties [MHN+11, HKM17, OT21a, OT21b, CDK23] as well as for their relations to metric space theory [Ver15, GP17, DH20], optimal transport [ÇJM+21] and physics, where they appear in the context of the Kuramoto synchronization model [CDM18, Che19] (see [DDM22] for a more detailed account of these connections).

Given a finite simple graph $G = ([n], E)$, the associated *symmetric edge polytope* $\mathcal{P}_G$ is defined as

$$\mathcal{P}_G = \mathrm{conv}(\pm(e_i - e_j) \ : \ ij \in E).$$

Symmetric edge polytope have been shown to exhibit several nice properties, independent of the underlying graph: all of these polytopes are known to admit a pulling regular unimodular triangulation [OH14, HJM19] and to be centrally symmetric, terminal and reflexive [Hig15]. In particular, by this latter property, it follows from work of Hibi [Hib92] that their h^*-vectors are palindromic. Thus, given the h^*-vector $h^*(\mathcal{P}_G) = (h^*_0, \ldots, h^*_d)$ of a symmetric edge polytope, one can define the γ-vector of $\mathcal{P}_G$ by applying the following change of basis:

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (t+1)^{d-2i} = \sum_{j=0}^{d} h^*_j t^j. \tag{1}$$

Obviously, $\gamma(\mathcal{P}_G) = (\gamma_0, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor})$ stores the same information as $h^*(\mathcal{P}_G)$ in a more compact form. More generally, in the same way, one can associate a γ-vector with any symmetric vector and this has been done and studied extensively in a lot of cases. One of the most prominent examples in topological combinatorics, which is strongly related to the just mentioned example of h^*-vectors of reflexive polytopes, are h-vectors of simplicial spheres. For *flag* spheres, Gal's conjecture [Gal05] states that their γ-vectors are nonnegative. Several related conjectures exist, including the Charney–Davis conjecture [CD95], claiming nonnegativity only for the last entry of the γ-vector, and the Nevo–Petersen conjecture [NP11] which even conjectures the γ-vector of a flag sphere to be the f-vector of a balanced simplicial complex.

If a polytope $\mathcal{P}$ admits a regular unimodular triangulation Δ, which is the case for symmetric edge polytopes, then the restriction of Δ yields a unimodular triangulation of the boundary complex of $\mathcal{P}$, as well. If, in addition, $\mathcal{P}$ is reflexive, it is well-known that the h^*-vector of $\mathcal{P}$ equals the h-vector of any unimodular triangulation Δ of its boundary, which in particular is a simplicial sphere. This provides a link between the study of the γ-vector of $\mathcal{P}_G$ and the rich world of conjectures on the γ-nonnegativity of simplicial spheres; however, note that the objects we are interested in will *not* be flag in general. Despite the lack of flagness, in all the cases known so far the γ-vector of $\mathcal{P}_G$ is nonnegative, and this brought Ohsugi and Tsuchiya to formulate the following conjecture, which is the starting point of this paper:

Conjecture 1. [OT21a, Conjecture 5.11] Let G be a graph. Then $\gamma_i(\mathcal{P}_G) \geq 0$ for every $i \geq 0$.

On the one hand, it is already known and follows e.g. from [BR07] that a weaker property, namely, unimodality of the h^*-vector holds. On the other hand, though it is tempting to hope that even the stronger property of the h^*-polynomial being real-rooted is true, this is not the case in general, as shown by the 5-cycle. The γ-nonnegativity conjecture above has been verified for special classes of graphs, mostly by direct computation: as shown in [OT21a, Section 5.3], such classes encompass cycles, suspensions of graphs (which include both complete graphs and wheels), outerplanar bipartite graphs and complete bipartite graphs. This last instance was originally proved in [HJM19] but was generalized in [OT21a] to bipartite graphs $\widetilde{H}$ obtained from another bipartite graph H as in [OT21a, p. 708].

The main goal of this work is to provide some supporting evidence to the γ-nonnegativity conjecture, independent of the graph. We take two different approaches: a *deterministic* and a *probabilistic* one.

In the deterministic part, we focus on the coefficient γ_2. Through some delicate combinatorial analysis, we are able to prove that γ_2 is always nonnegative. Moreover, we provide a characterization of those graphs for which $\gamma_2(\mathcal{P}_G) = 0$:

Theorem 2. *] Let $G = ([n], E)$ be a graph. Then $\gamma_2(\mathcal{P}_G) \geq 0$. Moreover, if G is 2-connected, then $\gamma_2(\mathcal{P}_G) = 0$ if and only if either $n < 5$, or $n \geq 5$ and G is isomorphic to one of the following two graphs:*

- *the graph G_n with edge set $\{12\} \cup \{1k, 2k \ : \ k \in \{3, \dots, n\}\}$; or*
- *the complete bipartite graph $K_{2,n-2}$.*

The "if" part of the equality statement can be deduced from the results in [HJM19] and [OT21a], where the authors compute explicitly the γ-vector of the families of graphs appearing in 2.

For the probabilistic approach, we consider the Erdős-Rényi model $G(n, p(n))$, which is one of the most popular and well-studied ways to generate a graph on the vertex set $[n]$ via a random process: for a graph $G \in G(n,p)$, the probability of ij with $1 \leq i < j \leq n$ being an edge of G equals $p(n)$, and all of these events are mutually independent. Our question is then: for an Erdős-Rényi graph $G \in G(n,p)$, how likely is it that the entries of the γ-vector of $\mathcal{P}_G$ are nonnegative? As an extension, we pose the question of how *big* those entries will most likely be. Our main result, answering both questions, is the following:

Theorem 3. *Let k be a positive integer. For the Erdős-Rényi model $G(n, p(n))$, where $p(n) = n^{-\beta}$ for some $\beta > 0$, $\beta \neq 1$, the following statements hold:*

- (subcritical regime) *if $\beta > 1$, then asymptotically almost surely $\gamma_\ell = 0$ for all $\ell \geq 1$;*
- (supercritical regime) *if $0 < \beta < 1$, then asymptotically almost surely $\gamma_\ell \in \Theta(n^{(2-\beta)\ell})$ for every $0 < \ell \leq k$.*

In particular, this shows that $\gamma_\ell \geq 0$ for $1 \leq \ell \leq k$ with high probability, thereby proving that (up to a fixed entry of the γ-vector) Gal's conjecture holds with high probability. To prove this result, we need to distinguish two regimes: subcritical ($\beta > 1$) and supercritical ($0 < \beta < 1$), the subcritical one being the easier one. Along the proof, we derive concentration inequalities for the number of non-faces and faces of the triangulation of $\mathcal{P}_G$ studied in [HJM19, Proposition 3.8]. Unfortunately, our approach does not give results for the critical regime.

This is joint work with Alessio D'Alí, Daniel Köhne and Lorenzo Venturello.

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Lorentzian polynomials

Hendrik Süss

In [1] Brändén and Huh introduced the notion of Lorentzian polynomials. This is a class of polynomials which generalizes the log-concavity properties of volume polynomials appearing in convex and algebraic geometry and behaves well with respect to many natural operations, such as multiplication, specialization and (positive) linear transformation. The theory of Lorentzian polynomials has been used to prove and reprove important conjectures in matroid theory, see [1, 2]. Moreover, many polynomials arising from representation theory are conjectured to be (denormalized) Lorentzian [3].

In my introductory talk I gave an overview of the definitions and basic theory of Lorentzian polynomials as presented in [1] and discussed the proof of the Strong Mason Conjecture also given in [1] as an exemplary application to matroid theory.

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Some analogies between valuation on convex bodies and algebraic cycles on varieties

Nguyen-Bac Dang

Fix $X = \mathbb{P}^d(\mathbb{C})$ the complex projective space of dimension $d \geq 1$ and a rational map $f\colon X \dashrightarrow X$ on X whose image is not contained in a hypersurface (i.e f is called dominant). Fix a Kähler form ω, for each $k \leq d$, a general problem is to estimate the growth of the sequence $(\deg_k(f^n))_{n\in\mathbb{N}}$ where

$$\deg_k(f^n) = \int_X (f^n)^*(\omega^k) \wedge \omega^{d-k}.$$

When the map f is holomorphic, these sequences can be understood on the cohomology of X. Namely, the element ω^k represents a class in the Dolbeaut cohomology $H^{k,k}(X)$ and f induces a pullback action this vector space by multiplication by $\deg_1(f)^k$. To tackle this problem for general meromorphic maps, the general strategy is to consider the pullback action of f on an infinite vector space:

$$\text{n-C}^k(\mathcal{X}) = \varinjlim H^{k,k}(Y),$$

where the inductive limit is taken over all birational models Y with a birational morphism $\pi\colon Y \to X$. More generally, one sees that the group of bimeromorphic transformations $Cr(d)$ of $\mathbb{P}^d(\mathbb{C})$ (the Cremona group) induces an action on the graded algebra:

$$Cr(d) \hookrightarrow \oplus_{k=0}^{d} \text{n-C}^k(\mathcal{X}). \qquad (1)$$

This viewpoint was very fruitful and allowed for example Cantat [Can11] to study group theoretic properties of the Cremona group of dimension 2.

One can then read the growth of the sequence $\deg_k(f^n)$ on the growth of the sequence of vectors $(f^n)^*\omega^k \in$ n-$C^k(\mathcal{X})$. In [BFJ08, DF21] a purely exponential growth of the sequence $\deg_1(f^n)$ was obtained under some conditions. The method was to complete the space n-$C^1(\mathcal{X})$ with a suitable Banach norm so that the sequence of classes $(f^n)^*\omega$ converges to a unique eigenvector for the operator f^*.

The situation is very well-understood when $k \geq 2$ if the map f is defined by monomials. Fix a matrix $A = (a_{ij}) \in GL_d(\mathbb{Z})$, the monomial map associated to A is:

$$f_A\colon (x_1, \ldots, x_d) \mapsto \left(y_1 = \prod_{j=1}^{d} x_j^{a_{1j}}, \ldots, y_d = \prod_{j=1}^{d} x_j^{a_{dj}} \right).$$

The map $A \in GL_d(\mathbb{Z}) \to f_A \in Cr(d)$ induces an injection of $GL_d(\mathbb{Z})$ in the Cremona group. In that case, this subgroup acts on the subspace:

$$\oplus_{k=0}^{d} \text{n-C}^k(\mathcal{X}_{tor}) = \oplus_k \varinjlim_{Y \text{toric}} H^{k,k}(Y),$$

where the injective limit is taken over all toric compactifications Y of $(\mathbb{C}^*)^d$. On one hand, elements of n-$C^k(\mathcal{X}_{tor})$ correspond to collection of classes of algebraic cycles living on a toric variety, but on the other hand, the theory of toric varieties allows one to view those as valuations on convex bodies. Namely, if P is the fundamental polytope of $\mathbb{R}^d$, then

$$\deg_k(f_A) = MV(A(P)[k], P[d-k]),$$

where $MV(A(P)[k], P[d-k])$ denotes the mixed volume of $A(P)$ taken k times and P taken $d-k$ times. Precisely, the class of ω^k is associated to the translation invariant valuation ϕ_{ω^k} homogeneous of degree $d-k$ such that

$$\phi_{\omega^k}(K) = MV(P[k], K[d-k]),$$

for all K convex body in $\mathbb{R}^d$. The action by $GL_d(\mathbb{Z})$ is then given by $A \cdot \phi(K) = \phi(A^{-1}(K))$. Denote by $Val_k(\mathbb{R}^d)$ vector space of translation invariant valuations of given degree k, one recovers an action on the graded vector space

$$GL_d(\mathbb{Z}) \hookrightarrow \oplus_k Val_k(\mathbb{R}^d). \tag{2}$$

Comparing (1) with (2), one sees that the previous space had a structure of graded algebra while in the second, it is only a graded vector space since the convolution between two valuations is not always well-defined. When $d = 2$, the analog in convex geometry of the norm defined by Boucksom-Favre-Jonsson is given by:

$$||\phi||^2 = 2\phi(B)^2 - \phi \star \phi \tag{3}$$

where B is a ball of volume 1 and where ϕ is a smooth valuation of degree 1. Taking the completion of smooth valuations for this norm yields a smaller space on which the convolution extends continuously. The fact that the above formula yields a norm is a consequence of Hodge-index theorem in algebraic geometry, and in convex geometry is the Legendre-Fenchel inequality or the Hodge-Riemann property for degree 1 valuations.

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Schur polynomials, positivity and the Hodge-Riemann property

Matei Toma

(joint work with Julius Ross)

We present recent joint work with Julius Ross, [1], [2], [3], showing that Schur polynomials evaluated on "positive" forms or cycle classes exhibit strong positivity properties themselves, such as the Hard Lefschetz property and the Hodge-Riemann property. Our work was motivated by the need to understand intersection properties of algebraic cycles on complex projective manifolds and was inspired by two parallel developments. On one hand, in algebraic geometry the extension of the classical Hard Lefschetz Theorem proved by Bloch and Gieseker, [4], paved the way towards the work of Fulton and Lazarsfeld, [5], on positivity of Schur classes of ample vector bundles. On the other hand, in Kähler geometry it was suggested by Gromov in [6] and proved by Dinh and Nguyen, [7], that the Hard Lefschetz Theorem and the Hodge-Riemann bilinear relations may be extended to a mixed situation, meaning by this that both work with a product of Kähler classes replacing the power of a single Kähler class in the classical statements. A natural question arises, whether other combinations of positive classes, besides those exhibited by the Bloch-Gieseker and Dinh-Nguyen theorems, have similar Hard Lefschetz and Hodge-Riemann properties. Our results, which we next describe, say that this is the case for two-codimensional Schur classes.

We will denote by $c_0, c_1, \dots, c_e \in k[x_1, \dots, x_e]$ the elementary symmetric polynomials, by $\Lambda(d, e)$ the set of partitions $\lambda = (\lambda_1, \dots, \lambda_N)$ of d with

$$0 \leq \lambda_N \leq \dots \lambda_1 \leq e, \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = d$$

and we will set

$$s_\lambda := \det \begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+N-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+N-2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{\lambda_N-N+1} & c_{\lambda_N-N+2} & \cdots & c_{\lambda_N} \end{pmatrix}.$$

The s_λ are called *Schur polynomials* and build a basis of the space $k[x_1, \dots, x_e]_d^{sym}$ of degree d homogeneous symmetric polynomials in e variables, when λ runs through $\Lambda(d, e)$.

Then we can prove the following three instances of Hard Lefschetz and Hodge-Riemann properties for "Schur classes" of degree $d = n - 2$.

Theorem (linear algebra case). Let $\omega_1, \dots, \omega_e$ be strictly positive $(1,1)$-forms on $V = \mathbb{C}^n$, λ be a partition in $\Lambda(n-2, e)$ and vol be the standard volume form on V. Then the linear map

$$\bigwedge_{\mathbb{R}}^{2} V^* \to \bigwedge_{\mathbb{R}}^{2n-2} V^*,\ \alpha \mapsto \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e)$$

is an isomorphism and the quadratic form

$$Q_{s_\lambda(\omega_1,\dots,\omega_e)} : (\bigwedge^{1,1} V^*)_{\mathbb{R}} \to \mathbb{R},\ \alpha \mapsto (\alpha \wedge s_\lambda(\omega_1, \dots, \omega_e) \wedge \alpha)/\mathrm{vol},$$

is non-degenerate of signature $(1, n^2 - 1)$.

Theorem (Kähler case). If $\omega_1, \dots, \omega_e$ are Kähler classes on a compact Kähler manifold X of dimension n and λ is a partition in $\Lambda(n-2, e)$, then the linear map

$$H^2(X, \mathbb{R}) \to H^{2n-2}(X, \mathbb{R}),\ \alpha \mapsto \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e)$$

is an isomorphism and the quadratic form

$$Q_{s_\lambda(\omega_1,\dots,\omega_e)} : H^{1,1}(X)_{\mathbb{R}} \to \mathbb{R},\ \alpha \mapsto \int_X \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e) \wedge \alpha,$$

is non-degenerate of signature $(1, h^{1,1} - 1)$.

Theorem (ample vector bundle case). If E is a rank e ample vector bundle on a complex projective manifold X of dimension n and $\lambda \in \Lambda(n-2, e)$, then the linear map

$$H^2(X, \mathbb{R}) \to H^{n-2}(X, \mathbb{R}),\ \alpha \mapsto \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e)$$

is an isomorphism and the quadratic form

$$Q_{s_\lambda(\omega_1,\dots,\omega_e)} : H^{1,1}(X)_{\mathbb{R}} \to \mathbb{R},\ \alpha \mapsto \int_X \alpha \wedge s_\lambda(\omega_1, \dots, \omega_e) \wedge \alpha,$$

is non-degenerate of signature $(1, h^{1,1} - 1)$.

We start by proving the "ample vector bundle case" and in doing so we make use of the Bloch-Gieseker Theorem and of the Fulton-Lazarsfeld cone construction. We then successively deduce the "linear algebra case" and the "Kähler case". Let us note that a different, more algebraic, approach to prove the "linear algebra case" has appeared in the meantime in [8]. It seems however that for the "ample vector bundle case" a geometric proof is needed.

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Bezout inequalities for mixed volumes

Maud Szusterman

Bezout inequality (in $\mathbb{P}^n$) and the Bernstein-Khovanskii-Kushnirenko (BKK) theorem allows to derive inequalities of mixed volumes

$$V(A_1, ..., A_r, \Delta)V(\Delta)^{r-1} \leq \prod_{i\leq r} V(A_i, \Delta[n-1]),$$

where A_i are arbitrary convex bodies in $\mathbb{R}^n$, and Δ is an n-simplex. Another consequence of the BKK theorem is

$$V(A_1, ..., A_n)V(\Delta) \leq V(A_2, ..., A_n, \Delta)V(A_1, \Delta[n-1]).$$

We introduce the affine invariant quantities $b_r(K)$ and $b(K)$ as the least $b_r, b \geq 1$ such that

$$V(A_1, ..., A_r, K)V(K)^{r-1} \leq b_r \prod_{i\leq r} V(A_i, K[n-1]), \quad \text{respectively}$$

$$V(A_1, ..., A_n)V(K) \leq bV(A_2, ..., A_n, K)V(A_1, K[n-1]),$$

holds true for any (A_i). In particular note that $1 \leq b_2(K) \leq b_r(K) \leq b(K)^{r-1}$ for any $n \geq 2$, and for any K.

In [1], C. Saroglou, I. Soprunov and A. Zvavitch have proven that $b(K) = 1$ characterizes the simplex among all convex bodies, and that $b_2(K) = 1$ characterizes the simplex among all n-polytopes: we shall review the proof of this latter characterization, and explain where it fails to generalize to the setting of convex bodies (if one uses Wulff-shape perturbations of K rather than "perturbated polytopes"). Moreover it follows from Fenchel's inequality, respectively from Diskant's inequality (see also [3]) that $b_2(K) \leq 2$ and $b(K) \leq n$ for all K (both constants are sharp, as shown by the cross-polytope for b_2, and by the unit cube for b). While the characterization of all K such that $b_2(K) = 2$ is known, that of all K such that $b(K) = n$ remains open.

This study of Bezout inequalities for mixed volumes was initiated by Soprunov and Zvavitch in [2], where they conjectured that the n-simplex is the only minimizer of b_2. Though this conjecture remains open, we will discuss recent progress on restricting the set of potential minimizers; namely we will present a necessary condition on the support of the surface area measure of K. In dimension 3, this necessary condition, together with previously established restrictions, is enough to answer positively Soprunov-Zvavitch's conjecture.

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Applications of Legendre transforms in Kähler geometry

Xu Wang

The Legendre transform of a generalized function $\phi : \mathbb{R}^n \to [-\infty, \infty]$ is defined by

$$\phi^*(y) := \sup_{x\in\mathbb{R}^n} x \cdot y - \phi(x).$$

It is one of the most important concepts in convex geometry. For instance, it can be used to define the interpolating family between two convex functions and prove that the mixed volume function is a polynomial for convex bodies (see formula (3.2) and Corollary 3.8 in [1]). It also plays an crucial role in the intersection theory in algebraic geometry. For example, it can be used to prove compactness of a Delzant toric manifold and the Bernstein-Kushnirenko inequality (see section 2 in [6]). In this talk, we will introduce a few recent applications of Legendre transforms in Kähler geometry. The first result is the following generalization [3] of McDuff–Polterovich's result [4] (for $\beta = (1, \cdots, 1)$, in which case $\epsilon_x(\omega; \beta)$ is called the Seshadri constant).

Theorem A. *Let (X, ω) be a compact Kähler manifold. Fix $x \in X$, we have*

$$\epsilon_x(\omega; \beta) = c_x(\omega; \beta),\ \ \beta = (\beta_1, \cdots, \beta_n), \beta_j > 0,\ \ 1 \leq j \leq n,$$

where the β-Seshadri constant of (X, ω) at $x \in X$ is defined by

$$\epsilon_x(\omega; \beta) := \sup\{\gamma \geq 0 : \textit{there exists } \psi \in \mathrm{PSH}(X, \omega) \textit{ such that } \psi = \gamma \log(|z_1|^{2/\beta_1} + \cdots + |z_n|^{2/\beta_n}) \textit{ near } x\},$$

and "$\psi \in \mathrm{PSH}(X,\omega)$" means that ψ is upper semi continuous on X and $\omega+dd^c\psi \geq 0$, $d^c := (\partial - \bar\partial)/(4\pi i)$, in the sense of currents on X. The β-Kähler width

$$c_x(\omega;\beta) := \sup\left\{\pi r^2 : B_r^\beta \hookrightarrow hol_x(X,\tilde\omega),\ \exists\, \tilde\omega \in \mathcal{K}_\omega\right\},$$

$$B_r^\beta := \left\{ z\in\mathbb{C}^n : \sum_{j=1}^n \beta_j |z_j|^2 < r^2 \right\},$$

where "$B_r^\beta \hookrightarrow hol_x(X,\tilde\omega)$" means that there exists a holomorphic injection $f : B_r^\beta \to X$ such that $f(0)=x$ and $f^*(\tilde\omega) = \frac{i}{2}\sum_{j=1}^n dz_j\wedge d\bar z_j$, $\mathcal{K}_\omega$ denotes the space of Kähler metrics in $[\omega]$.

The main ingredient of our proof is the following Legendre transform result.

Theorem B ([3, Theorem 3.7]). *Let ϕ be smooth strictly convex on $\mathbb{R}^n$ and $A\subset\mathbb{R}^n$ be closed. Put $\phi_A(x) = \sup_{y\in A} y\cdot x - \phi^*(y)$. If x satisfies $\phi_A(x) < \phi(x)$ then $\phi_A(x) = \sup_{y\in\partial A} y\cdot x - \phi^*(y)$, where ∂A denotes the boundary of A.*

Another application of Theorem B is the following Ross-Witt Nyström theorem [5].

Theorem C. *Let ϕ be a smooth strictly convex function on $\mathbb{R}^n$. Assume that $A := \nabla\phi(\mathbb{R}^n)$ is bounded. Fix a concave function u on A and assume that $u\in C^\infty(\mathbb{R}^n)$. Then for every $t>0$,*

$$(\phi^* - tu)^* = \sup_{\alpha\in\mathbb{R}}\{\phi_\alpha + t\alpha\}, \quad \phi_\alpha(x) := \sup_{u(y)\geq\alpha} y\cdot x - \phi^*(y).$$

The compact Kähler version of the above theorem is known as the Ross-Witt Nyström correspondence between the maximal test curves and geodesic rays.

Definition D. *Let $(L, e^{-\phi})$ be a positive line bundle over a compact complex manifold X. A map $\alpha\mapsto v_\alpha$ from $\mathbb{R}$ to $\mathrm{PSH}(X,\omega)$, $\omega := dd^c\phi$, is called a bounded test curve if*

(1) $\lambda_v := \inf\{\alpha\in\mathbb{R} : v_\alpha \equiv -\infty\} < \infty$;
(2) $\alpha\mapsto v_\alpha(x)$ is concave, decreasing and usc for any $x\in X$;
(3) $v_\alpha\equiv 0$ for $\alpha\leq 0$ and $\sup\{\alpha\in\mathbb{R} : v_\alpha\equiv 0\} = 0$.

A bounded test curve is said to be maximal if $P[v_\alpha] = v_\alpha$ for every $\alpha\in\mathbb{R}$, where

$$P[v_\alpha] := \sup^*\{v\in\mathrm{PSH}(X,\phi) : v\leq 0 \text{ and } v - v_\alpha \text{ is bounded on } X\},$$

is called the maximal envelope of v_α.

Definition E. *Let $(L,e^{-\phi})$ be a positive line bundle over a compact complex manifold X. A map $t\mapsto u_t$ from $(0,\infty)$ to $\mathrm{PSH}(X,\omega)$, $\omega := dd^c\phi$, is called a sub-linear sub-geodesic ray if*

(1) $\phi(x) + u_{-\log|\xi|^2}(x)$ is psh on $X\times\{\xi\in\mathbb{C} : |\xi|<1\}$;
(2) $u_t \geq \lim_{t\to 0} u_t = 0$ and $\lambda_u := \lim_{t\to\infty}\sup_X u_t/t < \infty$.

A sub-linear sub-geodesic ray u_t is called a geodesic ray if for every $0<a<t<b$ we have $u_t = \sup\{v_t\}$ where the supremum is taken over all sub-geodesics v_t with $\limsup_{t\to a,b} v_t \leq u_{a,b}$.

The main theorem in the Ross-Witt Nyström correspondence theory is the following result.

Theorem F ([5, Theorem 1.1])**.** *The α-Legendre transform $\hat{v}_t := \sup_{\alpha\in\mathbb{R}}\{v_\alpha + t\alpha\}$, $t > 0$, gives a bijective map, say $\mathcal{L}$, between*

(1) bounded test curves and sub-linear sub-geodesic rays;
(2) maximal bounded test curves and geodesic rays.

Moreover, we have $\lambda_v = \lambda_{\hat{v}}$ and $\mathcal{L}^{-1}(u_t)$ is given by the t-Legendre transform

$$\check{u}_\alpha := \inf_{t>0}\{u_t - t\alpha\}, \quad \alpha \in \mathbb{R}.$$

The above theorem implies the following Bergman kernel estimate in [2].

Theorem G. *Let $(L, e^{-\phi})$ be a positive line bundle over an n-dimentional compact complex manifold X. Assume that the Seshadri constant of L is $> n$ on X. Then $B_\phi \geq \mathrm{HS}_\phi$, where*

$$B_\phi(x) := \sup_{f\in H^0(X,\mathcal{O}(K_X+L))} \frac{i^{n^2} f(x)\wedge\overline{f(x)}e^{-\phi(x)}}{\int_X i^{n^2} f\wedge\bar{f}e^{-\phi}}, \quad \forall\ x \in X,$$

denotes the ϕ-weighted Bergman kernel form on X and

$$\mathrm{HS}_\phi(x) := \frac{(dd^c\phi)^n(x)}{\int_{T_xX} e^{-\phi_{L,x}}(dd^c\phi)^n(x)}, \quad \forall\ x \in X,$$

is called the Hele-Shaw form on X, where

$$\phi_{L,x} := \sup\left\{G_{hom,x} : G \in \mathrm{PSH}(X,\omega) \ \textit{with} \ \sup_X G = 0\right\}$$

is called the canonical growth condition [7] *of $\omega := dd^c\phi$ at x, here*

$$G_{hom,x}(w) := \limsup_{t\to 0}\{G(\exp_x(tw)) - \nu_x(G)\log(|t|^2)\}, \quad w \in T_xX,$$

$\exp_x$ *denotes the exponential map from T_xX to X with respect to ω and*

$$\nu_x(G) := \liminf_{z\to 0}\frac{G(z)}{\log(|z|^2)}$$

denotes the Lelong number of G at x.

The proof of Theorem G is to use an Ohsawa-Takegoshi extension theorem (see Theorem A in [2]) behind Theorem F.

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Towards Hodge theory for smooth translation-invariant valuations

Jan Kotrbaty

Let $A = \bigoplus_{k=0}^n A_k$ be a commutative, associative, graded algebra over $\mathbb{R}$ with $A_n \cong \mathbb{R}$ and a fixed cone $K \subset A_1$. Let k be any integer between 0 and $\frac{n}{2}$. We say that A satisfies

(1) *Poincaré duality* if for each $x \in A_k$ with $x \neq 0$ there exists $y \in A_{n-k}$ such that $x \cdot y \neq 0$;
(2) *hard Lefschetz theorem* if for each $x_1, \dots, x_{n-2k} \in K$, the map $A_k \to A_{n-k}$ given by $y \mapsto y \cdot x_1 \cdots x_{n-2k}$ is an isomorphism;
(3) *Hodge–Riemann relations* if for each $x_1, \dots, x_{n-2k+1} \in K$ and $y \in A_k$ such that $y \neq 0$ and $y \cdot x_1 \cdots x_{n-2k+1} = 0$ one has $(-1)^k\, y \cdot y \cdot x_1 \cdots x_{2n-k} > 0$.

A prototypical example of such an algebra—from which the terminology was inherited—is the subring $\bigoplus_k H^{k,k}$ of the Dolbeault cohomology of a compact Kähler manifold (M, ω). The statement is classical for the one-dimensional cone $K = \mathbb{R}_{>0}\omega$. However, the case when K is the full Kähler cone was proved only recently by Dinh–Nguyên [9]. A more elementary example is the linear counterpart $\bigoplus_k \bigwedge^{k,k}(\mathbb{C}^n)^*$ proven by Timorin [15]. Further examples are the McMullen's algebra $\Pi(P)$ generated by polytopes strongly isomorphic to a fixed simple polytope P [14] or the Chow ring of a matroid, as proven by Adiprasito–Huh–Katz [1]. Many more examples along with remarkable applications of these properties to combinatorics are listed in the excellent account of Huh [10].

Let $\mathcal{K}$ denote the space of convex bodies, i.e., compact convex subsets in $\mathbb{R}^n$. We call $\phi : \mathcal{K} \to \mathbb{R}$ a valuation if

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}$. The space Val of translation-invariant, continuous valuations is a Banach space. It carries a natural left $GL(n)$ action given by $g \cdot \phi = \phi \circ g^{-1}$. $GL(n)$-smooth vectors in Val are called smooth valuations. It follows from a classical result of McMullen [14] that the space of smooth valuations is graded by the degree of homogeneity of a valuation: $\mathrm{Val}^\infty = \bigoplus_{k=0}^n \mathrm{Val}_k^\infty$. Moreover, by a combination of results of Alesker and Bernig–Fu [2, 4, 8], Val^∞ is in fact a graded algebra satisfying Poincaré duality with respect to a natural product given as follows: Denoting the mixed volume on $\mathbb{R}^n$ by V and k copies of a convex body by $[k]$, one has $A \mapsto V(B_1, \dots, B_k, A[n-k]) \in \mathrm{Val}_{n-k}^\infty$ provided the convex bodies B_i are from $\mathcal{K}_+^\infty$, i.e., have smooth boundaries with positive curvature. Then we define

$$V(B_1,\dots,B_k,\bullet[n-k]) * V(C_1,\dots,C_l,\bullet[n-l]) \\ = c^n_{k,l} V(B_1,\dots,B_k,C_1,\dots,C_l,\bullet[n-k-l])$$

where $c^n_{k,l} = \frac{(n-k)!(n-l)!}{n!(n-k-l)!}$.

Motivated by the aforementioned results in other contexts and by known special cases listed below, the following conjecture was formulated in [11]:

Conjecture 1. The algebra Val^∞ satisfies the hard Lefschetz theorem and the Hodge–Riemann relations with respect to $K = \{V(C,\bullet[n-1]) \mid C \in \mathcal{K}^\infty_+\}$.

The conjecture is now known to hold for the one-dimensional cone

$$\{V(D,\bullet[n-1]) \mid D \in \mathcal{K}^\infty_+ \text{ is a Euclidean ball}\}.$$

In this case, the hard Lefschetz theorem was first showed by Alesker [3] for the subalebra of even valuations. Later on, Bernig–Bröcker [7] removed the evenness assumption and prove the statement for Val^∞. Similarly, the Hodge–Riemann relations were first proved in the even case by Kotrbatý [11]. Somewhat later, Kotrbatý–Wannerer [13] proved the Hodge–Riemann relations for Val^∞ and also gave a new proof of the hard Lefschetz theorem. The point of working with the Euclidean cone is that the Lefschetz map then commutes with the group $SO(n)$. This makes it possible to use representation theory, in particular the known decomposition of Val^∞ into $SO(n)$-types established by Alesker–Bernig–Schuster [6].

For the full cone K, Conjecture 1 is proven in general only for $k = 0, 1$. The former case is easily seen to be equivalent to non-negativity of the mixed volume. The latter was proved by Kotrbatý–Wannerer [12] (and observed independently by Alesker) by generalizing the Alexandrov's second proof of the Alexandrov–Fenchel inequality

$$V(A,B,C_1,\dots,C_{n-2})^2 \geq V(B,B,C_1,\dots,C_{n-2})V(A,A,C_1,\dots,C_{n-2}).$$

Conversely, it was first observed by Alesker that the Hodge–Riemann relations for valuations subsume geometric inequalities: Taking $n = 2$, $k = 1$, and $y = V(K,\bullet) - \frac{V(K,D)}{V(D,D)} V(D,\bullet)$, where $K \in \mathcal{K}^\infty_+$ is arbitrary and $D \in \mathcal{K}^\infty_+$ is a Euclidean ball, the Hodge–Riemann relations together with the definition of the product $*$ of valuations yield at once the isoperimetric inequality on the plane. More generally, the case $k = 1$ of Conjecture 1 implies for general n the Alexandrov–Fenchel inequality [11]. Moreover, Alesker [5] and Kotrbatý–Wannerer [13] deduced in this way from the Hodge–Riemann relations new inequalities for mixed volumes, apparently beyond the previously known geometric methods.

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Octonionic Monge-Ampère operator and its applications to valuations theory and PDE

Semyon Alesker

(1) In this talk I introduce an octonionic Monge-Ampère (MA) operator for 2 octonionic variables, apply it to a construction of translation invariant continuous valuations on $\mathbb{R}^{16}$, in particular a $Spin(9)$-invariant example. Then I introduce (jointly with Peter Gordon) an octonionic analogue of Kähler metrics on 16-torii and prove a Calabi-Yau type theorem for them. The latter states solvability of certain non-linear elliptic second order PDE.

(2) Let $\mathbb{O}$ be the (non-commutative, non-associative) field of octonions. Recall that any octonion $q \in \mathbb{O}$ can be written uniquely

$$q = \sum_{i=0}^{7} x_p e^p,$$

where $x_p \in \mathbb{R}$, and e^p are octonionic units such that $e^0 = 1$ and $(e^p)^2 = -1$ for $p > 0$. The conjugate is defined by

$$\bar{q} = x_0 - \sum_{i=1}^{7} x_p e^p.$$

(3) Let F be a smooth $\mathbb{O}$-valued function on $\mathbb{O} \simeq \mathbb{R}^8$. Define two operators

$$\frac{\partial F}{\partial \bar{q}} := \sum_{i=0}^{7} e^p \frac{\partial F}{\partial x_p}, \quad \frac{\partial F}{\partial q} := \sum_{i=0}^{7} \frac{\partial F}{\partial x_p} \bar{e}^p.$$

Such operator can be defined in the case of several octonionic variables for each variable.

(4) For a smooth function $f\colon \mathbb{O}^n \to \mathbb{R}$ define its octonionic Hessian

$$Hess_{\mathbb{O}}(f) = \left(\frac{\partial^2 f}{\partial \bar{q}_i \partial q_j} \right).$$

This $n \times n$ matrix is Hermitian, i.e. $a_{ij} = \bar{a}_{ji}$.

(5) In order to define the MA operator we need a notion of determinant. There is such a notion for 2×2 octonionic Hermitian matrices. A general such a matrix has the form

$$\begin{bmatrix} a & q \\ \bar{q} & b \end{bmatrix}, \quad a, b \in \mathbb{R}, q \in \mathbb{O}.$$

Its determinant is defined by the usual formula $ab - q\bar{q} = ab - \bar{q}q$.

Finally we define the octonionic MA operator for a C^2-smooth real valued function f by

$$MA_{\mathbb{O}}(f) := \det Hess_{\mathbb{O}}(f).$$

(6) We show that $MA_{\mathbb{O}}(f)$ can be defined by continuity (with respect to the uniform convergence) for arbitrary continuous plurisubharmonic (in particular for convex) functions on $^2 \simeq \mathbb{R}^{16}$ which is not necessarily C^0-smooth as a non-negative measure.

Theorem 1 (Alesker [1], 2008)**.** *Fix $\psi \in C_c^0(\mathbb{R}^{16}, \mathbb{R})$. Define the functional on the family of all convex compact subsets of $\mathbb{O}^2 \simeq \mathbb{R}^{16}$ by*

$$K \mapsto \int_{\mathbb{O}^2} \psi \cdot MA_{\mathbb{O}}(h_K),$$

where h_K is the supporting functional of K. This is a continuous translation invariant 2-homogeneous valuation.

Note that the valuation property follows from a version of the Blocki's formula saying that if u, v are continuous octonionic psh functions and $\min\{u, v\}$ is also psh then

$$MA_{\mathbb{O}}(\min\{u, v\}) + MA_{\mathbb{O}}(\max\{u, v\}) = MA_{\mathbb{O}}(u) + MA_{\mathbb{O}}(v).$$

If the function ψ is $O(16)$-invariant then the corresponding valuation is $Spin(9)$-invariant. For different such ψ's the corresponding valuations are proportional.

The same argument works to construct continuous valuations on the class of continuous octonionic psh (in particular, on convex) functions on $\mathbb{O}^2 \simeq \mathbb{R}^{16}$.

(7) P. Gordon and me introduced a class of metrics on $\mathbb{O}^2$ which are octonionic analogues of Kähler metrics and proved a Calabi-Yau type theorem for an octonionic MA equation on 16-torii for such metrics.

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Singularities of plurisubharmonic functions

Dano Kim

This talk was a survey on recent results in the study of singularities of plurisubharmonic functions, a topic which has seen active interactions among complex analysis, algebraic geometry and convex geometry.

A plurisubharmonic (psh for short) function φ on a complex manifold X is said to have *analytic singularities* (of type $\mathfrak{a}^c$) if it is locally of the form $\varphi = c \log \sum_{i=1}^{m} |g_i| + u$ where $c \geq 0$ is real, u bounded and $g_1, \ldots, g_m$ are local holomorphic functions generating a (global) coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$. Informally, let us say that such φ is *algebraic psh* in that its singularities are encoded in $\mathfrak{a}^c$ which is algebro-geometric data. Otherwise, let us say φ is *general psh*, which is a transcendental object.

General psh functions emerge in several different contexts in algebraic geometry: for example, from the study of graded sequence of ideal sheaves (cf. [5], [11]) or from local weight functions of singular hermitian metrics for a pseudoeffective line bundle (cf. [4]). In many concrete statements/results, one can observe two patterns. 1) A general psh function behaves very differently from algebraic ones. 2) A general psh function behaves similarly to algebraic ones.

An instance of 1) is a recent result [12, Thm. 5.7] joint with Hoseob Seo on psh functions with accumulation points of jumping numbers, which generalized a single initial example due to [8] to infinitely many examples, in fact characterizing them among all toric psh functions in dimension 2. Seo generalized this result to arbitrary dimension in [14]. Connection with convex analysis and geometry played an important role in these works. In this regard, another recent paper of Seo with An [1] developed further methods of using convex analysis to study equisingular approximation of psh functions.

On the other hand, as an instance of 2), the following result (joint with J. Kollár, in preparation) was announced. ($\mathcal{J}(\varphi)$ is the multiplier ideal sheaf of φ, cf. [4].)

Theorem 1. *Let X be a complex manifold and φ a quasi-plurisubharmonic function on X such that (X, φ) is log canonical. Then every point of X has a Stein open neighborhood $U \subset X$ with holomorphic functions g_i on U and real $c_i > 0$, such that $\psi := \sum_{i=1}^{m} c_i \log |g_i|$ is log canonical at every point of U, and $\mathcal{J}(\varphi) = \mathcal{J}(\psi)$.*

In the second part of this talk, we consider psh functions with isolated singularities at a point, say $0 \in \mathbb{C}^n$. For such psh functions $u_1, \ldots, u_n$, we denote their mixed Monge-Ampère mass at $0 \in \mathbb{C}^n$ by

$$m(u_1, \ldots, u_n) = \int_{\{0\}} (dd^c u_1) \wedge \ldots \wedge (dd^c u_n)$$

which is defined due to work of Demailly [3]. In the case when $u_k = \log|\mathfrak{a}_k|, k = 1, \dots, n$, for zero-dimensional ideals $\mathfrak{a}_k$ at 0, $m(u_1, \dots, u_n)$ is equal to the mixed multiplicity $\mu(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ of the ideals. In the joint work with Alexander Rashkovskii [10], we have the following Alexandrov-Fenchel inequality for mixed Monge-Ampère masses (generalizing a result of [5]).

Theorem 2. *Let $u_1, \dots, u_n$ be psh functions with isolated singularities at $0 \in \mathbb{C}^n$. Then we have the inequality*

$$m(u_1, u_1, u_3, \dots, u_n)\, m(u_2, u_2, u_3, \dots, u_n) \geq m(u_1, u_2, u_3, \dots, u_n)^2.$$

As a consequence for convex geometry, we derive an Alexandrov–Fenchel inequality for mixed covolumes [10, Cor. 1.5] using the case when $u_1, \dots, u_n$ are appropriate toric psh functions.

An important special case of mixed MA masses is 'higher Lelong numbers' of u defined as

$$e_k(u) = \int_{\{0\}} (dd^c u)^k \wedge (dd^c \log|z|)^{n-k} \tag{1}$$

for $k = 1, \dots, n$, generalizing the usual Lelong number $e_1(u)$. The main result of [5] (cf. [2]) is the following lower bound for the log canonical threshold $\mathrm{lct}(u)$ at 0,

$$\mathrm{lct}(u) \geq \frac{e_{n-1}(u)}{e_n(u)} + \frac{e_{n-2}(u)}{e_{n-1}(u)} + \dots + \frac{1}{e_1(u)}. \tag{2}$$

When u is algebraic psh associated to a zero-dimensional ideal $\mathfrak{a}$, this improves an earlier result of [7], $\mathrm{lct}(u) \geq n(\frac{1}{e_n(u)})^{\frac{1}{n}}$ which was applied in the topic of birational rigidity from birational geometry. The author does not yet know of an instance where (2) itself was used in birational geometry so far.

On the other hand, in a recent paper [9], we discovered an application of (2) to a completely different topic in algebraic geometry, namely hypersurface singularities. Let $(f = 0)$ be a germ of an isolated hypersurface singularity at $0 \in \mathbb{C}^n$. In [15], Teissier defined the *polar invariant* $\theta(f)$ which measures the rate of vanishing of the Jacobian ideal J_f of f with respect to that of the maximal ideal $\mathfrak{m}$ of $0 \in \mathbb{C}^n$. Also consider $\theta(f_1), \dots, \theta(f_{n-1})$ where f_j denotes the restriction of f to general j-codimensional planes containing $0 \in \mathbb{C}^n$. We have the following upper bound for the particular combination of these polar invariants from a question in [15].

Theorem 3. [9] *Let $\mathrm{lct}(\mathfrak{m} \cdot J_f)$ be the log canonical threshold at $0 \in \mathbb{C}^n$ of the product ideal $\mathfrak{m} \cdot J_f$. We have*

$$\frac{1}{1+\theta(f)} + \frac{1}{1+\theta(f_1)} + \dots + \frac{1}{1+\theta(f_{n-1})} \leq \mathrm{lct}(\mathfrak{m} \cdot J_f). \tag{3}$$

In fact, in his question [15, p.7], Teissier conjectured that one can put the *Arnold exponent* $\sigma(f)$ of f at 0, in the place of $\mathrm{lct}(\mathfrak{m} \cdot J_f)$ in (3). The Arnold exponent $\sigma(f)$ is related to log canonical thresholds by $\mathrm{lct}(f) = \min\{\sigma(f), 1\}$. The conjectured upper bound was proved by [6] whose methods are very different from [9] and based on more algebraic theories such as Saito's theory of mixed Hodge modules and the theory of Hodge ideals (cf. [13]).

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Weighted Ehrhart theory

Katharina Jochemko

The convex hull of finitely many points in the integer lattice $\mathbb{Z}^d$ is called a lattice polytope. Ehrhart [2] showed that for any lattice polytope $P \subset \mathbb{R}^d$, there is a polynomial $\mathrm{E}_P(n)$ such that $\mathrm{E}_P(n) = |nP \cap \mathbb{Z}^d|$ for all integers $n \geq 0$. The polynomial $\mathrm{E}_P(n)$ is called the Ehrhart polynomial and is the central object of study in Ehrhart theory. At the heart of Ehrhart theory are questions about the interpretation and characterization of the coefficients of the Ehrhart polynomial. A standard technique is to consider the h^*-polynomial $h_P^*(t)$, a linear transform of the Ehrhart polynomial, with many desirable properties. For a d-dimensional polytope P it is a polynomial of degree at most d given by the numerator of the generating series

$$\sum_{n\geq 0} \mathrm{E}_P(n)t^n \;=\; \frac{h_P^*(t)}{(1-t)^{d+1}}$$

A fundamental result by Stanley [4] states that the coefficients of the h^*-polynomial are always nonnegative and integers, in contrast to the coefficients of the Ehrhart polynomial which can be negative and rational in general. Another desirable property due to Stanley [5] is monotonicity of the coefficients, that is, for lattice polytopes $P, Q \subset \mathbb{R}^d$ with h^*-polynomials $h_P^*(t) = \sum_{i\geq 0} h_i^*(P)t^i$ and $h_Q^*(t) = \sum_{i\geq 0} h_i^*(Q)t^i$, if $P \subseteq Q$ then

$$h_i^*(P) \;\leq\; h_i^*(Q) \quad \text{for all } i \geq 0\,.$$

In this talk we present extension of Stanley's nonnegativity and monotonicity results [4, 5] to weighted lattice point enumeration. These results were obtained in joint collaboration with Esme Bajo, Robert Davis, Jesús A. De Loera, Alexey Garber, Sofía Garzón Mora and Josephine Yu [1].

A naive way to express the number of lattice points in a polytope P is $\sum_{\mathbf{x}\in P\cap\mathbb{Z}^d} 1$. We consider more general expressions of the form

$$\mathrm{ehr}(P, w) \;=\; \sum_{\mathbf{x}\in P\cap\mathbb{Z}^d} w(\mathbf{x})$$

where $w\colon \mathbb{R}^d \to \mathbb{R}$ is a polynomial function. Weighted sums of that type appear in various different areas, in particular, in enumerative combinatorics, optimization, convex geometry and statistics, see [1] and references therein. By results of Khovanskiĭ and Puklikov [3], Ehrhart's polynomiality result [2] extends to this weighted setup. More precisely, if w is a polynomial function of degree at most m and P a lattice polytope of dimension d then $\mathrm{ehr}(nP, w)$ is given by a polynomial of degree at most $d+m$ in the dilation factor $n \geq 0$. It follows that the corresponding generating series is again a rational function and we define the weighted h^*-polynomial of P, denoted $h_{P,w}^*(t)$, to be its numerator:

$$\sum_{n\geq 0} \mathrm{ehr}(nP, w)t^n \;=\; \frac{h_{P,w}^*(t)}{(1-t)^{d+m+1}}\,.$$

A natural question to ask is for which classes of polynomial functions w the weighted h^*-polynomial $h_{P,w}^*(t)$ satisfies nonnegativity and monotonicity of its coefficients. We consider two families of weights: sums of products of linear forms that are nonnegative on P, denoted R_P, and nonnegative sums of products of linear forms, denoted S_P. Clearly, $R_P \subset S_P$. In general, this inclusion is strict. For example, if $P = \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2) \subset \mathbb{R}^2$ then $w(x_1, x_2) = (x_1 - x_2)^2$ is in S_P but not in R_P. It is rather easy to find examples of non-homogeneous weight functions for which the weighted h^*-polynomial has negative coefficients, even if the value of the weight function at every point in the polytope is nonnegative [1]. We thus restrict to homogeneous polynomial weight functions.

We have the following nonnegativity results.

Theorem 1 ([1]). *Let $P \subset \mathbb{R}^d$ be a lattice polytope and let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial function.*

(i) If the weight w is a homogeneous element in R_P, then the coefficients of $h^*_{P,w}(t)$ are nonnegative.
(ii) If the weight w is a homogeneous element in S_P, then $h^*_{P,w}(t) \geq 0$ for all $t \geq 0$.

We remark that Theorem 1 (i) is rather sharp in the sense that it does in general not even extend to the case when w is the square of a single linear form, except for if P is a lattice polygon in $\mathbb{R}^2$ [1].

Further, we have the following monotonicity results.

Theorem 2 ([1]). *Let $P, Q \subset \mathbb{R}^d$ be a lattice polytopes, $P \subseteq Q$, and let $w\colon \mathbb{R}^d \to \mathbb{R}$ be a polynomial function.*

(i) If the weight w is a homogeneous element in R_P, then $h^*_{P,w}(t) \preceq h^*_{Q,w}(t)$ coefficient-wise.
(ii) If the weight w is a homogeneous element in S_P and $\dim P = \dim Q$, then $h^*_{P,w}(t) \geq 0$ for all $t \geq 0$.

Observe that in Theorem 2 (ii) the assumption $\dim P = \dim Q$ is necessary [1], in contrast to the classical monotonicity result by Stanley [5] and Theorem 2 (i) where P and Q may have different dimensions.

While the talk focusses on weighted Ehrhart polynomials of lattice polytopes, the theory and result hold more general for rational polytopes after suitable adjustments. See [1] for detailed statements.

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Uncertainty and quasianalyticity on higher grassmannians

Dmitry Faifman

1. Introduction

Recall two integral transforms playing important roles in integral geometry.

The Radon transform is defined for $p < k$ by

$$\mathcal{R}_{p,k} : C(Gr_p(\mathbb{R}^n)) \to C(Gr_k(\mathbb{R}^n)), \quad \mathcal{R}_{p,k} f(F) = \int_{Gr_p(F)} f(E)dE;$$

The cosine transform is given by

$$\mathcal{C}_k : C(Gr_k(\mathbb{R}^n)) \to C(Gr_k(\mathbb{R}^n)), \quad \mathcal{C}_k f(F) = \int_{Gr_k(\mathbb{R}^n)} |\cos(E, F)| f(E)dE.$$

Both admit natural extensions to the space of distributions.

For a geometric application, let $\mathcal{K}_s(\mathbb{R}^n)$ the centrally symmetric convex bodies, and by $\mathcal{S}_s(\mathbb{R}^n)$ the centrally-symmetric star-shaped sets. Let $A_k(E; S) := vol_k(E \cap S) \in C(Gr_k(\mathbb{R}^n))$ denote the k-section function of $S \in \mathcal{S}_s(\mathbb{R}^n)$, and $P_k(E; K)) := vol_k(\mathrm{Pr}_E(K)) \in C(Gr_k(\mathbb{R}^n))$ the k-projection function of $K \in \mathcal{K}(\mathbb{R}^n)$.

It then holds that $A_k(E; S) = \mathcal{R}_{1,k}(\frac{1}{k}\rho_S^k)(E)$, where ρ_S is the radial function of S. Far less obviously, it holds also that $P_k(E; K)$ lies in the closure of Image($\mathcal{C}_k$).

Two foundational results in geometric tomography are as follows.

Theorem 1 (Funk 1916)**.** *Fix $1 \le k \le n-1$. If $S, S' \in \mathcal{S}_s(\mathbb{R}^n)$ satisfy $A_k(E; S) = A_k(E; S')$ for all $E \in Gr_k(\mathbb{R}^n)$, then $S = S'$.*

Theorem 2 (Aleksandrov 1937 [1])**.** *Fix $1 \le k \le n-1$. If $K, L \in \mathcal{K}_s(\mathbb{R}^n)$ satisfy $P_k(E; K) = A_k(E; L)$ for all $P \in Gr_k(\mathbb{R}^n)$, then $K = L$.*

The former result is based on the injectivity of the Radon transform $\mathcal{R}_{1,k}$. The latter makes use of the injectivity of the cosine transform $\mathcal{C}_1$.

2. Results

In the following, T denotes either the Radon transform $\mathcal{R}_{p,k}$ with $\dim Gr_p < \dim Gr_k$, or alternatively $\mathcal{C}_k$ with $2 \le k \le n-2$. We will write $\mathrm{Image}_{C^N}(T) := \mathrm{Image}_{C^{-\infty}}(T) \cap C^N$, where $\mathrm{Image}_{C^{-\infty}}(T)$ is the image of T on distributions. In either cases, $\mathrm{Image}_C^N(T)$ is not dense in $C^N(Gr_k(\mathbb{R}^N))$ $N \in \{-\infty, 0, \infty\}$. A representation-theoretic description of the image is available, which we now recall.

Denote $\kappa = \min(k, n-k)$, and $\Lambda_\kappa = \{\lambda_1 \ge \cdots \ge \lambda\kappa : \lambda_i \in 2\mathbb{Z}_+\}$. One has the multiplicity-free decomposition [8]

$$L^2(Gr_k(\mathbb{R}^n)) = \oplus_{\lambda \in \Lambda_\kappa} V_\lambda,$$

where V_λ are certain pairwise distinct irreducible representations of $\mathrm{O}(n)$.

We then have

Theorem 3 (Gelfand-Graev-Rosu [5])**.** Image($\mathcal{R}_{p,k}$) *consists of those V_λ with $\lambda_{p+1} = 0$.*

Theorem 4 (Alesker-Bernstein [2])**.** Image($\mathcal{C}_k$) *consists of those* V_λ *with* $\lambda_2 \leq 2$.

It follows that any $f \in \text{Image}(T)$ has rather stringent restrictions on its spectrum. Our goal is to find a geometric rigidity manifestation of this spectral restriction. It will be realized through a quasianalyticity phenomenon. Generally speaking, a class of functions is called quasianalytic if it has a unique continuation property understood in broad terms: the values of a function from the class in an appropriate small set must determine the function uniquely.

Definition 5. Fix $F \in Gr_{n-k}(\mathbb{R}^n)$. The open Schubert cell $\Sigma_F^k \subset Gr_k(\mathbb{R}^n)$ is $\Sigma_F^k = \{E : E \cap F = \{0\}\}$. The Schubert equator $\mathcal{X}i_F^k$ is its complement, $\mathcal{X}i_F^k = \{E : E \cap F \neq \{0\}\}$.

Definition 6. A class of functions $\mathcal{A} \subset C(Gr_k(\mathbb{R}^n))$ is exp-$\mathcal{X}i$-quasianalytic if, whenever $f, g \in \mathcal{A}$ coincide exponentially on $\mathcal{X}i_F^k$, namely if for some $C, c > 0$ it holds for all E that

$$|f(E) - g(E)| \leq C \exp\left(-\frac{c}{d_{Gr}(E, \mathcal{X}i_F^j)}\right),$$

then $f = g$.
A class of distributions $\mathcal{A} \subset C(Gr_k(\mathbb{R}^n))$ is Bernstein-$\mathcal{X}i$-quasianalytic if, whenever $f, g \in \mathcal{A}$ coincide in a neighborhood of $\mathcal{X}i_F^k$, then $f = g$.

Our main result is as follows.

Theorem 7 (F [4])**.** *Let T denote either $\mathcal{R}_{p,k}$ with* $\dim Gr_p < \dim Gr_k$, *or* $\mathcal{C}_k$ *with* $2 \leq k \leq n-2$. *Then* $\text{Image}_{C^0}(T)$ *is exp-$\mathcal{X}i$-quasianalytic, and* $\text{Image}_{C^{-\infty}}(T)$ *is Bernstein-$\mathcal{X}i$-quasianalytic.*

This immediately implies sharper version of Funk's and Aleksandrov's theorems:

Theorem 8 (Sharper Funk, F [4])**.** *Fix $1 \leq k \leq n-1$. If for $S, S' \in \mathcal{S}_s(\mathbb{R}^n)$ it holds that $A_k(E; S)$ and $A_k(E; S')$ coincide on any single Schubert equator, then $S = S'$.*

Theorem 9 (Sharper Aleksandrov, F [4])**.** *Fix $1 \leq k \leq n-1$. If for $K, L \in \mathcal{K}_s(\mathbb{R}^n)$ it holds that $P_k(E; K)$ and $P_k(E; L)$ coincide on any single Schubert equator, then $K = L$.*

One similarly obtains a sharper version of the Klain injectivity theorem [9] in convex valuation theory, which can then be used to prove also a sharper version of the Schneider injectivity theorem [10].

3. Sketch of proof of Theorem 7

The idea of the proof is quite simple. Let us work with $T = \mathcal{C}_k$.

Assume by contradiction that a counterexample f exists, which for simplicity we assume to be supported inside Σ_F^k. The first step is producing a counterexample supported at a point. We use a trick going back to Gelfand-Graev-Rosu [5], writing $\mathcal{C}_k$ as a $\mathrm{GL}_n(\mathbb{R})$-equivariant transform between spaces of sections of certain line bundles. Considering f as such a section, this allows to take $g_\epsilon = \mathrm{Pr}_{F^\perp} + \epsilon \mathrm{Pr}_F \in \mathrm{GL}_n(\mathbb{R})$, and define $f_\epsilon = g_\epsilon^*(f)$. Evidently f_ϵ has support shrinking to $\{F^\perp\}$, and one can show that a sequence $c_\epsilon \to 0$ exists such that $c_\epsilon f_\epsilon$ converges to a distribution f_0 supported at $\{F^\perp\}$ which is still inside Image($\mathcal{C}_k$).

The second step consists of proving an uncertainty principle. Namely, we prove

Theorem 10 (F [4]). *Assume $1 \leq k \leq n-1$. Assume $f_0 \in C^{-\infty}(Gr_k(\mathbb{R}^n))$ is supported at one point. Consider $supp(\widehat{f_0}) = \{\lambda \in \Lambda_\kappa : \widehat{f_0}(\lambda) \neq 0\}$. Then*

$$\lim_{m\to\infty} \frac{\#\{\lambda \in supp(\widehat{f_0}) : \sum \lambda_i \leq 2m\}}{\#\{\lambda \in \Lambda_\kappa : \sum \lambda_i \leq 2m\}} = 1.$$

Moreover for $k \in \{1, n-1\}$, $supp(\widehat{f_0})$ must be co-finite.

However by the theorem of Alesker-Bernstein, that limit above must vanish, leading to a contradiction.

Let us conclude by remarking that by results of Grinberg [7], Gonzalez and Kakehi [6], the image of the Radon transform admits a description as the kernel of $\mathrm{SO}(n)$-invariant differential operator, while the image of the cosine transform lies in the kernel of another such operator by results of Alesker-Gourevitch-Sahi [3]. However, this quasianalyticity property does not appear to be a consequence of the PDEs. Furthermore, the methods and results above apply in greater generality to various $\mathrm{GL}_n(\mathbb{R})$-modules realized as spaces of sections of equivariant line bundles over the grassmannians, where no PDE description is known to exist.

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Valuations on Kähler manifolds

Gil Solanes
(joint work with Andreas Bernig, Joseph H.G. Fu, Thomas Wannerer)

Let M^n be a smooth manifold, and let $\mathcal{P}(M)$ be the class of compact submanifolds with boundary of M. A *smooth valuation* on M (cf. [1]) is a functional $\phi\colon \mathcal{P}(M) \to \mathbb{R}$ of the form

$$\phi(A) = \int_{N(A)} \omega + \int_A \eta$$

where $N(A)$ is the so-called *conormal cycle* of A, and $\omega \in \Omega^{n-1}(S^*M)$ is a differential form on the cosphere bundle S^*M of M, while $\eta \in \Omega^n(M)$.

Remarkably (cf. [2]), the space $\mathcal{V}(M)$ of smooth valuations on M has an algebra structure fulfilling $e^*(\phi \cdot \varphi) = e^*(\phi) \cdot e^*(\varphi)$ for all $\phi, \varphi \in \mathcal{V}(N)$ and any smooth embedding $e\colon M \to N$.

As a consequence of H. Weyl's tube theorem, every riemannian manifold M^n has a canonical subalgebra $\mathcal{LK}(M) \subset \mathcal{V}(M)$, called the *Lipschitz-Killing algebra*, characterized by the following facts:

i) if $e\colon M \to N$ is an isometric embedding between riemannian manifolds, then $e^*(\mathcal{LK}(N)) = \mathcal{LK}(M)$
ii) if M is euclidean space $\mathbb{R}^n$, then $\mathcal{LK}(M)$ is the full algebra $\mathrm{Val}^{O(n)}$ of isometry invariant valuations.

It follows that the algebra structure of $\mathcal{LK}(M)$ is universal: it depends only on the dimension of M. Another simple consequence is that the algebras of invariant valuations of euclidean space $\mathbb{R}^n$ and the round sphere S^n are isomorphic to each other.

It was realized in [3] that also the algebra of isometry invariant valuations of $\mathbb{C}P^n$ is isomorphic to the algebra $\mathrm{Val}^{U(n)}$ of valuations of $\mathbb{C}^n$ invariant under hermitian isometries. This suggested the possibility that an extened version of the Lipschitz-Killing algebra might be present on any Kähler manifold. This is precisely the content of the following theorem.

Theorem 1 ([4]). *To every Kähler manifold M^n there is an associated subalgebra $\mathcal{KLK}(M) \subset \mathcal{V}(M)$ isomorphic to $\mathrm{Val}^{U(n)}$ in such a way that $e^*(\mathcal{KLK}(N)) = \mathcal{KLK}(M)$ for every isometric holomorphic embedding $e\colon M \to N$ of Kähler manifolds.*

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Superforms and convex geometry

Bo Berndtsson

This lecture continues the first part of my introductory lecture. A superform on $\mathbb{R}^n$ is defined as a differential form on $\mathbb{C}^n$ whose coefficients depend only on $x = \operatorname{Re} z$;

$$\alpha = \sum \alpha_{JK}(x) dx_J \wedge d\xi_K.$$

The usual exterior differentiation operator acts on superforms, and we define

$$d^{\#} = \sum \partial/\partial x_j \, d\xi_j \wedge .$$

This coincides with the operator $d^c = i(\bar\partial - \partial)$ from complex analysis (the d-operator twisted with the complex structure), but we write $d^{\#}$ to emphazise that we consider only its action on superforms. If $\phi(x)$ is a function on $\mathbb{R}^n$ we have

$$d^{\#}\phi = \sum \phi_{jk} dx_j \wedge d\xi_k,$$

which makes sense as a current with measure coefficients for any convex (finite valued) function, and also for functions that can be written locally as the difference of two convex functions. By the theory of Bedford and Taylor (see the (abstract of) the introductory lecture), wedge products of such currents

$$\Omega_k = d^{\#}\phi_1 \wedge ... d^{\#}\phi_k$$

are also well defined. Taking $k = n$ and $\phi = \phi_1 = ... \phi_n$ we get the Monge-Ampère measure of ϕ, $MA(\phi)$.

The integral of a superform of maximal degree is defined as

$$\int a_0(x) dx_1 \wedge d\xi_1 \wedge ... dx_n \wedge d\xi_n := \int \alpha_0(x) dx,$$

meaning essentially that we define

$$\int d\xi_1 \wedge ... d\xi_n = \pm 1$$

(Berezin integration).

The main point of the lecture was to advocate the use of superforms for calculations involving the volume of convex bodies (this is partly based on previous work of my students A. Lagerberg and S. Larsson) , and we tried to illustrate that with a possibly new proof of the Alexandrov-Fenchel theorem. If ϕ is the support function of a convex body K, one finds that $MA(\phi)$ is a a Dirac measure at the origin of size $|K|$, the volume of K. More generally, if ϕ_j are support functions of convex bodies K_j;

$$d^{\#}\phi_1 \wedge ... d^{\#}\phi_n = V(K_1, ... K_n)\delta_0,$$

where $V(K_1, ... K_n)$ is the mixed volume of the K_j. More generally we can then define in the same way $V(u_1, ... u_n)$, where u_j are 1-homogeneous functions that can be written as differences of support functions of convex bodies. Fixing convex bodies $K_3, ... K_n$ and their support functions, we then get a quadratic form

$$Q(u, u) := V(u, u, \phi_3, ... \phi_n).$$

The essence of the Alexandrov-Fenchel theorem is that this form has Lorentzian signature, i. e. that it is positive somewhere, and seminegative on a subspace of codimension 1.

One approach to proving this (essentially Alexandrov's approach) is to note that

$$Q(u,v) = V(u,v,\phi_3 ...) = \int_{\partial U} d^{\#}u \wedge d^{\#}v \wedge \Omega_{n-2},$$

where U is any convex neighbourhood of the origin. After a rewrite this can be written as

$$\int_{\partial U} uA(v)dm$$

where dm is, say, surface measure and $A(v)$ is an elliptic second order differential operator. Alexandrov's proof proceeds by studying the eigenvalues of A. We sketched an alternative way, based on a study of Dirichlet problem for A on domains in ∂U of the form

$$D = \{x \in \partial U, x_1 > 0\}.$$

The main points were that the only function in D with zero boundary values, solving $A(u) = 0$, is $u = x_1$, and that this statement implies the Alexandrov-Fenchel theorem.

On the Adler-Taylor Gaussian kinematic formula

Joseph Fu

The statisticians R. Adler and J. Taylor have introduced a new type of kinematic formula based on the behavior of *Gaussian random fields* on a Riemannian manifold M^n: that is, smooth random functions $f = f_\omega, \omega \in \Omega$, on M whose value $f(x)$ is an $N(0,1)$ random variables for each $x \in M$, and which give an isometric embedding $M \to \mathbb{R}^\Omega$. Specifically, given d i.i.d. random functions of this type we obtain a random mapping $F : M \to \mathbb{R}^d$. For convex bodies $D \subset \mathbb{R}^d$, Adler-Taylor show that for nice objects $A \subset M$ that

$$\mathbb{E}[\chi(A \cap F^{-1}D)] = \sum c_i \mu_i(A)\gamma_i(D) \tag{1}$$

where the μ_i are the intrinsic volumes and the γ_i are the "Gaussian intrinsic volumes" on $\mathbb{R}^d$, viz. the Gaussian measure γ_0 and its derivatives with respect to metric expansion.

We propose that Adler-Taylor theory should be viewed as a chapter in the theory of valuations initiated by Alesker. To do so requires the resolution of two technical issues.

First, we recall that Faifman and Hofstätter [3] have shown that any π belonging to the algebra $\mathcal{V}(M)$ of smooth valuations on M may be expressed as

$$\pi = \int_{\mathcal{S}} \chi(\cdot \cap S)\, dS, \tag{2}$$

and subject to the multiplication formula

$$\nu \cdot \pi = \int_{\mathcal{S}} \nu(\cdot \cap S)\, dS \tag{3}$$

for any $\nu \in \mathcal{V}(M)$, where $(\mathcal{S}, dS)$ is a measured family of subsets of M. The left hand side of (1) is an expression of the type (2), and we expect that it is also subject to (3), but we do not yet have a full understanding of the conditions on the family $(\mathcal{S}, dS)$ that would ensure this. This is surely true of the (1), in view of the the higher Adler-Taylor formulas

$$\mu_j \cdot \mathbb{E}[\chi(\cdot \cap F^{-1}D)] = \mathbb{E}[\mu_j(\cdot \cap F^{-1}D)] = \sum c_{i,j}\mu_{i+j}(\cdot)\gamma_i(D) \tag{4}$$

Second, [2] proposes a proof of (1) via embeddings of M into spheres $\Sigma_N \subset \mathbb{R}^{N+1}$ of radius $\sqrt{N}$ and dimension N, obtaining the Gaussian projection of M into $\mathbb{R}^d$ by precomposing a given orthogonal projection $\mathbb{R}^{N+1} \to \mathbb{R}^d$ with a random rotation of Σ_N. This brings the spherical kinematic formula into play, which we understand thoroughly using the methods of algebraic integral geometry. To confirm (1) by this means requires a careful study of the resulting tube formulas in Σ_N.

Beyond these technical questions, Adler-Taylor theory also suggests an avenue towards resolving a central puzzle of integral geometry: the extension of the Weyl Tube Theorem (that the $\mu_i(M)$ are Riemannian invariants) to singular spaces such as convex hypersurfaces or sets with positive reach.

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Control Methods in Hyperbolic PDEs

Organized by
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Olivier Glass, Paris
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05 November – 10 November 2023

Abstract. Control of hyperbolic partial differential equations (PDEs) is a truly interdisciplinary area of research in applied mathematics nurtured by challenging problems arising in most modern applications ranging from road traffic, gas pipeline management, blood circulation, to opinion dynamics and socio-economical models, as well as in environmental and biological issues, or more recently in the analysis of deep learning and machine learning methods. The topic has gained an increasing attraction of researchers in the last twenty years due to fundamental theoretical as well as numerical advances achieved in the field of nonlinear hyperbolic PDEs. The hyperbolic and the control of PDEs communities, while pursuing separate interests in their respective range of action with a different focus and, often, with a different array of technical tools, do share a substantial body of common knowledge and background. We think the time is right and the momentum is propitious to bring those communities together at a joint workshop, to mutually stimulate each other and interact with one another, for a marked advancement of this area of research on a broad spectrum of control ranging from theoretical to numerical problems and covering also the emerging challenges involving the interplay between (topics of) control and learning. In order to also attract young scientists to this striving field we focus on selected lectures, in-depth discussions, transfer of information from senior to young researchers, and vice versa, and invited plenary talks.

Mathematics Subject Classification (2020): 35Lxx, 35L65, 35Q93, 49J20, 93B05, 93C20.

Introduction by the Organizers

The workshop *Control Methods in Hyperbolic PDEs*, organized by F. Ancona (Padua), B. Anvari (London), O. Glass (Paris) and M. Herty(Aachen) was well attended with over 40 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds. The presentations fostered discussions and exchange on emerging points of interest. Young researchers had also the opportunity to give ad hoc presentations in the afternoon to present their research and to benefit from discussions with colleagues in their field of research. The intention of the workshop has been to cover three topics that are currently very relevant for the future of the field.

First, hyperbolic equations on networks have been studied recently inspired by applications in infrastructure networks. Control questions arise naturally by considering for example nodal flow regulation or capacity planing problems. Usually, the problems lead to spatially one-dimensional hyperbolic equations that allow for a broader range of applicable tools. Natural boundary controllability questions need to be addressed. In the case of regular solutions very general results were obtained in the recent past while the case of entropy weak solutions is much less clear. For example, a very important question is to understand how these equations can be controlled when the control is applied only on one side of the boundary. New ideas of methods of vanishing viscosity and relative entropy analysis in order to understand the well-posedness and control for the dynamics shall be considered. The topic is strategically located in a dimension between one and two and bridges the rather broad knowledge on one-dimensional problems with the less established theory in multiple dimensions by studying problems on highly connected graphs.

Second, the topic of closed–loop feedback controls has been intensively studied in the context of ordinary differential equations but its development for hyperbolic problems is still in early stage besides strong progress on methods and tools in the case of smooth solutions. The rigorous analysis of control strategies, the incorporation of complex flow models in the numerical simulation, as well as, error estimates for the numerical approximation are not yet fully understood. Challenges like nonlinear models, online efficient control in particular for large scale networks are also currently absent. Feedback laws using methods of suitable Lyapunov functions have been applied to control smooth and mildly nonlinear flow patterns. Questions on how to extend this to strong nonlinearities and on how for example entropy and entropy flux pairs may be applied to obtain results also for weakly differentiable functions will be considered in this part. Closed–loop systems might be studied without relying on a sensitivity calculus and therefore we placed this topic in the middle between a pure Cauchy problem and a possible optimality system arising in open loop control. Techniques developed here might therefore shed light on possible ways to tackle problems in open loop control.

Third, apart from formal approaches, very little is known about the link of the sensitivity of particle based models with corresponding sensitivity of the formal kinetic equations. Presentations on individual-based models, their rigorous coarse-graining into macroscopic models and possible applications will be given. The agent based models have the advantage that they consists of finitely many ordinary differential equations such that control concepts are readily available. The coarse graining of those concepts to derive a suitable calculus on the kinetic level will be a first step towards a simi lar calculus on the macroscopic, i.e., hyperbolic level. Therefore, this topic might open new ways to derive suitable concepts for control questions by revisiting existing hierarchies from particles to kinetic and hyperbolic equations. In a similar spirit, recent advances in the (characteristic-like) Lagrangian representation techniques developed for nonlinear conservation laws could provide powerful tools to address controllability and optimality issues in mixed PDE-ODE problems arising for example in mixed traffic flow. This may include also control problems for PDE-ODE models and nonlocal models that naturally arise in pedestrian traffic and autonomous vehicles applications, as well as in supply-chain for complex production networks. Further, there are formulations of dense neural networks where coarse graining has been proposed to develop an efficient description that is amend- able for optimization and control methods. Those in turn could be beneficial in understanding training process of complex learning tasks.

The workshop succeeded in having talks on all fields as well as intensive discussions across those. The following extended abstracts illustrate this nicely and summarize very well the successful workshop.

Workshop: Control Methods in Hyperbolic PDEs

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Abstracts

Bilinear control of PDE's by bounded or unbounded scalar-input control and constructive algorithms for concrete applications

Fatiha Alabau-Boussouira
(joint work with Piermarco Cannarsa, Cristina Urbani)

Scalar-input bilinear controlled abstract PDE's systems take the form:

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, \ t \in [0, T], \\ u(0) = u_0 \in X, \end{cases} \tag{1}$$

where X is the state space, A stands for the diffusion operator, $A : D(A) \subset X \mapsto X$ is a linear unbounded operator and the infinitesimal generator of a $\mathcal{C}^0$ semigroup of bounded linear operators on X, B is a fixed given control operator with $B : D(B) \subset X \mapsto X$ is a linear (bounded or unbounded) operator on X. Here u stands for the state depending on time and taking values in X, and the control p satisfies $p : [0, T] \mapsto \mathbb{R}$, so that the control is a real scalar-valued function that appears as a multiplicative factor acting on the state u (see [7, 9, 10] and references therein).

Such abstract formulation have applications in quantum control (Schrödinger equation) or for the evolution of a probability density (Fokker-Planck equation), on the modeling of nuclear chain reactions for the processes of production of neutrons in fission (heat-based models) or for mechanical systems (such as beams) for describing the dynamics of smart materials. We consider mainly in the sequel heat-based models.

1. Local and semi-global exact controllability to the ground state

We consider (1) under the following assumptions: $(X, \langle\cdot,\cdot\rangle, ||\cdot||)$ is a separable Hilbert space, $B \in \mathcal{L}(X)$, $p \in L^1_{loc}(0, \infty); \mathbb{R})$, $A : D(A) \subset X \mapsto X$ is a densely defined linear operator satisfying:

$$\begin{cases} A \text{ is self-adjoint} \\ \exists\ \sigma > 0 : \ \langle Ax, x\rangle \geq -\sigma ||x||^2 \quad \forall\ x \in D(A) \quad (\textit{Assumptions on } A) \\ \exists\ \lambda > 0 : \ \text{such that } (\lambda I + A)^{-1} : X \mapsto X \text{ is compact} \end{cases} \tag{2}$$

so that X has an orthonormal basis $(\varphi_k)_{k\in\mathbb{N}^*}$ formed of eigenvectors of A, the eigenvalues $(\lambda_k)_{k\in\mathbb{N}^*}$ of A are bounded below by σ, ordered in a nondecreasing sequence converging to ∞ as k goes to infinity. Setting $\psi_j(t) = e{-}\lambda_j t\varphi_j$, we prove:

Theorem 1 (A.-B., Cannarsa, Urbani NoDEA 2022)**.** *Let A be a densely defined linear operator on X satisfying the above assumption, and $B \in \mathcal{L}(X)$. Assume that $\{A, B\}$ is $j-$null controllable in any time $T > 0$ for some $j \in \mathbb{N}^*$. Suppose that the control cost $N(\cdot)$ satisfies: $N(\tau) \leq e^{\nu/\tau} \quad \forall\ \tau \in (0, T_0] \quad (CCC)$, for some constants $\nu > 0$ and $T_0 > 0$. Then for any $T > 0$, there exists a constant $R_T > 0$*

such that for any $u_0 \in B(\varphi_j, R_T)$*, there exists a control* $p \in L^2(0,T)$ *such that the solution of* (1) *satisfies* $u(T; u_0, p) = \psi_j(T)$ *(with explicit estimates on the control cost).*

Theorem 2 (A.-B., Cannarsa, Urbani NoDEA 2022, with applications to the Fokker-Planck equation)**.** *Let* A *satisfy (Assumptions on* A*) and be such that the (Gap Condition holds). Let* $B \in \mathcal{L}(X)$ *be such that there exist* $b > 0$*,* $q > 0$ *such that the following non vanishing condition (NVC) (first assumption) and asymptotic behavior (second inequality) hold:*

$$(3) \qquad \langle B\varphi_j, \varphi_j \rangle \neq 0 \ (NVC) \ \& \ |\lambda_k - \lambda_j|^q |\langle B\varphi_j, \varphi_k \rangle| \geq b \quad \forall\, k \neq j \ (AB).$$

Then the pair $\{A, B\}$ *is* $j-$*null controllable in any time* $T > 0$ *and the control cost satisfies (Control Cost Condition) with explicit estimates.*

Thus, (1) is locally controllable to the jth eigensolution ψ_j in any time $T > 0$. Furthermore when A is an accretive operator, we prove additional two semi-global controllability results, showing that every $u_0 \in X \backslash \varphi_1^\perp$ is controllable to the evolution (i.e. the dynamical system without control) of its orthogonal projection along the ground state.

2. Unbounded control operator and applications

Let $(X, \langle \cdot, \cdot \rangle, ||\cdot||)$ be a separable Hilbert space. Let $A : D(A) \subset X \to X$ be a densely defined linear operator with the following properties:

$$\begin{array}{ll} (a) & A \text{ is self-adjoint}, \\ (b) & \langle Ax, x \rangle \geq 0, \ \forall\, x \in D(A), \quad (New\ Assumptions\ on\ A) \\ (c) & \exists\, \lambda > 0 : (\lambda I + A)^{-1} : X \to X \text{ is compact.} \end{array}$$

Under the above assumptions A is a closed operator and $D(A)$ is also a Hilbert space with respect to the scalar product $(x, y)_{D(A)} = \langle x, y \rangle + \langle Ax, Ay \rangle, \quad \forall\, x, y \in D(A)$. Note also that $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions on X which is also analytic and will be denoted by e^{-tA}. We also assume that $B : D(B) \subset X \to X$ be a linear unbounded operator such that $D(A^{1/2}) \hookrightarrow D(B)$ with continuous embedding.

Theorem 3 (A.-B., Cannarsa, Urbani Comptes Rendus Mathématique 2023)**.** *Let* A *and* $B : D(B) \subset X \to X$ *be a linear unbounded operator satisfying the above hypotheses. Let* $\{A, B\}$ *be* 1*-null controllable in any* $T > 0$ *with control cost* $N(\cdot)$ *such that there exist* $\nu, T_0 > 0$ *for which (CCC) holds. Then, for any* $T > 0$*, there exists a constant* $R_T > 0$ *such that, for any* $u_0 \in B_{R_T,1/2}(\varphi_1)$*, there exists a control* $p \in L^2(0,T)$ *for which* (1) *with initial data* $u_0 \in D(A^{1/2})$ *is locally controllable to the ground state solution in time* T*, that is,* $u(T; u_0, p) = \psi_1(T)$*.*

Remark 1. *We show that the above assumptions can be weakened in order to include the cases when* A *admits negative eigenvalues. We also prove two semi-global exact controllability results to the ground state, or to its orthogonal projection.*

3. An original methodology and algorithm

We developed a general method for producing infinite classes of potential functions that fulfill the non vanishing condition (NVC) holds (see the first condition in (3)), for the above results to hold. We also provide an algorithm to derive functions μ (the dipole moment for the Schrödinger equation for instance). The asymptotic lower estimates (AB) for an appropriate q can also be deduced thanks to our results. It also allows to derive new spaces X in which several of the above theorems hold.

Theorem 4 (A.-B., Urbani 2019-2020)**.** *Let $T > 0$ be given and $\mu \in (S)_{NVC}$ (as given in A.-B.-Urbani's algorithm). Then there exist $\delta > 0$ and a $\mathcal{C}^1$ map $\Gamma : \mathcal{R}_T \longrightarrow L^2(0,T)$ such that $\Gamma\Big(\psi_1(T)\Big) = 0$, and for all $u_f \in \mathcal{R}_T$, the solution of the Schrödinger equation with initial data $u_0 = \varphi_1 = \psi_1(0)$ and control $p = \Gamma(u_f)$ satisfies $u(T) = u_f$ where $\psi_1(t,x) = e^{-i\lambda_1 t}\varphi_1(x) \quad \forall\ t \geq 0, \forall\ x \in [0,1]$ and for $T > 0$ and $\delta > 0$, $\mathcal{R}_T = \{u_f \in \mathcal{S} \cap D(A), ||u_f - \psi_1(T)||_{H^2} < \delta\}$*

4. Some perspectives and open question

Whenever the moment method is used, it reduces the scope to $1D$ or radial results. This raises the question of proving $j-$null controllability, by other strategies for $2D$ and $3D$ controllability results such that the ones relying on (3). Extension to semi-linear parabolic dynamical systems, more complex models and to build efficient numerics are also of great interest. For the Fokker-Planck equation we cannot handle the case of perfectly reflecting boundary conditions for which $D(A)$ becomes a domain that both depends on the time and the control function p. It is an open question to treat such boundary conditions.

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Instabilities in machine learning and in PDEs

Giovanni S. Alberti

(joint work with Rima Alaifari, Tandri Gauksson)

Adversarial examples in classification. Deep neural networks (DNN) in classification have been shown to be susceptible to adversarial perturbations [8]: small changes in the inputs can lead to misclassification. In other words, given an image x that is correctly classified by the network, it is possible to construct an adversarial example x' that is visually indistinguishable from x but misclassified by the network. The standard measure of the distance between x and x' is given by a suitable norm of $x - x'$, e.g.

$$\|x - x'\|_\infty \leq \varepsilon.$$

Here, it is required that the maximum pixel perturbation in the adversarial example is smaller than or equal to ε. For this reason, x' is an adversarial (additive) *perturbation* of x.

It is natural to wonder about the best way to measure the distance between x and x'. While the above approach is easy to interpret and straightforward to implement, it is also clear that many small transformations between two images are not captured. For example, if x' is a small translation of x, their image content is unchanged, but $\|x - x'\|_\infty$ will not be small. This observation leads us to consider adversarial *deformations*. Given an image $x \in L^2([0,1]^2)$, we consider

$$x_\tau(p) = x(p + \tau(p)),$$

where $\tau\colon [0,1]^2 \to \mathbb{R}^2$ is a vector field determining the deformation. The distance between x and x_τ is measured by a suitable norm of τ, e.g. $\|\tau\|_{L^\infty([0,1]^2)}$. In [1], we design a simple iterative algorithm that constructs adversarial deformations, and show that neural networks that are commonly used in classification (for MNIST and ImageNet) are indeed vulnerable to adversarial deformations.

Adversarial examples in inverse problems. These observations indicate that DNN tend to be unstable, meaning that given two inputs that are very close, their outputs may be far apart. Another domain where stability plays a crucial role is inverse problems, where unknown physical quantities need to be reconstructed from indirect measurements. Since these measurements are typically noisy, it is important for any reconstruction method to be stable. In recent years, machine learning, and especially deep learning, has become a popular tool for solving inverse problems. Therefore, it is crucial to understand how stable these methods are. In [4], the authors demonstrate that several deep learning based methods developed for solving the inverse problem (undersampled Fourier measurements) in accelerated magnetic resonance imaging (MRI) are indeed vulnerable to adversarial perturbations. The reconstructions may contain hallucinations, namely, undesirable features that are completely created by the network and may not be easily identified. Another conclusion of [4] is that state-of-the-art reconstruction methods based on total variation (TV) regularization and with guarantees provided by the theory of compressed sensing are more stable and do not hallucinate.

A more quantitative analysis is provided in [5], where deep learning and TV regularization are compared by using the relative L^2 error of the reconstruction as a metric. The results show a similar vulnerability of the two methods to adversarial errors and to Gaussian errors. However, as discussed above, the L^2 norm does not fully capture the meaningful features of an image. For instance, given a (discretized) image x, one can construct two noisy perturbations:

$$x_1 = x + g, \qquad x_2 = x + a\chi_A,$$

where g is white Gaussian noise with $\|g\|_2 = \varepsilon$ and χ_A is the indicator function of a small region A, so that $\|a\chi_A\|_2 = \varepsilon$. As a consequence, we have

$$\|x - x_1\|_2 = \|x - x_2\|_2.$$

Thus, if we use the L^2 norm, these two perturbations are quantitatively identical. However, in terms of visual impact, if ε is small, x_1 will be visually indistinguishable from x (or the noise can easily be identified), while the feature A added in x_2 may be more problematic. A possible strategy to construct adversarial perturbations that are visually meaningful as well as quantitatively measurable is proposed in [2], where localized adversarial artifacts are constructed for the inverse problem of undersampled MRI. We show that TV regularization is more vulnerable than DNN-based methods. Furthermore, we show how the vulnerability to this type of attacks is inherently connected to the exact recovery guarantees given by compressed sensing theory for TV regularization.

Instabilities and adversarial examples in PDEs. The stability properties of DNN can also be analyzed through the lens of the theory of partial differential equations (PDEs). It was shown in [7] that when both the layers and the space variables are considered in the continuous limit, the action of a residual convolutional neural network (CNN) on an input f may be written as

$$f \longmapsto u(T),$$

where u is the solution of a (nonlinear) dynamic PDE of the form:

$$\partial_t u(t) = F_t(u(t)), \qquad u(0) = f,$$

where F_t is a, possibly nonlinear, differential operator. Whenever energy conservation is crucial, this *parabolic CNN* may be replaced by a *hyperbolic CNN*:

$$\partial_t^2 u(t) = F_t(u(t)), \qquad u(0) = f, \qquad \partial_t u(0) = 0.$$

In both cases, under suitable assumptions on the differential operators and on the nonlinearities included in F_t, it is possible to show that

$$\|u_1(T) - u_2(T)\|_2 \lesssim \|u_1(0) - u_2(0)\|_2.$$

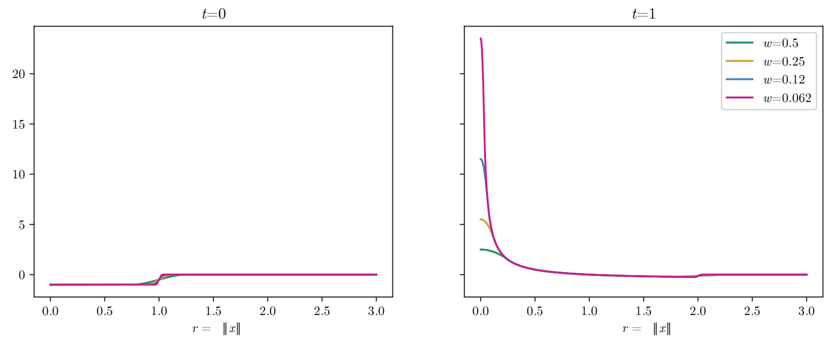

Figure 1. An example of a sequence as in (2). These two figures show the radial components of f_n and of $u_n(1,:)$, respectively. The initial states f_n, $n=1,2,3,4$, are continuously differentiable functions that transition from the value -1 to the value 0 in a window of width $w=2^{-n}$, centered at $r=1$.

Namely, these maps are Lipschitz stable.

These bounds are obtained by using standard energy estimates, and are specific to the L^2 norm. Because of the above discussions on the different possibile notions of perturbations, and in particular on the observations on the limited relevance of the L^2 norm in certain contexts, it is natural to wonder whether these stability estimates can be extended, for instance, to the L^∞ norm. The answer is negative, as a simple example shows. Consider the (linear) wave equation with constant coefficients in $\mathbb{R}^3$:

$$\partial_t^2 u - \Delta u = 0, \qquad u(0) = f, \qquad \partial_t u(0) = 0. \tag{1}$$

With a radial initial condition $f(x) = g(|x|)$, u is a spherical wave of the form

$$u(t,x) = \frac{1}{2|x|}\left(\varphi(|x|-t) + \varphi(|x|+t)\right), \qquad \varphi(r) = rg(r).$$

If we evaluate this expression at $x=0$, we obtain $u(t,0) = g(t) + t\cdot g'(t)$. Therefore, it is possible to find a sequence of adversarial examples f_n such that

$$\|f_n\|_\infty \le 1, \qquad \|u_n(1,\cdot)\|_\infty \to +\infty, \tag{2}$$

see Figure 1. In particular, the map $f \mapsto u(1,\cdot)$ is unstable with respect to the L^∞ norm.

From an abstract point of view, this instability can be analysed by using the framework of Fourier multipliers [6]. The map $f \mapsto u(1,\cdot)$ may be written as

$$B\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad f \mapsto \mathcal{F}^{-1}\left[\cos\left(2\pi\,|\cdot|\right)\mathcal{F}f\right],$$

and it is easy to see that this map is not bounded with respect to the L^∞ norm. In [3], we propose a method to regularize the operator B in order to obtain a family of approximations that are bounded as operators $L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. More

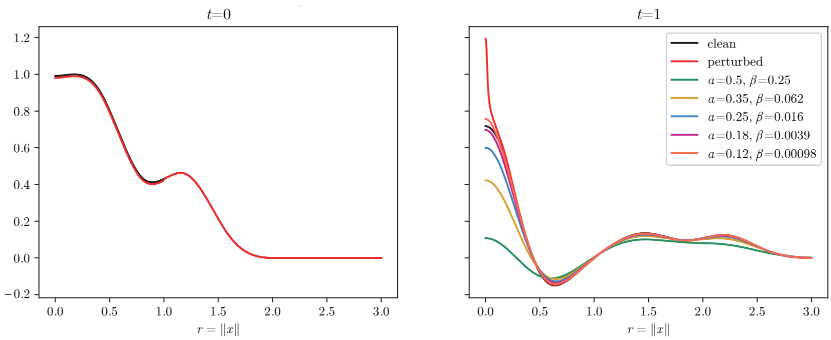

FIGURE 2. Radial component of spherical waves at times $t = 0$ and $t = 1$. Left: the clean initial state f and a perturbed initial state $f + r$, with $\|r\|_{L^\infty(\mathbb{R}^3)} = 0.01 \cdot \|f\|_{L^\infty(\mathbb{R}^3)}$. Right: the end states Bf, $B(f + r)$, and $B_{\alpha,\beta}(f + r)$.

precisely, let

$$B_{\alpha,\beta}(f) = \mathcal{F}^{-1}\left(\cos\left(2\pi\,|\cdot|\right) \cdot \kappa_\alpha \cdot \mathcal{F}\left(h_\beta f\right)\right),$$

with filters $\kappa_\alpha, h_\beta \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. The operator $B_{\alpha,\beta}\colon L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ is well-defined and bounded for every $p \in [2, +\infty]$. Moreover, if the family of filters k_α and h_β is suitably chosen, then $B_{\alpha,\beta} \to B$ in a suitable sense as $\alpha, \beta \to 0$.

An example of this behavior is shown in Figure 2. A radial input f is modified with a small (with respect to both the L^∞ and the L^2 norm) adversarial perturbation r. The quantity $B(f + r)$, namely, the solution to (1) with initial value $f + r$ calculated at time $t = 1$, presents a visible artifact in the origin. By using the regularized quantities $B_{\alpha,\beta}(f + r)$, it is possible to substantially reduce the artifact, while maintaining a good overall quality of the output. As is typical in regularization, this procedure requires a suitable choice of the parameters α and β. This regularization strategy turns out to be effective for this simple linear PDE, and it would be interesting to investigate whether similar ideas can be used also for more complicated nonlinear PDEs or for the corresponding neural networks.

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Source and subdomain control of scalar conservation laws

Boris Andreianov
(joint work with Shyam S. Ghoshal)

Given a trajectory $v : [s, T] \mapsto X$ for evolution equation of the form $\dot{u} + Au \ni g$ governed by an m-accretive densely defined operator on a Banach space X ([3]), given any initial state $u_0 \in X$, we provide avery simple construction of feedback control that permits to reach, at $t = T$, the final state $v(T)$ starting from u_0 at $t = 0$. The construction is reniniscent of Luenberger's observers (the nudging strategy) in Data Assimilation ([6]), but the exponential stabilisation to the target state at large times is replaced by exact control at final time T. In turn, this provides a distributed control in $L^1(0, T; X)$ where X is the state space. Therefore, if s a state v_T is attainable (with source) at some time τ, it is attainable - with source- from any initial/boundary states, at any time $T \geq \tau$.

The specific application we have in mind is to scalar conservation laws, possibly multi-dimensional and non-convex; moreover, replacing the abstract semigroup arguments by the standard PDE arguments, we can also handle e.g. the Cauchy-Dirichlet problem with time-dependent boundary conditions. The construction of the distributed control has a numerical counterpart, in the setting of Finite Volume approximation by a monotone scheme.

Backward constructions with source are proposed, showing that wide classes of data (including BV and many fractional Sobolev spaces) are attainable - with distributed source in L^1- at any time (cf. [5] for bounded source). In $1D$, a variant of backward front-tracking construction provides reconstructions under the form of a continuous juxtaposition of compression and rarefaction fans, for BV data.

Further, combining the idea of source control, now localized in a compact interval in space, with geometric observability-kind constructions in terms of backward characteristics ([2, 1]), we are also able to explore the framework of subdomain control (cf. [5]). This approach remains however restricted to a 1D, strictly convex scalar conservation law, in the Cauchy or the Cauchy-Dirichlet setting.

Extension to kinetic formulation systems of conservation laws makes sense ([4]).

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Non-admissibility of spiral-like strategies

Stefano Bianchini
(joint work with Martina Zizza)

1. Introduction

We study a dynamic blocking problem first proposed by Bressan in [3]. The problem is concerned with the model of wild fire spreading in a region of the plane $I\!R^2$ and the possibility to block it constructing some *barriers* in real time. If we denote by $R(t) \subset I\!R^2$ the region burned by the fire at time t, then we can describe it as the reachable set for a differential inclusion. More precisely, one considers the Cauchy Problem

$$\dot{x} \in F(x), \qquad x(0) \in R_0,$$

where the set $R_0 \subset I\!R^2$ represents the region burnt by the fire at the initial time $t = 0$, while the function F describes the speed of spreading of the fire. The set $R_0 \subset I\!R^2$ is assumed to be open, bounded, non-empty and connected with Lipschitz boundary, whereas the standard assumptions on $F : I\!R^2 \Rightarrow I\!R^2$, which is a Lipschitz-continuous multifunction, are:

(1) there exists $r > 0$ such that $B_r(0) \subset F(x) \quad \forall x \in I\!R^2$;
(2) $F(x)$ is compact and convex $\forall x \in I\!R^2$;
(3) $x \Rightarrow F(x)$ is Lipschitz-continuous in the Hausdorff topology.

If no barriers are present the *reachable set* for the differential inclusion is

$$R(t) = \Big\{ x(t), \quad x(\cdot) \text{ abs. cont.}, x(0) \in R_0, \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0,t] \Big\}.$$

When the fire starts spreading, a fireman can construct some barriers, modeled by a one-dimensional rectifiable set $\zeta \subset I\!R^2$, in order to block the fire. More in detail, we consider a continuous function $\psi : I\!R^2 \Rightarrow I\!R^+$ together with a positive constant $\psi_0 > 0$ such that $\psi \geq \psi_0$. If we denote by $\zeta(t) \subset I\!R^2$ the portion of the barrier constructed within the time $t \geq 0$, we say that ζ is an admissible barrier (or admissible strategy) if

(1) (H1) $\zeta(t_1) \subset \zeta(t_2), \forall t_1 \leq t_2$;
(2) (H2) $\int_{\zeta(t)} \psi d\mathcal{H}^1 \leq t, \quad \forall t \geq 0,$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Once we have an admissible strategy ζ, then we define the reachable set for ζ at time t the set

(1)
$$R^\zeta(t) = \Big\{x(t) : x \text{ abs. cont.}, \dot{x}(\tau) \in F(x(\tau)) \text{ a.e. } \tau \in [0,t], x(\tau) \notin \zeta(\tau)\ \forall \tau \in [0,t]\Big\}.$$

Definition. Let $t \Rightarrow \zeta(t)$ be an admissible strategy. We say that it is a *blocking* strategy if

$$R^\zeta_\infty \doteq \bigcup_{t\geq 0} R^\zeta(t)$$

is a bounded set.

We call *isotropic* the case in which the fire is assumed to propagate with unit speed in all directions, while the barrier is constructed at a constant speed $\sigma > 0$, namely

$$F \equiv \overline{B_1(0)}, \quad R_0 = B_1(0), \quad \psi \equiv \frac{1}{\sigma}, \tag{2}$$

where $\overline{B_1(0)}$ denotes the closure of the unit ball of the plane centered at the origin. We remark that in [5] there are comparison results between more general choices of the data R_0 and F and the isotropic problem for the study of the fire problem in a more general setting.

The existence of admissible blocking (or winning) strategies for the isotropic blocking problem is a very challenging open problem and it has been addressed mainly in [3],[5].[1] In particular, the following theorems hold:

Theorem. Assume that (2) hold. Then if $\sigma > 2$ there exists an admissible blocking strategy.

Theorem. Assume that (2) hold. Then if $\sigma \leq 1$ no admissible blocking strategy exists.

The two theorems are proved in [3] and they motivate the following Fire Conjecture [4]:

Conjecture. Let (2) hold. Then if $\sigma \leq 2$ no admissible blocking strategy exists.

In this talk we study spiral-like strategies: namely, admissible barriers that are constructed putting all the effort on a single branch. The study of spiraling strategies is of key importance in the complete solution of Bressan's Fire conjecture, indeed there is a strongly belief that these strategies are the best possible barriers that can be constructed when $\sigma \leq 2$.

We start giving the definition of spiral-like strategies:

[1] One can prove that the existence of blocking strategy does not depend on the starting set R_0 but only on the speed σ [2].

Definition. Let $Z = \zeta([0, S]) \subset I\!R^2$ be a strategy, where ζ is a parametrization by length. We say that it is a spiral-like strategy if it satisfies:

- $\zeta(0) = (1, 0)$ and $\zeta|_{[0,S)}$ is simple;
- $s \mapsto u \circ \zeta(s)$ is increasing.

Finally, we say that Z is an admissible spiral if it is a spiral-like strategy, the curve is locally convex, in the sense of the definition above and moreover it satisfies the following assumption

$$\text{(A1)} \qquad 0 \leq \angle(\mathbf{t}^+(0), \mathbf{e}_2) \leq \frac{\pi}{2},$$

where $\mathbf{e}_2$ is the vertical vector of the canonical base $\mathbf{t}(0)$ is the tangent vector of the spiral in the starting point (commonly $(1, 0)$) and $\angle$ denotes the angle between the two vectors. What really matters in the definition of spiral-like strategies is the requirement

$$s \Rightarrow u \circ \zeta(s) \qquad \text{is increasing,}$$

which corresponds to the fact that either a portion of the barrier lies on the level set $\{u =\}$ (so that the previous function is constant), or the fire can not burn two portions of the barrier simultaneously. We believe instead that the hypothesis of local convexity and the assumption (A1) are automatically satisfied by *optimal* spiral-like strategies, which is an open question.

In addition to the parametrization by arc-length, it is possible to parametrize any admissible spiral by $(r(\phi), \phi)$, where ϕ denotes the angle of rotation on the spiral, while $r(\phi)$ represents the length of the final segment of the fire ray reaching the point $(r(\phi), \phi)$.

The only results known on these barriers can be found in [5] and [8]. In the two papers it is proved independently and with different techniques the following

Theorem. *Let $\sigma > 2.6144..$ (critical speed). Then there exists a spiral-like strategy which confines the fire.*

This theorem inspires therefore the following

Conjecture. If $\sigma \leq 2.6144...$ then no spiral-like strategy is admissible.

A partial answer to this conjecture has been given in [8] where the authors use a geometric argument to prove that if $\sigma \leq \frac{1+\sqrt{5}}{2}$ then no spiral-like strategy is admissible.

We proved the following

Theorem. *No admissible spiral-like strategy confines the fire if $\sigma \leq 2.3$.*

The bound 2.3 is not sharp, since it is obtained by purely numerical computations. It could be an interesting question to investigate the numerical optimization of the parameter σ accordingly to the method we will propose for the solution of this problem. But unfortunately, even if the speed σ could be optimized, the critical case $\sigma = 2.6144..$ at the present time seems out of reach and very delicate. We remark that this theorem proves Bressan's Fire Conjecture in the case of spiral strategies.

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Optimal control strategies for moving sets

Alberto Bressan

We consider a family of control problems for a moving set. In applications, this set can describe the region infested by an invasive biological population, which can grow or shrink in time, depending on the control applied along the boundary. For example, a region infested by mosquitoes can be reduced in size by spraying pesticides along its boundary. Optimal strategies are sought, which minimize the contaminated area plus a cost for implementing the control.

In mathematical terms, given an initial bounded set $\Omega(0) = \Omega_0 \subset \mathbb{R}^2$, for $t \in [0,T]$ we seek a motion $t \mapsto \Omega(t)$ which minimizes a cost functional of the form

$$J \doteq \int_0^T \phi\big(\mathcal{E}(t)\big)\,dt + \int_0^T \text{meas}\big(\Omega(t)\big)\,dt + \kappa\,\text{meas}\big(\Omega(T)\big). \tag{1}$$

This accounts for a control cost, and a cost depending on the area of the contaminated region at various times.

Denoting by $\beta(t,x)$ the **inward-pointing velocity** of the boundary of the set $\Omega(t)$ at the point $x \in \partial\Omega(t)$, the **total control effort** at time t is measured by

$$\mathcal{E}(t) \doteq \int_{\partial\Omega(t)} E\big(\beta(t,x)\big)\,dx. \tag{2}$$

A natural choice for the effort function E in (2) is

$$E(\beta) = \max\{1+\beta, 0\}. \tag{3}$$

In other words, if no effort is made, the contaminated region expands with unit speed ($E = 0$ corresponds to $\beta = -1$). By applying a control along the boundary, this expansion can be reduced, or even reversed.

Possible choices for the function ϕ in (1) are:

$$\phi(s) \;=\; \frac{s^2}{2} \qquad\qquad \text{or} \qquad\qquad \phi(s) = \begin{cases} 0 & \text{if } s \le M, \\ +\infty & \text{if } s > M. \end{cases} \tag{4}$$

Notice that, if the second definition applies, then the first term in (1) has the meaning of a constraint: at every time $t \in [0,T]$ the total control effort must satisfy $\mathcal{E}(t) \le M$.

Optimization problems of the form (1) were first considered in [2], and formally derived from the control of a parabolic equation, by taking a sharp interface limit. Existence of optimal solutions was proved in [3]. Necessary conditions for optimality were also determined, in the form a Pontryagin maximum principle. A basic setting is the following:

(OP) Optimization Problem. *Given a bounded initial set $\Omega(0) = \Omega_0 \subset \mathbb{R}^2$, find a motion $t \mapsto \Omega(t) \subset \mathbb{R}^2$ that minimizes the cost*

$$\mathcal{J} \;=\; meas\big(\Omega(T)\big),$$

subject to $\beta(t,x) \ge -1$ and

$$\mathcal{E}(t) \;\doteq\; \int_{\partial\Omega(t)} \big(1 + \beta(t,x)\big)\, d\sigma \;\le\; M \qquad \textit{for every } t \in [0,T]. \tag{5}$$

Notice that in this case:

(i) With no control, the contaminated set $\Omega(t)$ expands with unit speed in all directions.

(ii) Implementing a control along the boundary, we can clear a region of area M per unit time.

In turn, this implies that the increase of the area of $\Omega(t)$ is determined by

$$\frac{d}{dt}\text{meas}\big(\Omega(t)\big) \;=\; \text{length}\big(\partial\Omega(t)\big) - M.$$

This suggest that, in order to reduce the area, it is always most convenient to reduce the perimeter as quickly as possible. A rigorous proof of this fact was recently given in [1].

Theorem 1. *In connection with the optimization problem* **(OP)***, assume that the initial set $\Omega(0) = \Omega_0 \subset \mathbb{R}^2$ is convex. Then, at each time $t \in [0,T]$, the optimal set $\Omega(t)$ is convex.*

The optimal control is active precisely along the portion of the boundary $\partial\Omega(t)$ where the curvature is maximum. This is a union of arcs of circumferences, all with the same radius $r(t)$.

At the present time, several related questions remain open. In particular:

(Q1) What is the regularity of the optimal sets $\Omega(t)$? We recall that, to derive the necessary conditions in [3], one needs to construct trajectories $t \mapsto x(t,\xi) \in \partial\Omega(t)$ orthogonal to the boundary. Can these trajectories be always well defined?

(Q2) If the initial set Ω_0 is not convex, what can one say about the optimal strategy? Under what conditions is it true that the sets $\Omega(t)$ are connected, for all $t \in [0, T]$?

(Q3) More generally, all the above problems can be formulated on $\mathbb{R}^n$, or even on an n-dimensional Riemann manifold. How much of the theory remains valid in a multidimensional setting? Does the convexity result stated in Theorem 1 remain true for optimal set motion in $\mathbb{R}^3$?

The case with geographical constraints, where the pest population needs to be eradicated from an island (a bounded open set V in the plane), is also of interest. As before it is assumed that, within V, the contaminated region expands with unit speed in all directions, while the control can "clean up" an area M per unit time. In this setting, two main problems arise.

Eradication problem. *Find an admissible strategy* $t \mapsto \Omega(t) \subseteq V$ *that completely eradicates the contamination in finite time. This means:* $\Omega(0) = V$, $\Omega(T) = \emptyset$, *and the following constraint is satisfied:*

$$\mathcal{E}(t) \doteq \int_{\partial\Omega(t)\cap V} \big(1 + \beta(t,x)\big)\, d\sigma \le M \qquad \text{for every } t \in [0,T].$$

The existence or non-existence of such a strategy can be determined by comparing the speed M with two geometric invariants of the set V. In the positive case, one can further consider:

Minimum time problem. *Among all strategies that eradicate the contamination, find one that minimizes the time* T.

The analysis of optimality conditions indicates that, in an optimal strategy, at each time $t > 0$ the interface between free and contaminated zone should be the union of

(i) arcs of circumferences, all with the same radius $r(t)$, where the control is active, together with
(ii) additional arcs where the control is not active, and the contamination expands with unit speed.

A direction of current research aims at understanding the structure of optimal strategies in two main cases:

(i) V is a polygon.
(ii) V is a generic convex set, with smooth boundary.

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A Front Tracking Approach to an Euler-type Flocking Model

Cleopatra Christoforou
(joint work with Debora Amadori)

The study of hydrodynamic models that emerged in the area of self-organization has received alot of attention in the recent years and many new challenges in partial differential equations have arisen that yield interesting questions in the mathematical community.

We consider the system

$$(1)\qquad \begin{cases} \partial_t \rho + \partial_x(\rho \mathtt{v}) = 0, \\ \partial_t(\rho \mathtt{v}) + \partial_x\left(\rho \mathtt{v}^2 + p(\rho)\right) = K \displaystyle\int_{\mathbb{R}} \rho(x,t)\rho(x',t)\left(\mathtt{v}(x',t) - \mathtt{v}(x,t)\right)\,dx' \end{cases}$$

with $(x,t) \in \mathbb{R} \times [0,+\infty)$. Here $\rho \geq 0$ stands for the density, $\mathtt{v}$ for the velocity, p for the pressure, given by

$$(2)\qquad p(\rho) = \alpha^2 \rho\,, \qquad \alpha > 0\,,$$

and $K > 0$ is a given constant. Having set $\mathtt{m} := \rho \mathtt{v}$ as the momentum variable, we consider the Cauchy problem (1) with the initial condition

$$(3)\qquad (\rho, \mathtt{m})(x,0) = (\rho_0(x), \mathtt{m}_0(x)) \qquad x \in \mathbb{R}\,,$$

and our aim is to formulate a problem to (1)–(3) with conditions appropriate for the models of self-organized systems and then seek weak solutions that admit time-asymptotic flocking.

The pioneering work of Cucker and Smale [5] led a major part of the mathematical community to conduct research intensively on this topic. Many mathematical models have arised and most work so far is on the behavior of the particle models, the kinetic equation and the hydrodynamic formulation. However, very little is done in this area on *weak solutions* and the scope of our work is to contribute in this direction of weak solutions in the set up of the Euler-type flocking system that can be derived using a hydrodynamic formulation. We refer to the reviews [4, 7, 8] and the references therein. We stand out the work of Karper, Mellet and Trivisa in [6], in which it is shown the convergence of weak solutions to the kinetic equation Cucker-Smale flocking model to strong solutions of an Euler-type flocking system of the form (1) with pressure of the form (2). Thus, we study this system with pressure as derived in [6] although pressureless systems have received most attention so far. Also, we are interested in solutions with discontinuities and most results so far deal with regular solutions.

Our analysis goes over the following steps: we first set up the problem with initial data appropriate for flocking models, that is the initial data having finite total mass confined in a bounded interval and initial density uniformly positive therein. Next, we introduce an appropriate notion of entropy weak solutions with concentration centred along two free boundaries emanating from the endpoints of the initial support. It is shown that under this notion, solutions conserve mass and momentum and the system reduces to a local one for an all-to-all interaction kernel.

The construction of the weak solution is obtained by transforming the problem into Lagrangian variables (cf. [9]) and employing the front tracking algorithm (cf. [3]). Showing that the linear functional is non-increasing, it allows us to pass to the limit and obtain an entropy weak solutions with concentration. Additional analysis at the level of approximate solution reveals a geometric wave decay and this yields an exponential decay of the total variation in time. Having this, we conclude unconditional time-asymptotic flocking, i.e. the support of the solutions remains bounded for all times and velocity alignment occurs without any further restrictions on the data.

There are many open problems on this topic arise that would be very interesting to be studied and contribute in the better understanding of flocking phenomena. It would be important to extend our result to the non-constant case of interaction kernel K that would include especially the singular kernels. Another direction would be to study control-type problems on this set-up, i.e. given a final state at time $t = T$, to determine the initial data for which the given profile is reached at time T. An alternative interesting direction is the study of the system with pressure of the form ρ^γ for $\gamma > 1$ and its relation with the solutions to (1), we have taking the limit $\gamma \to 1$.

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Optimal Control of Propagaton Fronts

Maria Teresa Chiri
(joint work with Alberto Bressan, Najmeh Salehi)

The control of parabolic equations is by now a classical subject [8, 9, 10, 12]. More specifically, several studies have been devoted to the optimal harvesting of spatially distributed populations [5, 6, 11]. Our recent interest in the control of reaction-diffusion equations is primarily motivated by models of pest eradication [1, 2, 7, 13]. The controlled spreading of a population, in a simplest form, can be described by a semilinear parabolic equation

$$u_t \;=\; f(u) + \Delta u - \alpha u. \tag{1}$$

Here $u = u(t, x)$ denotes the population density at time t, at a location $x \in \mathbb{R}^2$. The function f describes the reproduction rate, while $\alpha = \alpha(t, x)$ is a distributed control representing the quantity of pesticides sprayed at time t at location x, and $\alpha\, u$ describes the amount of population which is eliminated by this strategy. Given an increasing, convex cost function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, we can consider the following optimization problem

(OP1) Optimization Problem for a reaction-diffusion equation. *Given an initial profile $u(0, x) = u_0(x)$ and a time interval $[0, T]$, determine a control $\alpha = \alpha(t, x) \geq 0$ so that, calling $u(t, x)$ the corresponding solution to (1), the total cost*

$$\mathcal{J} \;\doteq\; \int_0^T \phi\left(\int \alpha(t,x)\,dx\right) dt + \kappa \int_0^T \int u(t,x)\,dx\,dt \tag{2}$$

is minimized.

We think of $\int \alpha(t, x)dx$ as the *total control effort* at time t. Standard results yield the existence of an optimal strategy, and necessary conditions for optimality. However, very rarely one can find explicit formulas for the optimal solution. Assuming that $f(0) = f(1) = 0$, and observing that solutions to (1) often develop stable traveling fronts, the parabolic problem **(OP1)** can be approximated with an optimal control problem for the moving set $\Omega(t) \;=\; \big\{x \in \mathbb{R}^2\,;\; u(t, x) \approx 1\big\}$, representing the contaminated region. In this case, the control function is the speed $\beta = \beta(t, x)$ at which the boundary $\partial\Omega(t)$ is pushed in the inward normal direction. This new problem has the form

(OP2) Optimization Problem for a moving set. *Let an initial set $\Omega_0 \subset \mathbb{R}^2$ and cost functions $E : \mathbb{R} \mapsto \mathbb{R}_+$, $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be given. Find a set-valued function $t \mapsto \Omega(t)$, with $\Omega(0) = \Omega_0$, which minimizes*

$$J \;=\; \int_0^T \phi\left(\int_{\partial\Omega(t))} E\big(\beta(t,x)\,d\sigma\right) dt + \kappa \int_0^T meas\big(\Omega(t)\big)\,dt. \tag{3}$$

In our work [3, 4] we investigated the underlying connection between the two above optimization problems. In particular, the effort $E(\beta)$, needed to achieve the inward normal speed β, can be uniquely determined by solving an optimal control problem for traveling wave profiles of (1). The cost for moving the interface at different speeds in the normal direction is determined through the analysis of traveling wave profiles for the PDE model, and justified via a sharp interface limit. More generally, the same approach remains valid for systems of (possibly degenerate) parabolic equations with spatial variable in $\mathbb{R}^n$.

A rigorous derivation of **(OP2)** would require a study of the Γ-limit of the functionals

$$\mathcal{F}_\varepsilon(u) \doteq \int_0^T \int \frac{[\varepsilon\Delta u + \varepsilon^{-1} f(u) - u_t]_+}{u}\, dx\, dt \tag{2}$$

as $\varepsilon \to 0$. Here $[s]_+ = \max\{s, 0\}$. As first step in this direction, we have proved that the cost J at (3) can be achieved as the limit of the cost (2), for a family of solutions to the rescaled parabolic equations

$$u_t^\varepsilon = \frac{1}{\varepsilon} f(u^\varepsilon) + \varepsilon\, \Delta u^\varepsilon - u^\varepsilon \alpha^\varepsilon, \qquad t \in [0, T],\ x \in \mathbb{R}^2.$$

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Renewal Equations: Models, Analysis and Control Problems

Rinaldo M. Colombo
(joint work with Mauro Garavello, Francesca Marcellini, Elena Rossi)

According to [13, § 3.1] by *renewal equation* the following initial boundary value problem is meant:

$$\begin{cases} \partial_t \rho + \partial_x \rho = 0 & (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ \rho(t,0) = \int_0^{+\infty} b(\xi)\, \rho(t,\xi)\, d\xi & t \in \mathbb{R}_+ \\ \rho(0,x) = \rho_o(x) & x \in \mathbb{R}_+ \,. \end{cases}$$

However, recently, this term has been used more and more also to refer to rather general problems, such as

$$(1) \qquad \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho\, V) = S & (t,x) \in \mathbb{R}_+ \times \Omega \\ \rho(t,\xi) = B(t,\xi) \quad \text{(if there is a boundary)} & (t,\xi) \in \mathbb{R}_+ \times \partial\Omega \\ \rho(0,x) = \rho_o(x) & x \in \Omega \end{cases}$$

that typically have in common the presence of non local terms and a motivation coming from some sort of biological model. In (1), the velocity V, the source S and the boundary term B may well depend – locally or not locally – on the unknown function ρ which, in turn, may well be both a scalar or a vector.

An attempt to obtain a rather general well posedness result for a renewal equation in the sense of (1) is presented in [10]. There, the precise form of the considered problem is

$$(2) \quad \partial_t u^h + \nabla_x \left(v^h(t,x)\, u^h\right) = p^h\left(t,x,u(t)\right) u^h + q^h\left(t,x,u,u(t)\right) \qquad h \in \{1,\ldots,k\}\,.$$

Here $t \in \mathbb{R}_+$ is time and the *"space"* variable x varies in $\mathbb{R}_+^m \times \mathbb{R}^n$. This choice allows to encompass in (2) also situations where where part of the independent variables are bounded below, while the remaining part varies in $\mathbb{R}^n$. The former variables may thus represent age or time since vaccination, for instance, see [4, 9], while the latter variables are typically space coordinates, for instance. Note also that in (2), the dependence on $u(t)$ stands for a dependence which is *non local* in the x variable. In the right hand side in (2) the choices of p^h and q^h are flexible, so that the provided estimates can be optimized for the specific model considered. Refer to [10] for specific models that fit into (2). Other well posedness results in similar settings are provided in [6, 7] and in [12], where the problem is set in the framework of evolution equations in metric spaces.

Renewal equations appear also in mixed systems devoted to some sort of predator prey dynamics, such as

$$(3) \qquad \begin{cases} \partial_t u + \nabla \cdot (u\, v(u)) = f(t,x,w)\, u + a \\ \partial_t w - \mu\, \Delta w = g(t,x,u,w)\, w + b \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \Omega$$

where u *"hunts"* w and the hunting term v is of the form

$$(4) \qquad v(u) = V\left(\nabla u * \eta\right)\,,$$

η being a smooth approximation of Dirac delta and V a Lipschitz continuous function. The case of Lotka–Volterra interaction is recovered by

$$f(t,x,w) = \alpha\, w - \beta \qquad \text{and} \qquad g(t,x,u,w) = \gamma - \delta\, u\,,$$

but further terms, for instance related to some sort of *capacity* are not excluded.

System (3)–(4) can be studied in both cases $\Omega = \mathbb{R}^n$, see [8], and Ω bounded, see [11] – in the latter case suitable boundary conditions need to be supplemented. The term $a = a(t,x)$, respectively $b = b(t,x)$, is a control parameter describing the amount of u, respectively w, that is deployed per unit time at position x and time t. Indeed, system (3)–(4) applies to cases, for instance, where the predator u is a parasitoid used against the propagation of the parasite w, see [5, 8] or also when the chemical substance w diffuses attracting and killing the pest u.

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Generalised characteristics of Hamilton-Jacobi equations, propagation of singularities, and long-time behaviour

Piermarco Cannarsa

A generalised characteristic (GC) is a solution of certain differential inclusions that play a crucial role for propagation of singularities of solutions to Hamilton-Jacobi equations $H(x, u(x), Du(x)) = 0$. GC's were introduced in [4], in the context of hyperbolic conservation laws and then adapted to Hamilton-Jacobi equations in [1] and [3]. In this talk, we will discuss several topics related to GC's including restricted classes of characteristics introduced in [5], uniqueness issues, continuation properties in connection with propagation of singularities [2].

We will then focus on classical mechanical systems on the torus with the Hamiltonian

$$H(x,p) = \frac{1}{2}|p|^2 + V(x), \qquad x \in \mathbb{T}^d, p \in \mathbb{R}^d$$

and consider the stationary Hamilton-Jacobi equation

$$H(x, Du(x)) = \frac{1}{2}|Du(x)|^2 + V(x) = \alpha[0], \qquad x \in \mathbb{T}^d, \tag{1}$$

where $\alpha(0)$ is the Mañé's critical value. For any semiconcave solution (or viscosity solution) u of (1), the set of the points of non-differentiability of u, $\mathrm{Sing}(u)$, is called the singular set of u. The singularities of a semiconcave solution u of (1) propagate along the generalized gradient flow defined by

$$\begin{cases} \dot{\mathbf{x}}(t,x) \in D^+u(\mathbf{x}(t,x)), & t \geq 0 \text{ a.e.} \\ \mathbf{x}(0,x) = x. \end{cases} \tag{2}$$

Denoting by $\mathbf{x}_u(t,x)$ the associated semi-flow for a semiconcave solution u of (2), for any $x \in \mathbb{T}^d$ and any $T > 0$ we introduce the (individual) invariant occupational measure for $\mathbf{x}_u(\cdot, x)$ as the Borel probability measure μ_x^T defined by

$$\int_{\mathbb{T}^d} f(y)\ d\mu_x^T(y) = \frac{1}{T}\int_0^T f\big(\mathbf{x}_u(t,x)\big)\ dt \quad \forall f \in C(\mathbb{T}^d).$$

Then, we call any weak limit of $\mu_x^{T_n}$, as $T_n \to \infty$, a limiting occupational measure of $\mathbf{x}_u(\cdot, x)$. We denote by $\mathcal{W}_u(x)$ be the family of all limiting occupational measures of $\mathbf{x}_u(\cdot, x)$. As we shall see, there are interesting connections among the critical set of u, the set of limiting occupational measures $\mathcal{W}_u(x)$ of $\mathbf{x}_u(\cdot, x)$, and $\mathrm{Sing}(u)$. We first show that $\mathcal{W}_u(x) \neq \emptyset$ and, by the Krylov-Bogoliubov argument, that each measure in $\mathcal{W}_u(x)$ is invariant under $\mathbf{x}_u(\cdot, x)$. Then we show that the critical set of u is an attractor for the semi-flow $\mathbf{x}_u(t,x)$ in the sense that, for any $\varepsilon > 0$, the probability for $\mathbf{x}_u(t,x)$ to be ε-close to the critical set of u, with t picked at random in $[0,T]$, tends to 1 as $T \to \infty$.

Given (a nonzero cohomology class) $c \in \mathbb{R}^d$, to extend the above results to the cell problem

$$\frac{1}{2}|Du(x) + c|^2 + V(x) = \alpha[c], \qquad x \in \mathbb{T}^d, \tag{3}$$

is a challenging open question that will definitely require new ides. Even more so, it would be extremely interesting to adapt the current approach to a general Hamilton-Jacobi equation

$$H(x, Du(x) + c) = \alpha[c], \qquad x \in \mathbb{T}^d \tag{4}$$

with H of Tonelli type.

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Some inverse problems for shock wave control

Gui-Qiang G. Chen

Controlling shock waves is crucial in applications in various fields of science and engineering, including aerodynamics, aerospace engineering, explosion mitigation, and shock tube experiments. In this talk, we present some inverse problems for controlling multi-dimensional steady shock waves and discuss some recent progress in controlling both the leading shock waves generated by wedges/conical bodies and associated fluid flows by designing the boundary geometries of the wedges/conical bodies with desired pressure distributions and/or leading shock locations through the inverse problems. Some further perspectives and open problems in this direction will also be addressed.

Hysteresis and string stability in traffic flows

Andrea Corli
(joint work with Haitao Fan)

In this talk we discuss some new macroscopic models of traffic flow [2, 4], whose aim is to model stop-and-go waves and related phenomena by means of a hysteretic term. The approach is inspired by the modeling of fluid flows in porous media [1], where hysteresis plays an important role. We also comment on the so-called string stability of the model, a notion of stability that is widely used in microscopic models but that is here extended to macroscopic models [3].

Stop-and-go waves are observed in real traffic flows but cannot be produced by the classical Lighthill–Whitham–Richards (LWR) model. To capture stop-and-go waves, we add hysteresis to the LWR model; we call HLWR such a model [2].

The model HLWR consists of two equations for the unknown functions ρ, the vehicle density, and h, the hysteresis variable. It is hyperbolic but it is not in conservation form, because there is no reason for the hysteresis to be conserved; moreover, it involves functions that are possibly discontinuous, and solutions are to be meant in the sense of [5]. For the model under consideration, we find all possible "viscous" waves as well as necessary and sufficient conditions for their existence. In particular, deceleration and acceleration shocks appear and stop-and-go waves are produced by pairs of deceleration and acceleration shocks completing a hysteresis cycle. We solve the Riemann problem for every Riemann data and show that, where hysteresis loops exist, a deviation in speed of a few vehicles in a uniform car platoon can generate stop-and-go waves. This analysis could be possibly useful for the control of traffic flows. We also discuss how the previous approach can be extended to the Aw-Rascle-Zhang (ARZ) model [4].

In microscopic models, string stability or instability is concerned with the propagation of oscillations in a car platoon caused by the leading vehicle. The issue is whether and when such oscillations are amplified or damped; in the former case, traffic jams occur. We propose a suitable notion of string stability for continuum models and show that the LWR and AWR models model are string stable for wide classes of perturbations. In the case of the previous hysteretic model, we show that string instability can occur for large perturbations, while, under small perturbations, examples as well as approximate solution analysis suggest that the hysteretic traffic flow modeled for instance by the HLWR model is string stable.

In this framework there are many open problems. First of all, the Riemann solvers are not unique, as it was already the case in [1]; is it possible to select a "rational" Riemann solver? Second, the analysis of the initial-value problem for general data with bounded variation is missing; we conjecture that such a solution exists, at least for small initial data. Third, it would be interesting to validate the model by analyzing real traffic flows. Indeed, the evidence of hysteresis loops in traffic flows is known in the applied literature since several decades, but the modeling of hysteresis as a variable of the models has never been checked with such real flows.

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Inverse design for the chromatography system

Carlotta Donadello
(joint work with Giuseppe M. Coclite, Nicola De Nitti and Florian Peru)

We present some results on the inverse design problem for the $n \times n$ chromatography system. The particular structure of the system, which can be written as a triangular system consisting of one autonomous strictly concave conservation law and $n-1$ linear continuity equations, allows to combine the backward reconstruction method introduced in [2] with recent results obtained by Colombo and Perrollaz, [3], and by Liard and Zuazua, [5, 4], on the inverse design problem for a strictly convex scalar conservation law. More precisely, in the case of two components, the original system

$$\begin{cases} \partial_t u_1 + \partial_x \left(\dfrac{u_1}{1+u_1+u_2} \right) = 0, & t>0,\ x \in \mathbb{R}, \\ \partial_t u_2 + \partial_x \left(\dfrac{u_2}{1+u_1+u_2} \right) = 0, & t>0,\ x \in \mathbb{R}, \end{cases}$$

rewrites as

$$\begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v} \right) = 0, & t>0,\ x \in \mathbb{R}, \\ \partial_t w + \partial_x \left(\dfrac{w}{1+v} \right) = 0, & t>0,\ x \in \mathbb{R}, \end{cases}$$

thanks to the change of variables

$$v := u_1 + u_2, \qquad w := u_1 - u_2.$$

For any $v_0 \in L^\infty(\mathbb{R})$, the Cauchy problem

$$\begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v} \right) = 0, & t>0,\ x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases}$$

admits a unique entropy solution in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$. From it we can define the vector field $\left(A(t,x) = v(t,x), B(t,x) = \dfrac{v(t,x)}{1+v(t,x)} \right)$, satisfying the hypothesis which, in [6], are necessary to prove the existence of a unique weak renormalized solution z of

$$\begin{cases} \partial_t (vz) + \partial_x \left(\dfrac{vz}{1+v} \right) = 0, & t>0,\ x \in \mathbb{R}, \\ z(0,x) = z_0(x), & x \in \mathbb{R}, \end{cases}$$

for any $z_0 \in L^\infty(\mathbb{R})$. In particular, z is time-reversible, in the sense that if $A(T,x)z(T,x) = A(T,x)z_T(x)$, then $t \mapsto z(T-t)$ is a generalized solution of

$$\begin{cases} \partial_t (A\rho) - \partial_x (B\rho) = 0, & t>0,\ x \in \mathbb{R}, \\ A(0,x)\rho(0,x) = A(0,x)z_T(x), & x \in \mathbb{R}. \end{cases}$$

Using this approach to implement a backward reconstruction, it was proven in [2] that the physically relevant attainable profiles for the chromatography system in the (u_1, u_2) variables are

$$\mathsf{A}_T(\mathbb{R}) = \Big\{(v_T, w_T) \,:\, v \in \mathcal{A}_T\left(\mathbb{R}, v \mapsto \frac{v}{1+v}\right) \text{ and there exists } z \in L^\infty(\mathbb{R}; [-1,1]) \text{ such that } w_T = z v_T\Big\},$$

where the set $\mathcal{A}_T$ contains the attainable profiles at time T for the first equation, as described in [1]. For a given positive time T and an attainable profile $V_T = (v_T = u_1^T + u_2^T, w_T = u_1^T - u_2^T) \in \mathsf{A}_T(\mathbb{R})$, the set of initial conditions leading to V_T can be easily characterized, and its topological properties inferred, thanks to a 1–to–1 correspondence with the set $I(v_T)$ of inverse designs for the scalar conservation law with strictly concave flux, [3, 5].

For a target profile V which is not attainable in time T, we recover the initial condition which would steer the system as close as possible to V in the L^2 norm thank to a minimization procedure analogous to the one in [4].

Building on the numerical scheme in [2] and results in [4], we implemented a finite volume numerical scheme which, for any attainable profile U_T and positive integer r, provides an initial condition leading to U_T and which first component (the initial condition for the nonlinear conservation law) suffers of exactly r discontinuities.

The final part of the presentation explains why these results are limited to a system with a very specific structure and cannot easily be generalized to the class of triangular systems considered in [2].

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Nonlocal macroscopic models of multi-population pedestrian flows for walking facilities optimization

Paola Goatin

(joint work with R. Bürger, D. Inzunza, E. Rossi, L. M. Villada)

We consider a class of nonlocal crowd dynamics models for N populations, $N \geq 1$, with different destinations trying to avoid each other in a confined walking domain $\Omega \subset \mathbb{R}^2$ and described by their densities $\boldsymbol{\rho} = (\rho^1, \ldots, \rho^N)^T$. This can be formalized in the following initial-boundary value problem:

$$(1) \quad \begin{cases} \partial_t \rho^k + \mathrm{div}_{\mathbf{x}} \left[f_k\left(\rho^k\right) \boldsymbol{\nu}^k\left(t, \mathbf{x}, \mathcal{J}^k[\boldsymbol{\rho}]\right)\right] = 0, & \mathbf{x} \in \Omega, t \geq 0,\ k = 1, \ldots, N, \\ \boldsymbol{\rho}(0, \mathbf{x}) = \boldsymbol{\rho}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \boldsymbol{\rho}(t, \mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases}$$

where $\boldsymbol{\nu}^k = (\nu_1^k, \nu_2^k)$ is the velocity vector field of the k-th population, $\mathcal{J}^k$ is a nonlocal operator, i.e. $\mathcal{J}^k[\boldsymbol{\rho}] = \left(\mathcal{J}^k[\boldsymbol{\rho}(t)]\right)(\mathbf{x})$, and $\boldsymbol{\rho}_0$ is a given initial datum. Usually, the vector fields $\boldsymbol{\nu}^k$ consist of a fixed smooth vector field of preferred directions (e.g. given by the regularized solution of an eikonal equation) together with nonlocal correction terms depending on the current density distribution.

We assume that $\Omega^c = \mathbb{R}^2 \setminus \Omega$ is a compact set consisting of a finite number $M \in \mathbb{N}$ of connected components $\Omega^c = \Omega_1^c \cup \ldots \cup \Omega_M^c$. To account for the presence of these obstacles, in [3] we proposed to evaluate the nonlocal operators on the extended density $\boldsymbol{\rho}_\Omega : \mathbb{R}^2 \to \mathbb{R}_+^N$ including the presence of obstacles:

$$\rho_\Omega^k(t, \mathbf{x}) := \begin{cases} \rho^k(t, \mathbf{x}) & \text{if } \mathbf{x} \in \Omega, \\ R_\ell & \text{if } \mathbf{x} \in \Omega_\ell^c, \end{cases}$$

with $R_\ell \geq R > 0$, $\ell = 1, \ldots, M$, big enough so that $\boldsymbol{\nu}^k\left(t, \mathbf{x}, \mathcal{J}^k[\boldsymbol{\rho}_\Omega]\right) \cdot \mathbf{n}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \partial\Omega$, $t \geq 0$, $\mathbf{n}$ being the outward normal to Ω. In this way, (1) can be rewritten as

$$(2) \quad \begin{cases} \partial_t \rho^k + \mathrm{div}_{\mathbf{x}} \left[f_k\left(\rho^k\right) \boldsymbol{\nu}^k\left(t, \mathbf{x}, \mathcal{J}^k[\boldsymbol{\rho}_\Omega]\right)\right] = 0, & \mathbf{x} \in \mathbb{R}^2, t \geq 0,\ k = 1, \ldots, N, \\ \boldsymbol{\rho}(0, \mathbf{x}) = \boldsymbol{\rho}_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2. \end{cases}$$

Under suitable regularity assumptions, well-posedness results for weak entropy solutions of (2) can be proved relying on [1, 2], see [3, 5].

The trick of incorporating the obstacles in the nonlocal operator allows to avoid including them in the vector field of preferred directions. In particular, we can address shape optimization problems aiming at finding the optimal position of the obstacles to minimize the total travel time, rewriting them as standard PDE-constrained optimization [4]. In addition, to accelerate the numerical optimization procedure, we propose to address the computational bottleneck represented by the convolution products by a Finite Difference scheme that couples high-order WENO approximations for spatial discretization, a multi-step TVD method for temporal discretization, and a high-order numerical derivative formula to approximate the derivatives of nonlocal terms, and in this way avoid excessive calculations.

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Boundary feedback control problems for systems of balance laws

Simone Göttlich
(joint work with Sonja Groffmann)

The main objective of this work is to investigate the exponential and Input-to-state (ISS) stability of solutions to linear hyperbolic balance laws that are paired with boundary feedback conditions. In particular, we consider the following $k \times k$ system of linear hyperbolic balance laws

$$\partial_t W(x,t) + \Lambda(x)\partial_x W(x,t) + \Pi(x)W(x,t) = 0, \qquad (x,t) \in [0,l] \times [0,+\infty) \tag{1}$$

for the state vector $W := W(x,t) : [0,l] \times [0,+\infty) \to \mathbb{R}^k$ over some space interval of length l and an open time horizon. The variable coefficients are chosen such that $\Pi(x)$ is a non-zero matrix in $\mathbb{R}^{k\times k}$ and $\Lambda(x)$ is a diagonal matrix with non-zero eigenvalues in $\mathbb{R}^{k\times k}$.

For an analysis of the hyperbolic system, characteristic variables are applied and $\Lambda(x)$ is split into a positive and a negative part according to the sign of its eigenvalues. Therefore, $\Lambda(x) = \mathrm{diag}\{\Lambda^+(x), -\Lambda^-(x)\}$, where $\Lambda^+(x) = \mathrm{diag}\{\lambda_i(x) > 0 : i = 1, ..., m\}$ and $-\Lambda^-(x) = \mathrm{diag}\{\lambda_i(x) < 0 : i = m+1, ..., k\}$ for some $m \in \mathbb{N}$ with $m \leq k$. Analogously, the state vector is split into the parts that correspond to the positive and negative eigenvalues of $\Lambda(x)$, respectively. Hence, $W = [W^+, W^-]^T$ with $W^+ \in \mathbb{R}^m$ and $W^- \in \mathbb{R}^{k-m}$.
The hyperbolic system of balance laws will be paired with an initial condition

$$W(x,0) = W_0(x), \qquad x \in (0,l), \tag{2}$$

for a function $W_0 : (0,l) \to \mathbb{R}^k$, as well as boundary conditions consisting of a disturbed linear feedback law

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + Mb(t), \qquad t \in [0,+\infty), \tag{3}$$

where $K, M \in \mathbb{R}^{k\times k}$ are constant matrices and $b : [0,+\infty) \to \mathbb{R}^k$ is a vector-valued function describing the disturbances. This model has been also studied in [4].

In case of exponential stability an undisturbed feedback boundary condition is required. Therefore, the following special case of (3) with $Mb(t) \equiv 0$ for all $t \in [0, +\infty)$ will sometimes be considered as well:

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}, \qquad t \in [0, +\infty). \tag{4}$$

Both of the above stated boundary conditions are called feedback laws, as the inflow into the system $W^+(0,t)$ and $W^-(l,t)$ at the spacial boundaries $x = 0$ and $x = l$ is a function of the outflow out of the system at the boundaries given by $W^+(l,t)$ and $W^-(0,t)$. The information about the outflow is returned via feedback to influence the inflow. Similar types of feedback boundary conditions are often employed for boundary control problems or boundary stabilization as presented for example in [2, 3].

The problem is now completed by the following assumptions which shall hold for all $x \in [0, l]$ and $t \in [0, +\infty)$.

- **A1** Λ is a real diagonal matrix of class $C^1([0, l])$, i.e., $\Lambda(x)$ is a function that is once continuously differentiable.
- **A2** Π is a real matrix of class $C^0([0, l])$, i.e., $\Pi(x)$ is a continuous function.
- **A3** b is a vector of boundary disturbances of class $C^0([0, +\infty))$, i.e., $b(t)$ is a continuous function.

Based on these assumptions, a detailed Input-to-state stability analysis in the L^2-norm has been provided in [4]. More precisely, the steady state $W \equiv 0$ of the system (1) with the boundary conditions (3) is Input-to-state stable in L^2-norm with respect to the disturbance function b if there exist positive real constants $\eta > 0$, $\xi > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every initial condition $W_0(x) \in L^2((0, l); \mathbb{R}^k)$, the L^2-solution to the system (1) with initial condition (2) and boundary conditions (3) satisfies for all $t \in \mathbb{R}^+$

$$\|W(\cdot,t)\|^2_{L^2((0,l);\mathbb{R}^k)} \leq C_1 \exp(-\eta t) \|W_0\|^2_{L^2((0,l);\mathbb{R}^k)} + \frac{C_2}{\eta}\left(1 + \frac{1}{\xi}\right) \sup_{s\in[0,t]} (|b(s)|^2). \tag{5}$$

If the disturbance function disappears, i.e., $b(t) \equiv 0$ for all $t \in [0, +\infty)$, the definition of exponential stability can be retrieved from the notion on Input-to-state stability. The latter notion is of course weaker than its exponential counterpart, as the disturbance term in inequality (5) counteracts the exponential decay of the first term and is highly dependent on the given disturbances.

A possible Lyapunov function candidate to study the exponential stability of hyperbolic balance laws has been originally introduced in [1], i.e.,

$$\mathcal{L}(W(\cdot,t)) = \int_0^l W^T P(x) W dx, \quad t \in [0, +\infty), \tag{6}$$

for continuously differentiable positive definite matrices $P(x)$. This Lyapunov function is then said to be an ISS-Lyapunov function for the system (1) with the boundary conditions (3) if there exist positive real constants $\eta > 0$, $\xi > 0$ and

$\nu > 0$ such that, for all functions $b(t) \in C^0([0,+\infty))$, for L^2-solutions of the system (1) satisfying the boundary conditions (3), and for all $t \in [0,+\infty)$,

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} \leq -\eta\mathcal{L}(W(\cdot,t)) + \nu\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}(|b(s)|^2). \tag{7}$$

It turns out that if the matrix

$$-\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi^T(x)P(x) + P(x)\Pi(x) \tag{8}$$

is positive definite for all $x \in [0,l]$ and the matrix

$$\begin{bmatrix}\Lambda^+(l)P^+(l) & 0\\ 0 & \Lambda^-(0)P^-(0)\end{bmatrix} - (1+\xi)K^T\begin{bmatrix}\Lambda^+(0)P^+(0) & 0\\ 0 & \Lambda^-(l)P^-(l)\end{bmatrix}K \tag{9}$$

is positive semi-definite, the steady state $W(x,t) \equiv 0$ of the system (1) with boundary conditions (3) is Input-to-state stable in the L^2-norm with respect to the disturbance function b.

A further approach intends to answer the question how the results from the continuous setting in terms of Input-to-state stability can be transferred to a discretized version of (1)-(3). Following the the numerical discretization presented in [3] based on the operator splitting technique

$$\partial_t W(x,t) + \Lambda(x)\partial_x W(x,t) = 0 \qquad (x,t) \in [0,l]\times[0,+\infty) \tag{10}$$

$$\partial_t W(\cdot,t) + \Pi(\cdot)W(\cdot,t) = 0 \qquad t \in [0,+\infty). \tag{11}$$

then allows for a rigorous numerical analysis of Input-to-state stability. In fact, a notable advantage of the discretized approach is the explicit computation of decay rates η that appear in (5). For an illustration of the results obtained so far, we make use of the Telegrapher's equations which is a 2×2 system of linear hyperbolic balance laws of the form

$$\partial_t W(x,t) + \Lambda\partial_x W(x,t) + \Pi W(x,t) = 0 \tag{12}$$

for $x \in [0,l]$, where Π and Λ are independent of x and defined as

$$\Pi := \frac{1}{2}\begin{bmatrix}RL^{-1}+ZC^{-1} & RL^{-1}-ZC^{-1}\\ RL^{-1}-ZC^{-1} & RL^{-1}+ZC^{-1}\end{bmatrix}, \tag{13}$$

$$\Lambda := \begin{bmatrix}\lambda^+ & 0\\ 0 & \lambda^-\end{bmatrix}, \text{ with } \lambda^{\pm} = \pm(\sqrt{LC})^{-1} \tag{14}$$

and positive constants R, L, Z, C. Applying the weighted matrices

$$P_j = \operatorname{diag}\{p_1\exp(-\mu_1 x_j), p_2\exp(\mu_2 x_j)\} \tag{15}$$

with $\mu_1, \mu_2 > 0$ and $p_1, p_2 > 0$ to the disctrization of (6), we are able to show that for the Telegrapher's equations given by (12) - (14) the decay rate η is defined by

$$\eta = \frac{1}{\Delta x\sqrt{LC}}(1-\exp(-\mu\Delta x)).$$

Future considerations will investigate the notion of Input-to-state stability for balance laws of type (1) on networks. This requires in particular a discussion on coupling conditions and their influence on the continuous and discretized problem, respectively.

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Isothermal flow in gas networks: Synchronization of observer systems

Martin Gugat

We study systems that are governed by neworked hyperbolic partial differential equations. As an example, think of gas pipeline networks.

We discuss the necessity to design observers for this type of system. They can be applied to produce an approximation of the full system state that can be used as input in a feedback law and also provide initial data for optimal control problems. We present an observer system that is also defined as a networked system of hyperbolic partial differential equations and is fed with pointwise measurements from the original system. These measurements are taken at a finite number of locations in the network. In our model, we consider the measurements as continuous in time. This is not completely realistic, and in future studies we will also consider time-discrete measurements and also an observer system that is defined on discrete times and is implementable on a computer.

We show that for a sufficiently large number of measurement locations, the observer system synchronizes exponentially fast with the original system, that is the error decays exponentially fast to zero. This is a local result that we can only proof under a number of smallness assumption for the state of the original system and also for the error of the estimate of the initial state that is used in the observer system. Since the semi-global classical solutions of quasilinear hyperbolic systems are a suitable framework to prove the synchronization results, we use an existence result for semi-global classical solutions in our analysis of the well-posedness of the system. These results have been investigated thoroughly in the group of Ta-Tsien Li.

We discuss a Lyapunov function with exponential weights that can be used for the proof of the exponential synchronization in the L^2–sense. Such exponential weights have been used very successfully by Jean-Michel Coron and his group.

As an outlook, we discuss the extension of the observer to the case where the gas is a mixture of hydrogen and natural gas. We consider a model of the following type that is similar as in [2]:

$$(1)\qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t c + v\,\partial_x(c) = 0 \\ \partial_t(\rho v) + \partial_x\left(\rho v^2 + \hat{p}(\rho, c)\right) = -\dfrac{\lambda^F}{2D}\rho|v|v \end{cases}$$

where ρ_1 and ρ_2 denote the densities of the two components, $\rho := \rho_1 + \rho_2$, v is the velocity of the gas mixture and $c := \rho_1/\rho$. Note that in order to close the system, the knowledge of the pressure law $\hat{p}(\rho, c)$ for the mixture is mandatory. The number λ^F is a friction parameter and D denotes the diameter of the pipe.

Finally we mention the challenge to deal with hydrogen embrittlement in steel pipelines. Since pressure fluctuations promote the damage of the pipes by hydrogen embrittlement, one aim of the operation control must be to mitigate these fluctuations. In the mathematical model this leads to new state constraints that have to be compatible with a realistic damage model, for example the rainflow-counting algorithm used for calculations of the fatigue. Problems of optimal boundary control for gas pipeline networks have been investigated in [3].

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Stabilization methods for evolution equations

Amaury Hayat

The topic of this talk is the stabilization of systems of (partial) differential equations. We focus on three parts: *Fredholm backstepping*, also known as *F-equivalence*, a powerful approach that allows the derivation of explicit feedback laws for a large class of systems of PDEs; *stabilization of traffic flow*, modeled by hyperbolic systems of PDEs, and the use of *Artificial Intelligence (AI) in mathematics.* The latter extends beyond the sole area of stabilization.

1. The F-equivalence method

We study the general stabilization problem for linear systems. Consider

$$(1)\qquad \partial_t y(t) = \mathcal{A}y(t) + Bu(t),$$

where $y(t) \in X$ describes the state of the system at time t, X is a Banach space, $\mathcal{A}$ is a differential operator with discrete spectrum and defined on a Banach space X with the domain $D(\mathcal{A})$, $B : H \to D(\mathcal{A}^*)'$ is a so-called control operator and $u(t) = Ky(t)$ is the control chosen as a feedback law, and K is consequently a feedback operator with values in H. The goal is to find K such that the system (1) is exponentially stable. Ideally, we would even want to ensure that, for any $\lambda > 0$, there exists K_λ such that the system (1) is exponentially stable with decay rate λ. When B is not bounded in X and H is finite dimensional, this becomes a complex problem. The F-equivalence approach is as follows: instead of trying to directly find a feedback operator K, one attempts to simultaneously find a mapping $\mathcal{T}$ and a feedback operator K such that $\mathcal{T}$ is an isomorphism and maps the system (1) to an exponentially stable target system

$$\partial_t z = \mathcal{A}' z, \tag{2}$$

where $\mathcal{A}'$ is a carefully chosen exponentially stable operator. If this can be done then, as a consequence, the original system (1) is exponentially stable (with the same decay rate).

The *backstepping method*, introduced in 2003 by Krstic and several collaborators, relied on this paradigm and suggested looking for $\mathcal{T}$ in the form of a Volterra transform, which led to many proven results. Another method, sometimes called *Fredholm backstepping*, and relying on more general transformations, was introduced in 2013 by Coron and Lu for a particular case. We present a new method [3] to generalize the F-equivalence approach to generic systems where $\mathcal{A}$ is skew adjoint with eigenvalues scaling more than linearly, B is admissible, and the system is exactly null controllable. In particular, this allow to solve an open question presented at the College de France in 2017. The hope is that the F-equivalence approach can provide quantitative estimates of T and K with respect to λ and can extend locally to nonlinear systems, even for systems where classical perturbation arguments do not apply.

2. Traffic flow stabilization

We examine a particular problem: the stabilization of traffic flow. In traffic, when the density of cars is high, the uniform flow steady-state becomes unstable and stop-and-go waves appear, resulting in a phenomenon commonly known as a traffic jam. Our goal is to stabilize these uniform steady-states. Our approach is to use autonomous vehicles (AVs) as controls on the traffic. The resulting mathematical system is modeled by one or several PDEs coupled with an ODE: the PDE(s) represent the bulk of traffic, while the ODE represents the location of the autonomous vehicle. The PDE model could be, for instance, the LWR model

$$\partial_t \rho + \partial_x(\rho V(\rho)) = 0, \tag{3}$$

where V is a function that is concave and C^2. A more realistic approach would be to consider a second-order model, for instance, the Generalized-ARZ (GARZ) equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho V(\rho, \omega)) = 0, \\ \partial_t(\rho\omega) + \partial_x(\rho\omega V(\rho, \omega)) = 0. \end{cases} \tag{4}$$

The dynamics of the AV's location $y(t)$ can be modeled by

$$\dot{y}(t) = \min\{u(t), V(\rho(t, y(t)+), w(t, y(t)+))\}, \tag{5}$$

where $u(t)$ refers to the control, and for a BV function f, $f(y(t)+)$ refers to the right limit at $y(t)$. The min comes from the fact that the AV cannot move faster than the traffic flow in front of it (to avoid a crash). This system has both practical and mathematical interests. From a practical perspective, reducing stop-and-go waves could lead to a significant reduction in fuel consumption and CO_2 emissions, as well as safer traffic. From a mathematical perspective, the PDE/ODE system modeling this situation is interesting in that the relevant physical solutions are not the entropic solutions, usually considered as the natural solutions for hyperbolic systems and studied for decades. More specifically, the physical solutions are not necessarily entropic at the AV's location. For this reason, we need a new condition to replace the entropy condition at the AV's location. We use a flux condition similar to the *Delle Monache-Goatin* flux condition

$$\rho(t, y(t))(V(\rho(t, y(t)), \omega(t, y(t))) - \dot{y}) \leq \alpha \max_{\rho,\omega}(\rho(V(\rho, \omega) - u(t))) \tag{6}$$

where $\alpha \in (0, 1)$. We present an existence result (in the class of BV solutions, entropic outside of the AV's location) for a solution to the Cauchy problem (4)–(6) for any initial condition (in the same class), provided that the control is constant [5]. The question of a time-varying control and, *a fortiori*, of the stabilization problem is widely open.

We also present field experiment results obtained in Nashville, TN in November 2022 as part of the CIRCLES project, where 100 autonomous cars were sent on a highway in dense traffic during peak hours, showing that our candidate controller drastically reduces the speed variance of the stop-and-go waves [4].

3. AI for mathematics

AI has seen many successes in the last decade, especially in natural language processing. A natural question we would like to ask is:

Can an AI learn mathematics in some sense?

We focus on two interpretations of this question:

- Can an AI predict the solution to an advanced mathematical problem?
- Can an AI prove a mathematical statement and provide a proof?

The answer to the first question seems to be yes. In several works, a trained AI model (a Transformer) managed to predict the solution to several mathematical problems, for instance, predicting explicit solutions to ODEs, the controllability of the linearized system given a nonlinear system, or a suitable feedback law [2]. We illustrate with two examples how this approach can assist mathematicians in solving open problems. The first one is a preliminary neural network trained to find Lyapunov functions for a nonlinear system, which is a general open question. The second one is a mix of deep reinforcement learning and mathematical analysis

[1] that allowed the finding of a control feedback law and the stabilization of a system for which finding such a feedback law was an open question until now.

Regarding the second question, we present a work [6] inspired by AlphaZero, where we train a neural network to demonstrate (small) mathematical statements and provide proof. This neural network is capable of demonstrating high school or undergraduate exercises and some exercises from the International Mathematical Olympiads.

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Convex Optimisation Methods for Variational Mean Field Games

Dante Kalise

(joint work with Luis Briceño-Arias and Francisco J. Silva)

We discuss the numerical approximation of stationary mean field games [1]

$$(MFG_\infty)\begin{cases} -\sigma^2\Delta v(x) + H(x,\nabla v(x)) - \lambda = f(x,m(x)) & \text{in } \mathbb{T}^d\,, \\ -\sigma^2\Delta m(x) - \nabla\cdot\big(\partial_p H(x,\nabla v(x))m(x)\big) = 0 & \text{in } \mathbb{T}^d\,, \\ \qquad m \geq 0, \quad \displaystyle\int_{\mathbb{T}^d} m\,dx = 1, \quad \int_{\mathbb{T}^d} v\,dx = 0\,, & \end{cases}$$

by convex optimisation methods. Under suitable hypotheses on the coupling term f and the Hamiltonian H, we begin by defining the momentum variable $w(x) = -\partial_p H(x,\nabla v(x))m(x)$ and reformulating the MFG system as a PDE-constrained optimisation problem

$$(OC_\infty):\begin{cases} \displaystyle\min_{(m,w)}\int_{\mathbb{T}^d} b(x,m(x),w(x))+F(x,m(x))\,dx \\ \text{s.t.}\quad -\sigma^2\Delta m(x)+\nabla\cdot(w(x))=0\,,\quad \text{and} \quad \displaystyle\int_{\mathbb{T}^d} m\,dx=1\,, \end{cases},$$

where

$$F(x,m):=\begin{cases}\int_0^m f(x,\mu)\,d\mu & \text{if } m\geq 0\,,\\ +\infty & \text{otherwise}\end{cases}$$

and

$$b(x,m,w):=\begin{cases} mH^*(x,-\frac{w}{m}) & \text{if } m>0\,,\\ 0 & \text{if } (m,w)=(0,0)\,,\\ +\infty & \text{otherwise}\,.\end{cases}$$

Here, $H^*(x,p^*)$ denotes the convex conjugate $H^*(x,p^*)=-\inf_p\{H(x,p)-\langle p^*,p\rangle\}$. Assuming $f(x,m(x))$ increasing in m, (OC_∞) is a convex optimisation problem [10, 11]. This variational formulation is related to other problems of interest, such as:

- the **Schrödinger Bridge problem** [4, 5]

  $$(SB)\begin{cases} \displaystyle\min_{(m,u)}\frac{1}{2}\int_0^1\int_{\mathbb{R}^d} m(x,t)|u(x,t)|^2dxdt \\ \text{s.t.} \\ \partial_t m(x,t)+\nabla\cdot(m(x,t)u(x,t))-\dfrac{\epsilon}{2}\Delta m(x,t)=0\,,\\ m(x,0)=m_0(x)\,,\quad m(x,1)=m_1(x)\,,\end{cases}$$

  which in the deterministic limit ($\sigma=0$) corresponds to the Benamou-Brenier formulation of the optimal transport problem [2, 3].
- the **JKO scheme for gradient flows** [6]

  $$\partial_t m=\nabla\cdot\left[\rho\nabla\left(U'(m)+V+W*m\right)\right]$$

  where, at every discrete time step, the mass is recovered as the solution of

  $$m_{\Delta t}^{n+1}\in\operatorname*{arginf}_m\left\{\frac{1}{\Delta t}\mathcal{W}_2(m_{\Delta t}^n,\,m)+\mathcal{F}(m;U,V,W)\right\}\,,$$

  and the Wasserstein distance $\mathcal{W}_2$ can be computed using a similar formulation as in (SB) above [7].

The construction of a numerical scheme for these problems begins with a suitable discretization of the transport equation and the cost, leading to

$$(OC^h_\infty):\begin{cases} \min\limits_{(m_h,w_h)} \sum\limits_{i,j} \hat{b}(x_{i,j},m_{i,j},w_{i,j}) + F(x_{i,j},m_{i,j}) \\ \text{s.t} \quad A_h m_h + B_h w_h = 0\,, \quad \forall i,j = 1,\dots,N\,, \\ \sum\limits_{i,j} h^2 m_{i,j} = 1\,, \end{cases}$$

where $\hat{b}(m,w) = b(m,w) + \iota_{\mathbb{R}\times K}(m,w)$, with $K := \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^+ \times \mathbb{R}^-$. Problem (OC^h_∞) belongs to a class of convex optimization problems of the form

$$\min_y \; \Phi(y) + \Psi \circ C(y)$$

where Φ, Ψ are convex, l.s.c, proper functions, and C is a linear operator. Algorithms for this class of problems rely on duality and on the proximal operator

$$\text{prox}_{\lambda f}(v) := \underset{x}{argmin} \left(f(x) + \frac{1}{2\lambda}||x-v||^2 \right).$$

Within a wide class of suitable convex optimisation algorithms, we focus on the application of the primal-dual algorithm proposed by Chambolle and Pock [8, 9]. Given $\gamma, \tau \geq 0$, such that $\gamma\tau < \|C\||^{-2}$, the iteration reads:

$$\begin{aligned} \sigma^{k+1} &= \text{prox}_{\gamma\Psi^*}(\sigma^k + \gamma C\bar{y}^k) \\ y^{k+1} &= \text{prox}_{\tau\Phi}(y^k - \tau C^* \sigma^{k+1}) \\ \bar{y}^{k+1} &= 2y^{k+1} - y^k \end{aligned}$$

We discuss convergence to the solution of (MFG_∞), assignments for Φ and Ψ, and the effective computation of proximal operators.

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Structure of non entropy solutions to scalar conservation laws

Elio Marconi

We consider bounded weak solutions of conservation laws of the form

$$\partial_t u + \mathrm{div}_x(F(u)) = 0, \qquad u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \quad F : \mathbb{R} \to \mathbb{R}^d. \tag{1}$$

In particular we are interested in solutions with finite entropy production, namely solutions for which for every smooth entropy-entropy flux pair (η, Q) the distribution

$$\mu_\eta := \partial_t \eta(u) + \mathrm{div}_x(Q(u))$$

is a locally finite Radon measure. Compared to the more usual notion of entropy solution we do *not* require that if η is convex, then μ_η is a negative measure.

These solutions arise in the study of large deviations for stochastic particle scheme approximations of entropy solutions of (1) (see [11] and the recent results in [10]). A variational point of view investigated in [3] relates this problem to the following control problem for conservation laws in dimension 1: given $\varepsilon > 0$ and u_ε, let E_ε be such that

$$\partial_t u_\varepsilon + \partial_x(F(u_\varepsilon)) = \varepsilon \partial_{xx} u_\varepsilon + \partial_x E_\varepsilon \tag{2}$$

and set $I_\varepsilon(u_\varepsilon) = \min \int_{\mathbb{R}^2} E_\varepsilon^2 dxdt$, where the minimum is taken among the functions E_ε satisfying (2).

It is well known that if u_ε solves (2) with $E_\varepsilon \equiv 0$, the family u_ε converges to the entropy solution of (1) as $\varepsilon \to 0$.

An interesting regime is when $\varepsilon^{-1} I_\varepsilon(u_\varepsilon)$ remains bounded: in this case it is shown in [3] that u_ε converges up to subsequences to a solution with finite entropy production. Moreover the Γ-convergence of $\varepsilon^{-1} I_\varepsilon$ is investigated: a natural candidate H is proposed as well as a proof of

$$H \le \Gamma - \liminf_{\varepsilon \to 0} \varepsilon^{-1} I_\varepsilon.$$

The functional H can be described in terms of the entropy production measures μ_η: in the case of the Burgers equation $F(u) = u^2$ we expect that it coincides with

$$H(u) = \frac{1}{6} \int_{J_u^+} |u^+ - u^-|^3 d\mathcal{H}^1,$$

where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure and J_u^+ is the subset of the jump set of u where the right trace u^+ is larger than the left trace u^-, namely the non-entropic shocks.

In order to complete the analysis we need to build a recovery sequence for this functional: it can be done for solutions with some additional structure (see the notion of entropy-splittable solution in [3]). It would be sufficient to check that these solutions are dense (in energy) in the class of solutions with finite entropy production, alternatively we should provide some different procedure to produce a recovery sequence directly for a general solution with finite entropy production.

In both cases it seems that a better understanding of the structure of solutions with finite entropy production is needed.

The general picture is that solutions with finite entropy production share several fine properties with solutions with bounded variation, even if in general they do not belong to BV. It is shown in [7] (in dimension 1) and in [5] in several space dimensions that under mild nonlinearity assumption on F for any solution of (1) with finite entropy production we can define a co-dimension 1 rectifiable jump set J with the following properties:

(1) every $(t, x) \notin J$ is a point of vanishing mean oscillation;
(2) for $\mathcal{H}^d$-a.e. $(t, x) \in J$ there are strong traces $u^\pm$ in L^1.

One may wonder if vanishing mean oscillation points are Lebesgue points (at least $\mathcal{H}^d$-a.e. as it happens for BV functions). A partial result in this direction has been obtained in [6] for $d = 1$ and in [8] in several space dimension: in both cases it is shown that the singular points has co-dimension (at least) 1.

Since the candidate Γ-limit functional H introduced in [3] can be written in terms of the measures μ_η, a desirable property that BV solutions enjoy and we would like to prove for general solutions with finite entropy production is that for every η the measure μ_η is concentrated on J_u. This is proven for Burgers equation in [9], and it is possible to extend this result to general conservation laws in the case $d = 1$ with similar techniques [2].

The results in [2, 8, 9] relies on a 'Lagrangian' description of the solutions of (1) developed in [4, 8]: inspired by the Lagrangian techniques introduced in [1] to study the linear continuity equation, we consider the kinetic formulation of (1) and we provide a decomposition along characteristics of the kinetic function associated to the solution of (1). A useful feature of this technique is that it allows to exploit some geometric constraint on the underlying characteristics structure of these solutions, despite they are a priori merely bounded. Since the geometry if $d = 1$ is simpler compared to the case of several space dimensions, this may explain why, using these techniques, we obtain better results when reducing to one space dimension.

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Optimization by kinetic equations

Lorenzo Pareschi

Since its introduction n the late 1980s [14], simulated annealing has become a popular optimization algorithm, and its applications have expanded to many fields, including artificial intelligence, machine learning, and operations research. The algorithm was developed as an extension of the Metropolis-Hastings algorithm, a Monte Carlo method used for simulating complex systems in physics [15], and adapts the concept of annealing to optimization problems by viewing the process of slowly cooling a material as a search for the lowest-energy state of the system.

Simulated annealing is similar to other metaheuristic algorithms such as genetic algorithms, ant colony optimization, particle swarm optimization and consensus based optimization in that it is based on the idea of exploring a large search space to find a global optimum without using gradient-based information. However, simulated annealing is distinct in that it uses a single solution-based probabilistic approach to accept worse positions in the hope of finding a better one, whereas other metaheuristic algorithms often use a population-based approach and other stochastic strategies.

By applying the above concepts, metaheuristic algorithms have been able to make significant advances in the search for valuable solutions to challenging optimization problems out of reach of traditional (gradient-based) methods. However, proving the rigorous convergence of metaheuristic optimization algorithms to the global minimum for non convex functionals, or to some reasonable approximation of it, remains a challenge. Indeed, metaheuristics involves the creative use of available resources to find efficient solutions without necessarily relying on a rigorous mathematical foundation that provides an analytical setting.

On the other hand, metaheuristics share similarities with statistical physics since they both deal with the complexities of large systems. The principles of statistical physics are versatile and powerful, providing insight into the behavior of large systems in a wide range of fields, from materials science to biophysics. By drawing upon the principles of statistical physics, it may be possible to provide a solid mathematical foundation to these class of optimization methods and develop more effective and efficient optimization algorithms that can handle increasingly complex problems and larger search spaces.

Mean field equations and kinetic equations are among the concepts in statistical physics that are most relevant to optimization. Mean field equations describe how each particle in a system interacts with a theoretical "average" field created by all the other particles in the system. This provides insight into the behavior of large systems, making it possible to predict macroscopic properties such as temperature or pressure. Kinetic equations, on the other hand, describe the evolution of a particle system, considering the interactions between particles as instantaneous, microscopic collisions [18]. Recently, these ideas have led to a new view of methaeuristic optimization by considering the corresponding continuous dynamics described by appropriate kinetic equations of Boltzmann type [3] and mean-field type [4, 6, 16, 13] even in constrained contexts [1, 11, 9, 10], multiobjective situations [2], or in generalizations to sampling [5]. See [17] for a recent survey.

In this talk we will focus our attention on one of the most notable examples of metaheuristics, namely the simulated annealing algorithm. This algorithm was inspired by the Monte Carlo algorithm developed by Metropolis et al. in the middle of last century. We show how classical tools of kinetic theory can be used to describe the Markov process which characterizes the method and show how its convergence to the global minimum is related to classical functional inequalities based on the so-called entropy method [8].

Furthermore, the continuous setting based on kinetic PDEs permits to investigate the relationships with Fokker-Planck equations describing the so-called mean-field Langevin dynamic [7, 12]. In particular, we illustrate how to formally derive the corresponding mean-field model taking a suitable scaling limit of the linear kinetic model describing the simulated annealing. This has been generalized to other types of kinetic equations describing variations of the simulated annealing method that avoid the acceptance-rejection process. Numerical evidence of such asymptotic behavior has also been discussed through simulation examples. From a mathematical viewpoint let us finally mention that several challenging questions remains open, like estimating the rate of convergence to equilibrium in the different functional spaces and analyzing the convergence properties of the Maxwellian variant here introduced. We leave these aspects to future researches.

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Peculiarities of non-homogeneous conservation laws

Vincent Perrollaz
(joint work with Rinaldo M. Colombo, Abraham Sylla)

It might be surprising to realize that in the context of

$$\begin{cases} \partial_t u + \partial_x f(x,u) = 0, \\ u(0,\cdot) = u_0. \end{cases} \tag{1}$$

the hypotheses of Kruzkov seminal existence result [4] may appear too restrictive. More specifically in the context of (1) — one space dimension and no source term — Kruzkov hypotheses include:

$$\sup_{(x,u)\in\mathbb{R}^2} -\partial^2_{xu} f(x,u) < +\infty. \tag{K}$$

Even within the context of regular flux functions f, some reasonable models are not directly covered. For instance, when considering a LWR model for traffic flow where the number of lanes and the speed limit may vary:

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) v_{\max}(x) \left(1 - \frac{\rho(t,x)}{\rho_{\max}(x)} \right) \right) = 0,$$

reasonable assumptions on the functions $v_{\max}$ and $\rho_{\max}$ could be

$$\forall x \in \mathbb{R}, \quad 0 < \underline{v} \le v_{\max}(x) \le \bar{v}, \quad 0 < \underline{\rho} \le \rho_{\max}(x) \le \bar{\rho} \tag{2}$$

with $\rho'_{\max} \in \mathcal{C}^2_c(\mathbb{R})$ and $v'_{\max} \in \mathcal{C}^2_c(\mathbb{R})$. This puts the model out of the scope of a direct application of Kruzkov existence result.

In [2], we provide an alternative framework of hypotheses on the flux f which allows us to still get a semigroup generated by the entropy solutions. More precisely we replace hypothesis (K) by

$$\forall \bar{f} \in \mathbb{R},\ \exists \bar{U} \in \mathbb{R},\ \forall (x,u) \in \mathbb{R}^2, \quad |f(x,u)| \le \bar{f} \implies |u| \le \bar{U}. \tag{C}$$

Let us briefly explain the role that this coercivity hypothesis play in our construction. The family of constant functions plays a central role in most techniques in the case of an x-independent flux. For instance, when combined with a locally contractive semigroup in L^1, it provides a priori L^∞ bounds. We detail, using our hypothesis (C), the construction of an alternative family of — possibly discontinuous — stationary solutions in the context of equation (1). We first make use of tools of differential topology to build such families for a special class of fluxes with nice geometric properties. We then use the method of compensated compactness to extend the result to a class of fluxes satisfying (C) (and some technical hypothesis which we expect could be relaxed).

In addition to the *a priori* L^∞ bounds granted by this family of stationary solutions, we use the method of vanishing viscosity and again compensated compactness to obtain the existence of a semigroup $(S^{CL}_t)_{t\ge 0}$ whose orbits are the unique maximal entropy solutions of (1). As a byproduct of this construction we also obtain the existence of a semigroup $(S^{HB}_t)_{t\ge 0}$ whose orbits are the unique maximal viscosity solutions of the following Hamilton-Jacobi equation

$$(3) \qquad \begin{cases} \partial_t U + f(x, \partial_x U) = 0, \\ U(0,\cdot) = U_0. \end{cases}$$

Of course, since they are both obtained as singular limits of regular viscous approximations, those semigroups are shown to be related according to the following commutative diagram

$$\begin{array}{ccc} U_o & \longrightarrow & S_t^{HJ} U_o \\ \partial_x \downarrow & & \downarrow \partial_x \\ u_o & \longrightarrow & S_t^{CL} u_o \end{array}$$

where the derivation ∂_x is taken in the distributional sense. A somewhat surprising fact is that the continuity properties and stability properties of the semigroups are not in full correspondence.

Finally, following [3], we describe why — even in the simplest case where the flux f is convex in u — the entropy semigroup is actually more singular than in the x-independent case. This phenomenon appears naturally when investigating the inverse design sets associated to (1)

$$I_T(w) := \{u_0 \in \mathrm{L}^\infty(\mathbb{R}) \,:\, S_T^{CL} u_0 = w\}.$$

The connection between entropy solutions of equation (1) and viscosity solutions of the Hamilton-Jacobi equation (3) is key for this analysis.

To be specific, we show in [1] for the case where the flux f does not depend on x that whenever $w \in \mathrm{L}^\infty(\mathbb{R})$ is such that $I_T(w) \neq \emptyset$ — i.e. w is reachable — then there is at least one initial data u_0 such that the solution $t \mapsto S_t^{CL} u_0$ is isentropic between $t = 0$ and $t = T$. Since the isentropic solutions are in some sense the topological closure of the classical solutions, this means that to understand the range of the semigroup one needs only to consider the classical solutions even though the blowup in time is generic with respect to initial data. In contrast, we show in [3] that when the flux f does depend on x then some states are reachable but a minimal entropy price has to be paid to reach them.

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Understanding Consensus-Based Optimization: Two Analytical Perspectives

Konstantin Riedl
(joint work with Massimo Fornasier and Timo Klock)

Consensus-based optimization (CBO) [1] is a multi-particle derivative-free optimization method capable of globally minimizing high-dimensional nonconvex and nonsmooth functions $\mathcal{E}:\mathbb{R}^d\to\mathbb{R}$, i.e., solving problems of the form

$$x^*\in\arg\min_{x\in\mathbb{R}^d}\mathcal{E}(x).$$

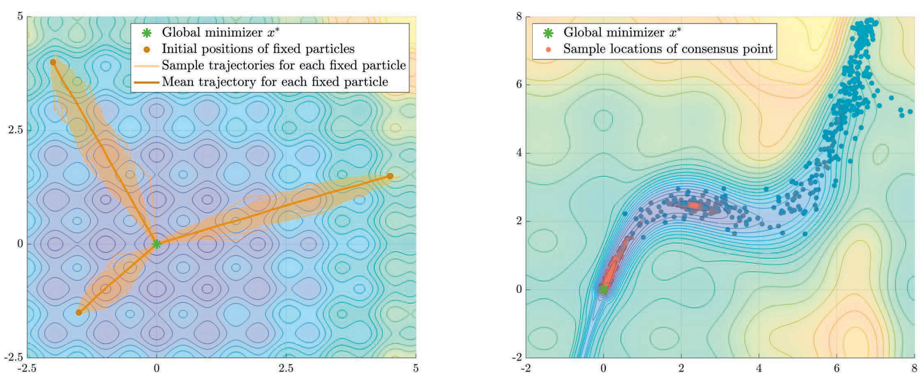
Inspired by consensus dynamics and opinion formation, CBO methods employ a finite number of agents $X^1,\dots,X^N$ to explore the domain and to form a consensus about the global minimizer x^* as time passes. More concretely, for a discrete time step size $\Delta t>0$ and user-specified parameters $\alpha,\lambda,\sigma>0$, the time-discrete evolution of the i-th particle X^i is defined according to the iterative update rule

$$X_k^i=X_{k-1}^i-\Delta t\lambda\big(X_{k-1}^i-x_\alpha^{\mathcal{E}}(\widehat\rho_{k-1}^N)\big)+\sigma\mathrm{diag}\big(X_{k-1}^i-x_\alpha^{\mathcal{E}}(\widehat\rho_{k-1}^N)\big)B_k^i, \tag{1}$$

where $\widehat\rho_k^N=\frac{1}{N}\sum_{i=1}^N\delta_{X_k^i}$ is the empirical measure of the particles at time step k and where B_k^i are i.i.d. Gaussian random vectors with zero mean and covariance $\Delta t\mathrm{Id}$. Moreover, $x_\alpha^{\mathcal{E}}$ denotes the so-called consensus point, a weighted average of the particles' positions, which is computed for a measure $\varrho\in\mathcal{P}(\mathbb{R}^d)$ according to

$$x_\alpha^{\mathcal{E}}(\varrho)=\int x\frac{\omega_\alpha^{\mathcal{E}}(x)}{\|\omega_\alpha^{\mathcal{E}}\|_{L_1(\varrho)}}\,d\varrho(x),\quad\text{with}\quad\omega_\alpha^{\mathcal{E}}(x):=\exp(-\alpha\mathcal{E}(x)). \tag{2}$$

In what follows and as illustrated in Figure 1, we provide insights into the internal mechanisms of CBO from two analytical perspectives.



(a) CBO convexifies any nonconvex problem in the mean-field limit, see [2, 3].

(b) CBO can be interpreted as a stochastic relaxation of gradient descent, see [4].

Figure 1. Illustrations of the internal mechanisms of CBO, which are responsible for the success of the method.

First, based on an experimentally supported intuition that, as the number of particles goes to infinity in the continuous-time analogous of (1), i.e., in the mean-field limit, which is captured by the nonlinear nonlocal Fokker-Planck equation

$$\partial_t \rho_t = \lambda \mathrm{div}\big(\big(x - x_\alpha^{\mathcal{E}}(\rho_t)\big)\,\rho_t\big) + \frac{\sigma^2}{2}\sum_{k=1}^{d} \partial_{kk}\left(\big(x - x_\alpha^{\mathcal{E}}(\rho_t)\big)_{kk}^2\,\rho_t\right), \tag{3}$$

CBO always performs a gradient descent of the Wasserstein distance to the global minimizer (see Figure 1a), we present a novel technique for proving global convergence in mean-field law for a rich class of objective functions. More precisely, when analyzing the quantity $W_2^2(\rho_t, \delta_{x^*})$ we observe the following.

Theorem 1 ([2, Theorem 12] and [3, Theorem 2])**.** *Let the objective $\mathcal{E} \in \mathcal{C}(\mathbb{R}^d)$ satisfy $\|x - x^*\|_\infty \leq (\mathcal{E}(x) - \inf \mathcal{E})^\nu/\eta$ for all $x \in \mathbb{R}^d$ with constants $\eta, \nu > 0$. Moreover, let $\rho_0 \in \mathcal{P}_4(\mathbb{R}^d)$ with $x^* \in \mathrm{supp}(\rho_0)$. Then, for any $\varepsilon > 0$, $\gamma \in (0,1)$ and with parameters λ, $\sigma > 0$ obeying $2\lambda > \sigma^2$, there exists $\alpha_0 = \alpha_0(\varepsilon, \gamma, \lambda, \sigma, d, \nu, \eta, \rho_0)$ such that for all $\alpha \geq \alpha_0$ a weak solution $(\rho_t)_{t\in[0,T^*]}$ to (3) satisfies $W_2^2(\rho_T, \delta_{x^*}) = \varepsilon$, where $T \in \left[\frac{1-\gamma}{1+\gamma/2}T^*, T^*\right]$ with $T^* = \frac{1}{(1-\gamma)(2\lambda-\sigma^2)} \log\left(W_2^2(\rho_0, \delta_{x^*})/\varepsilon\right)$. Furthermore, on the time interval $[0,T]$, it holds*

$$W_2^2(\rho_t, \delta_{x^*}) \leq W_2^2(\rho_0, \delta_{x^*}) \exp\left(-(1-\gamma)\left(2\lambda - \sigma^2\right) t\right). \tag{4}$$

From this result it becomes apparent that the hardness of any global optimization problem is necessarily encoded in the mean-field approximation, i.e., in the way how the empirical measure of the finite particle dynamics is used to approximate the mean-field limit. In consideration of the central significance of such approximation with regards to the overall computational complexity of the implemented numerical scheme, we discuss a probabilistic quantitative result about the convergence of the interacting particle system towards the corresponding mean-field dynamics, for which we refer to [2, Proposition 16]. While the observed convergence rate is of order N^{-1} in the number of particles N, the constant in this approximation depends exponentially on the parameter α, which in turn depends in worst-case scenarios linearly on the dimension d. Characterizing more insightfully the dependence of α on properties of specific classes of objectives remains an exciting open problem for future research. A combination of the former results yields a holistic convergence proof of CBO methods on the plane, see [2, Theorem 14]. This analytical framework has allowed to obtain convergence guarantees for several variants of CBO including CBO with memory effects and local gradients [5], CBO with truncated noise [6], constrained CBO [7], CBO for multi-objective optimization problems [8], CBO for saddle point problems [9], and FedCBO for clustered federated learning problems [10]. Moreover, it may permit to prove convergence for other metaheuristics, such as the particle swarm optimization method [11].

Second, by turning our back on the previous mean-field-focused analysis point of view and by leveraging a completely nonsmooth analysis, which combines a novel quantitative version of the Laplace principle (log-sum-exp trick) and the minimizing movement scheme (proximal iteration), we interpret CBO as a stochastic relaxation of gradient descent (see Figure 1b), thereby providing a novel analytical perspective on the theoretical understanding of gradient-based learning algorithms. We observe that through communication of the particles, CBO exhibits a stochastic gradient descent (SGD)-like behavior despite solely relying on evaluations of the objective function. More rigorously, it holds the following.

Theorem 2 ([4, Theorem 1]). *Let the objective $\mathcal{E} \in \mathcal{C}^1(\mathbb{R}^d)$ be L-smooth, Λ-convex and satisfy minimal regularity assumptions. Then, for $\tau > 0$ (satisfying $\tau < 1/(-2\Lambda)$ if $\Lambda < 0$) and with parameters $\alpha, \lambda, \sigma, \Delta t > 0$ such that $\alpha \gtrsim \frac{1}{\tau} d \log d$, the iterates $(x_k^{\mathrm{CBO}})_{k=0,\dots,K}$ with $x_k^{\mathrm{CBO}} := x_\alpha^{\mathcal{E}}(\widehat{\rho}_k^N)$ follow a stochastically perturbed GD, i.e., they obey*

$$x_k^{\mathrm{CBO}} = x_{k-1}^{\mathrm{CBO}} - \tau \nabla \mathcal{E}(x_{k-1}^{\mathrm{CBO}}) + g_k,$$

where g_k is stochastic noise fulfilling for each $k = 1, \dots, K$ with high probability the quantitative estimate $\|g_k\|_2 = \mathcal{O}\big(|\lambda - 1/\Delta t| + \sigma\sqrt{\Delta t} + \sqrt{\tau/\alpha} + N^{-1/2}\big) + \mathcal{O}(\tau)$.

The fundamental value of such link between CBO and SGD lies in the formerly established fact that CBO is provably globally convergent, hence, on the one side, offering a novel explanation for the success of stochastic relaxations of gradient descent, and, on the other side and contrary to the conventional wisdom for which zero-order methods ought to be inefficient or not to possess generalization abilities, unveiling an intrinsic gradient descent nature of such heuristics. With this we furnish insights that explain how stochastic perturbations of gradient descent overcome energy barriers and reach deep levels of nonconvex functions.

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Modeling and management of gas flows across junctions

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(joint work with Andrea Corli, Ulrich Razafison)

This presentation focuses on the mathematical theory of flows on networks, which finds diverse applications in areas such as vehicular traffic, supply chains, and data networks. Specifically addressing gas flows, the talk delves into the challenges of modeling and mathematically analyzing these flows at different nodes connecting pipes with possibly different sections. This includes the consideration of various devices like junctions, compressors, valves and control valves.

We underline that control and optimization theory plays a crucial role in gas flows on networks, given its wide-ranging applications in various engineering and industrial systems.

A common challenge encountered in these systems is the phenomenon of chattering of the devices, characterized by a rapid switching on and off at critical states. This behavior corresponds to the mathematical concept of coherence of the corresponding coupling Riemann solver (c-Riemann solver). In control theory, a similar phenomenon is represented by a bang-bang controller.

Our primary objective is to establish a general framework for constructing and studying properties of a c-Riemann solver. The solver's properties, including invariant domains, $\mathbf{L}^1_{\mathbf{loc}}$-continuity, consistency, and coherence, are examined and applied to widely used models.

We consider along the pipes an isothermal plug flow as described by the one-dimensional Euler equations, which express conservation of mass and linear momentum in the absence of viscous effects. The Riemann problem, a critical component in solving these equations, is introduced along with the Rankine-Hugoniot conditions and Lax curves. At the nodes we consider c-Riemann solvers, detailing the notation and definitions.

Special attention is given to self-similar c-Riemann solvers, particularly in scenarios where a gas flows through a device, causing a loss of momentum conservation. On the other the conservation of mass at the node is typically ensured, leading to the definition of coupling functions and the corresponding c-Riemann solvers.

The presentation explores then the coherence of c-Riemann solvers, emphasizing its importance as a stability property. Various properties and sufficient conditions for coherence are discussed, providing insights into the analytical and numerical stability of solutions. The talk concludes with a comprehensive overview of valves, showcasing examples and addressing challenges related to coherence, consistency, and $\mathbf{L}^1_{\mathbf{loc}}$-continuity. Theoretical and practical approaches to mitigate or to avoid chattering in valves are presented, offering a glimpse into the ongoing research endeavors in this fascinating and complex domain.

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The turnpike property for mean-field optimal control problems

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(joint work with Martin Gugat, Michael Herty)

In this work we study the turnpike phenomenon for optimal control problems with mean field dynamics that are obtained as the limit $N \to \infty$ of systems governed by a large number N of ordinary differential equations. We show that the optimal control problems with with large time horizons give rise to a turnpike structure of the optimal state and the optimal control. For the proof, we use the fact that the turnpike structure for the problems on the level of ordinary differential equations is preserved under the corresponding mean-field limit.

1. The optimal control problem

From a mathematical point of view, a multiagent control problem is described by minimization of an integral objective functional subject to a constraint that is the complex dynamic depicted by a system of ordinary differential equations (ODE). The formulation of an interacting particle system at a microscopic level requires the study of large-scale systems of agents (or particles) and it requires a considerable effort both from a theoretical and numerical point of view. We may consider a different level of description, that is the derivation of mesoscopic or mean-field approximations of the original dynamic. Here, the density of the particles is obtained as the number of particles tends to infinity. Of particular interest is therefore the design of controls in the mean-field control approaches. In this work, we focus on the turnpike phenomenon for optimal control problems. This topic has been studied recently for example in [1, 2, 3], and it concerns relations between the solutions of dynamic optimal control problems with objective functionals of tracking type and the corresponding static optimal control problems. The turnpike property states that the distance between the dynamic and the static optimal solution is small, in particular, for large time intervals. Hence, it allows to use this information about the structure of the dynamic optimal control to reduce the cost to obtain a numerical approximation by using the static optimal control that can be obtained more easily. In this work we consider the turnpike property with interior decay, which describes the situation that in the interior of the time interval, the distance between the dynamic optimal control/state pair and the corresponding static solution is often very small for sufficiently large time horizons. We are interested in particular on the question whether the turnpike property of a system persists in the limit of infinitely many ODEs and under which conditions such a turnpike property holds true on the mean-field level.

We consider the control of high-dimensional nonlinear dynamics accounting for the evolution of N agents at the microscopic level and, the mean-field dynamics given by a non-local transport equation for the density of particles at position $x \in \mathbb{R}^d$ and time $t \in \mathbb{R}^+$. The initial particle density $\mu^0(x)$ is given and the control action is modeled by an additive term in the partial differential equation

(PDE). More specifically, we consider a PDE of the type

$$\partial_t \mu(t,\, x) + \partial_x\Big(((P * \mu)(t,x) + u(t,x))\; \mu(t,\, x)\Big) = 0, \qquad \mu(a,x) = \mu^0(x), \tag{1}$$

where $*$ denotes the convolution operator, the function P is given, and the real positive number a is the initial time. We consider an optimal control problem for a finite large time horizon, subjected to system (1). The objective function that we want to minimize depends both on the control and the state

$$\mathbf{J}_{(a,\,b)}(\mu,\, u) = \int_a^b f(\mu(t,\, x),\, u(t,x))\, dt,$$

for a given real-valued function f

$$f(\mu,\, u) = \int_{\mathbb{R}^d} (L(x) + \Psi(u(t,x)))\; d\mu(t,\, x), \tag{2}$$

and a time interval $[a, b]$ with $a < b$ real positive numbers. We define the parametric mean-field optimization problem

$$\mathbf{Q}(a,\, b,\, \mu^0) : \min_u \mathbf{J}_{(a,\,b)}(\mu,\, u)$$

subject to (1). We define the optimal value of the mean-field limit problem $\mathbf{Q}(a,\, b,\, \mu^0)$ as $\mathbf{V}(a,\, b,\, \mu^0)$. The existence of solutions for $\mathbf{Q}(a,\, b,\, \mu^0)$ is guaranteed by Theorem 5.1 in [4].

2. The strict dissipativity inequality

We assume that the optimal control problem satisfies a strict dissipativity assumption, i.e. for all $\tau \in [a,\, b]$

$$\begin{aligned} &\int_a^\tau f(\mu(t,x),\, u(t,x))dt \\ &\quad \geq \int_a^\tau \int_{\mathbb{R}^d} \Big(\|x - \psi^{(\sigma)}\|^2 + \|u(t,x) - u^{(\sigma)}\|^2\Big)\; d\mu(t,\, x)\, dt, \end{aligned} \tag{3}$$

where f is the functional in (2).

3. The cheap control condition

For our analysis, a cheap control condition is essential. It requires that the optimal values are bounded in terms of the distance between the initial state and the desired static state. Given $\mathcal{C}_0 > 0$, for all initial times $a \geq 0$, terminal times $b > a$ and initial states $\mu(a,x) = \mu^0(x) \in P_1(\mathbb{R}^d)$, we have

$$\mathbf{V}(a,\, b,\, \mu^0) \leq \mathcal{C}_0 \int_{\mathbb{R}^d} \|x - \psi^{(\sigma)}\|\, d\mu^0(x) \tag{4}$$

with

$$\mathcal{C}_0 = \frac{1}{\beta}\Big(C_L + \beta C_\Psi + 2C_P C_\Psi\Big). \tag{5}$$

4. The turnpike property with interior decay

We present a turnpike property for the optimal control problem $\mathcal{Q}(N,\, a,\, b,\, \psi^0)$ that follows from the dissipativity inequality and the cheap control condition. As the name indicates, this property focuses on the situation that the set where the distance between the optimal dynamic and the optimal static solution is small for large b is located in final part of the time interval $[a, b]$.

Theorem 1. *Let $\lambda \in (0,\, 1)$ be given, and the interval $[a, b]$ with $b > 0$. Consider the quantity*

$$\boldsymbol{A}_*(b) = \int_{a+\lambda(b-a)}^{b} \int_{\mathbb{R}^d} \left(\|x - \psi^{(\sigma)}\|^2 + \|\hat{u}_{(a,b,\mu^0)}(t,x) - u^{(\sigma)}\|^2 \right) d\hat{\mu}_{(a,b,\mu^0)}(t,x)\, dt,$$

where we define as $\hat{\mu}_{(a,b,\mu^0)}(t,x)$ and $\hat{u}_{(a,b,\mu^0)}(t,x)$ the density and control respectively at time t with initial condition $\mu(a,x) = \mu^0(x) = \hat{\mu}_{(a,b,\mu^0)}(a,x)$. Then the optimization problem $\boldsymbol{Q}(a,\, b,\, \mu^0)$ has a turnpike property with interior decay in the sense that

$$\boldsymbol{A}_*(b) \leq \frac{\mathcal{C}_0^2}{\lambda(b-a)} \int_{\mathbb{R}^d} \|x - \psi^{(\sigma)}\|\, d\mu(a,x).$$

where $\mathcal{C}_0$ is as in (5).

Providing suitable assumptions to guarantee the existence of solutions in the mean-field limit, we have proven the turnpike property on a mean-field level. Possible future work includes the numerical simulation and the extension e.g. to the case that the microscopic model is governed by a second-order dynamics.

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Regularity and Control for Conservation Laws with Space Discontinuous Flux

Luca Talamini

(joint work with Fabio Ancona)

We consider the Cauchy problem for the scalar conservation law

$$(1) \qquad \begin{aligned} &\partial_t u(t,x) + \partial_x f(u(t,x),x) = 0, \qquad (t,x) \in [0,T] \times \mathbb{R}; \\ &u(0,x) = u_0(x), \qquad x \in \mathbb{R} \end{aligned}$$

where f is a discontinuous function

$$f(u,x) = \begin{cases} f_l(u), & x < 0, \\ f_r(u), & x > 0 \end{cases}$$

Here f_l, f_r are strictly convex maps. Conservation laws with discontinuous flux have numerous applications; two well known examples are traffic flow with heterogeneous road conditions and two phase flow in porous media.

The discontinuity of the flux naturally leads to the study of infinitely many $\mathbf{L}^1$ contractive semigroups $\mathcal{S}_t^{AB}$, each one associated to particular pair of values, a *connection* (A, B), such that $f_l(A) = f_r(B)$ and $f_l'(A) \leq 0 \leq f_r'(B)$ (this is a particular example of an $\mathbf{L}^1$ *dissipative germ*, see [5]). Connections (A, B) are introduced in [1], and the corresponding solutions are the ones that dissipate the additional generalized Kružkov entropy

$$\eta^{AB} = \begin{cases} |u - A|, & x < 0, \\ |u - B|, & x > 0. \end{cases}$$

In [3] and [4] we are mainly interested in a theoretical analysis of the solutions of (1) associated to a connection (A, B) and we address both control and regularity problems.

Regularity and Exact Controllability. The *attainable set* at time $T > 0$ is defined by

$$\mathcal{A}^{AB}(T) \doteq \left\{ \mathcal{S}_T^{AB} u_0 \;\middle|\; u_0 \in \mathbf{L}^\infty(\mathbb{R}) \right\}. \tag{2}$$

Oleinik Estimates and Regularity. To understand the structure of $\mathcal{A}^{AB}(T)$ we first prove some adapted Oleinik estimates. In order to fix the ideas, assume $x < 0$ and introduce the *auxiliary characteristics lines*:

$$\vartheta_x(t) \doteq \begin{cases} x - (T - t) \cdot f_l'(\omega(x)), & \text{if } \tau(x) \leq t \leq T, \\ (t - \tau(x)) \cdot f_r'\big(\pi_{r,-}^l(\omega(x))\big), & \text{if } 0 \leq t < \tau(x) \end{cases}, \qquad t \in [0, T].$$

These are lines which proceed with the characteristic speed until they are refracted by the interaction with the interface $\{x = 0\}$ at time $\tau(x)$, according to the unique transition map $\pi_{r,-}^l$ such that the Rankine-Hugoniot conditions are satisfied. It should be noted that these are not real characteristic of the solution due to the presence of undercompressive zones at $\{x = 0\}$.

A first point that we make is that if $\omega \in \mathcal{A}^{AB}(T)$, then the lines $\vartheta_x(t)$ are monotone in x, although they are not characteristics. In turn, this yields a bound of the form

$$\partial_x \omega \leq g_T \mathcal{L}^1 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^-) \tag{3}$$

where $g_T : \mathbb{R}^- \to \mathbb{R}^+$ is a continuous function with possibly a non integrable singularity in $x = 0$, i.e. it can happen that $\lim_{x \to 0-} g(x) = +\infty$ and moreover

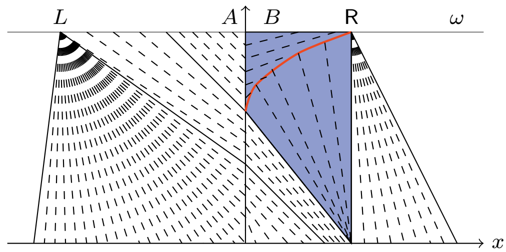

FIGURE 1. A shock in the solution is necessary to create the discontinuity at (T, R).

$g \notin \mathbf{L}^1(\mathbb{R}^-)$ (only when $f_l'(A) \cdot f_r'(B) = 0$). In fact in general $\omega \notin BV(\mathbb{R})$. Nevertheless, a calculation shows that

$$f_l'(x) \cdot g_T(x) \leq \mathcal{O}(1) \cdot \frac{1}{T|x|^{1/3}}$$

which is integrable. Repeating the argument for $x > 0$ yields $f(\omega) \in BV(\mathbb{R})$. From this one also deduces Lipschitz in time regularity for the solution map $t \mapsto u(t, \cdot)$ for uniformly positive times.

Structure of $\mathcal{A}^{AB}(T)$. As a second result, we fully characterize the attainable set $\mathcal{A}^{AB}(T)$ by the above Oleinik-like estimates plus additional geometric constraints. To give a flavour of the geometric constraints, consider a profile $\omega \in \mathcal{A}^{AB}(T)$ as in Figure 1. To create the discontinuity at the point (T, R), a shock must be present in the solution before time T. To construct this shock one needs the shaded area to be big enough, and this translates into a condition like $\omega(\mathsf{R}+) \leq u_{\mathsf{R}}$, for some state u_{R}. The tricky part is to characterize u_{R} and to show that $\omega(\mathsf{R}+) \leq u_{\mathsf{R}}$ is also a necessary condition. In [3] we characterize u_{R} by a duality procedure introducing a natural backward semigroup (see (4)) and we show that $\omega(\mathsf{R}+) \leq u_{\mathsf{R}}$ is also a necessary condition by using an elementary comparison argument.

Initial Data Identification and Backward Semigroup. For $\omega \in \mathbf{L}^\infty(\mathbb{R})$, our goal will be to characterize the set

$$\mathcal{I}_T^{AB}\omega = \Big\{u_0 \in \mathbf{L}^\infty(\mathbb{R}) \;\Big|\; \mathcal{S}_T^{AB} u_0 = \omega\Big\}.$$

In particular, in [4] we prove that $\mathcal{I}_T^{AB}\omega$ is either a singleton or an infinite dimensional cone. A distinctive feature is that this cone can be non-convex: this is in contrast with the classical case of a single conservation law with convex flux, or even a flux smoothly depending on space, in which the corresponding set of initial data is always a convex cone (see [6], [7]).

An important point is the construction of a *backward semigroup operator* $\mathcal{S}_T^{[AB]-}$, through which the vertex of the cone $\mathcal{I}_T^{AB}\omega$ can be characterized as the backward evolution of ω: we prove that the vertex is exactly $\mathcal{S}_T^{[AB]-}\omega$. In addition to helping in the proof, we believe that the backward operator has an independent theoretical interest by itself, therefore we sketch its construction. For a connection (A, B) we define the dual objects:

(i) a *dual flux*, defined by $\overline{f}(u, x) \doteq f(-x, u)$;
(ii) a *dual connection* $(\overline{A}, \overline{B})$, which is uniquely determined by being a connection for the flux $\overline{f}(u, x)$ with $f_r(B) = f_r(\overline{A})$;
(iii) and a *dual semigroup* $\widetilde{\mathcal{S}}_T^{\bar{B}\bar{A}}$, of $\overline{AB}$ entropy solutions for $\overline{f}$.

Then the *backward AB-semigroup* $\mathcal{S}_T^{[AB]-} : \mathbf{L}^\infty \to \mathbf{L}^\infty$ is defined by

$$\mathcal{S}_T^{[AB]-}\omega(x) := \widetilde{\mathcal{S}}_T^{\bar{A}\bar{B}}(\omega(-\cdot))(-x). \tag{4}$$

Finally, using the backward AB semigroup, we prove the equivalence

$$\omega \in \mathcal{A}^{AB}(T) \quad \Longleftrightarrow \quad \mathcal{S}_T^{AB} \circ \mathcal{S}_T^{[AB]-}\omega = \omega$$

providing a more intrinsic, alternative characterization of the attainable set.

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Optimal control of scalar conservation law with particle approximations

Oliver Tse

This talk reports on recent (unpublished) work that introduces a convenient approach to analyzing and numerically solving optimal control problems of the form

$$\inf \mathcal{J}(u) := \Phi(\rho(T)\mathsf{Leb}), \qquad u \in \mathscr{U}(\Omega, M), \tag{OCP}$$

where $[0, \infty) \ni t \mapsto \rho(t) \in L^1(\mathbb{R})$ is a solution to a scalar conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0, \qquad \rho(0) = u, \tag{SCL}$$

and

$$\mathscr{U}(\Omega, M) := \left\{ u \in BV(\Omega) \ : \ 0 \le u \le M, \ \int_\Omega u \, dx = 1 \right\}, \quad M \in (0, +\infty),$$

is the family of admissible controls. Here, the *terminal cost* $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ is a functional on the space of probability measures $\mathcal{P}(\mathbb{R})$ over $\mathbb{R}$, which we assume to be $\mathbb{W}_p$-continuous, i.e. continuous w.r.t. the p-Wasserstein metric, and $f(r) = r\beta(r)$ is the *flux* function, with $\beta : [0,\infty) \to [0,\infty)$ being a Lipschitz non-increasing *mobility* function with $\beta(0) = \beta_{\max} > 0$, $\beta \equiv 0$ on $[M, +\infty)$ and smooth on $[0, M)$.

An example of a terminal cost Φ and mobility β to keep in mind is

$$\mathcal{P}_p(\mathbb{R}) \ni \mu \mapsto \Phi(\mu) = \frac{1}{p}\mathbb{W}_p^p(\mu,\nu), \qquad \nu \in \mathcal{P}_p(\mathbb{R}),$$
$$[0,\infty) \ni r \mapsto \beta(r) = (1-r)^+,$$

where $\mathcal{P}_p(\mathbb{R})$, $p \in [1,\infty]$ is the space of probability measures with finite p-moments.

While the solution theory for (**SCL**) is by now rather mature, the development of optimal control theories for such equations has been slow due to issues regarding the non-differentiability, in L^1, of flows generated by solutions to (**SCL**), thus rendering standard PDE constrained optimal control theory incompatible. Indeed, although the map $L^1 \cap L^\infty(\mathbb{R}) \ni u \mapsto \rho \in \mathcal{C}([0,T]; L^1(\mathbb{R}))$ can be shown to be locally Lipschitz, it is in general not directionally differentiable if $\rho = \rho[u]$ contains shocks [8]. For this reason, new notions of differentiability (eg. shift-differentiability) had to be developed to deal with the issues [1, 2, 3, 6, 7], which subsequently led to rigorous studies of optimal control problems for conservation laws, and for conservation laws on networks used in modeling, i.a., traffic flow, gas networks, and product flow in supply chains. However, these new notions of differentiability are often difficult to work with and this study originates from the desire to address these issues.

Using a *(follow-the-leader) discrete particle approximation* of (**SCL**), one obtains a discrete optimal control problem

$$(\mathbf{OCP}_n) \qquad \inf_{\bar{\mathbf{x}} \in \mathscr{K}^n(\Omega,M)} \mathcal{J}^n(\bar{\mathbf{x}}) := \Phi(\mathfrak{p}^{\mathbf{x}}(T)\mathsf{Leb}), \qquad M \in (0,\infty),$$

where $\mathbf{x} = \mathbf{x}(t)$ is the unique solution of deterministic particle approximation

$$(\mathbf{DPA}_n) \qquad \begin{aligned} \dot{x}_i &= \beta(\rho_i^{\mathbf{x}}), \qquad i = 0,\dots,n-1\,, \\ \dot{x}_n &= \beta_{\max}\,, \end{aligned} \qquad \mathbf{x}(0) = \bar{\mathbf{x}},$$

and the *discrete density* is defined by

$$\mathfrak{p}^{\mathbf{x}} := \sum_{i=0}^{n-1} \rho_i^{\mathbf{x}} \mathbf{1}_{K_i^{\mathbf{x}}}, \qquad K_i^{\mathbf{x}} = [x_i, x_{i+1}),$$

where

$$\rho_i^{\mathbf{x}} := \frac{h}{x_{i+1} - x_i} \qquad i = 0,\dots,n-1, \qquad \rho_n = 0, \qquad h = 1/n.$$

Here,

$$\mathscr{K}^n(\Omega, M) := \Big\{ \mathbf{x} \in \mathbb{R}^{n+1} : M\,(x_{i+1} - x_i) \geq h,\ i = 0,\dots,n-1 \Big\}.$$

denotes the family of admissible controls $\bar{\mathbf{x}}$.

The discrete particle approximation ($\mathbf{DPA}_n$) is known to converge to the Kružkov entropy solution of (**SCL**) [4, 5] and allows one to establish stability estimates w.r.t. the p-Wasserstein metric. In particular, one obtains the following result:

Result 1: Let $\mathbf{x}, \mathbf{y}$ be solutions of ($\mathbf{DPA}_n$) with initial data $\bar{\mathbf{x}}, \bar{\mathbf{y}} \in \mathscr{K}^n(\Omega, M)$, respectively. Then,

$$\mathbb{W}_p(\mathfrak{p}^{\mathbf{y}}(t)\mathsf{Leb}, \mathfrak{p}^{\mathbf{x}}(t)\mathsf{Leb}) \leq \mathbb{W}_\infty(\mathfrak{p}^{\bar{\mathbf{y}}}(t)\mathsf{Leb}, \mathfrak{p}^{\bar{\mathbf{x}}}(t)\mathsf{Leb}) \qquad \forall\, t \geq 0.$$

The convergence of ($\mathbf{DPA}_n$) consequently provides the continuous counterpart of the stability estimate:

$$\mathbb{W}_p(S_t(w)\mathsf{Leb}, S_t(u)\mathsf{Leb}) \leq \mathbb{W}_\infty(w\,\mathsf{Leb}, u\,\mathsf{Leb}) \qquad \forall\, t \geq 0,$$

where $u \mapsto S_t(u)$ is the solution operator associated to (**SCL**).

Moreover, the discrete optimal control problem ($\mathbf{OCP}_n$) is shown to admit a minimizer $\bar{\mathbf{x}}^n \in \mathscr{K}^n(\Omega, M)$ for each $n \geq 1$ such that $\mathfrak{p}^{\bar{\mathbf{x}}^n} \in \mathscr{U}(\Omega, M)$. In addition, the sequence $(\mathfrak{p}^{\bar{\mathbf{x}}^n})_{n\geq 1}$ admits an accumulation point $u^* \in \mathscr{U}(\Omega, M)$ that turns out to be a minimizer of the continuous optimal control problem (**OCP**), which is a consequence of the following Γ-convergence result, justifying the role of the deterministic particle approximation ($\mathbf{DPA}_n$) as a surrogate model for the optimal control problem:

Result 2: The family of functionals $\{\widehat{\mathcal{J}}^n\}_{n\geq 1}$ defined by

$$\widehat{\mathcal{J}}^n(u) := \begin{cases} \mathcal{J}^n(\bar{\mathbf{x}}) & \text{if } u = \mathfrak{p}^{\bar{\mathbf{x}}},\ \bar{\mathbf{x}} \in \mathscr{K}^n(\Omega, M), \\ +\infty & \text{otherwise,} \end{cases} \qquad n \geq 1,$$

is (weakly) equi-coercive and $\Gamma_{\text{weak-}L^1}$-converges to $\mathcal{J}$.

In practice, one would like to numerically compute a minimizer of (**OCP**). The previous result suggests that the numerical solution of ($\mathbf{OCP}_n$) can be used as a proxy for obtaining a minimizer of (**OCP**). One way of obtaining a minimizer of ($\mathbf{OCP}_n$) is utilizing an adjoint-based approach, which can be easily executed due to the finite-dimensional nature of the problem.

The last part of the talk was devoted to an initial attempt to link the discrete adjoint with its continuous counterpart and highlight the issues encountered along the way. Under very restrictive assumptions on the behavior of the state ρ, one obtains an equation governing the continuous adjoint equation. This leads to the following conjecture for general states:

Conjecture: Let $u \in \mathscr{U}(\Omega, M)$ and ρ be its corresponding state satisfying (**SCL**). Then, the associated adjoint equation for the adjoint state η reads

$$\partial_t(\eta\rho) + \partial_x(\eta P(\rho)) = 0,$$

subjected to an appropriate terminal condition, where $P(s) = sf'(s) - f(s)$.

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Neural Network approaches for High-dimensional Optimal Control Problems

Deepanshu Verma

Optimal Control (OC) problems are pervasive in various fields, from finance to robotics, aiming to find a control policy minimizing a defined control objective functional. Traditionally, Dynamic Programming is employed to solve these problems, seeking the value function that assigns each system state the optimal cost-to-go and satisfies the Hamilton-Jacobi-Bellman (HJB) equation.

The challenge lies in solving the HJB PDE for the value function. Traditional numerical schemes suffer from the Curse-of- Dimensionality (CoD), where computational complexity increases exponentially with problem dimension. In order to mitigate the CoD, we not only need to alleviate the need for spatial discretization but also to effectively parameterize the value function in high-dimensions. This is achieved by parameterizing the value function using NNs.

In [1, 2], we present *neural-HJB* approach for solving high-dimensional OC (stochastic and deterministic) problems informed by control theory. Our method leverages Pontryagin's maximum principle to guide system sampling and obtain the optimal control in real-time from the value function via feedback form. Notably, our learning is *unsupervised*, requiring no prior data to learn value function. In comparison with Reinforcement Learning, a popular unsupervised learning approach for approximate policies. Our neural-HJB approach demonstrates improved accuracy, reduced time-to-solution, and fewer PDE solves compared to RL while approximating superior policies.

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Kinetic Modelling and Control of Multiagent Systems with Missing Information

Mattia Zanella

Kinetic equations play a leading role in the modelling of large systems of interacting particles/agents with a recognized effectiveness in describing real world phenomena ranging from plasma physics to multi-agent dynamics. The derivation of these models has often to deal with physical, or even social, forces that are deduced empirically and of which we have limited information [1]. To produce realistic descriptions of the underlying systems, it is of paramount importance to quantify the propagation of uncertain quantities across the scales.

We concentrate on the interplay of this class of models with collective phenomena in life and social sciences, where the assessment of uncertainties in data assimilation is crucial to design efficient interventions. Furthermore, to discuss the mathematical interface of this class of models with available data, we derive the evolution of observable quantities based on suitable macroscopic limits of classical kinetic theory [2, 3]. Finally, we analyze how the introduction of robust control strategies leads to the damping of the uncertainties characterizing the system at the macroscopic level [4].

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Sidewise control

Enrique Zuazua

Inspired on [1] and [4], we explored the lateral or sidewise control properties of 1-dimensional waves, a concept applicable to any spatial dimension. Drawing parallels with classical control and inverse problems in wave propagation, our focus lies in influencing the behavior of waves on a portion of the domain boundary through localized control actions on a distinct subset of the boundary. Unlike classical problems, our objective is not the control of wave dynamics within the domain but rather their boundary traces, constituting a goal-oriented controllability problem.

In the one-dimensional scenario, the typical aim is to govern the trace on one end of the string by means of a control action at the other end.

Utilizing duality, we reformulated the lateral control problem, into a pertinent observability inequality. This is applicable in any space dimension. This inequality involves estimating non-homogeneous boundary traces of waves on a specific subset of the boundary using measurements acquired on a different one. These inequalities pose novel challenges that diverge from traditional techniques in the field, such as Carleman inequalities, non-harmonic Fourier series, microlocal analysis, and multipliers.

We introduced a distinctive one-dimensional solution method grounded in sidewise energy propagation estimates, leading to a complete and precise solution. This methodology extends to address 1-dimensional wave equations featuring BV-variable coefficients. By combining it with fixed-point techniques as in [5], this allows handling 1-dimensional semilinear wave equations.

In the multi-dimensional scenario, building upon [3], we demonstrated how Fourier series decomposition facilitates addressing the problem, resulting in lateral controllability properties in rectangular domains. Controls are applied on one side, influencing dynamics on the opposite side. However, the attained results exhibit an infinite loss (in Sobolev terms) on observed norms and controlled sources. To deal with more general geometries, we introduced a geometric control condition of microlocal nature, ensuring control towards targets defined on concave subsets of the boundary of suitable domains.

Finally, we delved into the parabolic counterpart, introducing a novel transmutation formula and establishing a connection between wave and heat equations, as presented in [2].

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Mathematical Logic: Proof Theory, Constructive Mathematics

Organized by
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Rosalie Iemhoff, Utrecht
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Michael Rathjen, Leeds

12 November – 17 November 2023

Abstract. The Workshop 'Mathematical Logic: Proof Theory, Constructive Mathematics' focused on proof-theoretic research on the foundations of mathematics, on the extraction of explicit computational content from given proofs in core areas of ordinary mathematics using proof-theoretic methods as well as on topics in proof complexity. The workshop contributed to the following research strands:

- Interactions between foundations and applications.
- Proof mining.
- Constructive and semi-constructive reasoning.
- Proof theory and theoretical computer science.
- Structural proof theory.

Mathematics Subject Classification (2020): 03Fxx.

Introduction by the Organizers

The workshop *Mathematical Logic: Proof Theory, Constructive Mathematics* was held November 12-17, 2023 in a hybrid format due to the Corona pandemic. It had 46 participants at the Oberwolfach Institute and 4 virtual participants who were connected via ZOOM. The program consisted of 23 talks of 40 minutes (2 of which were given via ZOOM).

The purpose of the workshop was

To promote the interaction between the foundations of mathematics and applications to mathematics as done for example in the field of 'proof mining'. M. Neri,

P. Oliva and T. Powell talked on the very recent novel development of applying proof-theoretic proof mining techniques in the context of probability theory. N. Pischke extended the framework of previously existing logical metatheorems for proof mining to include concepts such as dual and bidual spaces of a Banach space, gradients of uniformly Fréchet differentiable convex functions and their Fenchel conjugates and, finally, Bregman distances which allows one to treat for the first time important algorithms in optimization which compute zeros of maximally monotone operators in Banach spaces. P. Pinto used a concrete proof mining (of a celebrated theorem of S. Reich) due to Kohlenbach and Sipoş to generalize Reich's result (together with a quantitative analysis) to a newly defined class of uniformly smooth and convex hyperbolic spaces (which covers CAT(0)-spaces as a special case). L. Leuştean gave a survey on recently extracted effective rates of asymptotic regularity in optimization with a special focus on case where linear rates can be obtained using proof-mining methods. This topic was further extended in the talk by H. Cheval who, moreover, discussed the potential use of proof assistants such as LEAN in partially automatizing parts of the mining process.

A. Sipoş gave a quantitative treatment of the class of super strongly nonexpansive mappings which was recently introduced by Liu et al. as a counterpart to maximally monotone and uniformly monotone operators. This leads to a quantitative inconsistent feasibility result which was even qualitatively new. Talks on the interplay between foundational research in the context of reverse mathematics (RM) and core mathematics where given by J. Aguilera, who spoke about recent results on the reverse mathematics of systems of determinacy provable in second-order arithmetic and on some which go beyond it, and by S. Sanders, who studied, in particular, the status of various weak forms of continuity in the context of higher order reverse mathematics. V. Brattka's talk discussed a number of uniform dichotomies for problems in the Weihrauch lattice. M. Baaz showed that a Skolemization method due to P. Andrews - and used prominently in the context of resolution - can have a non-elementary speed up over the standard Skolemization method. R. Kahle and I. Oitavem talked about a problem in the proof complexity of a Hilbert-type system for propositional logic and for combinatorial logic. S. Negri developed a natural deduction calculus for Gurevich logic and related it to a previously proposed cut-free sequent calculus to prove a normalization result.

To explore connections between proof theory, constructive formal systems and computer science. M. Fujiwara's talk investigated the formula classes U_k, E_k, introduced in 2004 by Akama et al., from the point of view of the standard transformation procedure for prenex normalization showing that they are exactly the classes of formulas induced by Σ_k and Π_k resp. via these transformations. M.E. Maietti proved that the formal system for the 'Minimalist Foundation for Constructive Mathematics', introduced in 2005 by herself and G. Sambin, is equiconsistent with its extension by the law of the excluded middle. I. van der Giessen presented an intuitionistic version of Gödel-Löb logic that includes both modalities Box and Diamond, and allows for a Gentzen-Gödel negative translation of its classical counterpart. P. Schuster (jww G. Fellin) talked about a generalization of Glivenko's theorem to an arbitrary nucleus and to an inductively generated abstract consequence relation. M. Zorzi presented extensional proof systems for modal logics, focussing on a "geometric" approach that entails a notion of position.

To investigate further the connections between logic and computational complexity. E. Jeřábek's talk addressed the question of characterizing and axiomatizing ordered rings that are existential integer parts of real-closed exponential fields, and especially the first-order theory of such rings. P. Pudlák discussed implicit proof systems for propositional logic, and the use of iterated implicit proof systems to capture self-consistency statements. N. Thapen presented first-order theories of bounded arithmetic for semi-algebraic reasoning about polynomial inequalities, such as used by the Sum-of-Squares (SoS) proof system. M. Müller presented a proof of the independence of circuit-lower bounds for nondeterministic exponential time from theories of bounded arithmetic.

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Workshop: Mathematical Logic: Proof Theory, Constructive Mathematics

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Abstracts

On a shortest proof of $\varphi \to \varphi$

Reinhard Kahle, Isabel Oitavem
(joint work with Paulo Guilherme Santos)

1. The standard proof of $\varphi \to \varphi$ in a Hilbert-style calculus

Let us consider the *Pure Positive Implication Propositional Calculus* in a Hilbert-style calculus, based on Frege's axioms for implication [2]:

$$\vdash \varphi \to (\psi \to \varphi) \qquad \text{(F1)}$$

$$\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \qquad \text{(F2)}$$

The only rule is Modus ponens (MP).

Although not an axiom, $\varphi \to \varphi$ is a derivable formula:

Theorem. $\varphi \to \varphi$ is derivable, for every formula φ.

Proof. Consider the derivation D_1:

1	$\vdash (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$	(F2)
2	$\vdash \varphi \to ((\varphi \to \varphi) \to \varphi)$	(F1)
3	$\vdash (\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$	MP[1,2]
4	$\vdash \varphi \to (\varphi \to \varphi)$	(F1)
5	$\vdash \varphi \to \varphi$	MP[3,4]

□

We call D_1 the *standard proof* of $\varphi \to \varphi$ (for the given axiomatization).

Some historical notes concerning the discovery of this proof can be found in [5].

Is D_1 the shortest proof of $\varphi \to \varphi$?

Of course, this question makes sense only, when the formal system is fixed, and when an appropriate measure of length is defined.

We consider the formal system described above, and we focus on the measure $\mathsf{M1}$ which counts the lines of the proof. For instance $\mathsf{M1}(D_1) = 5$.

It is an easy combinatorial exercise to see that there is no shorter proof of $\varphi \to \varphi$, for an arbitrary formula φ, in this formal system.

However, could there exist shorter proofs than D_1 for special instances of φ?

2. The special case $(\varphi \to \varphi) \to (\varphi \to \varphi)$

Consider D_2:

1	$\vdash (\varphi \to (\varphi \to \varphi)) \to ((\varphi \to \varphi) \to (\varphi \to \varphi))$	(F2)
2	$\vdash \varphi \to (\varphi \to \varphi)$	(F1)
3	$\vdash (\varphi \to \varphi) \to (\varphi \to \varphi)$	MP[1,2]

For the length we have, $\mathsf{M1}(D_2) = 3 < \mathsf{M1}(D_1)$.

Is this the only "special case"? We answer this question via *Combinatory Logic*.

3. Combinatory Logic and the Curry-Howard Correspondence

Schönfinkel [6] and Curry [1] developed the framework of Combinatory Logic which turned out to be a "computational counterpart" of the Hilbert-style calculus with Frege's axioms (F1) and (F2) for implication.

Combinatory terms are build inductively from the two constants, K and S, variables $(X, Y, \cdots)$, and closure under application: If X and Y are combinatory terms, then the application $(X \cdot Y)$ is also a combinatory term. As usual, the dot for application is often suppressed; and one uses left associativity to reduce parentheses.

Combinatory terms serve as a kind of programming language, when one considers the following equalities:

- $\mathsf{K}\,X\,Y = X$;
- $\mathsf{S}\,X\,Y\,Z = X\,Z\,(Y\,Z)$.

The combinators can be *typed* by formulas, such that the combinatory terms represent proofs of these formulas:

- $\mathsf{K}^{\varphi\to(\psi\to\varphi)}$ for the axiom (F1);
- $\mathsf{S}^{(\varphi\to(\psi\to\chi))\to((\varphi\to\psi)\to(\varphi\to\chi))}$ for the axiom (F2);
- Application relates to (an application of) *modus ponens*: $X^{\varphi\to\psi}Y^{\varphi}$ has type ψ.

In this way, the derivation D_1 can be written by the following (typed) combinatory term: $\mathsf{S}^{(\varphi\to((\varphi\to\varphi)\to\varphi))\to((\varphi\to(\varphi\to\varphi))\to(\varphi\to\varphi))}\,\mathsf{K}^{\varphi\to((\varphi\to\varphi)\to\varphi)}\,\mathsf{K}^{\varphi\to(\varphi\to\varphi)}$. This term has, indeed, type $\varphi \to \varphi$.

When taking the application dot into account, we have also a one-to-one correspondence between the number of lines of the proof and the length of the combinatorial term: $\mathsf{M1}(D_1) = 5 = lh(S \cdot K \cdot K)$.

4. Identity Combinators and Fixed Points

The identity combinator I with $\mathsf{I}\,X = X$ can be defined by $\mathsf{I} = \mathsf{S}\,\mathsf{K}\,\mathsf{K}$.

According to the Curry-Howard Correspondence, any identity combinator, i.e., a combinator M with $M\,X = X$, for all X, will give rise to a proof of (an instance of) $\varphi \to \varphi$. But it does not need to be an identity combinator.

Definition. Let M be a closed combinatory term of type $\varphi_0 \to (\cdots \to (\varphi_n \to \psi)\cdots)$, $n \geq 0$. X is a *fixed point*, if for all terms $Y_1, \ldots Y_n$:

$$M\,X\,Y_1\,\cdots\,Y_n = X\,Y_1\,\cdots\,Y_n.$$

Theorem. Let M be a closed combinatory term.

- If M has a fixed point, then M corresponds to a proof of an instance of $\varphi \to \varphi$.
- Moreover, the number of lines of that proof is $lh(M)$.

Considering only combinatory terms of length less than or equal to 5, we obtain the following *special cases*. For terms starting with K:

Comb. M	F.P.	Proof of $\varphi \to \varphi$ for φ being
$\mathsf{K\,K}$	K	$\varphi \to (\psi \to \varphi)$
$\mathsf{K\,S}$	S	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
$\mathsf{K\,(K\,K)}$	$\mathsf{K\,K}$	$\varphi \to (\psi \to (\chi \to \psi))$
$\mathsf{K\,(K\,S)}$	$\mathsf{K\,S}$	$\varphi \to ((\psi \to (\chi \to \tau)) \to ((\psi \to \chi) \to (\psi \to \tau)))$
$\mathsf{K\,(S\,K)}$	$\mathsf{S\,K}$	$(\varphi \to \psi) \to (\varphi \to \varphi)$
$\mathsf{K\,(S\,S)}$	$\mathsf{S\,S}$	$((\varphi \to (\psi \to \chi)) \to (\varphi \to \psi)) \to$ $((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$

For terms starting with S:

Combinator M	Fixed point	Proof of $\varphi \to \varphi$ for φ being
$\mathsf{S\,K}$	I	$\varphi \to \varphi$
$\mathsf{S\,S}$	*	$(\varphi \to (\psi \to \psi)) \to (\varphi \to \psi)$
$\mathsf{S\,K\,K}$	X	φ
$\mathsf{S\,K\,S}$	X	$\varphi \to (\psi \to \chi)$
$\mathsf{S\,(S\,K)}$	X	$(\varphi \to \psi) \to \varphi$

5. Further considerations

- $\mathsf{S\,S}$ does not has a fixed point in the sense defined above; the analysis of this case gives, indeed, reason for further considerations.
- Hindley [3] provided a typing algorithm for combinators. From this algorithm one obtains a more general type of $\mathsf{S\,K\,K}$ which is of interest when considering other measures (which, for instance, take the length of formulas in a proof into account).
- The present study is some ground work for more detailed investigations on *Hilbert's 24th problem* [4]. This problem, preserved in Hilbert's mathematical notebook, asks for criteria of simplicity of proofs, proposing, in particular, to take the length of proofs into account.

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Consistency, implicit proofs, and cut-elimination

Pavel Pudlák

Two computable operators have been conjectured to be jumps.

Definition 1 (consistency jump)**.** *For a proof system P, define $con(P)$,* the consistency jump, *to be the strongest proof systems Q such that $S^1_2 + Con(S^1_2 + Rfn(P))$ proves the reflection principle for Q.*

The second operator is based on implicit proofs.

Definition 2 (Krajíček [4], implicit proofs)**.** *Let P, Q be proof systems; we define a proof system $[P, Q]$ as follows. A $[P, Q]$-proof of ϕ is a pair (π, c), where*

- *c is a circuit that defines bits of a (possibly exponential size) Q-proof of ϕ,*[1] *and*
- *π is a P-proof of the fact above.*

We conjecture that $imp(P) := [P, P]$ is a jump.

We want to find connection between the two operators and believe that it could be proved by showing that cut-elimination produces implicit proofs in the sense of the above definition. The fact that elimination of one level of cuts produces exponential size proofs that have succinct representations has already been observed before, [1, 2]. The problem is, however, that we still do not fully understand the concept of an implicit proof. Part of the reason is that it is not a robust concept. For instance Khaniki proved under plausible complexity-theoretical assumption that there are two proof systems P and Q such that $P \equiv_p Q$, but $imp(P) \not\equiv imp(Q)$, (cf. [3]). Therefore we decided to first study a restricted version of implicit proofs.

A restricted kind of implicit proofs is defined by requiring that the circuit computes *formulas*, not single bits, see [4]. Thus the formulas must be of polynomial size (in the size of the implicit proof). Such proof systems are denoted by $[P, Q]^m$. (This operation is only defined when Q is a proof system based on formulas.) In our modification circuits compute *sequents*, because we want to use the sequent calculus. We will consider the following proof systems (cf. [5]):

- SF denotes the sequent calculus for propositional logic augmented with the *substitution rule*:

$$\frac{\Gamma \longrightarrow \Delta}{\Gamma[x/\alpha] \longrightarrow \Delta[x/\alpha]},$$

  where α is a Boolean formula.

[1] If c has n input bits, then it can define bits of a string of length 2^n.

- G denotes the *the quantified propositional sequent calculus.* E.g., the $\exists$-right rule is
$$\frac{\Gamma \longrightarrow \Delta, \phi(\alpha)}{\Gamma \longrightarrow \Delta, \exists x.\phi(x)},$$
where α is a Boolean formula.
- For $i \geq 1$, G_i denotes the Σ_i^q fragment of G.

Theorem 1.

(1) $[SF, SF]^m \equiv_p G_1$,

(2) $[SF, G_i]^m \equiv_p G_{i+1}$ *for* $i \geq 1$.

The more technical part of the proofs are polynomial simulations $[SF, SF]^m \geq_p G_1$ and $[SF, G_i]^m \geq_p G_{i+1}$. They are based on eliminating cuts with the highest quantifier complexity and showing that this produces implicit proofs.

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New applications of proof theory: Greedy algorithms, probability, and proof assistants

Thomas Powell

I will give a brief and high-level overview of some new research projects that I believe have the potential to yield exciting results over the next few years.

The first revolves around greedy approximation schemes in Hilbert and Banach spaces. This is an area replete with convergence results, proofs of which are often nonconstructive and hinge on geometric properties of the underlying space, such as uniform smoothness. I will present an initial case study and argue that the area in general may form a fertile ground for applied proof theory, with particular relevance at the moment given its connections to learning algorithms.

I will also present an overview of some ongoing work in probability theory (joint with Morenikeji Neri). My focus will be on our efforts to understand some of the basic notions of probabilistic convergence and the relationships between them from a computational perspective. Here things seem to get particularly interesting where uniform integrability plays a role, and implications between convergence statements seem computationally subtle. A much broader open question is how to formalise the underlying proofs in a suitable abstract system.

Finally, I will outline some broad goals in formalised mathematics and automated reasoning, which are relevant to both of the above themes and applied proof theory in general.

First-Order Reasoning and Efficient Semi-Algebraic Proofs

Neil Thapen

(joint work with Fedor Part, Iddo Tzameret)

Semi-algebraic proof systems such as sum-of-squares (SoS) [5] have attracted a lot of attention due to their relation to approximation algorithms: constant degree semi-algebraic proofs lead to conjecturally optimal polynomial-time approximation algorithms for important NP-hard optimization problems [1]. Motivated by the need to allow a more streamlined and uniform framework for working with SoS proofs than the restrictive propositional level, we initiate a systematic first-order logical investigation into the kinds of reasoning possible in algebraic and semi-algebraic proof systems. Specifically, we develop first-order theories that capture in a precise manner constant degree algebraic and semi-algebraic proof systems: every statement of a certain form that is provable in our theories translates into a family of constant degree polynomial calculus or SoS refutations, respectively; and using a reflection principle, the converse also holds.

This places algebraic and semi-algebraic proof systems in the established framework of bounded arithmetic, while providing theories corresponding to systems that vary quite substantially from the usual propositional-logic ones [2, 4, 6].

We give examples of how our semi-algebraic theory proves statements such as the pigeonhole principle, we provide a separation between algebraic and semi-algebraic theories, and we describe initial attempts to go beyond these theories by introducing extensions that use the inequality symbol, identifying along the way which extensions lead outside the scope of constant degree SoS. Moreover, we prove new results for propositional proofs, and specifically extend Berkholz's [3] dynamic-by-static simulation of polynomial calculus (PC) by SoS to PC with the radical rule.

An earlier version of this work appeared as [7].

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Quantitative Probability from a Logician's Perspective

Morenikeji Neri

Over the last few decades, proof mining has enjoyed many successes in numerous areas of mathematics, mostly within analysis. To date, there have only been a handful of papers that extend proof mining to probability and measure theory. On the other hand, probability theorists have been informally extracting quantitative bounds for many years, in particular, obtaining rates for probabilistic convergence theorems.

In this talk, I shall first discuss some results from quantitative probability theory obtained by logicians and probability theorists, giving an overview of the relevant notions from probability theory. I shall then present my own ongoing work in obtaining quantitative bounds from strong law of large numbers type results, that not only build on the existing body of work in the proof mining of probability theory literature but also extend work done by probability theorists obtaining quantitative results. Lastly, I shall look towards the future and introduce some questions in quantitative probability theory that one could potentially answer using ideas from the proof mining program.

On the consistency of circuit lower bounds for non-deterministic time

Moritz Müller
(joint work with Albert Atserias, Sam Buss)

We prove the first unconditional consistency result for superpolynomial circuit lower bounds with a relatively strong theory of bounded arithmetic. Namely, we show that the theory V_2^0 is consistent with the conjecture that NEXP $\not\subseteq$ P/poly, i.e., some problem that is solvable in non-deterministic exponential time does not have polynomial size circuits. We suggest this is the best currently available evidence for the truth of the conjecture. The same techniques establish the same results with NEXP replaced by the class of problems decidable in non-deterministic barely superpolynomial time such as NTIME($n^{O(\log\log\log n)}$). Additionally, we establish a magnification result on the hardness of proving circuit lower bounds.

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Structural proof theory for logics of strong negation

Sara Negri
(joint work with Norihiro Kamide)

Gurevich logic is an extended constructive three-valued logic obtained from intuitionistic logic by adding a connective $\sim$ of *strong negation*, with the following axiom schemata, where $\neg$ is intuitionistic negation:[1]

(1) $\sim\sim A \supset\subset A$,
(2) $\sim\neg A \supset\subset A$,
(3) $\sim A \supset \neg A$,
(4) $\sim(A \wedge B) \supset\subset \sim A \vee \sim B$,
(5) $\sim(A \vee B) \supset\subset \sim A \wedge \sim B$,
(6) $\sim(A \supset B) \supset\subset A \wedge \sim B$.

Nelson logic [11], also known as Nelson's constructive three-valued logic N3, is the intuitionistic negation-less fragment of Gurevich logic.

The primary formal difficulty in developing a natural deduction system for Gurevich logic, and more generally for logics that employ strong negation, lies in the requirement of having rules for $\neg$ and $\sim$ without $\bot$. This is solved using the rules of *explosion*, of $\neg$-*introduction*, and of *excluded middle*:[2]

$$\frac{\neg A \quad A}{C}\,\text{Exp} \qquad \frac{\begin{array}{cc}[A] & [A]\\ \vdots & \vdots \\ C & \neg C\end{array}}{\neg A}\,\neg\text{I} \qquad \frac{\begin{array}{cc}[\neg A] & [A]\\ \vdots & \vdots \\ C & C\end{array}}{C}\,\text{Em}$$

The natural deduction system for intuitionistic logic $\mathrm{NI}^\star$ is obtained replacing the rule of *ex falso quodlibet* of NI with the rule of explosion and adding rule $\neg$I, and the natural deduction system for classical logic $\mathrm{NK}^\star$ is obtained from $\mathrm{NI}^\star$ by adding the rule of excluded middle. Next, the natural deduction system for Gurevich logic NG is obtained from $\mathrm{NI}^\star$ by adding the following rules for strong negation:

$$\frac{\sim A \quad A}{C}\,{\sim}\text{Exp}$$

$$\frac{A}{\sim\sim A}\,{\sim\sim}\text{I} \qquad \frac{\sim\sim A}{A}\,{\sim\sim}\text{E} \qquad \frac{A}{\sim\neg A}\,{\sim\neg}\text{I} \qquad \frac{\sim\neg A}{A}\,{\sim\neg}\text{E}$$

$$\frac{A \quad \sim B}{\sim(A \supset B)}\,{\sim\supset}\text{I} \qquad \frac{\sim(A \supset B)}{A}\,{\sim\supset}\text{E}_1 \qquad \frac{\sim(A \supset B)}{\sim B}\,{\sim\supset}\text{E}_2$$

[1] Gurevich logic can also be obtained by adding intuitionistic negation to Nelson logic (N3) [1, 11], which, in turn, is obtained by adding the principle of explosion to Nelson's paraconsistent four-valued logic, N4 [1, 11]. In the original study by Gurevich [4], completeness with respect to three-valued Kripke semantics, embedding into intuitionistic logic, functional completeness, and duality theorems for Gurevich logic were proven using a Hilbert-style axiomatic system. Cut-free Gentzen-style sequent calculi for Gurevich logic have been introduced in [4, 6].

[2] Systems with primitive negation for intuitionistic logic have a long history, dating back to the the 1930s with the work of Heyting and Gentzen [5, 12].

$$\frac{\sim A}{\sim (A \wedge B)}\,{\sim}{\wedge}\mathrm{I}_1 \quad \frac{\sim B}{\sim (A \wedge B)}\,{\sim}{\wedge}\mathrm{I}_2 \quad \frac{\sim (A \wedge B) \quad \begin{matrix}[\sim A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[\sim B]\\ \vdots \\ C\end{matrix}}{C}\,{\sim}{\wedge}\mathrm{E}$$

$$\frac{\sim A \quad \sim B}{\sim (A \vee B)}\,{\sim}{\vee}\mathrm{I} \quad \frac{\sim (A \vee B)}{\sim A}\,{\sim}{\vee}\mathrm{E}_1 \quad \frac{\sim (A \vee B)}{\sim B}\,{\sim}{\vee}\mathrm{E}_2$$

The natural deduction system NN for Nelson logic N3 is obtained from NG by deleting Exp, $\neg$I, $\sim\neg$I, and $\sim\neg$E (i.e., NN is the $\neg$-less fragment of NG).

Equivalence between these natural deduction systems and correspondence with previously proposed cut-free Gentzen-style sequent calculi are proven and used to obtain normalization of the corresponding natural deduction systems. The normalization theorem for NK* cannot be obtained using the equivalence with LK, and therefore the single-succedent sequent calculus for classical logic LC originally introduced by von Plato in [13] (see also [9]) is used. In particular, an equivalence is established between NG and the previously proposed cut-free Gentzen-style sequent calculus LG for Gurevich logic, and this result is used to prove normalization for NG, and, as a bonus, also normalization for NN and NI*.

Next, G3-style sequent calculi are introduced for these logics and Avron and De-Omori logic. G3-style sequent calculi are sequent calculi with *all* structural rules admissible, not only cut but also weakening and contraction, and with all or most of the rules invertible. They are especially suited for root-first proof search and therefore useful for automated deduction, but also for meta-theoretical purposes because of their analyticity [9, 10]

First, the G3-style intuitionistic calculus with primitive negation G3ip$^\neg$ is obtained from G3ip by admitting an empty succedent and replacing the initial sequents $\bot, \Gamma \Rightarrow C$ for the falsity constant $\bot$ with the following rules for $\neg$:

$$\frac{\neg A, \Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow}\,\neg\mathrm{L} \quad \frac{A, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A}\,\neg\mathrm{R}$$

Then, the G3-style sequent calculus for Gurevich logic G3gv is obtained from G3ip$^\neg$ by adding the following initial sequents and rules for $\sim$, where γ represents a formula or the empty multiset:

$$\sim P, \Gamma \Rightarrow \sim P \;\; \mathrm{init}_2 \qquad \sim P, P, \Gamma \Rightarrow \;\; \mathrm{init}_3$$

$$\frac{A, \Gamma \Rightarrow \gamma}{\sim\sim A, \Gamma \Rightarrow \gamma}\,{\sim}{\sim}\mathrm{L} \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \sim\sim A}\,{\sim}{\sim}\mathrm{R}$$

$$\frac{A, \sim B, \Gamma \Rightarrow \gamma}{\sim(A \supset B), \Gamma \Rightarrow \gamma}\,{\sim}{\supset}\mathrm{L} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \supset B)}\,{\sim}{\supset}\mathrm{R}$$

$$\frac{\sim A, \Gamma \Rightarrow \gamma \quad \sim B, \Gamma \Rightarrow \gamma}{\sim(A \wedge B), \Gamma \Rightarrow \gamma}\,{\sim}{\wedge}\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \sim A}{\Gamma \Rightarrow \sim(A \wedge B)}\,{\sim}{\wedge}\mathrm{R}_1 \quad \frac{\Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \wedge B)}\,{\sim}{\wedge}\mathrm{R}_2$$

$$\frac{\sim A, \sim B, \Gamma \Rightarrow \gamma}{\sim(A \vee B), \Gamma \Rightarrow \gamma}\,{\sim}{\vee}\mathrm{L} \quad \frac{\Gamma \Rightarrow \sim A \quad \Gamma \Rightarrow \sim B}{\Gamma \Rightarrow \sim(A \vee B)}\,{\sim}{\vee}\mathrm{R}$$

$$\frac{A, \Gamma \Rightarrow \gamma}{\sim\neg A, \Gamma \Rightarrow \gamma} \sim\neg\mathrm{L} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \sim\neg A} \sim\neg\mathrm{R}$$

The G3-style sequent calculus for Nelson N3, G3n3, is obtained from G3gv by deleting the rules $\neg$L, $\neg$R, $\sim\neg$L, and $\sim\neg$R (i.e., as the $\neg$-less part of G3gv, and the calculus for Nelson N4, G3n4, is obtained from G3n3 by deleting init3.

Structural properties including cut elimination are established for these calculi and a Glivenko theorem for embedding G3gv into G3ip$^{\neg}$ is shown, providing at the same time an indirect alternative proof of the cut-elimination theorem for G3gv.

The G3-style sequent calculus G3cp$^{\neg}_{\sim}$ is obtained from the intuitionistic calculus turning it to a multisuccedent system. In G3cp$^{\neg}_{\sim}$, $\neg$ is equivalent to $\sim$. Thus, G3cp$^{\neg}_{\sim}$ is a redundant G3-style sequent calculus for classical propositional logic, however, the interest in this calculus lies in the fact that it provides a platform to obtain G3 calculi for a wealth of logical systems, already studied in the literature, that lacked a G3-style proof system: it is used to define G3-style sequent calculi for classical versions of N3 and N4, for Avron logic [2], and for De–Omori logic (the extension of Belnap–Dunn logic with classical negation) [3].

Finally, the explicit use of $\sim$ in G3cp$^{\neg}_{\sim}$ as an auxiliary connective makes it possible to prove a Glivenko theorem for embedding G3cp$^{\neg}_{\sim}$ into G3gv.

For the details, cf. [7, 8].

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Determinacy and $\Pi^1_n-\mathsf{CA}_0$

Juan P. Aguilera

There is an extremely large body of work on the metamathematics of determinacy principles in the context of set theory and reverse mathematics. From the perspective of the former, it was known from work of Steel, Tanaka, Heinatsch-Möllerfeld, Montalbán-Shore, and Nemoto that most of the usual subsystems of second-order arithmetic, such as WKL_0, ACA_0, ACA_0^+, ATR_0, $\Pi^1_1-\mathsf{CA}_0$, $\Pi^1_2-\mathsf{CA}_0$, and $Z_2 = \Pi^1_\infty-\mathsf{CA}_0$, are equiconsistent with schemata of axioms asserting the determinacy of games with complexity at various levels of the hierarchy of continuous or Lipschitz reducibility. It was open whether the same result is true for the missing subsystems $\Pi^1_n-\mathsf{CA}_0$, where $2 < n < \infty$.

In this talk, we mentioned the main ingredients of the proof behind the theorem asserting that the systems $\Pi^1_n-\mathsf{CA}_0$ are not equiconsistent with any schema of determinacy assertions when $n \neq 1, 2, \infty$. The main tool was the representation of the Wadge classes between the levels of the difference hierarchy over the $G_{\delta,\sigma}$ sets in terms of *separated Boolean* connectives in the style of Louveau, together with an argument by transfinite induction employing an abstract determinacy transfer theorem which is provable from hypotheses asserting the existence of certain nonstandard models of Kripke-Platek set theory admitting infinitely nested sequences of elementarity gaps of various kinds. This type of determinacy transfer theorem, although provable in the weak theory RCA_0, also has applications in the context of ZFC and its extensions. The specific theorem mentioned in the talk was:

Theorem. *Suppose that every $x \in \mathbb{R}$ belongs to a nonstandard β_m-model M of Kripke-Platek set theory satisfying $V = L$ and Γ-determinacy, where Γ is a Borel Wadge class, and such that there exists a sequence $\{(\zeta_i, s_i) : i \in \mathbb{N}\}$ of M-ordinals for which the following hold for all i:*

(1) $\zeta_i < \zeta_{i+1} \in wfp(M)$,
(2) $s_{i+1} < s_i$,
(3) $M \models L_{\zeta_i} \prec_{\Sigma_{m+1}} L_{s_i}$,
(4) $M \models L_{s_{i+1}} \prec_{\Sigma_{m-1}} L_{s_i}$.

Then, all games in the class $\mathsf{LU}(\Sigma^0_2, \Gamma, m{-}\Sigma^0_3)$ are determined. This is the class of all sets of the form

$$W = \bigcup_{i\in\mathbb{N}} \Big(A_i \cap C_i\Big) \cup B \setminus \bigcup_{i\in\mathbb{N}} C_i,$$

where $A_i \in \Gamma$, $C_i \in \Sigma^0_2$, $B \in m{-}\Sigma^0_3$, and $W \cap C_i = A_i \cap C_i$ for all $i \in \mathbb{N}$.

Although quite technical, the theorem is very powerful. Relating the sets provided by the theorem to those considered in Louveau's analysis of the Wadge ranks of Borel sets and using an analog of his of the Hausdorff-Kuratowski theorem, one can then transfinitely iterate the theorem inside $\Pi^1_n-\mathsf{CA}_0$, leading to the following dichotomy:

Theorem. *Suppose that* $\Gamma = \bigcup_{i\in\mathbb{N}} \Gamma_i$ *is a Borel Wadge class, provably so in* $\Pi^1_{n+3}{-}\mathsf{CA}_0$. *Write* Γ*-Determinacy for the schema* $\{\Gamma_i$*-Determinacy*: $i \in \mathbb{N}\}$. *Then, one of the following holds:*

(1) $\Pi^1_{n+3}{-}\mathsf{CA}_0 \vdash \Gamma$*-Determinacy* & $\mathsf{con}(\Gamma$*-Determinacy*$)$*; or*
(2) $\mathsf{RCA}_0 + \Gamma$*-Determinacy* $\vdash \mathrm{con}(\Pi^1_{n+3}{-}\mathsf{CA}_0)$.

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Proof Mining and duality in Banach spaces

Nicholas Pischke

We present a proof-theoretically tame approach for treating the dual space of an abstract Banach space in systems amenable to proof mining metatheorems on bound extractions, unlocking a major branch of functional analysis to these methods. The approach relies on using intensional methods to deal with the high quantifier complexity of the predicate defining the dual space as well as on a novel treatment of suprema over certain bounded sets in normed spaces to deal with the norm induced on the functionals of the dual. Beyond this, we provide an overview of the many possible extensions and concrete applications to core mathematics obtainable from this (which in particular includes a theory of convex functions and corresponding Fréchet derivatives and their duality theory through Fenchel conjugates, together with the associated Bregman distances).

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Proof mining and asymptotic regularity

Laurenţiu Leuştean
(joint work with Horaţiu Cheval, Paulo Firmino, Ulrich Kohlenbach, Pedro Pinto)

Proof mining is a research program that consists in the extraction of new information from mathematical proofs by applying proof-theoretic techniques. This program was systematically developed beginning with the 1990s by Kohlenbach and collaborators, in connection with applications to approximation theory, nonlinear analysis, ergodic theory, topological dynamics, Ramsey theory, (partial) differential equations, and convex optimization. Kohlenbach's monograph [11] is the standard reference for proof mining.

Asymptotic regularity is a very useful property in the study of the asymptotic behaviour of nonlinear iterations, introduced in the 1960s by Browder and Petryshyn [3] for the Picard iteration and extended to general iterations by Borwein, Reich, and Shafrir [1]. If (x_n) is a sequence in a metric space (X,d), $\emptyset \neq C \subseteq X$, and $T : C \to C$, then (x_n) is said to be asymptotically regular if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and T-asymptotically regular if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It turns out that in numerous results on the weak or strong convergence of a nonlinear iteration (x_n), the first step is to prove the $(T$-)asymptotic regularity of (x_n). Usually one proves first that (x_n) is asymptotically regular and afterwards that (x_n) is T-asymptotically regular.

A mapping $\varphi : \mathbb{N} \to \mathbb{N}$ is said to be a rate of asymptotic regularity of (x_n) if φ is a rate of convergence of $(d(x_n, x_{n+1}))$ towards 0, that is

$$\forall k \in \mathbb{N}\, \forall n \geq \varphi(k) \left(d(x_n, x_{n+1}) \leq \frac{1}{k+1} \right).$$

One defines similarly the notion of a rate of T-asymptotic regularity of (x_n). As pointed out in [14], the notion of T-asymptotic regularity can be extended to countable families of mappings. Thus, if $(T_n : C \to C)$ is such a family, then we say that (x_n) is (T_n)-asymptotically regular with rate φ if $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$ with rate of convergence φ.

In this talk I present recent applications of proof mining consisting in quantitative asymptotic regularity results for different nonlinear iterations.

In [5] we define the Tikhonov-Mann iteration as a generalization to W-hyperbolic spaces [11] of a modified Mann iteration studied by Yao, Zho, and Liou [18] and rediscovered by Boţ, Csetnek, and Meier [2]. Applying proof mining, we compute uniform rates of $(T$-)asymptotic regularity for the Tikhonov-Mann iteration. Furthermore, we prove in [4] that there is a strong relation between the Tikhonov-Mann iteration and the modified Halpern iteration introduced by Kim and Xu [10]. Thus, asymptotic regularity and strong convergence results can be translated from one iteration to the other and the translation holds also for quantitative versions of these results, providing rates of $(T$-)asymptotic regularity and rates of metastability. As an application of a lemma on real sequences due to Sabach and Shtern [16] we also obtain in [4] linear rates of $(T$-)asymptotic regularity for both the Tikhonov-Mann and the modified Halpern iterations for a special choice of the parameter sequences.

Dinis and Pinto introduced recently [7] the alternating Halpern-Mann iteration as an iterative scheme associated with two mappings T, U that alternates between the well-known Halpern and Mann iterations. They proved, in the setting of CAT(0) spaces, quantitative results that provide rates of $(T, U$-)asymptotic regularity and rates of metastability for this iteration by using proof mining techniques developed in [8]. In [15], we show that the quantitative $(T, U$-)asymptotic regularity results obtained in [7] can be extended to UCW-hyperbolic spaces [12, 13], a class of W-hyperbolic spaces that generalize both CAT(0) spaces and uniformly convex normed spaces. Moreover, we apply again Sabach and Shtern's lemma to compute for the alternating Halpern-Mann iteration linear rates of asymptotic regularity in W-hyperbolic spaces and quadratic rates of T, U-asymptotic regularity in CAT(0) spaces, for a special case of the scalars.

In [6] we show that Sabach and Shtern's lemma can be applied to compute linear rates of (T-)asymptotic regularity or ((T_n)-)asymptotic regularity for other Halpern-type iterations studied in optimization and nonlinear analysis.

The viscosity approximation method (VAM), associated to resolvents $J^A_{\lambda_n}$ ($\lambda_n \subseteq (0,\infty)$) of an accretive operator A in a Banach space X, was studied by Xu et al. in a recent paper [17], where they prove results on the convergence of VAM to a zero of the operator A. We obtain in [9] quantitative versions of the asymptotic regularity results from [17] and, as a consequence, we compute uniform rates of (($J^A_{\lambda_n}$)-)asymptotic regularity for VAM. Sabach and Shtern's lemma gives us again linear rates when we consider a particular case of the parameters.

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A New Intuitionistic Version of Gödel-Löb Logic: Box and Diamond

Iris van der Giessen
(joint work with Anupam Das, Sonia Marin)

We introduce an intuitionistic version of Gödel-Löb modal logic GL (the provability logic of Peano Arithmetic) in the style of Simpson [7]. We develop a non-wellfounded labelled proof theory and coinciding birelational semantics, and we call the resulting logic IGL. While existing intuitionistic versions of GL are typically defined over only the box (and not the diamond), IGL includes both modalities. One of its interests is that it allows for the Gödel-Gentzen negative translation into GL which is promising to recover a computational interpretation of classical GL.

Semantics for IGL

Well-known intuitionistic modal logic iGL is sound and complete with respect to *birelational models* $(W, \leq, R, V)$ such that $(\leq; R) \subseteq R$ and R is transitive and conversely wellfounded [8]. The valuation V is *persistent*, i.e., monotone in $\leq$. To interpret the $\Diamond$, the models for iGL are too restrictive. In this work we adopt the same frame conditions as [7], i.e., $(R^{-1}; \leq) \subseteq (\leq; R^{-1})$ and $(R; \leq) \subseteq (\leq; R)$, and further require R to be transitive and $(R; \leq)$ to be conversely wellfounded. We call this class of models $\mathscr{B}$IGL.

One can view (this form of intuitionistic) modal logic as a fragment of (intuitionistic) predicate logic under the standard translation, cf. [7]. In this sense, we obtain another intuitionistic reading of GL, by interpreting the converse wellfoundedness of $(R; \leq)$ within a predicate Kripke models. We denote this class by $\mathscr{P}$IGL.

Proof theory for IGL

To obtain intuitionistic versions of classical modal logics, it typically suffices to restrict a 'standard' calculus, to having one formula on the right of a sequent. For GL, restricting the sequent calculus in [1] and cyclic sequent system in [6], yields calculi for logic iGL [3, 4]. For our setting, labelled systems admitting independent treatments of $\Box$ and $\Diamond$ have been fruitful to define intuitionistic calculi [7]. We develop a labelled calculus for GL taking inspiration from *non-wellfounded proof theory*, where (co)induction principles are devolved to the proof structure rather than explicit rules or axioms. Note that, in contrast to the labelled system for GL in [5], we do not modify the usual labelled rules for $\Box$ and $\Diamond$. From this we define a single-succedent and a multi-succedent non-wellfounded labelled system for IGL, denoted ℓIGL and mℓIGL, respectively.

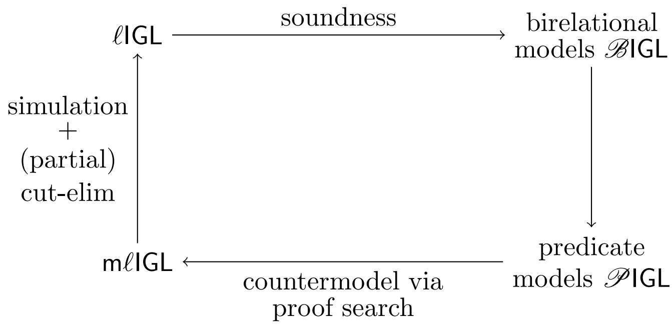

Figure 1. Summary of main results. All arrows denote inclusions of modal logics, so the four characterisations coincide.

Results

Our main result is that these notions coincide as depicted in Figure 1. Soundness for both aforementioned classes of models is readily established via an infinite descent argument by contradiction that is now standard in non-wellfounded proof theory. For completeness, we provide a predicate countermodel construction from a failed proof search in the multi-succedent calculus $\mathsf{m}\ell\mathsf{IGL}$ by appealing to the (lightface) analytic determinacy result for the corresponding 'proof search game'. Simulations using cuts show the equivalence between $\ell\mathsf{IGL}$ and $\mathsf{m}\ell\mathsf{IGL}$ concluding our result. All results can be found in [2].

In future work we would like to establish an explicit axiomatisation for the logic introduced. At the same time it would also be pertinent to investigate the complexity of our logic, given our hitherto non-finitary-presentations. Finally, we would like to examine the role of our logic as a logic of provability in appropriate models of Heyting Arithmetic.

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The Biggest Five of Reverse Mathematics

Sam Sanders
(joint work with Dag Normann)

1. The Biggest Five phenomenon and its limits

The aim of the program *Reverse Mathematics* (RM for short) is to find the minimal axioms needed to prove a given theorem of ordinary mathematics. The *Big Five phenomenon* of RM is the observation that many (perhaps even 'most') theorems are equivalent to one of four logical systems, assuming a weak logical system called the *base theory*. These five systems are called the *Big Five*.

In [7, 12], the Big Five phenomenon is greatly **extended** by establishing numerous equivalences involving the **second-order** Big Five on one hand, and well-known **third-order** theorems from analysis about discontinuous functions on the other hand, working in Kohlenbach's base theory RCA_0^ω from [3, §2]. By [7, §2.8], *slight* variations/generalisations of these third-order theorems cannot be proved from the Big Five and *much* stronger systems. A basic example is as follows.

- Over RCA_0^ω, WKL_0 is equivalent to the *supremum principle* for any of the following: Baire 1, cadlag, quasi-continuity, normal bounded variation.
- Over RCA_0^ω, the Big Five (and much stronger[1] systems like Z_2^ω) cannot prove the supremum principle for any of the following: bounded variation, regulated, cliquish, semi-continuity, Baire 2.

The supremum principles and associated function classes in the first item are called **second-order ish**: although they are third-order in nature, they can be proved from second-order comprehension principles (only). While second-order RM generally deals with countable and separable constructs, quasi-continuity is much wilder[2], yet part of the RM of WKL_0, which is perhaps unexpected.

Many similar examples exist, including for the other Big Five, e.g. the supremum principle for *effectively Baire 2* functions, the Jordan decomposition, and basic properties of the Riemann integral. A full(er) list may be found in [7, 12].

Finally, Rathjen states in [8] that Π^1_2-CA_0 *dwarfs* Π^1_1-CA_0 and Martin-Löf talks of a *chasm* and *abyss* between these two in [4]. The previous examples show that small variations of second-order-ish theorems go *far* beyond the Big Five and Π^1_2-CA_0, far beyond the aforementioned abyss.

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[1] The system Z_2^ω proves the same second-order sentences as Z_2 ([2]). Here, Z_2^ω is RCA_0^ω extended with, for each $k \geq 1$, the functional S_k^2 which decides Π^1_k-formulas.

[2] If $\mathfrak{c}$ is the cardinality of $\mathbb{R}$, there are $2^{\mathfrak{c}}$ non-measurable quasi-continuous $[0,1] \to \mathbb{R}$-functions and $2^{\mathfrak{c}}$ measurable quasi-continuous $[0,1] \to [0,1]$-functions (see [1]).

2. Exploring the Abyss: Kleene's quantifiers

The results in Section 1 are based on the RM of Kleene's quantifiers $(\exists^2)$ and $(\exists^3)$, which is interesting in its own right, and discussed in this section.

First of all, Kohlenbach proves the equivalence between the following in [3, §2].

- Kleene's $(\exists^2) : (\exists E : \mathbb{N}^\mathbb{N} \to \mathbb{N})(\forall f \in \mathbb{N}^\mathbb{N})(E(f) = 0 \leftrightarrow (\exists n \in \mathbb{N})(f(n) = 0)$.
- *There exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$.*

Moreover, $(\exists^2)$ is also equivalent to the following (see [7, §2] for a complete list).

- There exists a function $f : [0, 1] \to \mathbb{R}$ that is not Baire 1.

There are *many* similar equivalences, but following surprise also lies in wait: the system Z_2^ω, a conservative extension of Z_2, cannot prove that

There exists a function $f : [0, 1] \to \mathbb{R}$ that is not Baire 2.

We invite the reader to contemplate the meaning of 'a code for a Baire 3 function' in light of the previous result. Since it is consistent with Z_2^ω that all functions are Baire 2, we find there to be very little meaning in this coding construct.

Secondly, while at the far edges of the subject, the RM of $(\exists^3)$ can be surprisingly basic, as follows. Now, there are dozens (hundreds?) of **decompositions of continuity**, where continuity is shown to be equivalent to the combination of two or more weak continuity[3] notions, going back to Baire, as follows:

continuity $\leftrightarrow$ *weak continuity notion* A plus *weak continuity notion* B. (D)

It is then a natural question whether these weak continuity notions are as *tame* as continuity, e.g. how hard is it to find the supremum of weakly continuous functions? We note that Kohlenbach in [3, §3] singles out this supremum functional as an interesting object of study.

Now, most of these weak continuity notions are rather *tame*: working in $\mathsf{RCA}_0^\omega + (\exists^2)$, one can define the supremum functional $\lambda p, q, f. \sup_{y \in [p,q]} f(y)$ restricted to f satisfying the weak continuity notion at hand.

By contrast, there are seven weak continuity notions that are rather exceptional. In particular, over RCA_0^ω, the following are equivalent.

- Kleene's $(\exists^3)$: $(\exists E)(\forall Y : \mathbb{N}^\mathbb{N} \to \mathbb{N})(E(Y) = 0 \leftrightarrow (\exists f \in \mathbb{N}^\mathbb{N})(Y(f) = 0)$,
- Kleene's quantifier $(\exists^2)$ plus the existence of a supremum functional for any of these classes: the Young condition, almost continuity (Husain), graph continuity, not of Cesàro type, peripheral, pre-, or $\mathcal{C}$-continuity.

These weak continuity notions exist in the literature, side-by-side with the tame ones, and two go back over a hundred years.

[3] We note that *weak and generalised continuity* come with its own AMS code, namely 54C08, i.e. weak continuity is not a fringe topic in mathematics.

3. New Big systems

We list four third-order theorems that boast many equivalences, similar to the original Big Five, with some hints on the kind of principles involved.

- The *uncountability of* $\mathbb{R}$ ([6, 10, 12]) is equivalent to basic properties of regulated and bounded variation functions.
- The *Jordan decomposition theorem* ([5, 12]) is equivalent to the fact that countable sets can be enumerated.
- The *Baire category theorem* ([11, 12]) is equivalent to basic properties of semi-continuous functions.
- The *pigeon-hole principle* for the Lebesgue measure ([11, 12]) is equivalent to one direction of the Vitali-Lebesgue theorem.

For the first two items, the following definition of 'countable set' is used. No elegant equivalences are known for the usual definition based on injections to $\mathbb{N}$.

Definition 1. *A set* $A \subset \mathbb{R}$ *is* height-countable *if there is a* height *function* $H : \mathbb{R} \to \mathbb{N}$ *for* A*, i.e. for all* $n \in \mathbb{N}$*,* $A_n := \{x \in A : H(x) < n\}$ *is finite.*

Definition 2 (Finite set)**.** *Any* $X \subset \mathbb{R}$ *is* finite *if there is* $N \in \mathbb{N}$ *such that for any finite sequence* $(x_0, \ldots, x_N)$ *of distinct reals, there is* $i \leq N$ *such that* $x_i \notin X$*.*

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On the theory of exponential integer parts

Emil Jeřábek

An *integer part (IP)* of an ordered ring R is a discretely ordered subring $I \subseteq R$ such that every $x \in R$ is within distance 1 from I. (By abuse of language, we will conflate a discretely ordered ring I with the ordered semiring $I_{\geq 0}$.) A classical result of Shepherdson [5] characterizes models of IOpen (= Robinson's arithmetic + induction for open formulas in the language $\mathcal{L}_{\mathrm{OR}} = \langle 0, 1, +, \cdot, < \rangle$):

Theorem 1. *Integer parts of real-closed fields are exactly the models of* IOpen.

Let an *exponential field* be an ordered field R endowed with an isomorphism $\exp\colon \langle R, 0, 1, +, < \rangle \to \langle R_{>0}, 1, 2, \cdot, < \rangle$, optionally satisfying the *growth axiom (GA)* $\exp(x) > x$. Introduced by Ressayre [4], an *exponential integer part (EIP)* of an exponential ordered field $\langle R, \exp \rangle$ is an IP $I \subseteq R$ such that $I_{\geq 0}$ is closed under exp. We are interested in the question of characterizing (non-negative parts of) ordered rings that are EIP of real-closed exponential fields (RCEF), and in particular, what is the first-order theory of such rings. This problem (and in particular, the question whether this theory properly extends IOpen) was raised by Jeřábek [2], who provided an upper bound: all countable models of the bounded arithmetical theory VTC^0 in $\mathcal{L}_{\mathrm{OR}}$ are EIP of RCEF.

Extensions of Theorem 1 to exponential ordered fields were previously studied by Boughattas and Ressayre [1] and Kovalyov [3], but they focussed on generalizing the other direction of the theorem (e.g., what additional properties of RCEF ensure that their EIP are models of open induction in a language with exponentiation?). Moreover, they were mostly concerned with EIP in a language with the binary powering operation $x^y = \exp(y \log x)$. Since $\langle I, +, \cdot, <, x^y \rangle$ can define approximations of exp on its fraction field F, we can canonically extend exp to the completion of F; but no such direct construction seems possible for EIP in $\mathcal{L}_{\mathrm{OR}}$ or $\mathcal{L}_{\mathrm{OR}} \cup \{2^x\}$, hence our arguments will be of different nature.

The main goal of this talk is to present complete axiomatizations of the first-order theories of EIP of RCEF in $\mathcal{L}_{\mathrm{OR}} \cup \{2^x\}$, $\mathcal{L}_{\mathrm{OR}} \cup \{P_2\}$ (where P_2 is a predicate for the image of 2^x), and $\mathcal{L}_{\mathrm{OR}}$, and determine some properties of these theories.

Our first result can be proved by an easy application of Robinson's joint consistency theorem:

Theorem 2. *The theory* TEIP_{2^x} *of EIP of RCEF in* $\mathcal{L}_{\mathrm{OR}} \cup \{2^x\}$ *is axiomatized over* IOpen *by*

$$\begin{aligned} &x > 0 \to \exists y\, x < 2^y \leq 2x, \\ &2^{x+y} = 2^x 2^y, \\ &2^x \neq 0. \end{aligned}$$

The theory of EIP of RCEF satisfying GA is $\mathsf{TEIP}_{2^x} + \mathsf{GA}$.

Next, we treat the language with a predicate for powers of 2:

Theorem 3. *The theory* TEIP_{P_2} *of EIP of RCEF, with or without GA, in* $\mathcal{L}_{\mathrm{OR}} \cup \{P_2\}$ *is axiomatized over* IOpen *by*

$$\begin{aligned} &x > 0 \to \exists u\, (P_2(u) \wedge u \leq x < 2u), \\ &P_2(u) \wedge P_2(v) \wedge u \leq v \to \exists w\, (P_2(w) \wedge uw = v). \end{aligned}$$

The conservativity of TEIP_{2^x} over TEIP_{P_2} is, again, proved by a simple application of joint consistency; for $\mathsf{TEIP}_{2^x} + \mathsf{GA}$, we need a rather more complex back-and-forth argument on a countable recursively saturated model of TEIP_{P_2}.

We mention here that Shepherdson's [5] model of IOpen expands to a model of TEIP_{P_2}, but not to a model of TEIP_{2^x}.

For any $\mathfrak{M} \vDash \mathsf{IOpen}$ and $n \in \mathbb{N}$, the *power-of-two game* $\mathrm{PowG}_n(\mathfrak{M})$ is played between two players, Challenger (C) and Powerator (P), in n rounds: in each round $0 \leq i < n$, C picks $x_i \in M_{>0}$, and P responds with $u_i \in M_{>0}$ such that $u_i \leq x_i < 2u_i$. C wins if $u_i u_j < u_k < 2u_i u_j$ for some $i, j, k < n$, otherwise P wins. (While not part of the official rules, we may note that if $u_i < u_j$ but $u_i \nmid u_j$ for some i, j, C can force a win in the next round by playing $\lfloor u_j / u_i \rfloor$.)

The motivation for the game is that if $\langle \mathfrak{M}, P_2 \rangle \vDash \mathsf{TEIP}_{P_2}$, then "play $u_i \in P_2$" is a winning strategy for P. The theory of EIP in the basic language $\mathcal{L}_{\mathrm{OR}}$ is now axiomatized by a schema asserting that Powerator has a winning strategy in PowG for an arbitrary number of rounds:

Theorem 4. *The theory* TEIP *of EIP of RCEF (with or without GA) in* $\mathcal{L}_{\mathrm{OR}}$ *is axiomatized over* IOpen *by the sentences*

$$\forall x_0\, \exists u_0 \dots \forall x_{n-1}\, \exists u_{n-1} \Big(\bigwedge_{i<n} (x_i > 0 \to u_i \leq x_i < 2u_i) \wedge \bigwedge_{i,j,k<n} \neg (u_i u_j < u_k < 2u_i u_j) \Big)$$

for all $n \in \mathbb{N}$.

The idea of the proof is that if $\mathfrak{M} \vDash \mathsf{TEIP}$ is countable and recursively saturated, then P has a winning strategy in "$\mathrm{PowG}_\omega(\mathfrak{M})$", and if we let C enumerate all elements of M, the responses of P form a set P_2 such that $\langle \mathfrak{M}, P_2 \rangle \vDash \mathsf{TEIP}_{P_2}$.

We mention that Svenonius [7] gave a general construction of an axiomatization of a reduct of a given theory by means of sentences expressing the existence of winning strategies in a certain game, mimicking a Henkin completion procedure. However, this axiomatization is rather opaque; in contrast, our game is explicit enough that we are able to derive useful properties of TEIP from it.

First, using the existence of a nonstandard model of IOpen that is a UFD (Smith [6]), we can show that TEIP properly extends IOpen:

Theorem 5. *The following consequence of* TEIP *is not provable in* IOpen*:*

$$\forall x\, \exists u > x\, \forall y\, \big(0 < y < x \to \exists v\, (v \leq y < 2v \wedge v \mid u)\big).$$

We also make partial progress on the main remaining problem about TEIP:

Question 6. *Is* TEIP *finitely axiomatizable over* IOpen*?*

Let us write $\mathsf{TEIP} = \mathsf{IOpen} + \{\forall x_0 > 0\, \exists u_0\, \big(u_0 \leq x_0 < 2u_0 \wedge \theta^1_n(u_0)\big) : n \in \mathbb{N}\}$, where $\theta^1_n(u_0)$ denotes

$$\forall x_1\, \exists u_1 \dots \forall x_{n-1}\, \exists u_{n-1} \Big(\bigwedge_{1 \leq i < n} (x_i > 0 \to u_i \leq x_i < 2u_i) \wedge \bigwedge_{i,j,k<n} \neg(u_i u_j < u_k < 2u_i u_j) \Big).$$

If $\{\theta^1_n : n \in \mathbb{N}\}$ contained only finitely many inequivalent formulas, then TEIP would be finitely axiomatizable over IOpen, but this is not the case:

Theorem 7. *The formulas* θ^1_n *form an infinite hierarchy over* $\mathrm{Th}(\mathbb{N})$.

We show this by analysis of the power-of-two game. Let $\mathrm{PowG}^1_n(u)$ denote the game $\mathrm{PowG}_n(\mathbb{N})$ where the first round is fixed such that P plays $u_0 = u$ (x_0 does not matter). If u is not a power of 2, then C has a winning strategy in $\mathrm{PowG}^1_{n+1}(u)$ for sufficiently large n; let the smallest such n be denoted $c(u)$. Then Theorem 7 amounts to $\sup\{c(u) : u \text{ not a power of } 2\} = +\infty$, which follows from:

Theorem 8. *Let* $u = 2^{\nu_2(u)} v^r$*, where* $v > 1$ *is not a perfect power. Then*

$$c(u) \leq \log\log\log\min\{\nu_2(u), r\} + O(1) \leq \log\log\log\log u + O(1);$$

more precisely, $c(u) \leq \log\log d + O(1)$ *for any* $d \nmid r$*. On the other hand,*

$$c(u) \geq \min\left\{\log\log\log\frac{\nu_2(u)}{\log v}, \log\log d : d \nmid r\right\} + O(1).$$

For example, this shows that $c\big(6^{2^{2^k}!}\big) = k + O(1)$.

Another consequence of Theorem 8 is that there are models $\langle \mathfrak{M}, P_2 \rangle \vDash \mathrm{Th}(\mathbb{N}) + \mathsf{TEIP}_{P_2}$ such that P_2 is distinct from the set of "oddless numbers" (i.e., whose all nontrivial divisors are even); indeed, $u \in P_2$ may even be divisible by 3.

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Quantitative Analysis of Stochastic Approximation Methods

Paulo Oliva
(joint work with Rob Arthan)

In an ongoing case study in Stochastic Approximation Theory, Rob Arthan and I have been working on a quantitative version of Derman-Sachs' proof [3] of Dvoretzky's theorem [4], a vast generalisation of the well-known Robins-Monro seminal stochastic approximation method [5]. Our current proof mining builds on our recent quantitative analysis of the Borel-Cantelli lemmas [1] – one of the ingredients in Derman-Sachs proof. This case study has been proven to be extremely interesting for several reasons.

Firstly, arguments in Probability Theory (and also Measure Theory) look a priori extremely ineffective and non-computational. Most arguments rely on uses of set comprehension to form increasing or decreasing sequences of events, or the axiom of *countable additivity*, which does not seem to have a clear constructive interpretation. Examples of these are the Continuity from Above/Below Lemma and Egorov's Theorem. We rely on recent work of Avigad et. al. [2] and interpret almost sure convergence statements about sequences of random variables

$$\mathbb{P}[\{\omega \mid \forall \varepsilon > 0 \exists N \forall i, j \geq N(|X_i(\omega) - X_j(\omega)| \leq \varepsilon)\}] = 1$$

via a λ-uniform ε-convergent modulus Φ, i.e.

$$\forall \varepsilon, \delta > 0\, (\mathbb{P}[\{\omega \mid \forall i, j \geq \Phi(\varepsilon, \delta)(|X_i(\omega) - X_j(\omega)| \leq \varepsilon)\}] \geq 1 - \lambda)\,.$$

Egorov's Theorem states that a certain sequence of random variables converges, and our proof mining extracts an explicit λ-uniform ε-convergent modulus Φ for the sequence.

Secondly, Derman-Sachs' proof is also extremely interesting in that it uses several subtle lemmas about sequences of real numbers, which as far as we know have not been proof mined yet. The main ones are:

(1) If $\sum a_n$ is a convergent series and $\{b_n\}$ is monotone and bounded then $\sum a_n b_n$ is also a convergent series (Abel's test),
(2) If $\{b_n\}$ is a sequence of non-negative reals such that the series $\sum b_n$ converges then the sequence $\{1/B_n\}$ also converges, where $B_n = \prod_{i \leq n}(1+b_i)$,

(3) If $\{b_n\}$ is a sequence of non-negative reals such that the series $\sum b_n$ converges then there exists a sequence a_n which converges to 0 such that $\sum b_n/a_n^2$ still converges.

It seems to us that a shared repository of results about converging or diverging sequences and series of real numbers which have already been "mined" would be a very useful resource.

Finally, Derman-Sachs relies on a form a "Transfer Principle", whereby the almost sure convergence of a sequence of random variables $\{X_n\}$ is proven by finding a suitable event E where for $\omega \in E$ the convergence of the sequence of real numbers $x_n = X_n(\omega)$ can be derived. Ensuring that the rates on the convergence of the sequences of reals are uniform enough for this transfer to be possible is part of the the challenge in this proof mining case study.

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Double negation and conservation

Peter Schuster
(joint work with Giulio Fellin)

1. Heuristics

Recall that from derivability in minimal logic $\vdash_m$ one obtains derivability first in intuitionistic logic $\vdash_i$ and then in classical logic $\vdash_c$ by allowing as additional axioms finitely many instances of (in first-order logic: the universal closures of) *ex falso quodlibet* $\bot \to B$ and *tertium non datur* $B \vee \neg B$, respectively: that is,

$$\Gamma \vdash_i A \equiv \Gamma, \mathrm{EFQ} \vdash_m A\,, \qquad \Gamma \vdash_c A \equiv \Gamma, \mathrm{TND} \vdash_i A\,.$$

With double negation, Glivenko's theorem [4] for propositional logic can be put as

$$\Gamma \vdash_c A \Longrightarrow \Gamma \vdash_i \neg\neg A\,.$$

In view of $\vdash_m \neg\neg(B \vee \neg B)$, Glivenko's theorem follows from Brouwer's lemma:

$$\Delta, D \vdash_* \neg C \Longrightarrow \Delta, \neg\neg D \vdash_* \neg C \quad (* \in \{m, i\})\,.$$

This and Odintsov's [6] have brought us to analyse the conclusion of Glivenko's theorem in terms of $\vdash_m$:

$$\Gamma \vdash_i \neg\neg A \iff \Gamma, \mathrm{EFQ} \vdash_m \neg\neg A \iff \Gamma, \neg\neg\mathrm{EFQ} \vdash_m \neg\neg A\,.$$

Since $\not\vdash_m \neg\neg(\bot \to B)$ in general, if $\vdash_i$ were replaced by $\vdash_m$, then Glivenko's theorem would fail already for $A \equiv \bot \to B$.

Lemma 1. $\neg\neg$EFQ *is equivalent, over* $\vdash_m$*, to the* double negation shift for $\to$*:*

$$\mathrm{DNS}_{\to}: \quad (B \to \neg\neg C) \to \neg\neg(B \to C).$$

So Glivenko's theorem can alternatively be put with $\vdash_m$ as follows:

$$\Gamma \vdash_c A \Longrightarrow \Gamma, \mathrm{DNS}_{\to} \vdash_m \neg\neg A.$$

While $\mathrm{DNS}_{\to}$ is provable with $\vdash_i$ and thus has hitherto remained invisible in Glivenko's theorem, it is in analogy to

(1) the double negation shift for $\forall$, viz.

$$\mathrm{DNS}_{\forall}: \quad \forall x \neg\neg Cx \to \neg\neg\forall x Cx\,,$$

in Kuroda's [5] generalisation of Glivenko's theorem to first-order logic:

$$\Gamma \vdash_c A \Longrightarrow \Gamma, \mathrm{DNS}_{\forall} \vdash_i \neg\neg A\,;$$

(2) the double negation shift for $\bigwedge_{\mathbb{N}}$, viz.

$$\mathrm{DNS}_{\bigwedge_{\mathbb{N}}}: \quad \bigwedge_{n\in\mathbb{N}} \neg\neg Cn \to \neg\neg \bigwedge_{n\in\mathbb{N}} Cn\,,$$

in Tesi's [7] counterpart of Kuroda's theorem for infinitary logic:

$$\Gamma \vdash_c A \Longrightarrow \Gamma, \mathrm{DNS}_{\bigwedge_{\mathbb{N}}} \vdash_i \neg\neg A\,.$$

2. Conservation for nuclei

Let S be a set and $\rhd$ an *inductively generated* single-succedent *entailment relation*: that is, $\rhd \subseteq \mathcal{P}_{<\omega}(S) \times S$ is the least such relation which satisfies certain generating axioms and rules on top of the following three *structural rules*:[1]

$$\text{reflexivity: } \frac{}{U, a \rhd a} \qquad \text{monotonicity: } \frac{U \rhd a}{U, V \rhd a} \qquad \text{transitivity: } \frac{U \rhd a \quad V, a \rhd b}{U, V \rhd b}$$

By a *nucleus* over $\rhd$ we understand a map $j\colon S \to S$ satisfying

$$U, a \rhd jb \iff U, ja \rhd jb.$$

We consider two entailment relations which contain $\rhd$:

- the *weak* or *Kleisli extension* is defined by $U \rhd_j a \equiv U \rhd ja$;
- the *strong* or *stable extension* $\rhd^j$ is inductively generated by the same axioms and rules as $\rhd$ plus the axiom of *stability* $ja \rhd a$.

[1] By an axiom we understand a premissless rule; for instance, reflexivity is an axiom. Unless one needs to distinguish axioms from rules, one may subsume the former under the latter.

Note that always $\rhd_j \subseteq \rhd^j$. If $\rhd \equiv \vdash_i$ and $j \equiv \neg\neg$, then $\rhd^j \equiv \vdash_c$, and Glivenko's theorem means *conservation*: that is, $\rhd_j \supseteq \rhd^j$.

While the stable extension $\rhd^j$ by its very inductive definition satisfies all axioms and rules of $\rhd$, the Kleisli extension $\rhd_j$ *a priori* satisfies—in addition to the structural rules—only all axioms of $\rhd$.

Theorem 1. *$\rhd_j \supseteq \rhd^j$ if and only if $\rhd_j$ satisfies all (non-axiom) rules of $\rhd$.*

In fact, stability is automatic for $\rhd_j$, because $ja \rhd_j a \equiv ja \rhd ja$.

Corollary 1. *$\rhd_j = \rhd^j$ whenever $\rhd$ is inductively generated by axioms only.*

We hasten to add that for applying Theorem 1 and Corollary 1 it is irrelevant which axioms and rules we take for the inductive generation of $\rhd$. In fact, collections $\mathcal{R}$ and $\mathcal{R}'$ of axioms and rules generate the same $\rhd$ precisely when every member of $\mathcal{R}$ is the composition of members of $\mathcal{R}'$ and vice versa; and "to hold for the Kleisli extension $\rhd_j$" is closed under composition of rules.

3. Applications to logic

Let $\rhd$ be $\vdash_m$. For propositional logic this is generated by the axioms and rules

$$\frac{}{A \wedge B \rhd A}\text{L}\wedge_1 \qquad \frac{}{A \wedge B \rhd B}\text{L}\wedge_2 \qquad \frac{}{A, B \rhd A \wedge B}\text{R}\wedge$$

$$\frac{}{A \vee B, A \to C, B \to C \rhd C}\text{L}\vee \qquad \frac{}{A \rhd A \vee B}\text{R}\vee_1 \qquad \frac{}{B \rhd A \vee B}\text{R}\vee_2$$

$$\frac{}{A \to B, A \rhd B}\text{L}\to \qquad \frac{\Gamma, A \rhd B}{\Gamma \rhd A \to B}\text{R}\to \qquad \frac{}{\rhd \top}\text{R}\top$$

From this variant of minimal propositional logic one obtains

(1) minimal first-order logic by adding the axioms and rules

$$\frac{}{\forall x\, A \rhd A[t/x]}\text{L}\forall \qquad \frac{\Gamma \rhd A[y/x]}{\Gamma \rhd \forall x\, A}\text{R}\forall \quad (y \text{ fresh})$$

$$\frac{}{\exists x\, A, \forall x(A \to B) \rhd B}\text{L}\exists \quad (x \notin \text{FV}(B)) \qquad \frac{}{A[t/x] \rhd \exists x\, A}\text{R}\exists$$

(2) minimal infinitary logic by adding the axioms and rules

$$\frac{}{\bigwedge_{i\in\mathbb{N}} A_i \rhd A_n}\text{L}\bigwedge_n \ (n \in \mathbb{N}) \qquad \frac{\{\Gamma \rhd A_n : n \in \mathbb{N}\}}{\Gamma \rhd \bigwedge_{i\in\mathbb{N}} A_i}\text{R}\bigwedge$$

$$\frac{}{\bigvee_{i\in\mathbb{N}} A_i, \bigwedge_{i\in\mathbb{N}}(A_i \to B) \rhd B}\text{L}\bigvee \qquad \frac{}{A_n \rhd \bigvee_{i\in\mathbb{N}} A_i}\text{R}\bigvee_n \ (n \in \mathbb{N})$$

The only non-axiom rules of the calculi above are R$\to$, R$\forall$ and R$\bigwedge$.

Now let j be a nucleus—compatible with substitution for first-order logic [8]:

$$j(A[t/x]) = (jA)[t/x]\,.$$

Lemma 2. *Each of* R$\to$, R$\forall$ *and* R$\bigwedge$ *holds for $\rhd_j$ if any only if the variant of* $\text{DNS}_\to$, $\text{DNS}_\forall$ *and* $\text{DNS}_\bigwedge$, *respectively, obtains in which $\neg\neg$ is replaced by j.*

logic	**non-axiom rule**	R_ **holds for** $\rhd_j$ **iff**	**case** $j \equiv \neg\neg$
propositional	R→	$B \to jC \rhd j(B \to C)$	$\text{DNS}_{\to}$
first-order	R∀	$\forall x\, jB \rhd j\forall x\, B$	$\text{DNS}_{\forall}$
infinitary	R⋀	$\bigwedge_{n\in\mathbb{N}} jB_n \rhd j \bigwedge_{n\in\mathbb{N}} B_n$	$\text{DNS}_{\bigwedge}$

As for the true DNS, the converse $\lhd$ is automatic in the third column. E.g. R→ holds for $\rhd_j$ if and only if j commutes with every open nucleus [1, 8].

Generalisations include Glivenko-style conservation theorems for the translations ascribed to Kolmogorov, Gentzen and Kuroda in place of double negation.

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Prenex normalization and the hierarchical classification of formulas

Makoto Fujiwara

(joint work with Taishi Kurahashi)

In this workshop, I gave a talk about my recent work [3] on the prenex normalization of first-order formulas by the standard reduction procedure without any reference to the notion of derivability, as well as some ongoing attempt after the work.

The prenex normal form theorem states that for any first-order theory based on classical logic, every formula is equivalent (over the theory in question) to some formula in prenex normal form. This theorem is verified by using the fact that several transformations of formulas moving quantifiers in the formula from inside to outside in a suitable way are admissible in first-order classical logic. For example, if x is not contained in δ, then $\forall x\xi(x) \to \delta$ is transformed into $\exists x(\xi(x) \to \delta)$ with preserving classical validity because they are classically equivalent. For each first-order formula, one can obtain an equivalent formula in prenex normal form by the following procedure:

(1) Apply the above mentioned transformations finitely many times to the subformulas of the form $A \circ B$ with A and B in prenex normal form where $\circ \in \{\wedge, \vee, \to\}$, and transform the subformulas into equivalent formulas in prenex normal form;
(2) Repeating this procedure until when all subformulas become to be in prenex normal form.

Akama, Berardi, Hayashi and Kohlenbach [1] introduced the classes E_k and U_k of formulas defined by counting the number of the alternations of quantifiers in a given formula (the formal definitions are given in [2]). The class E_k (resp. U_k) is intended to form the class of formulas which are classically equivalent to some Σ_k-formula (resp. Π_k-formula). In addition, as mentioned in [1], the class P_k is intended to represent the set of Δ_{k+1}-formulas, namely, formulas which is equivalent to some Σ_{k+1}-formula and also to some Π_{k+1}-formulas. Note that every formula with quantifier occurrences is classified into exactly one of E_{k+1}, U_{k+1} and P_{k+1}.

In [3], Kurahashi and the author gave a proper justification for the hierarchical classes. They formalized the above mentioned procedure for prenex normalization and investigated the relation between the classes of prenex formulas and the hierarchical classes in [1, 2] modulo the transformation procedure in a general language of a first-order theory. In particular, they showed that a formula is in E_k^+ (resp. U_k^+) if and only if it can be transformed into a formula in Σ_k^+ (resp. Π_k^+) by the transformation procedure, where $\mathrm{E}_k^+, \mathrm{U}_k^+, \Sigma_k^+$ and Π_k^+ are cumulative variants of $\mathrm{E}_k, \mathrm{U}_k, \Sigma_k$ and Π_k, respectively. By the results for the cumulative classes, it also follows that non-cumulative classes E_k, U_k and P_k (except P_0) are the cumulative counterparts of Σ_k, Π_k and Δ_{k+1} respectively modulo the transformation procedure.

In addition, I have presented an ongoing attempt about a classification of first-order formulas based on hierarchical prenex normalization procedures restricted to those which are admissible in intuitionistic and semi-classical theories. For that purpose, we introduce new hierarchical classes of first-order formulas and characterize the classes by the hierarchical prenex normalization procedures restricted to some of those classes.

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Proof mining, applications to optimization, and interactive theorem proving

Horaţiu Cheval

Let H be a Hilbert space, $(T_n : H \to H)$ be a family of nonexpansive mappings and consider the problem of finding a common fixed point $x \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$. Boţ, and Meier [1] introduced an iterative method for finding such a point, which proceeds by constructing the sequence (x_n) via

$$x_{n+1} = (1-\lambda_n)\beta_n x_n + \lambda_n T_n(\beta_n x_n), \tag{1}$$

where $(\lambda_n), (\beta_n)$ are sequences in $[0, 1]$, and $x_0 \in H$ is arbitrary. The main results of [1] show that, under certain conditions on (λ_n), (β_n) and (T_n), it holds that

- $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$;
- $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$;
- $\lim_{n\to\infty} x_n = x$, for some $x \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(T_n)$.

The first two results are also known as the asymptotic (resp. (T_n)-asymptotic) of (x_n).

We present [3] an extension of this iteration from the setting of Hilbert spaces to the nonlinear case of W-hyperbolic spaces, in the sense of [7]. For X a W-hyperbolic space and $(T_n : X \to X)$ a family of nonexpansive self-mappings thereof, we define its associated *Tikhonov-Mann iteration* (x_n) by

$$x_{n+1} = (1-\lambda_n)u_n + \lambda_n T_n u_n, \quad \text{where} \tag{2}$$

$$u_n = (1-\beta_n)u + \beta_n x_n, \tag{3}$$

with $(\lambda_n), (\beta_n)$. This simultaneously generalizes (1), as well as the single mapping case studied in W-hyperbolic spaces in [2]. As the main results of [3], we show that the asymptotic regularity of (x_n) still holds in this setting, i.e. that, under certain conditions on (λ_n), (β_n), (T_n) we have that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$ and that for any $m \in \mathbb{N}$, $\lim_{n\to\infty} d(x_n, T_m x_n) = 0$.

Furthermore, the convergence theorems obtained are enriched with quantitative information, in the form of rates of ((T_n)-, T_m-)asymptotic regularity, which display a high degree of uniformity with respect to the space X and the mappings (T_n). We also present work in progress regarding the generalization of the strong convergence of (x_n) from Hilbert to CAT(0) spaces. This can be carried out effectively, adapting arguments from [4–6, 9]. These results are part of the program of *proof mining* [8].

Finally, in the second part, we discuss work in progress and research ideas for using the Lean interactive theorem prover in proof mining, which can be found at `https://github.com/hcheval/`. This includes the formalization of mathematical results used and obtained in the context proof mining (for example quantitative versions of lemmas widely used in optimization about sequences of real numbers), ideally in the form of a unified library. See also `https://github.com/Kejineri` for such formalizations.

A different direction is the implementation of the general logical metatheorems from proof mining which guarantee the possibility of extracting quantitative content from certain classes of formal proofs. Given the constructive character of these metatheorems, they could be built into automatic program extraction tools, which could then be applied to proofs already formalized in Lean, in order to obtain strengthened variants thereof.

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Equiconsistency of the Minimalist Foundation with its classical version

Maria Emilia Maietti

In our Oberwolfach talk we showed that the Minimalist Foundation, which is a foundation for constructive mathematics, is equiconsistent with its classical version, obtained by extending the underlying logic with the law of excluded middle.

The Minimalist Foundation, for short **MF**, was initially conceived in 2005 in collaboration with Giovanni Sambin in [MS05] and further developed into a comprehensive two-level system in 2009 in [Mai09].

This two-level structure comprises an intensional level, referred to as **mtt**, which is envisioned as a theory possessing sufficiently decidable properties to serve as a foundation for a proof assistant which at the same time might be enriched with a mechanism of program extraction from its proofs. Additionally, there is an extensional level, named **emtt**, which is formulated in a language closely aligned with that of traditional mathematics. Then, **emtt** is interpreted within the intensional level, **mtt**, through the utilization of a quotient model.

One of the main novelties of **MF** is that of serving as a shared core among significant foundations for mathematics. Notably, its estensional level **emtt** is compatible with several prominent mathematical foundations, including the standard axiomatic set theory ZFC, Aczel's Constructive Zermelo-Fraenkel set theory, the general theory of elementary toposes, as shown in [Mai09] (see also [MS22]), and more recently, Homotopy Type Theory and Voedvosky's Univalent Foundations as shown in [CM23]. Instead, its intensional level **mtt** is compatible with Martin-Löf's intensional type theory, Coquand-Huet-Paulin's Calculus of Inductive Constructions, as shown in [Mai09], and again Homotopy Type Theory as shown in [CM23].

When we say that a theory is "compatible" with another theory, we mean that there exists a translation preserving the meaning of logical and set-theoretic operators from the first theory to the latter (and, for example, this implies that the translation commutes with the embedding of Heyting arithmetics with finite types in each of the mentioned theories if the first theory includes it).

As a byproduct **MF** is both constructive and predicative. In particular, the computational contents of proofs developed within **MF** and further extensions with inductive and coinductive topological definitions has been made explicit through realizability models described in [IMMS18, MMR21, MMR22].

In our Oberwolfach talk we showed the remarkable property that both levels of **MF** are still predicative and equiconsistent with the addition of the law of the excluded middle and are all mutually equiconsistent.

It is worth mentioning two key steps to prove our claim.

The first key step of our proof is that we can smoothly extends Goedel-Gentzen's double negation translation of classical Peano arithmetics into the intuitionistic one (for example in [Tv88]), to show that the intensional level **mtt** with the addition of proof-irrelevance for propositions is equiconsistent with its classical version obtained with the further addition of the law of excluded middle. This works because the elimination rule of the propositional identity of **mtt** is equivalent to the usual replacement rules of first-order equality. Proof-irrelevance is then needed to interpret the universe of small classical propositions as the subtype of the **mtt**-universe of small propositions that are $\neg\neg$-stable.

The second key step to show our claim is that **mtt** (with or without the addition of proof-irrelevance for propositions) is equiconsistent with the extensional level **emtt** of **MF** through the use of the quotient model and of canonical isomorphisms in [Mai09]. The proof of this fact extends smoothly to show the equiconsistency of **mtt** (with or without proof-irrelevance) with **emtt** when the law of excluded middle is added to each of them.

We also mention that Pietro Sabelli in his forthcoming PhD's thesis shows that Goedel-Gentzen's double negation translation can be extended to provide a direct interpretation of the classical version of **emtt** into **emtt** (whilst the propositional equality of **emtt** has stronger rules that the usual first-order equality) thanks to the fact that the propositional equality of **emtt** ground types is $\neg\neg$-stable and that

emtt-type-theoretic constructors preserve the ¬¬-stability of their propositional equality.

We conclude by underlying that the predicativity and equiconsistency of the classical version of **MF** with **MF** itself is a peculiarity of **MF** since the other well known constructive and predicative foundations mentioned above, namely Martin-Löf's intensional type theory, Aczel's Constructive Zermelo-Fraenkel set theory and Homotopy Type Theory, do not satisfy this property because they become impredicative when the law of excluded middle is added to their underlying logic.

We leave to future research to investigate whether the extensions of **MF** with inductive and coinductive topological definitions in [MMR21] and [MMR22] are still equiconsistent with their classical version or, at least, are still predicative.

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The computational content of super strongly nonexpansive mappings

Andrei Sipoş

Strongly nonexpansive mappings are a core concept in convex optimization. Recently, they have begun to be studied from a quantitative viewpoint: U. Kohlenbach has identified in [2] the notion of a 'modulus' of strong nonexpansiveness, which leads to computational interpretations of the main results involving this class of mappings (e.g. rates of convergence, rates of metastability). This forms part of the greater research program of 'proof mining', initiated by G. Kreisel and highly developed by U. Kohlenbach and his collaborators, which aims to apply proof-theoretic tools to extract computational content from ordinary proofs in mainstream mathematics (for more information on the current state of proof mining, see the book [1] and the recent survey [3]). The quantitative study of strongly nonexpansive mappings has later led to finding rates of asymptotic regularity for the problem of 'inconsistent feasibility' [4, 7], where one essential ingredient has been a computational counterpart of the concept of rectangularity, recently identified in [5] as a 'modulus of uniform rectangularity'.

Last year, Liu, Moursi and Vanderwerff [6] have introduced the class of 'super strongly nonexpansive mappings', and have shown that this class is tightly linked to that of uniformly monotone operators. What we do is to provide a modulus of super strong nonexpansiveness, give examples of it in the cases e.g. averaged mappings and contractions for large distances and connect it to the modulus of uniform monotonicity. In the case where the modulus is supercoercive, we give a refined analysis, identifying a second modulus for supercoercivity, specifying the necessary computational connections and generalizing quantitative inconsistent feasibility.

The results in this talk may be found in the paper [8].

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Sunny nonexpansive retractions in nonlinear spaces

Pedro Pinto

Undoubtedly, one of the most complicated instances of proof mining to date is the proof-theoretical analysis of Reich's theorem, one of the most pivotal results in functional analysis, carried out in [2].

In this talk, we introduce the notion of a nonlinear smooth space generalizing both CAT(0) spaces as well as smooth Banach spaces [3]. Concretely, we say that a hyperbolic space (X, d, W) (in the sense of [1]) is a *smooth hyperbolic space* if there exists a function $\pi : X^2 \times X^2 \to \mathbb{R}$ satisfying for all $x, y, z, u, v \in X$

(P1) $\pi(\overrightarrow{xy}, \overrightarrow{xy}) = d^2(x, y)$
(P2) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) = -\pi(\overrightarrow{yx}, \overrightarrow{uv}) = -\pi(\overrightarrow{xy}, \overrightarrow{vu})$
(P3) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) + \pi(\overrightarrow{yz}, \overrightarrow{uv}) = \pi(\overrightarrow{xz}, \overrightarrow{uv})$
(P4) $\pi(\overrightarrow{xy}, \overrightarrow{uv}) \leq d(x, y)d(u, v)$

and for any $\lambda \in [0, 1]$

$$\text{(P5)}\ \ d^2(W(x, y, \lambda), z) \leq (1 - \lambda)^2 d^2(x, z) + 2\lambda\pi\left(\overrightarrow{yz}, \overrightarrow{W(x, y, \lambda)z}\right).$$

Moreover, we say that (X, d, W, π) is a *uniformly smooth hyperbolic space* if it satisfies additionally

$$\text{(P6)}\ \ \begin{cases} \forall \varepsilon > 0\ \forall r > 0\ \exists \delta > 0\ \forall a \in X\ \forall u, v \in \overline{B}_r(a) \\ \quad d(u, v) \leq \delta \to \forall x, y \in X\ (|\pi(\overrightarrow{xy}, \overrightarrow{ua}) - \pi(\overrightarrow{xy}, \overrightarrow{va})| \leq \varepsilon \cdot d(x, y)). \end{cases}$$

Formally we can consider these spaces in a extension of the system $\mathcal{A}^\omega[X, d, W]$ from [1] where we have a further constant π of type $1(X)(X)(X)(X)$ governed by the axioms (P1)–(P6). Clearly by (P4) π is majorizable, and using (P6) the system proves the extensionality of π. Thus, if we include a modulus of uniform continuity ω_X in the sense of providing a witness for δ in (P6), we have a metatheorem for the extraction of bounds from (formalizable) proofs in this new class of nonlinear spaces.

We discuss that this notion allows for a unified treatment of several mathematical proofs in functional analysis. In particular, we show that Kohlenbach's and Sipoş's treatment of Reich's result can be appropriately discussed in this nonlinear setting.

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Extensional Proof-Systems for Modal Logics

Margherita Zorzi
(joint work with S. Guerrini, S. Martini, A. Masini)

1. Introduction

Designing a robust proof theory for modal logics is a subtle task. The difficulty lies not merely in establishing deductive systems; rather, the real challenge is in formulating a concrete structural proof theory, in which the objects of study are (not only) modal formulas, but also modal proofs. A well defined systems satisfies some desirable properties, such as the the syntactical study of cut elimination/normalization theorem and its consequences (the sub-formula property and the consistency theorem, see [1])). Furthermore, if feasible, one could attempt to define an *extensible* system – a system capable of capturing not only a single logic but an entire family.

In the literature, several deductive styles and approaches to modal proof theory have been introduced. We recall *multidimensional systems*, where the primary concept involves equipping formulas with an index or position, (offering a kind of "spatial coordinate") and Labeled Deductive Systems, where the rules that model the accessibility relationship are explicitly integrated into the syntactical deductive instruments. In this abstract we will focus on natural deduction and on a family of multidimentional systems based on the notion of *position*. The main ideas of our frameworks are the following: formulas are marked with a spatial coordinate; only one introduction rule and one elimination rule per connective; no additional structural rules; no explicit reference to the accessibility relation; only modal operators can "change" the spatial position of the formulas and are treated in analogy of first-order quantifiers. Refer to [2–4] for complete technical details, comprehensive references to related work, and a thorough comparison with the state of the art.

2. From K to S4: the system $\mathcal{N}_{pos}$

A *position-formula* is an expression of the form A^{α}, where A is a modal formula and α is a position. Positions are constructed based on *tokens*, which are essentially uninterpreted symbols. According to the definition of position we adopt (a sequence, a set, a singleton set) we are able to characterize different modal systems.

The classical natural deduction system $\mathcal{N}_{pos}$ captures the normal extension of the logic K by incorporating the basic axiom $K \equiv \Box(A \to B) \to (\Box A \to \Box B)$ and one or more of the following axioms: $D \equiv \Box A \to \Diamond A$, $T \equiv \Box A \to A$, $4 \equiv \Box A \to \Box\Box A$. This results in K (containing only the basic axiom K) D (K+D), T (K+T), $\mathsf{K4}$ (K+4), $\mathsf{D4}$ (K+D+4), and **S4** (K+T+4).

In $\mathcal{N}_{pos}$ positions are interpreted as *sequences* of tokens (with related operations such as concatenation).

Rules for modal operator are designed as much as possible in analogy with fist-order logic quantifiers:

$$\frac{\begin{array}{c}\vdots\\ A^{\alpha x}\end{array}}{\Box A^{\alpha}}\ (\Box I) \qquad \frac{\begin{array}{c}\vdots\\ \Box A^{\alpha}\end{array}}{A^{\alpha\beta}}\ (\Box E)$$

$$\frac{\begin{array}{c}\vdots\\ A^{\alpha\beta}\end{array}}{\Diamond A^{\alpha}}\ (\Diamond I) \qquad \frac{\begin{array}{c}\vdots\\ \Diamond A^{\alpha}\end{array} \quad \begin{array}{c}[A^{\alpha x}]\\ \vdots\\ C^{\beta}\end{array}}{C^{\beta}}\ (\Diamond E)$$

In the rule $\Box I$, one has $\alpha x \notin \mathfrak{Init}[\Gamma]$, where Γ is the set of (open) assumptions on which $A^{\alpha x}$ depends and $\mathfrak{Init}[\Gamma] = \{\beta : \exists A^{\alpha} \in \Gamma.\ \beta \sqsubseteq \alpha\}$. In the rule $\Diamond E$, one has $\alpha x \notin \mathfrak{Init}[\beta]$ and $\alpha x \notin \mathfrak{Init}[\Gamma]$, where Γ is the set of (open) assumptions on which C^{β} depends, with the exception of the discharged assumptions $A^{\alpha x}$. The system K and $\mathsf{K4}$ are partial logics we use existence predicates (à la Scott) for formulating sound deduction rules to deal with partial systems.

All the logical systems share the rules above. To obtain a specific logic, one can "tune" some syntactic constraints, described in the following tables:

name of the calculus	**constraints on the rules $\Box E$ and $\Diamond I$**
$\mathcal{N}_{\mathsf{S4}}$	no constraints
$\mathcal{N}_{\mathsf{T}}$	$\beta = \langle\ \rangle$
$\mathcal{N}_{\mathsf{D}}$	β is a singleton sequence $\langle z \rangle$
$\mathcal{N}_{D4}$	β is non empty
name of the calculus	**constraints on the rules $\Box E$ and $\Diamond I$**
$\mathcal{N}_{\mathsf{K4}}$	β is a non empty sequence
$\mathcal{N}_{\mathsf{K}}$	β is a singleton sequence $\langle z \rangle$

Following Prawitz's original proof for first-order logic, one proves a *Normalization Theorem*: for each derivation Π there exists a derivation Π' s.t. $\Pi \overset{*}{\succ} \Pi'$ and Π' is in normal form. As a Corollary, one obtains the *Consistency* of the system(s) (by syntactical arguments): for each position α, $\nvdash_{\mathcal{N}_{Pos}} \perp^{\alpha}$.

The formal definition of semantics of $\mathcal{N}_{Pos}$ is very technical but intuitive. Positions are mapped into nodes of a tree-like Kripke structure (and hence sublists of a position will range on paths of nodes). Each system captured by $\mathcal{N}_{Pos}$ is sound and complete w.r.t. its standard Hilbert-style axiomatization.

3. Beyond S4: the logics S4.2 and S5

What's happen to if we relax the "structure" of the positions?

If we release the ordering and the multiplicity of tokens, then we move from sequences to *sets*, we obtain a (classical) natural deduction system for the logic **S4**.2. The logic **S4**.2 is employed in different settings, ranging from epistemology to the metamathematics of set theory and algebraic structures. It can be derived by adding to **S4** the axiom $2 \equiv \Diamond\Box A \to \Box\Diamond A$. We do not add any additional constraints to the rules except for the usual ones on $\Box I$ and $\Diamond E$. The resulting system $\mathcal{N}_{S4.2}$ is sound and complete w.r.t. the standard Hilbert-style axiomatization of **S4**.2. Moreover, we can prove a Normalization theorem and, as a consequence, a Consistency theorem (again, by means of purely syntactical arguments).

Regarding semantics, the interpretation of position formulas requires an interesting observation. It is well-know that **S4**.2 is characterized (at the level of the accessibility relation) by direct partial preorders. If we were to consider this characterization, in attempting to assign semantics to the positions, we would encounter a non-trivial problem. We have to decide which point in the Kripke model could be uniquely associated with a set of tokens $\{x_1, \ldots, x_n\}$. The standard, naive choice would be to take one of the upper bounds of the worlds associated with each token, but this choice would not be unique, and in a direct pre-order the supremum of a finite set of elements might not exist. However, we can use some results of Goldblatt and Shetmann, which imply that **S4.2** is also characterized by a class of ordered structures different than direct pre-orders, that of semilattices with a minimum element, where the problem disappears. We can now interpret a position (i.e. a set) $\{x_1, \ldots, x_k\}$ as the *least upper bound* of the points (in a space) $x_1, \ldots, x_k$.

We have interpreted positions as general sets. Now, let's restrict the definition of positions to *singleton sets.* What we obtain is $\mathcal{N}_{S5}$ a *indexed* natural deduction for **S**5 logic. The logic **S**5 is obtained by adding the axiom $B \equiv \phi \to \Box\Diamond\phi$ to **S4** and is characterized by a universal semantics (this means that the accessibility relation is an equivalence relation). We do not add any additional constraints to the rules except for an adaptation of the usual ones on $\Box I$ and $\Diamond E$. Both the classical and intuitionistic version $\mathcal{N}_{S5}$ enjoys expected good properties such as the soundness and completeness w.r.t. their Hilbert-style axiomatization, the Normalization Theorem and its consequences. In the case of intuitionistic logic, instead of the universal semantics, it is more interesting to explore a BHK interpretation, that interpret modal operators as follows: a proof of $\Box A^x$ is a construction that for each y gives a proof $f(y)$ of A^y and a proof of $\Diamond A^x$ is a pair (y, a) such that a is a proof of A^y. The BHK interpretation, via the natural deduction system, induces a Curry–Howard Isomorphism in the usual sense. The resulting calculus is similar to λ-P, i.e. the typed lambda-calculus for the negative fragment of first order intuitionistic logic in the so called Barendregt-cube.

The described research is open to various investigations, including the study of the intuitionistic version of the $\mathcal{N}_{Pos}$ system from a Curry-Howard perspective and the extension of techniques to infinitary logics.

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Dichotomies in Weihrauch Complexity

Vasco Brattka

We discuss a number of uniform dichotomies for problems in the Weihrauch lattice. Such dichotomies have the common form that a problem is either quite well-behaved (continuous, measurable of some form, etc.) or already relatively badly behaved. We show that often such dichotomies also have non-uniform versions in terms of computable reducibility and we indicate how computability concepts such as Turing jumps, Weak Kőnig's Lemma, diagonal non-computability, etc., occur naturally in these non-uniform versions. This leads, for instance, to first-order characterizations of continuity in terms of Turing degrees. We also discuss how some known dichotomies from descriptive set theory, such as Solecki's dichotomy, can be seen in this context. The talk is based on ongoing research, but some of the discussed results are published in [1].

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Mini-Workshop: Combinatorial and Algebraic Structures in Rough Analysis and Related Fields

Organized by
Carlo Bellingeri, Berlin
Yvain Bruned, Vandœuvre-lès-Nancy
Ilya Chevyrev, Edinburgh
Rosa Preiß, Potsdam

26 November – 2 December 2023

Abstract. Recent years have seen an explosion of algebraic methods to study singular stochastic and rough dynamics. These include developments in geometric rough path theory based on the algebra of words, the introduction of decorated trees in regularity structures, and the recent approach to singular stochastic partial differential equations based on multi-indices. These developments have furthermore led to important links with numerical analysis, machine learning, stochastic quantisation, and the study of symmetries of physical systems. The aim of this mini-workshop was to bring together experts working on these fields using algebraic structures that appear in rough dynamics. The goal was to facilitate the exchange of ideas and encourage further connections to be established.

Mathematics Subject Classification (2020): 60L10, 60L20, 60L30, 60L70, 16T05, 16S10, 18M60, 18G45, 18M60, 65M12.

Introduction by the Organizers

Organizational details

The mini-workshop *Combinatorial and Algebraic Structures in Rough Analysis and Related Fields*, organised by Yvain Bruned (Université de Lorraine), Carlo Bellingeri (TU Berlin), Ilya Chevyrev (University of Edinburgh) and Rosa Preiß (University of Postdam) was attended by 16 participants currently based in France, Germany, Norway, Poland and the UK. The program consisted of 16 talks (45 minutes each), each being followed by a discussant's presentation (15 minutes each), leaving sufficient time for additional questions from the audience.

Due to some participants becoming ill at short notice in connection with the Covid-19 pandemic, this event took place in a hybrid format having 4 participants attending online. In accordance with Oberwolfach's tradition, the schedule was not known in advance by the participants. The days' schedules were sent each evening to the group. Further informal discussions took place in between and after the talks. The Zoom session was managed with the precious help of Carlo Bellingeri, and Usama Nadeem took care of the the report.

Motivation

The main purpose of the mini-workshop was to gather together early career researchers working in the development of new algebraic structures to study non-linear singular random dynamics arising from rough analysis and connected areas. In particular, we wanted to encourage collaborative work and the sharing of recent contributions among different research groups, including groups working on SPDEs with regularity structures and multi-indices, signatures, numerical analysis, data science, and operad theory. Combinatorial and algebraic structures arise naturally in non-linear dynamics when we want to describe in a compact way higher order expansions of differential equations.

Consider for instance a autonomous system

$$x'(t) = f(x(t)), \quad x(0) = x_0\,.$$

Then, applying iteratively the Taylor formula, we can write x as the asymptotic series indexed by trees. The same trees can be used also to describe higher order Runge–Kutta methods to solve the system numerically [3]. More generally, by introducing appropriate algebraic structures, an important example of which is the Butcher–Connes–Kreimer Hopf algebra [5], it is possible to derive a consistent theory of numeric integration for ordinary differential equations [10] and to renormalise Feynman diagrams in quantum field theory. Moving beyond numerical analysis, formal expansions can be used analytically to establish well-posedness of singular stochastic dynamics. We refer principally to rough differential equations (RDEs)

$$dY_t = g(Y_t)dW_t, \quad Y(0) = Y_0$$

and singular stochastic partial differential equations (SPDEs) of the form

$$(\partial_t - L)u = F(u, \nabla u)\xi, \,, \quad u(0) = v\,.$$

Both systems are characterised by the presence of noise terms, which are associated respectively to a highly oscillating driving noise W and random distribution ξ, making the equation singular. The resolution of these systems is performed in a series of papers [8, 2, 4, 1], taking their roots in Lyons theory of rough paths [11, 6, 7], and that have culminated in the formation of the recent field of rough analysis, see [9]. In this context, combinatorial and algebraic structures are adopted to construct truncated Taylor-type expansions of the solutions of the previous equations.

We should indeed mention the following offshoots which have been widely discussed by the participants:

- Multi-indices, which are a different way to encode the expansions for solutions of singular SPDEs. The idea is to index the expansion according to the elementary differentials (coefficients arising from the nonlinearities) instead of the iterated integrals. Talks on the subject were given by Bruned, Linares and Tempelmayr.
- Regularity Structures via decorated trees where the characterisation of symmetries in a general combinatorial context remains a challenge. For example, there is no unification between the chain rule in the geometric and quasilinear KPZ equations and gauge-covariance in Yang-Mills. Talks were given on this topic by Chevyrev and Nadeem.
- Numerical Analysis for dispersive PDEs where a resonance analysis allows us to get low regularity schemes for a large class of equations. The combinatorial structure used is very similar to decorated trees developed for singular SPDEs. Talks were given by Alama Bronsard and Schratz.
- Rough paths where its geometry is much better understood via the isomorphisms between words and trees. Talks were given on this topic by Bellingeri, Ferrucci, Rahm and Tapia.
- The potential of these combinatorial structures could be seen via other fields of application brought by the participants such as Algebraic geometry for rough paths (Preiß), Algebraic operads (Tamaroff), Perturbative Quantum Field Theory (Klose), Stochastic analysis in Frobenius manifold (Combe) and 2D signature via multiparemeter iterated integrals (Diehl).

Acknowledgement: The workshop organizers would like to thank MFO for the nice environment provided for this event.

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Timetable of the Mini-workshop

	Monday	Tuesday	Wednesday	Thursday	Friday
9h30	Bellingeri	Combe	Schratz	Tamaroff	Nadeem
11h00	Klose	Chevyrev	Diehl	Preiß	Bruned
15h30	Tempelmayr	Rahm		Tapia	
17h00	Linares	Alama Bonsard		Ferrucci	

Speakers and discussants

Speaker	Title	Discussant
Bellingeri	Algebraic structures in the rough change of variable formula	Alama Bronsard
Klose	Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras	Tempelmayr
Tempelmayr	Recentering for rough paths and regularity structures via multi-indices	Tamaroff
Linares	Algebraic renormalization of rough paths and regularity structures based on multi-indices	Bellingeri
Combe	Semimartingales with values in a (pre-)Frobenius manifolds	Ferrucci
Chevyrev	Symmetries in stochastic Yang–Mills equations	Linares
Rahm	Planarly Branched Rough Paths Are Geometric	Combe
Alama Bronsard	Numerical approximations to rough solutions of dispersive equations	Diehl
Schratz	Resonances as a computational tool	Tapia
Diehl	Multiparameter iterated integrals	Schratz
Tamaroff	From bialgebras to algebraic operads	Nadeem
Preiß	An algebraic geometry of (rough) paths	Rahm
Tapia	Branched Itô formula	Klose
Ferrucci	Natural Itô-Stratonovich isomorphism	Bruned
Nadeem	Solution theory for quasilinear generalised KPZ Equation	Chevyrev
Bruned	Novikov algebras and multi-indices in regularity structures	Preiß

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Abstracts

Algebraic structures in the rough change of variable formula

Carlo Bellingeri

Given a smooth function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ and a continuous bounded variation path $x\colon [0,T] \to \mathbb{R}^d$, $x = (x^1, \dots, x^d)$ the fundamental theorem of calculus tells us the well-known identity

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} dx^i_r \,.$$

This formula is a cornerstone of standard calculus. In particular, when x does not satisfy this property, the integral in might not be well defined because x is not a.e. differentiable and Lebesgue integration theory is not useful any more. Surprisingly, thanks to the theory of rough paths [5] it is still possible to write a similar change of variable formula. However, in this case, the formula is not unique, depending on the underlying *algebraic theory* defining the integrals. The goal of this talk is to fully explore the possible identities known in the theory and prepare the discussion for the talks of Emilio Ferrucci and Nikolas Tapia on [2].

A first possibility is represented by introducing the *Young integral*, see [11], defined as the limit of the Riemann-type sum

$$\int_s^t f(x_r) d\hat{x}^i_r : \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} f(x_u)(x^i_v - x^i_u) \,,$$

where π is a generic partition of $[s,t]$ with size $|\mathcal{P}|$. This sum converges if and only if x is γ-Hölder with $\gamma \in (1/2, 1)$ and one has the formula

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} d\hat{x}^i_r \,.$$

More generally, using the standard theory of *geometric rough paths*, see [10], the starting point is not anymore a path but an *extended path* $X\colon [0,T]^2 \to \mathcal{G}(\mathbb{R}^d)$ with values in the character group of the shuffle Hopf algebra $(T(\mathbb{R}^d), \sqcup\!\sqcup, \Delta_c)$. Using the additional components of X we can indeed define for any $\gamma \in (0,1)$ the *geometric rough integral*

$$\int_s^t f(X_r) dX^i_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \sum_{|w|<N-1} \frac{\partial f(X_u)}{\partial x_{i_1} \dots \partial x_{i_w}} \langle wi, X_{u,v}\rangle \,,$$

where we sum over a set of words with a length smaller than a finite N depending on γ. It follows from elementary considerations of Taylor's formula and the shuffle product that in this case one has the identity

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} dX^i_r \,.$$

Similar computations were also provided in [1] where the shuffle product is deformed into a quasi-shuffle product [8]. The main feature of this approach starts with some *apriori* relations among the components of X and then one derives the formula using standard combinatorial relations.

In case we not want to assume any apriori relations we need to start from a *branched rough paths* [6, 7] i.e. our starting path will take value in the character group of the Butcher–Connes–Kreimer Hopf algebra $(\mathcal{H}(\mathbb{R}^d), ., \Delta_{ck})$ [3] where the product is free. A first general theory to express these identities was given in the last chapter of David Kelly's PhD Thesis [9, Chap. 5]. This condition allows us to obtain an extremely general formula but at the same time, this notion requires to satisfy some additional properties and it is not unique, which makes this definition more arduous for applications. Some parts of [2] are dedicated to providing a new formula in this context. Interestingly we will have a final identity of the form

$$\varphi(x_t) - \varphi(x_s) = \sum_{i=1}^{d} \int_s^t \frac{\partial \varphi(x_r)}{\partial x_i} dX_r^i + \text{``higher order integrals''}\,.$$

where in the remaining integral we will integrate not just with respect to the component of X but the value of X on its primitive elements, whose properties were deeply analysed in [4].

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Perturbation theory for the Φ^4_3 measure, revisited with Hopf algebras

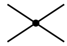
Tom Klose
(joint work with Nils Berglund)

The Φ^4_3 model, defined on the 3-dimensional torus $\mathbb{T}^3$, is probably one of the simplest non-trivial models in Euclidean quantum field theory. At cut-off scale $N \in \mathbb{N}$, it can be written as

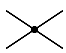
$$\mu^N_{\Phi^4_3}(\mathrm{d}\phi) = \frac{1}{Z_N(\varepsilon)} \exp\Big(-\int_\Lambda \Big(\frac{\|\nabla\phi(x)\|^2}{2} + \frac{1-\varepsilon^2 C_N^{(2)}}{2}{:}\,\phi(x)^2{:} + \frac{\varepsilon}{4}{:}\,\phi(x)^4{:} + \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}\Big)\mathrm{d}x\Big)\mathrm{d}\phi$$

where $C_N^{(k)}$, $k = 1, \dots, 4$ are suitable explicit renormalisation constants and ${:}\cdot{:}$ is the Wick product w.r.t. the covariance $C_N^{(1)}$. The purpose of this talk is to revisit perturbation theory for the renormalised log partition function

(1)
$$-\log\frac{Z_N(\varepsilon)}{Z_N(0)} = \gamma - \log \mathbb{E}^{\mu_N}\left[e^{-\alpha X - \beta Y}\right] = \gamma - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!},$$

$\kappa_n = \mathbb{E}_c^{\mu_N}\Big[\Big(\alpha$  $+\,\beta$ $\Big)^n\Big]$

associated with this measure, where μ_N is the Gaussian measure with covariance $(-\Delta+1)^{-1}$, regularised at scale N,

(2) $X \equiv$  $\equiv \int_{\mathbb{T}^3} {:}\,\phi(x)^4{:}\mathrm{d}x, \quad Y \equiv$ $\equiv \int_{\mathbb{T}^3} {:}\,\phi(x)^2{:}\mathrm{d}x,$

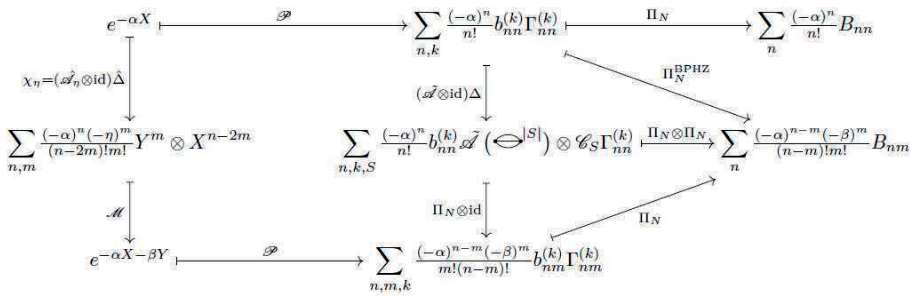
and the parameters α, β, and γ are defined as $\alpha := \varepsilon/4$, $\beta := \frac{\varepsilon^2}{2}C_N^{(2)}$, and $\gamma := \varepsilon^2 C_N^{(3)} - \varepsilon^3 C_N^{(4)}$. The last equality in (1) is an expansion in terms of *cumulants* κ_n and it is well-known (see, e.g. [10]) that they can be expressed in terms of *connected* Feynman diagrams $\Gamma^{(k)}_{nm}$ with m vertices of valency 4, $n-m$ vertices of valency 2, and $n+m$ edges. These diagrams come with a degree $\deg(\Gamma^{(k)}_{nm}) = 2n-m-3$ for all k and are associated with a real number via a canonical valuation map Π_N. Even though all diagrams Γ with valuation $\Pi_N(\Gamma) \le 0$ are divergent as $N \to \infty$, it is unfortunately *not* the case that all the diagrams with positive valuation converge; this is known as the problem of (nested) *subdivergences*. It has been overcome in the celebrated work by Bogoliubov, Parasiuk, Hepp, and Zimmermann [3, 9, 11], who constructed a renormalised *BPHZ valuation* map Π_N^{BPHZ} for which $\deg(\Gamma) > 0$ implies that $\Pi_N^{\mathrm{BPHZ}}(\Gamma)$ is uniformly bounded in N; see also the recent work by Hairer [8] for a self-contained formulation. The main result of our work [2, Thm. 3.5] is the following theorem:

Theorem 1. *The following equality holds in the sense of formal power series:*

(3)
$$\gamma - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!} = \sum_{p=4}^{\infty}\frac{1}{p!}(-\alpha)^p \sum_k b^{(k)}_{pp}\,\Pi_N^{\mathrm{BPHZ}}\left(\Gamma^{(k)}_{pp}\right)$$

where the $b_{pp}^{(k)}$'s denote combinatorial factors. Since $\deg(\Gamma_{pp}^{(k)}) = p - 3 \geq 1$ *for* $p \geq 4$*, this implies that all terms in the perturbative cumulant expansion* (1) *are bounded uniformly in the cut-off parameter* N*.*

This result is already known but our proof is new. The theorem follows from the commutativity of the diagram below which we establish in [2, Sec. 3.6].



In this diagram, the RHS is well-known. Since we work with connected diagrams, note that the only possible divergent sub-diagram in our setting is the "bubble" .

Furthermore, the co-product Δ describes the extraction-contraction procedure due to Connes and Kreimer [5, 6], $\tilde{\mathscr{A}}$ is the twisted antipode w.r.t. the BPHZ valuation, and $\mathscr{C}_S\Gamma$ is the graph Γ with all bubbles with labels in S contracted to a vertex, see [2, Sec. 3.3 and 3.4] and the references therein for details.

Inspired by the work of Ebrahimi-Fard et al. [7], the main novelty of our approach is represented on the LHS of the diagram: We consider the polynomial Hopf algebra H spanned by the monomials X and Y given in (2), equipped with the classical co-product $\hat{\Delta}$; on top of that, we build a map $\hat{\mathscr{A}}_\eta$ that resembles the twisted antipode $\tilde{\mathscr{A}}$ such that the map χ_η then resembles the BPHZ renormalisation procedure of Feynman diagrams on the RHS. The map $\mathscr{M}$ describes the multiplication $\mathscr{M} : H \otimes H \to H$ and the connection between the LHS and the RHS is given by the map $\mathscr{P}$, which formalises the pairings in Wick's formula and projects onto connected diagrams, see [2, Sec. 3.5] for details.
Interaction with the other participants. The discussant, Markus Tempelmayr, did a wonderful job and raised several interesting questions. The first question concerns the generality of our approach, in particular with regards to

- the *full subcritical regime* of the $\Phi^4_{4-\delta}$ model that was recently investigated in [4] or even
- the case of the *critical* Φ^4_4 model, the triviality of which was established by Aizenman and Duminil-Copin [1].

A potential answer to this question is linked to another question raised by the discussant, namely: Can we characterise the algebraic structure on the LHS of the commutative diagram above? In our work, we have left that problem open but we believe that one should be able to recast (a modification of) our construction in the language of the above-mentioned work by Ebrahimi-Fard et al. [7]. While this question remains open for now, the workshop has provided the author with the opportunity to initiate a discussion with Nikolas Tapia, another participant and co-author of the article [7], which could potentially lead to a follow-up project.

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Recentering for rough paths and regularity structures via multi-indices

Markus Tempelmayr
(joint work with Pablo Linares and Felix Otto)

Following [3, Section 6], we review the Hopf algebra structure underlying recentering in multi-index based regularity structures introduced in [4]. To simplify this exposition, we consider instead of a PDE the rough differential equation

$$\frac{d}{dt}u(t) = a(u(t))\xi(t), \quad u(t=0) = 0. \tag{1}$$

Here, we think of $a : \mathbb{R} \to \mathbb{R}$ as being a smooth nonlinearity, and of ξ as a random Schwartz distribution, say $\xi \in C^{\alpha-1}$ for $\alpha \in (0, 1)$. For such ξ, the product $a(u)\xi$ is a product of a function with a distribution which falls short of the Young regime.

The basic idea to develop a solution theory in [4] is to parameterize the model, which captures the local solution behaviour, by partial derivatives w.r.t. the nonlinearity a. We thus make the ansatz

$$u(t) - u(s) = \sum_\beta \Pi_{s\beta}(t) \prod_{k=0}^{\infty} \Big(\frac{1}{k!}\frac{d^k a}{du^k}\big(u(s)\big)\Big)^{\beta(k)}$$

for a base point $s \in \mathbb{R}$, where $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ is a multi-index. With help of the coordinates z_k on the space of nonlinearities given by $\mathsf{z}_k[a] := \frac{1}{k!}\frac{d^k a}{du^k}(0)$, the above ansatz takes the more compact form of

$$u(t) - u(s) = \sum_\beta \Pi_{s\beta}(t)\, \mathsf{z}^\beta[a(\cdot + u(s))],$$

where the monomials z^β are given by $\mathsf{z}^\beta := \prod_{k=0}^\infty \mathsf{z}_k^{\beta(k)}$. This power series does in general not converge. We thus "algebraize" our ansatz by not evaluating the coordinates at a nonlinearity a, and consider instead formal power series in the abstract variables $\{\mathsf{z}_k\}_{k=0}^\infty$,

$$\Pi_s(t) := \sum_\beta \Pi_{s\beta}(t)\, \mathsf{z}^\beta \in \mathbb{R}[[\mathsf{z}_k]].$$

Plugging this ansatz into (1) and comparing coefficients yields

$$\frac{d}{dt}\Pi_{s\beta}(t) = \sum_{k=0}^\infty \sum_{e_k+\beta_1+\cdots+\beta_k=\beta} \Pi_{s\beta_1}(t)\cdots\Pi_{s\beta_k}(t)\xi(t), \quad \Pi_s(t=s) = 0,$$

where we denote by e_k the multi-index mapping l to δ_k^l. Some examples are

$$\tfrac{d}{dt}\Pi_{s\,e_0} = \xi, \quad \tfrac{d}{dt}\Pi_{s\,e_0+e_1} = \Pi_{s\,e_0}\xi, \quad \tfrac{d}{dt}\Pi_{s\,2e_0+e_2} = \Pi_{s\,e_0}^2\xi,$$
$$\tfrac{d}{dt}\Pi_{s\,2e_0+e_1+e_2} = \Pi_{s\,2e_0+e_2}\xi + 2\Pi_{s\,e_0}\Pi_{s\,e_1}\xi.$$

Comparison to rough paths. We compare this construction to branched rough paths [1]. For rooted trees $\tau_1, \ldots, \tau_k$ and a tree $\tau = [\tau_1 \cdots \tau_k]$ the rough path $\mathbb{X}(\tau)$ is recursively defined by

$$\frac{d}{dt}\mathbb{X}_{s,t}(\tau) = \mathbb{X}_{s,t}(\tau_1)\cdots\mathbb{X}_{s,t}(\tau_k)\xi, \quad \mathbb{X}_{s,t=s}(\tau) = 0.$$

Some examples are

$$\tfrac{d}{dt}\mathbb{X}_s(\bullet) = \xi, \quad \tfrac{d}{dt}\mathbb{X}_s([\bullet]) = \mathbb{X}_s(\bullet)\xi, \quad \tfrac{d}{dt}\mathbb{X}_s([\bullet\bullet]) = \mathbb{X}_s(\bullet)\mathbb{X}_s(\bullet)\xi,$$
$$\tfrac{d}{dt}\mathbb{X}_s([[\bullet\bullet]]) = \mathbb{X}_s([\bullet\bullet])\xi, \quad \tfrac{d}{dt}\mathbb{X}_s([\bullet[\bullet]]) = \mathbb{X}_s(\bullet)\mathbb{X}_s([\bullet])\xi.$$

As these examples correctly suggest, every model component $\Pi_{s\beta}$ can be expressed as a linear combination of rough paths $\mathbb{X}_s(\tau)$.

Proposition 1. For every β we have

$$\Pi_{s\beta}(t) = \sum_{\tau\in\mathcal{T}_\beta} \tfrac{\sigma(\beta)}{\sigma(\tau)}\, \mathbb{X}_{s,t}(\tau),$$

where $\mathcal{T}_\beta$ is the set of all trees having $\beta(k)$ nodes with k children for all $k \in \mathbb{N}_0$, $\sigma(\beta) := \prod_{k=0}^{\infty}(k!)^{\beta(k)}$ is a symmetry factor of a multi-index, and $\sigma(\tau)$ is the symmetry factor of the tree τ.

This induces a dictionary ϕ from (linear combinations of) multi-indices T to (linear combinations of) trees $\mathcal{T}$, given by $\phi(\beta) = \sum_{\tau\in\mathcal{T}_\beta} \frac{\sigma(\beta)}{\sigma(\tau)}\,\tau$, such that

$$\Pi = \mathbb{X} \circ \phi.$$

Recentering. We turn to recentering, and aim to relate Π_s to $\Pi_{\bar{s}}$. Observe that for given $u_{\bar{s}} \in \mathbb{R}$ and $u[a(\cdot + u_{\bar{s}})]$ the solution to (1) with a replaced by $a(\cdot + u_{\bar{s}})$,

$$\bar{u} := u\big[a(\cdot + u_{\bar{s}})\big] + u_{\bar{s}}$$

satisfies (1) with initial condition $\bar{u}(0) = u_{\bar{s}}$. Allowing the shift to depend on a, we might hope to choose $u_{\bar{s}}[a]$ such that $\bar{u}(\bar{s}) = 0$. We thus informally identified the transformation $\Gamma^*_{\bar{s}0}$ that recenters solutions from 0 to $\bar{s}$ by

$$(\Gamma^*_{\bar{s}0}u)[a] = u\big[a(\cdot + u_{\bar{s}}[a])\big] + u_{\bar{s}}[a].$$

The goal is to translate this to the level of Π, where the above can not directly be applied as Π is not a well-defined functional of a (only a formal power series!). Instead, we consider the infinitesimal generator D of the u-shift of a defined by

$$(Du)[a] := \tfrac{d}{dv}\big|_{v=0} u[a(\cdot + v)],$$

which as a derivation is well defined on formal power series $\mathbb{R}[[\mathsf{z}_k]]$. Analogously, the generator of the a-dependent u-shift is given by $\mathsf{z}^\beta D \in \mathrm{Der}(\mathbb{R}[[\mathsf{z}_k]])$. The linear span $\mathsf{L} := \mathrm{span}\{\mathsf{z}^\beta D \,|\, \beta \text{ multi-index}\}$ is then a pre-Lie algebra when equipped with

$$\mathsf{z}^\beta D \triangleright \mathsf{z}^\gamma D := (\mathsf{z}^\beta D.\mathsf{z}^\gamma)D,$$

the dot meaning the application of the derivation $\mathsf{z}^\beta D$ to the power series z^γ. Note that its universal enveloping algebra $U(\mathsf{L})$ is naturally a Hopf algebra. We define the space of "forests" of multi-indices T^+ via the pairing

$${}_{U(\mathsf{L})}\langle \mathsf{z}^{\beta_1}\cdots\mathsf{z}^{\beta_k} D\cdots D, \gamma_1\cdots\gamma_l\rangle_{\mathsf{T}^+} = \delta^l_k \delta^{\gamma_1}_{\beta_1}\cdots\delta^{\gamma_k}_{\beta_k},$$

where $\mathsf{z}^{\beta_1}\cdots\mathsf{z}^{\beta_k} D\cdots D$ can be given a sense in $U(\mathsf{L})$ by help of the pre-Lie product $\triangleright$, which crucially does not depend on any order as the multiplication of monomials z^β commutes. This turns T^+ into a Hopf algebra. From its character group, we can analogous to the theory of regularity structures [2] with help of a comodule build a (structure) group G containing endomorphisms $\Gamma_{s\bar{s}}$, such that their duals $\Gamma^*_{s\bar{s}} \in \mathrm{End}(\mathbb{R}[[\mathsf{z}_k]])$ satisfy

$$\Gamma^*_{s\bar{s}}\Pi_{\bar{s}} = \Pi_s.$$

By working on the "dual side", we thus obtained a geometric interpretation of the Hopf algebra at play in regularity structures for recentering.

Comparison to rough paths again. On the tree-side we consider the Grossman-Larson pre-Lie algebra $\mathcal{L} := \mathrm{span}\{\tau \,|\, \tau \text{ tree}\}$ equipped with

$$\tau_1 \rightsquigarrow \tau_2 := \sum_\tau n(\tau_1, \tau_2, \tau)\tau,$$

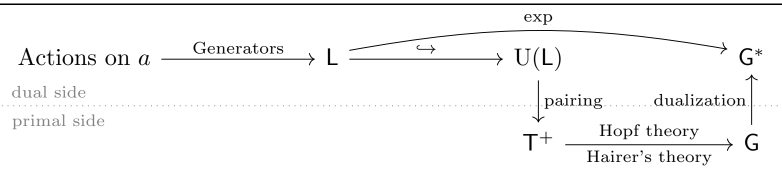
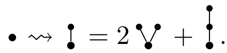


FIGURE 1. Algebraic construction of the group G^*.

where $n(\tau_1, \tau_2, \tau)$ is the number of single cuts performed on τ with branch τ_1 and trunk τ_2, e.g.



The dictionary ϕ can by the pairings ${}_{\mathsf{T}}\langle \beta, \mathsf{z}^\gamma D\rangle_{\mathsf{L}} = \delta_\beta^\gamma$ and ${}_{\mathcal{T}}\langle \tau_1, \tau_2\rangle_{\mathcal{L}} = \delta_{\tau_1}^{\tau_2}$ be transposed to obtain $\phi^\dagger : \mathcal{L} \to \mathsf{L}$, given by $\phi^\dagger(\tau) = \frac{\sigma(\beta)}{\sigma(\tau)} \mathsf{z}^\beta D$ provided $\tau \in \mathcal{T}_\beta$.

Proposition 2. $\phi^\dagger$ is a pre-Lie morphism, i.e.

$$\phi^\dagger(\tau_1 \rightsquigarrow \tau_2) = \phi^\dagger(\tau_1) \triangleright \phi^\dagger(\tau_2).$$

By the universality property it lifts to a Hopf algebra morphism $\phi^\dagger : U(\mathcal{L}) \to U(\mathsf{L})$. Yet another pairing on forests of trees between the Grossman-Larson Hopf algebra $U(\mathcal{L})$ and the Connes–Kreimer Hopf algebra $\mathcal{H}$ defined by

$${}_{U(\mathcal{L})}\langle \tau_1 \cdots \tau_k, \sigma_1 \cdots \sigma_l\rangle_{\mathcal{H}} = \delta_k^l \delta_{\tau_1}^{\sigma_1} \cdots \delta_{\tau_k}^{\sigma_k},$$

allows to transpose $\phi^\dagger$ once more and yields a Hopf algebra morphism $\phi : \mathsf{T}^+ \to \mathcal{H}$.

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Algebraic renormalization of rough paths and regularity structures based on multi-indices

PABLO LINARES

The theories of rough paths [13], [8, 9] and regularity structures [10] provide local well-posedness results for RDEs (respectively SPDEs): The fundamental analytic objects they are based on take the form of local expansions with respect to nonlinear functionals of the driving noise, collected in what is called *rough path* (respectively *model*). In their usual approaches, these local expansions and their natural transformations (re-expansions, multiplication, renormalization) are encoded in Hopf algebras of trees, leading to a systematic treatment of semi-linear subcritical singular SPDEs [2, 4, 6]. More recently, and in the context of quasi-linear SPDEs, [14] obtained a priori bounds in a regularity structures set-ups based on multi-indices instead of trees: A deeper algebraic understanding of this new bookkeeping, and particularly of the recentering operation based on multi-indices, was later given in [12].

The purpose of this talk, based on [5, 11], is to provide a complete description of the algebraic structures emerging from multi-indices for a general class of semi-linear equations of the form

$$\mathcal{L}u = \sum_{\mathfrak{l}\in\mathfrak{L}} a_{\mathfrak{l}}(\mathbf{u})\xi^{\mathfrak{l}}, \;\; u:\mathbb{R}^d\to\mathbb{R}, \;\; \mathbf{u} = \big(\tfrac{1}{\mathbf{m}!}\partial^{\mathbf{m}}u\big)_{\mathbf{m}\in\mathbb{N}_0^d}.$$

More precisely, we give a systematic construction of a regularity structure based on multi-indices, including a careful study of the recentering transformations leading to the so-called structure group; we introduce finite counterterms in the original equation and describe a recursive procedure to construct an algebraically renormalized smooth model; and we provide, in the rough path case, the construction of a renormalization group based on multi-indices.

The basis of the constructions consists of variables $\{\mathsf{z}_{(\mathfrak{l},k)}\}$, $\mathfrak{l}\in\mathfrak{L}$, $k\in M(\mathbb{N}_0^d)$, which are placeholders for the derivatives of the nonlinearities $a_{\mathfrak{l}}$; and $\{\mathsf{z}_{\mathbf{n}}\}$, $\mathbf{n}\in\mathbb{N}_0^d$, which represent Taylor coefficients of a local parameterization of the manifold of solutions. Multi-indices arise when considering monomials in these variables z^{β}. Next to this, we have infinitesimal generators of shifts in the space of solutions (denoted $D^{(\mathbf{n})}$, $\mathbf{n}\in\mathbb{N}_0^d$), which act like pre-Lie products; and of shifts in space-time (denoted $\boldsymbol{\partial}_i$, $i=1,...,d$), which are commutative linear maps. Appealing to the construction of Guin and Oudom [7], combined with suitable grading properties, we derive a Lie algebraic PDE in mild form describing a smooth model, namely

$$\Pi_x = K*\rho\big(\exp(\boldsymbol{\Pi}_x)\big)\sum_{\mathfrak{l}\in\mathfrak{L}}\mathsf{z}_{(\mathfrak{l},0)}\xi^{\mathfrak{l}} + \sum_{\mathbf{n}\in\mathbb{N}_0^d}\mathsf{z}_{\mathbf{n}}(\cdot-x)^{\mathbf{n}},$$

where exp is a symmetric exponential and ρ some action onto the algebra of $\{\mathsf{z}_{(\mathfrak{l},k)}\}$ $\cup\{\mathsf{z}_{\mathbf{n}}\}$. Similar techniques give rise to the structure group as a group of symmetric exponentials.

In the description of an algebraically renormalized model, the introduction of a counterterm is reflected in a shift of the form $\mathsf{z}_{(0,0)}\mapsto\mathsf{z}_{(0,0)}+c$, where we think of $\xi^0=1$ and c is a polynomial of $\{\mathsf{z}_{(\mathfrak{l},k)}\}\cup\{\mathsf{z}_{\mathbf{n}}\}$; this, in particular, connects with translation of rough paths [3] and preparation maps [1]. It is also possible to characterize the infinitesimal generators of local counterterms to seek a non-recursive construction of renormalized models; these generators create another pre-Lie algebra, which in the simpler rough path case allows to write translation maps as symmetric exponentials (the general SPDE case would require an enlargement of the structure via extended decorations, cf. [4]).

As reflected in [5, 11, 12], multi-indices encode linear combinations of trees, and have been proven more efficient e. g. when bookkeeping renormalization constants. Understanding if there are any analytic or stochastic interpretations in this grouping of trees is an open problem which was brought up in the posterior discussion.

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Symmetries in stochastic Yang–Mills equations

Ilya Chevyrev

(joint work with Ajay Chandra, Martin Hairer, and Hao Shen)

Recent works [2, 3, 5] have made sense of the stochastic quantisation equations of Yang–Mills (YM) on the torus $\mathbb{T}^d$, $d = 2, 3$, that read (in the DeTurck gauge)

$$\partial_t A^\varepsilon = -\,\mathrm{d}^*_{A^\varepsilon} F_{A^\varepsilon} - \mathrm{d}_{A^\varepsilon}\,\mathrm{d}^* A^\varepsilon + C^\varepsilon A + \xi^\varepsilon = \Delta A^\varepsilon + A^\varepsilon \partial A^\varepsilon + (A^\varepsilon)^3 + C^\varepsilon A^\varepsilon + \xi^\varepsilon \;.$$

Here $A^\varepsilon \colon \mathbb{R}_+ \times \mathbb{T}^d \to \mathfrak{g}^d$ is a 1-form and $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, ξ^ε is an adapted mollification at scale $\varepsilon > 0$ of a $\mathfrak{g}^d$-valued white noise ξ on $\mathbb{R} \times \mathbb{T}^d$, and $\{C^\varepsilon\}_{\varepsilon>0} \subset L(\mathfrak{g}, \mathfrak{g})$ are renormalisation counterterms. In the final expression we write the heuristic form of the non-linearities in the equation.

For $d = 2$ or $d = 3$, there exist choices for C^ε such that, as $\varepsilon \to 0$, the solutions A^ε converge (modulo blow-up) to a space-time distribution A that we call a solution to the stochastic YM equations (SYM) with mass $\{C^\varepsilon\}_{\varepsilon>0}$.

In this report, we describe the argument in [3] based on small-noise limits that shows there is distinguished choice for C^ε such that the solution A is *gauge-covariant* in the following way: if $A(t)$ and $\bar{A}(t)$ are solutions of SYM with mass $\{C^\varepsilon\}_{\varepsilon>0}$ and gauge equivalent initial conditions $A(0) \sim \bar{A}(0)$, then $[A(t)]$ is equal in law to $[\bar{A}(t)]$ (modulo blow-up). Here $[A] = \{B : B \sim A\}$ is the gauge orbit of A where $\sim$ denotes gauge equivalence which, roughly speaking, means that there exists $g\colon \mathbb{T}^d \to G$ such that $A^g := \mathrm{Ad}_g A - (\mathrm{d}g)g^{-1} = B$. Since solutions to SYM are distributions, gauge equivalence needs to be interpreted appropriately.

This result is shown for $d = 2$ and $d = 3$ in [2] and [3] respectively (see also [4] for a survey). It in particular implies that the projected process $[A(t)]$ is Markov. In the case $d = 2$, one can furthermore choose $C^\varepsilon \equiv C$ independent of ε, which is 'atypical' for a singular stochastic PDE. We also mention that in [5] it is shown, for $d = 2$, that the Markov process $[A(t)]$ has a unique invariant measure which is the YM measure on $\mathbb{T}^2$ associated with trivial principal G-bundle and that the operator $C^\varepsilon \equiv C \in L(\mathfrak{g}, \mathfrak{g})$ with the above property is unique.

The first step in both [2, 3] in the proof of the gauge-covariance proprety is the following result that follows from the general theory of regularity structures.

Proposition 1. *There exist operators $C^\varepsilon_{\mathrm{BPHZ}}, \tilde{C}^\varepsilon, \tilde{C}^{0,\varepsilon} \in L(\mathfrak{g}, \mathfrak{g})$ such that, for any fixed $\mathring{C}_1, \mathring{C}_2 \in L(\mathfrak{g}, \mathfrak{g})$, the solutions to*

$$(1) \quad \partial_t B = \Delta B + B\partial B + B^3 + \mathrm{Ad}_g \xi^\varepsilon + (C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1)B + (\tilde{C}^\varepsilon + \mathring{C}_2)(\mathrm{d}g)g^{-1} \ ,$$

$$(2) \quad \partial_t \bar{A} = \Delta \bar{A} + \bar{A}\partial \bar{A} + \bar{A}^3 + (\mathrm{Ad}_{\bar{g}} \xi)^\varepsilon + (C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1)\bar{A} + (\tilde{C}^{0,\varepsilon} + \mathring{C}_2)(\mathrm{d}\bar{g})\bar{g}^{-1} \ ,$$

converge to the same limit in probability as $\varepsilon \downarrow 0$, where g and $\bar{g}$ solve

$$\partial_t g = \Delta g - (\partial_j g)g^{-1}(\partial_j g) + [Z_j, (\partial_j g)\bar{g}^{-1}]g \ \textit{ with initial condition } g(0)$$

with Z taken as B and $\bar{A}$ respectively.

The relevance of this result is that, if we choose $C^\varepsilon = C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1$, then $B := A^g$ solves (1) provided that $\tilde{C}^\varepsilon + \mathring{C}_2 = C^\varepsilon$. On the other hand, provided that $\tilde{C}^{0,\varepsilon} + \mathring{C}_2 = 0$, then by Itô isometry, since $(\mathrm{Ad}_{\bar{g}}\xi)^\varepsilon$ is equal in law to ξ^ε (which is where we use that $(\mathrm{Ad}_{\bar{g}}\xi)^\varepsilon$ and thus $\bar{g}$ are adapted), $\bar{A}$ is equal in law to A. The gauge-covariance property would thus follow once we show that the limits

$$(3) \qquad \lim_{\varepsilon\downarrow 0} \tilde{C}^\varepsilon - C^\varepsilon_{\mathrm{BPHZ}} \ \text{ and } \ \lim_{\varepsilon\downarrow 0} \tilde{C}^{0,\varepsilon} \ \text{ exist.}$$

This is because, if these limits exist, then we can choose $\mathring{C}_2 = -\lim_{\varepsilon\downarrow 0} \tilde{C}^{0,\varepsilon}$ and $\mathring{C}_1 = \lim_{\varepsilon\downarrow 0}\{\tilde{C}^\varepsilon - C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_2\}$ to satisfy the the above conditions with $C^\varepsilon = C^\varepsilon_{\mathrm{BPHZ}} + \mathring{C}_1$. However, since renormalisation constants generically diverge, it is not clear a priori that (3) holds.

In [2] for $d = 2$, the claim (3) is shown by direct calculation since the number of diverging diagrams involved is rather small (three to be precise).

For $d = 3$, the argument in [3] is different and inspired by the work [1] on manifold-valued stochastic heat equations. We demonstrate this method by showing that $\limsup_{\varepsilon\downarrow 0} |\tilde{C}^{0,\varepsilon}| < \infty$ without knowing the precise form of $\tilde{C}^{0,\varepsilon}$.

Arguing by contradiction, suppose $\limsup_{\varepsilon\downarrow 0} |\tilde{C}^{0,\varepsilon}| = \infty$ and let $\tilde{C}^{0,\varepsilon}_\sigma$ denote the renormalisation constant arising from a rescaled noise $\sigma\xi$. It is not difficult to see that there exist $\sigma_\varepsilon \downarrow 0$ such that $\tilde{C}^{0,\varepsilon}_{\sigma_\varepsilon} \to \hat{C} \neq 0$ as $\varepsilon \downarrow 0$ along a subsequence. Take now bare masses $\mathring{C}_1 = 0$, $\mathring{C}_2 = -\hat{C}$ in the equation for $\bar{A}$. Then, by continuity in the noise, $\bar{A}$ converges to the solution of $\partial_t \bar{A} = \Delta \bar{A} + \bar{A}\partial\bar{A} + \bar{A}^3 - \hat{C}\,\mathrm{d}\bar{g}\bar{g}^{-1}$. On the other hand, $\bar{A}$ is equal in law to the solution of

$$\partial_t \tilde{A} = \Delta \tilde{A} + \tilde{A}\partial\tilde{A} + \tilde{A}^3 + C^{\varepsilon}_{\mathrm{BPHZ},\sigma_\varepsilon}\tilde{A} + \sigma_\varepsilon \tilde{\xi}^\varepsilon + (\tilde{C}^{0;\varepsilon}_{\sigma_\varepsilon} - \hat{C})\,\mathrm{d}\tilde{g}\tilde{g}^{-1} \ .$$

Treating $\tilde{C}^{0;\varepsilon}_{\sigma_\varepsilon} - \hat{C}$ as a bare mass that converges to 0, by joint continuity in noise and bare mass, $\tilde{A}$ converges to the deterministic YM heat flow $\partial_t \tilde{A} = \Delta\tilde{A} + \tilde{A}\partial\tilde{A} + \tilde{A}^3$. The limits of $\bar{A}$ and $\tilde{A}$ are not equal since $\hat{C} \neq 0$, which yields a contradiction.

A similar but slightly different argument based on gauge-covariance of the deterministic YM heat flow shows that $\limsup_{\varepsilon\downarrow 0} |\tilde{C}^\varepsilon - C^\varepsilon_{\mathrm{BPHZ}}| < \infty$. With further work, one can show that the limits in (3) actually exist, completing the proof of gauge-covariance. This argument raises the natural question of whether there is an algebraic framework to describe and unify the symmetries appearing in [1, 2, 3].

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Semimartingales with values in a (pre-)Frobenius manifolds

Noémie C. Combe

In the sixties, Pierre Cartier proposed a generalisation of probability theory on richer structures such as manifolds. In this paper we follow this idea. We show that there exists a class of symmetric spaces of Cartan–Hadamard type for which Itô's integrals of 1-forms along semimartingales with value in such a manifold have no divergences. In particular, one can omit the approach relying on perturbative expansion of the functional integral appearing as a sum labelled by Feynman graphs. This is explained by the fact that the manifolds investigated below are Hessian manifolds satisfying the properties of a pre-Frobenius potential manifold and they contain a submanifold which is a Frobenius manifold.

For this class of manifolds, covariant derivatives form a pre-Lie algebra. The fibres of the Frobenius manifold's tangent bundle have the structure of a Frobenius algebra. The fact that one can omit perturbative expansions here relates—among others—to the phenomenon that F-manifold algebras are the corresponding semi-classical limits of pre-Lie formal deformations of commutative associative algebras. Moreover, by [6], the class of Frobenius algebras is a class closed under deformations. Finally, applying the geometric flavoured argument (the "no-go theorem") of [4] ends the discussion.

1. A new approach

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered complete probability space. For X to be a martingale with values in a vector space V with some given connection ∇, it is necessary and sufficient that for any 1-form ω, $Y_t = \nabla \int_{X_0^t} \omega$ is a real local martingale.

We consider the problem of defining semimartingales with values in a Riemannian manifold $(\mathcal{M}, g)$. Let $(\mathcal{M}, g)$ be a Riemannian manifold of dimension n and consider its corresponding frame bundle, with frame $(H_i)_{i=1}^n$. An $\mathcal{M}$-valued semimartingale X is defined throughout the set $Z = (Z^i)_{i=1}^n$ of real semimartingales such that the Stratonovitch formula is

$$Z_t^i = \int_{X_0^t} \omega^i, \tag{1}$$

where $\omega^i \in T^*\mathcal{M}$ is a 1-form in the cotangent bundle $T^*\mathcal{M}$ (see [5] for a precise definition of (1)).

A classical problem in stochastic differential equations is to understand how to reconstruct X from a real semimartingale Z. That is, given a real valued semimartingale Z one looks for the $\mathcal{M}$-valued semimartingale X with given X_0 and satisfying

$$d^2 X_t = h_{X_t}(d^2 Z_t),$$

where h is defined by putting $dx^i = h_a^i \omega^a$.

The Stratonovitch like formula leads to the Itô formula. This step leads to highlighting relations to connections on a manifold:

$$X_t^i - X_0^i = \int_{X_0^t} dx^i = \int_{X_0^t} h_a^i \omega^a = \int_0^1 h_a^i(X_b) \cdot dZ_s^a.$$

Indeed, this implies that $dX_t^i = h_a^i(X_s)dZ_s^a + \frac{1}{2} d\langle h_a^i(X_s), Z_s^a\rangle$. But since

$$d\langle h_a^i(X_s), Z_s^a\rangle = D_j h_a^i(X_s) d\langle X_s^a, Z_s^a\rangle\,, \quad d\langle X^j, Z^a\rangle = h_b^j(X_s) d\langle Z^b, Z^a\rangle_s\,,$$

so $dX_t^i = h_a^i(X_s)dZ_s^a + \frac{1}{2}(D_j h_a^i(X_s) h_b^j(X_s) d\langle Z_s^b, Z_s^a\rangle)$. Symbolically this amounts to writing

$$dX_t = dZ_t^a \cdot H_a(X).$$

2. Results on $(\mathcal{M}, g)$-valued semimartingales

Theorem 1. *Consider one of the following symmetric Riemannian manifolds $GL_n(\mathbb{R})/SO_n$, $GL_n(\mathbb{C})/SU_n$, $GL_n(\mathbb{H})/Sp_n$ $GL_3(\mathbb{O})/F_4$ and $O(1, n-1)/O(n-1) \oplus \mathbb{R}$ (or a linear combination of those), where $n \geq 2$ is a positive integer. If $\mathcal{M}$ is one of those manifolds then it possess all required conditions for $(\mathcal{M}, g)$-valued semimartingales to be well defined.*

The proof of this relies on the approaches of L. Schwartz, M. Emery, P A. Meyer. We illustrate this on the following fact. Let $X^1, \cdots, X^n$ be continuous semimartingales and $f \in C^2(\mathbb{R}^n)$. Then $Y = f(X)$ is a semimartingale and $dY = \sum_i D_i f(X) dX^i + \frac{1}{2} \sum_{i,j} D_{ij} f(X) d\langle X^i, X^j\rangle$. The rightmost part of the equation is ruled by the connection of the manifold. Connections for the listed above manifolds are torsion-free and the covariant derivatives form a pre-Lie algebra. By [1] those manifolds are potential pre-Frobenius manifolds. This implies that there exists everywhere locally a potential function (given by the Koszul–Vinberg characteristic function) such that the Hessian is non-degenerate.

Furthermore, following [1] any of the spaces listed above obey to a decomposition into two submanifolds:

(1) a flat torus being a totally geodesic submanifold of $(\mathcal{M}, g)$. It carries the structure of a Frobenius manifold; all geodesics lie in that subspace.
(2) A homogeneous Hadamard space, having strictly negative sectional curvature.

This implies the following.

Proposition 1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold. Suppose that $(\mathcal{M}, g)$ is one of the following $GL_n(\mathbb{R})/SO_n$, $GL_n(\mathbb{C})/SU_n$, $GL_n(\mathbb{H})/Sp_n$ $GL_3(\mathbb{O})/F_4$ and $O(1, n-1)/O(n-1) \oplus \mathbb{R}$ (or a linear combination of those), where $n \geq 2$ is a positive integer. Consider $\overline{\mathcal{M}}$ the Frobenius manifold (a flat torus) in $(\mathcal{M}, g)$. Each point of $\overline{\mathcal{M}}$ has an open neighborhood $U \subset \mathcal{M}$ such that for every U-valued martingale X with $X_1 \in \overline{\mathcal{M}}$ a.s the whole process $(X_t)_{0\leq t\leq 1}$ lives in the Frobenius manifold $\overline{\mathcal{M}}$.*

The proof is based on works of M. Emery [3] and of the theorem in [1].

Proposition 2. *The Frobenius manifold in $(\mathcal{M}, g)$ (where $(\mathcal{M}, g)$ is defined as above) is the locus in which exist pure fluctuations / local martingales.*

Remark 1. *The above Riemannian manifolds parametrise the space of Wishart probability distributions. Wishart laws being exponential we can proceed to a direct application of our statements above and of the main theorem of* [2]. *In the latter, the existence of a Frobenius manifold in a space of probability distributions of exponential type is shown.*

So, as a corollary we have:

Corollary 1. *Let $(\mathcal{M}^W, g^W)$ be a manifold of Wishart distributions (finite dimensional). Then, there exists $\overline{\mathcal{M}}^W$ a Frobenius manifold of $(\mathcal{M}^W, g^W)$ such that for each point of $\overline{\mathcal{M}}$ one has an open neighborhood $U \subset \mathcal{M}$ and for every U-valued martingale X with $X_1 \in \overline{\mathcal{M}}^W$ a.s the whole process $(X_t)_{0\leq t\leq 1}$ lives in the Frobenius submanifold $\overline{\mathcal{M}}^W$.*

Conclusion. We have explored some aspects of the question raised by P. Cartier. Further developments concerning a discussion and classification of manifolds satisfying good properties for semimartingales is expected.

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Planarly Branched Rough Paths Are Geometric

Ludwig Rahm

(joint work with Kurusch Ebrahimi-Fard)

In 2002 Foissy published the important work [3], where he characterized all finite-dimensional comodules and all endomorphisms of the Butcher–Connes–Kreimer Hopf algebra $\mathcal{H}_{BCK} = (\mathcal{F}, \odot, \Delta_{BCK})$ of non-planar rooted trees. He furthermore constructed a recursively defined projection map onto the primitive elements of the Hopf algebra, and showed a Hopf algebra ismomorphism to the shuffle Hopf algebra generated by the primitives. Almost all of his proofs were based on the so-called natural growth operation

$$\top : \mathcal{H}_{BCK} \otimes \mathcal{H}_{BCK} \to \mathcal{H}_{BCK},$$

and its relation to the reduced coproduct:

$$\hat{\Delta}_{BCK}(x \top y) = x \otimes y + x^{(1)} \otimes x^{(2)} \top y, \tag{1}$$

where y is a primitive element and we use Sweedler's notation for the reduced coproduct

$$\hat{\Delta}_{BCK}(x) = x^{(1)} \otimes x^{(2)}.$$

The Hopf algebra isomorphism constructed by Foissy was later used by Boedihardjo and Chevyrev to interpret branched rough paths as being geometric rough paths [1]. This allowed the authors to consider important results on the well-studied theory of geometric rough paths, and obtain the same results for branched rough paths.

Rough path theory is a very successful theory for solving rough differential equations. A rough path is a two-parameter path taking values in the character group of a Hopf algebra. A branched rough path takes values in the BCK Hopf algebra, and a geometric rough path takes values in a shuffle Hopf algebra. Both of these rough path theories are used for rough differential equations on Euclidean spaces. In [2], the authors constructed so-called planarly branched rough paths to solve rough differential equations on homogeneous spaces. These rough paths are valued in the Munthe-Kaas-Wright Hopf algebra $\mathcal{H}_{MKW} = (\mathcal{OF}, \sqcup\!\sqcup, \Delta_{MKW})$.

In this talk we note that the MKW Hopf algebra can be endowed with a natural growth operation, meaning a map that satisfies equation (1) for the reduced MKW coproduct $\hat{\Delta}_{MKW}$. This lets us apply the results Foissy obtained for $\mathcal{H}_{BCK}$, to $\mathcal{H}_{MKW}$. In particular, we obtain a Hopf algebra isomorphism between $\mathcal{H}_{MKW}$ and a shuffle Hopf algebra. We also obtain a way to find the primitive elements via a recursively defined projection map. Following the approach of Boedihardjo and Chevyrev, we can then interpret planarly branched rough paths as being geometric rough paths by using the Hopf algebra isomorphism. Results for geometric rough paths can then be transfered to results for planarly branched rough paths. As an example of this, we obtain the result that two planarly branched rough paths have the same signature if and only if they are tree-like equivalent.

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Numerical approximations to rough solutions of dispersive equations

Yvonne Alama Bronsard

An introduction was given on *resonance-based schemes*, a class of schemes which allows for the approximation at low-regularity to the following class of nonlinear dispersive equation:

$$
\begin{aligned}
&i\partial_t u(t,x) + \mathcal{L}(\nabla)\, u(t,x) = p\,(u(t,x), \overline{u}(t,x)) \\
&u(0,x) = u_0(x), \quad x \in \mathbb{T}^d,
\end{aligned}
\tag{1}
$$

with $\mathcal{L}$ real operator, p polynomial nonlinearity. The idea behind their construction was illustrated on the prototypical Nonlinear Schrödinger equation (NLS):

$$
i\partial_t u(t,x) = -\Delta u(t,x) + |u(t,x)|^2 u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d, \ + \text{I.C},
$$

where one goes to Fourier variables in space to carefully extract dominant and lower-order contributions appearing from the interaction of the linear evolution and the nonlinearity. This idea was worked out by [6] in the first order case. A generalization was made by [7] which allows for first and second order low-regularity approximation to a class of nonlinear evolution equation set on more general domains. These new schemes, together with their optimal local error, allow for convergence under lower regularity assumptions than required by classical methods, such as exponential integrator or splitting methods.

Higher order extensions were then presented, following new techniques based on decorated trees series inspired by singular SPDEs via regularity structures. The work of [4] was first presented, where the authors derive resonance-based schemes up to arbitrary order for solving the class of equations (1).

We then presented the work [2] which considers the case of a randomized initial condition of the form:

$$u(0,x) = v^{\eta}(x) = \sum_{k\in\mathbb{Z}^d} v_k \eta_k(\omega) e^{ikx}, \tag{2}$$

with $(\eta_k)_{k\in\mathbb{Z}}$ i.i.d standard complex Gaussians. By letting u be solution of (1) starting from the randomized initial data (2), we obtained higher order approximations to the second moment $\mathbb{E}(|u_k(t,v^{\eta})|^2)$, together with a formal local error bound. This second order moment is a central quantity of interest for the derivation of the Wave Kinetic equation. This equation is widely used in oceanography for the forecasting of waves in the ocean.

A limitation of the former resonance-based approaches was, since the algorithm for extracting dominant parts depended on Fourier computations, the method is restricted to spatial domains which are periodic. In the work [3] we consider systematizing the higher order derivation of low-regularity schemes for the following class of nonlinear evolution equations set on more general domains:

$$\partial_t u - \mathcal{L}u = \sum_{\mathfrak{l}} f_{\mathfrak{l}}(u,\overline{u})V_{\mathfrak{l}}, \qquad (t,x)\in\mathbb{R}\times\Omega,\ \Omega\subseteq\mathbb{R}^d.$$

This work was inspired by the work [1] which dealt with the first and second order low-regularity approximation to the Gross-Pitaevski equation. In the work [3] we extended it to higher orders and for a more general class of nonlinear evolution equations, using the commutators and filtering functions introduced in [7].

We finished by presenting a symmetric low-regularity schemes for the NLS equation, which exactly conserves time-reversibility of the underlying equation. We explained on the one hand the construction of this symmetric scheme, which inherits much better structure preserving properties on the discrete level than previous low-regularity schemes. On the other hand we presented rigorous low-regularity error analysis results.

Higher order construction of a class of symmetric schemes using the previously introduced tree formalism was briefly discussed through our recent joint work [4].

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Resonances as a computational tool

Katharina Schratz

A large toolbox of numerical schemes for dispersive equations has been established, based on different discretization techniques such as discretizing the variation-of-constants formula (e.g., exponential integrators) or splitting the full equation into a series of simpler subproblems (e.g., splitting methods). In many situations these classical schemes allow a precise and efficient approximation. This, however, drastically changes whenever non-smooth phenomena enter the scene such as for problems at low regularity and high oscillations. Classical schemes fail to capture the oscillatory nature of the solution, and this may lead to severe instabilities and loss of convergence. In this talk I present a new class of resonance based schemes. The key idea in the construction of the new schemes is to tackle and deeply embed the underlying nonlinear structure of resonances into the numerical discretization.

Let me explain the key idea behind resonances as a computational tool on the nonlinear PDE[1]

$$\partial_t u(t,x) + i\mathcal{L}\left(\nabla, \varepsilon^{-1}\right) u(t,x) = f\big(u(t,x)\big), \quad u(0,x) = u_0(x) \tag{1}$$

which covers a large class of important models, e.g., Schrödinger ($\mathcal{L} = -\Delta$), KdV ($\mathcal{L} = -i\partial_x^3$) and half-wave ($\mathcal{L} = \sqrt{-\Delta}$) equations, wave maps, Zakharov, Kadomtsev–Petviashvili, and many more systems.

The symmetric differential operator $\mathcal{L}\left(\nabla, \varepsilon^{-1}\right)$ thereby triggers oscillations (in space and/or in time) and, unlike for parabolic problems, no smoothing can be expected. At low regularity, e.g., for rough solutions and in highly oscillatory regimes $\varepsilon \to 0$, it is therefore crucial to capture these oscillations numerically. Most classical schemes were originally developed for linear problems and fail to resolve the nonlinear frequency interactions in system (1).

The key idea to overcome this is to understand, control and deeply embed the nonlinear resonance structure (driven by the nonlinear frequency interaction of the operator $\mathcal{L}$ and nonlinearity f in (1)) into the numerical discretisation. In order to achieve this we have to first understand the behaviour of the nonlinear PDE (1). Duhamel's formula (suppressing the x-dependence) reads

[1]We include the parameter ε^{-1} to also cover relativistic regimes, e.g., relativistic Klein–Gordon with $\mathcal{L} = \varepsilon^{-1}\sqrt{\varepsilon^{-2} - \Delta}$

$$u(t) = e^{-it\mathcal{L}}u(0) + \int_0^t \mathrm{e}^{-i(t-\xi)\mathcal{L}} f(u(\xi))\mathrm{d}\xi \tag{2}$$

with the next iteration (i.e., using that $u(\xi) = \mathrm{e}^{-i\xi\mathcal{L}}u(0) + \int_0^\xi \dots \mathrm{d}\xi_1$) given by

$$u(t) = e^{-it\mathcal{L}}u(0) + \int_0^t \mathrm{e}^{-i(t-\xi)\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right)\mathrm{d}\xi + \int_0^t \int_0^\xi \dots \mathrm{d}\xi\mathrm{d}\xi_1. \tag{3}$$

At first order we can neglect the higher order terms (i.e., the double integral) and observe that the underlying structure of the solution is driven by the nonlinear frequency interaction of $\mathcal{L}$ and f with central oscillations of the form

$$\mathrm{e}^{i\xi\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right). \tag{4}$$

Classical numerical methods are based on linear frequency approximations, (e.g., splitting schemes, Gautschi-type, exponential and Lawson methods with possible filter functions, and in general neglect the nonlinear interactions in (4). For instance, in case of splitting or an exponential approach the underlying frequency approximations read

$$\text{(splitting)} \qquad \mathrm{e}^{i\xi\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right) \approx f\left(u(0)\right) \tag{5}$$

While such linearised frequency approximations are computational very handy (as on the right-hand side of (5) no oscillations anylonger appear), they dramatically destroy the underlying structure of the PDE (1). This is due to the fact that nonlinear frequency interactions play an essential role (especially on bounded domains, where no dispersion can be expected) and can heavily impact the solution: Note that while the influence of $i\mathcal{L}$ can be small, the influence of the interaction of $+i\mathcal{L}$ with $-i\mathcal{L}$ can be huge, and *vice versa.* The central idea lies in a new nonlinear approach: Instead of linearising the frequency interactions in the central oscillations (4) (as done in (5)) the key idea is to filter out the dominant parts of the oscillations and solve them exactly while only approximating the lower order terms in spirit of

$$\mathrm{e}^{i\xi\mathcal{L}} f\left(\mathrm{e}^{-i\xi\mathcal{L}}u(0)\right) \approx \left[\mathrm{e}^{i\xi\mathcal{L}_{\mathrm{dom}}} f_{\mathrm{dom}}(u(0))\right] f_{\mathrm{noc}}(u(0)) + \text{lower order terms}. \tag{6}$$

Here, $\mathcal{L}_{\mathrm{dom}}$ denotes a suitable dominant part of the high frequency interactions and f_{noc} the corresponding non-oscillatory part. A first attempt of so-called *resonance-based schemes* (Schratz et al. [4]), based on the approximation (6), was profoundly inspired by major breakthroughs in the theoretical analysis of dispersive equations at low regularity (Bourgain [3], Tao [8]) and rough path theory (Gubinelli [6]) and provides a powerful tool which in many situations allows for approximations in a much more general setting (i.e., for rougher data) than classical schemes (e.g., Splitting with $\mathcal{L}_{\mathrm{dominant}} = 0$ cf. (5)), see also the recent important works [2, 5, 7, 9, 10] and references therein.

The severe shortcoming of the approach (6), however, lies in the fact that the corresponding resonance-based schemes are *not* structure preserving as they do not take the underlying geometric structure of PDEs into account. Lack of structure preservation is also observed drastically in numerical experiments and, as for classical schemes, breaks down the earlier and earlier the rougher the solutions becomes.

This is an open question, and up to now only symmetric schemes could be found [1].

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Multiparameter iterated integrals

Joscha Diehl

Iterated sums and integral have in the last decade found great success in data science applications. Whereas the original domain of their application is to data indexed by one parameter, i.e. time series, there are recent investigations of multi-parameter generalizations [2, 8, 4, 3].

The success of iterated sums/integrals is partly explained by the fact that their calculation is possible in linear time, owing to a dynamic programming principle. It finds its algebraic counterpart in *Chen's formula*, which establishes a connection between the concatenation of words and the concatenation of time series.

For multi-parameter objects, the situation is more complicated. There is no canonical way to concatenate two objects, and, apart from special cases, none of the algebraic structures in the mentioned papers is compatible with the different concepts of concatenation. This has the consequence that the calculation of the multiparameter sums from [2] or the multiparameter integrals from [3] is, in general, not possible in linear time (lower complexity bounds for special cases are established in [2]).

Let us illustrate the problem with a simple example. Consider the following integral of a two-parameter function $Z : [0,1]^2 \to \mathbb{R}$:

$$\int_{\substack{0 \le r_1^1 < r_1^2 \le t_1 \\ 0 \le r_2^1 < r_2^2 \le t_2}} Z_{r_1^1, r_2^1} Z_{r_1^2, r_2^2} dr$$

Now, we try to split this integral in the horizontal direction at some point $u_1 < t_1$:

$$\ldots = \int_{\substack{0 \le r_1^1 < r_1^2 < u_1 \le t_1 \\ 0 \le r_2^1 < r_2^2 \le t_2}} Z_{r_1^1, r_2^1} Z_{r_1^2, r_2^2} dr + \int_{\substack{u_1 < r_1^1 < r_1^2 \le t_1 \\ 0 \le r_2^1 < r_2^2 \le t_2}} Z_{r_1^1, r_2^1} Z_{r_1^2, r_2^2} dr$$
$$+ \int_{\substack{0 \le r_1^1 < u_1 < r_1^2 \le t_1 \\ 0 \le r_2^1 < r_2^2 \le t_2}} Z_{r_1^1, r_2^1} Z_{r_1^2, r_2^2} dr.$$

Note that the last term presents an issue, since the integral cannot be split into a product of two integrals, as it would in the one-parameter case.

The problem of (naive) non-multiplicativity of multi-parameter integrals is well-known in category theory and it has been addressed with techniques from higher categories, see for example [1, 7] for entry points into the literature.

In the work in progress presented, which is joint with Ilya Chevyrev, Kurusch Ebrahimi-Fard, and Nikolas Tapia, we build on the work of [5] to realize an analog to the classical iterated-integrals signature that *does* satisfy a Chen-like identity and allows for a linear-time calculation. An important ingredient is the notion of crossed modules of Lie algebras, in particular the free crossed module of Lie algebras over the free Lie algebra over $\mathbb{R}^n$. Here n is the dimension of the ambient space of the data.

The techniques are closely related to the recent work [6], but the two approaches differ in at least two aspects:

(1) We work with the free crossed module of Lie algebras, whereas [6] works with a specific crossed module. Our current expectation is that the object obtained by us is universal in the sense that any "surface development" in another crossed module can be arbitrarily well approximated by terms in our object.

(2) We consequently do calculations in the Lie *algebra*, in what can be considered a Magnus-like expansion.

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From bialgebras to algebraic operads

Pedro Tamaroff

Algebraic homotopy theory explains that, in order to find new robust and more general definitions of algebras (algebras up to homotopy), one can find (possibly non-minimal) models for operads. These also allows us understand the homotopy theory of their (co)algebras and, in particular, define their deformation theory. This has been done in many situations: for associative algebras [1], commutative algebras, Lie algebras [2], Gerstenhaber and Poisson algebras [3], and Batalin–Vilkovisky algebras [4, 5, 6], among others. In the context of the current workshop, it is useful to remark that pre-Lie algebras fall within the scope of Koszul duality theory, through which most of the previous examples are handled, and other interesting recent generalizations due to P. Laubie [7] involving Greg trees decorated by coalgebras, have been proved to also fall within the scope of Koszul duality.

The present talk explained how to take an operadic point of view of the well known fact that, for any bialgebra H, the category of left H-modules admits an internal tensor product —defined through the so called diagonal action of H— coming from its coproduct. This means that it makes sense to consider associative algebras in the category ${}_H\mathrm{mod}$ of left H-modules, which we show are controlled by an algebraic operad. There is a functor $H \longmapsto \mathsf{Ass}_H$ that assigns to each weight graded bialgebra H a weight graded operad Ass_H so that an associative algebra in ${}_H\ \mathrm{mod}$ is the same as an Ass_H-algebra. The idea of producing such functors from certain "amenable" categories to study operads and related structures, or even producing endofunctors on operads themselves, has already appeared several times in the literature, see [18, 11, 19] and [13]*Chapter 4, for example.

Unraveling the definitions, we see that the way the associative product x_1x_2 of an Ass_H-algebra and an operation T_h coming from $h \in H$ behave with respect to each other is dictated by the coproduct of H: using Sweedler notation, we require that the following compatibility relation holds (á la Boardmann–Vogt [8]) $T_h(x_1x_2) = T_{h_{(1)}}(x_1)T_{h_{(2)}}(x_2)$. This relation is not quadratic, so the operad Ass_H falls outside the scope of the theory of Koszul duality, in strong contrast to the examples we mentioned above. To counter this, we use the methods of V. Dotsenko [19] (word operads) and pertubation theoretic methods in the spirit of B. Vallette and S. Merkulov [9] to show how to obtain a minimal model of Ass_H from an associated quadratic operad $q\mathsf{Ass}_H$ which, in case H is Koszul, is itself a Koszul operad. Moreover, we showed that this functor behaves well with respect to Gröbner bases: one can directly compute one of Ass_H in case H admits a Gröbner basis. In particular, we showed that $q\mathsf{Ass}_H$ is *strongly Koszul* —that it admits a quadratic Gröbner basis— in case H is itself strongly sKoszul.

The takeaway is that we can explicitly describe the differential of the minimal model of Ass_H provided we can do this for the Koszul model of H and its coproduct. This problem, pertaining to the domain of algebras and coalgebras, is usually a simpler problem to tackle, so our result gives a useful bridge to solve from a much familiar problem a seemingly more complicated one. The theory of Koszul duality for usual associative algebras, on the other hand, has existed for almost five decades since its inception in [10], and now extensive literature and methods exist to deal with them and with many of their variants; see for example [12, 14, 15, 16, 17].

As a by product, the talk introduced key concepts in the study of algebraic operads and their theory, which lead to and allowed for a detailed discussion of the results of P. Laubie [7] regarding families of pre-Lie algebras with a common Lie bracket by participant U. Nadeem. In particular, Koszulness of the operads constructed by Laubie, which follow from the existence of Gröbner bases for them, were discussed, and compared to the results presented in the talk.

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An algebraic geometry of (rough) paths

Rosa Preiss

In previous work, see e.g. [1], the complex projective Zariski closure of the finite dimensional semialgebraic set that is $\sigma^{(k)}(\mathcal{X}_\ell)$, where $\mathcal{X}_\ell$ is piecewise linear paths/polynomial paths/log-linear rough paths of order ℓ.

Our new approach, however, is to introduce a Zariski topology and algebraic geometry on the infinite dimensional path space itself, see the preprint [2].

The algebraic and combinatorial structure we are working with is

$$(T(\mathbb{R}^d), ⧢, \Delta_\bullet, \mathcal{A}, \succ, \prec),$$

where $(T(\mathbb{R}^d), ⧢, \Delta_\bullet, \mathcal{A})$ is the well known shuffle-deconcatenation Hopf algebra. Let the right $\succ$ and left $\prec$ halfshuffles be recursively defined by

$$w \succ \mathtt{i} := w\mathtt{i}, \qquad \mathtt{i} \prec w := \mathtt{i}w$$
$$w \succ v\mathtt{i} := (w \succ v + v \succ w)\mathtt{i}, \qquad \mathtt{i}v \prec w := \mathtt{i}(w \prec v + v \prec w)$$

Then $x ⧢ y = x \succ y + y \succ x = x \prec y + y \prec x$ and $\mathcal{A}(x \succ y) = \mathcal{A}y \prec \mathcal{A}x$, $\mathcal{A}(x \prec y) = \mathcal{A}y \succ \mathcal{A}x$. Let $\langle W \rangle_\succ$ denote the $\succ$-ideal generated by W.

In classical algebraic geometry, affine varieties in $\mathbb{R}^d$ are sets of the form $V(P) = \{x \in \mathbb{R}^d | p(x) = 0\, \forall p \in P\}$, where P is a set of polynomials $p : \mathbb{R}^d \to \mathbb{R}$.

Similarly, we now consider varieties in the space $C^{2^- \text{-var}}(\mathbb{R}^d)$ of continuous paths in $\mathbb{R}^d$ with finite p-variation for some $p < 2$. We call an *affine path variety* any subset of the form

$$\mathcal{V}(W) := \{X \in C^{2^- \text{-var}}(\mathbb{R}^d) | \langle \sigma(X), x\rangle = 0\, \forall x \in W\}, \quad W \subseteq T(\mathbb{R}^d)$$

They form the closed sets of what we introduce as the *path Zariski topology*. Path varieties are in 1-to-1 correspondance to the 2^--var 'radical' shuffle ideals

$$\mathcal{I}(U) := \{x \in T(\mathbb{R}^d) | \langle \sigma(X), x\rangle = 0\, \forall X \in U\}, \quad U \subseteq C^{2^- \text{-var}}(\mathbb{R}^d).$$

$\mathcal{V} \circ \mathcal{I}$ is the closure operator, and $\mathcal{I} \circ \mathcal{V}$ is the 2^--var radical operator. Our first main result is the following.

Theorem 1. *Whenever a set of paths U contains history, i.e. all left subpaths of reduced paths, $\mathcal{I}(U)$ is a $\succ$-ideal. Whenever I is a $\succ$-ideal, $\mathcal{V}(I)$ contains history.*

The next corollary is of key importance.

Corollary 1. *Let $p : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial map with $p(0) = 0$. Then $\mathcal{V}(\langle \varphi(p_i), i\rangle_\succ)$ is the variety $P_{\in M}$ of all paths X such that $\check{X} - X_0$ lies in the point variety M defined by the vanishing of all p_i.*

This allows us to define rough paths on point varieties! Indeed, it makes sense to demand that geometric rough paths living on an affine point variety should be those which are limits of smooth paths living on that point variety. This is a strictly stronger property than just the underlying path living on that point variety!

Our second main result concerns another way of using the time ordered aspect of paths, through concatenation.

Theorem 2. *If $M \subseteq C^{2^- \text{-}var}(\mathbb{R}^d)$ is a set of paths closed under concatenation, then the variety $\bar{M}$ is closed under concatenation, time reversal and taking admissible roots, and $\mathcal{I}(M)$ is a Hopf ideal.*

Corollary 2. *The set of lattice paths $\mathfrak{L}$ is Zariski dense in $C^{2^- \text{-}var}(\mathbb{R}^d)$.*

To summerize, if an affine path variety V contains history then $\mathcal{I}(V)$ is a half-shuffle ideal, and thus its coordinate ring $\mathbb{R}[V] := T(\mathbb{R}^d)/\mathcal{I}(V)$ is a Zinbiel algebra again.

If an affine path variety V is stable under concatenation, then $\mathcal{I}(V)$ is a Hopf ideal, and this means $\mathbb{R}[V] := T(\mathbb{R}^d)/\mathcal{I}(V)$ is a Hopf algebra.

An important remark, however, is that to understand the geometrical structure of V, we need the algebraic structure of the coordinate ring $\mathbb{R}[V]$ **plus** the 2^--var radical operator on the power set of $\mathbb{R}[V]$. At least until we can find a purely algebraic characterization of $\mathcal{I} \circ \mathcal{V}$, and can answer whether the radical operator can be derived from the ring structure of $\mathbb{R}[V]$ alone, or not.

In the discussion led by Ludwig Rahm we answered that while understanding $\mathcal{I} \circ \mathcal{V}$ for $C^{2^- \text{-var}}(\mathbb{R}^d)$ is a very hard problem, the solution to which would in particular solve the important open problem about how to characterize the image of the signature, understanding the radical operator for piecewise linear paths and polynomial paths should be feasible much earlier. Furthermore, as also brought up in a question by Ludwig Rahm, generalizations of our approach to maps from subsets of $\mathbb{R}^n$ to $\mathbb{R}^d$, instead of just time dependend paths, will become relevant. Finally, as asked by Ilya Chevyrev, a generalization of the notion of variety to vanishings $\langle \sigma(X), x \rangle = 0$ for infinite series $x \in T((\mathbb{R}^d))$ which can be paired with the signature is another opportunity for future work.

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Branched Itô formula

Nikolas Tapia
(joint work with Carlo Bellingeri, Emilio Ferrucci)

Branched rough paths were introduced by Gubinelli [8], as an extension of Lyons' original approach [9], in order to encode iterated integrals of processes that do not satisfy an integration-by-parts rule. These are defined as families of characters over the Connes–Kreimer Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ [3] of non-planar decorated rooted trees satisfying certain regularity and compatibility conditions. By leveraging Foissy's decomposition of this Hopf algebra in terms of its primitive elements [6] and iterations of the *natural growth* operator [2], we show that nonetheless an integration by parts rule is still satisfied. Primitive elements can be interpreted as higher-order variations of the process, analogous to the stochastic bracket present in classical Itô calculus, in the sense that they describe the correction terms in said formula. The algebraic structure precisely describing this new integration-by-parts identities is that of a $\mathbf{B}_\infty$-algebra [7].

Let $\mathcal{P}$ denote the space of primitive elements, $\pi\colon \mathcal{H}_{\mathrm{CK}} \to \mathcal{P}$ be Foissy's projection and $\mathcal{Q} := \mathrm{im}(\pi)^\perp$. Denote by $\mathcal{F}_+$ the set of non-empty forests. We define rough differential equations with drifts, as solutions to RDEs of the form

$$\mathrm{d}y = \sum_{f\in\mathcal{F}_+} F_{\pi^*(f)}(Y)\,\mathrm{d}X^{\pi(f)},$$

where $F \in \mathcal{L}(\mathcal{Q}, C^\infty(\mathbb{R}^n,\mathbb{R}^n))$ is a given collection of vector fields, and show they satisfy the following change of variable formula: there exists a family of differential operators $\mathbf{F}\colon \mathcal{L}(\mathcal{Q},\mathrm{Diff}(\mathbb{R}^n))$ such that for any smooth observable $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ we have

$$\varphi(y_t) = \varphi(y_0) + \sum_{f\in\mathcal{F}_+} \int_0^t \mathbf{F}_{\pi^*(f)}\varphi(y_u)\,\mathrm{d}\mathbf{X}_u^{\pi(f)}, \tag{1}$$

where the integrals are defined in the rough sense.

In the case of quadratic drift, i.e., an Itô SDE, (1) coincides with the classical Itô formula. The definition of $\mathbf{F}_q$ fully relies on the pre-Lie structure of vector fields on Euclidean space, and the proof relies on an extended form of Davie expansion including corrections induced by the drifts.

We also show that quasi-geometric rough paths correspond to a particular quotient of branched rough paths, and therefore an analog of formula (1) holds in that case. This is connected to already-known formulas [1, 5].

Question 1. Any coalgebra equipped with a family of 1-cocycles indexed by its primitive elements is cofree. What other kinds of rough paths can be show to satisfy integration-by-parts, and therefore an Itô formula? This question has been partially answered by K. Ebrahimi-Fard and L. Rahm.

Question 2. In regularity structures Hopf algebras also play an important role. Is it possible to obtain a similar decomposition for positive and/or negative renormalization? If so, what would be the interpretation of primitive elements in that context?

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Natural Itô-Stratonovich isomorphism

Emilio Ferrucci
(joint work with Carlo Bellingeri, Nikolas Tapia)

Following the theory introduced by Nikolas Tapia in a previous talk, we consider the Connes–Kreimer Hopf algebra $\mathcal{H}_{\mathrm{CK}}$ as a commutative $\mathbf{B}_\infty$-algebra over its primitive elements $\mathcal{P}$. After introducing the Eulerian idempotent of a Hopf algebra and some of its properties, we use it, together with Foissy's idempotent $\pi\colon \mathcal{H}_{\mathrm{CK}} \to \mathcal{P}$, to define an explicit Hopf isomorphism from the shuffle algebra over $\mathcal{P}$ to $\mathcal{H}_{\mathrm{CK}}$. This isomorphism can be used to transform branched rough paths to geometric ones of inhomogeneous regularity over a larger space. Compared to the work of [1, 2], who considered this problem previuosly, our isomorphism has the distinguishing property of being a natural transformation when $\mathcal{H}_{\mathrm{CK}}$ and the shuffle algebra are viewed as a covariant functor in the decorating vector spaces. The motivation for naturality comes from, among other things, the requirement that our theory continue to work when shifted to the setting of smooth manifolds. We compare our isomorphism with Hoffman's exponential [3], which can be obtained from it, but which contains strictly fewer terms: those not present come from interactions between forests that may contain edges. Our work has since appeared on arXiv [4].

During the Q&A, the question of uniqueness (subject to naturality) came up. During the discussion portion, Yvain Bruned brought up the question of how the theory might develop along similar lines when $\mathcal{H}_{\mathrm{CK}}$ is replaced with more recently-introduced Hopf algebras that appear in the context of regularity structures. When cofreeness no longer holds, it might not be reasonable to find an isomorphism, and a natural epimorphism may be the next best thing.

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Solution theory for quasilinear generalised KPZ Equation

Muhammad Usama Nadeem
(joint work with Yvain Bruned and Mate Gerencser)

In this work, we provide a (local-in-time) solution theory for the so called quasi-linear generalised KPZ equation defined on the 1-d torus, that takes the following form:

$$\partial_t u - a(u)\partial_x^2 u = \underbrace{f(u)(\partial_x u)^2 + k(u)\partial_x u + h(u)}_{F_1} + \underbrace{g(u)}_{F_\Xi}\xi. \tag{1}$$

Here f, k, h, and g are assumed to be smooth (although one may be able to survive with functions regular enough), a is smooth with the additional constraint of being bounded by some $c \in \mathbb{R}_+$, and ξ is a random spacetime distribution - the quintessential example being that of the spacetime white noise. This equation falls under the umbrella of singular Stochastic Partial Differential Equation (SPDEs) of the parabolic type and as such the regularising effect of the dynamics fall short of facilitating the pathwise understanding of certain products in the equation. In the equation above and the sort of random fields we are after, $g(u)(\partial_x u)^2$ for example, does not make sense.

In the semilinear case (i.e. when $a(u) \equiv 1$) the advent of the theories such as regularity structures [7] have provided a definite solution to this problem. A major component of this theory is the notion of (negative) renormalisation, which amounts to subtracting infinite constants (dubbed renormalisation constants) from the equation to cure the divergence caused by the ill-defined products in the equation. From a generalisation of Hairer's work [2] we can quote the renormalised equation for the semilinear gKPZ:

$$\partial_t u_\varepsilon - \partial_x^2 u_\varepsilon = f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + k(u_\varepsilon)\partial_x u_\varepsilon + h(u_\varepsilon) + \sum_\tau \frac{\Upsilon_F[\tau]}{S(\tau)} C_\varepsilon(\tau) + g(u_\varepsilon)\xi_\varepsilon, \tag{2}$$

where ξ_ε is a mollified version of the noise, τ denotes rooted trees that encode multiple stochastic integral, $S(\tau)$ is a symmetry factor, $C_\varepsilon(\tau)$ are the renormalisation constants, and $\Upsilon_F[\tau]$ are elementary differentials that are defined by taking derivatives of F_1 and F_Ξ. This has also inspired investigation into potential adaptation of these techniques to quasilinear problems. We refer the reader to [5, 8, 9, 10, 11] for the existing state of this work. The present work is an extension of the work [5], wherein the authors employ an innovative extension of the Hairer's original work. What the authors realise in [5], is that the local-in-time solution of (1) solves the following the system of equations:

$$
\begin{aligned}
&u = I(a(u), \hat{F}), \qquad v_\alpha = I_\alpha(a(u), \hat{F}), \\
&\hat{F} = \left[q(f - a') + a(a')^2 v_{(2,0)} + aa'' v_{(1,0)}\right](\partial_x u)^2 \\
&\qquad + 2(aa')(u)(\partial_x u) v_{(1,1)} + a'(u)(\partial_x u) v_{(0,1)} + \hat{g}(u)\xi.
\end{aligned} \tag{3}
$$

The benefit of this reformulation is that the existing results of the semilinear SPDEs can be applied with only minor changes. Unfortunately, due to the restrictions of the methodology (1) remains out of reach of this work. We ameliorate this condition by introducing some new abstract derivatives ∂_{v_α} for $\alpha \in \mathbb{N}^2$, wherewith we define new elementary differential equations: $\Upsilon_{\hat{F}}$ and Υ_{V_α}. This allows us to prove the following theorem:

Theorem 1. *The renormalised version of* (1) *is given by:*

$$
\begin{aligned}
\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + k(u_\varepsilon)\partial_x u_\varepsilon + h(u_\varepsilon) + g(u_\varepsilon)\xi_\varepsilon \\
+ \sum C_\varepsilon^c(\tau) \frac{\Upsilon_{\hat{F}}[\tau](u_\varepsilon)}{S(\tau)}.
\end{aligned} \tag{4}
$$

Also, the local solutions u_ε on $\mathbb{T}$ endowed with an initial condition $u_\varepsilon(0,\cdot) = \varphi \in \mathcal{C}^{2\delta}(\mathbb{T})$, converge in probability in $\mathcal{C}_\star^\delta$ to a nontrivial limit u.

The defect that [5] suffers from, and by extension our work, is that there is no systematic way of proving that the renormalisation constants C_ε^c do not depend on the non-local terms v_α that were introduced into the equation when transforming the equation into the non-divergence form. The way this is dealt with by them, is to prove that the constants satisfy some integration by parts [4, Lemma 2.4], and then use this result to check for each τ that the non-local terms cancel out [4, Section 3.4]. The problem with this approach is that it very easily becomes unwieldy, due to the amount of calculations involved. Our solution to this problem is to recognise that the integration-by-parts formula is just a specific case of the chain rule that was derived in [3]. To leverage the results of that paper we need to specify a set of covariant derivatives that are capable of generating the space of τ, and at the same time are independent of the non-local terms. By positing these, we suspect the following result is immediate from the arguments in [3]:

Conjecture 1. *The renormalised equation of* (1) *is given by:*

$$
\begin{aligned}
&\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = \\
&f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + k(u_\varepsilon)\partial_x u_\varepsilon + h(u_\varepsilon) + g(u_\varepsilon)\xi_\varepsilon + \sum_\tau C_\varepsilon^{a(u_\varepsilon)}(\tau) \frac{\Upsilon_F[\tau](u_\varepsilon)}{S(\tau)}
\end{aligned}
$$

where the $C_\varepsilon^{a(u)}(\tau)$ satisfy certain chain rule identities. Moreover u_ε converges in the same sense as before.

Some possible open problems in this programme include that of global-in-time solutions, identification of the Butcher series for these problems á la [1], and finally one could look at the same problem in some other manifold.

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Novikov algebras and multi-indices in regularity structures

Yvain Bruned
(joint work with Vladimir Dotsenko)

We are looking at the class of subcritical semi-linear stochastic partial differential equations (SPDEs) of the form

$$(\partial_t - \mathcal{L})\, u = \sum_{\mathfrak{l} \in \mathfrak{L}^-} a^{\mathfrak{l}}(\mathbf{u}) \xi_{\mathfrak{l}}. \tag{1}$$

where $\mathfrak{L}^-$ is a finite set, $\mathcal{L}$ is a differential operator, $\xi_{\mathfrak{l}}$ are space-time noises and $a^{\mathfrak{l}}(\mathbf{u})$ are non-linearities depending on the solution u and its derivatives. This class of equations have been successfully treated via the theory of Regularity Structures [9, 3, 1, 6]. The resolution is based on new Taylor expansions whose monomials are recentered iterated integrals that can be described in a systematic way via decorated trees in [3]. More recently, another index set has been proposed in [13, 11] for quasi-linear SPDEs. It has been extended in [4] for covering equations of the form (1). The simplest possible instance of multi-indices corresponds to considering a set of abstract variables $(z_k)_{k\in\mathbb{N}}$, where the variable z_k encodes nodes of the tree that have k children. Multi-indices β over $\mathbb{N}$ can be represented as monomials

$$z^\beta := \prod_{k\in\mathbb{N}} z_k^{\beta(k)}.$$

The pre-Lie product on the vector space of such monomials is defined as

$$z^\beta \triangleright z^{\beta'} = z^\beta D(z^{\beta'}), \quad D = \sum_{k\in\mathbb{N}} (k+1) z_{k+1}\partial_{z_k}.$$

The action of this operator corresponds to adding one child to one of the nodes of our tree in all possible ways. We focus on multi-indices satisfying the so called "populated" condition [11]:

$$\sum_{k\in\mathbb{N}} (1-k)\beta(k) = 1.$$

It was conjectured by Dominique Manchon that populated multi-indices form the free Novikov algebra. A Novikov algebra is a vector space equipped with a bilinear product $x, y \mapsto x \triangleright y$, satisfying the identities

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),$$
$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright y.$$

This type of algebras was considered in [8, 5, 12]. It turns out that the corresponding theorem does exist in the literature; it goes back to [7].

Theorem 1. [7, 10, 2] *The Novikov algebra of populated multi-indices is isomorphic to the free algebra on one generator.*

One first extends this theorem to general multi-indices defined using formal variables of the form $z_{(\mathfrak{l},w)}$ with $\mathfrak{l}$ belongs to $\mathfrak{L}^-$ and w is a commutative monomial in the alphabet $A = \mathbb{N}^{d+1}$. One can define a collection of derivations $D^{(\mathbf{n})}$ indexed by A. These very general multi-indices have been proposed in [4]. One needs a new structure for these multi-indices called multi-Novikov algebra which is a vector space equipped with bilinear products $x, y \mapsto x \triangleright_a y$ indexed by a set A

$$(x \triangleright_a y) \triangleright_b z - x \triangleright_a (y \triangleright_b z) = (y \triangleright_a x) \triangleright_b z - y \triangleright_a (x \triangleright_b z),$$
$$(x \triangleright_a y) \triangleright_b z - x \triangleright_a (y \triangleright_b z) = (x \triangleright_b y) \triangleright_a z - x \triangleright_b (y \triangleright_a z),$$
$$(x \triangleright_a y) \triangleright_b z = (x \triangleright_b z) \triangleright_a y,$$

for all $a, b \in A$. This is analogue to the generalisation from pre-Lie algebras to multi-pre-Lie algebras in [1]. One gets a new version of Theorem 1.

Theorem 2. [2] *The multi-Novikov algebra of populated general multi-indices is isomorphic to free algebra generated by the set $\mathfrak{L}^-$.*

For capturing the complexity of the multi-indices for singular SPDEs, one has to introduce other derivations ∂_i, $0 \le i \le d$, that satisfy, together with the derivations $D^{(\mathbf{n})}$, the following relations:

$$D^{(\mathbf{n})} D^{(\mathbf{m})} = D^{(\mathbf{m})} D^{(\mathbf{n})}, \quad \partial_i \partial_j = \partial_j \partial_i$$
$$D^{(\mathbf{n})} \partial_i = \partial_i D^{(\mathbf{n})} + n_i D^{(\mathbf{n}-e_i)},$$

where e_i is the standard basis vector of $\mathbb{N}^{d+1}$. There is a corresponding generalisation of multi-indices which we shall call SPDE multi-indices.

Theorem 3. [2] *The multi-Novikov algebra of populated SPDE multi-indices is isomorphic to free algebra generated by the set* $\mathbb{N}^{d+1} \times \mathfrak{L}^-$.

After free multi-pre-Lie, one has a new free structure useful for expanding solutions of singular SPDEs. They are several applications/open problems to such a result:

- One can try to find other combinatorial sets and their free structures that will be different from multi-indices and decorated trees.
- One can get a more operadic perspective as it was initiated in [14] that recovers as an example the multi-Novikov algebra.
- One can study symmetries in the contex of multi-indices like the chain rule or Itô isometry by defining maps from the free Novikov structure.

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Mini-Workshop: Homological aspects of TDLC-groups

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27 November – 01 December 2023

Abstract. This mini-workshop aimed at bringing together experts and early career researchers on finiteness conditions for discrete groups, and experts on varying aspects of locally compact groups to find a common framework to develop a systematic theory of homological finiteness conditions for totally disconnected locally compact groups. Whereas the homological theory of finiteness conditions of discrete groups is well developed and the structure theory of totally disconnected locally compact groups has seen some important breakthroughs in the last decade, the homological theory for (non-compact) totally disconnected locally compact groups is an emerging research area. Specific topics include finiteness conditions for locally compact groups, Mackey functors and Bredon cohomology for topological groups, connections to condensed mathematics, connections to ℓ^2-invariants and Σ-invariants.

Mathematics Subject Classification (2020): 18Gxx, 20Exx, 20Fxx, 20Jxx, 22Dxx, 57Txx.

Introduction by the Organizers

The class of locally compact (= LC) groups plays a central role among topological groups. With the solution of Hilbert's 5th problem, the understanding of the structure of connected LC-groups has significantly increased. Since every LC-group is an extension of a connected LC-group by a totally disconnected LC-group, the contemporary structure problem focuses on totally disconnected LC-groups (= TDLC-groups). In the last decades there has been a significant progress in the study of the structure theory of TDLC-groups, see e.g. [2, 4, 14, 10, 12, 13]. However, the study of homological finiteness conditions has, so far, been rather disjointed and piece-meal. Stefan Witzel gave a 3-lectures survey on finiteness properties for LC-groups, discussing the notions introduced by Castellano–Corob Cook [6] and Abels–Tiemeyer [1] and stressing further on the missing pieces of the theory. Ged Corob Cook, during his talk, suggested a new strategy to investigate finiteness properties for TDLC-groups based on new model structures on k-spaces and simplicial k-spaces [8]. Homological finiteness conditions are very well understood for discrete groups [3], and there is a rich theory for cohomology (both discrete and profinite) for profinite groups. A link, however, to the theory of TDLC-groups, so far, is only very superficial. In recent years, partly due to the theory developed by Castellano–Weigel [7], the study of cohomological finiteness conditions for TDLC-groups has had a little bit of a resurgence [6, 5, 11]. Thomas Weigel's lectures introduced the state of the art of the rational discrete cohomology for TDLC-groups [7], and highlighted the connection with zeta functions for groups. Stable categories for the rational discrete modules of a TDLC-group were considered by Rudradip Biswas, whereas Sofiya Yatsyna introduced Gedrich and Gruenberg invariants for TDLC-groups. Bianca Marchionna presented her recent work concerning double coset zeta functions of TDLC-groups acting on trees, and Laura Bonn discussed the relation between the finiteness properties of a discrete group and those of its Schlichting completion. Peter Kropholler offered three lectures related to finiteness properties of discrete groups that culminated in a lecture on condensed mathematics and profinite groups. He highlighted the advantages of condensed mathematics over other theories: homological algebra in condensed mathematics takes place in abelian categories, which provides an apparently easier approach, though there is still much work to do in that area. Ian Leary presented several embeddings theorems for discrete groups whose TDLC analogue is still unknown, and Lewis Molyneux discussed finiteness properties of groups generalizing Richard Thompson's group F. Dawid Kielak [9] and Yuri Santos Rego offered different perspectives on profinite rigidity.

This mini-workshop was attended by the 16 invited participants, who all travelled to Oberwolfach. Amongst the 20h lectures, there were four mini series of 3h each given by four experienced mathematicians, eight 1h talks, mostly given by early career researchers, and one very lively problem session.

The staff of the Mathematisches Forschungsinstitut Oberwolfach have excelled, providing all the support that we could have wished, and all in a very courteous manner. We are very grateful for the additional funding for 2 young PhD students through Oberwolfach-Leibniz-Fellowships. We strongly believe that such opportunities enable Ph.D. and junior researchers to get integrated within the research community at an early stage in their academic career, and broaden their networking activities. In conclusion, the meeting was a success.

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Abstracts

Geometric and Cohomological Finiteness Conditions in Group Theory

Peter H. Kropholler

The background to Leary's results on groups of type FP_2**.** In 1937, Bernhard Neumann proved that there are uncountably many 2-generator groups. Subsequently, in 1949, Higman, Neumann, and Neumann [3] introduced what are now known as HNN extensions and used them to prove that every countable group can be embedded in a 2-generator group. A parallel can be drawn with Liouville's discovery of a transcendental number in 1844. In essence Liouville showed that the number

$$\sum_{n=1}^{\infty} 10^{-n!}$$

is far better approximated by rational numbers than is possible for any irrational root of a polynomial with integer coefficients, and therefore this number must be transcendental. Liouville's result was trumped by Cantor's proof that there are $2^{\aleph_0}$ transcendental numbers but only $\aleph_0$ algebraic numbers. Of course, Cantor's extraordinary insight took many year to develop because Set Theory was in its infancy and the theory of infinite cardinals needed to be developed. In particular, clear proofs (avoiding the axiom of choice) of the Schroeder–Bernstein Theorem (which is needed to show that the order relation on cardinal numbers satisfies the law of trichotomy) did not emerge until the turn of the century and until that matter was settled, Cantor's diagonal argument showing that the set of real numbers is uncountable remained one piece of a jig-saw.

Liouville's number admits many variations and it is easy to generate $2^{\aleph_0}$ numbers with the essential property concerning rational approximation. In 1844, Liouville and others would have been aware that there was now a whole family of transcendental numbers but would not have been able to formulate this in terms of countability or uncountability.

By 1937, Bernhard Neumann's construction of $2^{\aleph_0}$ finitely generated groups was a milestone similar in nature to Liouville's discovery of $2^{\aleph_0}$ transcendental numbers and 1949 paper [3] cements this discovery with the more remarkable embedding theorem which can be compared with Cantor's discover that almost all real numbers are transcendental.

Since there are only countably many finitely presented groups up to isomorphism the question arises: which finitely generated groups can be embedded into finitely presented groups. Higman answered this in 1961 by exhibiting a remarkable connection with logic, [2]: a finitely generated group admits a finitely presented overgroup if and only if it is recursively presentable.

By this point, cohomological finiteness conditions emerged in the work of Serre. A group G if of type FP_n if there is a projective resolution $P_* \to \mathbb{Z}$ of the trivial $\mathbb{Z}G$-module with P_j finitely generated for $j < n+1$. It was easy to see that type

FP_1 is equivalent to finite generation, and that finitely presented groups are of type FP_2. Therefore the natural question arose:

Do there exist groups of type FP_2 that are not finitely presented? The question was answered in 1997 by the celebrated work of Bestvina and Noel Brady, [1]. They developed a combinatorial form of Morse theory suited to cube complexes and were able to exhibit whole families of examples. However, unlike the case of Louiville's numbers where we can see now that Liouville has a continuum of examples of similar kinds of number, the Bestvina–Brady examples were only countable in number. This fact seemed to pass unnoticed but around 2014 at a gathering in the Oxford Mathematical Institute, Charles Miller III raised the question. By 2018, Ian Leary had the answer both in the spirit of Bernhard Neumann's theorem [5]:

There are uncountably many groups of type FP. And in the spirit of the Higman–Neumann–Neumann theorem [4]:

Every countable group can be embedded in a group of type FP_2. So arguably, Leary's results build on Bestvina and Brady's work in the same way that Cantor's diagonal argument transcends Liouville's examples of 1844.

This raises a number of questions. The obvious one, and only one I will mention in this short abstract is:

For which n is it possible to embed every countable group into a group of type FP_n? It is natural to suspect that the answer is $n = \infty$ but there is not method known at present even to address the case $n = 3$.

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Discrete cohomology, the Hattori-Stallings rank, the Euler characteristic and formal Dirichlet series for t.d.l.c. groups

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(joint work with Ilaria Castellano, Gianmarco Chinello, Bianca Marchionna and George Willis)

In recent years totally disconnected locally compact (= t.d.l.c.) groups have raised much attention. As reflected by D. van Dantzig's theorem (cf. [6]), which states that every t.d.l.c. group contains a compact open subgroup, the structure theory of t.d.l.c. groups is significantly different from the theory of Lie groups. T.d.l.c. groups arise in many different areas in Mathematics.

Example 1. (A) If X is an affine group scheme defined over $\mathbb{Z}$ and $\mathbb{F}$ is a t.d.l.c. field, then $X(\mathbb{E})$ carries naturally the structure of a t.d.l.c. group.
(B) If $\mathscr{T}$ is a locally finite tree then $\mathrm{Aut}(\mathscr{T})$ carries naturally the structure of a t.d.l.c. group.
(C) Let Λ be a connected graph and let $\mathscr{G}$ be a graph of groups based on Λ such that $\mathscr{G}_x$ are profinite groups for all $x \in V(\Lambda) \cup E(\Lambda)$, and that $\alpha_e \colon \mathscr{G}_e \to \mathscr{G}_{t(e)}$ are open immersions for all edges $e \in E(\Lambda)$. Then $\pi_1(\mathscr{G}, \Lambda, x_0)$ carries naturally the structure of a t.d.l.c. group.
(D) If $\mathbb{F}$ is a field of characteristic 0, and $\mathbb{E}/\mathbb{F}$ is a field extension of finite transcendence degree then $\mathrm{Aut}_{\mathbb{F}}(\mathbb{E}) = \{\, \alpha \in \mathrm{Aut}(\mathbb{E}) \mid \alpha|_{\mathbb{F}} = \mathrm{id}_{\mathbb{F}} \,\}$ carries naturally the structure of a t.d.l.c. group (cf. [1, §6.3]).
(E) For every crystallographic Coxeter group (W, S) there exists a simply-connected root group datum $\mathscr{D}$ and a Tits functor $X_{\mathscr{D}}$. Evaluating this functor on a finite field $\mathbb{F}$ and taking the completion with respect to its action on the positive part of the twin building $\Delta_{\pm}$ one obtains the topological Kac-Moody group $\hat{X}_{\mathscr{D}}(F)$.

1. Discrete cohomology. Let G be a t.d.l.c. group, let $R \in \{\mathbb{Z}, \mathbb{Q}\}$ and let M be a left $R[G]$-module. Then

$$\mathrm{d}M = \{\, m \in M \mid \mathrm{stab}_G(m) \text{ open in } G \,\} \tag{1}$$

is an $R[G]$-submodule of M, the largest discrete left $R[G]$-submodule of M. One calls the left $R[G]$-module *discrete*, if $M = \mathrm{d}M$. The full subcategory ${}_{R[G]}\mathbf{dis}$ of ${}_{R[G]}\mathbf{mod}$, the objects of which are discrete left $R[G]$-modules, is an abelian category with enough injectives and thus allows to define cohomology with coefficients in ${}_{R[G]}\mathbf{dis}$ by $\mathrm{d}H^\bullet(G, \_) = \mathscr{R}^\bullet(\_^G)$. For $R = \mathbb{Z}$ these cohomology groups are quite difficult to compute. Nevertheless, an interesting question in this context which has not yet obtained the attention it deserves, is the following:

Question 1. *Let* $G = \mathrm{Aut}_{\mathbb{F}}(\mathbb{E})$, *where* $\mathbb{F}$ *is a field of characteristic* 0 *and let* $\mathbb{E}/\mathbb{F}$ *be a field extension of finite transcendence degree over* $\mathbb{F}$. *What is* $\mathrm{d}H^1(G, \mathbb{E}^\times)$*?*

In [3] the authors addressed many problems concerning the category ${}_{\mathbb{Q}[G]}\mathbf{dis}$. However, several questions remained unanswered. E.g.:

Question 2. *Let G be a t.d.l.c. group. For which closed subgroups H of G is $\mathrm{res}^G_H(\_)$ mapping injectives to injectives. Of particular interest would be the case when H is co-compact in G, or when H is discrete in G.*

For $R = \mathbb{Q}$ calculations of $\mathrm{d}H^\bullet(G, \_)$ become easier due to the following fact

Fact 1. *The category ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is an abelian category with enough projectives. Indeed for any compact, open subgroup $\mathcal{O}$ of G the left $\mathbb{Q}[G]$-permutation module $\mathbb{Q}[G/\mathcal{O}]$ is a projective object in ${}_{\mathbb{Q}[G]}\mathbf{dis}$.*

If M and N are two rational discrete left G-modules it is straightforward to verify that $M \otimes_{\mathbb{Q}} N$ is again a rational discrete left G-module. Neverthelesss the following question remained unanswered.

Question 3. *Let P be a projective object in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ and let M be a rational discrete left G-module. Is $P \otimes_{\mathbb{Q}} M$ necessarily a projective object in ${}_{\mathbb{Q}[G]}\mathbf{dis}$?*

The existence of enough projectives in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ allows one to define discrete rational homology $\mathrm{d}H_\bullet(G, \_)$ with coefficients in ${}_{\mathbb{Q}[G]}\mathbf{dis}$, and yields also a natural notion of being rationally of type FP_∞. The setup allows one to define the rational discrete cohomological dimension of a t.d.l.c. group G by

$$\mathrm{cd}_{\mathbb{Q}}(G) = \sup\{\, n \in \mathbb{N} \mid \mathrm{d}H^{n+1}(G, \_) = 0 \,\} \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

E.g., for a discrete group G this number coincides with the cohomological $\mathbb{Q}$-dimension of G. It is a direct consequence of Bass-Serre theory that the t.d.l.c. groups $\pi = \pi_1(\mathscr{G}, \Lambda, x_0)$ described in Example (C) satisfy $\mathrm{cd}_{\mathbb{Q}}(\pi) \le 1$. Let G be a t.d.l.c. group and let μ be a fixed left-invariant Haar measure on G. One says that G is *c/o-bounded*, if there exists a positive real number c such that for every compact open subgroup $\mathcal{O}$ of G one has $\mu(\mathcal{O}) \le c$. The following theorem can be seen as a second[1] t.d.l.c. version of the Stallings-Swan theorem (cf. [4]).

Theorem 2 (I. Castellano, B. Marchionna, T.W.)**.** *Let G be a unimodular c/o-bounded, compactly generated t.d.l.c. group satisfying $\mathrm{cd}_{\mathbb{Q}}(G) \le 1$. Then there exists a graph of groups $\mathscr{G}$ like in Example (C) based on a finite connected graph Λ such that $G \simeq \pi_1(\mathscr{G}, \Lambda, x_0)$.*

Question 4. *Does Theorem 2 remain true without the hypothesis of c/o boundedness, and/or unimodularity?*

Although there is no group algebra one may associated to the abelian category ${}_{\mathbb{Q}[G]}\mathbf{dis}$, there is a canonical rational discrete bimodule

$$\mathrm{Bi}(G) = \varinjlim_{\mathcal{O} \subseteq_{c/o} G} \mathbb{Q}[G/\mathcal{O}] \tag{3}$$

which plays a similar role as the integral group algebra for discrete groups. E.g., the t.d.l.c. group G is said to be a *rational duality group of dimension $d \in \mathbb{N}$*, if

(a) G is rationally of type FP_∞,
(b) $\mathrm{cd}_{\mathbb{Q}}(G) = d$,

[1] A first version has been obtained by I. Castellano in [2].

(c) $\mathrm{d}H^k(G, \mathrm{Bi}(G)) = 0$ for $k \neq d$.

From work of Michael Davis one concludes that the t.d.l.c. group $G = \hat{X}(\mathbb{F})$ of Example 1(E) is a rational duality groups of dimension $d \geq 1$ if, and only if, (W, S) is a $\mathbb{Q}$-duality group of dimension d (cf. [3]). Although many features of Coxeter groups have been studied in detail, the author could not find a satisfactory[2] answer to the following question.

Question 5. *What crystallographic Coxeter groups are $\mathbb{Q}$-duality groups of dimension $d \geq 1$?*

2. The Hattori-Stallings rank and the Euler-Poincaré characteristic. In case that G is unimodular, the Hom-$\otimes$ identity in combination with the evaluation morphism $\phi_{\mathcal{O}} : P \otimes_G \mathrm{Hom}_G(P, \mathrm{Bi}(G)) \to \mathbb{Q} \cdot \mu$ assigns do the identity of every finitely generated projective object P in ${}_{\mathbb{Q}[G]}\mathbf{dis}$ a rational multiple $\mathbf{hs}(P) \in \mathbb{Q} \cdot \mu$ of a normalized Haar measure μ, and thus can be considered as a generalized Hattori-Stallings rank (cf. [5]), e.g., $\mathbf{hs}(\mathbb{Q}[G/\mathcal{O}]) = \mu_{\mathcal{O}}$, where $\mu_{\mathcal{O}}$ is the Haar measure on G which restriction to $\mathcal{O}$ is a probability measure. The following theorem may be considered as a t.d.l.c. version of a theorem of I. Kaplansky (cf. [5]).

Theorem 3 (I. Castellano, G. Chinello, T.W.)**.** *Let G be a t.d.l.c. group, let $\mathcal{O}$ be a compact open subgroup of G, and let $P \in \mathrm{ob}({}_{\mathbb{Q}[G]}\mathbf{dis})$ be projective. Then $\mathbf{hs}(P) \in \mathbb{Q}_0^+ \cdot \mu_{\mathcal{O}}$. In particular, $\mathbf{hs}(P) = 0$ if, and only if, $P = 0$.*

Let G be a t.d.l.c. group which is

(i) unimodular,
(ii) rationally of type FP_∞,
(iii) of finite rational cohomological dimension.

For such a group let $(P_\bullet, \partial_\bullet)$ be a finite and finitely generated projective resolution of the trivial left $\mathbb{Q}[G]$-module in the category ${}_{\mathbb{Q}[G]}\mathbf{dis}$. Then one defines the Euler-Poincaré characteristic χ_G of G by

$$\chi_G = \sum_{k \in \mathbb{N}_0} (-1)^k \cdot \mathbf{hs}(P_k). \tag{4}$$

Example 2. (A) If $G = \pi_1(\mathscr{G}, \Lambda, x_0)$ is the fundamental group of a profinite graph of groups based on the finite graph Λ (cf. Ex.1(C)), one has

$$\chi_G = \sum_{v \in V(\Lambda)} \mu_{\mathscr{G}_v} - \sum_{\mathbf{e} \in \mathbb{E}^g(\Lambda)} \mu_{\mathscr{G}_{\mathbf{e}}} \tag{5}$$

(B) If $G = \hat{X}_{\mathscr{D}}(\mathbb{F})$ for some finite field of cardinality q (cf. Ex.1(E)) one obtains

$$\chi_G = \tfrac{1}{p_{W,S}(q)} \cdot \mu_{\mathrm{Iw}}, \tag{6}$$

where Iw is the stabilizer of a chamber in the building Δ_+.

[2]Obviously, affine and crystallographic hyperbolic Coxeter groups share this property, but one may speculate whether the class of examples is much larger or not.

3. Formal Dirichlet series associated to t.d.l.c. groups. Let G be a t.d.l.c. group, let μ be a left-invariant Haar measure of G, let $\mathcal{O} \subseteq_{\mathrm{c/o}} G$ and let $\mathscr{R} \subset G$ be a set of coset representatives for $\mathcal{O}\backslash G/\mathcal{O}$. One says that G has *bounded coset growth with respect to* $\mathcal{O}$, if for all $n \in \mathbb{N}$ one has

$$a_n = |\{\, r \in \mathscr{R} \mid \mu(\mathcal{O}r\mathcal{O}) = n \cdot \mu(\mathcal{O}) \,\}| < \infty. \tag{7}$$

If G has bounded coset growth with respect to some compact open subgroup, then it has bounded coset growth with respect to all compact open subgroups. For such a t.d.l.c. group G one defines the formal Dirichlet series

$$\zeta_{G,\mathcal{O}}(s) = \sum_{n\in\mathbb{N}} a_n \cdot n^{-s}. \tag{8}$$

In many cases one verifies that $\zeta_{G,\mathcal{O}}$ defines a meromorphic function $\hat{\zeta}_{G,\mathcal{O}} \colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. In this case one calls $(G, \mathcal{O})$ a *meromorphic pair*. A question we have investigated for many examples is the following:

Question 6. *For which merohoric pairs* $(G, \mathcal{O})$ *satisfying (i)-(iii) is it true that*

$$\chi_G = \tfrac{1}{\zeta_{G,\mathcal{O}}(-1)} \cdot \mu_{\mathcal{O}} \tag{9}$$

Question 6 has an affirmative answer for $(\hat{X}_{\mathscr{D}}(\mathbb{F}), \mathrm{Iw})$ for every crystallographic Coxeter group (W, S) (cf. Ex. 1(E)). The same is true if $G = X(\mathbb{E})$ for a Chevalley group scheme X, a t.d.l.c. field $\mathbb{E}$ and $\mathcal{O} \subset G$ a parahoric subgroup of G (cf. [5]). However, recently B. Marchionna found examples of merohoric pairs which do not satisfy (4) and also many merohoric pairs satisfying (4) (for t.d.l.c. groups G without Bruhat decomposition).

The abscissa of convergency $a = \mathbf{abs}(\zeta_{G,\mathcal{O}})$ as well as the order of the pole $\mathbf{ord}(\zeta_{G,\mathcal{O}})$ at a, do not depend on the choice of compact open subgroup, and thus are invariants of G.

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Cohomological finiteness conditions and stable categories on rational discrete modules over TDLC groups

Rudradip Biswas

1. Objective. We want to associate to the abelian category of rational discrete modules over a TDLC group a "stable category" with good behaviour (i.e. satisfying the equivalences shown in **Theorem C**).

2. A useful homological result. For any TDLC group G, it is known (Proposition 3.2 of [3]) that the rational discrete modules form an abelian category with enough projectives and enough injectives - we will be denoting this category by $\mathfrak{A}_G$. Denote the supremum over all $\mathfrak{A}_G$-objects with finite projective (resp. injective) dimension by $\operatorname{Fin}\operatorname{ProjDim}(\mathfrak{A}_G)$ (resp. $\operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G)$).

Theorem A. (partly new, partly inspired by [5], and partly covered by Thm. VII .2.2 of [2]) The following are equivalent for any TDLC group G.

a) Every object in $\mathfrak{A}_G$ admits a complete projective resolution, i.e. every object has a projective resolution that eventually agrees with a totally acyclic complex of projectives.

b) $\operatorname{silp}(\mathfrak{A}_G)$ (defined as the supremum over the injective dimension of $\mathfrak{A}_G$-projectives) and $\operatorname{spli}(\mathfrak{A}_G)$ (defined as the supremum over the projective dimension of $\mathfrak{A}_G$-injectives) are finite.

c) Every object in $\mathfrak{A}_G$ admits a complete injective resolution, i.e. every object has an injective resolution that eventually agrees with a totally acyclic complex of injectives, and $\operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G) < \infty$.

d) Complete cohomology computed with projective resolutions agrees with complete cohomology computed with injective resolutions (in the style of Nucinkis [5]).

Proposition B. When any of the equivalent statements of **Theorem A** are satisfied (an easy example is when $G = \mathrm{SL}_n(\mathbb{Q}_p)$ as it has finite virtual cohomological dimension), we have

$\operatorname{silp}(\mathfrak{A}_G) = \operatorname{spli}(\mathfrak{A}_G) = \operatorname{Fin}\operatorname{ProjDim}(\mathfrak{A}_G) = \operatorname{Fin}\operatorname{InjDim}(\mathfrak{A}_G) < \infty$

We can add some more invariants here like the finitistic and the global Gorenstein dimensions. The main use of **Theorem A** is that it gives very neat conditions on when every object has complete resolutions which is useful in constructing a well-behaved stable category as we describe below.

3. Candidates for stable categories and equivalences.

Theorem C. (new, in the spirit of [1, 4]) Let G be a TDLC group such that $\operatorname{silp}(\mathfrak{A}_G)$ and $\operatorname{spli}(\mathfrak{A}_G)$ are finite. Then, the following triangulated categories are equivalent (and are therefore equivalently adequate candidates for our stable category):

(i) $(\mathfrak{A}_G, \widehat{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ (here, the objects are all modules in $\mathfrak{A}_G$ and the Hom-sets are given by the zero-th complete cohomology groups computed with projective resolutions)

(ii) $(\mathfrak{A}_G, \widetilde{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ (here, the objects are all modules in $\mathfrak{A}_G$ and the Hom-sets are given by the zero-th complete cohomology groups computed with injective resolutions)

(iii) $\mathcal{D}^b(\mathfrak{A}_G)/\operatorname{K}^b(\operatorname{Proj}\text{-}\mathfrak{A}_G)$ (the Verdier quotient of the derived bounded category on $\mathfrak{A}_G$ and the homotopy category of bounded complexes of $\mathfrak{A}_G$-projectives)

(iv) $\mathcal{D}^b(\mathfrak{A}_G)/\operatorname{K}^b(\operatorname{Inj}\text{-}\mathfrak{A}_G)$ (the Verdier quotient of the derived bounded category on $\mathfrak{A}_G$ and the homotopy category of bounded complexes of $\mathfrak{A}_G$-injectives)

(v) The homotopy category of totally acyclic complexes of $\mathfrak{A}_G$-projectives.

(vi) The homotopy category of totally acyclic complexes of $\mathfrak{A}_G$-injectives.

(vii) $\underline{\operatorname{GProj}}(\mathfrak{A}_G)$ (Gorenstein projectives of $\mathfrak{A}_G$, i.e. $\mathfrak{A}_G$-objects arising as cycles in totally acyclic complexes of $\mathfrak{A}_G$-projectives, form a Frobenius category with the class of projective-injectives given by the $\mathfrak{A}_G$-projectives; $\underline{\operatorname{GProj}}(\mathfrak{A}_G)$ denotes its stable category where we keep all Gorenstein projectives as objects and kill all morphisms that factor through an $\mathfrak{A}_G$-projective). To get to (iii) from (i), consider a module as a complex concentrated in degree 0; for (iii) to (v), take complete projective resolutions (possible due to Thm A); for (v) to (vii), take the zero-th syzygy functor (see Def. 3.7 of [4]); and for (vii) to (i), use the inclusion functor. Composing these, get $(i) \cong (iii) \cong (v) \cong (vii)$. Repeat the analogous treatment with "module as a deg 0 concentrated complex", "taking complete injective resolutions" (again, possible due to Thm A), and the zero-th cosyzygy functor, to get $(ii) \cong (iv) \cong (vi)$. $(i) \cong (ii)$ by Thm A, and $(iii) \cong (iv)$ as $\operatorname{silp}(\mathfrak{A}_G), \operatorname{spli}(\mathfrak{A}_G) < \infty$.

Note that all the categories in **Theorem C** except two, namely $(\mathfrak{A}_G, \widehat{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$ and $(\mathfrak{A}_G, \widetilde{\operatorname{Ext}}^0_{\mathfrak{A}_G}(\_,\_))$, are clearly triangulated categories. Without the assumption that $\operatorname{silp}(\mathfrak{A}_G)$ and $\operatorname{spli}(\mathfrak{A}_G)$ are finite, these two categories need not even be triangulated. Since we know very little about $\mathfrak{A}_G$-injectives, the presence of (vi) above is noteworthy.

4. Finishing remarks. We are insisting on these equivalences because they are useful for making progress on stratification questions in the spirit of, for example, Benson-Iyengar-Krause or Barthel-Heard-Sanders. This is why even if we can replace $\mathfrak{A}_G$ with a more refined abelian category associated to G using the full force of *Condensed Maths*, our use of and dependence on complete resolutions (both projective and injective) to establish "good behaviour" of our stable category will remain.

Ending open question. Can we achieve **Theorem C** with a finiteness assumption on just one of $\operatorname{silp}(\mathfrak{A}_G)$ and $\operatorname{spli}(\mathfrak{A}_G)$?

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Finiteness properties and locally compact groups

Stefan Witzel

Topological finiteness properties of discrete groups are studied for two principal reasons. The classical one is that they give sufficient conditions for group cohomology to be effectively computable using a CW-model for a classifying space. The modern one is that certain finiteness properties are coarse invariants and therefore allow to distinguish the large-scale geometry of groups. Finiteness properties for locally compact groups have been proposed. What is missing is the universal object whose finiteness they are supposed to describe.

1. Discrete groups

Topological finiteness properties are about the existence of classifying spaces with good finiteness properties so we start with these. If G is a discrete group, a $K(G,1)$ is a homotopy type CG of a pointed CW-complex such that $\pi_1(CG) = G$ and the map $[(Z,z_0), CG] \to \operatorname{Hom}(\pi_1(Z,z_0), G)$ is bijective. Another perspective is via principal bundles: a principal G-bundle $p\colon EG \to BG$ is universal if every numerable principal G-bundle arises as a pullback $f^*(p)$ along a map $f\colon Z \to Y$.

For a discrete group G, a principal G-bundle $X \to Y$ is the same as a normal covering with group of deck transformations G which by covering space theory corresponds to a homomorphism $\pi_1(Y) \to G$. In particular, the universal bundle $EG \to BG$ can be recovered from BG and $K(G,1)$ and BG are the same thing. This is specific to discrete groups and for non-discrete groups it may be more reasonable to generalize EG.

A discrete group is *of type F_n* if it admits a CW-model for EG whose n-skeleton is compact modulo G, equivalently if acts freely and cocompactly on an $(n-1)$-connected CW complex. It is of type F_∞ if it is of type F_n for all n.
A basic fact about these finiteness properties is that if $1 \to N \to G \to Q \to 1$ is an extension in which N is of type F_n then G of type F_{n+1} implies Q of type F_{n+1} and Q of type F_n implies G of type F_n.

The main tool in determining finiteness properties is Brown's criterion [3, Theorems 2.2, 3.2]:

Theorem. *Fix G and n. Let X be an $(n-1)$-connected G-CW complex. Let $(X_i)_i$ be a filtration by G-CW subcomplexes. Assume that G_σ is of type $F_{n-\dim\sigma}$ for all σ. Then G is of type F_n if and only if $(X_i)_i$ is essentially $(n-1)$-connected.*

The filtration is essentially n-connected if the directed system $(\pi_k(X_i))_i$ is essentially trivial for every $k \le n$ which in turn means that for every i there exists a

j such hat $\pi_k(X_i \to X_j) = 0$. Brown's criterion can be decomposed into two parts: the stabilizer part says that a CW-complex on which G acts can be replaced by one on which G acts freely without affecting cocompactness on the n-skeleton provided the stabilizers have the right finiteness properties; and the filtration part says that if G acts freely on an $(n-1)$-connected CW complex, its finiteness properties are detected by any cocompact filtration.

2. Universal spaces for locally compact groups

Moving on to locally compact groups, we would like to define finiteness properties F_n that capture the finiteness of some universal free G-space EG. It turns out that is relatively easy to agree on what the properties F_n will be but not so clear what the universal space EG is that they describe finiteness of, a basic problem being that non-discrete groups do not act freely and continuously on CW-complexes (in contrast there is no problem to define a classifying space for proper actions of a tdlc group, for instance). One may hope for EG to be an object in a model category for topological spaces with continuous G-actions that has a notion of dimension and then F_n would be universality with respect to n-dimensional objects. Milnor's EG, the infinite join of G with itself appropriately topologized, is bound to be model for EG in the sense to be established, but studying finiteness properties becomes interesting only once one can vary the model within its (equivariant) homotopy class. Corob Cook [4] has results in this direction with the additional ambition of recovering G from BG as a topological group, but their universal properties are unclear and connected groups will likely interfere with the homotopy theory.

Finiteness properties of locally compact groups should generalize the notions of being compactly generated (F_1) and of being finitely presented (F_2) and these special cases are instructive: G is compactly generated if there is a compact subset C that generated G as an abstract group. This can be reformulated to say that G acts cocompactly on a topological graph with vertex set G and edge set $G \times C$ whose *underlying discrete graph* is connected. A natural extension to G being of type F_n would be to ask for the existence of a simplicial space Δ (consisting of topological spaces $\Delta[k]$ of k-simplices and continuous face and degeneracy maps) on which G acts freely such that the action on $\Delta[k], k \leq n$ is cocompact and that the geometric realization of the underlying simplicial set $|F\Delta|$ is $(n-1)$-connected (where F is the forgetful functor from topological spaces to topological sets). Or alternatively to act freely on a topological CW-complex (whose n-cells are indexed by a topological space rather than a set) cocompactly on k-cells, $k \leq n$, and that the underlying CW-complex (with discrete set of cells) be $(n-1)$-connected. But for now these are just ad-hoc notions without justification by some form of universality.

There are a few more lessons to be learned from looking at compact generation. First, the models for EG we are looking for will not be G-CW complexes, which are built out of equivariant cells $G/H \times D^k$: while the 0-skeleton of the topological graph is a single equivariant cell $G/\{1\} \times D^0$, already the edges are parametrized by $G \times C$, so a single G/H-factor is not sufficient to capture the amount of non-discreteness. Second, a connected group G is not compactly generated simply because $G \to G\backslash G$ is a bundle with connected total space but rather because it is generated by any identity neighborhood and this neighborhood can be taken to be compact. This leads to a warning when moving beyond the locally compact setting: if $G\backslash X$ is compact it may not be true that there is a compact $C \subseteq X$ with $G.C = X$.

3. Finiteness properties for locally compact groups

The reason that the correct notion of finiteness properties for locally compact groups is uncontroversial is that certain simple assumptions determine them uniquely. For instance:

Observation. *Suppose $(T_n)_{n\in\mathbb{N}}$ are properties of groups (and T_∞ means T_n for all n) such that the following hold:*

(1) If $1 \to N \to G \to Q \to 1$ is an extension with N of type T_∞ then G is T_n if and only if Q is T_n.
(2) If G acts properly and transitively on a contractible manifold then it is of type T_∞.

Then G is T_n if and only if $G/G^{(0)}$ is T_n. If in addition

(3) Brown's criterion holds (at least for proper actions)

then the properties T_n are uniquely determined.

Proof. If G is compact then it is T_∞ by (2). If G is connected Lie then it admits a maximal compact subgroup C [7, Theorem 14.1.3] and $C\backslash G$ is contractible [7, Theorem 14.3.11] so G is T_∞ by (2). If G is connected then by the Gleason–Yamabe theorem it is pro-Lie. Since the Lie groups involved have bounded dimension, the inverse system eventually consists of coverings. Since the fundamental group of a Lie group is finitely generated abelian [7, Theorem 12.4.14], only finitely many of these can be infinitely-sheeted. It follows that G is (pro-finite)-by-Lie and hence of type T_∞ by (1). If G is a general locally compact group with connected component $G^{(0)}$ it follows by another application of (1) that G is T_n if and only if $G/G^{(0)}$ is, reducing to the tdlc case. Finally if G is tdlc then by van Dantzig's theorem there is a compact open subgroup C. Then G acts properly on the free simplicial set over G/C, which is contractible, so it is of type T_n if and only if some/any cocompact filtration is essentially $(n-1)$-connected. □

The existing notions of finiteness properties for locally compact groups stipulate some assumption of this form to get a definition. The compactness properties C_n by Abels–Tiemeyer [2] stipulate that the filtration part of Brown's criterion should hold for the filtration of the free simplicial set over G filtered by G-orbits of free simplicial sets over compact subsets of G. The finiteness properties F_n by Castellano–Corob-Cook [5], which are only defined for tdlc groups, stipulate that compact groups should be F_∞ and that the stabilizer part of Brown's criterion should hold for proper actions. Applying [2, Theorem 3.2.2] and [5, Theorems 4.7,4.10] to the free simplicial set over G/C, C a compact open subgroup, one finds that both notions coincide on tdlc groups.

4. Examples

Examples of locally compact groups with interesting finiteness properties exist in the literature although they are usually formulated for discrete groups because of the unclear meaning of finiteness properties of locally compact groups. This is true specifically of the solvable groups discussed in [12, 10]. Note that Brown's criterion together with the fact that arithmetic groups are F_∞ allows to reduce the determination of finiteness properties of S-arithmetic groups to that of finiteness properties of algebraic groups over local fields, see [11, Theorem 3.1]. For instance finiteness properties of $A(\mathbb{Z}[1/p])$ are equivalent to compactness properties of $A(\mathbb{Q}_p)$. Beyond this equivalence, however, the proof that $A(\mathbb{Z}[1/p])$ is of type F_{n-1} but not of type F_n by filtering a Bruhat–Tits building applies in verbatim to prove that $A(K)$ is of type F_{n-1} but not of type F_n even if K is local field of positive characteristic, using the version [5, Theorems 4.7,4.10] of Brown's criterion.

5. Coarse geometry

The second motivation for studying finiteness properties mentioned in the introduction is coarse geometry. In this context the situation is much clearer, even beyond locally compact groups. A metric space X is *coarsely n-connected* if the Vietoris–Rips filtration $\mathrm{VR}_r(X)$ is essentially n-connected. Alonso [1] observed that this is a coarse invariant and by Brown's criterion being of type F_n coincides with being coarsely $(n-1)$-connected for countable groups. A locally compact σ-compact group G carries an adapted pseudo-metric that is unique up to coarse equivalence. If G is compactly generated then the metric can be taken to be coarsely geodesic and is then unique up to quasi-isometry (see [6, Milestones 4.A.8 and 4.B.13]. This generalizes statements for discrete groups that are countable and finitely generated, respectively. Thus from a geometric perspective the natural notion for a locally compact (σ-compact) group to be of type F_n is to be coarsely $(n-1)$-connected. If one is willing to work with coarse structures that need not be metrizable (see [8]) one can go further: Rosendal [9] defines a coarse structure on every topological group that is unique up to coarse equivalence.

An analogous form of coarse and large-scale geometry also exists for approximate groups leading to similar questions. In particular, it would be interesting to know how coarse connectivity relates to finiteness of cohomology.

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Profinite rigidity, homology, and Coxeter groups

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(joint work with Petra Schwer)

Dropping 'locally' from TDLC, a (topological, Hausdorff) group G is called *profinite* if it is totally disconnected (TD) and compact (C). Many infinite groups have TDC counterparts. Given $G \in \mathcal{FGRF}$ = class of finitely generated residually finite (discrete) groups, its *profinite completion* $\widehat{G}$ is the topological closure

$$\widehat{G} := \overline{\iota(G)} \leq \prod_{N \trianglelefteq G \,\&\, [G:N]<\infty} G/N, \text{ where } \iota(g) = (gN)_N \text{ is the diagonal embedding}$$

and each finite quotient G/N is given the discrete topology (thus $\widehat{G}$ is TDC).

1. Profinite rigidity and homological aspects

To what extent does the collection of (isomorphism classes of) finite quotients of an infinite group G determine its algebraic structure? This problem, whose origin is traced back to questions of Grothendieck and others in the 1970s, is a common point of interest for geometric group theory and the theory of TDLC groups. An ambitious first version is whether finite quotients determine isomorphism types.

Definition 1. Given a subclass $\mathcal{C} \subseteq \mathcal{FGRF}$ we say $G \in \mathcal{FGRF}$ is *profinitely rigid relative to* $\mathcal{C}$ if $(\widehat{G} \cong \widehat{H} \implies G \cong H)$ holds for all $H \in \mathcal{C}$. If $\mathcal{C} = \mathcal{FGRF}$ we call G *absolutely* profinitely rigid. If (up to isomorphism) there are only finitely many $H \in \mathcal{C}$ with $\widehat{H} \cong \widehat{G}$ but $H \not\cong G$, we call G *almost* profinitely rigid (rel. $\mathcal{C}$).

Below is a widely nonexhaustive list around the current state of knowledge. We tacitly assume our discrete groups to lie in $\mathcal{FGRF}$ unless explicitly said otherwise.

(1) (Folklore) $\mathbb{Z}$ and the infinite dihedral group are absolutely profinitely rigid.
(2) (Baumslag) $\exists B_1, B_2 \in \mathcal{FGRF}$ metacyclic with $\widehat{B_1} \cong \widehat{B_2}$ but $B_1 \not\cong B_2$.
(3) (Pickel) Nilpotent groups are almost absolutely profinitely rigid.
(4) (Remeslennikov; open problem) Are free groups absolutely profinitely rigid?
(5) (Wilton) Free groups are profinitely rigid relative to limit groups.
(6) (Liu) Fundamental groups of hyperbolic 3-manifolds of finite volume are almost profinitely rigid relative to 3-manifold groups.

Adapting the question, what kinds of features are witnessed by finite quotients?

Definition 2. A group-theoretic property (P) and a group-theoretic invariant $\eta(-)$ are said to be *profinite relative to* $\mathcal{C} \subseteq \mathcal{FGRF}$ in case the implications

- (G has property (P), G, $H \in \mathcal{C}$, and $\widehat{G} \cong \widehat{H}$) $\implies$ H has property (P),
- (G, $H \in \mathcal{C}$, and $\widehat{G} \cong \widehat{H}$) $\implies$ $\eta(G) = \eta(H)$

hold, respectively. If $\mathcal{C} = \mathcal{FGRF}$ we call the property (P) (resp. $\eta(-)$) *profinite*.

For instance, the first integral homology group $H_1(-, \mathbb{Z})$ is a profinite invariant. In fortunate cases, further homological information is detected by completions, or homological tools aid in computing completions, motivating the following (broad) program: Given a class of groups $\mathcal{C} \subseteq \mathcal{FGRF}$, ...

(1) ...find (co)homological invariants relative to $\mathcal{C}$,
(2) ...use (co)homological methods to check whether $\widehat{G} \cong \widehat{H}$ for $G, H \in \mathcal{C}$.

Here we mention some important contributions in this direction.

Theorem 3 (Platonov–Tavgen)**.** *Consider $G_1, G_2 \in \mathcal{FGRF}$ and suppose there exists a finitely presented group $Q \notin \mathcal{FGRF}$ for which $\widehat{Q} = 1$ and $H_2(Q, \mathbb{Z}) = 0$ and such that there are epimorphisms $\pi_1 : G_1 \to Q$ and $\pi_2 : G_2 \to Q$. Then the fiber product associated to π_1 and π_2 has the same profinite completion as $G_1 \times G_2$.*

Theorem 4 (Lück; Bridson–Conder–Reid)**.** *The rational Euler characteristic is a profinite invariant relative to $\mathcal{C} =$ lattices in $\mathbb{P}\mathrm{SL}_2(\mathbb{R})$.*

Theorem 5 (Kammeyer–Kionke–Raimbault–Sauer)**.** *Let $\mathcal{C}$ be the class of arithmetic groups with the congruence subgroup property. Then the rational Euler characteristic* is not *a profinite invariant relative to $\mathcal{C}$, though its sign* is so.

Theorem 6 (Jaikin-Zapirain; Hughes–Kielak)**.** *Let $\mathcal{C}$ be the class of finitely presented subgroup-separable groups. Then the property "the BNS invariant $\Sigma^1(-; R)$ contains antipodal points (for any commutative unital ring R)" is profinite rel. $\mathcal{C}$.*

2. Coxeter groups

Profinite topics have been actively studied for groups of strong geometric flavor. Surprisingly, not much is known for the (arguably) core examples of such groups: a *Coxeter group* W (of rank n) is a group admitting a *Coxeter presentation*

$$W \cong \langle s \in S \mid (st)^{m_{s,t}} \text{ for every pair } s, t \in S \text{ with } m_{s,t} < \infty \rangle,$$

i.e., $|S| = n$ and the orders $m_{s,t}$ satisfy $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

Example 7. $\mathrm{Sym}(n)$ is a prototypical finite Coxeter group, and $D_\infty \cong C_2 * C_2$ is the 'smallest' infinite Coxeter group. The following Coxeter group of rank 5,

$$W \cong \langle a, b, c, d, e \mid a^2, b^2, c^2, d^2, e^2, (ab)^3, (bc)^5, (ad)^2, (bd)^2, (be)^2, (ce)^2 \rangle,$$

is also isomorphic to the (hyperbolic) Coxeter triangle group $\Delta(6, 10, \infty)$.

To our knowledge, profinite literature around the class $\mathcal{G} :=$ Coxeter groups of finite rank $\subset \mathcal{FGRF}$ is at most a decade old. We record:

(Kropholler–Wilkes [4]; Corson–Hughes–Möller–Varghese [3]) Right-angled Coxeter groups are profinitely rigid relative to $\mathcal{G}$.
(Bridson–McReynolds–Spitler–Reid [2]) 14 hyperbolic Coxeter triangle groups are known to be absolutely profinitely rigid.
(Möller–Varghese [5]) Relative to $\mathcal{G}$, irreducible and affine imply rigidity.

Let us highlight two results addressing our motivating program for Coxeter groups.

Theorem 8 (Corson–Hughes–Möller–Varghese [3])**.** *The right-angled Coxeter group* $(C_2 * C_2 * C_2 * C_2) \times (C_2 * C_2 * C_2 * C_2)$ *is not absolutely profinitely rigid.*

Proof sketch. Apply Theorem 3 of Platonov–Tavgen taking $G_1 = G_2 = C_2 * C_2 * C_2 * C_2$ and using R. Thompson's simple group V as the quotient Q, and check that the corresponding fiber product is not isomorphic to $G_1 \times G_2$. □

Theorem 9 (Santos Rego–Schwer [6])**.** *Write* $\mathcal{G}_{\leq 3}$ *for the class of Coxeter groups that admit some Coxeter presentation of rank three or less. Then every* $W \in \mathcal{G}_{\leq 3}$ *is profinitely rigid relative to* $\mathcal{G}_{\leq 3}$*. Moreover, Coxeter triangle groups satisfy*

$$\widehat{\Delta(p, q, r)} \cong \widehat{\Delta(p', q', r')} \iff \{p, q, r\} = \{p', q', r'\}.$$

Proof strategy. Rule out spherical groups and clear the affine case by looking at $H_1(-, \mathbb{Z})$ and comparing completions of virtually abelian groups. In the hyperbolic case the presence of von Dyck subgroups implies that Theorem 4 still applies. Compare Euler characteristics and invoke profinite techniques of [1] to finish. □

Theorem 9 applies to some groups of higher rank, see Example 7. Besides using homological tools, Theorem 9 shows that the geometry of the given groups (e.g., covolume, having cusps, being hyperbolic) is encoded by finite quotients. We pose:

Problem 10. *Which (co)homological properties and invariants are profinite relative to the class* $\mathcal{G}$ *of Coxeter groups of finite rank?*

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Double-coset zeta functions for groups acting on trees

Bianca Marchionna

Double cosets play a prominent role in the study of totally disconnected locally compact (= t.d.l.c.) groups, e.g. in Representation Theory or Geometric Group Theory. From now on, we focus on double cosets of a t.d.l.c. group G with respect to a compact open subgroup $K \leq G$. In this case, each double coset KgK has

(1) $$\mu_K(KgK) = |K : K \cap gKg^{-1}| \in \mathbb{Z}_{\geq 1},$$

where $\mu_K(\_)$ is the left Haar measure on G such that $\mu_K(K) = 1$. For a given pair (G, K), consider the following (formal) Dirichlet series:

(2) $$\zeta_{G,K}(s) := \sum_{KgK \in K\backslash G/K} \mu_K(KgK)^{-s},$$

which is called the *double-coset zeta function associated to* (G, K) (cf. [2]).

The key for studying $\zeta_{G,K}(s)$ is to find a favourable enumeration of the K-double cosets and an explicit formula for each $\mu_K(KgK)$. Consider for instance $G = \mathrm{SL}_2(\mathbb{Q}_p)$ and $K = \mathrm{SL}_2(\mathbb{Z}_p)$, where $\mathbb{Q}_p$ is the p-adic field and $\mathbb{Z}_p$ is its ring of integers. By the Cartan decomposition of G with respect to K one has

(3) $$G = \bigsqcup_{d \in \mathbb{Z}_{\geq 0}} K \cdot \underbrace{\mathrm{diag}(p^{-d}, p^{d})}_{=: g_d} \cdot K.$$

Via a direct computation, $\mu_K(Kg_dK) = |K : K \cap g_d K g_d^{-1}|$ is 1 if $d = 0$, and equals $(p+1)p^{2d-1}$ otherwise. Therefore,

(4) $$\zeta_{G,K}(s) = \sum_{d=0}^{+\infty} \mu_K(Kg_dK)^{-s} = 1 + \frac{(p+1)^{-s}p^{-s}}{1 - p^{-2s}}.$$

There is another (geometric) way to obtain the same result, which goes beyond matrix computations and opens to further generalizations. It is based on the fact (cf. [6, Ch. II, § 1]) that $G = \mathrm{SL}_2(\mathbb{Q}_p)$ acts on a $(p+1)$-regular (simplicial) tree T_{p+1} and $K = \mathrm{SL}_2(\mathbb{Z}_p)$ is the stabilizer of a vertex, say v_0. Remarkably, the G-action on T_{p+1} is *locally* ∞*-transitive*, i.e., every vertex-stabilizer G_v acts transitively on the set of geodesics $\{[v, w] \subset T_{p+1} : \mathrm{length}([v, w]) = k\}$, for every $k \geq 0$. Hence there is a 1-to-1 correspondence mapping Kg_dK, $d \geq 0$, to the orbit $K \cdot [v_0, g_d \cdot v_0] = \{[v_0, w] \subset T_{p+1} : \mathrm{length}([v_0, w]) = 2d\}$. Moreover, by (1) one has

$$\mu_K(Kg_dK) = |K \cdot [v_0, g_d \cdot v_0]| = |\{[v_0, w] \subset T_{p+1} : \mathrm{length}([v_0, w]) = 2d\}.$$

By the regularity of T_{p+1}, one recovers the formula of $\mu_K(Kg_dK)$ claimed before.

This second argument easily extends to every *t.d.l.c. group G acting locally ∞-transitively on a locally finite tree T with compact open vertex-stabilizers*, taking K as a vertex-stabilizer (or, up to minor changes, an edge-stabilizer).

More generally, we can assume that G acts *weakly locally ∞-transitively* on T, i.e., every vertex-stabilizer G_v acts transitively on the set of geodesics from v in T having the same image via the quotient map $\pi : T \longrightarrow \Gamma := G\backslash T$. If the G-action on T is edge-transitive, this condition coincides with locally ∞-transitivity. In general, however, it comprises many other (non-edge transitive) examples, like the groups of automorphisms of locally finite trees preserving a vertex-coloring (cf. [7, § 5]) or universal groups associated to certain local action diagrams (cf. [5]).

In this more general setting, let $K = G_v$ be a compact open vertex-stabilizer. Then, each KgK corresponds to a loop at $\pi(v)$ in Γ, i.e., $\pi([v, g\cdot v])$, and $\mu_K(KgK)$ is the number of geodesics from v in T lifting the loop $\pi([v, g\cdot v])$ via π. A similar argument can be found in [1, § 3]. Hence, for computing $\zeta_{G,K}(s)$, we need only two tools: the quotient graph $\Gamma = G\backslash T$ regarded as a *Serre-graph*[1], and a weight $\omega(e) \in \mathbb{Z}_{\geq 1}$ on each $e \in \mathrm{Edg}(\Gamma)$ giving the number of edges in the Serre-graph associated to T lifting e and with a common origin. With a similar argument as in [3], one deduces what follows.

Theorem A. *Let G be a t.d.l.c. group acting weakly locally ∞-transitively on a locally finite tree T with compact open vertex-stabilizers, and let $K = G_v$ be a vertex-stabilizer. Let $\Gamma = G\backslash T$ be finite, and $\omega(e) \geq 3$ for every $e \in \mathrm{Edg}(\Gamma)$. Then $\zeta_{G,K}(s)$ converges at some $s \in \mathbb{C}$ and it can be meromorphically continued to $\mathbb{C}$ as*

$$\zeta_{G,K}(s) = \frac{\det(I - W(s) + U_{\pi(v)}(s))}{\det(I - W(s))}.$$

Here, $W(s)$ and $U_{\pi(v)}(s)$ are $|\mathrm{Edg}(\Gamma)|$-dimensional matrices whose entries are entire functions in $s \in \mathbb{C}$ depending only on Γ and $\omega(\_)$.

Following [2], we can often recover the *Euler characteristic*[2] χ_G of the group G from the meromorphic continuation of $\zeta_{G,K}(s)$ as follows.

Theorem B. *Let (G, K) as in Theorem A. If Γ is a tree, then $\chi_G = \zeta_{G,K}(-1)^{-1}\mu_K$.*

Unlike the examples studied in [2], if Γ is not a tree, the conclusion of Theorem B is no longer true in general (cf. [4]). At the current stage, however, there is not a complete characterization of all pairs (G, K) for which $\chi_G = \zeta_{G,K}(-1)^{-1}\mu_K$ holds yet.

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Profinite rigidity of fibring

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(joint work with Sam Hughes)

The starting point for this series of three lectures is the theorem of Jaikin-Zapirain [5].

Theorem 1. *Let M and N be compact connected orientable aspherical three-manifolds. If their fundamental groups are profinitely isomorphic, then M fibres over the circle if and only if N does.*

Here, being *profinitely isomorphic* means precisely that the profinite completions of the groups are isomorphic. When the groups are finitely generated, as is the case for the groups above, this amounts to saying that for every finite group, it is a quotient of one of the groups if and only if it is a quotient of the other.

For irreducible three-manifolds, fibring over the circle can be understood algebraically: it was shown by Stallings [7] that it is equivalent to *algebraic fibring*, that is, admitting an epimorphism to $\mathbb{Z}$ with finitely generated kernel (this fact has two more modern proofs [3, 1]). Hence, it becomes natural to ask under what circumstances does being algebraically fibred pass between groups with the same profinite completions.

Theorem 2 (Hughes–K. [4])**.** *Let G and H be two finitely presented LERF groups, and suppose that the profinite completions $\widehat{G}$ and $\widehat{H}$ are isomorphic. If G is algebraically fibred, then so is H.*

Recall that a group G is *LERF (locally extended residually finite)* if and only if for every finitely generated subgroup $A \leqslant G$ and an element $b \in G \smallsetminus A$, there exists a quotient map $\rho\colon G \to Q$ with finite image such that $\rho(b) \notin \rho(A)$.

The above is actually an instant of a more general result. To state it, we need to introduce the concept of BNS-invariants. Given a ring R, the *nth BNS invariant of G over R*, denoted $\Sigma^n(G; R)$, is a subset of the set of non-zero homomorphisms $G \to \mathbb{R}$ consisting of maps $\phi\colon G \to \mathbb{R}$ for which

$$\mathrm{H}_i(G; \widehat{RG}^{\phi}) = 0$$

for all $i \leqslant n$. Here $\widehat{RG}^{\phi}$ is the *Novikov ring* associated to ϕ, defined as the ring of functions $G \to R$ whose support intersects $\phi^{-1}((-\infty, \kappa))$ in a finite set for every $\kappa \in \mathbb{R}$.

The BNS-invariants are related to our previous discussion, since for a character $\phi\colon G \to \mathbb{Z}$, lying in $\Sigma^n(G;R) \cap -\Sigma^n(G;R)$ is equivalent to having kernel of type $\mathtt{FP}_n(R)$.

A character $G \to \mathbb{Z}$ lying in $\Sigma^n(G;R) \cup -\Sigma^n(G;R)$ will be called *n-semi-fibred.*

In [4] we introduced the following definition.

Definition 3. Let R be an integral domain. A group G lies in $\mathrm{TAP}_n(R)$ if and only if n-semi-fibred characters are precisely the characters for which all twisted Alexander polynomials over R in dimensions i for all $i \leqslant n$ do not vanish.

The twisting considered above is that by an epimorphism from G to a finite group. For every group, if we are given an n-semi-fibred character, then its twisted Alexander polynomials over R in dimensions i for all $i \leqslant n$ do not vanish; the interesting part of the definition is the reverse implication.

It was first observed by Friedl–Vidussi [2] that twisted Alexander polynomials are important in recognising fibred 3-manifolds. In the language we just introduced, the main theorem of [2] states that fundamental groups of connected compact orientable three-manifolds with empty or toroidal boundary lie in $\mathrm{TAP}_1(R)$ for every Noetherian UFD R.

We now have more examples of such groups.

Theorem 4 (Hughes–K. [4])**.** *Let R be a commutative ring.*

- *If G is a LERF group of type $\mathtt{FP}_2$ over any commutative ring, then G lies in $\mathrm{TAP}_1(R)$.*
- *The class $\mathrm{TAP}_1(R)$ is closed under finite products.*
- *Products of limits groups lie in $\mathrm{TAP}_\infty(R)$.*

Once a group is shown to lie in $\mathrm{TAP}_1(\mathbb{F})$ over a finite field $\mathbb{F}$ we can use it to study profinite rigidity of BNS-invariants, thanks to the following result, heavily inspired by ideas of Jaikin-Zapirain [5] and Liu [6].

Theorem 5 (Hughes–K. [4])**.** *Let G and H be groups of type $\mathtt{FP}_n(\mathbb{Z})$ that are n-good in the sense of Serre, and that have isomorphic profinite completions. If G lies in $\mathrm{TAP}_n(R)$ and $\Sigma^n(G;R) = \emptyset$ then $\Sigma^n(H;R) = \emptyset$ as well.*

The class $\mathrm{TAP}_1(R)$ remains quite mysterious.

Conjecture 6. *Do all {finitely generated free}-by-cyclic groups lie in $\mathrm{TAP}_\infty(R)$?*

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Classifying Spaces

Ged Corob Cook

An important tool for studying abstract groups G is the use of group actions on spaces. When we care about understanding these actions up to homotopy, we typically work in the categories like CW-complexes or G-CW-complexes. There is a CW-complex BG which is a classifying space for G: it classifies principal G-bundles, it is an Eilenberg-MacLane space for G, and so on. As noted by Stefan Witzel at this workshop, we would like to phrase questions about homological finiteness conditions for totally disconnected, locally compact groups (type FP_n, cohomological dimension) in terms of properties of such a classifying space. When group theorists think of doing this for topological groups, they tend to take the approach that, at any rate, there can be no Eilenberg-MacLane space for a topological group, since the homotopy group(oid)s of such a space would be abstract groups. The purpose of this talk is to argue that most, if not all, of the work done by classifying spaces for abstract groups can be recovered for topological groups.

This generalisation is done by category theory in [1]. First, we replace the category of topological spaces with a convenient category of spaces, that is, a cartesian closed category. In my existing work, this is the category $\mathcal{U}$ of compactly generated, weakly Hausdorff spaces (k-spaces); in future iterations, it will probably be condensed sets. The first benefit of this comes when we want to define topological homotopy groups of topological spaces X: putting the compact-open topology on the set of (pointed) continuous maps $S^n \to X$, and the quotient topology from this on $\pi_n(X)$, does not give a topological group in general. The problem is precisely the failure of topological spaces to be cartesian closed. But using the version of the compact-open topology internal to $\mathcal{U}$ (that is, the k-ification of the usual compact-open topology) instead, $\pi_n(X)$ becomes an internal group object of $\mathcal{U}$: a k-group.

We can put a model structure on $\mathcal{U}$, the CH-structure, such that the fibrant-cofibrant objects (that is, our analogue of CW-complexes) are retracts of KW-complexes, spaces built by 'attaching spaces of n-cells' in dimension n, instead of attaching a discrete sets of n-cells. Formally, a KW-complex X is a colimit (in k-spaces) of a sequence

$$X^0 \to X^1 \to^2 \to \cdots$$

in which $X^0 = T_0$ is a disjoint union of compact Hausdorff spaces, and each maps $X^n \to X^{n+1}$ is given by a pushout of a diagram

$$X^n \leftarrow S^n \times T_{n+1} \to B^{n+1} \times T_{n+1},$$

where S^n is the n-sphere, B^{n+1} the $n+1$-ball, and T_{n+1} is a disjoint union of compact Hausdorff spaces.

This model structure has fine enough weak equivalences that weak equivalences induce isomorphisms of all homotopy k-groups. On the other hand, maps that induce such isomorphisms are not weak equivalences in the CH-structure in general. For this reason, we also need to track a weaker structure, called the regular structure, in which the weak equivalences are those which induce isomorphisms of homotopy group objects in the Barr-exact completion of $\mathfrak{U}$. Both the CH-structure and the regular structure have analogous versions in $s\mathfrak{U}$, the category of simplicial objects in $\mathfrak{U}$, which we will also need.

Finally, we can use this category-theoretic work to start constructing classifying spaces for totally disconnected, locally compact groups, in [2]. For an abstract group G, the classifying space can be constructed as the geometric realisation of a simplicial set S with S_n given by the n-fold product G^n, and we copy this construction for k-groups, building the classifying space BG from a simplicial k-space S with $S_n = G^n$. By van Dantzig's theorem, S_n is a disjoint union of compact Hausdorff spaces when G is totally disconnected, locally compact, so BG is a KW-complex. (For k-groups more generally, one takes a cofibrant replacement of S before the geometric realisation.) The main technical result is the following:

Theorem 1. *Suppose C is an open cover of $X \in \mathfrak{U}$, closed under intersections; we think of C as a poset ordered by inclusion. Then $\underline{\mathrm{Sing}}(X)$ is weakly equivalent (in the regular structure on $s\mathfrak{U}$) to the homotopy colimit (in the CH-structure on $s\mathfrak{U}$) of $\{\underline{\mathrm{Sing}}(U)\}_{U \in C}$.*

From this, we show the main result.

Theorem 2. *If a k-group G is totally path-disconnected, BG is an Eilenberg–Mac Lane space $K(G, 1)$ for G.*

This applies, in particular, to totally disconnected, locally compact groups. I conjecture that if one replaces $\mathfrak{U}$ with the category of condensed sets, the weak equivalences in the CH-structure and the regular structure will be the same; this should strengthen the equivalent of Theorem 1 and allow further results to be proved in this direction.

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From discrete to t.d.l.c.

Laura Bonn

For discrete groups there are a lot of results about finiteness conditions, see [3]. The question is now, how we can transform these results into the world of totally disconnected locally compact groups. As a first step we generalize the concept of normal subgroups.

Definition. (also see [1]) Let Γ be a group and $\Lambda \leq \Gamma$. Then Λ is called a **commensurated** subgroup of Γ, if $\Lambda \cap g\Lambda g^{-1}$ has finite index in Λ and in $g\Lambda g^{-1}$.

In the following we want Γ to be a discrete group and $\Lambda \subseteq \Gamma$ to be a commensurated subgroup. Then Γ acts on Γ/Λ by left multiplication, such that we can define a map $\alpha\colon \Gamma \to \mathrm{Sym}(\Gamma/\Lambda)$. We can equipped $\mathrm{Sym}(\Gamma/\Lambda)$ with the topology of pointwise convergence and then we define the following group.

Definition. The group $\Gamma//\Lambda := \overline{\alpha(\Gamma)}$ is called the **Schlichting completion**.

Sometimes this construction is called the profinite completion of Γ relative to Λ.

Remark. (also see [2])

- If Λ is a normal subgroup, then $\Gamma//\Lambda = \Gamma/\Lambda$.
- $\Gamma//\Lambda$ is a totally disconnected locally compact group.
- If Λ is a not normal subgroup, then $\Gamma \hookrightarrow \Gamma//\Lambda$ is a dense embedding.
- $\overline{\alpha(\Lambda)}$ is a compact open subgroup of $\Gamma//\Lambda$.

For the Schlichting completion the following lemma holds.

Lemma 1. [1, Lemma 6.3 and 6.4] $\Gamma//\Lambda = \overline{\alpha(\Lambda)}\alpha(\Gamma)$ *and* $\overline{\alpha(\Lambda)} \cap \alpha(\Gamma) = \alpha(\Lambda)$.

We want to transform some finiteness conditions of the discrete group along the Schlichting completion to the tdlc case.

Theorem. [1, Theorem 6.1] *If Γ is finitely presented and $\Lambda \subseteq \Gamma$ is a finitely generated commensurated subgroup then $\Gamma//\Lambda$ is compactly presented.*

Sketch of proof: For the full proof see [1, Theorem 6.1].
The compact generation set is $\overline{\alpha(\Lambda)} \cup S$, where S is a finite generation set of $\alpha(\Gamma)$. Give four types of relations, G inherits all relations of $\alpha(\Gamma)$, the intersection of $\overline{\alpha(\Lambda)}$ and S gives the second type, the two last types comes from lemma 1. $\square$

In the setting of discrete groups the following result about finiteness properties is known [4, Section 6], for a short exact sequence $0 \to N \to G \to H \to 0$, with N is of type F_{n-1} and G of type F_n, then H is of type F_n and if N and H of type F_n, then G is of type F_n, too.
Here we have seen, if Γ is of type F_2 and the commensurated subgroup Λ is of type F_1, then $\Gamma//\Lambda$ is of type F_2.
I am currently working on generalizing Le Boudec's theorem to higher finiteness conditions.

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Some cohomological invariants for tdlc-groups

Sofiya Yatsyna

When looking at the homological aspects of totally disconnected locally compact (tdlc) groups, one may come across rational discrete cohomology theory introduced by Castellano–Weigel in [1]. Specifically, given a commutative ring R with identity and tdlc group G, let $R[G]$ denote the R-group algebra, and ${}_{R[G]}\mathbf{mod}$ – the abelian category of left $R[G]$-modules. A left $R[G]$-module D is *discrete* if and only if for each $d \in D$, the stabilizer $\mathrm{stab}_G(d)$ is an open subgroup of G. In [1], the authors establish ${}_{R[G]}\mathbf{dis}$, the full subcategory of ${}_{R[G]}\mathbf{mod}$, whose objects are discrete left $R[G]$-modules; whereby in the case of $R = \mathbb{Q}$, ${}_{\mathbb{Q}[G]}\mathbf{dis}$ is an abelian category with enough injectives, and rational discrete cohomology theory for tdlc groups can be defined.

The question naturally arises: *Can this cohomology theory be used to find analogous tdlc versions of known results?* One such interesting result is given by Gedrich–Gruenberg in [3] looking at two homological finiteness conditions on a ring R: the supremum of projective lengths (dimensions) of injective R-modules ($\mathrm{spli}(R)$) and the supremum of injective lengths of projective R-modules ($\mathrm{silp}(R)$). By way of these invariants, one can show the following:

Theorem (Gedrich–Gruenberg [3]). *Let R be a commutative noetherian ring of finite R-injective dimension t. If Λ is a R-projective Hopf R-algebra, then*

$$\mathrm{silp}\,\Lambda \leq \mathrm{spli}\,\Lambda + t.$$

Results of Cornick–Kropholler in [2] expand on the relationship of the above Gedrich–Gruenberg invariants with the finitistic dimension when R is the group algebra of a hierarchically decomposable group. Using rational discrete cohomology, it would be interesting to develop the theory of tdlc analogues. Furthermore, it turns out that ${}_{\mathbb{Z}[G]}\mathbf{dis}$ does not have enough projections (unlike its ${}_{\mathbb{Q}[G]}\mathbf{dis}$ counterpart) – whether the above cohomological invariants could be defined for ${}_{\mathbb{Z}[G]}\mathbf{dis}$ is also interesting and worth exploring.

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Finiteness Properties of Algebraic Bieri-Strebel Groups

Lewis Molyneux
(joint work with Brita Nucinkis, Yuri Santos Rego)

Given a subinterval of the real numbers I, a subgroup of positive real numbers under multiplication P, and a $\mathbb{Z}[P]$-submodule of the real numbers A, the Bieri-Strebel group $G(I, A, P)$ is the group of piecewise linear, orientation preserving homeomorphisms of I, with slopes in P and breakpoints in A. Bieri and Strebel constructed these groups[1] as a generalisation of Thompson's group F. We consider groups of the form $G([0,1], \mathbb{Z}[\beta], \langle\beta\rangle)$, where β is the positive real root of the polynomial $(\sum_{i=1}^{n} a_i x^i) - 1, a_i \in \mathbb{N}$. We call these Algebraic Bieri-Strebel groups.

A frequently useful property of Thompson's group F is the ability to express each element of the group as an ordered pair of rooted binary trees. [2] This representation of elements has proven useful in proofs of properties of Thompson's group, particularly finiteness properties such as F_n (and in particular F_∞) and the BNSR-invariant. For instance, Thompson's group can be shown to be F_∞ via its action on a space of pairs of rooted binary forests and rooted binary trees, as summarised by Zaremsky [3]. In addition, while the initial calculation of the BNSR invariant of Thompson's group was performed by Bieri, Geoghegan and Kochloukova [4], Zaremsky and Witzel were able to recalculate the invariant using Morse theory and an adaptation of the previous forest-tree space [5].

For polynomials of the form $a_2x^2 + a_1x - 1$, Winstone was able to show that tree-pair representations for all elements of the associated group are only possible when $a_2 \leq a_1$ [6]. Initially, Cleary was able to demonstrate the F_∞ property for groups with associated polynomial of the form $x^2 + nx - 1, n > 0$ [8]. This proof has since been generalised using tree-pair representations, demonstrating the F_∞ property for all quadratic Bieri-Strebel groups with complete tree-pair representations.

Molyneux, Nucinkis and Santos Rego [7] were able to apply Bieri, Geoghegan and Kochloukova's method in order to calculate the BNSR-invariant of F_τ, but the BNSR-invariant for Algebraic Bieri-Strebel groups in general remains an open problem. For those with complete tree-pair representations, A complex similar to that constructed by Stein and Farley is producible, and the Morse Theory used by Zaremsky and Witzel should be applicable.

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Embedding theorems for discrete groups

Ian J. Leary

This talk gave a survey of three embedding theorems and discussed some related open questions. The theorems, together with their dates of publication, are stated below.

Theorem 1 (Higman–Neumann–Neumann [4], 1949)**.** *Every countable group embeds in a 2-generator group.*

Theorem 2 (Higman [3], 1961)**.** *A finitely generated group embeds in a finitely presented group if and only if it is recursively presented.*

Theorem 3 ([6], 2018)**.** *Every countable group embeds in a group of type FP_2.*

Remark 4. *The questions of whether every finitely presented group embeds in a group of type F_3 and whether every group of type FP_2 embeds in a group of type FP_3 remain open. Arguably the question for FP_2 and FP_3 should be easier because computability seems not to arise.*

Remark 5. *Can one state and either prove or find a counterexample to a version of Theorem 1 for tdlc groups? Ilaria Castellano points out that free products, which are used in the proof of Theorem 1, are not available in this context* [9]*.*

Although the statement of Theorem 3 is similar to that of Theorem 1, the proof is modelled closely on Valiev's proof of the Higman embedding theorem (Theorem 2) [11, 7]. The only recent ingredient needed in the proof is the existence of a family of groups of type FP indexed by subsets of $\mathbb{Z}$ [5].

Even the most streamlined versions of the proof of the Higman embedding theorem are difficult. Very roughly there are three steps: reduction to subgroups of the free group $\mathbb{F}_2$ using Theorem 1; reduction to subsets of $\mathbb{N}$; encoding suitable subsets of $\mathbb{N}$ inside finitely presented groups.

A key definition is that of a *benign* subgroup $H \leq G$ of a finitely generated group G. This is a subgroup H such that the HNN-extension

$$\langle G, t \ : \ tht^{-1} = h \ \ h \in H \rangle$$

can be embedded in a finitely presented group.

Higman's rope trick states that if H is a benign normal subgroup of finitely generated G then G/H embeds in a finitely presented group. With the rope trick and the HNN-embedding theorem, one is reduced to showing that every recursively generated subgroup of the free group $\mathbb{F}_2$ is benign.

Words in two generators and their inverses are encoded as subsets of $\mathbb{N}$ via a Gödel numbering, with the digits $1, 2, 3, 4$ standing for the generators and their inverses, concatenation being unchanged, and 0 standing for the empty word. This gives a coding process that replaces *subsets* of $\mathbb{F}_2 = \langle a, b \rangle$ by *subgroups* of $\mathbb{F}_3 = \langle c, d, e \rangle$ of the form $\langle c^n d e^n \,:\, n \in S \rangle$ for some $S \subseteq \mathbb{N}$ in an algorithmic way.

After this step, one is reduced to showing that any recursively enumerable subset of $\mathbb{N}$ can be 'encoded' within a finitely presented group. For this the proof in Lyndon and Schupp [7] uses a technique that was not available to Higman in 1961: Matiyasevich's theorem (building on work of Davis, Putnam and Robinson) that recursively enumerable subsets of $\mathbb{Z}$ are Diophantine, i.e., of the form

$$\{x \in \mathbb{Z} \,:\, \exists y_1, \ldots, y_n \in \mathbb{Z}\ f(x, y_1, \ldots, y_n) = 0\}$$

for some n and for some integer polynomial f [8, 2, 10].

In the 1990's Bestvina and Brady constructed the first groups of type FP that are not finitely presented [1], resolving a well-known problem that had been open for at least 30 years. Around 20 years later, I discovered a way to generalize the Bestvina–Brady construction to produce an uncountable family of groups of type FP; in particular for any $S \subseteq \mathbb{Z}$ with $0 \in S$ I could construct a group $J = J(S)$ of type FP and elements $j_1, \ldots j_4 \in J$ so that $j_1^n j_2^n j_3^n j_4^n = 1$ iff $n \in S$ [5]. This showed that the class of subgroups of groups of type FP_2 is larger than the class of subgroups of finitely presented groups. Note that the map $S \mapsto J(S)$ encodes *any* subset of $\mathbb{Z}$ that contains 0 inside the presentation of a group of type FP.

This also suggested the possibility of proving the new Theorem 3, by modifying Valiev's proof of Theorem 2. Define a subgroup H of a finitely generated group G to be *homologically benign* if the HNN-extension $\langle G, t \rangle$ as defined earlier can be embedded in a group of type FP_2. Next check that there is a homological version of the Higman rope trick: if $H \leq G$ is homologically benign, then G/H embeds in a group of type FP_2. Just as in the proof of the Higman embedding theorem, this plus the HNN-embedding theorem gives a reduction: to prove Theorem 3 it suffices to show that every normal subgroup of the free group $\mathbb{F}_2$ is homologically benign. The encoding of subsets of $\mathbb{F}_2$ via subgroups of $\mathbb{F}_3$ and then subsets of $\mathbb{N}$ can be used essentially unchanged. Since we already know how to encode arbitrary subsets of $\mathbb{N} \subseteq \mathbb{Z}$ inside presentations of groups of type FP_2, this gives Theorem 3.

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Open problems on TDLC-groups

The mini-workshop also featured a discussion session on the open problems and further questions related to totally disconnected locally compact groups. In addition to problems already stated in the extended abstracts presented in this report, this section compiles further problems that remain unresolved and are ordered by those who have expressed interest in them.

1. Dawid Kielak

Question 1. *Let G be a group of type $\mathrm{FP}_n(\mathrm{F})$, for $\mathrm{F} \in \{\mathbb{Q}, \mathbb{F}_p : p \text{ prime}\}$. In general, G is not of type $\mathrm{FP}_n(\mathbb{Z})$ $(n \geq 1)$, but there exist examples where it holds (e.g. right-angled Artin group (RAAG)).*

(i) In which classes of groups is it true?
(ii) Is it true if there exists an exact sequence $G \to H \to \mathbb{Z}$, where H of type F?

2. Rudradip Biswas

Theorem 1. [1, Theorem 19.1] *Let $\Gamma \in \mathbf{LH}\mathfrak{F}$ (locally in Kropholler's Hierarchy with all finite groups as the base class), and suppose Γ is of type $\mathrm{FP}_\infty(\mathbb{Z})$. Then Γ has only finitely many conjugacy classes of finite elementary abelian p-groups.*

Question 2. *Is it possible to formulate a TDLC version of this theorem?*

Remark 2. *It follows from the discussion that it may be useful to consider groups of type $\mathrm{FP}_\infty(\mathbb{Q})$ or F_∞.*

3. Ilaria Castellano

Question 3. *Let G, H be compactly generated TDLC groups, and suppose G is quasi-isometric to H, with G and H both having finite cohomological dimension over $\mathbb{Q}$. Does $\mathrm{cd}_\mathbb{Q}(G) = \mathrm{cd}_\mathbb{Q}(H)$?*

Question 4. *Is there a suitable TDLC analogue of right-angled Artin groups (RAAGs)?*

4. Ian Leary & Ilaria Castellano

Question 5. *Suppose G is a σ-compact TDLC group. Does G embed into a compactly generated TDLC group?*

5. Roman Sauer

Question 6. *Does a version of the Atiyah conjecture hold for TDLC groups? Or do there exist examples of TDLC groups G with irrational L^2-Betti numbers, i.e. $b_n^{(2)}(G,\mu) \notin \mathbb{Q}$, where μ is the Haar measure normalised to be 1 on a compact-open subgroup of G.*

Remark 3. *For a locally compact group G and lattice $\Gamma < G$, then $b_n^{(2)}(\Gamma) = \mathrm{covol}(\Gamma) b_n^{(2)}(G)$. There exist non-compactly generated examples with irrational covolume.*

Question 7. *Are there examples of compactly generated TDLC groups with lattices of irrational covolume?*

6. Yuri Santos Rego

Question 8. *Under which conditions are Coxeter groups virtually residually finite rationally solvable (RFRS)?*

Remark 4. *Kielak in* [2] *shows that for a finitely generated virtually RFRS group G, virtual fibering is equivalent to the vanishing of the first L^2-Betti number, i.e.* $b_1^{(2)}(G) = 0$.

Question 9. *Do there exist Coxeter groups W_1, W_2 with isomorphic profinite completions $\widehat{W_1} \cong \widehat{W_2}$ but different rational Euler characteristic $\chi(W_1) \neq \chi(W_2)$?*

7. Thomas Weigel

Question 10. *Let N_q be the Neretin group acting on the q-regular rooted tree.*
 (i) Is $H^k(N_q, \mathrm{Bi}(N_q)) = 0$ *for all k?*
 (ii) Is $H_c^k(\underline{E}_{\mathcal{O}} N_q, \mathbb{Q}) = 0$ *for all k?*

Question 11. *Are there "non-good" Coxeter groups?*

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Mini-Workshop: Flavors of Rabinowitz Floer and Tate Homology

Organized by
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Nathalie Wahl, Copenhagen

27 November – 1 December 2023

Abstract. Rabinowitz Floer homology originated 15 years ago in symplectic geometry. Recent developments have related it to algebraic topology via string toplogy and Tate homology, and to mirror symmetry via Fukaya categories. This mini-workshop brought together researchers from these different communities, in order to foster exchange and collaborations across research fields.

Mathematics Subject Classification (2020): 55P50.

Introduction by the Organizers

The mini-workshop *Flavors of Rabinowitz Floer and Tate homology* was organized by Kai Cieliebak, Alexandru Oancea and Nathalie Wahl. Its goal was to bring together specialists from symplectic geometry, topology, and algebra, in order to discuss recent algebraic structures that emerged in parallel in these different fields.

The workshop was structured as follows. The mornings were generally dedicated to individual talks by the participants, with an intended duration time of 30 min. and 30 min. discussion time for each talk. The afternoons were generally dedicated to discussions on topics that arose during the morning talks. This ensured an intense atmosphere of exchange during the whole duration of the workshop. Each of the participants in the mini-workshop gave a talk, and we also had a special guest talk by Peter Kropholler, who was participating in a parallel mini-workshop during the same week. The 16 participants in the workshop covered a large age spectrum and included 3 Ph.D. students, and also a large geographic area with participants from 8 countries. We had also a reasonable gender balance with 5 women and 11 men.

The first day started with an Introductory opening talk by Kai Cieliebak. He outlined the various symplectic, algebraic, and topological constructions that we were planning to discuss during the week, as well as the interactions between them, both those that are already understood and those that are not yet understood and that formed part of the discussion material for the workshop. The abstract of his talk can serve as a concise guide to the topics that were subsequently discussed during the week. Then followed two talks by Ph.D. students. Shuaipeng Liu explained in his talk *Introduction to symplectic homology* the fundamentals of Floer homology, with the specific goal of building common mathematical ground for the participants in the workshop. Zhen Gao explained in his talk *Calabi–Yau algebras* the fundamentals of Calabi–Yau structures. The talk was geared towards the participants that were not specialists in algebra, and it had a similar goal of building common mathematical ground. The afternoon of the first day was dedicated to discussing various flavors of Calabi–Yau algebras, arising both in the context of Hochschild homology and in the context of string topology. In the afternoon, Inbar Klang gave a talk on *String topology category as a Calabi–Yau category*, which put the previous notions in a topological context.

During the second day we steered towards various version of the Tate construction. Alexandru Oancea's first talk in the morning on *Cone perspective on Rabinowitz–Floer homology* explained a Tate-type construction in Floer theory. More specifically, he focused on algebraic structures arising from mixing together at chain level a product and a coproduct. Alice Hedenlund followed with a talk on the *Tate construction*, both in the classical setting of finite groups, and in the much more general setting of spectra. In the afternoon, Peter Kropholler gave a guest talk on *Tate homology without complete resolutions*, and Alex Takeda explained during the first discussion session various other flavors of Tate constructions arising in the context of Hochschild homology. The second part of the afternoon featured an intense discussion on the precise interpretation of Rabinowitz Floer homology as a Tate construction. Also during the second day we had an evening talk by Mohammed Abouzaid on *Symplectic cohomology with supports and framed E_2-algebra structures*, in which he explained how to build suitable models for symplectic cochains, strictly compatible both with the structure of framed E_2-algebra, and with Viterbo restriction maps.

The third day of the workshop started with a talk by Noémie Legout on *Rabinowitz Floer homology/category from the perspective of Symplectic Field Theory.* Her construction uses pseudo-holomorphic curves in symplectizations and complements geometrically the constructions inspired by the wrapped Fukaya category, as the one presented by Hanwool Bae later in the day. The second talk that morning was given by Urs Frauenfelder, on *Spectral jumps in Rabinowitz–Floer–Tate homology.* He emphasized certain spectral jump phenomena akin to ones encountered in quantum mechanics, which become visible once the classical Rabinowitz–Floer homology is enhanced by an additional construction of Tate flavor with respect to the natural circle action. The third talk in the morning was given by Hanwool Bae, on *Calabi–Yau structures on Rabinowitz Fukaya categories.* His talk complemented the one by Noémie Legout by presenting a construction of that category that relies heavily on methods from symplectic homology and wrapped Floer homology.

On Wednesday afternoon the whole group went on the traditional hike to Sankt Roman. It was a beautiful sunny afternoon.

The fourth day started with the talk by Amanda Hirschi, in which she presented the recent *Counterexamples to Donaldson's 4-6 question* that she discovered in joint work with Luya Wang. This is a classical question in 4-manifold topology, asking whether the diffeomorphism type of symplectic 4-manifolds is detected by the symplectic deformation class after stabilization with S^2. The second talk in the afternoon by Ph.D. student Colin Fourel was on *Sheaf and singular models for* ∞*-groupoid cohomology.* He explained how to prove the equivalence between singular cohomology of a space X and sheaf cohomology of the constant sheaf, based on the analogy between X and BG, with $G = \Omega X$. This point of view was directly relevant to the theme of the workshop, during which the based loop space played a prominent role. The third talk was given by Andrea Bianchi, on *String topology and graph cobordisms*, in which he explained how to generalize the fundamental operations from string topology to spaces of maps in a functorial way. The afternoon was dedicated to phrasing some key questions and discussing possible answers and future directions of research. We discussed the construction of operations in Floer theory, the interpretation of the Rabinowitz Floer chain complex as a classifying space for Tate homology, explicit computations for free loop spaces of spheres, as well as S^1- and $O(2)$-equivariant aspects. Koszul duality between $C_*(\Omega M)$ and $C^*(M)$ for a simply connected manifold M was discussed in several instances.

The last half-day of the workshop featured three talks. Alex Takeda explained the *Categorical formal punctured neighborhood at* ∞, a categorical construction that gives an open-string description of Rabinowitz Floer homology. Nathalie Wahl gave a talk on *Spaces of operations*, in which she explained Sullivan diagrams as a model for moduli spaces of Riemann surfaces, and how these give rise to algebraic operations on Hochschild complexes. The last talk of the workshop was by Georgios Dimitroglou Rizzell on *Relative Calabi–Yau structure from acyclic Rabinowitz–Floer complexes of Legendrians.* This was a beautiful conclusion to the workshop, putting to work all the algebraic structures that had been seen over the week in a geometric context.

At the end of the week all the participants were exhausted, but happy. One Ph.D. student said: "I learnt more mathematics this week than during a whole semester!" We would like to interpret that as a sign of success for the workshop.

Mini-Workshop: Flavors of Rabinowitz Floer and Tate Homology

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Abstracts

Three flavors of Rabinowitz Floer and Tate homology

Kai Cieliebak

Rabinowitz Floer homology appears in three flavours: symplectic, topological, and algebraic. The goal of this talk is to describe these aspects and discuss their relationships, with an emphasis on open problems. The talk has three parts.

Part 1 is devoted to the following isomorphisms for a closed oriented n-dimensional manifold M:

$$SH_*(D^*M) \cong H_*(\Lambda M) \cong HH_*\big(C_*(\Omega M)\big) \overset{\pi_1 M=0}{\cong} HH^*\big(C^*(M)\big).$$

Here the first term is the symplectic homology of the unit disk cotangent bundle D^*M. This is a purely symplectic invariant which is defined more generally for any Liouville domain V. The second term is of topological nature and denotes the singular homology of the free loop space ΛM. The third and fourth terms are of algebraic nature. Here $HH_*(A)$ and $HH^*(A)$ denote the Hochschild homology and cohomology of a differential graded algebra A (or, more generally, an A_∞-algebra or an A_∞-category). These are applied, respectively, to the singular chains on the based loop space ΩM and the singular cochains on M. All (co)homology groups are taken with $\mathbb{R}$-coefficients, and for the last isomorphism we assume that M is simply connected.

All four (co)homology groups are BV algebras in a natural way, and the isomorphisms are expected to respect this structure. In the first two groups the BV structure arises from the pair-of-pants product resp. the Chas-Sullivan loop product and the circle action. In the last two groups it arises from the cup product on Hochschild cohomology and the Connes operator, using the fact that $C_*(\Omega M)$ is smooth Calabi–Yau and (a suitable Poincaré model of) $C^*(M)$ is proper Calabi–Yau. The first isomorphism is known as the Viterbo isomorphism, and the last isomorphism arises from the fact that $C^*(M)$ is the Koszul dual of $C_*(\Omega M)$. The isomorphism $SH_*(D^*M) \cong HH_*\big(C_*(\Omega M)\big)$ is a special case of the isomorphism

$$SH_*(V) \cong HH_*\big(\mathcal{W}\mathrm{Fuk}(V)\big)$$

for any Weinstein domain V, where $\mathcal{W}\mathrm{Fuk}(V)$ denotes its wrapped Fukaya category.

Part 2 concerns the extension of the previous structures by coproducts. On the symplectic side, this requires the passage to reduced symplectic homology $\overline{SH}_*(V) = \mathrm{coker}(c_*)$ with respect to the canonical chain map

$$c : SC^{-*}(V) \to SC_*(V)$$

from symplectic cochains to chains. For a suitable class of Weinstein domains (including unit disk cotangent bundles), $\overline{SH}(V)$ carries a secondary pair-of-pants coproduct defining together with the pair-of-pants product the structure of a unital infinitesimal antisymmetric bialgebra. On the topological side, such a structure exists on reduced loop homology $\overline{H}_*(\Lambda M) = H_*(\Lambda M)/\chi(M)[\mathrm{pt}]$, the quotient of loop homology by the Euler characteristic times the point class. Here the coproduct is an extension of the Sullivan-Goresky-Hingston coproduct and the Viterbo isomorphism descends to an isomorphism of bialgebras

$$\overline{SH}_*(D^*M) \cong \overline{H}_*(\Lambda M).$$

The relation of this structure to similar structures on the algebraic side appears not yet to be fully understood.

Part 3 concerns the partly conjectural isomorphisms

$$RFH_*(S^*M) \cong \widehat{H}_*(\Lambda M) \cong \widehat{HH}_*\big(C_*(\Omega M)\big) \overset{\pi_1 M=0}{\cong} \widehat{HH}^*\big(C^*(M)\big).$$

Here the first term is the Rabinowitz Floer homology of the unit sphere cotangent bundle S^*M. This is a purely symplectic invariant which is defined more generally for the boundary ∂V of any Liouville domain V. One of its definitions is as the homology of the cone of the map $c : SC^{-*}(V) \to SC_*(V)$ from above. Similarly, Rabinowitz loop homology $\widehat{H}_*(\Lambda M)$ can be defined as the homology of the cone of the map $c : C^{-*}(\Lambda M) \to C_*(\Lambda M)$ multiplying the point class by $\chi(M)$. On the algebraic side, $\widehat{HH}_*(A)$ and $\widehat{HH}^*(A)$ denote the Tate Hochschild homology and cohomology of a differential graded Frobenius algebra A, applied to suitable models for $C_*(\Omega M)$ and $C^*(M)$ (where to our knowledge only the latter has been defined so far).

All four (co)homology groups are expected to be graded Frobenius algebras and the isomorphisms are expected to respect this structure. However, this appears to be proved only for the first two groups and their isomorphism. The isomorphism $RFH_*(S^*M) \cong \widehat{HH}_*\big(C_*(\Omega M)\big)$ is expected to generalize to isomorphisms

$$RFH_*(\partial V) \cong \widehat{HH}_*\big(\mathcal{W}\mathrm{Fuk}(V)\big) \cong HH_*\big(\mathcal{RW}\mathrm{Fuk}(V)\big)$$

for certain Weinstein domains V, where $\mathcal{RW}\mathrm{Fuk}(V)$ denotes the Rabinowitz (wrapped) Fukaya category.

Further open questions concern the description of the above groups in terms of symplectic field theory, their S^1-equivariant versions, the underlying chain-level structures, their relation to Varolgunes' version of symplectic homology, and their role in semiclassical quantization.

Symplectic Homology and Rabinowitz Floer Homology Revisited

Shuaipeng Liu

Symplectic homology is defined in a Liouville domain $(W, d\lambda, \partial W = M, \lambda)$ with symplectic completion $\widehat{W} = W \sqcup_M ([1,\infty) \times M, d(r\lambda))$ by attaching the positive symplectization along the boundary M, where the Liouville vector field defined by $\iota_X d\lambda = \lambda$ transversally points outwards along M.

As a filtered version of Floer homology, the filtered chain group $CF_*^{<a}(H)$ is the $\mathbb{Q}-$vector space generated by the critical points of the Hamiltonian action functional of a loop $x : S^1 \to \widehat{W}$,

$$\mathcal{A}_H(x) := \int_{S^1} x^*\lambda - \int_0^1 H(t, x(t))dt,$$

and graded by the Conley-Zehnder index $CZ(x)$, where the filtration is given by the bounded action $\mathcal{A}_H(x) < a$. The differential operators $\partial_k : CF_k(H) \to CF_{k-1}(H)$ are defined by algebraically counting the number of the unparametrized moduli space of Floer trajectories,

$$\partial_k x := \sum_{CZ(y)=k-1} \#\mathcal{M}(y, x)y, \ x \in CF_k(H)$$

decreasing the action. By setting an action window $-\infty \leq a < b \leq \infty$, they restrict to differential operators $\partial_*^{(a,b)}$ on $CF_*^{(a,b)} := CF_*^{<b}/CF_*^{<a}$. Then the filtered Floer homology group is

$$FH_*^{(a,b)}(H) := H_*(CF^{(a,b)}, \partial^{(a,b)}).$$

The symplectic homology is defined as the direct limit of filtered Floer homologies with respect to Hamiltonians of an admissible class, which will be non-negative on W and grow linearly for r large enough in the completion $\widehat{W}$ with only nondegenerate 1-periodic orbits. By standard argument about the continuation map, i.e. a monotone increasing homotopy $\widehat{H} : H_- \to H_+$, one can define the symplectic homology by taking direct limit via the monotone homotopy,

$$SH_k^{(a,b)} := \varinjlim FH_k^{(a,b)}(H).$$

Likewise, one can dually define the symplectic cohomology by taking the inverse limit.

Symplectic homology and symplectic cohomology cannot be related directly by an isomorphism. The main result to describe the relationship is by a new version of Floer homology constructed via a new class of admissible Hamiltonians required addionally to be non-negative in some tubular neighborhood of the boundary M and positive elsewhere in the Liouville domain W, thus they vividly look like with the shape $\bigvee$. The new version is called the Rabinowitz Floer Homology, denoted by $\check{SH}$. And the main result is as follows

Theorem [1] There exists a long exact sequence

$$\cdots \longrightarrow SH^{-*}(W) \longrightarrow SH_*(W) \longrightarrow \check{SH}_*(W) \longrightarrow SH^{-*+1}(W) \longrightarrow \cdots$$

In the talk, I will briefly explain the notions mentioned in the definition of symplectic homology as a revisit to Floer-like theory.

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Flavors of Calabi–Yau structures

Zhen Gao

The notion of Calabi–Yau structure in the algebraic context was initially noticed by Maxim Kontsevich and formally introduced by Victor Ginzburg [1], then further studied and developed by many others, e.g. Michel Van den Bergh, Bertrand Toën, and Bernhard Keller. Calabi–Yau structures have played a prominent role in algebraic geometry, noncommutative geometry, and representation theory. Recently, Calabi–Yau structures are emerging in string topology and symplectic topology on the relevant homological algebraic invariants.

Let $\mathcal{A}$ be an DG/A_∞-category over a field $\mathbb{K}$. Denote by $\mathcal{A}_\Delta$ the diagonal $\mathcal{A}$-bimodule, and $\mathcal{A}^! := \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e)$ the inverse dualizing bimodule of $\mathcal{A}$ regarded as diagonal bimodule. There are following absolute Calabi–Yau structures:

Smooth n-CY: Suppose $\mathcal{A}$ is (homologically/locally) smooth, i.e. $\mathcal{A}$ is a perfect $\mathcal{A}$-bimodule, a *weak smooth Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a Hochschild class $[\xi_{\mathcal{A}}] \in HH_n(\mathcal{A})$ such that the induced map $[\hat{\xi}_{\mathcal{A}} \circ \Sigma^{-n}] : A^![n] \to A$ is an isomorphism in the derived category $\mathcal{D}(\mathcal{A}^e)$. A *strong smooth Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a negative cyclic class $[\tilde{\xi}_{\mathcal{A}}] \in HC_n^-(\mathcal{A})$ whose underlying Hochschild class $[\xi_{\mathcal{A}}] := h([\tilde{\xi}_{\mathcal{A}}]) \in HH_n(\mathcal{A})$ is a weak smooth n-Calabi–Yau structure.

Proper n-CY: Suppose $\mathcal{A}$ is (locally) proper, i.e. $\mathcal{A}_\Delta$ is a proper $\mathcal{A}$-bimodule, a *weak proper Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a degree n chain map $\mathrm{tr} : C_\bullet(\mathcal{A}, \mathcal{A}) \to \mathbb{K}[n]$ inducing an isomorphism $\mathcal{A} \xrightarrow{\sim} (\mathcal{A}^{\mathrm{op}})^*[-n]$ in derived category $\mathcal{D}(\mathcal{A}^e)$. A *strong proper Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a factorization of weak proper n-Calabi–Yau structure through the projection to the cyclic chain complex $\widetilde{\mathrm{tr}} : CC_\bullet(\mathcal{A}) \to \mathbb{K}[-n]$.

One of the significant consequences of the presence of smooth Calabi–Yau structure is the Poincaré duality between Hochschild cohomology and homology first observed by Michel Van den Bergh [2], see also [3], and naturally from which a Batalin–Vilkovisky algebra structure on Hochschild cohomology follows, e.g. [4].

Examples.

- In string topology, let X be a topological space, the mostly concerned DG/A_∞ algebras are chains of based loop space $C_\bullet(\Omega X; \mathbb{K})$ and singular cochains $C^\bullet(X; \mathbb{K})$. When X is a n-dimensional Poincaré duality space over characteristic 0 field $\mathbb{K}$, then $C_\bullet(\Omega X; \mathbb{K})$ resp. $C^\bullet(X; \mathbb{K})$ is strong smooth resp. proper n-CY.

- In symplectic topology, let X be a Liouville manifold, e.g. cotangent bundle T^*Q of a closed oriented smooth manifold Q, the relevant algebraic invariant is some A_∞-category called Fuakaya category, e.g. wrapped Fukaya category $\mathcal{W}(X)$ and its proper full subcategory $\mathcal{F}(X)$, introduced and established by Mohammend Abouzaid and Paul Seidel. When X is non-degenerate Liouville manifold, there is geometric strong smooth Calabi–Yau structure on $\mathcal{W}(X)$ resp. strong proper Calabi–Yau structure on $\mathcal{F}(X)$. C.f. [5],[6].

Relative Calabi–Yau structures are introduced by Christopher Brav and Tobias Dyckerhoff in [7] for a DG functor between DG categories $F : \mathcal{A} \to \mathcal{B}$. Definitions are similar to absolute cases hence generalizing the notions to relative sense, despite in addition the relevant Hochschild classes in $HH_\bullet(\mathcal{A})$ should also induces isomorphisms in derived category $\mathcal{D}(\mathcal{A}^e)$ between some distinguished triangles for the homotopy cofiber and fiber of certain induced maps $\gamma_F^!$ and γ_F from the functor F. In very recent work, Christopher Brav and Nick Rozenblyum have shown that in the compactly generated DG categories setting, there is framed E_2-algebra structure on the chain-level of Hochschild cohomology given a relative Calabi–Yau structure.

Examples.

- In string topology, key examples are $C_\bullet(\Omega\partial Q) \hookrightarrow C_\bullet(\Omega Q)$ and $C_\bullet(\partial Q) \hookrightarrow C_\bullet(Q)$ where Q is taken to be compact oriented smooth manifold with boundary ∂Q.
- In symplectic topology, Gergeois Dimitroglou Rizell and Noémie Legout reveal that Chekanov-Eliashberg DG algebra with coefficient as based loop DG algebra for some Legendrian submanifold carries relative smooth Calabi–Yau structure.

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The string topology category as a Calabi–Yau category

Inbar Klang

The string topology category $\mathcal{S}_M$ of a connected, closed, oriented manifold M was defined by Blumberg–Cohen–Teleman in [1]. The objects of this category are some collection of connected, closed, oriented submanifolds $N \subseteq M$ that all contain some chosen point $q \in M$. The morphisms between N_1 and N_2 consist of chains on the space of paths P_{N_1,N_2} that start in N_1 and end in N_2, shifted by the dimension of N_1.

The composition in this category is given by intersecting and concatenating, similar to the Chas–Sullivan product on homology of a free loop space of a manifold. To define this more precisely, one uses Poincaré duality on N_1 to rewrite the morphism complexes as a derived hom,

$$Rhom_{C_*(\Omega M)}(C_*(P_{q,N_1}), C_*(P_{q,N_2}))$$

Here $C_*(\Omega M)$ denotes the based loop space, $Map_*(S^1, M)$. This embeds the string topology category as a full subcategory of $Perf_{C_*(\Omega M)}$, the category of perfect modules over $C_*(\Omega M)$. In fact, if $\{q\} \in M$, the string topology category includes the generator of $Perf_{C_*(\Omega M)}$, and is Morita equivalent to it.

The category $Perf_{C_*(\Omega M)}$, and in the above case also the string topology category, are smooth Calabi–Yau categories. This comes from the fact that $C_*(\Omega M)$ is a smooth Calabi–Yau algebra. Roughly, A is a smooth Calabi–Yau algebra over k if it is smooth (a perfect $A \otimes A^{op}$ module) and has an S^1-invariant "fundamental class" in the Hochschild chains of A, evaluation on which gives an equivalence between the Hochschild chains and cochains of A (with an appropriate shift.)

In the case $A = C_*(\Omega M)$, this fundamental class comes from the fundamental class of M in $C_*(M)$, which can then be mapped to the Hochschild chains of $C_*(\Omega M)$, which agree with $C_*(Map(S^1, M))$. Since Hochschild cochains always have a shuffle product, this gives a product on (a shift of) $C_*(Map(S^1, M))$, which agrees with the Chas–Sullivan product.

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Cone perspective on Rabinowitz Floer homology

Alexandru Oancea
(joint work with Kai Cieliebak, Nancy Hingston)

In joint work with Kai Cieliebak and Nancy Hingston [1] we studied Rabinowitz Floer homology and cohomology $RFH_*(V)$ of a Liouville domain V. One of our key results is that they both carry the structure of graded Frobenius algebras and that they are related by a Poincaré duality isomorphism.

In the particular case when the Liouville domain is the unit disc cotangent bundle D^*Q of a closed smooth manifold Q, its symplectic topology is related to the topology of the free loop space $\Lambda = \mathrm{Map}(S^1, Q)$. Let $\Lambda_0 \subset \Lambda$ be the subspace of constant loops. We define the *Rabinowitz loop homology of* Q to be

$$\widehat{H}_*\Lambda = RFH_*(D^*Q).$$

Let $H_*\Lambda$ denote the homology of the free loop space of Q, and $H^*\Lambda$ its cohomology. Consider the map $\varepsilon : H^{-*}\Lambda \to H_*\Lambda$ that is everywhere zero, except in degree 0 in the component of contractible loops, where it is multiplication by the Euler characteristic $\chi(Q)$. The *reduced loop homology and cohomology groups*

$$\overline{H}_*\Lambda := \operatorname{coker}\varepsilon, \qquad \overline{H}^*\Lambda := \ker\varepsilon$$

therefore differ from $H_*\Lambda$ and $H^*\Lambda$ only by $\chi(M)$ times the point class.

Theorem [1, 2, 3]. *(i) The Chas-Sullivan product on $H_*\Lambda$ descends to $\overline{H}_*\Lambda$. The Goresky-Hingston product on $H^*(\Lambda, \Lambda_0)$ extends (canonically if $H_1Q = 0$) to $\overline{H}^*\Lambda$.*

(ii) We have a short exact sequence in which ι is a ring map

$$0 \to \overline{H}_*\Lambda \xrightarrow{\iota} \widehat{H}_*\Lambda \xrightarrow{\pi} \overline{H}^{1-*}\Lambda \to 0, \tag{1}$$

which splits (canonically if $H_1Q = 0$) via a ring map $\overline{H}^{1-*}\Lambda \xrightarrow{\bar{i}} \widehat{H}_*\Lambda$. The product on $\widehat{H}_*\Lambda$ restricts to the Chas-Sullivan product on $\overline{H}_*\Lambda$, and to the extended Goresky-Hingston product on $\overline{H}^{1-*}\Lambda$.

To prove this theorem, we developed in joint work with K. Cieliebak [2] a theory of multiplicative structures on cones. Indeed, the previous theorem can be proved by describing Rabinowitz loop homology at chain level as the "cone of ε".

The general setup for multiplicative structures on cones is that of a chain complex $(\mathcal{A}, \partial)$ and a chain map $c : \mathcal{A}^\vee \to \mathcal{A}$. We prove in [2] that a multiplicative structure on $\mathrm{Cone}(c) = \mathcal{A} \oplus \mathcal{A}^\vee[-1]$ can be obtained from the data of an A_2^+-*structure* on $\mathcal{A}$, a notion that we define. This consists of the chain map c, a homotopy between $c^\vee$ and c, a degree 0 product on $\mathcal{A}$ and a degree 1 coproduct on $\mathcal{A}$, satisfying certain relations. The product on the cone is obtained by dualizing the product and coproduct in all possible ways at their inputs and outputs. The axioms of an A_2^+-structure ensure that the resulting operation is a chain map. It is an open problem to develop the theory of A_3^+-structures (associativity), and indeed A_∞^+-structures (associativity up to homotopy).

In the talk I have explained the notion of an A_2^+-structure, how it determines a product structure one the cone, and how that articulates with Rabinowitz Floer homology. I have also argued that this construction can be interpreted as a chain level counterpart of a classical construction called *the Drinfeld double* [4].

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Tate Cohomology of Finite Groups and the Tate Construction

Alice Hedenlund

1. Tate Cohomology of Finite Groups

Let k be some commutative ring. Classically, *Tate cohomology* of the finite group G with coefficients in the G-module M is defined as

$$\hat{H}^*(G;M) = \widehat{\operatorname{Ext}}^*_{kG}(k, M)$$

where $\widehat{\operatorname{Ext}}$ denotes the *complete Ext* of Mislin [3]. In practice, the Tate cohomology groups are computed via the *complete resolution*

$$\hat{P}_* := (\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_0^\vee \longrightarrow P_1^\vee \longrightarrow \cdots), \qquad P_0 \xrightarrow{\ \epsilon\ } k \cong k^\vee \xrightarrow{\ \epsilon^\vee\ } P_0^\vee$$

where $\epsilon : P_* \to k$ is a projective resolution of k as a G-module and $P_i^\vee$ denotes the k-linear dual of P_i as in [1]. We note that this splices together group homology and cohomology together via the *norm map*

$$\mathrm{Nm}_G : M_G \longrightarrow M^G\,, \quad m \mapsto \sum_{g \in G} gm\,.$$

2. The Tate Construction

In this section, G will be a topological group. We denote by BG a fixed classifying space. Letting $\mathcal{D}k$ denote the derived ∞-category over k, we consider objects in the category $\mathrm{Fun}(BG, \mathcal{D}k)$ which we call *chain complexes with G-action.* If G is a finite group, then a G-module M can be viewed as an object in this category, and we have that

$$M_{hG} = \operatorname*{colim}_{BG} M \simeq k \otimes^L_{kG} M \quad \text{and} \quad M^{hG} = \lim_{BG} M \simeq R\mathrm{Hom}_{kG}(k, M)\,,$$

whose homology groups recover group homology and cohomology, respectively.

The analogue of the Tate construction is obtained by considering a generalization of the norm map. If X is equipped with a G-action, we can consider it as equipped with a $G\times G$-action by adding a trivial right action. Consider the chains $C_*(G;k)$, which is a chain complex with $G\times G$-action by the natural action of G on itself from the right and the left. Under the appropriate identifications the *norm map* is simply the colimit-limit exchange map

$$\begin{array}{ccc} \mathrm{colim}_{BG}\lim_{BG}(X\otimes C_*(G;k)) & \xrightarrow{\kappa} & \lim_{BG}\mathrm{colim}_{BG}(X\otimes C_*(G;k)) \\ \downarrow\simeq & & \downarrow\simeq \\ (X\otimes D_{BG})_{hG} & \xrightarrow{\mathrm{Nm}_{BG}} & X^{hG} \end{array}$$

where D_{BG} is the dualizing spectrum of G by Klein [2]. The Tate construction on X is defined as the cofibre

$$X^{tG} = \mathrm{cofib}(\mathrm{Nm}_{BG} : (X\otimes D_{BG})_{hG} \to X^{hG})\,.$$

If G is a finite group and M is a G-module, then the homology groups of the Tate construction in the above sense recover the Tate cohomology groups of G with coefficients in M, as in the previous section.

Let us finally outline how the Tate construction is related to Poincaré duality. If $G=\Omega Q$ where Q is an closed n-dimensional manifold, then D_Q can be identified with the Spivak normal bundle of Q. The norm map on homology groups is then

$$H_{*+n}(M;\omega_M) \longrightarrow H^{-*}(M;k)\,,$$

where ω_M is the orientation bundle associated to M. This is the same map that appears in the statement of twisted Poincaré duality. In this case, the norm map is an equivalence, so that the Tate construction vanishes.

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Tate Cohomology — with or without Complete Resolutions

Peter H. Kropholler

Fix an associative ring with one [8] and denoted R. Let ${}_R\mathbf{Mod}$ be the category of left R-modules. A *cohomological functor* H^* with domain ${}_R\mathbf{Mod}$ consists of a family $(H^n)_{n\in\mathbb{Z}}$ of functors $H^n\colon {}_R\mathbf{Mod}\to\mathbf{Ab}\,(={}_{\mathbb{Z}}\mathbf{Mod})$ such that there are natural connecting homomorphisms $\delta: H^n(N')\to H^{n+1}(N'')$ associated to any short exact sequence $0\to N''\xrightarrow{\iota} N\xrightarrow{\pi} N'\to 0$ yielding a long exact sequence

$$\cdots\to H^{n-1}(N')\xrightarrow{\delta} H^n(N'')\xrightarrow{\iota_*} H^n(N)\xrightarrow{\pi_*} H^n(N')\xrightarrow{\delta} H^{n+1}(N'')\to\cdots.$$

Chain complexes of projective modules provide a source of cohomological functors. Let P_* be a chain complex of projective modules. That the assignment $N \mapsto H^n(\hom_R(P^*, N))$ defines a cohomological functor rests on two key properties P^*: firstly it is a chain complex so that $\hom_R(P^*, N)$ is a cochain complex; secondly each P_n is projective ensuring the existence of long exact sequences. There is no requirement that the chain complex P^* be exact (i.e. having homology everywhere zero) or acyclic (i.e. having the homology of a point). Any chain complex of projective modules will serve as a foundation for a cohomological functor.

We define a *Tate cohomological functor* to be a cohomological functor which vanishes on projective modules in all dimensions. Mislin [7] shows that to any cohomological functor H^* there is a Tate cohomological functor $\widehat{H}^*$ together with a map $H^* \to \widehat{H}^*$ so that the following universal property holds: for any map ν from H^* to a Tate cohomological functor K^* there is a unique map $\widehat{H}^* \to K^*$ so that ν factorises as the composite $H^* \to \widehat{H}^* \to K^*$.

Projective modules. The importance of projective modules here is paramount. Recall the classical definition that a module P is *projective* if every map from P to the codomain of an epimorphism factors through the domain. Since ${}_R\mathbf{Mod}$ is an abelian category we can reformulate this definition: the modern definition might read: a module P is *projective* when the functor $\hom(P, \;) : {}_R\mathbf{Mod} \to \mathbf{Ab}$ commutes with finite colimits. This is equivalent to the classical definition: in effect the classical definition tells us that $\hom(P, \;)$ commutes with coequalisers but it automotically commutes with all limits and since finite coproducts in an abelian category are naturally identified with finite products we deduce that a classically project P yields a functor $\hom(P, \;)$ that commutes with finite coproducts and with coequalisers and therefore with all finite colimits. This philosophy holds for many abelian categories including categories of sheaves over a space or site. The category ${}_R\mathbf{Mod}$ admits a forgetful functor to set which has a left adjoint: the free module on a set. A map of modules is an epimorphism if it surjective on the underlying sets (to put this in modern language we may say that the forgetful functor reflects epimorphisms) and so free modules are projective. This leads to the characterisation that a module is projective if and only if it is a direct summand of some free module.

Mislin's approach to Tate Cohomology via Satellites. For a fixed R-module N choose any projective resolution $P_* \to N \to 0$. Let $\Omega^0 N$ denote N and let $P_{-1} = 0$. For $n \geq 1$, let $\Omega^n N$ denote the kernel of the map $P_{n-1} \to P_{n-2}$. Then we have short exact sequences $\Omega^n N \to P_{n-1} \to \Omega^{n-1} N$ for $n \geq 1$. We have a sequence of connecting homomorphisms

$$H^n(N) \xrightarrow{\delta} H^{n+1}(\Omega N) \xrightarrow{\delta} H^{n+2}(\Omega^2 N) \xrightarrow{\delta} H^{n+3}(\Omega^3 N) \xrightarrow{\delta} H^{n+4}(\Omega^4 N) \xrightarrow{\delta} \cdots .$$

The colimit of this sequence is the nth Tate cohomology group $\widehat{H}^n(N)$.

There are two other accounts by Goichot–Vogel and by Benson–Carlson published at around the same time and based on chain maps and chain homotopies. See the work of Cornick (some joint with the author) [3, 4, 5, 2] for further information.

Benson–Carlson [1]**, Goichot–Vogel** [6]**.** Tate cohomology can be defined using almost chain maps modulo almost chain homotopies. Let $C_\bullet$ and $C'_\bullet$ be chain complexes. An *almost chain map* $\phi : C_\bullet \to C'_\bullet$ *of degree* j is a family of maps $(\phi_* : C_* \to C'_{*+j})$ such that for all sufficiently large n the square starting at C_n commutes. An *almost chain homotopy* from an almost chain map ϕ of degree j to an almost chain map ψ of degree j is a map s of degree $j+1$ such that $ds+sd = \psi-\phi$ at the square starting at C_n for sufficiently large n. Using this approach we define the Tate Ext groups $\widehat{\mathrm{Ext}}^j_R(M,N)$ where M and N are two R-modules by first choosing two projective resolutions $P^* \to M \to 0$ and $Q^* \to N \to 0$ and then defining two projective chain complexes $P_\bullet$ and $Q_\bullet$ by removing M and N and defining P_n and Q_n to be zero when $n < 0$. Using these projective chain complexes we can define a cohomology theory by considering almost chain maps modulo almost chain homotopies from $P_\bullet$ to $Q_\bullet$. This is essentially the treatment advocated by Goichot [6] and attributed to Vogel and it is described in these terms by Benson–Carlson [1]. Crucially this definition produces a cohomological functor isomorphic to Mislin's construction when applied to $\mathrm{Ext}^*_R(M,\ \ )$. As an elegant consequence we have the

Lemma. *For any R-module M, $\widehat{\mathrm{Ext}}^0_R(M,M) = 0$ if and only if M has finite projective dimension over R.*

Tate and Farrell Cohomology. Historically, Tate cohomology was introduced first for finite groups having its origins in algebraic number theory. It concerns a finite Galois group G and for any G-module N there are isomorphisms

$$\widehat{H}^n(G,N) \simeq \begin{cases} H^n(G,N) & n \geq 1 \\ H_{-n-1}(G,N) & n \leq -1 \end{cases}$$

showing that the Tate cohomology conveniently records the ordinary cohomology in positive degrees and the ordinary homology in degrees ≤ -2. In dimension 0 there is the norm map $H_0(G,N) \to H^0(G,N)$ and the Tate cohomology groups $\widehat{H}^{-1}(G,N)$ and $\widehat{H}^0(G,N)$ are the kernel and cokernel of this map. Farrell, interested in generalising Tate cohomology to a wider class of groups, used the idea of virtual cohomological dimension. This conveniently applies to arithmetic groups such as $\mathrm{GL}_n(\mathbb{Z})$ that have torsion free subgroups of finite index. His theory of Tate cohomology produces a theory which coincides with ordinary cohomology in dimensions greater that the cohomological dimension.

Complete projective resolutions. It turns out the for Tate cohomology of finite groups one can take a projective resolution of the trivial module and then extend to the right to make a complete resolution that computes the Tate cohomology in all degrees. The same conclusion holds for Farrell's generalization but one has to perform the surgery a little way along the resolution beyond the virtual cohomological dimension. A study of when there is a complete resolution can be found in [4]. There is a connection between the existence of complete resolutions and the presence of certain finiteness conditions of which finite virtual cohomological dimension is a special case. But as remarked at the outset, one really just needs a projective chain complex P_* and to have one that determines the Tate cohomology does not in general require exactness. At first sight, all that is really needed in order to define a Tate cohomology is a projective chain complex P_* such that for any projective module Q, $\hom_R(P_*, Q)$ is exact. This holds in situations that go far beyond the Tate–Farrell cases. For example Richard Thompson's group F has cohomology that vanishes everywhere on projective modules so an ordinary projective resolution computes the Tate cohomology (the ordinary cohomology and the Tate completion of it coincide everywhere for this group, including in degrees 0 and -1). Another simpler example where this happens is a free abelian group of infinite rank.

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The framed E_2 structure on symplectic cohomology

Mohammed Abouzaid

The construction fo operations on symplectic cohomology has so far relied on adhoc methods relying on inductive choices of Floer data to define operations corresponding to moduli spaces of pseudo-holomorphic curves with increasing number of inputs, or with increasing energy. With Groman and Varolgunes, we developed a method that does not rely on choices, yielding a completely functorial invariant at the chain level. The key idea of the construction is to incorporate all possible choices of Floer data required to define an operation in an algebraic package, which takes the form of a topologically enriched multicategory, and the essential lemma to prove is that the choice of such data, lying over an abstract Riemann surface, is contractible.

We apply this method to construct a chain model for symplectic cohomology with support, which carries an algebra structure over an operad weakly equivalent to the framed E_2 operad. Our construction associates such an algebra to each compact subset of a symplectic manifold which is tame at infinity, and should in particular specialise to Rabinowitz Floer homology. The resulting invariant is defined over the Novikov ring, is strictly functorial under inclusions, and satisfies the Mayer-Vietoris property for Varolgunes covers.

An A_∞-category of Lagrangian cobordisms

Noémie Legout

Using techniques of Symplectic Field Theory, we define a Floer complex $RFC(\Sigma_0, \Sigma_1)$ associated to a pair of exact Lagrangian cobordisms in the symplectization of a contact manifold (Y, α). We describe higher order operations on this complex, leading to the definition of a cohomologically unital A_∞-category: the Fukaya category of $\mathbb{R} \times Y$. Namely, we have:

Theorem 0.1. *There exists a unital A_∞-category $\mathcal{F}uk(\mathbb{R} \times Y)$ whose objects are exact Lagrangian cobordisms equipped with augmentations of its negative ends and whose morphism spaces in the cohomological category satisfy*

$$H^*(\hom_{\mathcal{F}uk(\mathbb{R}\times Y)}(\Sigma_0, \Sigma_1)) \cong H^*(RFC(\Sigma_0, \Sigma_1)),$$

whenever Σ_0 and Σ_1 are transverse.

1. The Rabinowitz bimodule

Let $\Lambda_0^\pm, \Lambda_1^\pm \subset Y$ be Legendrian submanifolds of (Y, α) and denote $\mathcal{C}_0, \mathcal{C}_1$ the Chekanov-Eliashberg DGA over $\mathbb{Z}_2$ of Λ_0^- and Λ_1^- respectively, i.e.

$$\mathcal{C}_i = \left(\mathbb{Z}_2\langle \text{Reeb chords of } \Lambda_i^- \rangle = \mathbb{Z}_2 \oplus C_i \oplus C_i^{\otimes 2} \oplus C_i^{\otimes 3} \oplus \ldots, \partial\right),$$

where C_i is the $\mathbb{Z}_2$-vector space generated by Reeb chords of Λ_i^-.

Given two transverse exact Lagrangian cobordisms $\Sigma_0, \Sigma_1 \subset (\mathbb{R} \times Y, d(e^t\alpha))$ from Λ_0^- to Λ_0^+ and Λ_1^- to Λ_1^+ respectively, the Rabinowitz complex denoted $(RFC(\Sigma_0, \Sigma_1), \mathfrak{m}_1)$ is a DG $(\mathcal{C}_1, \mathcal{C}_0)$-bimodule generated by three types of generators, namely:

$$RFC(\Sigma_0, \Sigma_1) = C(\Lambda_1^+, \Lambda_0^+) \oplus CF(\Sigma_0, \Sigma_1) \oplus C(\Lambda_0^-, \Lambda_1^-)$$

where $C(\Lambda_1^+, \Lambda_0^+), CF(\Sigma_0, \Sigma_1)$ and $C(\Lambda_0^-, \Lambda_1^-)$ are $(\mathcal{C}_1, \mathcal{C}_0)$-bimodules generated respectively by Reeb chords from Λ_0^+ to Λ_1^+, intersection points in $\Sigma_0 \cap \Sigma_1$, and Reeb chords from Λ_1^- to Λ_0^-.

The differential $\mathfrak{m}_1$ is defined by a count of pseudo-holomorphic discs with boundary on Σ_0 and Σ_1, and with punctures asymptotic to Reeb chords and intersection points. See Figure 1, where each disc can have extra negative asymtptotics to Reeb chords of $\Lambda_0^\pm$ and $\Lambda_1^\pm$, which all become bimodule coefficients (using the functoriality of the Chekanov-Eliashberg DGA via cobordism).

Transversality and compactness results on the moduli spaces imply that $\mathfrak{m}_1^2 = 0$, i.e. $\mathfrak{m}_1$ is a differential.

At first glance, the Rabinowitz complex looks similar to the Cthulhu complex $Cth(\Sigma_0, \Sigma_1)$ defined by Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [1], but it has actually different properties. For example, it is not always acyclic when the contact manifold Y is the contactization of a Liouville manifold. Moreover, it admits a product structure and a *continuation element* with respect to this product.

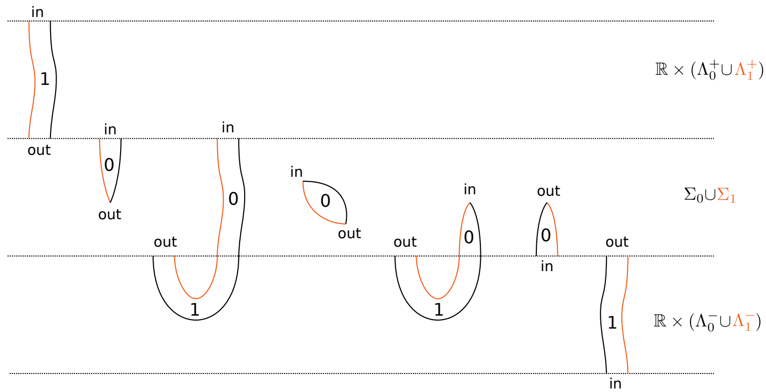

FIGURE 1. Pseudo-holomorphic discs contributing to the differential $\mathfrak{m}_1$, where "in" stands for "input" and "out" for "output".

2. The product structure

Given a triple $\Sigma_0, \Sigma_1, \Sigma_2$ of transverse exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, $i = 0, 1, 2$, such that the Chekanov-Eliashberg DGAs $\mathcal{C}_i$ of Λ_i^- admit augmentations (in such a way, the bimodules $RFC(\Sigma_i, \Sigma_j)$ can be turned into $\mathbb{Z}_2$-vector spaces), we define a map

$$\mathfrak{m}_2 \colon RFC(\Sigma_1, \Sigma_2) \otimes RFC(\Sigma_0, \Sigma_1) \to RFC(\Sigma_0, \Sigma_2)$$

by a count of pseudo-holomorphic discs, and show that it satisfies the Leibniz rule $\mathfrak{m}_1 \circ \mathfrak{m}_2 + \mathfrak{m}_2(\mathfrak{m}_1 \otimes 1) + \mathfrak{m}_2(1 \otimes \mathfrak{m}_1) = 0$. We then show:

Theorem 2.1. *When Σ_1 is a negative perturbed copy of Σ_0, there exists an element $e_{01} \in RFC(\Sigma_0, \Sigma_1)$ such that the map*

$$\mathfrak{m}_2(\cdot, e_{01}) \colon RFC(\Sigma_1, \Sigma_2) \to RFC(\Sigma_0, \Sigma_2)$$

is a quasi-isomorphism.

We construct more generally a family of maps $\{\mathfrak{m}_d\}_{d \geq 1}$ satisfying the A_∞-equations, which together with Theorem 2.1 are used to construct the category $\mathcal{F}uk(\mathbb{R} \times Y)$ by localization. It is expected (but not proved) that this category is equivalent to the Rabinowitz wrapped Fukaya category defined recently by Ganatra, Gao, Venkatesh [2] using Hamiltonian techniques (and under the hypotheses that the contact manifold Y is fillable.)

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Spectral Jumps in Rabinowitz Tate Homology

Urs Frauenfelder

We consider m exact symplectic manifolds $(M_i, \omega_i = d\lambda_i)$ with $1 \leq i \leq m$. Each loop space $C^\infty(S^1, M_i)$ is endowed with a circle action obtained by reparametrisation of its domain. This endows the loop space of the product manifold $M = M_1 \times \ldots \times M_m$

$$C^\infty(S^1, M) = C^\infty(S^1, M_1) \times \ldots \times C^\infty(S^1, M_m)$$

with an action of the m-dimensional torus T^m. We further assume that each symplectic manifold is endowed with a smooth function

$$H_i \colon M_i \to \mathbb{R}.$$

This gives rise to a smooth function

$$H \colon M \to \mathbb{R}^m, \quad (x_1, \ldots, x_m) \to \big(H(x_1), \ldots, H(x_m)\big).$$

There is further given a smooth function

$$f \colon \mathbb{R}^m \to \mathbb{R}.$$

so that the composition leads to a smooth function

$$f \circ H \colon M \to \mathbb{R}.$$

Abbreviating

$$\lambda = \lambda_1 \oplus \ldots \oplus \lambda_m \in \Omega^1(M)$$

we have two Rabinowitz action functionals

$$\mathcal{A} \colon C^\infty(S^1, M) \times \mathbb{R}, \quad (v, \tau) \mapsto \int v^*\lambda - \tau \int f \circ H(v) dt$$

and

$$\widetilde{\mathcal{A}} \colon C^\infty(S^1, M) \times \mathbb{R}, \quad (v, \tau) \mapsto \int v^*\lambda - \tau f \circ \int H(v) dt.$$

Both functionals have the same critical points on which they attain the same critical values. However, the second one is invariant under the action of the torus T^m, while the first one not necessarily is. Hence for the second functional we can consider Rabinowitz Tate homology for the torus action. Already for the case of a harmonic oscillator its chain complex is extremely rich and has nonvanishing homology classes. We discuss how the double filtration in Rabinowitz Tate homology leads to the phenomenon that their spectral numbers can jump from minus infinity to plus infinity.

Calabi–Yau structures on Rabinowitz Fukaya categories

Hanwool Bae
(joint work with Wonbo Jeong, Jongmyeong Kim)

Rabinowtiz Floer homology, introduced by Cieliebak and Frauenfelder [1], is a Floer homology associated to a symplectic manifold with contact boundary. As its open-string analogue, Rabinowitz Floer homology can be also associated to a Lagrangian submanifold with Legendrian boundary. It was shown in [5] that the Rabinowitz Floer homology can be defined as the homology of the mapping cone of a continuation map from the Floer complex for symplectic cohomology to that for symplectic homology. Following this idea, Ganatra-Gao-Venkatesh [6] introduced the Rabinowitz Fukaya category of a Liouville domain, which can be said to be a categorification of Rabinowitz Floer homology of Lagrangian submanifolds. Indeed, for given two Lagrangian submanifolds L_0 and L_1 of V, the morphism space $\mathrm{RFC}^*(L_0, L_1)$ is defined by the mapping cone of a continuation map from the Floer complex for wrapped Floer homology to that for wrapped Floer cohomology.

On the other hand, it has been shown by Cieliebak-Oancea([4, 5]) and Cieliebak-Hingston-Oancea([3]) that Rabinowitz Floer homology of a Louville domain (or a Lagrangian submanifold) has a duality that extends the classical Poincaré duality of its boundary. It was further shown that such a duality comes from a Frobenius algebra structure on Rabinowitz Floer homology. Consequently, it is natural to ask if the Frobenius nature of Rabinowitz Floer homology extends to the level of category.

As an answer to this question, I and collaborators(Jeong and Kim) proved that the Rabinowitz Fukaya category $\mathcal{RW}(V)$ of a Liouville domain (V, λ) of dimension $2n$ has a $(n-1)$-Calabi–Yau structure under a degree-wise finiteness assumption on Rabinowitz Floer homologies between generators. In particular, this means that, for every pair (X, Y) of objects of the derived Rabinowitz Fukaya category and every integer k, there is an isomorphism

$$\mathrm{RFH}^k(X, Y) \cong \mathrm{RFH}^{n-1-k}(Y, X)^\vee,$$

where RFH denotes the homology of the chain complex RFC and $^\vee$ is the linear dual.

To be more precise, we have shown that there is a $\mathcal{RW}(V)$-$\mathcal{RW}(V)$-bimodule quasi-isomorphism between $\mathcal{RW}(V)$ and $(\mathcal{RW}(V)^{\mathrm{op}})^\vee[1-n]$ if

- there are at most countable Lagrangian submanifolds $\{L_i\}_{i\in I}$ of V generating the wrapped Fukaya category of V and
- the dimension $\dim \mathrm{RFH}^k(L_i, L_j)$ is finite for all $i, j \in I$ and $k \in \mathbb{Z}$.

This can be proved by constructing a bimodule homomorphism from $\mathcal{RW}(V)$ to $(\mathcal{RW}(V)^{\mathrm{op}})^\vee[1-n]$ extending the natural Poincaré duality between Floer homologies.

For example, if a Liouville domain (V, λ) is given by the disk cotangent bundle $(D^*Q, \lambda_{\mathrm{can}})$ of a simply-connected smooth closed manifold Q, then the above two requirements are satisfied and therefore the corresponding Rabinowitz Fukaya category is Calabi–Yau.

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Some counterexamples to the Donaldson 4-6 question

Amanda Hirschi

(joint work with Luya Wang)

The following question, credited to Donaldson, concerns the uniqueness of symplectic structures and their relation to the smooth topology of the underlying manifold.

Conjecture 1. *Let (X_1, ω_1) and (X_2, ω_2) be two closed (simply connected) symplectic 4-manifolds such that X_1 and X_2 are homeomorphic. Then the product symplectic manifolds $(X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}})$ and $(X_2 \times S^2, \omega_2 \oplus \omega_{\text{std}})$ are deformation equivalent if and only if X_1 and X_2 are diffeomorphic.*

Two symplectic structures σ_1 on X_1 and σ_2 on X_2 are *deformation equivalent* if there exists a diffeomorphism $\varphi\colon X_1 \to X_2$ and a path $\{\sigma'_t\}_{t\in[0,1]}$ of symplectic structures on X_1 with $\sigma'_0 = \varphi^*\sigma_2$ and $\sigma'_1 = \sigma_1$.

If the conjecture were to be true, it would be a symplectic analogy of the fact that given two smooth simply-connected homeomorphic 4-manifolds X_1 and X_2, the products $X_1 \times S^2$ and $X_2 \times S^2$ are diffeomorphic. It holds for certain classes of symplectic 4-manifolds by [2] and [1]. However, Smith, [3], and Vidussi, [4], constructed symplectic forms on the same smooth 4-manifold that are distinguished by their first Chern classes. We show that this difference is preserved after taking the product with S^2.

Theorem 1. *There exist closed simply connected symplectic 4-manifolds (X_1, ω_1) and (X_2, ω_2), such that X_1 is diffeomorphic to X_2, while $(X_1 \times S^2, \omega_1 \oplus \omega_{\text{std}})$ and $(X_2 \times S^2, \omega_2 \oplus \omega_{\text{std}})$ are deformation inequivalent.*

This shows that the symplectic geometry of a product remembers more about the symplectic geometry of the factors than is true for the smooth structures.

While the proof of this result only uses classical invariants, Gromov–Witten invariants can be used in combination with [1] to prove the following partial converse.

Theorem 2. *Given two simply connected symplectic* 4*-manifolds* (X_0, ω_0) *and* (X_1, ω_1) *so that* $(X_0, \omega_0) \times (S^2, \omega_{\mathrm{std}})$ *and* $(X_1, \omega_1) \times (S^2, \omega_{\mathrm{std}})$ *are deformation equivalent and* $\sigma(X_i) \neq 0$*, there exists a homeomorphism* $X_0 \to X_1$ *relating their Seiberg-Witten invariants.*

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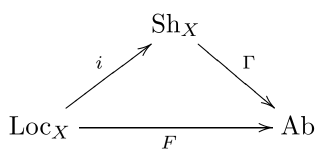
Sheaf and singular models for ∞-groupoïd cohomology

Colin Fourel

The goal of the talk was to explain how one can use group cohomology to prove that sheaf and singular cohomology are isomorphic on CW complexes.

Let G be a discrete group, then the cohomology of G coincides with the singular cohomology of any connected CW complex satisfying $\pi_1 = G$ and $\pi_i = 0$ for $i \geq 2$. Let X be such a CW complex, we also have the following commutative diagram of abelian categories



where Loc_X denotes the category of local systems over X, Sh_X that of sheaves over X, i the inclusion, Γ the global sections functor and $F(M) = M^G$. The statement that the sheaf cohomology groups with coefficients in local systems over X are isomorphic to the corresponding cohomology groups of G, is equivalent to the commutativity of this diagram at the level of derived functors.

Now, this is in turn equivalent to the fact that whenever I is an injective object of Loc_X, then $i(I)$ is an acyclic sheaf. Let us give an independent proof of that. Denote $\pi : \widetilde{X} \to X$ a universal cover of X. Since the total space of π is contractible, and its fiber are discrete, and since $\pi^*(i(I))$ is constant, the Leray spectral sequence implies that $\pi_*\pi^*(i(I))$ is acyclic. The unit of the adjunction between π_* and π^* gives an injective map of local systems $I \to \pi_*\pi^*(i(I))$ which, by injectivity of I, has a retract. Hence I is acyclic. We thus recover the isomorphism between the sheaf and singular cohomologies of X.

Let us now assume that X is any connected CW complex. Let us introduce the ∞-category of ∞-*local systems* over X, denoted $\infty\mathrm{Loc}_X$, which has the following three equivalent descriptions

(1) $D(C_*(\Omega X, \mathbb{Z}))$, the derived ∞-category of dg modules over chains over the based loop space of X with coefficients in $\mathbb{Z}$,
(2) $\mathrm{LC}(X; D(\mathbb{Z}))$, the ∞-category of locally constant sheaves on X with values in $D(\mathbb{Z})$,
(3) $\mathrm{Fun}(\Pi_\infty(X), D(\mathbb{Z}))$, the ∞-category of functors from the fundamental ∞-groupoïd of X to $D(\mathbb{Z})$.

The equivalence between (1) and (2) is proven in [2] (theorem 6.26), the equivalence between (2) and (3) is proven in [1] (theorem A.4.19).

Let $K \in \infty\mathrm{Loc}_X$. Using description (3), we define the i^{th} ∞-*groupoïd cohomology group of* $\Pi_\infty(X)$ with coefficients in K as:

$$H^i(\Pi_\infty(X), K) = H^i(\lim K).$$

Consider the constant ∞-local system $\mathbb{Z}$ on X. Using description (1) we have $H^i(\Pi_\infty(X), \mathbb{Z}) = \mathrm{Ext}^i_{C_*(\Omega X)}(\mathbb{Z}, \mathbb{Z})$, which is isomorphic to $H^i_{\mathrm{sing}}(X, \mathbb{Z})$ (see [3], Theorem B and Proposition 11.7). On the other hand, using description (2) we have $H^i(\Pi_\infty(X), \mathbb{Z}) = H^i(\Gamma(\mathbb{Z}))$, which is isomorphic to the usual sheaf cohomology group $H^i(X, \underline{\mathbb{Z}})$ ([4], Proposition 10 and Corollary 11). We thus recover the isomorphism between the sheaf and singular cohomologies of X.

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String topology and graph cobordisms

Andrea Bianchi

String topology is broadly concerned with the study of invariants of mapping spaces of the form $M^X = \mathrm{map}(X, M)$, where X is a topological space and M is a smooth, closed manifold of some dimension $d \geq 1$. More specifcually, we want to study, for a commutative ring R, the homology $H_*(M^X; R)$, in the assumption that M is R-oriented. A case of particular interest is $X = S^1$, recovering the free loop space LM: the homology $H_*(LM; \mathbb{Z})$ agrees with the (suitably twisted) symplectic homology of the Liouville domain T^*M, and the topology of LM can be used to study the existence and the number of closed geodesics on M, when we endow M with a Riemannian metric.

It is convenient to study the homology groups $H_*(M^X;R)$ for fixed M and varying X, as one can describe several *string operations* relating different homology groups. The most basic operations are:

(1) for a map $Y \to X$, we get a restriction map $H_*(M^X;R) \to H_*(M^Y;R)$;
(2) for $Y = X \sqcup *$, we get map $H_*(M^X;R) \overset{-\times[M]}{\to} H_{*+d}(M^X \times M;R) = H_{*+d}(M^Y;R)$ by cross product with the fundamental class of M;
(3) for $Y = X \sqcup_{\partial I} I$, i.e. Y is obtained from X by attaching a 1-cell, Chas and Sullivan [1] constructed a natural operation $H_*(M^X;R) \to H_{*-d}(M^Y;R)$.

The notion of graph cobordism gives a common denominator to (1)-(3). A *graph cobordism* between X and Y is a cospan of spaces $X \hookrightarrow W \leftarrow Y$, together with a finite cell structure of W relative to X consisting only of 0-cells and 1-cells. Each graph cobordism gives, by combining the above basic operations, an operation $H_*(M^X;R) \to H_{*+d\cdot\chi(W,X)}(M^Y;R)$.

I define a moduli space $\mathfrak{M}_{\mathrm{Gr}}(X,Y)$ of graph cobordisms from X to Y, by taking the classifying space of a suitable topological category $\mathrm{Gr}(X,Y)$ of graph cobordisms, with morphisms given by forest collapses. I also define a coefficient system ξ_d over $\mathfrak{M}_{\mathrm{Gr}}(X,Y)$, taking values in homologically graded R-modules, whose fibre over $X \hookrightarrow W \leftarrow Y$ is (non-canonically) isomorphic to $R[-d\cdot\chi(W,X)]$.

The main stated theorem is an extension of (1)-(3) to a chain map

$$C_*(M^X;R) \otimes_R C_*(\mathfrak{M}_{\mathrm{Gr}}(X,Y);\xi_d) \to C_*(M^Y;R).$$

The entire construction can in fact be generalised in the case in which R is an E_∞-ring spectrum: in this case ξ_d is a parametrised R-module of rank 1 over $\mathfrak{M}_{\mathrm{Gr}}(X,Y)$, and we obtain a map of R-modules

$$(R \otimes X) \otimes_R (\mathrm{colim}_{\mathfrak{M}_{\mathrm{Gr}}(X,Y)}\xi_d) \to (R \otimes Y).$$

The construction can be further generalised to the case in which M is an R-oriented Poincaré duality space; in particular all string operations arising in this way are invariant under homotopy equivalences of manifolds.

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Efimov's categorical formal punctured neighborhood of infinity, Rabinowitz Fukaya category, CY and pre-CY structures

Alex Takeda

The purpose of this talk is to propose the construction of the "categorical formal punctured neighborhood of infinity" [5] as an organizing principle to understand the relationship between duality structures on (usual) Floer theory and Rabinowitz Floer theory. This is a purely algebraic construction, that applied to a dg-category $\mathcal{C}$, produces a dg category $\widehat{\mathcal{C}}_\infty$, called the *categorical formal punctured neighborhood of $\mathcal{C}$*.

This category $\mathcal{C}$ should be thought of as an algebraic incarnation of the following geometric construction: for a smooth but non-compact algebraic variety Y, one chooses a compactification $\overline{Y} = Y \cup D$ and looks at perfect complexes (of coherent sheaves) supported on the punctured formal neighborhood of D. Suitably defined, this category $\mathrm{Perf}(\widehat{\overline{Y}}_D \setminus D)$ is independent of the choice of compactification $\overline{Y}$. Efimov's construction is a noncommutative version of this operation in the sense that if one takes $\mathcal{C}$ to be (a dg enhancement of) the bounded derived category of X, then $\widehat{\mathcal{C}}_\infty \cong \mathrm{Perf}(\widehat{\overline{Y}}_D \setminus D)$.

This construction has been extended to A_∞-categories and applied to symplectic topology by [7], where it is proven that, given any nondegenerate Liouville manifold X, there is an A_∞-equivalence $\mathcal{RW}(X) \to (\widehat{\mathcal{W}(X)})_\infty$, from the Rabinowitz Fukaya category of X, to the categorical formal punctured neighborhood of the wrapped Fukaya category. The former category has as morphism spaces the 'open string' version of Rabinowitz Floer homology [2], with composition maps as in [4]. As a result of this identification, together with some yet-unpublished work of Rezchikov, one gets an identification of Rabinowitz Floer cohomology $RFH^*(X)$ with

$$HH_*(\mathcal{W}(X), \mathcal{RW}(X)) = HH_*(\mathcal{W}(X), \widehat{\mathcal{W}(X)}_\infty),$$

that is, Hochschild homology of the wrapped Fukaya category with coefficients in the Rabinowitz Fukaya category. Moreover, this relation recovers the 'Tate construction' perspective [3] on RFH^*, since the complex calculating $HH_*(\mathcal{W}(X), \widehat{\mathcal{W}(X)}_\infty)$ is obtained by a cone construction.

After introducing these constructions and results, I explained in my talk a sketch of how this perspective could be used to understand the origin of products on RFH^*, as well as the 'Frobenius' property described by [1]. For example, in my own work with Rivera and Wang [8], we study products on $HH_*(\mathcal{C}, \widehat{\mathcal{C}}_\infty)$ constructed from a type of structure on some category $\mathcal{C}$ called a pre-Calabi–Yau structure, which in particular can be produced for the wrapped Fukaya category as a consequence of its smooth Calabi–Yau structure together with the results in [6]; it is reasonable to conjecture that the geometrically-defined product on RFH^* arises in such a way.

Lastly, the relation between such a description and the products constructed by [9] on 'singular Hochschild cohomology' should be given by some sort of Koszul duality. In some cases, where the wrapped Fukaya category $\mathcal{W}(X)$ has a 'proper Koszul-dualizing subcategory', as defined in [7], combining all the above results, one gets an equivalence between $\mathcal{RW}(X)$ and the derived category of singularities of a certain dg algebra. I ended my talk with the conjecture that all the product and duality structures above, should match under the many dualities and identifications. If proven, this would mean that they all encode the same data, given just by the smooth Calabi–Yau structure on the wrapped Fukaya category.

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Spaces of operations by example: two BV structures on the Hochschild homology of symmetric Frobenius algebras

Nathalie Wahl

Given a differential graded associative algebra, let $C_*(A,A)$ denote its Hochschild complex, $\overline{C}_*(A,A) = C_*(A,A)/C_{0,0}(A,A)$ the reduced complex where the copy of A_0 in Hochschild degree 0 has been killed, and $HH_*(A,A)$, $\overline{HH}_*(A,A)$ the corresponding homology groups.

A BV-algebra is a commutative differential graded algebra V_* equipped with an operator $\Delta : V_* \to V_{*+1}$ satisfying the BV-relation

$$\begin{aligned}\Delta(abc) = \Delta(ab)c &+ (-1)^{|a|}a\Delta(bc) + (-1)^{|a|(|b|+1)}b\Delta(ac)\\ &+ \Delta(a)bc + (-1)^{|a|}a\Delta(b)c + (-1)^{|a|+|b|}ab\Delta(c).\end{aligned}$$

A BV-algebra of dimension d is a BV-algebra with a product of degree $\pm d$ and appropriately modified signs in the commutativity and BV-relation; we refer to [8, Sec 6.3] for a systematic way to define a "dimension d" version of this type of algebraic structure.

Recall that the Hochschild complex is endowed with a degree 1 operator $B : C_*(A,A) \to C_{*+1}(A,A)$, the Connes-Rinehart operator. When A is a symmetric Frobenius algebra of dimension $d > 0$, the (long proven) cyclic Deligne conjecture states that this operator B, together with the dual of the cup product, defines a coBV-structure of dimension d on $H_*(A,A)$, induced from a chain-level structure. It corresponds to the string topology BV-structure of Chas-Sullivan when $A \simeq C^*M$, see [2]. From the papers [1, 4, 5], one can also deduce that the same operator B, together with a product corresponding to the dual of the Goresky-Hingston string topology coproduct when $A \simeq C^*M$ (see [6]), endows the reduced

Hochschild homology $\overline{HH}_*(A,A)$ with a 1-suspended BV–algebra structure of dimension d.

One can in principle prove the above stated results in homology by direct computation, as the product, coproduct and the operator B have explicit descriptions, but it is very difficult to get the signs right when checking the relations! We explain here how these statements follow from a more general result, and comes from two different embeddings of the BV-operad in a prop acting on the Hochschild complex of symmetric Frobenius algebras.

Recall that there is an isomorphism of operads $BV \cong H_*(fE_2)$ between the operad BV governing BV–algebras and the homology of the framed E_2–operad, an operad that is also equivalent to the cactus operad. We will here denote by $Cact$ the chain operad of normalised cacti, as defined in [3], with $H_*(Cact) = BV$. The above BV and co-BV structures are a consequence of the following chain level statement:

Theorem 1.

(1) [7, 8] *The Hochschild complex $C_*(A,A)$ of a symmetric Frobenius dg algebra A of dimension d admits an action of the dg-prop SD_d of degree d–shifted Sullivan diagrams.*
(2) [4, 8] *There are inclusions $Cact(n) \hookrightarrow SD(1,n)$ and $Cact(n) \times \Delta^{n-1} \hookrightarrow SD(n,1)$ compatible with composition.*
(3) [5] *The resulting action of $Cact(n) \times \Delta^{n-1}$ on $C_*(A,A)$ descends to an action of $Cact(n) \times \Delta^{n-1}/\partial\Delta^{n-1}$ on the reduced Hochschild chains $\overline{C}_*(A,A)$.*

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Relative Calabi–Yau structure from acyclic Rabinowitz–Floer complexes of Legendrians

Georgios Dimitroglou Rizell
(joint work with N. Legout)

1. Outline

In this joint work with Legout we establish a geometric incarnation of morphisms of distinguished triangles of bimodules, realised through the Legendrian invariant of Rabinowitz Floer complex. The morphism is a quasi-isomorphism if and only if this complex is acyclic, which is equivalent to the existence of a Calabi–Yau structure in the sense of Brav–Dyckerhoff [BD19].

2. The morphism of triangles

Consider the canonical inclusion $\iota\colon \mathcal{A}_* \hookrightarrow \mathcal{C}_* = \mathcal{C}_*(\Lambda;\mathcal{A})$ of the DGA of chains of the based loop space $\mathcal{A}_* = C_{-*}(\Omega\Lambda;\mathbf{k})$ into

$$\mathcal{C}_* = (\mathcal{A}_*\langle \text{Reeb chords of } \Lambda\rangle = \mathcal{A} \oplus Q \oplus (Q\otimes_{\mathcal{A}} Q) \oplus (Q\otimes_{\mathcal{A}} Q \otimes_{\mathcal{A}} Q)\oplus \ldots, \partial),$$

i.e. the Chekanov–Eliashberg DGA of a closed $\mathbf{k}$-oriented Legendrian submanifold $\Lambda^n \subset (Y^{2n+1},\alpha)$ of a contact manifold over the chains of the based loop space $\mathcal{A}_*$.

For a DGA $\mathcal{B}_*$, denote by $\mathcal{B}_\Delta$ the so-called diagonal (non-free) left $\mathcal{B}^e = \mathcal{B}\otimes_{\mathbf{k}} \mathcal{B}^{op}$-module (equivalently: left $\mathcal{B}-$bimodule) given by $\mathcal{B}$ endowed with the canonical bimodule structure coming from DGA-multiplication.

The DG-morphism ι induces a canonical map

$$\mu\colon \iota_!(\mathcal{A}_\Delta) \coloneqq \mathcal{C}^e \otimes^{\mathbb{L}}_{\mathcal{A}^e} \mathcal{A}_\Delta = \mathcal{C}\otimes_{\mathcal{A}} \mathcal{C} \xrightarrow{\mu} \mathcal{C}_\Delta$$

of left $\mathcal{C}^e-$modules induced by the multiplication $\mathcal{C}\otimes_{\mathcal{A}}\mathcal{C} \xrightarrow{\mu} \mathcal{C}$ of the DGA.

Theorem 2.1. *There is a quasi-isomorphism of the distinguished triangles*

$$\begin{array}{ccccccc}
\iota_!(\mathcal{A}) & \xrightarrow{\quad\mu\quad} & \mathcal{C}_\Delta & \longrightarrow & \mathrm{cof}(\mu) & \dashrightarrow & \cdots \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
C_*(\Lambda;\mathcal{C}^e) & \hookrightarrow & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e) & \twoheadrightarrow & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)/C_*(\Lambda;\mathcal{C}^e) & \dashrightarrow & \cdots,
\end{array}$$

The upper row is induced by $\mathcal{A}_* \hookrightarrow \mathcal{C}_*$, while the lower row is a short exact sequence induced by the action filtration in Legendrian contact homology.

Here the complex $LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)$ denotes the Legendrian contact homology complex with coefficients in $\mathcal{C}^e$, which is a projective left $\mathcal{C}^e-$module generated by the Reeb chords from Λ to its small Reeb push-off Λ^+. Note that this push-off creates also a small set of Reeb chords which are in bijection with the critical points of a small function; thus we get an inclusion of the Morse complex

$$C_*(\Lambda;\mathcal{C}^e) = \mathcal{C}^e \otimes_{\mathcal{A}^e} C_*(\Lambda;\mathcal{A}^e) \subset LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e).$$

Recall that $C_*(\Lambda;\mathcal{A}^e) \simeq \mathcal{A}_\Delta$ is the Morse homolgoy of Λ with $\mathcal{A}^e$ as a two-sided derived local system.

The bimodule dual $(-)^! := R\hom_{\mathcal{C}^e}(-, \mathcal{C}^e)$ is an endofunctor $(-)^! \colon D^b(\mathcal{C}^e) \to D^b(\mathcal{C}^e)$ which preserves semi-free $\mathcal{C}^e-$modules. There is a chain map

$$b\colon LCC_*(\Lambda, \Lambda^+; \mathcal{C}^e) \to LCC_*(\Lambda^+, \Lambda; \mathcal{C}^e)^![n+1]$$

defined by counting "bananas" in the symplectisation with two positive punctures. The co-domain of b, which is a Legendrian contact cohomology complex generated by Reeb chords from Λ_+ to Λ (note the order!) can be seen to be isomorphic to $(LCC_*(\Lambda, \Lambda^+; \mathcal{C}^e)/C_*(\Lambda; \mathcal{C}^e))^![n+1]$ by invariance under Legendrian isotopy.

Theorem 2.2. *The map b extends to a morphism of distinguished triangles*

$$\begin{array}{ccccccc} C_*(\Lambda;\mathcal{C}^e) & \overset{\mu}{\hookrightarrow} & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e) & \twoheadrightarrow & \mathrm{cof}(\mu) & \overset{\Sigma}{\dashrightarrow} & \cdots \\ \simeq\downarrow \widetilde{\mathcal{CY}} & & \downarrow b & & \downarrow b^! & & \\ (C_*(\Lambda;\mathcal{C}^e))^![n] & \overset{\Sigma}{\dashrightarrow} & \mathrm{cof}(\mu)^![n+1] & \hookrightarrow & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)^![n+1] & \twoheadrightarrow & \cdots \end{array}$$

Here the leftmost vertical map is a quasi-isomorphism that is induced by the *absolute* n-Calabi–Yau structure

$$\mathcal{A}_\Delta \simeq C_*(\Lambda; \mathcal{A}^e) \xrightarrow{\mathcal{CY}} (C_*(\Lambda; \mathcal{A}^e))^![n] \simeq \mathcal{A}^!_\Delta[n]$$

by tensoring $\mathcal{C}^e \otimes_{\mathcal{A}^e} (-)$. See [Gan13] or [Leg23] for the latter quasi-isomorphism. The Rabinowitz–Floer complex is the Legendrian isotopy invariant Cone(b), i.e.

$$RFC_*(\Lambda, \Lambda^+; \mathcal{C}^e) := \left(LCC_*(\Lambda, \Lambda^+; \mathcal{C}^e) \oplus LCC_*(\Lambda^+, \Lambda; \mathcal{C}^e)^![n], \begin{bmatrix} \partial & b \\ 0 & \partial^! \end{bmatrix} \right).$$

Theorem 2.3. *The Rabinowitz–Floer complex is acyclic when:*

- $Y = \partial_\infty(P \times \mathbb{C})$ *is the contact boundary of a subcritical Weinstein domain;*
- $\Lambda \subset Y$ *can be displaced from its Reeb trace by a contact isotopy; or*
- $Y = J^1S^2$ *with a non-trivial bulk-deformation by the H_2-class.*

The acyclicity of the Rabinowitz–Floer complex is equivalent to Theorem 2.2 being a quasi-isomorphism of triangles. This translates into the property that the morphism $\iota\colon \mathcal{A}_* \to \mathcal{C}_*$ of DGAs is a **relative** $(n+1)$**-Calabi–Yau pair** as defined by Brav–Dyckerhoff [BD19]. This can be seen as a generalisation of Sabloff duality [EES09] from augmentations to general DG-bimodules.

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Variational Methods for Evolution

Organized by
Franca Hoffmann, Pasadena
Alexander Mielke, Berlin
Mark Peletier, Eindhoven
Dejan Slepčev, Pittsburgh

5 – 8 December, 2023.

Abstract. Variational principles for evolutionary systems arise in many settings, both in those describing the physical world and in man-made algorithms for data science and optimization tasks. Variational principles are available for Hamiltonian systems in classical mechanics, gradient flows for dissipative systems, as well as in time-incremental minimization techniques for more general evolutionary problems. Additional challenges arise via the interplay of two or more functionals (e.g. a free energy and a dissipation potential), thus encompassing a large variety of applications in the modeling of materials and fluids, in biology, and in multi-agent systems.

Variational principles and associated evolutions are also at the core of the modern approaches to machine learning tasks, since many of them are posed as minimizing an objective functional that models the problem. The discrete and random nature of these problems and the need for accurate computation in high dimension present a set of challenges that require new mathematical insights. Variational methods for evolution allow for the usage of the rich toolbox provided by the calculus of variations, metric-space geometry, partial differential equations, and other branches of applied analysis.

The variational methods for evolution have seen a rapid growth over the last two decades. This workshop continued the successful line of meetings (2011, 2014, 2017, and 2020), while evolving and branching into new directions. We have brought together a wide scope of mathematical researchers from calculus of variations, partial differential equations, numerical analysis, and stochastics, as well as researchers from data science and machine learning, to exchange ideas, foster interaction, develop new avenues, and generally bring these communities closer together.

Mathematics Subject Classification (2020): 35-XX, 37Nxx, 49-XX, 62Gxx, 68T07, 82Dxx.

Introduction by the Organizers

The workshop *Variational Methods for Evolution*, organized by Franca Hoffmann (Caltech), Alexander Mielke (Berlin), Mark Peletier (Eindhoven), and Dejan Slepčev (Pittsburgh) brought together researchers with variety of backgrounds and from a geographically diverse set of academic institutions.

Hamiltonian systems, gradient systems, or mixtures of these two extreme types are almost ubiquitous in applications. They have been considered in connection with many real-world models such as fluid dynamics, phase transitions, thin films, quantum models, nonlinear diffusion and transport problems, chemical reactions, rate-independent phenomena, material modeling, and many others. Variational approaches to such evolutionary systems provide a powerful set of tools and methods, and the past years have seen impressive growth of this area, with the development of generalized gradient flows in Banach spaces and gradient flows in metric spaces, the characterisation of a very wide range of systems as variational evolutions, the study of the interplay between energy landscape and the dissipation geometry, the connections to stochastic particle systems, and many others.

These variational-evolution methods have recently found new applications in the rapidly developing field of data science. Many of the models of data science are variational in nature: to formulate a machine learning task one often creates an objective functional that describes the desired properties of the solution sought and then minimizes the functional. Many of these involve minimization over the probability measures and function spaces, whose minimization is closely connected to variational evolutions of the relevant functionals. The discrete nature, randomness, and high-dimensionality of the data create challenges that call for new mathematical approaches.

For instance, one task is to utilize the geometry of the data distribution carried by the available random samples. This leads to questions about evolutions on graphs and their many nodes limits. The desire for high-dimensional computations leads to questions about geometries for gradient flows that can be estimated accurately in high dimensions, and are robust to noise. Mean-field limits of neural networks (including deep ones) show promising connections to PDE and evolutionary problems. Likewise, sampling problems and generative models of learning have evolutionary descriptions that raise important questions.

In this workshop we sought to bring together mathematicians studying variational evolutions with researchers from the data science community for a stimulating exchange of ideas. We invited a selected group of experts and young researchers from both communities to work together to recognize the common mathematical structures, formulate the most important mathematical questions, and exchange ideas. Many participants said towards the end of the workshop that they had found the mix of topics particularly motivating; it is clear that this aim of bringing people together from different areas of mathematics was successful in creating a productive scientific meeting.

Another aim of the workshop was to offer the chance to many young and talented researchers that have started in this promising area, to get exposed to broad set of relevant ideas and have scientific discussions with the leaders in the field. Again, this seems to have been successful, based on the observation that a number of young researchers are now in contact with more established members of this community, and various plans for follow-up visits and research activities have already been made.

The workshop was purely on-site and in-person, despite two disruptive events: (i) a large snow storm in South East Germany on the Saturday evening before complicated the arrival of several participants and (ii) a strike involving the Deutsche Bahn on Friday forced many participants to invest time in rearranging their departure. Thanks to collective efforts alternative transport was arranged and most participants were able to stay on Friday. Despite the adversity, the workshop had an excellent atmosphere, featured exciting talks and lively scientific discussions; the participants were uniformly positive about the event when they left.

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Workshop: Variational Methods for Evolution

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Abstracts

Entropic interpolations are geodesics

Christian Léonard
(joint work with Marc Arnaudon, Giovanni Conforti)

Entropic interpolation. Let $\Omega = \{\text{paths}\}$ be the set of all continuous paths from the time interval $[0, T]$ to the state space $\mathbb{R}^n$ and denote $\mathrm{P}(\Omega)$ and $\mathrm{M}(\Omega)$ the sets of all probability measures and all positive measures on Ω. We choose as a reference path measure $R \in \mathrm{M}(\Omega)$ the law of the solution of the following stochastic differential equation on $\mathbb{R}^n$:

$$\begin{cases} dX_t = -a\nabla U(X_t)/2\, dt + \sqrt{a} dB_t, & 0 < t \leq T, \\ X_0 \sim m := \exp(-U)\,\mathrm{Leb}, & t = 0, \end{cases}$$

where X_t is the random position at time t, $a > 0$ is a positive number, B is a standard Brownian motion, $U : \mathbb{R}^n \to \mathbb{R}$ is a scalar potential and Leb stands for Lebesgue measure. Not only m is an invariant measure, but also R is reversible: X and $X^* : s \in [0, T] \mapsto X^*_s := X_{T-s}$, are statistically indistinguishable.

The relative entropy of $P \in \mathrm{P}(\Omega)$ with respect to R is

$$H(P|R) := \int_\Omega \log\left(\frac{dP}{dR}\right) dP$$

and the Schrödinger problem is

$$\inf H(P|R); \quad P \in \mathrm{P}(\Omega) : P_0 = \alpha, P_T = \beta$$

where $P_0, P_T \in \mathrm{P}(\mathbb{R}^n)$ are the initial and final marginals of P and $\alpha, \beta \in \mathrm{P}(\mathbb{R}^n)$ are prescribed. As a strictly convex problem, it admits a unique solution Q (if any) which is called the *Schrödinger bridge* between α and β and whose time marginal flow is called the *entropic interpolation* between α and β. This problem was addressed by Schrödinger in 1931 [9,10]. For a review see [7] for instance, and for its applications to computational optimal transport see [8].

Bridges. Any Schrödinger bridge Q is a mixture of bridges $R^{ab}(\cdot) := \mathbb{P}(X \in \cdot \mid X_0 = a, X_T = b)$ of R, that is: $Q(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} R^{ab}(\cdot)\, Q_{0T}(dadb)$ where Q_{0T} is the joint law under Q of the endpoint positions. One can extend the above definition of entropic interpolation to the time marginal flow of any bridge R^{ab}. The results below remain unchanged provided we restrict our attention to the open time interval $(0, T)$.

Principle of least action. Any Schrödinger bridge inherits the Markov property from R. Restricting, without loss of generality, the Schrödinger problem to Markov path measures P allows to write

$$H(P|R) = \mathcal{F}(\mu_T) - \mathcal{F}(\mu_0) + \int_{[0,T]\times\mathbb{R}^n} \frac{|v^{\mathrm{cu}}|^2 + |v^{\mathrm{os}}|^2}{2a}(t,x)\,\mu_t(dx)dt,$$

where $(\mu_t)_{0\le t\le T}$ is the time marginal flow of P, $\mathcal{F}(\gamma) = H(\gamma|m)/2$, $\gamma \in \mathrm{P}(\mathbb{R}^n)$ plays the role of a free energy functional, the current velocity field v^{cu} satisfies the continuity equation

$$\partial_t \mu_t + \nabla \cdot (v^{\mathrm{cu}}\mu) = 0,$$

and the osmotic velocity field is given by the time reversal formula

$$v^{\mathrm{os}}(t,x) = a\nabla \log \sqrt{\frac{d\mu_t}{dm}}(x).$$

The idea of the proofs of these expressions dates back to Föllmer [6]. The main difficulty in the present setting is to give sense to the above expressions while the only assumption $H(P|R) < \infty$ does not imply much regularity. This is done in [3].

One can also show that the current velocity field v^{cu} of the Schrödinger bridge is a gradient field (in some weak sense). Restricting, without loss of generality, our attention to such Markov path measures P, the above continuity equation permits us to interpret v_t^{cu} as the tangent vector $\dot\mu_t$ at μ_t of the marginal flow $(\mu_s)_{0\le s\le T}$, in the sense of the Otto-Wasserstein geometry, see [11, Ch. 15].

On the other hand, plugging the time reversal formula into the expression of $H(P|R)$, noting that as regards the Schrödinger problem $\mu_0 = \alpha$ and $\mu_T = \beta$ are prescribed, and multiplying by a, we arrive at the following least action principle

$$\inf \mathcal{A}(\mu); \quad \mu := (\mu_t)_{0\le t\le T} : \mu_0 = \alpha, \mu_T = \beta,$$

with

$$\mathcal{A}(\mu) = \int_0^T \Big(\|\dot\mu_t\|^2_{\mu_t}/2 + a^2 \mathcal{I}(\mu_t) \Big)\, dt$$

where $\|\dot\mu_t\|^2_{\mu_t} = \int_{\mathbb{R}^n} |v_t^{\mathrm{cu}}|^2\, d\mu_t$ is the Otto-Wasserstein squared norm of the tangent vector $v_t^{\mathrm{cu}} = \dot\mu_t$ at μ_t, and

$$\mathcal{I}(\gamma) := \int_{\mathbb{R}^n} \frac{1}{2} \left| \nabla \log \sqrt{\frac{d\gamma}{dm}} \right|^2 \, d\gamma$$

is the Fisher information of $\gamma \in \mathrm{P}(\mathbb{R}^n)$ with respect to m.

Newton's equation. The action $\mathcal{A}$ is analogous to a usual classical mechanical action on a Riemannian manifold M with Lagrangian $\mathcal{L}(\dot\gamma, \gamma) = \|\dot\gamma\|^2_\gamma/2 + a^2\mathcal{I}(\gamma)$ instead of the classical Lagrangian $L(q,v) := |v|_q^2/2 - V(q)$. Since L gives rise to the Newton equation:

$$\ddot{x}_t = -\mathrm{grad}_{x_t} V,$$

where $\ddot{x}_t = \nabla_{\dot{x}_t}\dot{x}_t$ is the acceleration of the trajectory $t \mapsto x_t$, one can show similarly [4, 5] that the entropic interpolation μ solves the Newton equation

$$\ddot{\mu}_t = a^2 \mathrm{grad}^{\mathrm{OW}}_{\mu_t} \mathcal{I}$$

with respect to the Otto-Wasserstein geometry. The main issue when extending the results of [4] to those of [5] is to overcome the lack of regularity under the weak assumption $H(P|R) < \infty$. In the special case (to keep the writing easy) where R is the reversible Brownian path measure (i.e. $U = 0$), we have

$$\mathrm{grad}^{\mathrm{OW}}_{\gamma} \mathcal{I} = \nabla Q_\gamma \quad \text{where} \quad Q_\gamma := -\frac{\Delta}{2}\sqrt{\gamma}\Big/\sqrt{\gamma}$$

is the quantum potential and γ also stands for the density of the measure γ.

Geodesic in spacetime. On the other hand, it is known since Cartan's article [2] in 1923 that any solution of Newton's equation $\ddot{x}_t = -\mathrm{grad}_{x_t} V$, is such that (t, x_t) is a geodesic in the curved spacetime $\mathbb{R} \times M$ with some Riemann curvature tensor built on the original curvature tensor of M, plus an additional curvature tensor built with the Hessian of the potential V. Similarly, it is proved in [1] that the same property holds for entropic interpolations in a curved spacetime $\mathbb{R} \times \mathrm{P}(\mathbb{R}^n)$ whose curvature tensor is the sum of a curvature coming from the Otto-Wasserstein geometry and a curvature tensor built with the Hessian of $\mathcal{I}$. In the above special case where $U = 0$,

$$\mathrm{Hess}^{\mathrm{OW}}_{\gamma} \mathcal{I}(\nabla\theta, \nabla\theta) = \int_{\mathbb{R}^n} \left(\mathrm{Hess}\, Q_\gamma(\nabla\theta, \nabla\theta) + \left| \frac{\Delta}{2}\nabla\theta + \nabla \log \sqrt{\gamma} \cdot \nabla\theta \right|^2 \right) d\gamma.$$

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Discrete-to-continuum limits of graph-based gradient flows

Yves van Gennip

(joint work with Yoshikazu Giga, Jun Okamoto, Samuel Mercer)

We are interested in discrete-to-continuum limits of graph-based gradient flows. Such flows are of interest to image analysis, machine learning, and other graph-based problems. A well-known example is the gradient flow based on the Allen–Cahn (or Ginzburg–Landau) functional for image segmentation, as proposed by Bertozzi and Flenner [2]. This treats the image segmentation problem as a graph classification problem, to be solved by minimizing

$$\frac{1}{2}\sum_{i,j\in V}\omega_{ij}(u_i-u_j)^2+\frac{1}{\varepsilon}\sum_{i\in V}W(u_i)+\frac{1}{2}\sum_{i\in Z}\mu_i(u_i-f_i)^2$$

over real-valued functions u defined on the node set V of a graph with edge weights ω_{ij}. The double-well potential $W:\mathbb{R}\to\mathbb{R}$ has wells at 0 and 1, f contains *a priori* known labels on a subset $Z\subset V$ and ε and μ_i are parameters to be chosen. Such a minimization problem can be tackled by computing a gradient flow.

By establishing a continuum limit, *consistency* of the method in the limit $|V|\to\infty$ is shown. In the main part of this talk we discuss explicit interpolation methods on a periodic grid to establish the continuum limit for total variation flow and for one-dimensional Allen–Cahn flow. A key ingredient in the proofs of these results is the variational inequality formulation for gradient flows.

In the latter part of the talk we present ideas from ongoing research into semi-group methods for establishing convergence of gradient flows if Γ-convergence of the underlying functionals is known. These methods can be applied to flows on random geometric graphs, but (at the moment) still demand stronger convexity requirements on the functionals than the variational-inequality-based results.

The main part of this talk is based on the work in [4]. Given a Hilbert space $(H,\|\cdot\|)$ and $\lambda\in\mathbb{R}$, a function $\Phi:H\to\mathbb{R}\cup\{+\infty\}$ is called geodesically λ-convex if $\Phi(\cdot)-\frac{\lambda}{2}\|\cdot\|^2$ is convex. In this case, a gradient flow of Φ with respect to $\|\cdot\|$ is defined to be a locally absolutely continuous curve $u:(0,\infty)\to H$ which satisfies, for almost all $t>0$ and for all v in the domain of Φ, the evolution variational inequality

$$\frac{1}{2}\frac{d}{dt}\|u(t)-v\|^2+\frac{\lambda}{2}\|u(t)-v\|^2\le\Phi(v)-\Phi(u(t)).$$

From the perspective of generalisability this formulation is interesting, because by replacing the norms $\|u(t)-v\|$ with a general distance $d(u(t),v)$, it allows us to define gradient flows on (non-normed) metric spaces as well[1]. For the purposes of this talk, it suffices to restrict ourselves to the Hilbert space setting. In this setting, the evolution variational inequality is equivalent to the, perhaps more commonly used, differential-inclusion-based gradient flow definition:

[1]In which case we also need to define geodesic λ-convexity in this generalised setting.

$$u'(t) \in -\partial \left(\Phi(u(t)) - \frac{\lambda}{2}\|u(t)\|^2 \right) - \lambda u(t).$$

Here ∂ denotes the subdifferential.

We wish to compare gradient flows on graphs with continuum gradient flows. The evolution variational inequality allows us to do so, once we have determined a way to embed these flows into the same space. Thus we consider a sequence of Hilbert spaces $(H_n, \|\cdot\|_n)$ indexed by a parameter $n \in \mathbb{N}$, which in our setting are going to be spaces of functions defined on the node set (of size n) of a graph. We require an embedding $i_n : H_n \to H$ and a corresponding 'projection' $p_n : H \to H_n$ such that $p_n \circ i_n$ is the identity map on H_n. We assume moreover that the maps i_n are isometries and that the maps p_n are 1-Lipschitz continuous.

Given functions $\Phi_n : H_n \to \mathbb{R} \cup \{+\infty\}$, we require each Φ_n as well as Φ to be geodesically λ-convex for some $\lambda \leq 0$ (the same λ for each function), lower semicontinuous, not identically equal to $+\infty$ and locally bounded below at some point in their domains (not necessarily the same point for each function). By Ambrosio *et al.* [1, Theorem 4.0.4], these conditions guarantee the unique existence of gradient flows of Φ_n and Φ for given initial conditions in the closure of the domains of the respective functions. We assume that these closures of the domains are equal to the whole spaces H_n and H, for Φ_n and Φ, respectively.

Writing u_n for the gradient flow of Φ_n with initial condition $u_n^0 \in H_n$ and u for the gradient flow of Φ with initial condition $u^0 \in H$, the evolution variational inequality allows us, in [4], to prove the following two theorems.

Theorem 1. *Assume the following three conditions are satisfied, for all $n \in \mathbb{N}$, all $v \in H_n$ and all $w \in H$: $\Phi(i_n v) \leq \Phi_n(v)$, $\Phi_n(p_n w) \leq \Phi(w)$, and*

$$\|v - p_n w\|^2 + \|i_n p_n w - w\|^2 = \|i_n v - w\|^2. \tag{1}$$

Then $i_n u_n$ is the gradient flow of Φ with initial condition $i_n u_n^0$ and, for all $t > 0$,

$$\|i_n u_n(t) - u(t)\|^2 \leq e^{-2\lambda t}\|i_n u_n^0 - u^0\|^2.$$

Theorem 2. *Assume that the equality in* (1) *is satisfied and the following hold.*

- (a) *For all $w \in H$, $\limsup_{n\to\infty} \Phi_n(p_n w) \leq \Phi(w)$.*
- (b) *There exist $T > 0$, $\delta > 0$, and a nonnegative function $\Psi : H \to \mathbb{R} \cup \{+\infty\}$ such that $\Psi(u(\cdot)) \in L^1(0,T)$ and, for all $w \in H$ and n large enough $\Phi_n(p_n w) \leq \Psi(w)$.*
- (c) *For all $t \in [0,T]$ and n large enough, $\Phi\big(i_n u_n(t)\big) \leq \Phi_n\big(u_n(t)\big) + o(1)$.*

If $i_n u_n^0 \to u^0$, then

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|i_n u_n(t) - u(t)\| = 0.$$

We apply these theorems in a setting in which H_n is the space of real-valued functions on the node set V_n of the graph we obtain by discretising the flat d-dimensional torus by a regular grid with n^d nodes.

Theorem 1 can be applied to the total variation flow, with

$$\Phi_n(u) = \frac{1}{2} \sum_{z \in V_n} \sum_{\substack{\tilde{z} \in V_n \\ \tilde{z} \sim z}} n^{1-d} |u(z) - u(\tilde{z})|, \qquad \Phi(u) = \int_{\mathbb{T}^d} |Du|_{\ell^1}.$$

The embedding maps i_n are constructed via piecewise-constant embedding and the 'projection' maps p_n by averaging over grid cells.

We apply Theorem 2 to the one-dimensional (i.e. $d = 1$) Allen–Cahn flow with

$$\Phi_n(t) = \frac{1}{4} \sum_{z \in V_n} \sum_{\substack{\tilde{z} \in V_n \\ \tilde{z} \sim z}} n^{2-d} \big(u(z) - u(\tilde{z})\big)^2 + \sum_{z \in V_n} n^{-d} W(u(z)),$$

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u(x)|^2 \, dx + \int_{\mathbb{T}^d} W(u(x)) \, dx.$$

In this case the embedding operator i_n is given by linear interpolation with p_n a corresponding 'orthogonal projection' map. Since these i_n are not an isometries, Theorem 2 cannot be applied directly. Instead, the inner product structure on H_n is adapted such that the i_n become isometries and it is shown that the resulting gradient flows of Φ_n do not differ much from the original gradient flows of Φ_n, for large n.

In the final part of the talk, which is based on work with Samuel Mercer which is currently in preparation, we wish to prove convergence of gradient flows in cases where Γ-convergence of the functions Φ_n is known, in the tradition of Sandier and Serfaty's work [5]. Moreover, we wish to be able to apply this in settings such as random geometric graphs in which no regular grid is available and interpolation techniques for embedding discrete functions into continuum function spaces requires more attention. Preliminary results, based on semigroup techniques and an extension of theorem by Brezis and Pazy [3, Theorem 3.1] to sequences of Banach spaces suggest this is possible, yet potentially at the cost of requiring stronger convexity of the functionals than geodesic λ-convexity.

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Thermodynamic limit of stochastic particle systems via EDP convergence

André Schlichting

(joint work with Chun Yin Lam)

We consider N particles on a lattice of L sites. The state of the system is given by the occupation numbers $\eta = (\eta_x)_{x\in 1,\dots,L}$ of each site, and the time evolution is modelled by a continuous-time Markov process $(\eta(t))_{t\geq 0}$ on the set $V^{N,L} := \{\eta \in \mathbb{N}_0^L : \sum_{x=1}^L \eta_x = N\}$ of configurations with a fixed number N of particles. The process is characterized by the generator

$$(1) \qquad \mathcal{L}f(\eta) = \frac{1}{L-1}\sum_{x=1}^{L}\sum_{y=1}^{L} K(\eta_x,\eta_y)\big(f(\eta^{x,y}) - f(\eta)\big) , \qquad f \in C(V^{N,L}) ,$$

where $\eta^{x,y}$ denotes the configuration obtained from η after one particle jumps from x to y.

This is a particular model class of stochastic particle systems (SPS), which has been introduced in [3] under the name *misanthrope processes*. The process is irreducible and has a unique canonical stationary measure $\pi^{N,L}$ on $V^{N,L}$.

We consider condensation as a phase separation phenomenon in the *thermodynamic limit* $N, L \to \infty$ with $N/L \to \rho \geq 0$: If the particle density ρ exceeds a critical value ρ_c, the system phase separates into a homogeneous bulk and a condensate, where a finite fraction of particles accumulates on a vanishing volume fraction of sites. Mathematically, we say that an SPS with canonical measures $(\pi^{N,L})$ exhibits *condensation* (in the thermodynamic limit) if the single-site marginals converge narrowly

$$\pi_{L,N}[\eta_x \in .] \Rightarrow \nu^\rho , \quad \text{a measure on } \mathbb{N}_0 \text{ with } \quad \textstyle\sum_{k\geq 0} k\nu^\rho(k) < \rho = \lim N/L .$$

The condensation threshold mass is denoted by $\rho_c \in [0,\infty]$, the largest ρ for which no condensation occurs. Condensation, i.e. $\rho_c < \infty$, in SPS of type (1) has been studied extensively (see e.g. [2] and references therein) and particular models include zero-range processes [8] with bounded kernels of the form

$$(2) \qquad K(k,l) = u(k) = 1 + \frac{b}{k^\gamma} \quad \text{with parameters } b > 0, \gamma \in (0,1] ,$$

or various models with product kernels $K(k,l) = k^\lambda(d+l^\lambda)$ for parameters $d, \lambda > 0$.

Spatially homogeneous SPS with these kernels are known to have stationary measures of product form

$$(3) \qquad \pi^{N,L}(\eta) = \frac{1}{Z_{N,L}}\prod_{x=1}^{L} Q_{\eta_x} \quad \text{with normalization} \quad Z_{N,L} = \sum_{\eta\in V^{N,L}}\prod_{x=1}^{L} Q_{\eta_x} ,$$

and stationary weights $Q : \mathbb{N}_0 \to (0,\infty)$, playing the role of the chemical potential.

The condensation transition has been established rigorously in the thermodynamic limit for zero-range processes of type (2) (see e.g. [8]), where the condensate consists only of a single site.

The configuration $\eta \in V^{N,L}$ of a mean-field SPS can be characterized by the *empirical cluster distribution*

$$F_k^L(\eta) := \frac{1}{L}\sum_{x=1}^{L} \delta_{k,\eta_x} \in [0,1] \ , \quad k \geq 0 \ . \tag{4}$$

In the thermodynamic limit these observables converge under quite general conditions, forming the basis of a mesoscopic description of the dynamics. The law of large numbers was obtained in [7]: Let the process $(\eta(t))_{t\geq 0}$ be given by the generator (1) for a kernel with at most linear growth, i.e.

$$0 \leq K(k,l-1) \leq C_K\, k\, l \qquad \text{for } k,l \geq 1 \ , \tag{K$_1$}$$

If $F_k^L(\eta(0)) \to c_k(0)$ satisfies some suitable tightness assumptions, then $c_k^L(\eta(0)) \to c_k(t)$ converges weakly in the thermodynamic limit $L,N \to \infty$, $N/L \to \rho \geq 0$ to the cluster concentrations $c_k(t)$ solving the (deterministic) *mean-field rate equations*

$$\begin{aligned} \dot{c}_k \ &= \ \sum_{l\geq 1} K(l,k-1)c_l c_{k-1} - \sum_{l\geq 1} K(k,l-1)c_k c_{l-1} \\ &\quad - \sum_{l\geq 1} K(l,k)c_l c_k + \sum_{l\geq 1} K(k+1,l-1)c_{k+1}c_{l-1} \ , \qquad \text{for } k \geq 0 \ . \end{aligned} \tag{EDG}$$

Note that the deterministic set of equations (EDG) can formally be obtained from (5) by mass-action kinetics, and describe the time evolution of concentrations of finite clusters, i.e. the bulk of the system, on a *mesoscopic scale*. This description, also known as *exchange-driven growth* [1]. Basic mathematical properties regarding the well-posedness and the longtime behavior of the EDG model in the form of (EDG) are investigated in [4, 6, 15].

Although the exchange-driven growth process is not necessarily realized by chemical kinematics, it is convenient to be interpreted as a reaction network of the form

$$\{k-1\} + \{l\} \underset{K(k,l-1)}{\overset{K(l,k-1)}{\rightleftharpoons}} \{k\} + \{l-1\} \ , \qquad \text{for} \quad k,l \geq 1 \ . \tag{5}$$

Hereby, clusters of integer size $k \geq 1$ are denoted by $\{k\}$ and the variable $\{0\}$ represents empty volume. The kernel $\big(K(k,l-1)\big)_{k,l\geq 1}$ encodes the rate of the exchange of a single monomer from a cluster of size k to a cluster of size $l-1$. Note that no mass is created or destroyed in the reaction.

In the present work, we lift the law of large numbers result to statement on the convergence of gradient structures related to the large deviation rate functional of the stochastic particle system. For the description and convergence, we use the recent framework of gradient flows in continuity equation format established in [13, 14]. The law of the empirical cluster distribution (4) of the SPS, denoted with $\mathbb{C}^{N,L} \in \mathcal{P}(\mathcal{P}(\mathbb{N}_0))$ is associated with a discrete continuity equation encoding the two conserved quantities of the system given in a suitable weak form of

$$\partial_t \mathbb{C}_t^{N,L} + \widehat{\operatorname{div}}\, \mathbb{J}_t^{N,L} = 0, \tag{6}$$

where the flux is a measure in $\mathbb{J}^{N,L} \in \mathcal{M}(\mathcal{P}(\mathbb{N}_0) \times \mathbb{N} \times \mathbb{N}_0)$ and $\widehat{\operatorname{div}}$ is the adjoint operator to the discrete gradient $\widehat{\nabla} f(c)(k,l-1) = f(c^{k,l-1}) - f(c)$ with $c^{k,l-1} = c + \frac{1}{L}\gamma^{k,l-1}$ and $\gamma^{k,l-1} = e_{k-1} + e_l - e_k - e_{l-1}$. For the specific absolutely continuous flux $\mathrm{d}\overline{\mathbb{J}}_t^{N,L}(c,k,l-1) = \frac{L^2}{L-1} c_k\big(c_{l-1} - \frac{\delta_{k,l-1}}{L}\big) K(k,l-1) \mathrm{d}\mathbb{C}_t^{N,L}(c)$, the solution of (6) is exactly the forward Kolmogorov equation for the SPS-generator (1). Under the detailed balance condition

$$\text{(BDA)} \qquad \frac{K(k,l-1)}{K(l,k-1)} = \frac{K(k,0)\,K(1,l-1)}{K(l,0)\,K(1,k-1)}$$

the system has a also the formulation as a generalized gradient flow by the theory developed in [11], which amounts to the fact, that the rate function $\mathbb{L}^{N,L}(\mathbb{C}^{N,L}, \mathbb{J}^{N,L})$ takes the form

$$(7)\quad \mathbb{F}^{N,L}(\mathbb{C}_t^{N,L})\Big|_{t=0}^{T} + \int_0^T \Big[\mathcal{R}^{N,L}(\mathbb{C}_t^{N,L}, \mathbb{J}_t^{N,L}) + \mathcal{R}^{N,L^*}\big(\mathbb{C}_t^{N,L}, -\overline{\nabla} D\mathcal{F}^{N,L}(\mathbb{C}_t^{N,L})\big)\Big]\mathrm{d}t,$$

where the free energy $\mathbb{F}$ is the relative entropy with respect to the equilibrium cluster distribution $F_\sharp^L \pi^{N,L}$ of the SPS from (3) and the functional $\mathcal{R}$ and $\mathcal{R}^*$ are dual dissipation functionals of cosh-type, which are typical for jump processes [9, 11, 13, 14]. In the form (7), a passage to the thermodynamic limit $N, L \to \infty$ of the gradient structure with $N/L \to \rho \geq 0$ is possible via the notion of *EDP-convergence*, also called *evolutionary* Γ*-convergence* [5, 10, 12, 14]. The strategy is to exploit suitable compactness for curves $(\mathbb{C}^{N,L}, \mathbb{J}^{N,L})$ solving (6) such that along converging subsequences $(\mathbb{C}^{N,L}, \mathbb{J}^{N,L}) \rightharpoonup (\mathbb{C}, \mathbb{J})$ the following Γ-lim inf statement holds

$$\liminf_{N/L \to \rho} \mathbb{L}^{N,L}(\mathbb{C}^{N,L}, \mathbb{J}^{N,L}) \geq \mathbb{L}^{\rho \wedge \rho_c}(\mathbb{C}, \mathbb{J}).$$

The limit functional has a density with respect to the limit measure $\mathbb{C}$ and one arrives at the diagram:

$$\begin{array}{ccc}
(\mathbb{C}^{N,L}, \overline{\mathbb{J}}^{N,L}) \text{ solves (6)} \iff \mathbb{L}^{N,L}(\mathbb{C}^{N,L}, \mathbb{J}^{N,L}) = 0 & \xrightarrow[N,L\to\infty,\ N/L\to\rho]{\text{EDP}} & \mathbb{L}^{\rho\wedge\rho_c}(\mathbb{C}, \mathbb{J}) = 0 \\
\text{Evolution of the law of} \Big\uparrow & & \Big\Downarrow \text{ implies via superposition} \\
\text{cluster distribution } (F_k^L(\eta(t)))_{t \geq 0} & & \mathsf{L}^{\rho\wedge\rho_c}(c, j(c)) = 0 \iff c \text{ solves (EDG).}
\end{array}$$

In the EDP convergence statement the choice of the topology is crucial and we equip the space $\mathcal{P}_{<\infty}(\mathbb{N}_0)$ with the distance

$$(8)\quad d_{\mathrm{Ex}}(\mu^0, \mu^1) = |\mathrm{T}(\mu^0) - \mathrm{T}(\mu^1)|_{\ell^1(\mathbb{N})} \quad \text{with} \quad \mathrm{T}_k(\mu) = \sum_{l=k}^{\infty} \mu_l \ \ \textit{tail distribution.}$$

The EDP convergence statement is then formulated in the topology $(\mathcal{P}_{<\infty}, d_{\mathrm{Ex}})$ and for the sake of brevity, we state only the Γ-convergence for the free energy.

Theorem (Γ-convergence of free energy)**.** *In the thermodynamic limit* $\frac{N}{L} \to \rho$, *the free energy* Γ *converges*

$$\mathbb{F}^{N,L}(\mathbb{C}^{N,L}) \xrightarrow{\Gamma} \int \mathcal{H}(c|\nu^{\rho\wedge\rho_c}) \mathrm{d}\mathbb{C}^\rho \qquad \textit{in the narrow topology on } (\mathcal{P}_{<\infty}, d_{\mathrm{Ex}}).$$

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Normal form and the Cauchy problem for cross-diffusive mixtures

Katharina Hopf

(joint work with Pierre-Étienne Druet, Ansgar Jüngel)

Irreversible physical processes compatible with the second law of thermodynamics can be modelled using the Onsager approach, which is based on a formal gradient-flow ansatz in the dual form

$$\dot{u} = -\mathcal{K}(u)D\mathcal{H}(u). \tag{1}$$

Here, $u = u(t)$ denotes the state, $\dot{u} = \frac{\mathrm{d}}{\mathrm{d}t}u$, $\mathcal{H}$ a differentiable driving functional and $\mathcal{K}$ the Onsager operator, a symmetric and positive semi-definite linear operator, whose symmetry property reflects the Onsager reciprocal relations. We are interested in diffusive processes formally obtained by choosing $u = u(t,x) \in \mathcal{O}$,

$t > 0, x \in \mathbb{T}^d$, for a convex domain $\mathcal{O} \subset \mathbb{R}^n$, $\mathcal{H}(u) = \int_{\mathbb{T}^d} h(u)\,dx$ with $h : \mathcal{O} \to \mathbb{R}$ smooth and strongly convex such that $\mathbb{H} := D^2 h > 0$ in $\mathcal{O}$. The Onsager operator is assumed to take the form $\mathcal{K}(u)\xi = -\nabla \cdot (\mathbb{M}(u)\nabla\xi)$ with $\mathbb{M}(u) \in \mathbb{R}^{n\times n}$ symmetric and positive semidefinite. Inserting these choices into (1) gives the quasi-linear second-order system

$$\partial_t u = \nabla \cdot (\mathbb{A}(u)\nabla u), \qquad \mathbb{A}(u) = \mathbb{M}(u)\mathbb{H}(u). \tag{2}$$

While the matrix $\mathbb{A}(u)$ need not be symmetric, the positive definiteness of $\mathbb{H}(u)$ and the positive semi-definiteness of $\mathbb{M}(u)$ ensure that it is diagonalisable over $\mathbb{R}$ and all its eigenvalues are non-negative. If in addition $\operatorname{rank}\mathbb{A}(u) = n$, the PDE system (2) is parabolic in the sense of Petrovskii, rendering the Cauchy problem locally well-posed for sufficiently regular data.

The present note is motivated by an application in population dynamics determined by the choice

$$h(u) = \sum_{i=1}^{n} \frac{1}{\lambda_i} u_i(\log u_i - 1), \qquad \mathbb{M}_{ij}(u) = u_i \mathbb{B}_{ij}\lambda_j u_j, \quad i,j \in \{1,\dots,n\}, \tag{3}$$

with $\mathcal{O} = (0,\infty)^n$ and where $\mathbb{B} = (\mathbb{B}_{ij}) \in \mathbb{R}^{n\times n}$, $\lambda = (\lambda_i) \in (0,\infty)^n$ are such that the product $\mathbb{B}\mathbb{D}(\lambda)$ is symmetric positive semidefinite, $\mathbb{D}(\lambda) := \operatorname{diag}(\lambda_1,\dots,\lambda_n)$. Thus, in this application, $\mathbb{A}_{ij}(u) = u_i\mathbb{B}_{ij}$, and the system is no longer parabolic if $\operatorname{rank}\mathbb{B} < n$. We are thus faced with a borderline case, where local well-posedness cannot directly be inferred from classical literature, but might still be expected given the non-negativity of all eigenvalues of $\mathbb{A}(u)$. To gain insights in the Cauchy problem, it is necessary to better understand the structure of the system. In the context of fluid dynamics a systematic procedure has been developed by Kawashima and Shizuta [2] for quasi-linear second-order systems with an entropy structure, who introduced a normal form, i.e. a change of the dependent variables that brings the PDE system in the form of a composite symmetric hyperbolic–parabolic system. The classical theory on normal forms strongly relies on a null-space/range invariance property of the matrix associated with the diffusive effects, which is not satisfied in the above model (with $\operatorname{rank}\mathbb{B} < n$) because $\operatorname{range}\mathbb{M}(u) = \operatorname{range}\mathbb{D}(u)\mathbb{B}$ depends on the state u. Nevertheless, in the specific example considered above, explicit calculations detailed in [1] allow us to identify a change of variables $u \mapsto w$ that brings system (2) in the form of a symmetric hyperbolic–parabolic system

$$\mathbb{A}_0^{\mathsf{I}}(w)\partial_t w_{\mathsf{I}} + \sum_{\nu=1}^{d} \mathbb{A}_1^{\mathsf{I}}(w, \partial_{x_\nu} w_{\mathsf{II}})\,\partial_{x_\nu} w_{\mathsf{I}} = f^{\mathsf{I}}(w, \nabla w_{\mathsf{II}}),$$
$$\mathbb{A}_0^{\mathsf{II}}\partial_t w_{\mathsf{II}} - \nabla \cdot \Big(\mathbb{A}_*^{\mathsf{II}}(w)\nabla w_{\mathsf{II}}\Big) = 0,$$

where the matrices $\mathbb{A}_0^{\mathsf{I}}(w) \in \mathbb{R}^{(n-r)\times(n-r)}$, $\mathbb{A}_0^{\mathsf{II}}, \mathbb{A}_*^{\mathsf{II}}(w) \in \mathbb{R}^{r\times r}$, $r := \operatorname{rank}\mathbb{B}$, are symmetric positive definite, and $\mathbb{A}_1^{\mathsf{I}}(w, \partial_{x_\nu} w_{\mathsf{II}}) \in \mathbb{R}^{(n-r)\times(n-r)}$ is symmetric. At this point, established methods for symmetric hyperbolic and symmetric parabolic systems can be applied separately to the respective subsystem in order to construct short-time classical solutions emanating from initial data in $H^s(\mathbb{T}^d)$, $s > \frac{d}{2} + 1$, that are positive componentwise.

Finally, consider more generally system (2) with $\operatorname{rank} \mathbb{A}(u) = r < n$. The following question arises naturally: under which conditions can it be recast in a normal form that ensures local well-posedness for smooth data?

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The large-data limit of the MBO scheme for data clustering

Tim Laux

(joint work with Jona Lelmi)

The MBO scheme is an efficient algorithm for data clustering, the task of partitioning a given dataset into several meaningful clusters. Vaguely speaking, a clustering is considered meaningful if all elements in a given cluster are similar to each other while they differ from those in others. Quantitatively, this is often interpreted as finding minimal cuts in an associated graph. However, nonlinear methods (like finding minimal graph cuts) have the disadvantage of being computationally inefficient, sometimes even giving rise to NP-hard problems. On the other hand, there are plenty linear algorithms, such as k-Means, which find some clustering, but cannot resolve the possibly nonlinear structure of the data set without suitable pre-processing of the data. The MBO scheme mediates between those two extreme cases: One merely solves a linear problem and then applies a pointwise nonlinearity which is computationally trivial. Therefore, it is as performant as a linear method but is not blind to nonlinear effects in the data structure. In this talk, I present the first rigorous analysis of this scheme in the large-data limit.

Given a point cloud $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, we encode a clustering by a partition $X = \Omega_1 \cup \ldots \cup \Omega_P$ for some $P \in \mathbb{N}$. Equipping the set X with a graph structure (for example by setting $x \sim y$ if and only if $|x-y| < \varepsilon$ for some fixed scale $\varepsilon > 0$), one can exploit the (say, random walk) graph Laplacian Δ to understand the geometry of the data set X.

The MBO scheme improves an initial guess (for example given by k-Means or a random assignment) by alternating between linear diffusion and pointwise thresholding. More precisely, given an (artificial) time-step size and an initial clustering $\Omega_1^{(0)} \cup \ldots \cup \Omega_P^{(0)}$, for $\ell = 1, 2, \ldots$, compute

$$\text{(1)} \qquad \text{Diffusion: } u_i^{(\ell)} := e^{-h\Delta} \chi_{\Omega_i^{(\ell-1)}} \quad (1 \le i \le P),$$

$$\text{(2)} \qquad \text{Thresholding: } \Omega_i^{(\ell)} := \left\{ x \in X : u_i^{(\ell)}(x) = \max_{1 \le j \le P} u_j^{(\ell)}(x) \right\} \quad (1 \le i \le P),$$

until some stopping criterion is met, such as only few points changing their label from Step $L-1$ to Step L. The partition in the last step then is the proposed clustering $X = \Omega_1^{(L)} \cup \ldots \cup \ldots \Omega_P^{(L)}$.

The starting point of the analysis is that each iteration of the MBO scheme can be viewed as one step of minimizing movements for the thresholding energy on the similarity graph of the dataset, i.e., writing $\Omega = (\Omega_1, \ldots, \Omega_P)$, the combination of (1) and (2) is equivalent to

$$\Omega^{(\ell)} \in \arg\min_{\Omega} \Big\{ E_h^{N,\varepsilon}(\Omega) + \frac{1}{2h}\big(d_h^{N,\varepsilon}\big(\Omega, \Omega^{(\ell-1)}\big)\big)^2 \Big\},$$

where the energy $E_h^{N,\varepsilon}$ is defined on partitions $\Omega = (\Omega_1, \ldots, \Omega_P)$ of X via

$$E_h^{N,\varepsilon}(\Omega) := \frac{1}{\sqrt{h}} \sum_{1 \le i < j \le P} \langle \chi_{\Omega_j}, e^{-h\Delta}\chi_{\Omega_j} \rangle \tag{3}$$

with $\langle \cdot, \cdot \rangle$ a suitable scalar product on functions on X that makes the graph Laplacian $\Delta = \Delta_{N,\varepsilon}$ self-adjoint, and $d_h^{N,\varepsilon}$ is a suitable distance function on partitions.

It is then natural to think that outcomes $\Omega^{(L)}$ of the MBO scheme are (local) minimizers of this energy. In [2], we prove that for large data sets the algorithm is consistent with the original task of finding minimal cuts in the sense that these (local) minimizers converge to (local) minimizers of the optimal partition problem given by the continuum limit of the minimal cut problem.

More precisely, we employ the so-called *manifold assumption* postulating that the points $(x_n)_n$ are independent samples of some probability measure $\mu = \rho \operatorname{Vol}_M$, where (M, g) is a closed k-dimensional submanifold of the high-dimensional feature space $\mathbb{R}^d$ and $\rho \colon M \to (0, \infty)$ is a smooth function. Then, the first result in [2] establishes the large-data limit of the energies for fixed time-step size h.

Theorem. *Under the manifold assumption, as the sample size N goes to infinity, almost surely and in a suitable scaling regime for the length scale $\varepsilon_N \to 0$, we have $E_h^{N,\varepsilon_N} \to E_h$ in the sense of Γ-convergence w.r.t. the weak TL^2-topology.*

Here, the continuum energy E_h is defined on relaxed partitions, i.e., maps $u \colon M \to [0,1]^P$ such that $\sum_i u_i = 1$, and is of the form

$$E_h(u) = \frac{1}{\sqrt{h}} \sum_{1 \le i < j \le P} \int_M u_i e^{-h\Delta_{\rho^2}} u_j \, \rho^2 d\operatorname{Vol}_M$$

(modulo some constants), where $\Delta_{\rho^2} f = -\frac{1}{\rho^2} \nabla \cdot (\rho^2 \nabla f)$ is the natural Laplacian on the weighted manifold (M, g, ρ^2).

The main ingredient for this result is the following natural fact that the diffusion equation upgrades weak to strong convergence.

Proposition. *In the situation of the theorem above, for any $t > 0$, we have*

$$u_N \rightharpoonup u \text{ weakly in } TL^2 \quad \Longrightarrow \quad e^{-t\Delta_{N,\varepsilon_N}} u_N \to e^{-t\Delta_{\rho^2}} u \text{ strongly in } TL^2. \tag{4}$$

Indeed, the proposition implies that the Γ-convergence in the theorem is in fact continuous convergence: Every weakly converging sequence is a recovery sequence for its limit. Furthermore, the proposition even implies that the whole minimizing movements functional (3) Γ-converges and hence we answer positively a question of Bertozzi:

Theorem. *Under the assumption of the above theorem, the iterates of the MBO scheme on the graph converge to the corresponding iterates of the MBO scheme on the weighted data manifold.*

The proposition is shown in [2] via the stability principle of gradient flows and exploiting the fact that the diffusion equation on the weighted manifold (M, g, ρ^2) is well-behaved so that the chain rule holds.

Finally, in the limit of vanishing time-step size, the problem converges to the desired optimal partition problem.

Theorem. *As $h \downarrow 0$, we have $E_h \to E$ in the sense of Γ-convergence w.r.t. the L^1 topology, where the sharp-interface energy is the following weighted optimal partition energy*

$$E(u) = \sum_{1 \le i < j \le P} \int_{\partial^* \Omega_i \cap \partial^* \Omega_j} \rho^2 \, d\mathcal{H}^{k-1} \quad \text{if } u = (\chi_{\Omega_1}, \ldots, \chi_{\Omega_P}) \tag{5}$$

and $E(u) = +\infty$ otherwise.

This confirms that the MBO scheme indeed places small cuts in regions of low data density. The work [2] is the first result on the large-data limit of the MBO scheme and still the only one valid for more than two clusters. In the case of two clusters, however, one can use the theory of viscosity solutions to get a more precise understanding of the dynamics, see [1]. This is a crucial next step as the non-convex energy (5) has many local minimizers. Understanding the effective behavior of the dynamics of the scheme gives insight into the path taken by the scheme in the energy landscape and therefore the selection of local minimizers.

The main ingredients for this analysis are (i) a new abstract convergence result for arbitrary discrete structures based on quantitative estimates for heat operators and (ii) the derivation of these estimates in the setting of random geometric graphs.

Overall, the results in [1] roughly state that the following.

Theorem. *Under the manifold assumption and in the joint limit $N \to \infty$, $\varepsilon \to 0$, $h \to 0$, in a suitable scaling regime, the MBO scheme for two clusters converges to the viscosity solution of mean curvature flow in the weighted data manifold (M, g, ρ^2), satisfying the level set equation*

$$\partial_t u = \frac{1}{\rho^2} \nabla \cdot \left(\rho^2 \frac{\nabla u}{|\nabla u|} \right) = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + \frac{\nabla u}{|\nabla u|} \cdot \nabla \log \rho^2.$$

Formally, this means that the limit is a solution to the geometric evolution equation

$$V = -H - \nu \cdot \nabla \log \rho^2, \tag{6}$$

which shows that the evolution is driven by both surface tension and data density. Naturally, this flow is the L^2-gradient flow of (the two-phase version of) the energy (5) which wants to straighten the cut and move it to low-density regions.

Remarkably, this proof also applies in case of a frequency cut-off, i.e., when replacing the diffusion semigroup $e^{-h\Delta}$ with the computationally much simpler projected version $e^{-h\Delta}P_{\langle\psi_1,\dots,\psi_K\rangle}$, where ψ_k denotes the k-th eigenfunction of $\Delta_{N,\varepsilon}$. The lower bound for the frequency cut-off which still guarantees convergence to (6) is of the form $K \gtrsim (\log N)^q$ for some (explicit) exponent $q > 0$.

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Globally Lipschitz transport maps

Max Fathi

(joint work with Dan Mikulincer, Yair Shenfeld)

This talk presented some results of [1] on existence of globally Lipschitz transport maps between probability measures, including in the Riemannian setting, as well as some conjectures on global Lipschitz regularity for optimal transport maps. An extended abstract on these results previously appeared in [2]

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On convergence of the fully discrete JKO scheme

Anastasiia Hraivoronska

(joint work with Filippo Santambrogio)

We study the convergence of the JKO scheme discretized on a regular lattice, motivated by application to developing numerical schemes. The JKO scheme introduced in [1] proved to be a powerful tool for analysis of evolutionary equations with gradient structure in the space of probability measures $\mathcal{P}(\Omega)$ endowed with the L^2-Wasserstein distance W_2. We recall that it is an iterative scheme that for a given energy functional $\mathcal{F} : \mathcal{P}(\Omega) \to \mathbb{R} \cup \{+\infty\}$, initial datum $\rho_0 \in \mathcal{P}(\Omega)$, and a time step $\tau > 0$ produces a sequence of probability measures $\{\rho_k^\tau\}$ as

$$\rho_{k+1}^\tau \in \arg \min_{\rho \in \mathcal{P}(\Omega)} \Big\{ \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \Big\}. \tag{JKO}$$

The theory initiated in [1,2] and further developed in [3] allows to prove under appropriate assumptions on $\mathcal{F}$ that the sequence of minimizers from (JKO) converges to a solution of

$$\partial_t \rho - \operatorname{div}\Big(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho}\Big) = 0 \qquad (0,T)\times\Omega, \tag{1}$$

with non-flux boundary condition on $\partial\Omega$. In this work, we focus on the energy functionals including internal and potential energy:

$$\mathcal{F}(\rho) = \begin{cases} \int_\Omega f\Big(\frac{d\rho}{d\mathcal{L}^d}\Big) d\mathcal{L}^d + \int_\Omega V\, d\rho, & \rho \ll \mathcal{L}^d, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

The JKO scheme is a natural time discretization that preserves structural features of the equation such as conservation of mass and energy dissipation, as well as some properties of solutions of the corresponding PDEs. It is tempting to come up with a numerical scheme based on (JKO) that enjoys similar properties. The challenging part is dealing with the Wasserstein distance term. Existing approaches to this problem include the methods exploiting the Benamou-Brenier dynamic formulation [4], using entropic regularisation and Sinkhorn algorithm [5, 6], and semi-discrete approaches [7]. We explore a new approach based on discretizing the JKO scheme on a regular lattice.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain discretized with $\mathcal{T}^h := h\mathbb{Z}^d \cap \Omega$. We notice that any probability measure on $\mathcal{T}^h$ can be represented as

$$\rho^h = \sum_{x\in\mathcal{T}^h} \rho^h_x \delta_x, \qquad \sum_{x\in\mathcal{T}^h} \rho^h_x = 1.$$

We call *the fully discrete JKO scheme*, the iterative scheme

$$\rho^{h,\tau}_{k+1} \in \arg \min_{\rho\in\mathcal{P}(\mathcal{T}^h)} \Big\{ \mathcal{F}_h(\rho^h) + \frac{1}{2\tau} W_2^2(\rho^h, \rho^{h,\tau}_k) \Big\}. \tag{JKO$_h$}$$

The problem we want to address is convergence of the sequence of minimizers in (JKO$_h$) to a solution of (1) in a joint limit $h \to 0$ and $\tau \to 0$. The first question is what is an appropriate relation between h and τ. We illustrate the importance of this relation for convergence on a toy example with the potential energy.

Example (Movement driven by a potential)**.** Let the energy functional include only potential energy with $V \in C^{1,1}(\mathbb{R}^d)$:

$$\mathcal{F}_h(\rho^h) = \sum_{x\in\mathcal{T}^h} V(x)\rho^h_x.$$

In this case, it is reasonable to consider separately the movement of the Dirac masses $\rho^h_0(x)\delta_x$ for $x \in \operatorname{spt}(\rho^h_0)$, because they move independently in absence of diffusion. Consider the movement of δ_{x_0}, $x_0 \in \mathcal{T}^h$. If x_1 is the minimizer of $V(x) + |x-x_0|^2/2\tau$ restricted to $\mathcal{T}^h$, then

$$V(x_0) \geq V(x_1) + \frac{|x_1 - x_0|^2}{2\tau},$$

which implies $h/\tau \le 2\|\nabla V\|_{\mathrm{Lip}}$. We see that if asymptotically $h/\tau > 2\|\nabla V\|_{\mathrm{Lip}}$, then we cannot expect convergence to the continuous solution, because every subsequent minimizer is equal to x_0 and the discrete evolution is "frozen".

Moreover, one can derive that the accumulated error between minimizers $\{x_k\}$ restricted to $\mathcal{T}^h$ and minimizers on the full space $\{\overline{x}_k\}$ for $T = k\tau$ is bounded as

$$|x_k - \overline{x}_k| \le Ckh = CT\frac{h}{\tau}.$$

Therefore, the convergence holds only if $h/\tau \to 0$.

Now we turn to a more interesting case of (JKO$_h$) with the internal energy

$$\mathcal{F}_h(\rho^h) = \sum_{x\in\mathcal{T}^h} f\Big(\frac{\rho^h_x}{h^d}\Big)h^d,$$

with convex and differentiable f such that f' is monotone. Let $\{\rho_k^{h,\tau}\}_{k=0,\dots,N}$ be a sequence of minimizers of (JKO$_h$) and $T = N\tau$. The goal is to prove convergence of $\{\rho_k^{h,\tau}\}_{k=0,\dots,N}$ to a solution of (1). Our strategy is to show that there exists a limit curve $[0,T] \ni t \mapsto \rho_t$ which is a solution of (1) in the EDI sense. This means that there exists a velocity field v such that (ρ, v) satisfies the continuity equation

$$\partial_t\rho + \operatorname{div}(\rho v) = 0 \qquad \text{on } (0,T)\times\Omega$$

and the energy-dissipation inequality (EDI) holds true

$$\text{(3)}\quad \mathcal{F}(\rho_T) - \mathcal{F}(\rho_0) + \frac{1}{2}\int_0^T\Big\{\int_\Omega |v_t|^2 d\rho_t + \int_\Omega |\nabla\ell(u_t)|^2 d\mathcal{L}^d\Big\}dt \le 0, \quad \rho_t = u_t\mathcal{L}^d,$$

where ℓ related to the energy density f in the following way: $\sqrt{s}f''(s) = \ell'(s)$ and $\ell(0) = 0$.

For the standard JKO scheme, the analogous convergence result is proven using the variational interpolation [3, Chapter 3]. The idea is to prove the inequality

$$\text{(4)}\qquad \mathcal{F}(\rho^\tau_{k+1}) - \mathcal{F}(\rho^\tau_k) + \frac{W_2^2(\rho^\tau_k, \rho^\tau_{k+1})}{2\tau} + \int_0^\tau \frac{W_2^2(\rho^\tau_k, \rho^\tau_r)}{2r^2}dr \le 0,$$

where ρ^τ_r is variational interpolant between ρ^τ_k and ρ^τ_{k+1}. Combining (4) with the lower bound on the Wasserstein distance with a slope of the energy

$$\text{(5)}\qquad \frac{1}{\tau}W_2(\rho^\tau_{k+1}, \rho^\tau_k) \ge \mathrm{Slope}\mathcal{F}(\rho^\tau_{k+1}),$$

one gets a sharp inequality which is convenient to pass to the limit to recover (3).

The crucial step in the discrete setting is to find an appropriate replacement for (5). Note that we cannot use the metric slope, because it blows up as $h \to 0$. Instead of the slope, we use the discrete Fisher information defined as

$$\mathcal{S}_h(\rho^h) := \frac{1}{4}\sum_{x\in\mathcal{T}^h}\sum_{y\sim x}\frac{|\ell(u^h_y) - \ell(u^h_x)|^2}{h^2}h^d, \qquad u^h = \frac{\rho^h}{h^d}.$$

Second, we do not expect the discrete counterpart of (5) to hold exactly. An intuitive reason for that is that we know that we expect the inequality to fail if h/τ does not tend to 0.

The lower bound on the Wasserstein distance with the discrete Fisher information we find for the fully discrete case is presented in the following lemma.

Lemma. Let $\rho_0^h \in \mathcal{P}(\mathcal{T}^h)$ be given and $\rho^{\tau,h}$ is the minimizer of (JKO$_h$). Then

$$\frac{1}{\tau^2} W_2^2(\rho^{\tau,h}, \rho_0^h) \geq \left(1 - \frac{h}{2\tau}\right) \mathcal{S}_h(\rho^{\tau,h}) - \frac{dh}{2\tau}.$$

This abstract presents ideas on convergence of the fully discrete JKO scheme. There are plenty of related questions that have to be explored, in particular: extend the result to different energies such as interaction energy and energy appearing in crowd motion models, where f becomes a constraint $\rho \leq 1$; establish the rate of convergence; and develop the numerical algorithm.

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Stability in Gagliardo-Nirenberg-Sobolev inequalities.

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(joint work with Matteo Bonforte, Jean Dolbeault and Bruno Nazaret)

In some functional inequalities, best constants and minimizers are known. The next question is stability: suppose that a function "almost attains the equality", in which sense it is close to one of the minimizers? We will address a recent result on the quantitative stability of a subfamily of Gagliardo-Nirengerg-Sobolev. The approach is based on the entropy method for the fast diffusion equation and allows us to obtain completely constructive estimates.

We consider the family of Gagliardo-Nirenberg-Sobolev inequalities given by

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p) \, \|f\|_{2p} \quad \forall\, f \in \mathcal{H}_p(\mathbb{R}^d)\,, \tag{1}$$

for simplicity we focus on the case $d \geq 3$, but $d = 1, 2$ can be treated (see [1]). The invariance of (1) under dilations determines the exponent

$$\theta = \frac{d\,(p-1)}{\big(d+2-p\,(d-2)\big)\,p}\,,\text{where} \quad p \in (1, p^\star] \quad \text{and} \quad p^\star := \tfrac{d}{d-2}\,.$$

The space $\mathcal{H}_p(\mathbb{R}^d)$ is defined as the completion of $C_c^\infty(\mathbb{R}^d)$, with respect to the norm $f \mapsto (1-\theta)\|f\|_{p+1} + \theta\|\nabla f\|_2$, where $\|f\|_q = \Big(\int_{\mathbb{R}^d} |f|^q dx\Big)^{1/q}$ for any $q > 1$. In the limit case where $p = p^\star$, for which $\theta = 1$, we are left with the space $\mathcal{H}_{p^\star}(\mathbb{R}^d) := \Big\{ f \in \mathrm{L}^{2\,p^\star}(\mathbb{R}^d) \,:\, |\nabla f| \in \mathrm{L}^2(\mathbb{R}^d) \Big\}$ and the Sobolev's inequality

$$\|\nabla f\|_2 \geq \mathsf{S}_d \,\|f\|_{2\,p^\star} \quad \forall\, f \in \mathcal{H}_{p^\star}(\mathbb{R}^d)\,. \tag{2}$$

Optimality in both (1) and (2) is achieved on the manifold of the *Aubin-Talenti functions* (see, for instance [2] and [3])

$$\mathfrak{M} := \Big\{ g_{\lambda,\mu,y} \,:\, (\lambda,\mu,y) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \Big\}$$

$$\text{where} \quad \mathsf{g}(x) = \Big(1+|x|^2\Big)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d\,,$$

and $g_{\lambda,\mu,y}(x) := \lambda^{\frac{d}{2p}}\,\mu^{\frac{1}{2p}}\,\mathsf{g}\big(\lambda\,(x-y)\big)$ with the convention $\mu^q = |\mu|^{q-1}\,\mu$ if $\mu < 0$. We can rewrite inequalities (1) and (2) in the form of a *positive*, *non-scale-invariant* functional which we shall call the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\tfrac{d-p\,(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathcal{K}_{\rm GNS}\,\|f\|_{2\,p}^{2\,p\,\gamma} \geq 0$$

with $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$ and $\mathcal{K}_{\rm GNS}$ chosen so that $\delta[\mathsf{g}] = 0$. Up to a scaling, the fact that $\delta[f] \geq 0$ is equivalent to (1) and (2) with optimal constants. In particular $\mathcal{K}_{\rm GNS}$ can be computed in terms of $\mathcal{C}_{\rm GNS}$.

Let us explain how *fast diffusion equation* enter into play. In *self-similar variables*, the fast diffusion equation, posed on $\mathbb{R}^d$, $d \geq 3$, with exponent $m \in [m_1, 1)$ and $m_1 := 1 - 1/d$, is

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big(v\,\nabla v^{m-1}\Big) = 2\,\nabla \cdot (x\,v)\,, \quad v(t=0,\cdot) = v_0\,. \tag{FDE}$$

By applying this flow to the *relative entropy* (see [2])

$$\mathcal{F}[v] := \frac{1}{m-1}\int_{\mathbb{R}^d}\Big(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v-\mathcal{B})\Big)\,dx$$

$$\text{where} \quad \mathcal{B}(x) := \big(1+|x|^2\big)^{\frac{1}{m-1}}\,,$$

we have $\frac{d}{dt}\mathcal{F}[v(t,\cdot)] = -\,\mathcal{I}[v(t,\cdot)]$ where the *relative Fisher information* functional $\mathcal{I}$ defined by

$$\mathcal{I}[v] := \frac{m}{1-m}\int_{\mathbb{R}^d} v\,\big|\nabla v^{m-1} - \nabla \mathcal{B}^{m-1}\big|^2 dx\,.$$

It is a key step to recognise that we are dealing with the same quantities as in the variational approach. With

$$p = \frac{1}{2\,m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2\,p}\,, \quad v = f^{2p}\,, \quad \mathcal{B} = \mathsf{g}^{2p}$$

and in particular with the condition $1 < p \leq d/(d-2)$, $d \geq 3$, which is equivalent to $m_1 \leq m < 1$. Indeed, it turns out that , as observed in [2],

$$\text{(EEP)} \qquad \frac{p+1}{p-1}\,\delta[f] = \mathcal{I}[v] - 4\,\mathcal{F}[v] \geq 0$$

for $v = |f|^{2\,p}$. Inequality (EEP) is called *entropy-entopy production* inequality and its optimal constant is 4. In particular, one of the main observations of [2] is that inequalities (1) and (2) are equivalent to (EEP). At the same time, by applying (EEP) and Gronwall's lemma, we get

$$\text{(3)} \qquad \mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-4t} \quad \forall\,t \geq 0$$

if v solves (FDE) . Here the main observation is that the exponential decay estimate with factor 4 in (3) is equivalent to the optimal constant in (EEP), see [1]. In the same spirit, if we are able to obtain a better convergence rate than the one in (3) (for instance under some moment condition), this would translate into an improved entropy-entropy prodcution inequality and, therefore, into a stability result.

Our overall strategy is now to prove that under some moment conditions on v, we can improve the decay with the rate $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-(4+\zeta)t}$ for all $t \geq 0$ using the properties of (FDE). In a word, we look for improved decay rates of the entropy in order to establish an *improved entropy - entropy production inequality.* Details are given in [1, Chapter 2]. Why is it that we can expect to obtain an improved decay rate of $\mathcal{F}[v(t,\cdot)]$? This can be obtained by a careful analysis of the *asymptotic time layer* regime (that is, as $t \to +\infty$). It is of standard knowledge, see for instance [4], that solutions to (FDE) converge to $\mathcal{B}$ in strong topologies. Hence, it makes sense to consider the Taylor expansions of the entropy and the Fisher information around $\mathcal{B}$. This expansion give us two quadrativ forms defined by

$$\mathsf{F}[h] = \lim_{\varepsilon\to 0} \varepsilon^{-2}\,\mathcal{F}\big[\mathcal{B} + \varepsilon\,\mathcal{B}^{2-m}\,h\big] \quad \text{and} \quad \mathsf{I}[h] = \lim_{\varepsilon\to 0} \varepsilon^{-2}\,\mathcal{I}\big[\mathcal{B} + \varepsilon\,\mathcal{B}^{2-m}\,h\big]\,.$$

By a *Hardy-Poincaré inequality* detailed in [1, Chapter 2], we have

$$\mathsf{I}[h] \geq \Lambda\,\mathsf{F}[h]$$

with $\Lambda = 4$ if $\int_{\mathbb{R}^d} h\,\mathcal{B}^{2-m} dx = 0$ and $\Lambda = 4\big(1 + d\,(m - m_1)\big)$ if, additionally, we assume that $\int_{\mathbb{R}^d} x\,h\,\mathcal{B}^{2-m} dx = 0$. In other words, the optimal decay rate of $\mathcal{F}[v(t,\cdot)]$ is characterized in the asymptotic time layer regime as $t \to +\infty$ by the spectral gap $\Lambda = 4$. Under the additional moment condition on the center of mass, we obtain $\zeta = \Lambda - 4 > 0$ if $m > m_1$. Recall that $m > m_1$ means $p < d/(d-2)$ and covers the whole subcritical range of inequality (1), inequality (2) can also be treated but the method is more involved. Altogether, we have an improved decay rate on an *asymptotic time layer* $[T_\star, +\infty)$, that has been explored in [4] and subsequent papers. An important feature is that the estimates on Λ are explicit but require strong regularity conditions, i.e., $(1-\varepsilon)\mathcal{B}(x) \leq v(t,x) \leq (1+\varepsilon)\mathcal{B}(x)$ for all $x \in \mathbb{R}^d$ and $t \geq T_\star$. This condition is ensured only if the initial datum v_0 satisfies the following moment condition (see [1, Chapters 3 and 7]) $\sup_{R>0} R^{\frac{2}{1-m}-d} \int_{|x|>R} v_0(x)dx \leq A < \infty$.

Once an improved decay rate is obtained in the asymptotic time layer, by using a nonlinear nonlinear generalization of the *carré du champ method* of D. Bakry and M. Emery, we are able to obtain an imporved decay rate in the *initial time layer* $[0, T_\star]$, which is also explicit. Combining the two layers, we are able to obtain the *improved* entropy-entropy production inequality $\mathcal{I}[v] \geq (4+\zeta)\,\mathcal{F}[v]$ for a functions v which satisfy the above moment conditions.

In terms of the variational language introduced in the beginning, we can say that for $d \geq 3$ and $1 < p < p^\star$, for any $f \in \mathrm{L}^{2p}(\mathbb{R}^d)$ with $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ such that $A := \sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} |f|^{2p}\,dx < \infty$ we have the estimate

$$\delta[f] \geq \kappa \inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

for some explicit positive constant κ which depends only on d, p, $\|f\|_{2p}$, A, and takes positive values on $\mathfrak{M}$. In the case $p = p^\star$ the above result still holds true under a stonger moment assumption.

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Minimal Acceleration for the Multi-Dimensional Isentropic Euler Equation

Michael Westdickenberg

We consider the multi-dimensional isentropic Euler equations

$$\left.\begin{aligned} \partial_t\varrho + \nabla\cdot(\varrho\mathbf{u}) &= 0 \\ \partial_t(\varrho\mathbf{u}) + \nabla\cdot(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\pi &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbf{R}^d, \tag{1}$$

$$(\varrho,\mathbf{u})(0,\cdot) =: (\bar\varrho,\bar{\mathbf{u}}) \quad \text{initial data.}$$

This system expresses local conservation of mass and momentum. To close system (1) one needs to specify the pressure. We consider polytropic gases, for which

$$\pi(t,\cdot) = P\big(r(t,\cdot)\big)\,\mathcal{L}^d \quad \text{for all } t \in [0,\infty),$$

where $U(r) := \kappa r^\gamma$ with constants $\kappa > 0$ and $\gamma > 1$ and

$$P(r) = U'(r)r - U(r) \quad \text{for } r \geq 0.$$

Here r is the Radon-Nikodym derivative of ϱ w.r.t. the Lebesgue measure $\mathcal{L}^d$.

Smooth solutions $(\varrho, \mathbf{u})$ of (1) satisfy the additional conservation law

$$\partial_t\Big(\tfrac{1}{2}\varrho|\mathbf{u}|^2 + U(\varrho)\Big) + \nabla\cdot\Big(\Big(\tfrac{1}{2}\varrho|\mathbf{u}|^2 + U'(\varrho)\varrho\Big)\mathbf{u}\Big) = 0, \tag{2}$$

which expresses local conservation of total energy

$$E(\varrho,\mathbf{u}) := \tfrac{1}{2}\varrho|\mathbf{u}|^2 + U(\varrho),$$

which is the sum of kinetic and internal energy. Since solutions of (1) may become discontinuous in finite time, solutions must be considered in the weak sense and energy conservation (2) must be relaxed to an $\leq$ inequality.

Global existence of weak solutions to (1) is still an open problem in several space dimension. A useful relaxation with guaranteed existence is the notion of dissipative solutions, introduced by [1]. Dissipative solutions are defined as tuples of $(\varrho, \mathbf{m})$ and defect measures $\mathbf{R}, \phi$ that satisfy the continuity equation and

$$\partial_t \mathbf{m} + \nabla\cdot\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + P(\varrho)\mathbf{1}\right) + \boxed{\nabla\cdot(\mathbf{R}+\phi\mathbf{1})} = 0,$$

$$\frac{d}{dt}\int_{\mathbf{R}^d}\left(\tfrac{1}{2}\varrho|\mathbf{u}|^2 + U(\varrho) + \boxed{\frac{1}{2}\mathrm{tr}(\mathbf{R}) + \frac{1}{\gamma-1}\phi}\right)(t,dx) \leq 0.$$

Here $\mathbf{R}$, ϕ are measures taking values in the symmetric, positive semidefinite matrices and the non-negative numbers, which form closed convex cones. Dissipative solutions become weak solutions of (1) iff the defect measures vanish.

The construction of infinitely many weak solutions to (1), pioneered by De Lellis-Székelyhidi [2], starts from so-called subsolutions, which can be interpreted as dissipative solutions with defect measures nonvanishing in open sets. Superimposing over $(\varrho, \mathbf{m})$ highly oscillatory waves, one can then remove the discrepancy between dissipative and weak solutions. To the extent that abstract arguments like the Baire category theorem are used to ensure the convergence of the iterative procedure, this result is based on the axiom of choice.

In contrast, our goal is to construct dissipative solutions to the isentropic Euler equations (1) that minimizes the defect measures from the start. It may very well be possible that for certain configurations, such as Kevin-Helmholtz instabilities, nonvanishing defect measures must occur. Indeed, since no viscosity is present, oscillatory features may persist at arbitrarily small length scales. In such cases, the best one can hope for is to construct solutions that are as close to being a weak solution as possible. One can speculate that in regions where defect measures do not vanish a variant of the De Lellis-Székelyhidi method could be used to repair the dissipate solution to become a (or infinitely many) weak solution(s).

In order to construct dissipative solutions with minimal defect measures we consider the acceleration functional, defined as

$$|\mathbf{m}'|(t) = \int_{\mathbf{R}^d} \mathrm{tr}\big(\mathbb{U}(t,dx)\big) \quad \text{for a.e. } t \in [0,\infty), \tag{3}$$

with momentum flux

$$\mathbb{U} := \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + P(\varrho)\mathbf{1}\right) + \boxed{\mathbf{R} + \phi\mathbf{1}}.$$

As the notation suggests, (3) can be understood as the metric derivative of the momentum curve $t \mapsto \mathbf{m}(t,\cdot)$ with respect to the dual Lipschitz norm, which is the natural topology for the momentum field, given its finiteness in total variation due to the energy bound. Notice that (3) is nonnegative because the defect measures $\mathbf{R}$ and ϕ are in closed convex cones. Since minimizing (3) for all times amounts to a multi-objective optimization problem, which typically does not have minimizers, we instead look for Pareto-optimal solutions, i.e., for minimal elements with respect to a suitable quasi-order defined in terms of comparing the acceleration (3) of different dissipative solutions at all times. A quasi-order is a binary relation that is reflexive and transitive, but not necessarily antisymmetric. If this quasi-order is compatible with a topology, one can use the following result by Wallace [3]:

Theorem (Wallace). Suppose that X is a nonempty compact set with a quasi-order R such that the set of predecessors $P(x)$ of x is closed for every $x \in X$. Then X has a minimal element, i.e., an element $m \in X$ with the property that,

$$\text{if } y \in X \text{ and } m \text{ can be compared at all, then } (m,y) \in R.$$

This result can be applied with X the set of dissipative solutions of (1) to given initial data, and with the quasi-order defined in terms of the acceleration functional (3). A suitable topology can be chosen as weak* convergence of Young measures. Note that Wallace's existence result constructs minimal elements starting from totally ordered subsets of X, which exist because of the Hausdorff maximal principle. Ultimately, it it therefore again an application of the axiom of choice.

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Transport meets variational inference

Nikolas Nüsken
(joint work with Francisco Vargas, Shreyas Padhy and Denis Blessing)

Given probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and a fixed terminal time $T > 0$, our objective is to (algorithmically) construct vector fields $a : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ with appropriate growth and regularity properties, such that the diffusion process

$$\mathrm{d}X_t = a_t(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 \sim \mu \tag{1}$$

transports μ to ν, that is, $X_T \sim \nu$. More precisely, we aim to construct functionals $\mathcal{L} : a \mapsto \mathbb{R}_{\geq 0}$ whose minimisers provide solutions to the stated transport problem. Clearly, neither a nor $\mathcal{L}$ will be unique without imposing further constraints. Building on a parameterisation $\theta \mapsto a_\theta$, typically in terms of neural networks, such functionals allow us to approximate transporting diffusions of the form (1) by applying gradient-descent type algorithms to $\theta \mapsto \mathcal{L}(a_\theta)$.

In the recent preprint [1], we propose a framework based on augmenting (1) to forward and reverse time diffusions,

$$\mathrm{d}X_t = a_t(X_t)\,\mathrm{d}t + \overrightarrow{\mathrm{d}}W_t, \qquad X_0 \sim \mu, \tag{2a}$$

$$\mathrm{d}X_t = b_t(X_t)\,\mathrm{d}t + \overleftarrow{\mathrm{d}}W_t, \qquad X_T \sim \nu, \tag{2b}$$

where $\overrightarrow{\mathrm{d}}$ and $\overleftarrow{\mathrm{d}}$ denote forward and backward Itô integrals, respectively. [1] The diffusions (2a) and (2b) induce path measures $\overrightarrow{\mathbb{P}}^{\mu,a}, \overleftarrow{\mathbb{P}}^{\nu,b} \in \mathcal{P}(C([0,T];\mathbb{R}^d))$, and we consider mappings of the form

$$(a,b) \mapsto D(\overrightarrow{\mathbb{P}}^{\mu,a} | \overleftarrow{\mathbb{P}}^{\nu,b}), \tag{3}$$

where D is a divergence (meaning that $D(\mathbb{Q}|\mathbb{P}) \geq 0$ for all $\mathbb{Q}, \mathbb{P} \in \mathcal{P}(C([0,T];\mathbb{R}^d))$, with equality if and only if $\mathbb{P} = \mathbb{Q}$), for example the Kullback-Leibler divergence

$$D_{KL}(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_{X\sim\mathbb{Q}}\left[\log \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(X)\right].$$

The simple but key observation is that $D(\overrightarrow{\mathbb{P}}^{\mu,a} | \overleftarrow{\mathbb{P}}^{\nu,b}) = 0$ if and only if the pair (a,b) produces diffusions that transport μ to ν (and back), and therefore modifications of (3), such as imposing further constraints on a and b, allow us to approach the transport problem stated at the beginning. In [1], we thereby recover entropic interpolations, stochastic optimal control problems, as well as the recently introduced score matching and action matching objectives from machine learning. We also develop a novel loss functional for the Bayesian sampling problem,

$$\phi \mapsto \mathbb{E}\left[\int_0^T |\nabla \log \pi_t(X_t)|^2\,\mathrm{d}t + \frac{1}{\sqrt{2}}\int_0^T (\nabla \log \pi_t - \nabla\phi_t)(X_t)\cdot \overleftarrow{\mathrm{d}}W_t - \log \pi_T(X_T)\right].$$

[1]The notions of stochastic integration in (1) and (2a) are the same; we use $\overrightarrow{\mathrm{d}}$ in (2a) to promote the symmetry of the framework.

In the above, $(\pi_t)_{0\leq t\leq T} \subset \mathcal{P}(\mathbb{R}^d)$ is a fixed curve of probability measures, and at optimality, the diffusion driven by $a = \nabla\phi^*$ reproduces these time-marginal laws. Motivated by excellent numerical results and relationships to the Crooks and Jarzinksy identities from statistical physics, future work will aim at a deeper understanding of this nonstandard control functional (nonstandard because of the backward Itô integral), and extensions to kinetic diffusions.

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Gradient flow characaterisation of the heat flow with Dirichlet boundary conditions

Matthias Erbar

(joint work with Giulia Meglioli)

In a bounded domain $\Omega \subset \mathbb{R}^d$ we consider the porous medium equation

$$(1) \qquad \begin{cases} \partial_t\rho = \Delta\rho^\alpha & \text{in } (0,\infty)\times\Omega; \\ \rho(0,\cdot) = \rho_0 & \text{in } \Omega \\ \rho = \lambda & \text{on} (0,\infty)\times\partial\Omega\ , \end{cases}$$

with constant Dirichlet boundary condition $\lambda \in (0,\infty)$ and $\alpha \geq 1$. Our goal is to give a variational characterisation of solutions in terms of gradient flows in the space of measures. While a large number of results characterising various evlutionary PDEs with Neumann boundary conditions as gradient flow w.r.t. the Wasserstein distance is available, little is known to date concerning other types of boundary conditions. Figalli and Gigli [1] have introduced a variant of the Wasserstein distance allowing for change of mass by letting the boundary $\partial\Omega$ act as a reservoir. For μ, ν positive measures in $\mathcal{M}_p(\Omega) := \{\mu \in \mathcal{M}_+(\Omega) : \int_\Omega d(\cdot,\partial\Omega)^p d\mu < \infty\}$ they define a distance $Wb_p(\mu,\nu)$ via

$$Wb_p(\mu,\nu)^p := \inf_{\gamma\in Adm(\mu,\nu)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^p d\gamma(x,y)\ ,$$

where $Adm(\mu,\nu)$ is the set of admissible transport plans and consists of all $\gamma \in \mathcal{M}_+(\overline{\Omega}\times\overline{\Omega})$ such that $\pi^1_\#\gamma|_\Omega = \mu$ and $\pi^2_\#\gamma|_\Omega = \nu$.

Figalli and Gigli [1] for the heat flow ($\alpha = 1$) and later Kim, Koo, and Seo [2] for the porous medium equation ($\alpha > 1$) showed the following. Consider the internal energy functional

$$E_\alpha(\mu) = \begin{cases} \int_\Omega U_\alpha(\rho)dx\ , & \mu = \rho\mathsf{Leb}|_\Omega\ , \\ +\infty\ , & \text{else}\ , \end{cases}\ ,$$

with

$$U_\alpha(s) = \begin{cases} s\big[\log s - \log\lambda - 1\big] + \lambda\ , & \alpha = 1, \lambda > 0\ , \\ \frac{s}{\alpha-1}\big[s^{\alpha-1} - \alpha\lambda^{\alpha-1}\big] + \lambda^\alpha\ , & \alpha > 1, \lambda \geq 0\ . \end{cases}$$

Then we have

Theorem 1 ([1,2]). *Solutions of the JKO-scheme*

$$\rho^\tau_{n+1} = \underset{\rho}{\operatorname{argmin}}\ E_\alpha(\rho) + \frac{1}{2\tau} Wb_2(\rho, \rho_n)^2$$

converge in Wb_2 as the time step τ goes to zero to weak solutions $(\rho_t)_t$ of (1), i.e. $t \mapsto \rho^{\alpha-1/2} - \lambda^{\alpha-1/2}$ belongs to $L^2_{\mathsf{loc}}\big([0,\infty), H^1_0(\Omega)\big)$ and it holds

$$\int_\Omega \phi(\rho_t - \rho_s) = \int_s^t \int_\Omega \Delta\phi \rho^\alpha_r dr \quad \forall \phi \in C^\infty_c(\Omega), s < t\ .$$

This is strong evidence that (1) should be regarded as the gradient flow of E_α with respect to Wb_2. We also mention the work of Profeta and Sturm [3] who give a description of the heat flow with boundary condition $\lambda = 0$ as a contraction of a larger auxiliary system of positive and negative densities which can be characterised as a gradient flow.

We show that the porous medium equation can indeed be characterised as the gradient flow of E_α w.r.t. Wb_2 in the sense of curves of maximal slope. To this end, we first give a dynamic characterisation of the transport distance Wb_2. We denote by $\mathcal{CE}^\Omega_T$ the set of all pairs (μ, v) of time-dependent measures and vectorfields such that

(i) $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}_2(\Omega)$ is vaguely continuous,
(ii) $\int_0^T \int_A |v_t| d\mu_t dt < \infty$ for all compact $A \subset \Omega$,
(iii) the continuity equation holds in the following sense:

$$\frac{d}{dt}\int \phi d\mu_t = \int \nabla\phi v_t d\mu_t \quad \forall \phi \in C^\infty_c(\Omega)\ .$$

Note that the choice of test functions in the continuity equation above allows for transport to and from the boundary. We obtain the following characterisation of absolutely continuous curves w.r.t. the distance Wb_p in terms of solutions to the continuity equation.

Theorem 2. *A curve $(\mu_t)_{t\in[0,T]}$ in $(\mathcal{M}_p(\Omega), Wb_p)$ is absolutely continuous if and only if there exists a Borel family $(v_t)_t$ of vector fields with*

$$\int_0^T \int |v_t|^p d\mu_t dt < \infty\ ,$$

such that $(\mu, v) \in \mathcal{CE}^\Omega_T$. In this case, the family of vector fields with minimal L^p-norm satisfies $|\mu'|(t) = ||v_t||_{L^p(\mu_t)}$ for a.e. $t \in [0,T]$, where $|\mu'|$ denotes the metric derivative w.r.t. Wb_p.

As an immediate consequence we obtain a dynamic characterisation of the distance Wb_p in the spirit of the Benamou-Brenier formula for the Wasserstein distance.

Corollary 1 (Benamou-Brenier formula)**.** *For $\mu_0, \mu_1 \in \mathcal{M}_p(\Omega)$ we have*

$$Wb_p(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int |v_t|^p d\mu_t dt \right\}$$

where the infimum is taken over all pairs $(\mu, v) \in \mathcal{CE}_1$ connecting μ_0 and μ_1.

We define the energy dissipation functional $\mathcal{I}_\alpha : \mathcal{M}_2(\Omega) \to [0, +\infty]$ as follows:

$$\mathcal{I}_\alpha(\mu) := \begin{cases} C(\alpha) \int_\Omega \left|\nabla\left(\rho^{\alpha-1/2}\right)\right|^2 & \text{if } \mu = \rho \mathsf{Leb}|_\Omega \text{ and } \rho^{\alpha-1/2} - \lambda^{\alpha-1/2} \in H_0^1(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

for a numerical constant $C(\alpha)$. Note that the boundary condition λ for the density of ρ of μ is encoded in finiteness of $\mathcal{I}_\alpha(\mu)$. We then obtain the following variational characterisation of the porous medium equation (1) with Dirichlet boundary condition λ.

Theorem 3. *For any curve $(\mu_t)_{t\in[0,T]}$ in $(\mathcal{M}_2(\Omega), Wb_2)$ with $E_\alpha(\mu_0) < +\infty$ we have*

$$\mathcal{L}_T(\mu) := E_\alpha(\mu_T) - E_\alpha(\mu_0) + \frac{1}{2}\int_0^T \Big[|\mu'|^2(r) + \mathcal{I}_\alpha(\mu_r)\Big]\, dr \geq 0\,.$$

Moreover, $\mathcal{L}_T(\mu_t) = 0$ if and only if $\mu_t = \rho_t \mathsf{Leb}|_\Omega$ with (ρ_t) a weak solution to the porous medium equation (1)*.*

In the framework of gradient flows in metric spaces the claim that $\mathcal{L}_T(\mu) \geq 0$ states that $\mathcal{I}_\alpha$ is a strong upper gradient of the functional E_α. The second claim states that weak solutions to (1) are precisely the curves of maximal slope w.r.t. this strong upper gradient. Note that the Dirichlet boundary condition is encoded through finiteness of $\mathcal{L}$ though the appearance of $\mathcal{I}_\alpha$. This is consistent with the observation that the De Giorgi functional $\mathcal{L}$ of a gradient flow PDE is strongly related with the path level large deviation rate functional of an underlying particle dynamics. In boundary driven particle systems leading to a macroscopic limit described by a PDE with Dirichlet boundary conditions, the rate function is typically infinite unless the boundary condition is satisfied for positive times.

We conclude by noting that the dissipation functional can be related to the metric slope of E_α w.r.t. Wb_2.

Proposition 1. *For any $\mu \in \mathcal{M}_2(\Omega)$ we have*

$$\mathcal{I}_\alpha(\mu) \leq |\nabla^- E_\alpha|(\mu) := \limsup_{\nu\to\mu} \frac{\big(E_\alpha(\mu) - E_\alpha(\nu)\big)^+}{Wb_2(\mu,\nu)}$$

In particular, this shows that finiteness of the metric slope $|\nabla^- E_\alpha|(\mu)$ implies that $\mu = \rho\mathsf{Leb}|_\Omega$ and ρ satisfies the Dirichlet boundary condition. Moreover, we note that by abstract results for gradient flows in metric spaces together with the last Proposition allow us to recover Theorem 1, i.e. the convergence of the JKO scheme to a a weak solution, from the variational characterisation in Theorem 3.

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Poincaré and Logarithmic Sobolev Inequalities for Brownian Motion with Sticky-Reflecting Boundary Diffusion

Max von Renesse

(joint work with Marie Bormann and Feng-Yu Wang)

Brownian motion on domains with sticky-reflecting boundary diffusion appears naturally as a microscopic model for heat flow in solids with surface coating or in interacting particle systems with sticky-reflecting zero-range interaction. A rigorous construction of the such processes can be given efficiently via Dirichlet forms, where both the invariant measure and the energy form are mixtures from corresponding bulk and boundary contributions [5, 6]. The question of the speed of convergence to equilibrium arises naturally. For convergence in quadratic mean this question was addressed in a previous work [10], where we derived upper bounds for the Poincaré constant under strict positivity assumptions on the Ricci curvature of the manifold and the second fundamental form of the boundary. The central method is an interpolation in the decomposition of the total variance into partial variances. The latter can then be estimated by the bulk energy through (variants of) the Steklov eigenvalue problem. In positive curvature one can get explicit quantitative bounds from application of the Reilly formula to the corresponding minimizers.

The talk presents new work [2] which extends the previous estimates in two ways. First we extend the interpolation method to the case of general curvature bounds. Instead of the Reilly formula the main tool in this case is based on integration by parts with a properly chosen test function of specific boundary behaviour and controlled energy contribution in the interior. As a side result we obtain new explicit estimates for the Steklov eigenvalue in this case. The second extension gives also bounds for the logarithmic Sobolev constant, where a similar type of interpolation in the decomposition of the entropy of mixtures is used. As another side result we obtain new explicit estimates for the norm of the boundary trace operator for Sobolev functions and a corresponding boundary trace logarithmic Sobolev inequality which was studied before in the special case of Euclidan balls by Beckner [1].

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Regularized Stein Variational Gradient Flow

Bharath K. Sriperumbudur

(joint work with Ye He, Krishnakumar Balasubramanian, Jianfeng Lu)

Given a potential function $V : \mathbb{R}^d \to \mathbb{R}$, the sampling problem involves generating samples from the density

$$\pi(x) := Z^{-1} e^{-V(x)}, \qquad \text{with} \qquad Z := \int_{\mathbb{R}^d} e^{-V(x)}\, dx$$

being the normalization constant, which is typically assumed to be unknown or hard to compute. The task of sampling arises in several fields of applied mathematics, including Bayesian statistics and machine learning in the context of numerical integration. There are two widely-used approaches for sampling: (i) diffusion-based *randomized* algorithms, which are based on discretizations of certain diffusion processes, and (ii) particle-based *deterministic* algorithms, which are discretizations of certain *approximate* gradient flows. A central idea connecting the two approaches is the seminal work [1] which provides a variational interpretation of the Langevin diffusion as the Wasserstein Gradient Flow (WGF),

$$\partial_t \mu_t = \nabla \cdot \big(\mu_t \, \nabla_{W_2} F(\mu_t)\big) = \nabla \cdot \left(\mu_t \, \nabla \log \frac{\mu_t}{\pi}\right),$$

where the term $\nabla_{W_2} F(\mu_t) = \nabla \log \frac{\mu_t}{\pi}$ is the Wasserstein gradient of the relative entropy functional (also called as the Kullback–Leibler divergence), defined by

$$F(\mu_t) = \mathrm{KL}(\mu_t|\pi) := \int_{\mathbb{R}^d} \log \frac{\mu_t(x)}{\pi(x)} \mu_t(x) dx,$$

evaluated at μ_t. This leads to the idea that sampling could be viewed as *optimization on the space of densities/measures.*

The Wasserstein gradient of the relative entropy, i.e., $\nabla \log \frac{\mu_t}{\pi}$ is related to the *Stein operator* by noting that, for any $v \in L_2^d(\mu_t)$,

$$\begin{aligned}
&\langle \nabla_{W_2} \mathrm{KL}(\mu_t|\pi), v \rangle_{L_2^d(\mu_t)} \\
&= \left\langle \nabla \log \frac{\mu_t}{\pi}, v \right\rangle_{L_2^d(\mu_t)} = \langle \nabla \log \mu_t, v \rangle_{L_2^d(\mu_t)} - \langle \nabla \log \pi, v \rangle_{L_2^d(\mu_t)} \\
&= -\int_{\mathbb{R}^d} \big(\nabla \cdot v + \langle \nabla \log \pi, v \rangle_2 \big) \, \mu_t(x) \, dx =: -\int_{\mathbb{R}^d} \mathcal{S}_\pi v \, d\mu_t,
\end{aligned}$$

where $\mathcal{S}_\pi$ is called the *Stein operator* and $L_2^d(\mu_t) := \{f = (f_1, \dots, f_d), f_i \in L_2(\mu_t), \forall i : \sum_{i=1}^d \|f_i\|^2_{L_2(\mu_t)} < \infty\}$. Since

$$\mathrm{KL}\left((I + hv)_\# \mu_t | \pi\right) = \mathrm{KL}(\mu_t|\pi) + h \langle \nabla_{W_2} \mathrm{KL}(\mu_t|\pi), v \rangle_{L_2^d(\mu_t)} + o(h),$$

we have

$$\nabla_{W_2} \mathrm{KL}(\mu_t|\pi) = -\arg \inf_{\|v\|_{L_2^d(\mu_t)} \le 1} \mathrm{KL}\big((I + hv)_\# \mu_t | \pi\big) = \arg \sup_{\|v\|_{L_2^d(\mu_t)} \le 1} \int_{\mathbb{R}^d} \mathcal{S}_\pi v \, d\mu_t.$$

Recently, in the machine learning community, the Stein Variational Gradient Descent (SVGD) [2,3] is proposed as a deterministic space-time discretization—in contrast to the Langevin diffusion which is a randomized space-time discretization of WGF—of the Stein Variational Gradient Flow (SVGF) [4] defined as

$$\partial_t \mu_t = \nabla \cdot \left(\mu_t \ \mathcal{T}_{k,\mu_t} \nabla \log \frac{\mu_t}{\pi} \right),$$

where $\mathcal{T}_{k,\mu} : L_2^d(\mu) \to L_2^d(\mu)$ is the integral operator defined as $\mathcal{T}_{k,\mu} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y) \mu(y) dy$ for any function $f \in L_2^d(\mu)$, and $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the reproducing kernel (r.k.) of a reproducing kernel Hilbert space, $\mathcal{H}$. By defining $\mathrm{id}_\mu : \mathcal{H}^d \to L_2^d(\mu)$, $f \mapsto f$ as the inclusion operator, it can be shown that $\mathcal{T}_{k,\mu} = \mathrm{id}_\mu \mathrm{id}_\mu^*$, which yields $\mathcal{T}_{k,\mu_t} \nabla \log \frac{\mu_t}{\pi} = \mathrm{id}_{\mu_t} \mathrm{id}_{\mu_t}^* \nabla \log \frac{\mu_t}{\pi}$, where

$$-\mathrm{id}_{\mu_t}^* \nabla \log \frac{\mu_t}{\pi} = \arg \sup_{\|v\|_{\mathcal{H}^d} \le 1} \int_{\mathbb{R}^d} \mathcal{S}_\pi v \, d\mu_t.$$

SVGD given by

$$x_i^{(n+1)} = x_i^{(n)} - \frac{h}{N} \sum_{j=1}^N k(x_i^{(n)}, x_j^{(n)}) \nabla V(x_j^{(n)}) - \nabla k(x_i^{(n)}, x_j^{(n)}), \ i = 1, \dots, N$$

is an interactive particle system (unlike Langevin diffusion), where N is the number of particles, $h > 0$ is the step-size, and n is the time index. However, SVGD (which is based on SVGF) only provides a discretization of a *constant-order approximation* to WGF due to the presence of the kernel integral operator in its vector field. Indeed, if $\operatorname{supp}(\mu_t) = \mathbb{R}^d$ and k is a bounded continuous translation invariant *characteristic* kernel [5] on $\mathbb{R}^d$ (e.g., Gaussian and Laplacian kernels), then

$$\|\mathcal{T}_{k,\mu_t} - I\|_{\mathrm{op}} = \sup\{\|\mathcal{T}_{k,\mu_t} f - f\|_{L_2^d(\mu_t)} : \|f\|_{L_2^d(\mu_t)} = 1\} \geq \|\mathcal{T}_{k,\mu_t}\mathbf{1} - \mathbf{1}\|_{L_2^d(\mu_t)}$$
$$\geq \|1 - \textstyle\int k(\cdot, x)\mu_t(x)\, dx\|_{L_2(\mu_t)} > 0,$$

where $\mathbf{1} = (1, \overset{d}{\ldots}, 1)^\top$.

To overcome the above issue with the SVGF, we propose the Regularized Stein Variational Gradient Flow (R-SVGF) [6] where the vector field is obtained as

$$-\arg \sup_{(1-\nu)\|v\|^2_{L_2^d(\mu_t)} + \nu\|v\|^2_{\mathcal{H}^d} \leq 1} \int_{\mathbb{R}^d} \mathcal{S}_\pi v \, d\mu_t, \tag{1}$$

where $0 \leq \nu \leq 1$ interpolates between WGF and SVGF. Clearly, $\nu = 0$ corresponds to the vector field in WGF while $\nu = 1$ yields that of SVGF. The vector field in (1) can be shown to be $((1-\nu)\mathcal{T}_{k,\mu_t} + \nu I)^{-1}\mathcal{T}_{k,\mu_t}\nabla \log(\mu_t/\pi)$ when seen as an element of $L_2^d(\mu_t)$, which satisfies

$$\|((1-\nu)\mathcal{T}_{k,\mu_t} + \nu I)^{-1}\mathcal{T}_{k,\mu_t}\nabla \log(\mu_t/\pi) - \nabla \log(\mu_t/\pi)\|_{L_2^d(\mu_t)} \to 0 \quad \text{as} \quad \nu \to 0$$

if $\nabla \log(\mu_t/\pi) \in \overline{\operatorname{Ran}(\mathcal{T}_{k,\mu_t})}$. Additionally, if $\nabla \log(\mu_t/\pi)$ is sufficiently smooth, i.e., there exists $\gamma \in \left(0, \frac{1}{2}\right]$ such that $\nabla \log(\mu_t/\pi) = \mathcal{T}^\gamma_{k,\mu_t} h$, for some $h \in L_2^d(\mu_t)$, then

$$\|((1-\nu)\mathcal{T}_{k,\mu_t} + \nu I)^{-1}\mathcal{T}_{k,\mu_t}\nabla \log(\mu_t/\pi) - \nabla \log(\mu_t/\pi)\|_{L_2^d(\mu_t)} = O(\nu^{2\gamma}) \quad \text{as} \quad \nu \to 0.$$

In other words, $((1-\nu)\mathcal{T}_{k,\mu_t} + \nu I)^{-1}\mathcal{T}_{k,\mu_t}\nabla \log(\mu_t/\pi)$ is a good approximation to $\nabla \log(\mu_t/\pi)$ for small ν. With this motivation, the corresponding gradient flow

$$\partial_t \mu_t = \nabla \cdot \left(\mu_t \ \left((1-\nu)\mathcal{T}_{k,\mu_t} + \nu I\right)^{-1} \mathcal{T}_{k,\mu_t}\left(\nabla \log \frac{\mu_t}{\pi}\right)\right), \tag{2}$$

is referred to as R-SVGF, where $\nu \in (0, 1]$. Clearly, R-SVGF interpolates between WGF and SVGF. The key advantage is that (2), which approximates WGF, can be discretized to yield a deterministic interacting particle system (similar to that of SVGD but with modifications involving the inverse of regularized Gram matrix), R-SVGD:

$$\bar{x}_{n+1} = \bar{x}_n - h_{n+1}\mathbf{K}_n^{-1}\left(\frac{1}{N}K_n(L_n\nabla V) - \frac{1}{N}\sum_{j=1}^N L_n \nabla k(x_j^{(n)}, \cdot)\right)$$

where $(h_n)_{n=1}^\infty$ is the sequence of step-sizes, $\bar{x}_n = [x_1^{(n)}, \cdots, x_N^{(n)}]^T$,

$$\mathbf{K}_n := \left(\frac{(1-\nu_{n+1})}{N}K_n + \nu_{n+1}I_N\right)$$

with K_n being the Gram matrix, $(K_n)_{ij} = k(x_i^{(n)}, x_j^{(n)})$ for all $i, j \in \{1, \ldots, N\}$, and $L_n f := [f(x_1^{(n)}), \ldots, f(x_N^{(n)})]^\top$ for $f : \mathbb{R}^d \to \mathbb{R}^N$.

Our contributions in this work [6] are as follows:

(1) For the R-SVGF, we provide rates of convergence to the target density, π in two cases: (i) in the Fisher Information metric under no further assumptions on π and (ii) in the relative entropy under an LSI (log Sobolev inequality) assumption on π. We also establish similar results for the time-discretized R-SVGF.
(2) We characterize the existence, uniqueness, and stability of the solutions to the R-SVGF in the mean-field limit.
(3) We provide preliminary numerical experiments demonstrating the superiority of R-SVGD over SVGD in estimating certain functionals involving π based on their respective particle approximations.

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The superposition principle for BV curves of measures

Riccarda Rossi

(joint work with Stefano Almi, Giuseppe Savaré)

The evolution equations of diffusive type whose gradient-flow structure, in the space of probability measures metrized by the Wasserstein distance, was unveiled by Jordan, Kinderleher & Otto more than 25 years ago, all share a common structure. The cornerstone of such structure is the continuity equation

$$\partial_t \mu + \mathrm{div}(\boldsymbol{\nu}) = 0 \qquad \text{in } (0,T)\times\mathbb{R}^d, \tag{1a}$$

where $\mu = (\mu_t)_{t\in(0,T)}$ is a Borel family of probability measures on $(0,T)$, and the flux measure $\boldsymbol{\nu}$ disintegrates into a family of measures absolutely continuous with respect to μ_t, namely

$$\boldsymbol{\nu}_t = \boldsymbol{v_t}\mu_t \qquad \text{for } \mathscr{L}\text{-a.a. } t\in(0,T). \tag{1b}$$

The vector field $\boldsymbol{v} : (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ is usually referred to as *velocity field*; equation (1a) is understood in the sense of contributions. The central role of (1) in the variational approach to diffusion has motivated a thorough study of its properties. In particular, we mention the deep results in [1, Chap. 8], where *(i)* it was proved that *The continuity equation characterizes absolutely continuous curves of measures with values in Wasserstein spaces*. More precisely, it was shown in [1, Thm. 8.3.1] that, in the case $p > 1$, for any given curve of probability measures (with finite pth-moment) $\mu : [0,T] \to \mathcal{P}_p(\mathbb{R}^d)$, p-absolutely continuous w.r.t. to the Wasserstein metric W_p with (W_p)-metric derivative $|\mu'| \in \mathrm{L}^p(0,T)$, there exists a velocity field $\boldsymbol{v} : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ such that $\boldsymbol{v}_t \in \mathrm{L}^p(\mathbb{R}^d;\mu_t)$ for $\mathcal{L}$-a.a. $t \in (0,T)$, the pair $(\mu, \boldsymbol{v})$ solves the continuity equation (1), and the velocity field satisfies the 'optimality condition'

$$\|\boldsymbol{v}_t\|_{\mathrm{L}^p(\mathbb{R}^d;\mu_t)} \le |\mu'|(t) = \lim_{h\to 0} \frac{W_p(\mu_t, \mu_{t+h})}{|h|} \qquad \text{for } \mathcal{L}\text{-a.a. } t \in (0,T). \tag{2}$$

Conversely, in [1, Thm. 8.3.1] it was also proved that for any solution $(\mu, \boldsymbol{v})$ of the continuity equation, the curve $(0,T) \ni t \mapsto \mu_t \in \mathcal{P}_p(\mathbb{R}^d)$ is p-absolutely continuous and (2) holds as an equality.

(ii) *A probabilistic representation of solutions of the continuity equation via the superposition principle* was provided.

We have extended the above results to curves of measures with values in $W_1(\mathbb{R}^d)$, that are just with bounded variation as functions of time. Simple examples show that it is not to be expected that, with a curve $\mu \in \mathrm{BV}([0,T]; W_1(\mathbb{R}^d))$, a flux measure $\boldsymbol{\nu}$ *absolutely continuous* w.r.t. μ as in (1b) may be associated. We have thus focused on the investigation of (1a) *per se*, understanding as solution of (1a) a pair $(\mu, \boldsymbol{\nu})$ such that

- μ is a finite positive Borel measure on $(0,T) \times \mathbb{R}^d$;
- the flux measure $\boldsymbol{\nu}$ has finite variation on $(0,T) \times \mathbb{R}^d$;
- $(\mu, \boldsymbol{\nu})$ solve (1a) in the distributional sense.

Hence, we have proved the following analogue of [1, Thm. 8.3.1], namely that *For any $\mu \in \mathrm{BV}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ there exists a Borel measure $\boldsymbol{\nu} \in \mathcal{M}([0,T]\times\mathbb{R}^d; \mathbb{R}^d)$ solving the continuity equation* (1a) *in the sense above specified, such that, moreover,*

$$|\boldsymbol{\nu}|([0,T]\times\mathbb{R}^d) = \mathrm{Var}_{W_1}(\mu; [0,T]) \tag{3}$$

(which is the counterpart to (2)*), and the singular part of $\boldsymbol{\nu}$ w.r.t. μ, $\boldsymbol{\nu}^\perp$, is minimal in a suitable sense. Conversely, let $\mu \in \mathcal{M}^+([0,T]\times\mathbb{R}^d)$ and $\boldsymbol{\nu} \in \mathcal{M}([0,T]\times\mathbb{R}^d; \mathbb{R}^d)$ solve the continuity equation* (1a) *in the* BV *sense. Then, μ disintegrates w.r.t. the Lebesgue measure $\mathcal{L}$ on $(0,T)$, i.e. $\mu = \int_0^T \mu_t \, \mathrm{d}t$, such that the curve $(0,T) \ni t \mapsto \mu_t$ is in $\mathrm{BV}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$, and* (3) *holds.*

We have also provided a 'BV' counterpart to the superposition principle, by

- associating with (1a) a continuity equation in an augmented state space,
- resorting to the superposition principle for the 'augmented continuity equation'
- obtaining therefrom a probabilistic representation for the measures μ and $\boldsymbol{\nu}$ in terms of trajectories with values in the extended phase space.

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Graph limits for nonlocal interaction PDEs

Antonio Esposito

(joint work with Georg Heinze, Francesco Patacchini, André Schlichting, Dejan Slepčev)

The study of evolution equations on graphs and networks has been receiving increasing interest in view of possible applications in several real-world phenomena where individuals interact if they are interconnected in specific ways. In social networks, for example, one can model the spread of opinions, or behaviours, by assigning probabilities for individuals to adopt certain attitudes based on their neighbours' choices. This is useful to model polarisation and formation of echo chambers, cf. for example [1]. Another possible application concerns transportation networks, where the flux from one vertex to a connected one depends on some scalar quantities at the neighbour vertices, see e.g. [7]. Graphs are also used in applications to data science, as they are indeed a suitable mathematical structure to classify and represent data by studying clustering, as in [6, 8] and the references therein. In [5], we introduce nonlocal dynamics relevant to detecting local concentrations in networks. The class of partial differential equations (PDEs) we consider can be specified through three elements: a *nonlocal continuity equation*, an *upwind flux interpolation*, and a *constitutive relation* for a nonlocal velocity. The nonlocal continuity equation is concerned with the time-evolution of a probability measure $\rho_t \in \mathcal{P}(\mathbb{R}^d)$, for $t \in [0,T]$, where mass located at a vertex $x \in \mathbb{R}^d$ can be nonlocally transported to $y \in \mathbb{R}^d$ along a channel with capacity, referred to as weight, given by an edge weight function $\eta : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to [0,\infty)$. The nonlocal continuity equation on a time interval $[0,T]$ is of the form

$$\partial_t \rho_t + \overline{\mathrm{div}} j_t = 0, \qquad \text{with} \quad \overline{\mathrm{div}} j_t(dx) = \int_{\mathbb{R}^d \setminus \{x\}} \eta(x,y)\, dj_t(x,y), \tag{1a}$$

where the flux is a time-dependent antisymmetric measure, $j_t \in \mathcal{M}(G)$, on the set $G = \{(x,y) \in \mathbb{R}^{2d} \setminus \{x = y\} : \eta(x,y) > 0\}$.

The relation constituting the flux depends on a σ-finite absolutely continuous measure $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, wherein μ acts as an abstract notion of vertices of a graph. More precisely, we associate to a nonlocal time-dependent velocity field $v_t : G \to \mathbb{R}$ the induced flux by using an upwind interpolation as follows

$$dj_t(x,y) = v_t(x,y)_+ d(\rho \otimes \mu)(x,y) - v_t(x,y)_- d(\mu \otimes \rho)(x,y). \tag{1b}$$

Here, for $a \in \mathbb{R}$, we denote with $a_+ = \max\{a, 0\}$ and $a_- = \max\{-a, 0\}$ the positive and negative part, respectively. Intuitively, the support of μ defines the underlying set of vertices, i.e. $V = \operatorname{supp} \mu$. In particular, any finite graph can be represented by choosing $\mu = \mu^N = \delta_{x_i}/N$, for $x_1, x_2, \dots, x_N \in \mathbb{R}^d$.

The last element is the identification of the velocity field in terms of a symmetric interaction potential $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and a potential $P : \mathbb{R}^d \to \mathbb{R}$ by

$$v_t(x,y) = -\overline{\nabla} K * \rho_t(x,y) - \overline{\nabla} P(x,y), \tag{1c}$$

where the nonlocal gradient is defined by $\overline{\nabla} f(x,y) := f(y) - f(x)$.

In [5] we show that system (1) is a *Finslerian* gradient flow of the interaction energy

$$\mathcal{E}(\rho) = \frac{1}{2} \iint_{\mathbb{R}^{2d}} K(x,y) d\rho(y) d\rho(x) + \int_{\mathbb{R}^d} P(x) d\rho(x), \tag{2}$$

with respect to a nonlocal Wasserstein quasi-metric based on the upwind interpolation. In this framework we show existence of weak solutions, curves of maximal slope with respect to a specific strong upper gradient, and estabilish a discrete-to-continuum limit as the number of vertices n goes to ∞, so called *graph limit*. Different types of flux interpolations are considered in [4].

An intriguing problem is to understand the limiting behaviour of weak solutions to (1) as the graph structure localises, i.e. the range of connection between vertices decreases, while the weight of each connecting edge increases, so called *graph-to-local limit*. One expects to approximate weak solutions of the more standard nonlocal interaction equation on $\mathbb{R}^d$. However, the intrinsic geometry of the graph impacts the limiting gradient structure of the equation. Accordingly, the main goal of [3] is to provide a rigorous proof of the local limit of the system (1) along a sequence of edge weight functions $\eta^\varepsilon : \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \to [0, \infty)$ defined by

$$\eta^\varepsilon(x,y) := \frac{1}{\varepsilon^{d+2}} \vartheta\left(\frac{x+y}{2}, \frac{x-y}{\varepsilon}\right), \tag{3}$$

in terms of a reference connectivity $\vartheta : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ satisfying suitable assumptions. The scaling in (3) leads to the local evolution

$$\partial_t \rho_t = \operatorname{div}(\rho_t \mathbb{T}(\nabla K * \rho_t + \nabla P)), \tag{$\mathsf{NLIE}_{\mathbb{T}}$}$$

where the *tensor* $\mathbb{T} : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ depends on the nonlocal structure encoded through the reference measure μ and the connectivity ϑ.

Following a heuristic argument based on several approximations and smoothness assumptions, which are not a priori satisfied by solutions to (1) and ($\mathsf{NLIE}_{\mathbb{T}}$), one can show the link between the two equations. This is made rigorous in [3] by using a variational framework, allowing to handle measure-valued solutions.

An interesting byproduct of this result is the link between Finslerian and Riemannian gradient flows. More precisely, (1) is shown to be a gradient flow of the nonlocal interaction energy in the infinite-dimensional Finsler manifold of probability measures endowed with a nonlocal upwind transportation quasi-metric, $\mathcal{T}$, peculiar of the upwind interpolation (1b). Due to the loss of symmetry the underlying structure of $\mathcal{P}(\mathbb{R}^d)$ does not have the formal Riemannian structure, but Finslerian instead. On the other hand, following [9], we establish a chain-rule inequality for the nonlocal interaction energy in a 2-Wasserstein space defined over

$\mathbb{R}^d_{\mathbb{T}}$, which is $\mathbb{R}^d$ endowed with a metric induced by $\mathbb{T}^{-1}$. Upon considering the corresponding *Wasserstein scalar product* on the tangent space of $\mathcal{P}_2(\mathbb{R}^d_{\mathbb{T}})$, at some probability measure with bounded second moment, one can notice the underlying Riemannian structure, thereby making the connection between the weak and variational formulations of ($\mathsf{NLIE}_{\mathbb{T}}$). We stress that not only do we connect the graph and tensorized local gradient structures using the notion of curves of maximal slope for gradient flows after De Giorgi, but, upon identifying weak solutions of ($\mathsf{NLIE}_{\mathbb{T}}$) with curves of maximal slopes, we also obtain an existence result for ($\mathsf{NLIE}_{\mathbb{T}}$) via stability of gradient flows. This is indeed another interesting property of the graph, as it represents a valuable space-discretisation for the PDE under study, working in any dimension, in addition to other methods, e.g. particle approximations and tessellations. Indeed, our result can be also seen as a deterministic approximation of ($\mathsf{NLIE}_{\mathbb{T}}$). The results in [3] are extended to the multi-species case in [2].

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An adversarial mean curvature flow

Leon Bungert

(joint work with Tim Laux, Kerrek Stinson)

In this talk we discuss how mean curvature flows appear in the context of training adversarially robust classifiers in machine learning. Such classifiers can be obtained using an algorithm called adversarial training. To set the scene, let $\Omega \subset \mathbb{R}^d$ denote an open and bounded set and let $\mu \in \mathcal{P}(\Omega \times \{0, 1\})$ be a probability measure, modeling the distribution of training data with the labels 0 and 1 in Ω. Adversarial training finds a binary classifier A in the Borel subset $\mathfrak{B}(\Omega)$ of Ω by solving the following robust optimization problem

$$\inf_{A\in\mathfrak{B}(\Omega)} \mathbb{E}_{(x,y)\sim\mu}\left[\sup_{\tilde{x}\in B_\varepsilon(x)} |1_A(\tilde{x})-y|\right], \tag{1}$$

parameterized by the so-called adversarial budget $\varepsilon > 0$.

$$\mathbb{E}_{(x,y)\sim\mu}\left[\sup_{\tilde{x}\in B_\varepsilon(x)} |1_A(\tilde{x})-y|\right] = \mathbb{E}_{(x,y)\sim\mu}\left[|1_A(x)-y|\right] + \varepsilon\,\mathrm{Per}_\varepsilon(A;\mu), \tag{2}$$

where $\mathrm{Per}_\varepsilon(A;\mu)$ denotes a nonlocal perimeter functional. This rewriting, the asymptotic results in [3], and the fact that $\mathrm{Per}_\varepsilon(\cdot;\mu)$ Gamma-converges to a local perimeter as $\varepsilon \to 0$ [2] suggest that (1) can be interpreted as time discretization of mean curvature flow, where ε acts both as time step and non-locality scale of the perimeter functional. To make this connection rigorous, we consider the following iteration for $k \in \mathbb{N}_0$:

$$A_0 \in \operatorname*{arg\,min}_{A\in\mathfrak{B}(\Omega)} \mathbb{E}_{(x,y)\sim\mu}\left[|1_A(x)-y|\right] \tag{3a}$$

$$A_{k+1} \in \operatorname*{arg\,min}_{A\in\mathfrak{B}(\Omega)} \int_\Omega |1_A(x)-1_{A_k}(x)|\,\mathrm{dist}(x,\partial A_k)\,\mathrm{d}\varrho(x) + \varepsilon\,\mathrm{Per}_\varepsilon(A;\mu). \tag{3b}$$

Here the set A_0 in (3a) is a so-called Bayes classifier which acts as initial condition. Starting from there, the iteration (3b) performs adversarial training using the label distribution from the previous classifier A_k and modifying (2) by means of the distance function $\mathrm{dist}(\cdot,\partial A_k)$ to the decision boundary of the previous classifier. The probability measure $\varrho \in \mathcal{P}(\Omega)$ in (3b) is the first marginal of μ, that is $\varrho := \mu(\cdot \times \{0,1\})$. The presence of the distance function is necessary to obtain the correct normal velocity for mean curvature flow [7]. Since the minimization problem in (3b) does not have unique solutions, we take the approach of [5] to select a solution using a strongly convex minimization problem. For this we replace (3b) by $A_{k+1} := T_\varepsilon(A_k)$, where the operator $T_\varepsilon : \mathfrak{B}(\Omega) \to \mathfrak{B}(\Omega)$ is defined as

$$\begin{aligned} &T_\varepsilon(A) := \{u^* \le 0\} \qquad \text{where } u^* \text{ solves} \\ &u^* := \operatorname*{arg\,min}_{u\in L^2(\Omega)} \frac{1}{2}\int_\Omega |u(x)-\mathrm{sdist}(x,A)|^2\,\mathrm{d}\varrho(x) + \varepsilon\,\mathrm{TV}_\varepsilon(u;\mu). \end{aligned} \tag{4}$$

Here $\mathrm{TV}_\varepsilon(u;\mu) := \int_{\mathbb{R}} \mathrm{Per}_\varepsilon(\{u \ge t\};\mu)\,\mathrm{d}t$ denotes a total variation functional and $\mathrm{sdist}(\cdot,A_k) := \mathrm{dist}(\cdot,A_k) - \mathrm{dist}(\cdot,\mathbb{R}^d \setminus A_k)$ is the signed distance function of the set A_k. It can indeed be shown that $A_{k+1} := T_\varepsilon(A_k)$ is a solution of the minimization problem in (3b).

The goal is to prove that (4) is a monotone and consistent scheme for the weighted mean curvature flow $t \mapsto A(t)$ with normal velocity

$$v(t) := -\frac{1}{\varrho}\,\mathrm{div}\left(\varrho\,\nu_{A(t)}\right), \tag{5}$$

where $\nu_{A(t)}$ is the outer unit normal to the boundary $\partial A(t)$. The abstract results of [6] then imply that (3) converges to a so-called barrier solution of the weighted mean curvature flow as $\varepsilon \to 0$.

Monotonicity of (4) in the sense of set inclusion (i.e., $A \subset B$ implies $T_\varepsilon(A) \subset T_\varepsilon(B)$) is a straightforward consequence of a comparison principle for the minimization problem in (4).

To verify consistency, one works with smooth super- / subflows, i.e., smooth evolutions of smooth sets $t \mapsto A(t)$ which move strictly faster / slower than mean curvature flow. Consistency of T_ε then means that for $\varepsilon > 0$ small enough it holds $T_\varepsilon(A(t)) \supset A(t+\varepsilon)$, meaning that the superflow also moves strictly faster than the scheme (and vice versa for subflows).

To show this we follow the strategy developed in [4] and utilize a superflow $t \mapsto A(t)$ in order to construct a supersolution of the minimization problem in (4). The signed distance function $d(t,x) := \operatorname{sdist}(x, A(t))$ of a smooth superflow satisfies the partial differential inequality

$$\partial_t d(t,x) > \frac{1}{\varrho(x)} \operatorname{div}\left(\varrho(x)\nabla d(t,x)\right), \quad x \in \partial A(t). \tag{6}$$

Considering the rescaled function $v_\varepsilon(x) := \psi(d(t+\varepsilon, x))$ with an appropriately chosen function $\psi : \mathbb{R} \to \mathbb{R}$ that satisfies $\psi(s) \geq s$ for all $s \in \mathbb{R}$ and $\psi(s) = s$ for $|s|$ small, one then gets for small $\varepsilon > 0$ that:

$$v_\varepsilon(x) - d(t,x) > \frac{1}{\varrho(x)} \operatorname{div}\left(\varrho(x)\frac{\nabla v_\varepsilon(x)}{|\nabla v_\varepsilon(x)|}\right). \tag{7}$$

The main ingredient for proving consistency is a careful analysis of the subdifferential of the total variation $u \mapsto \mathrm{TV}_\varepsilon(u;\mu)$ and the proof that for functions u with non-vanishing gradient and $\varepsilon \to 0$ it is consistent with the 1-Laplace operator $-\operatorname{div}\left(\varrho\frac{\nabla u}{|\nabla u|}\right)$. Applying this expansion to the function $u = v_\varepsilon$ yields that on a neighborhood of the interface $\partial A(t)$ we have

$$\left(v_\varepsilon(x) - d(t,x)\right)\varrho(x) + \varepsilon p(x) \geq 0 \tag{8}$$

for a subgradient $p \in \partial\, \mathrm{TV}_\varepsilon(v_\varepsilon;\mu)$. Inequality (8) together with a careful analysis of boundary conditions imply that v_ε is a supersolution of the problem (4) which implies $T_\varepsilon(A(t)) \supset \{v_\varepsilon \leq 0\} = \{d(\cdot, t+\varepsilon) \leq 0\} = A(t+\varepsilon)$ and hence consistency.

These results have interesting theoretical and practical implications: First, the minimizing movement scheme (3) can be modified to build discrete approximations to mean curvature flow on grids. The corresponding discrete perimeter functional was already investigated in [2]. Following a similar approach as in the continuum setting, we expect that such schemes are also monotone and consistent with mean curvature flow, yielding a novel discretization method. Second, on the applied side, the schemes (3b) or (4) give rise to novel adversarial training methods that involve the distance function to the decision boundary. The interpretation of (3b) is that—in contrast to (2)—the adversarial budget ε is replaced by a data dependent budget $\frac{\varepsilon}{\operatorname{dist}(x,\partial A_k)}$ which becomes large for points x which are close to the decision boundary and small for points far away.

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Minimizing movements along families of energies and dissipations

Andrea Braides

This is a short user's guide for the analysis of evolution for families of energies and dissipations, and its connection with Γ-convergence. For applications I have found it convenient to use a flexible version of the minimizing-movement approach as in the following definition.

Definition (see e.g. [5]). Given sequences $\varepsilon = \varepsilon_k \to 0$ and $\tau = \tau_k \to 0$ of positive numbers, X a topological space, $F_\varepsilon\colon X \to \mathbb{R}\cup\{+\infty\}$ and $D_\varepsilon\colon X\times X \to [0,+\infty]$, a *minimizing movement along F_ε with dissipations D_ε at scale τ* with initial data $u_0^\varepsilon \to u_0$ is a function $u\colon [0,+\infty) \to X$ such that $u(t)$ is the pointwise limit of $u^k(t)$ for all $t\geq 0$, and $u^k(t) = u^k_{\lfloor t/\tau_k\rfloor}$, where $u_0^k = u_0^{\varepsilon_k}$ and u_i^k is a minimizer of $v\mapsto F_{\varepsilon_k}(v) + \frac{1}{\tau_k}F_{\varepsilon_k}(v, u^k_{i-1})$.

We note that the usual conditions ensuring the existence of a minimizing movement for a single functional and dissipation (see [2]) allow to prove the existence of a minimizing movement along F_ε with dissipations D_ε. In particular this is achieved when (X,d) is a complete metric space, F_ε are lower semicontinuous and coercive, and $D_\varepsilon(u,v) = \frac12 d^2(u,v)$ (in which case we use the shorthand $D_\varepsilon = \frac12 d^2$). However that is an extremely abstract result and must be coupled with some characterization of u. A way to characterize u is in terms of curves of maximal slope.

Theorem (commutativity in terms of curves of maximal slope). *Let (X,d) be a complete metric space, $D_\varepsilon = \frac12 d^2$, F_ε a family of functionals, and F_0 a functional on X such that the following property holds.*

(H) *if $v_\varepsilon \to v$ is such that $\sup_\varepsilon(F_\varepsilon(v_\varepsilon) + |\partial F_\varepsilon|(v_\varepsilon)) < +\infty$, then we have $\lim_{\varepsilon\to0} F_\varepsilon(v_\varepsilon) = F_0(v)$ and $\liminf_{\varepsilon\to0} |\partial F_\varepsilon|(v_\varepsilon) \geq |\partial F_0|(v)$.*

Then every minimizing movement along F_ε is a curve of maximal slope for F_0.

We note that, while implied by convexity (and valid also under more general assumptions), condition (H) is unlikely to hold when we have many local minima

v_ε, for which $|\partial F_\varepsilon|(v_\varepsilon) = 0$. Indeed, if such families are dense, we obtain that $|\partial F_0|$ is identically 0, and F_0 is a constant.

Link with Γ-convergence. If $|\partial F_\varepsilon|$ are equibounded in a neighbourhood of v then $F_0(v)$ coincides with the Γ-limit of F_ε at v. An example by M. Solci shows that this equality in general may fail at all points even if $|\partial F|$ is everywhere finite (see [5]). Here and below we give as understood that Γ-limits are computer with respect to the topology of X.

We now consider the minimizing-movement scheme in terms of convergence of minimum problems, which are compatible with Γ-convergence.

Theorem (extreme regimes). *Let (X, d) be a complete metric space, let F_ε be a equi-coercive family, $D_\varepsilon = \frac{1}{2}d^2$, and $u^0_\varepsilon \to u^0$, and let u be a minimizing movement along F_ε with dissipations D_ε at scale τ with initial data $u^\varepsilon_0 \to u_0$. Then*

(i) *there exists $\underline{\tau}_\varepsilon$ such that if $\tau_k \leq \underline{\tau}_{\varepsilon_k}$ then any such minimizing movement u is a limit of minimizing movements for F_{ε_k} with initial datum $u^{\varepsilon_k}_0$;*

(ii) *there exists $\overline{\tau}_\varepsilon$ such that, if $\tau_k \geq \overline{\tau}_{\varepsilon_k}$ and F_0 is the Γ-limit of F_{ε_k}, any such minimizing movement u is a minimizing movements for F_0 with initial datum u_0.*

Critical scales. If the minimizing movements in the two extreme cases described by items (i) and (ii) above do not coincide, then there exist one or more *critical scales* at which we have a "change of regime". The simplest such case is when the domain of F_ε is a discrete space, in which the only possible minimizing movements in regime (i) are constant (*pinning*). In this case there exists a minimal scale $\tau = \tau_\varepsilon$ for which the evolution is not trivial for some initial datum (*depinning regime*). Conversely, if the minimizing movements in cases (i) and (ii) coincide, it is not clear if all possible u are characterized by (ii) (or (i)).

The following result states that in the convex case minimizing movements are independent of the scale (see [3]).

Theorem (the convex case) *Let (X, d) be a complete metric space, let F_ε be a equi-coercive family of convex energies and $D_\varepsilon = \frac{1}{2}d^2$, and $u^0_\varepsilon \to u^0$. Let F_0 be the Γ-limit of F_{ε_k}. Then every minimizing movement u along F_ε with dissipations D_ε at scale τ with initial data $u^\varepsilon_0 \to u_0$ is a minimizing movement u for F_0 with initial datum u_0, and is also a limit of minimizing movements for F_{ε_k} with initial data $u^{\varepsilon_k}_0$.*

This theorem states in a sense that the convex case is 'trivial' since the limit is the same at all scales. Nevertheless it may be useful to characterize the limits of gradient flows of convex energies through the study of their discrete-in-time approximations obtained by solving the Euler-Lagrange equations of the incremental problems. As such it has been applied for example to prove the convergence of non-local gradient flows to standard parabolic equations [1], and of gradient flows of double-porosity models to parabolic equations with memory [4].

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Diffusive transport: geodesics, convexity, and gradient flows

Daniel Matthes

(joint work with Eva-Maria Rott, André Schlichting, Giuseppe Savaré)

1. The diffusive transport metric

On the space $X := \{\rho \in L^1(\mathbb{S}^1) | \rho \geq 0, \int \rho\, dx = 1\}$ of probability densities on the circle, introduce the Hellinger distance $\mathbb{H}$, the L^2-Wasserstein metric $\mathbb{W}$, and the diffusive transport distance $\mathbb{D}$, respectively, by

$$\mathbb{H}(\rho_0, \rho_1)^2 = \inf \left\{ \int_0^1 \int_{\mathbb{S}^1} \frac{w_s^2}{\rho_s}\, dx\, ds \,\middle|\, \partial_s \rho_s - w_s = 0 \right\},$$

$$\mathbb{W}(\rho_0, \rho_1)^2 = \inf \left\{ \int_0^1 \int_{\mathbb{S}^1} \frac{w_s^2}{\rho_s}\, dx\, ds \,\middle|\, \partial_s \rho_s + \partial_x w_s = 0 \right\},$$

$$\mathbb{D}(\rho_0, \rho_1)^2 = \inf \left\{ \int_0^1 \int_{\mathbb{S}^1} \frac{w_s^2}{\rho_s}\, dx\, ds \,\middle|\, \partial_s \rho_s - \partial_{xx} w_s = 0 \right\},$$

where the infima are taken over all parametrized pairs $(\rho_s, w_s)_{s\in[0,1]}$ of probability densities ρ_s and Radon measures w_s on $\mathbb{S}^1$, respectively, that connect ρ_0 to ρ_1 by means of the (generalized) continuity equation. It is known that $(X, \mathbb{H})$ is a complete metric space with the L^1-topology, and that $(\bar{X}, \mathbb{W})$ is a complete metric space, where the completion $\bar{X}$ is the space of probability measures on $\mathbb{S}^1$, with the narrow topology. We show:

Theorem 1. $(\bar{X}, \mathbb{D})$ *is a complete metric space with the narrow topology. More precisely:*

$$\big\|\mu_1 - \mu_0\big\|_{(\dot{W}^{2,\infty}(\mathbb{S}^1))'} \leq \mathbb{D}(\rho_0, \rho_1) \leq \frac{2}{-\log \big\|\mu_1 - \mu_0\big\|_{(\dot{H}^1(\mathbb{S}^1))'}}.$$

Geodesics w.r.t $\mathbb{D}$ are currently little understood. Formally, the geodesic equations for $\mathbb{H}$, $\mathbb{W}$ and $\mathbb{D}$ read, respectively, as follows:

$$\partial_s \rho_s - \rho_s \,\psi_s = 0, \quad \partial_s \psi_s + \frac{1}{2}\psi_s^2 = 0,$$

$$\partial_s \rho_s + \partial_x(\rho_s \,\partial_x \psi_s) = 0, \quad \partial_s \psi_s + \frac{1}{2}(\partial_x \psi_s)^2 = 0,$$

$$\partial_s \rho_s - \partial_{xx}(\rho_s \,\partial_{xx} \psi_s) = 0, \quad \partial_s \psi_s + \frac{1}{2}(\partial_{xx} \psi_s)^2 = 0.$$

While the first system is solvable by plain linear interpolation w.r.t. $\sqrt{\rho_s}$, and the second one is solvable in principle by the method of characteristics, the third one appears inaccessible to explicit solution.

2. Contractive and gradient flows

Observation 1. *The linear diffusion equation $\partial_t \rho = \partial_{xx}\rho$ induces on X ...*

- *... a contractive flow w.r.t. $\mathbb{H}$,*
- *... a contractive gradient flow w.r.t. $\mathbb{W}$,*
- *... a contractive flow w.r.t. $\mathbb{D}$.*

The contractivity properties are essentially consequences of Jensen's inequality and the fact that linear diffusion is a linear averaging process. The potential for the gradient flow w.r.t. $\mathbb{W}$ is Boltzmann's entropy functional $\mathcal{H}(\rho) = \int \rho \log \rho \, dx$.

Observation 2. *The DLSS equation $\partial_t \rho = -\partial_{xx}(\rho \,\partial_{xx} \log \rho)$ induces on X ...*

- *... a contractive flow w.r.t. $\mathbb{H}$ [2],*
- *... a (non-contractive) gradient flow w.r.t. $\mathbb{W}$ [1],*
- *... a (non-contractive) gradient flow w.r.t. $\mathbb{D}$ [5].*

There is apparently no easy explanation for the contractivity in $\mathbb{H}$. The potentials for the gradient flows w.r.t. $\mathbb{W}$ and $\mathbb{D}$ are, respectively, the Fisher information $\mathcal{F}(\rho) = \int \rho \,(\partial_x \log \rho)^2 \, dx$ and the entropy $\mathcal{H}$.

3. Discretization

Consider an equidistant discretization of $\mathbb{S}^1$ of mesh width $\delta > 0$, denote the space of piecewise constant probability densities by X^δ. A mere restriction of the distances $\mathbb{W}$ or $\mathbb{D}$ to X^δ would produce metric spaces with pathological properties. Instead, the definitions of $\mathbb{H}$, $\mathbb{W}$ and $\mathbb{D}$ can be modified to provide adapted distances $\mathbb{H}^\delta$, $\mathbb{W}^\delta$ and $\mathbb{D}^\delta$ on X^δ: replace the derivative(s) in the continuity equations by difference quotients, and replace the denominator in w_s^2/ρ_s by a suitable mean value of the neighboring densities — simply ρ_k for $\mathbb{H}^\delta$, a two-point average $\mathbf{m}(\rho_{k-1/2}, \rho_{k+1/2})$ for $\mathbb{W}^\delta$, and a three-point average $\mathbf{M}(\rho_{k-1}, \rho_k, \rho_{k+1})$ for $\mathbb{D}^\delta$.

Observation 3. *The discretization $\dot\rho_k = (\rho_{k+1} - 2\rho_k + \rho_{k-1})/\delta^2$ of the linear diffusion equation by central finite differences induces on X^δ ...*

- *... a contractive flow w.r.t. $\mathbb{H}^\delta$,*
- *... a contractive gradient flow w.r.t. $\mathbb{W}^\delta$ [3, 4]*
- *... a contractive flow w.r.t. $\mathbb{D}^\delta$ [5].*

Contractivity follows again by the linear averaging effect of the (discretized) diffusion. For the appropriate mean in the definition of $\mathbb{W}^\delta$, one uses the logarithmic mean $\mathbf{m}(\rho_{\kappa-1/2},\rho_{\kappa+1/2}) = (\rho_{\kappa+1/2}-\rho_{\kappa-1/2})/\log(\rho_{\kappa+1/2}-\log\rho_{\kappa-1/2})$, and in the definition of $\mathbb{D}^\delta$, one uses $\mathbf{M}(\rho_{k-1},\rho_k,\rho_{k+1}) = \rho_k$.

Observation 4 ([5]). *The following discretization of the DLSS equation*

$$\dot\rho_k = \big(F_{k+1}-2F_k+F_{k-1}\big)/\delta^2, \quad F_\ell = \big(\sqrt{\rho_{\ell+1}\rho_{\ell-1}}-\rho_\ell\big)/\delta^2 \tag{1}$$

induces on X^δ *...*

- *... a contractive flow w.r.t.* $\mathbb{H}^\delta$
- *... a (non-contractive) gradient flow w.r.t.* $\mathbb{W}^\delta$
- *... a (non-contractive) gradient flow w.r.t.* $\mathbb{D}^\delta$.

Differently from Observation 3, we choose $\mathbf{m}(\rho_{\kappa-1/2},\rho_{\kappa+1/2}) = \sqrt{\rho_{\kappa+1/2}\rho_{\kappa-1/2}}$ for $\mathbb{W}^\delta$, and for $\mathbb{D}^\delta$:

$$\mathbf{M}(\rho_{k-1},\rho_k,\rho_{k+1}) = \frac{\sqrt{\rho_{k+1}\rho_{k-1}}-\rho_k}{\log\sqrt{\rho_{k+1}\rho_{k-1}}-\log\rho_k}.$$

These choices of $\mathbf{m}/\mathbf{M}$ appear to be crucial to guarantee the contractivity in $\mathbb{H}^\delta$. Indeed, the proof uses that (1) can be re-formulated as

$$\partial_t\sqrt{\rho_k} = -\frac{u_{k+1}-2u_k+u_{k-1}}{\delta^2}+\frac{u_k^2}{\sqrt{\rho_k}} \quad\text{with}\quad u_k = \frac{\sqrt{\rho_{k+1}}-2\sqrt{\rho_k}+\sqrt{\rho_{k-1}}}{\delta^2}.$$

Our main result is about the convergence of the scheme (1).

Theorem 2 ([5]). *Let an initial condition* $\hat\rho\in X$ *be given. For each mesh width* δ*, consider a strictly positive approximation* $\hat\rho^\delta\in X^\delta$ *of* $\hat\rho$*. Then the initial value problem for* (1) *possesses a unique solution* $\rho^\delta:[0,\infty)\to X^\delta$*, and*

$$\rho^\delta\to\rho^* \quad in \quad L^1_{loc}\big((0,\infty)\times\mathbb{S}^1\big)\cap C^\alpha\big([0,\infty);(W^{2,\infty}(\mathbb{S}^1))'\big) \quad as\ \delta\to 0,$$

where ρ^* *is a weak solution to the DLSS equation.*

The proof heavily uses the properties stated in Observation 4, particularly the contractivity in $\mathbb{H}^\delta$ and the monotonicity of $\mathcal{H}$. The key a priori estimate is

$$-\frac{d}{dt}\mathcal{H}(\rho^\delta)\ge\delta\sum_k\left(\frac{\sqrt{\rho_{k+1}}-2\sqrt{\rho_k}+\sqrt{\rho_{k+1}}}{\delta^2}\right)^2,$$

which provides weak compactness of the $\sqrt{\rho^\delta}$ in $L^2\big((0,\infty);H^2(\mathbb{S}^1)\big)$.

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Damage in viscoelastic materials at finite strains

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(joint work with Manuel Friedrich, Martin Kružík, and Riccarda Rossi)

This contribution reports on an ongoing work in progress dedicated to the mathematical analysis of a model for the evolution of damage in viscoelastic materials with physical and geometrical nonlinearities and under the influence of dynamic effects due to the propagation of elastic waves.

1. Challenges related to dynamic effects at finite strains

As has been already observed in existing literature, cf. e.g., [1, 2], one major challenge in this setting is the correct treatment of the axiom of material frame indifference ensuring that a model is independent of orthogonal rotations of the chosen coordinate system. Firstly, this requires static material frame indifference, i.e., $W(QF) = W(F)$ for all $(d \times d)$-matrices $F \in \mathrm{GL}^+(d)$ and $Q \in \mathrm{SO}(d)$, for a hyperelastic material with a stored elastic energy density $W : \mathrm{GL}^+(d) \to \mathbb{R}$. Yet, due to the presence of dynamic effects, this static condition is not enough to ensure the independence of the model of orthogonal rotations. Additionally, also dynamic material frame indifference is required, cf. [1], i.e., $V(QF; \partial_t(QF)) = V(F; \partial_t F)$ for all sufficiently smooth maps $F : [0, T] \to \mathrm{GL}^+(d)$ and $Q : [0, T] \to \mathrm{SO}(d)$, and where $V : \mathrm{GL}^+(d) \times \mathbb{R}^{d\times d} \to [0, \infty]$ denotes a dissipation potential to account for viscous effects of Kelvin-Voigt-type rheology. A simple, suitable choice is given by

$$V(F; \dot{F}) = \frac{1}{2}\mathbb{V}(F)\partial_t(F^\top F) : \partial_t(F^\top F)\,, \tag{1}$$

with $\mathbb{V}(F)G : G \geq c_{\mathbb{V}}|G|^2$ for all $F, G \in \mathbb{R}^{d\times d}$ and with a constant $c_{\mathbb{V}} > 0$. Suppose now that certain a priori estimates result in a bound on the corresponding integral functional, i.e. that $\mathcal{V}(\nabla\varphi; \nabla\dot{\varphi}) := \int_\Omega V(\nabla\varphi; \nabla\dot{\varphi})\,\mathrm{d}x \leq C$. Then (1) directly results in the bound $\frac{c_{\mathbb{V}}}{2}\|\dot{F}^\top F + F^\top \dot{F}\|^2_{L^2} \leq C$, but it does not provide a separate estimate on the partial time derivative $\dot{F} = \partial_t F = \nabla\dot{\varphi}$. In turn, this can be achieved thanks to generalized Korn's inequalities [3, 4] of the form

$$\|\nabla\dot{\varphi}(t)\|_{L^2} \leq C_{\mathrm{K}}\|\nabla\dot{\varphi}^\top(t)\nabla\varphi(t) + \nabla\varphi^\top(t)\nabla\dot{\varphi}(t)\|_{L^2} \tag{2}$$

with a constant $C_{\mathrm{K}} > 0$. Yet, (2) to be valid requires that, firstly, $\nabla\varphi(t) \in C^0(\Omega; \mathbb{R}^{d\times d})$ with $\|\nabla\varphi(t)\|_\infty \leq C$, and secondly, that

$$\det \nabla\varphi(t) \geq c > 0 \quad \text{on } \Omega\text{, uniformly for all } t \in [0, T]. \tag{3}$$

The first condition can be achieved by adding a higher order gradient term to the energy density in the spirit of second grade non-simple materials, i.e., a term

$$\mathcal{H}(\varphi) := \int_\Omega H(\nabla^2\varphi)\,\mathrm{d}x \tag{4}$$

will ensure the required regularity. Secondly, the term (4), given that $W + H$ additionally satisfies a growth estimate of the form

$$W(F) + H(G) \geq c_W\big(|F|^s + \frac{1}{(\det F)^q}\big) + c_H|G|^p \tag{5}$$

for all $F \in \mathrm{GL}^+(d)$ and $G \in \mathbb{R}^{d\times d\times d}$ with fixed constants $c_W, c_H > 0$ and fixed exponents $p > d$, $q \geq \frac{pd}{p-d}$, $s > 1$, will also provide condition (3) thanks to a result by Healey and Krömer [5]. The above considerations motivate the structure of the stored elastic energy density and of the viscoelastic dissipation potential.

2. The damage model

The effects of an evolving damage process on the elastic behavior of a body with reference configuration $\Omega \subset \mathbb{R}^d$ are further modeled with the aid of a damage variable $z : [0,T] \times \Omega \to [0,1]$, where $z(t,x) = 1$ means that the material is undamaged and $z(t,x) = 0$ that the material is maximally damaged in the material point $x \in \Omega$ at time $t \in [0,T]$. The energy functional is of the form

$$\mathcal{E}(t,z,\varphi) := \begin{cases} \int_\Omega \big(E_1(z,\varphi,\nabla\varphi,\nabla^2\varphi) - \langle \ell(t),\varphi\rangle + E_2(z,\nabla z)\big)\,\mathrm{d}x \\ \quad \text{if } E_1(z,\varphi,\nabla\varphi,\nabla^2\varphi) - \langle \ell(t),\varphi\rangle + E_2(z,\nabla z) \in L^1(\Omega)\,, \\ \infty \quad \text{otherwise,} \end{cases} \tag{6}$$

with $E_1(z,\varphi,\nabla\varphi,\nabla^2\varphi) := W(z,\nabla\varphi) + H(z,\nabla^2\varphi)$, $E_2(z,\nabla z) := \frac{1}{2}|\nabla z|^2 + \phi(z)$. Here, the energy term E_2 serves as a regularization for the damage variable and the function ϕ is chosen such that $z \in [0,1]$ can be ensured for a solution of the problem. The densities W and H are assumed to be suitably smooth, equipped with suitable analytical and physically reasonable growth properties, e.g., in the line of (5) and to ensure that a decrease of the damage variable (corresponding to an increase of damage) leads to a decrease of the stored elastic energy and, hence, the elastic stresses. A further ingredient to the model is the dissipation potential for the damage variable, which is assumed to be of the form

$$\mathcal{R}(\dot z) := \int_\Omega \big(R_1(\dot z) + R_2(\dot z) + I_{(-\infty,0]}(\dot z)\big)\mathrm{d}x \tag{7}$$

with $R_1(\dot z) := a_1|\dot z|$, $R_2(\dot z) := \frac{a_2}{2}|v|^2$, with constants $a_1, a_2 > 0$, and the indicator function $I_{(-\infty,0]}$, i.e., $I_{(-\infty,0]}(\dot z) = 0$ if $\dot z \in (-\infty,0]$ and $I_{(-\infty,0]}(\dot z) = \infty$ otherwise. The presence of two convex but non-smooth terms, the rate-independent dissipation R_1 and the indicator function $I_{(-\infty,0]}$ to prevent healing of damage, leads to an evolution law for the damage variable in terms of a subdifferential inclusion

$$a_1\mathrm{Sign}(\dot z) + a_2\dot z + \partial I_{(-\infty,0]}(\dot z) + \phi'(z) + \mathrm{D}_z W(z,\nabla\varphi) - \Delta z \ni 0\,, \tag{8}$$

where Sign and $\partial I_{(-\infty,0]}$ denote the subdifferentials of the absolute value function and the indicator function in the sense of convex analysis.

3. Existence of weak solutions and improved results

By means of a staggered time-discrete scheme one can prove the existence of weak solutions (z,φ), which are defined by the following three ingredients:

1. Weak formulation of the momentum balance:

$$\begin{aligned}\langle\ddot\varphi(t),\eta\rangle_{W^{2,p}} + \int_\Omega\big(\mathrm{D}_{\dot F}V(z(t),\nabla\varphi(t);\nabla\dot\varphi(t)) + \mathrm{D}_F W(z(t),\nabla\varphi(t))\big):\nabla\eta\,\mathrm{d}x \\ + \int_\Omega \mathrm{D}_G H(z(t),\nabla^2\varphi(t)) : \nabla^2\eta\,\mathrm{d}x = \langle\ell(t),\eta\rangle_{W^{2,p}}\end{aligned} \tag{9a}$$

for almost all $t\in(0,T)$ and for all $\eta\in W^{2,p}(\Omega;\mathbb{R}^d)\cap H^1_0(\Omega;\mathbb{R}^d)$, together with

$$\min_{(t,x)\in[0,T]\times\overline\Omega}\det(\nabla\varphi(t))>0 \quad\text{and}\quad \varphi(t)|_{\partial\Omega}(t,\cdot)=\mathrm{Id}\quad\text{for all } t\in[0,T].$$

2. Damage flow rule in terms of a one-sided variational inequality, cf. also [6]:

$$\begin{aligned}\int_\Omega\big(a_1 + a_2\dot z(t) + \phi'(z(t)) + \mathrm{D}_z W(z(t),\nabla\varphi(t)) + \mathrm{D}_z H(z(t),\nabla^2\varphi(t))\big)\zeta\,\mathrm{d}x \\ + \int_\Omega \nabla z(t)\cdot\nabla\zeta\,\mathrm{d}x \ge 0\end{aligned} \tag{9b}$$

for almost all $t\in(0,T)$ and for all $\zeta\in H^1(\Omega)\cap L^\infty(\Omega)$ with $\zeta\le 0$ a.e. in Ω.

3. Upper energy-dissipation estimate:

$$\begin{aligned}&\mathcal{E}(t,\varphi(t),z(t)) + \mathcal{K}(\dot\varphi(t)) \\ &\qquad + 2\int_s^t\big(\mathcal{V}(z(r),\varphi(r);\dot\varphi(r)) + \mathcal{R}_2(\dot z(r))\big)\,\mathrm{d}r + \int_s^t \mathcal{R}_1(\dot z(r))\,\mathrm{d}r \\ &\le \mathcal{E}(s,\varphi(s),z(s)) + \mathcal{K}(\dot\varphi(s)) + \int_s^t \partial_t\mathcal{E}(r,\varphi(r),z(r))\,\mathrm{d}r\,,\end{aligned} \tag{9c}$$

for all $t\in[0,T]$ and almost all $s\in[0,t)$ and where $\mathcal{K}(\dot\varphi) := \int_\Omega \frac{\rho}{2}|\dot\varphi|^2\,\mathrm{d}x$ denotes the kinetic energy with a constant mass density $\rho>0$.

Under the additional assumption that $H:(z,G)\mapsto H(z,G)$ is convex, it can be further shown that the inequality (9c) improves to an equality. Moreover, if the regularization term H does not depend on z, then one obtains $z\in H^1(0,T;H^1(\Omega))$ and (9b) can be replaced by the subdifferential inclusion (8).

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Gradient flow: yes or no?

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(joint work with Morris Brooks)

Let $X \in \Gamma(TM)$ be a vector field on a smooth manifold M and let $f : M \to \mathbb{R}$ be a smooth function. Does there exist a Riemannian metric g on M such that the evolution equation $\dot{u} = X(u)$ is the gradient flow equation for f with respect to the metric g? In order words, using standard index notation, does there exist a metric $g_{\alpha\beta}$ such that $g_{\alpha\beta}X^{\beta} = -D_{\alpha}f$?

Some assumptions are clearly needed:

Firstly, Df should be zero at every stationary point of the evolution: the vector field X and the co-vectorfield Df should have the same set of zeroes.

Secondly, X and $-Df$ should "agree on the sign" outside the set where they vanish: for all $x \in M$ with $X(x) \neq 0$, they should satisfy $X^{\alpha}D_{\alpha}f(x) < 0$ (since $X^{\alpha}D_{\alpha}f = -g_{\alpha\beta}X^{\alpha}X^{\beta}$ and g is positive definite.) This requirement reflects the fact that f should decrease along the evolution.

Thirdly, at every point x where $X(x) = 0$, one should have that $D_{\alpha}D_{\gamma}f = \bar{g}_{\alpha\beta}D_{\gamma}X^{\beta}$, for some scalar product $\bar{g}_{\alpha\beta}$ on T_xM. This somewhat less obvious condition is obtained by differentiating the equation $D_{\alpha}f = g_{\alpha\beta}X^{\beta}$ at x, using the assumption that $X(x) = 0$.

Our main result asserts that these conditions are not only necessary, but also sufficient, under mild regularity conditions.

Theorem 1. *Let $f : M \to \mathbb{R}$ be a function and $X^{\alpha} \in \Gamma(TM)$ be a vector field. We assume that f and X are real-analytic (in some coordinate chart). Suppose further that Df has a unique zero, $\bar{x} \in M$, at which f attains its minimum. Then there exists a Riemannian metric $g_{\alpha\beta} \in \Gamma(T^*M \otimes T^*M)$ satisfying*

$$\nabla_{\beta}f = g_{\alpha\beta}X^{\alpha},$$

if and only if the following conditions hold:

(1) *$D_{X^{\alpha}}f(x) < 0$ for all $x \in M$ with $x \neq \bar{x}$;*
(2) *$X^{\alpha}|_{\bar{x}} = 0$;*
(3) *The linear map $\Lambda := D_{\alpha}X^{\beta}|_{\bar{x}} : T_{\bar{x}}M \to T_{\bar{x}}M$ is positive and symmetric with respect to the Hessian scalar product $h_{\alpha\beta} := D_{\alpha}D_{\beta}f|_{\bar{x}}$ on $T_{\bar{x}}M$.*

In fact, [3] contains a more general version of this result, in which Df is replaced by an arbitrary co-vector field Y. We also prove a variant of this result in which X and Y are of class C^{k+1} for $k \geq 0$. In this case, the metric g is of class C^k. The case $k = 0$ was proved earlier in [2].

The existence of a metric with the desired properties is easy to prove outside the set of critical points; see, e.g., [1]. The nontrivial part of the proof is to establish the existence of a smooth metric in a neighbourhood of every point where X vanishes. This is done using a power series construction by an iterative argument, in which each iterative step involves the solution of a certain tensor equation.

As an application of Theorem 1 we solve a problem that arose in joint work with Carlen on gradient flow formulations of Lindblad equations, which describe the time-evolution of open quantum systems. It was shown earlier [4, 9] that Lindblad equations with a certain symmetry condition (*GNS-detailed balance*) can be formulated as gradient flow equation for the quantum relative entropy. The notion of GNS-detailed balance is one among several quantum generalisations of the notion of detailed balance for classical Markov chains. Subsequently, a different notion of detailed balance (*BKM-detailed balance*) was shown to be necessary for the existence of an entropic gradient flow structure for Lindblad equations [5]. However, as the notion of BKM-detailed balance is strictly weaker than the notion of GNS-detailed balance, there was a gap between the necessary and sufficient conditions above. As a consequence of Theorem 1, we close this gap: the notion of BKM-detailed balance is also sufficient for the existence of an entropic gradient flow structure. This result provides a quantum analogue of earlier work on the sufficiency of detailed balance [6] in gradient flow structures for classical Markov chains [7, 8].

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Shape Optimisation for nonlocal anisotropic energies

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(joint work with R. Cristoferi, M.G. Mora)

In this work we consider shape optimisation problems for sets of prescribed mass, where the driving energy functional is nonlocal and *anisotropic*. More precisely, for a given mass $m > 0$, we are interested in the minimisation of the energy functional

$$\mathcal{I}(\Omega) = \int_\Omega \int_\Omega \left(W(x-y) + \frac{1}{2}|x-y|^2 \right) dxdy, \tag{1}$$

over the class of sets with mass m,

$$\mathcal{A}_m = \left\{ \Omega \subset \mathbb{R}^d : \ \Omega \ \text{measurable}, \ |\Omega| = m \right\},$$

for $d = 2, 3$. In (1), the interaction potential W is defined for $x \neq 0$ as

$$W(x) = \begin{cases} -\log|x| + \kappa\left(\dfrac{x}{|x|}\right) & \text{if } d = 2, \\ \dfrac{1}{|x|}\kappa\left(\dfrac{x}{|x|}\right) & \text{if } d = 3, \end{cases} \tag{2}$$

and $W(0) = +\infty$. For the profile $\kappa : \mathbb{S}^{d-1} \to \mathbb{R}$ we require that it is even, and that both W and $\widehat{W}$ are continuous on $\mathbb{S}^{d-1}$. Additionally, if $d = 3$, κ is assumed to be strictly positive on $\mathbb{S}^{d-1}$. The potential W is an anisotropic extension of the classical, radially symmetric Coulomb potential, which corresponds to the special case of a constant profile κ. The anisotropy is fully encoded in the profile κ, which introduces an additional dependence on the directions of interaction.

The energy $\mathcal{I}$ is the sum of two competing terms: an attractive, quadratic interaction, that dominates at large distances, and a repulsive, Coulomb-like interaction, driven by the anisotropic potential W. The additional positivity requirement for κ in the three-dimensional case is there to preserve the repulsive nature of W; this is not needed for $d = 2$ since κ is bounded, and hence at short range the repulsive nature of $-\log|\cdot|$ is not affected by the additional anisotropy κ.

1. Main result

Our main result is the characterisation of the minimiser of $\mathcal{I}$ in the class of sets $\mathcal{A}_m$, for any mass $m > 0$. This is done under the sole assumption that the Fourier transform $\widehat{W}$ of the potential W on the sphere $\mathbb{S}^{d-1}$ is nonnegative.

In fact we have two main results, depending on whether $\widehat{W}$ is strictly positive or not. In the first case we show that above a given threshold for the mass the unique minimiser of $\mathcal{I}$ is a d-dimensional ellipsoid. Uniqueness has to be intended up to translations, since the functional $\mathcal{I}$ is translation-invariant.

In the case of degeneracy of $\widehat{W}$, instead, we have the following dichotomy: either there exists a threshold value for the mass as in the case above, or the minimiser is an ellipsoid for any positive value of the mass.

The occurrence of one or the other possibility is related to the minimisation problem for the energy (1) in the wider class of measures (rather than sets) with prescribed mass (see [2, 3, 5]).

2. Method of proof

For the proof of existence, we consider the relaxed energy

$$\mathcal{I}(\rho) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(W(x-y)+\frac{1}{2}|x-y|^2\right)\rho(x)\rho(y)\,dxdy, \tag{3}$$

which extends (1) to the class of densities

$$\mathcal{A}_{m,1} = \left\{\rho \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d):\ \|\rho\|_{L^1} = m,\ 0\le \rho \le 1 \text{ a.e.}\right\}. \tag{4}$$

It was proved in [1] that a set $\Omega \in \mathcal{A}_m$ is a minimiser of (1) if and only if its characteristic function $\chi_\Omega \in \mathcal{A}_{m,1}$ is a minimiser of the relaxed energy (3), and the same holds true in our case. Since for small mass m the minimising densities are not the characteristic functions of a set, our original problem on sets can only have a solution for large enough mass.

For large mass we then (equivalently) study the problem on densities, for which existence and compact support of minimisers can be proved by standard arguments. Uniqueness, up to translations, follows by the sign condition on the Fourier transform of W, which implies that the energy $\mathcal{I}$ is strictly convex (on measures with barycentre at the origin). Strict convexity of the energy, in its turn, guarantees that the minimiser can be characterised as the only solution of the Euler-Lagrange optimality conditions.

Motivated by the results in [2,3] and [5] we look for a candidate ellipsoid $E \subset \mathbb{R}^d$ centred at the origin, with $|E| = m$, such that its characteristic function χ_E satisfies the Euler-Lagrange conditions

$$(W * \chi_E)(x) + m\frac{|x|^2}{2} = \lambda \qquad \text{if } x \in \partial E, \tag{5}$$

$$(W * \chi_E)(x) + m\frac{|x|^2}{2} \le \lambda \qquad \text{if } x \in E^\circ, \tag{6}$$

$$(W * \chi_E)(x) + m\frac{|x|^2}{2} \ge \lambda \qquad \text{if } x \in \mathbb{R}^d\setminus E, \tag{7}$$

for a constant $\lambda \in \mathbb{R}$. To evaluate the potential of a generic ellipsoid E we use the representation of the potential in Fourier form proved in [5, 6] for $d = 2, 3$. Following [5, 6] one can see that condition (7) is automatically satisfied by any solution E of (5)–(6).

The key idea to solve (5)–(6) is to rewrite (5) as the stationarity condition for an auxiliary scalar function f defined on symmetric and positive definite matrices M (encoding the information on the semi-axes and orientation of E), under the determinant constraint $\det M = \frac{m^2}{|B|^2}$ (encoding the mass constraint $|E| = m$). One of the main advantages of this alternative formulation is that (6) corresponds to a condition on the sign of the Lagrange multiplier associated to the constraint.

The strategy is then to first show that the auxiliary minimisation problem for f obtained by replacing the equality constraint for the determinant with the unilateral condition $\det M \geq \frac{m^2}{|B|^2}$ admits a solution. As a final step we show that this solution in fact satisfies the equality constraint. This immediately gives the required sign condition for the multiplier, and concludes the proof of (5)–(6).

2.1. **Motivation and comparison with the radially symmetric case.** The problem we consider can be interpreted as a first shape optimisation result for nonlocal anisotropic energies with competing attractive and repulsive terms.

The isotropic counterpart of this problem is well-studied. The closest analogue to our energy $\mathcal{I}$ is the energy considered in [1, 4], namely

$$\mathcal{E}(\Omega) = \int_\Omega \int_\Omega K(x-y)\,dxdy, \quad \Omega \in \mathcal{A}_m, \tag{8}$$

where K is a power-law potential of the form

$$K(x) = \frac{|x|^q}{q} - \frac{|x|^p}{p}, \quad -d < p < 0, \quad q > 0. \tag{9}$$

In the special case of Coulomb repulsion and quadratic attraction there is a threshold for the mass, given by the volume of the unit ball B, such that the energy $\mathcal{E}$ admits no minimiser if $m < |B|$, while for $m \geq |B|$ the minimiser of $\mathcal{E}$ is a ball of mass m. While this is similar to our main result, the corresponding proofs are substantially different. In particular, the radial symmetry of the interactions in $\mathcal{E}$ allows immediately to identify a (unique) ball as the candidate minimiser and greatly simplifies the proof.

Another important class of isotropic attractive/repulsive energies is given by

$$\mathcal{E}_{\mathrm{P}}(\Omega) = \int_\Omega \int_\Omega \frac{1}{d-2}\frac{1}{|x-y|^{2-d}}\,dxdy + \mathrm{Per}(\Omega), \quad \Omega \in \mathcal{A}_m,$$

where Per denotes the classical perimeter. The energies $\mathcal{E}_{\mathrm{P}}$ have been first introduced by Gamow in his liquid drop model and widely studied since.

Considering an anisotropic analogue of $\mathcal{E}_{\mathrm{P}}$ is a very natural direction of investigation, but this is not a direction we will pursue in this work.

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Existence and uniqueness in law for some doubly nonlinear SPDEs

Ulisse Stefanelli
(joint work with Carlo Orrieri, Luca Scarpa)

Assume to be given a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^d$, and a cylindrical Wiener process W on $L^2(\mathcal{O}) =: H$. The progressive σ-algebra associated with $(\mathcal{F}_t)_{t\geq 0}$ is indicated by $\mathcal{P}$. We are interested in the doubly nonlinear parabolic SPDE

$$\mathrm{d}u = (\partial_t u^d)\mathrm{d}t + G\,\mathrm{d}W, \tag{1a}$$

$$\alpha(\partial_t u^d) - \Delta u + \beta(u) = f(u). \tag{1b}$$

Relation (1a) entails that u^d is the absolutely continuous part of the Ito process u. The Hilbert-Schmidt operator $G \in \mathcal{L}^2(H, H)$ is given. Moreover, α, $\beta \in C^0(\mathbb{R})$ are nondecreasing, with β bounded, $\alpha^{-1} \in C^{0,\eta}(\mathbb{R})$ for some $\eta \in (0, 1)$, and $r \in \mathbb{R} \mapsto \alpha^{-1}(r) - \kappa r$ bounded for some $\kappa > 0$. This in particular entails that α is nondegenerate: there exists $c > 0$ such that $\alpha(r)r \geq cr^2 - 1/c$ for all $r \in \mathbb{R}$. Eventually, we ask $f \in W^{1,\infty}(\mathbb{R})$.

Equation (1b) is posed in the space-time cylinder $\mathcal{O} \times (0, \infty)$ and is complemented with boundary and initial conditions

$$u = 0 \quad \text{on } \partial\mathcal{O} \times (0, \infty), \tag{1c}$$

$$u(\cdot, 0) = u^0 \quad \text{in } \mathcal{O}. \tag{1d}$$

In particular, the Laplacian $-\Delta$ in (1b) is seen as an unbounded, linear, selfadjoint operator in H with domain $D(-\Delta) := H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$. Via spectral decomposition, one classically defines the powers $(-\Delta)^\sigma$ and the corresponding domains $D((-\Delta)^\sigma)$ for any $\sigma > 0$. Eventually, we ask G to have $\ker G = \{0\}$ and to commute with $-\Delta$. Problem (1) is a concrete example for the abstract theory developed in [4,5], where indeed the above assumptions are somewhat generalized.

In the deterministic case $G \equiv 0$, a general well-posedness theory covering (1) has been obtained by Akagi [2], see also Colli & Visintin [3] for the unperturbed case of $f \equiv 0$. Solutions to the deterministic problem are unique for β Lipschitz continuous and α strongly monotone. In case α is not strongly monotone, uniqueness may fail, also for $\beta = 0$, see [1].

In the stochastic case $G \not\equiv 0$, we are able to give two distinct results, relating to two different regularity setting for the initial datum u^0 and the noise G. Correspondingly, we consider two different type of solutions to (1), both of probabilistic-weak type. For more regular data, we focus on analytically strong solutions. In the less regular setting, we resort to Friedrichs weak solutions instead. We record our findings in the two theorems below.

Theorem 1 (More regular setting)**.** *Let $u^0 \in H^1_0(\mathcal{O})$ and $G \in \mathcal{L}^2(H, H^1(\mathcal{O}))$. Then, there exists an* analytically strong solution *to* (1)*, namely,*

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, u, u^d)$$

where $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *is a filtered probability space,* W *is a cylindrical Wiener process on* H*, and, for all* $T > 0$*,* $u \in L^2_{\mathcal{P}}(\Omega; C^0([0,T]; H) \cap L^\infty(0,T; H^1_0(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O})))$*, and* $u^d \in L^2_{\mathcal{P}}(\Omega; H^1(0,T; H))$ *solve*

$$u(t) = u^0 + \int_0^t \partial_t u^d(s)\,\mathrm{d}s + \int_0^t G\,\mathrm{d}W(s) \quad a.e.\ in\ \mathcal{O},\ \forall t \geq 0,\ \mathbb{P} - a.s.,$$

$$\alpha(\partial_t u^d) - \Delta u + \beta(u) = f(u) \quad a.e.\ in \quad \mathcal{O} \times (0, +\infty),\ \mathbb{P} - a.s.$$

In addition, if α *is strongly monotone and* β *is Lipschitz continuous we have that analytically strong solutions are unique and* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and* W *can be a-priori chosen.*

This result is proved in [5] by means of an approximation procedure: for all $\lambda > 0$ one solves the approximate problem

$$\mathrm{d}u_\lambda = (\partial_t u^d_\lambda)\mathrm{d}t + G\,\mathrm{d}W, \tag{2a}$$

$$\lambda \partial_t u^d + \alpha_\lambda(\partial_t u^d) + B_\lambda u = f(u), \tag{2b}$$

along with the boundary and initial conditions (1c)-(1d). In (2b), B_λ is the Yosida approximation of $-\Delta + \beta$ at level $\lambda > 0$: for all $u \in H$ we define $B_\lambda u = (u - v_\lambda)/\lambda$, where $v_\lambda \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$ is the unique solution to $v_\lambda - \lambda\Delta v_\lambda + \lambda\beta(v_\lambda) = u$ a.e. in $\mathcal{O}$. The well-posedness of (2) for all $\lambda > 0$ follows from the Lipschitz continuity of $(\lambda\,\mathrm{id} + \alpha_\lambda)^{-1}$, B_λ, and f. One derives λ-independent estimates on u_λ and u^d_λ, extracts suitably converging subsequences, and passes to the limit $\lambda \to 0$ in (2a)-(2b) obtaining an analytically strong solution. Pathwise uniqueness in case α is strongly monotone and β is Lipschitz is straightforward. Based on such uniqueness, the possibility of a-priori fixing $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and W follows by classical arguments.

Before moving on, let us remark that (1) can be equivalently rewritten as

$$\mathrm{d}u - \kappa\Delta u\,\mathrm{d}t + \kappa\beta(u)\,\mathrm{d}t = \kappa f(u)\,\mathrm{d}t + C(u)\,\mathrm{d}t + G\,\mathrm{d}W$$

where the operator $C : H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})$ is defined as

$$C(u) = \alpha^{-1}(f(u) + \Delta u - \beta(u)) - \kappa(f(u) + \Delta u - \beta(u)).$$

Note that the range of C is bounded in H. For all $\lambda > 0$, we also define $C_\lambda(u) = \alpha^{-1}(f(u) - B_\lambda u) - \kappa(f(u) - B_\lambda u)$ for all $u \in H$.

Theorem 2 (Less regular setting). *Let* $u^0 \in H$ *and* $G(H) \subset D((-\Delta)^\sigma)$ *for some* σ *with*

$$0 < \sigma < \min\left\{\frac{\eta}{4 - 2\eta}, \frac{1}{6}\right\}. \tag{3}$$

Then, for any sequence $\lambda_n \to 0$ *there exists a* Friedrichs-weak solution *to* (1)*, namely,* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, u, y)$ *where* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *is a filtered probability space satisfying the usual conditions,* W *is a cylindrical Wiener process on* H*, and, for all* $T > 0$*,* $u \in L^2_{\mathcal{P}}(\Omega; C^0([0,T]; H) \cap L^2(0,T; H^1_0(\mathcal{O})))$*, and* $y \in L^\infty_{\mathcal{P}}(\Omega \times (0,T); H)$ *such that there exists a not relabeled subsequence* λ_n*, a sequence of data*

$(u_n^0, G_n) \in H_0^1(\mathcal{O}) \times \mathcal{L}^2(H, H_0^1(\mathcal{O}))$, and a sequence of analytically strong solutions to

$$\mathrm{d}u_n - \kappa\Delta u_n \,\mathrm{d}t + \kappa\beta(u_n)\,\mathrm{d}t = \kappa f(u_n) + C_{\lambda_n}(u_n)\,\mathrm{d}t + G_n\,\mathrm{d}W$$

with boundary condition (1c) and initial condition $u_n(\cdot, 0) = u_n^0$ such that, for all $T > 0$,

$$u_n^0 \to u^0 \quad \text{in } H, \quad \lambda_n^{-1/2-\sigma}\|G_n - G\|_{\mathcal{L}^2(H,H)} \to 0,$$
$$u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^2_{\mathcal{P}}(\Omega; L^\infty(0,T;H) \cap L^2(0,T;H_0^1(\mathcal{O}))),$$
$$u_n \to u \quad \text{in } L^2(0,T;H) \quad \mathbb{P}\text{-a.s.}, \quad C_{\lambda_n}(u_n) \overset{*}{\rightharpoonup} y \quad \text{in } L^\infty_{\mathcal{P}}(\Omega \times (0,T); H).$$

Two such Friedrichs-weak solutions u_1 and u_2 with the same $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, W, and initial datum u^0 coincide in law in $C^0([0,T]; H)$, namely,

$$\mathbb{E}[g(u_1(t))] = \mathbb{E}[g(u_2(t))] \quad \forall g \in C_b(H),\ \forall t \geq 0, \tag{4}$$

where $C_b(H)$ are the bounded continuous functions on H and $\mathbb{E}$ denotes the expectation w.r.t. $\mathbb{P}$.

This result is proved in [4], as a subcase of a more general abstract theory. The existence statement follows by an approximation argument via analytically strong solutions. The uniqueness-in-law statement results from the analysis of the associated Kolmogorov equation. For given $\gamma > 0$ and $g \in C_b(H)$ one considers

$$\gamma\varphi(h) - \frac{1}{2}\mathrm{Tr}(G^*GD^2\varphi(h)) = g(h) + \int_{\mathcal{O}} (\Delta h - \beta(h) + f(h))D\varphi(h)\,\mathrm{d}x,$$

to be solved for all $h \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. Assuming $\varphi \in C_b^2(H)$ to solve such Kolmogorov equation, an application of the Ito formula to $\varphi \circ u$ on the time interval $[0, t]$ and a limit for $t \to \infty$ formally entail that

$$\int_0^\infty \mathrm{e}^{-\gamma s}\mathbb{E}[g(u(s))]\,\mathrm{d}s = \varphi(u^0).$$

By establishing the latter for any Friedrichs-weak solution, as $\gamma > 0$ is arbitrary, basic properties of the Laplace transform and the a.s. continuity of $g \circ u$ imply (4). In order to make the above argument rigorous, one has to argue at the approximate λ_n level, where the corresponding Kolmogorov equation is solved by φ_{λ_n}. Then, one proves estimates on φ_{λ_n} and its derivatives, applies the Ito formula to $\varphi_{\lambda_n} \circ u_n$, and passes to the limit. In the process, the qualification (3) on σ is used.

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Quantum computing with Rydberg-atom quantum processors: A personal journey

Oliver Tse
(joint work with Robert de Keijzer, Servaas Kokkelmans, Luke Visser)

Recent advancements in Rydberg atoms along with the spectacular degree of experimental control of state-of-the-art platforms have made it possible to realize quantum gates with high fidelity, thereby drawing the advent of universal quantum computers closer to reality. Yet, quantum computing is still in the so-called noisy intermediate-scale quantum (NISQ) era, where the number of available error-free qubits is modest, and quantum algorithms have yet to outperform their classical counterparts in practice.

This talk introduces and reports on progress in the following research topics:

Variational Quantum Optimal Control (VQOC). The development of hybrid and near-term quantum algorithms, such as Variational Quantum Eigensolvers (VQEs) based on digital quantum circuits, has been progressing at an enormous pace to allow for quantum advantage in the NISQ era. This development, however, has mostly been independent of the developments in quantum computing hardware, where the physical control of qubits in Rydberg systems is governed by inherently analog laser pulses. In this talk, we introduce the VQOC framework, which brings together recent progress in the understanding and control of Rydberg platforms and the well-developed theory of quantum optimal control, and show applications of VQOC on examples related to the electronic structure problem.

Learning quantum channels. The state of a closed quantum system evolves under the Schrödinger equation, where the reversible evolution of the state is propagated from initial time by an action of a unitary operator. However, realistic quantum systems are open, i.e. they interact with their environment, resulting in non-reversible evolutions, described by quantum semigroups on density matrices. To simulate an open quantum system using an *ideal* quantum computer, which is intrinsically closed, thus requires one to model an open quantum system with a closed one. We do this by invoking the Stinespring dilation theorem, allowing us to learn a target quantum semigroup by approximating equivalent unitary evolutions on an extended system. We further report on an experimentally feasible method to extrapolate the quantum evolution at later times using only data from the first few time steps.

Towards understanding noisy qubits. Noise on a controlled quantum system is generally introduced via the non-reversible Lindblad equation. This equation describes the average state of the system via the density matrix. One way of deriving this Lindblad equation is by taking a sample average of states evolving under the stochastic Schrödinger equation (SSE) driven by white noise. However, white noise, where all noise frequencies contribute equally in the power spectral density, is not a realistic noise profile as lower frequencies commonly dominate the spectrum. For this reason, we provide analytical solutions to the full fidelity distribution for important cases of the SSE driven by more realistic noise. This allows for predictions of the mean, variance, and higher-order moments of the fidelities of these qubits, which can be of value when deciding on the allowed noise levels for future quantum computing systems, e.g. deciding what quality of control systems to procure. Furthermore, these methods will prove to be integral in the optimal control of qubit states under (classical) control system noise.

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Discrete to continuous crystalline curvature flows

Antonin Chambolle

(joint work with Daniele DeGennaro, Massimiliano Morini)

The talk described a work in progress, where we investigate a fully discrete version of the Almgren-Taylor-Wang / Luckhaus-Sturzenhecker scheme [1, 7] for building mean curvature flows. This scheme, after some rewriting, can be described as follows: given a set E^0, and d_{E^0} the *signed distance function* to its boundary, we solve in $\mathbb{R}^d$, for $h > 0$ small and each $n \geq 1$:

$$\begin{cases} -h\operatorname{div} z^n + u^n = d_{E^{n-1}}\,, \\ |z^n| \leq 1, \quad z^n \cdot Du^n = |Du^n| \end{cases} \tag{1}$$

which is formally the Euler-Lagrange equation of

$$\min_u \int |Du| + \frac{1}{2h}\int (u - d_{E^{n-1}})^2 dx$$

(yet this energy is infinite in the whole space). We let then $E^n = \{x : u^n(x) \leq 0\}$. By translational invariance and comparison, u^n is trivially 1-Lipschitz (since $d_{E^{n-1}}$ is), in particular the second condition in (1) reads $z \in \partial|\cdot|(\nabla d)$ a.e. in $\mathbb{R}^d$ (the subgradient of the Euclidean norm). One also deduces that $d_{E^n} \geq u^n$ in $\{u^n > 0\}$, and $d_{E^n} \leq u^n$ in $\{u^n < 0\}$. Hence,

$$\frac{d^n - d^{n-1}}{h} \geq \operatorname{div} z^n$$

out of E^n. Getting some control on d^n in time and $\operatorname{div} z^n$ in space allows then to pass to the limit and deduce the existence of $E \subset \mathbb{R}^d \times [0, \infty)$ (the Hausdorff limit of $\bigcup_{n\geq 0} E^n \times \{nh\}$) a closed set such that

$$\partial_t d \geq \operatorname{div} z \quad \text{in} \quad \mathcal{D}'((\mathbb{R}^d \times (0,\infty)) \setminus E), \qquad z \in \partial|\cdot|(\nabla_x d) \text{ a.e.} \tag{2}$$

with $d(x,t) = \operatorname{dist}(x, E(t))$ for all x, t. Reasoning with the complement, one finds a similar equation for $A \subset E$, the complement of the Hausdorff limit of $\bigcup_{n\geq 0}(\mathbb{R}^d \setminus E^n) \times \{nh\})$.

This equation, which holds in the distributional or measure sense, is seen to hold also in the viscosity sense [5, 6] and hence characterizes the mean curvature flow (with a possible, but exceptional, fattening of the set $E \setminus A$), as shown in [8]. An important step in proving the convergence is an estimate of the solution of (1) with the right-hand side replaced with $|x|$, first computed in [3], this is crucial to estimate the variation of d^n in time as well as $\operatorname{div} z^n$ from above where $d^n > 0$.

Now, in [4, 6], it is also shown that the same scheme (and the same proof) can be applied to build and characterize anisotropic, or crystalline flows. Sticking to the simpler case of [6], and given φ a convex norm (with possibly polyhedral level sets) we replace Du above with $\varphi(Du)$, the distance with the φ° distance ($\varphi^\circ(x) = \sup\{x \cdot \nu : \varphi(\nu) \leq 1\}$), the condition on z^n in (1) with $z^n \in \partial\varphi(\nabla u^n)$ and end up with a distributional definition of a well posed *crystalline* mean curvature flow (see [6] for a comparison result which guarantees the uniqueness, in general, of the limit—when A is the interior of E). In this case, the motion is still described by (2) yet the second condition is $z \in \partial\varphi(\nabla_x d)$ and d is the φ°-distance to E.

In this work, with M. Morini (Parma) and our student D. DeGennaro (Ceremade), we propose a to solve a fully discrete equation, which reads (for $h, \varepsilon > 0$, small time and space steps)

$$\begin{cases} h(D^* z^n)_i + u_i^n = d_i^{n-1} & \text{for } i \in \varepsilon\mathbb{Z}^d \\ |z_{i,j}^n| \leq \beta_{\frac{j-i}{h}}, \quad z_{i,j}^n(u_j - u_i) = \beta_{\frac{j-i}{h}} |u_j - u_i| & \end{cases}$$

where $D : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ is defined by $(Du)_{i,j} = (u_j - u_i)/h$, D^* is its adjoint (for the standard scalar product), and β_k, $k \in \mathbb{Z}^d$, is a finitely supported family of positive weights (positive at least on a basis of $\mathbb{Z}^d$). One then sends $h, \varepsilon \to 0$. In case $\varepsilon \ll h$, but more interestingly and somewhat surprisingly in case $\varepsilon = h$, one may then adapt the techniques above to show again the convergence to (2), for the crystalline anisotropy $\varphi(p) = \sum_{k\in\mathbb{Z}^d} \beta_k |k \cdot p|$. Interestingly, in case $\varepsilon = h$, we may define d^n from u^n with a sort of interpolation scheme (defined by suitable inf/sup convolutions with the distance φ°), so that the limiting evolution is precisely given by (2) without any drift, contrarily to the dicrete scheme introduced previously in [2], which was the starting point for our study, and where a rounding occurs at each step which accumulates in the limit.

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Coarse-graining to GENERIC

Johannes Zimmer
(joint work with Alexander Mielke, Mark A. Peletier)

We study a model system on two different scales, called microscopic and macroscopic, with the aim of deriving the macroscopic description from the microscopic one.

The *macroscopic* description is governed by a thermodynamic formulation in form of GENERIC, the *General Equation for the Non-Equilibrium Reversible-Irreversible Coupling* [3], sometimes also called metriplectic evolution. It takes the form

$$\dot{y} = \mathbb{J}(y)D\mathcal{E}(y) + \mathbb{K}(y)D\mathcal{S}(y); \tag{1}$$

here $y \in \mathbf{Y}$ is the macroscopic variable defined on a state space $\mathbf{Y}$, $\mathbb{J} \colon \mathbf{Y}^* \to \mathbf{Y}$ is a symplectic operator, $\mathbb{K} \colon \mathbf{Y}^* \to \mathbf{Y}$ is a positive semidefinite operator, $\mathcal{E}, \mathcal{S} \colon \mathbf{Y} \to \mathbb{R}$ are the energy and entropy functionals and D denotes a derivative. The structure of GENERIC immediately implies that the energy $\mathcal{E}$ is constant along trajectories and the entropy $\mathcal{S}$ is non-decreasing. Thus, GENERIC ensures thermodynamic consistency, and very different thermodynamic systems can be put in this framework [2].

The aim is to derive (1) as macroscopic description of a microscopic model, i.e., give a mathematically rigorous coarse-graining procedure. We choose a classic microscopic system, which has been investigated in detail [1], though to our knowledge not in connection with GENERIC. The model is purely Hamiltonian, to reflect the description of atoms and molecules by Newtonian mechanics. It consists of a finite-dimensional Hamiltonian system (System $\mathcal{A}$) coupled to an infinite-dimensional heat bath (System $\mathcal{B}$) via a coupling $\mathcal{C}$. Both System $\mathcal{B}$ and the coupling are linear, while system $\mathcal{A}$ can be nonlinear. Specifically, with $z \in \mathbf{Z}$ denoting the state of system $\mathcal{A}$ and $\eta \in \mathbf{H}$ denoting the state of system $\mathcal{B}$, the total Hamiltonian of the microscopic system is

$$\mathcal{H}_{\text{total}}(z, \eta) = \mathcal{H}_{\mathcal{A}}(z) + \mathcal{H}_{\mathcal{B}}(\eta) + \mathcal{H}_{\mathcal{C}}(z, \eta), \tag{2}$$

where the Hamiltonian for the heat bath is

$$\mathcal{H}_{\mathcal{B}}(\eta) = \frac{1}{2}\|\eta\|^2_{\mathbf{H}} \quad \text{for all } \eta \in \mathbf{H},$$

and the coupling is described by the Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = (\mathbb{A}z|\mathbb{P}\eta)_{\mathbf{H}}\,;$$

here $(\cdot|\cdot)_{\mathbf{H}}$ is the inner product in $\mathbf{H}$, $\mathbb{A}\colon \mathbf{Z} \to \mathbf{H}$ is a linear embedding operator and $\mathbb{P}$ is an orthogonal projection operator discussed later. Below, we write $\mathbb{B} := \mathbb{P}\mathbb{A}$.

One can show that, due to the linearity of the heat bath, the corresponding evolution satisfies for $t \in \mathbb{R}$

$$\dot{z}(t) = \mathbb{J}_{\mathcal{A}}(D\mathcal{H}_{\mathcal{A}}(z(t)) + \mathbb{B}^*\eta(t)), \tag{3a}$$

$$\eta(t) = \mathrm{e}^{\mathbb{J}_{\mathcal{B}}t}\left(\eta(0) + \mathbb{B}z(0)\right) - \mathbb{B}z(t) + \int_0^t \mathrm{e}^{\mathbb{J}_{\mathcal{B}}(t-s)}\mathbb{B}\dot{z}(s)\,\mathrm{d}s, \tag{3b}$$

where $\mathbb{J}_{\mathcal{A}}$ and $\mathbb{J}_{\mathcal{B}}$ are the sympletic operators associated with $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$.

So far, we have not specified the initial condition of the heat bath $\mathcal{B}$. While the coupled microscopic system evolves deterministically, we introduce randomness through the initial data $\eta(0)$ for the heat bath. Then (3b) involves a memory term, namely the time integral in the right, and a stochastic term stemming from the initial data. One can show that one can rephrase this equation as a generalized (i.e., non-Markovian) Langevin equation.

Such non-Markovian equations are often encountered in a Mori-Zwanzig reduction procedure, where suitable projections of an infinite-dimensional microscopic system are considered [6]. Often, when dealing with non-Markovian systems, one tries to rephrase them as Markovian ones by augmenting the state space.

The GENERIC equation (1) is Markovian (in the sense that it is a nonlinear semigroup, hence in particular local, i.e., memoryless in time). Abstractly, due to the presence of the dissipative term $\mathbb{K}(y)D\mathcal{S}(y)$, one expects GENERIC systems to be given by a (nonlinear) contraction semigroup. The evolution given by (3), however, corresponds to a unitary group on $\mathbf{Z} \times \mathbf{H}$. To link these two, we use the concept of compressions. *Compressions* can be seen as the 'inverse' of dilations; the theory of dilations provides an embedding of a given contraction semigroup $\mathrm{e}^{-t\mathbb{D}}$ defined on a Hilbert space $\mathbf{W}$ into a strongly continuous unitary group $\mathrm{e}^{t\mathbb{J}}$ defined on a bigger space such that the restriction of the unitary group to $\mathbf{W}$ agrees with the given contraction semigroup,

$$\mathbb{P}\mathrm{e}^{t\mathbb{J}}\big|_{\mathbf{W}} = \begin{cases} \mathrm{e}^{-t\mathbb{D}} & t \geq 0 \\ \mathrm{e}^{t\mathbb{D}^*} & t \leq 0 \end{cases} \qquad \text{on } \mathbf{W}, \tag{4}$$

where $\mathbb{P}$ is the orthogonal projection on $\mathbf{W}$. A minimal dilation is unique up to Hilbert space isomorphism [5, Chapter 1].

The existence of a compression means that given the unitary group $\mathrm{e}^{t\mathbb{J}}$, a subspace $\mathbf{W}$ and an orthogonal projection $\mathbb{P}$ onto $\mathbf{W}$ and a contraction semigroup $\mathrm{e}^{-t\mathbb{D}}$ exist such that (4) holds. For existence of compressions we refer to [4, Section 5].

The coarse-graining to go from the microscopic evolution (3), i.e., the unitary group associated with (2), to a GENERIC evolution, i.e., a macroscopic evolution which can be cast in the general form (1), that involves as key step finding a (finite-dimensional) compression subspace $\mathbf{Y}$ of $\mathbf{Z} \times \mathbf{H}$ such that the compression of the unitary group on $\mathbf{Y}$ is defined. It is natural that the compression subspace $\mathbf{Y}$ contains $\mathbf{Z}$.

Here a compression of the heat bath can be interpreted as reduction to observables, as elements of $\mathbf{W}$. For the coupled system, a compression subspace is given by $\mathbf{Z} \times \mathbf{W}$, where $\mathbf{W} = \mathbb{P}\mathbf{H}$. Thus observables are (z, w) with $w = \mathbb{P}\eta$. A GENERIC evolution can be formulated if this state space is augmented by a suitably defined energy e. The GENERIC form has the following structure. For the contraction semigroup $\mathbb{D}$, we consider the split in symmetric and skew-symmetric parts,

$$\mathbb{D} = \mathbb{D}_{\mathrm{sym}} + \mathbb{D}_{\mathrm{skw}} \quad \text{with } \mathbb{D}_{\mathrm{sym}} = \frac{1}{2}(\mathbb{D}+\mathbb{D}^*) \;\text{ and }\; \mathbb{D}_{\mathrm{skw}} := \frac{1}{2}(\mathbb{D}-\mathbb{D}^*).$$

Then the symplectic operator $\mathbb{J}$ of the GENERIC evolution (1) in $y := (z, w, e)$ involves $\mathbb{J}_{\mathcal{A}}$ and $\mathbb{D}_{\mathrm{skw}}$, while $\mathbb{D}_{\mathrm{sym}}$ enters the positive semi-definite operator $\mathbb{K}$ of GENERIC. The random initial data of the heat bath enters the entropy S and the dissipative operator $\mathbb{K}$. In passing from the stochastic microscopic system via compression to GENERIC, an interim stage is a stochastic version of GENERIC, which includes a noise term Σ. One can show that a fluctuation-dissipation statement linking $\mathbb{K}$ and Σ holds.

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Analytic properties of the sliced Wasserstein distance

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(joint work with Dejan Slepčev)

Given two probability measures $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d) := \{\mu \in \mathscr{P}(\mathbb{R}^d) \ : \ \int |x|^2 d\mu(x) < \infty\}$, recall that the 2-Wasserstein distance W_2 between them is defined as follows:

$$W_2(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, d\gamma(x,y) \right)^{1/2}$$
$$\text{where } \Gamma(\mu,\nu) = \left\{ \gamma \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d) : \, \pi^1_{\#}\gamma = \mu, \, \pi^2_{\#}\gamma = \nu \right\}.$$

The sliced Wasserstein distance, introduced by Rabin, Peyré, Delon, and Bernot [4], compares probability measures on $\mathbb{R}^d$ by taking averages of the Wasserstein distances between projections of the measures to each 1-dimensional subspaces of $\mathbb{R}^d$. To be more precise, for each $\theta \in \mathbb{S}^{d-1}$ define the projection $\pi^\theta : \mathbb{R}^d \to \mathbb{R}$ by

$$\pi^\theta(x) = \theta \cdot x.$$

The 2-sliced Wasserstein distance SW_2 is defined by

$$SW_2(\mu,\nu) = \left(\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} W_2^2(\pi^\theta_{\#}\mu, \pi^\theta_{\#}\nu) \, d\theta \right)^{\frac{1}{2}}$$

where $_{\#}$ denotes the pushforward of a measure.

Thanks to its lower sample and computational complexities relative to the Wasserstein distance especially in high dimensions, the sliced Wasserstein distance has recently expanded its applications in statistics and machine learning as a tool to compare measures and construct paths in spaces of measures.

In this talk we presented a number of analytic properties of the SW_2 and the sliced Wasserstein length metric ℓ_{SW}, defined as the infimum of the lengths of curves between measures in the SW_2-space. Moreover, we discussed their implications on the sliced Wasserstein gradient flows and statistical estimation rates in the metrics.

Comparison of sliced Wasserstein metric with negative Sobolev norms and Wasserstein metric. To understand the metric properties of the sliced Wasserstein distance, we establish the comparison theorems of SW_2 with negative Sobolev norms near absolutely continuous measures and comparisons of SW_2 with the Wasserstein metric W_2 near discrete measures. In particular, consider an absolutely continuous measure μ bounded away from zero and infinity on some bounded open convex domain Ω. For all measures μ, ν which are within constant multiples of the Lebesgue measure restricted to Ω, we show

(1)
$$\|\mu-\nu\|_{\dot{H}^{-(d+1)/2}(\mathbb{R}^d)} \lesssim SW_2(\mu,\nu) \le \ell_{SW}(\mu,\nu) \lesssim SW_2(\mu,\nu) \lesssim \|\mu-\nu\|_{\dot{H}^{-(d+1)/2}(\mathbb{R}^d)},$$

where the rightmost inequality additionally requires ν to coincide with μ near the boundary of Ω. In other words, near μ, SW_2 is equivalent to $\dot{H}^{-(d+1)/2}$.

On the other hand, we show that

$$SW_2(\mu^n,\nu) \le \ell_{SW}(\mu^n,\nu) \le \frac{1}{d}W_2(\mu^n,\nu) \le (1+o(1))SW_2(\mu^n,\nu) \tag{2}$$

for ν near discrete measures of the form $\mu^n = \sum_{i=1}^n m_i \delta_{x_i}$.

These two results provide interesting insights about the SW_2 measure. Near smooth measures it behaves like a highly negative Sobolev space, in contrast to the Wasserstein metric which for such measures behaves like the $\dot{H}^{-1}$ norm as noted by Peyre [3], while near discrete measures SW_2 behaves like the Wasserstein distance.

Approximation by discrete measures in sliced Wasserstein length. It is known that finite-sample estimation of measures with respect to maximum mean discrepancy (MMD) also enjoys parametric rate [5, Theorem 3.3]. MMD distance is nothing but the norm in the dual of a reproducing kernel Hilbert space (RKHS). In particular the results of [5] apply to the dual of the Sobolev space H^s with $s > \frac{d}{2}$ (when the spaces embeds in the spaces of Hölder continuous functions and are RKHS). The comparison (1) says that near absolutely continuous measures, SW_2 behaves like $\dot{H}^{-(d+1)/2}$-norm; as the associated norm $\|\cdot\|_{H^{-(d+1)/2}(\mathbb{R}^d)}$ is an MMD, we can formally understand SW_2 to exhibit behaviors like an MMD. Thus MMD parametric estimation can be seen as a tangential or a linearized analogue of the finite sample estimation rates in SW_2 distance. Indeed, Manole, Balakrishnan, and Wasserman [2, Proposition 4] have shown that a finite random sample (i.e. the empirical measure of the set of n random points) of a probability measure on $\mathbb{R}^d$ estimates the measure in the sliced Wasserstein distance at a parametric rate, $\frac{1}{\sqrt{n}}$, for a large class of measures.

We establish that finite sample approximation in ℓ_{SW} happens at the parametric rate up to a logarithmic correction, namely that

$$SW_2(\mu,\mu^n) \le \ell_{SW}(\mu,\mu^n) \lesssim \sqrt{\frac{\log n}{n}} \quad \text{with high probability,}$$

where $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ with $X_i \overset{i.i.d.}{\sim} \mu$. This is in stark contrast with the Wasserstein distance where the approximation rate is poor in high dimensions and scales like $n^{-\frac{1}{d}}$.

Implications on gradient flows. The comparison results on ℓ_{SW}, SW_2 can be used to obtain comparisons for the metric slopes. Given a metric space (X,m), recall that metric slope $|\partial\mathcal{E}|_m$ of a functional $\mathcal{E}: X \to \mathbb{R}$ is defined by

$$|\partial\mathcal{E}|_m(u) = \limsup_{v \xrightarrow{m} u} \frac{[\mathcal{E}(u)-\mathcal{E}(v)]_+}{m(u,v)}. \tag{3}$$

Consider the potential energy $\mathcal{V}(\mu) := \int_{\mathbb{R}^d} V(x)\,d\mu(x)$. When V is smooth and compactly supported, for suitable absolutely continuous $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ it holds that

$$|\partial\mathcal{V}|_{\dot{H}^{(d+1)/2}(\mathbb{R}^d)}(\mu) \lesssim |\partial\mathcal{V}|_{\ell_{SW}}(\mu) \le |\partial\mathcal{V}|_{SW}(\mu) \lesssim |\partial\mathcal{V}|_{\dot{H}^{(d+1)/2}(\mathbb{R}^d)}(\mu) \tag{4}$$

whereas the slope behaves quite differently at discrete measures, $\mu^n = \sum_{i=1}^n m_i \delta_{x_i}$, namely that

$$|\partial \mathcal{V}|_{SW_2}(\mu^n) = |\partial \mathcal{V}|_{\ell_{SW}}(\mu^n) = \sqrt{d}\, |\partial \mathcal{V}|_W(\mu^n). \tag{5}$$

Hence $|\partial \mathcal{V}|_{SW_2}$ (resp. $|\partial \mathcal{V}|_{\ell_{SW}}$) is not lower-semicontinuous in SW_2 (resp. ℓ_{SW}) in general, even when $V \in C_c^\infty(\mathbb{R}^d)$. This implies that the potential energy is not λ-geodesically convex in $(\mathscr{P}_2(\mathbb{R}^d), \ell_{SW})$. Consequently, the curves of maximal slope in the Wasserstein space starting from discrete measures with finite number of particles, after a constant rescaling of time, is the curve of maximal slope in SW_2 space.

On the other hand, for smooth measures, the curves of maximal slope with respect to the Wasserstein metric are not curves of maximal slope in SW_2 space. We formally show that SW_2 gradient flow of potential energy is a higher order equation given by a pseudodifferential operator of order d, which is consistent with the rigorous results (4).

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Entropic propagation of chaos for population dynamics

Jasper Hoeksema

Interacting particle systems where particles can be created and deleted form the backbone of several models in ecology, with the particular example of the Bolker-Pacala-Dieckmann-Law (BPDL) model [1], which was originally introduced to study the evolution of and pattern formation in populations of plants, but which turned out to accurately describe models involving mutation of traits.

Various methods exist to derive mean-field limits for these systems, but these sometimes require stringent assumptions on the interactions and use weak notions of convergence. In this talk, we discuss past [6] and current work to alleviate both these restrictions for weakly interacting birth/death processes where, using techniques inspired by convergence of gradient flows for interacting particle systems [3–5], we prove entropic propagation of chaos for the BPDL model.

We model the particles as a collection of points $\{X_t^{n,1}, \dots, X_t^{n,N_t}\}$ in a compact Polish space $\mathcal{X}$, where the parameter n will control the order of the number of particles in the system. We are interested in convergence in a suitable sense of the rescaled empirical measure $\nu_t^n \in \Gamma := \mathcal{M}(\mathcal{X})$ given by

$$\nu_t^n := \frac{1}{n}\sum_{i=1}^{N_t} \delta_{X_t^{n,i}}.$$

Formally, ν_t^n is a measure-valued jump process with generator

$$Q_n F = n\int_{\mathcal{X}} \left(F(\nu + \tfrac{1}{n}\delta_x) - F(\nu)\right)\chi_\nu^+(dx) + n\int_{\mathcal{X}} \left(F(\nu - \tfrac{1}{n}\delta_x) - F(\nu)\right)\chi_\nu^-(dx)$$

with $F \in C_c(\Gamma)$ and the measure-dependent birth/death rates $\chi_\nu^\pm$, which in the case of the BPDL model looks like

$$\chi_\nu^+(dx) = \left(\int_{\mathcal{X}} m(x,y)\nu(dy)\right)\gamma(dx), \qquad \chi_\nu^-(dx) = \left(\int_{\mathcal{X}} c(x,y)\nu(dy)\right)\nu(dx)$$

where m, c are the mutation and competition kernels, and $\gamma \in \Gamma$ is some reference measure.

It is the corresponding forward Kolmogorov equation that is our object of study. It describes the law of ν_t^n, and for a path of measures $(\mathsf{P}_t^n)_{t\in[0,T]}$ over some time horizon $[0,T]$ it satisfies

$$\partial_t \mathsf{P}_t^n = Q_n^* \mathsf{P}_t^n.$$

After a Taylor expansion of $Q_n F$, with F a suitable cylindrical function, one can expect that under suitable conditions

$$\lim_{n\to\infty} Q_n F = Q_\infty F = \int_{\mathcal{X}} (\nabla_\Gamma F)(\nu, x) V_\nu(dx),$$

where $V_\nu(dx) := \chi_\nu^+(dx) - \chi_\nu^-(dx)$, and would surmise that $\lim_{n\to\infty} \mathsf{P}_t^n = \mathsf{P}_t$ narrowly, where P_t satisfies the corresponding Liouville equation with velocity field V. In particular, if $\mathsf{P}_0^n \to \mathsf{P}_0 := \delta_{\bar\nu_t}$ then one would expect that

$$\mathsf{P}_t^n \to \delta_{\bar\nu_t} \tag{1}$$

where $\bar\nu_t$ satisfies the mean-field equation

$$\partial_t \bar\nu_t = V_{\bar\nu_t}.$$

The convergence (1) is known as *propagation of chaos*, and implies narrow convergence of the corresponding correlation functions. However, we are interested in a strictly stronger notion, called *entropic propagation of chaos*, which implies vanishing relative entropy. For suitable bounds on $\bar\nu_t$ the latter can be shown to be equivalent to the statement

$$\lim_{n\to\infty} \frac{1}{n}\mathcal{E}\mathrm{nt}(\mathsf{P}_0^n|\Pi_n) = \int_\Gamma \mathcal{E}\mathrm{nt}(\nu|\gamma)\mathsf{P}_0(d\nu) \Longrightarrow \lim_{n\to\infty} \frac{1}{n}\mathcal{E}\mathrm{nt}(\mathsf{P}_t^n|\Pi_n) = \int_\Gamma \mathcal{E}\mathrm{nt}(\nu|\gamma)\mathsf{P}_\mathsf{t}(d\nu)$$

for all $t \in [0,T]$, with Π_n a rescaled Poisson point measure induced by γ.

In our work we prove this property, using large deviation techniques for the rescaled entropies, lower semicontinuity of the entropy dissipation, and the Sandier-Serfaty approach [2] to obtain convergence.

A key tool is the fact that the rescaled entropy dissipation, in the reversible setting where $m = c$ and $c(x,x) = 0$ for all $x \in \mathcal{X}$, reduces to (with a slight abuse of notation)

$$\mathcal{K}^n(\mathsf{P}) = 2\mathcal{E}\mathrm{nt}\left(\mathsf{P}(d\nu)\chi^+_\nu(dx)\middle|\mathsf{P}\left(d(\nu+\tfrac{1}{n})\right)\chi^-_{\nu+\frac{1}{n}}(dx)\right),$$

and is related to the convergence of the associated gradient flow structures as shown in [6]. In current work we extend this to the irreversible setting.

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Geometric, Algebraic and Topological Combinatorics

Organized by
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Isabella Novik, Seattle
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10 December – 15 December 2023

Abstract. The 2023 Oberwolfach meeting "Geometric, Algebraic, and Topological Combinatorics" was organized by Gil Kalai (Jerusalem), Isabella Novik (Seattle), Francisco Santos (Santander), and Volkmar Welker (Marburg). It covered a wide variety of aspects of Discrete Geometry, Algebraic Combinatorics with geometric flavor, and Topological Combinatorics. Some of the highlights of the conference were (1) Federico Ardila and Tom Braden discussed recent exciting developments in the intersection theory of matroids; (2) Stavros Papadakis and Vasiliki Petrotou presented their proof of the Lefschetz property for spheres, and, more generally, for pseudomanifolds and cycles (this second part is joint with Karim Adiprasito); (3) Gaku Liu reported on his joint work with Spencer Backman that establishes the existence of a regular unimodular triangulation of an arbitrary matroid base polytope.

Mathematics Subject Classification (2020): 05Exx, 52Bxx, 13Dxx,13Fxx, 14Txx, 52Cxx, 57Qxx, 90Cxx.

Introduction by the Organizers

The 2023 Oberwolfach meeting "Geometric, Algebraic, and Topological Combinatorics" was organized by Gil Kalai (Hebrew University, Jerusalem), Isabella Novik (University of Washington, Seattle), Francisco Santos (University of Cantabria, Santander), and Volkmar Welker (Philipps-Universität Marburg, Marburg).

The conference featured three 1-hour talks by Federico Ardila on "Intersection theory of matroids", Eran Nevo on "Rigidity expander graphs", and by Tom Braden on "The intersection cohomology module of a matroid", two back-to-back 35-minute talks by Vasiliki Petrotou and Stavros Papadakis "Lefschetz properties via anisotropy", a 50-minute talk by Nati Linial on "Some stories about graphs and geometry", and 23 additional talks, ranging from 30 to 40 minutes. On Thursday evening we held a problem session. After and before the lectures many small groups embarked in discussions, some of which initiated new collaborations. All together it was a very productive, intense and enjoyable week.

The conference covered a broad spectrum of topics from Algebraic Combinatorics (intersection cohomology modules, Lefschetz theorems, Koszul duality), Topological Combinatorics (configuration spaces, envy-free partitions, random complexes), and Geometric Combinatorics (face enumeration, polytope theory, matroid polytopes, lattice polytopes, rigidity theory).

In the next paragraphs we summarize the richness and depth of the work and the presentations, concentrating on some of the highlights.

The first lecture on Monday, by Federico Ardila (based on his Clay lecture at the British Combinatorial Conference 2024, see F. Ardila-Mantilla, Intersection theory of matroids: Variations on a theme, in: Surveys in Combinatorics 2024, pp. 1-30, Cambridge University Press, 2024) discussed four different ways to define the Chow ring of a toric variety due to Billera, Brion, Fulton–Sturmfels, and Allermann–Rau. Federico also explained how the different representations of the Chow ring enable different proofs of recent spectacular combinatorial results such as unimodality of the coefficients of chromatic polynomials.

Gaku Liu's talk then presented an ingenious inductive proof that every matroid base polytope has a regular unimodular triangulation.

The rest of Monday was devoted to a variety of topics in algebraic and geometric combinatorics. For instance, Eran Nevo discussed a proof of the existence of an infinite family of k-regular d-rigidity-expander graphs for every $d \geq 2$ and $k \geq 2d + 1$.

Tuesday morning focused on topological combinatorics. Florian Frick talked about topological methods in zero-sum Ramsey theory. Pablo Soberon discussed high-dimensional envy-free partitions. Kevin Piterman talked about fixed-point-free actions of finite groups on contractible spaces. More specifically, Kevin presented a solution to a central problem about the existence of fixed points for every finite group acting on a compact 2-complex. Finally, Roy Meshulam's lecture on random balanced Cayley complexes was a very rich blend of combinatorial, topological, Fourier-theoretical, and algebraic methods.

On Tuesday afternoon we had several talks related to polytope theory and in particular to lattice polytopes.

Wednesday morning started with an hour lecture by Tom Braden. This lecture complemented Ardila's talk from Monday morning reporting on recent fascinating developments in the matroid theory; this time via the lens of Algebraic Geometry.

The second part of Wednesday morning consisted of two back-to-back talks by Stavros Papadakis and Vasiliki Petrotou. They discussed their notion of anisotropicity of simplicial spheres which led to their proof of the Lefschetz property for spheres, and, more generally, for pseudomanifolds and cycles (this second part is joint with Adiprasito). The Lefschetz property, in turn, leads to a simpler proof of the g-conjecture for spheres. Their talks were followed by Christos Athanasiadis' talk on face enumeration and real-rootedness.

On Thursday, the focus returned to topological questions, a highlight being Geva Yashfe's talk about the number of triangulations of homology 3-spheres.

Friday morning was devoted to a mixture of topics in polyhedral geometry and hyperplane arrangements. The final lecture of the conference was given by Nati Linial who discussed recent progress on geodetic and metrizable graphs.

It bears repeating that numerous breakthrough results were announced and presented during the conference.

We are extremely grateful to the Oberwolfach institute, its directorate and to all of its staff for providing a perfect setting for an inspiring, intensive week of "Geometric, Algebraic, and Topological Combinatorics".

Gil Kalai, Isabella Novik, Francisco Santos, Volkmar Welker
Jerusalem/Seattle/Santander/Marburg, April 2024

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Workshop: Geometric, Algebraic and Topological Combinatorics

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Abstracts

Intersection theory of matroids: techniques and examples

Federico Ardila

Chow rings of toric varieties, which originate in intersection theory, feature a rich combinatorial structure of independent interest. We survey four different ways of computing in these rings, due to Billera, Brion, Fulton–Sturmfels, and Allermann–Rau. We illustrate the beauty and power of these methods by giving four proofs of Huh and Huh–Katz's formula $\mu^k(M) = deg_M(\alpha^{r-k}\beta^k)$ for the coefficients of the reduced characteristic polynomial of a matroid M as the mixed intersection numbers of the hyperplane and reciprocal hyperplane classes α and β in the Chow ring of M. Each of these proofs sheds light on a different aspect of matroid combinatorics, and provides a framework for further developments in the intersection theory of matroids. Our presentation is combinatorial, and does not assume previous knowledge of toric varieties, Chow rings, or intersection theory.

Face enumeration of order complexes and real-rootedness

Christos A. Athanasiadis
(joint work with Katerina Kalampogia-Evangelinou)

Given a finite poset P, the order complex $\Delta(P)$ is the abstract simplicial complex which consists of all chains in P. Order complexes of Cohen–Macaulay posets form a class of flag simplicial complexes with especially nice properties [12, Section III.4]. Somewhat unexpectedly, their face enumeration is far from being well understood. We aim to show that their f-polynomials (equivalently, their h-polynomials) tend to be real-rooted surprisingly often by discussing examples and methods that can be applied.

Let us denote by $c_k(P)$ the number of k-element chains in P. The f-polynomial and the h-polynomial of $\Delta(P)$ are then defined as

$$f(\Delta(P),x) = \sum_{k=0}^{n} c_k(P)x^k,$$
$$h(\Delta(P),x) = \sum_{k=0}^{n} c_k(P)x^k(1-x)^{n-k} = (1-x)^n f\left(\frac{x}{1-x}\right),$$

where n is the largest cardinality of a chain in P. The polynomial $f(\Delta(P),x)$ is also called the chain polynomial of P. We recall that $h(\Delta(P),x)$ has nonnegative coefficients for every Cohen–Macaulay poset P (see [12, Chapter II]) and that $f(\Delta(P),x)$ is real-rooted (meaning, all its roots are real) if and only if so is $h(\Delta(P),x)$.

Our motivation comes from the following two conjectures. The first was posed as a question by Brenti–Welker [9] and claims that barycentric subdivisions of convex polytopes have real-rooted h-polynomials.

Conjecture 1. (cf. [9, Question 1]) *The polynomial $h(\Delta(P), x)$ is real-rooted if P is the face lattice of a convex polytope.*

Conjecture 2. ([3, Conjecture 1.2]) *The polynomial $h(\Delta(P), x)$ is real-rooted for every geometric lattice P (equivalently, if P is the lattice of flats of a matroid).*

The latter conjecture would imply the unimodality of the h-polynomials of order complexes of geometric lattices. These two conjectures naturally raise the following question.

Question 3. ([3, Question 1.1]) *For which finite Cohen–Macaulay posets P is $h(\Delta(P), x)$ real-rooted? Equivalently, which finite Cohen–Macaulay posets have a real-rooted chain polynomial?*

Let us briefly discuss some answers to Question 3 which are known in interesting special cases. For distributive lattices the question is known to be equivalent to the Neggers conjecture [10] (see also [5] [11, Conjecture 1]), which claims the real-rootedness of poset Eulerian polynomials. Thus, there exist distributive lattices which fail to have real-rooted chain polynomials [13]. On the other hand, classes of Cohen–Macaulay posets with real-rooted chains polynomials include some classes of distributive lattices [8, 14] and:

- Cohen–Macaulay simplicial posets [9] (in particular, face lattices of simplicial or simple polytopes) and all their rank-selected subposets [4];
- CL-shellable cubical posets [2] (in particular, face lattices of cubical polytopes);
- the face lattices of the pyramid and the prism over polytopes which have a face lattice with real-rooted chain polynomial [3];
- partition lattices of types A and B and subspace lattices [3];
- the lattices of flats of paving matroids [7] and those of near-pencils, uniform matroids and all matroids on at most nine elements [3];
- all noncrossing partition lattices associated to irreducible finite Coxeter groups [4].

A popular method of proof is to express $h(\Delta(P), x)$ as a nonnegative linear combination of real-rooted polynomials with positive leading coefficients which form an interlacing sequence (or, more generally, which have a common interleaver); see [6, Section 7.8] and references therein for the relevant background. We illustrate this method in two cases in which it has been successful, namely those of simplicial and cubical posets (see [12, Section II.6] and [1] for background on simplicial and cubical posets and their h-vectors).

Theorem 4. (cf. [9, Theorems 1 and 2]) *For every positive integer n, there exists an interlacing sequence $(p_{n,k}(x))_{0 \le k \le n}$ of real-rooted polynomials with nonnegative coefficients such that*

$$h(\Delta(P), x) = \sum_{k=0}^{n} h_k(P) p_{n,k}(x)$$

for every simplicial poset P of rank n, where $(h_k(P))_{0\le k\le n}$ is the simplicial h-vector of P. In particular, $h(\Delta(P),x)$ is real-rooted for every simplicial poset P with nonnegative simplicial h-vector.

The polynomial $p_{n,k}(x)$ can be defined by the formula

$$\sum_{m\ge 0} m^k(1+m)^{n-k}x^m = \frac{p_{n,k}(x)}{(1-x)^{n+1}},$$

or as the descent enumerator of permutations w of $\{1,2,\ldots,n+1\}$ such that $w(1)=k+1$.

Theorem 5. ([2]) *For every nonnegative integer n, there exists an interlacing sequence $(p^B_{n,k}(x))_{0\le k\le n+1}$ of real-rooted polynomials with nonnegative coefficients such that*

$$h(\Delta(Q),x) = \sum_{k=0}^{n+1} h_k(Q)p^B_{n,k}(x)$$

for every cubical poset Q of rank $n+1$, where $(h_k(Q))_{0\le k\le n+1}$ is the cubical h-vector of Q. In particular, $h(\Delta(Q),x)$ is real-rooted for every cubical poset Q which has a nonnegative cubical h-vector.

The polynomials $p^B_{n,k}(x)$ can be defined by the formula

$$\frac{p^B_{n,k}(x)}{(1-x)^{n+1}} = \begin{cases} \sum_{m\ge 0}(2m+1)^n x^m, & \text{if } k=0,\\ \sum_{m\ge 0}(4m)(2m-1)^{k-1}(2m+1)^{n-k}x^m, & \text{if } 1\le k\le n,\\ \sum_{m\ge 1}(2m-1)^n x^m, & \text{if } k=n+1. \end{cases}$$

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Convex partitions in a slice

Pavle V. M. Blagojević
(joint work with Michael C. Crabb)

1. Convex partitions of Euclidean spaces

Problems related to the existence of convex partitions of a Euclidean space of a desired type have a long and rich history, starting with the 1930's ham-sandwich theorem of Steinhaus and Borsuk as the most famous example. The ham-sandwich theorem claims that for every collection of d proper convex bodies $C_1, \dots, C_d$ in $\mathbb{R}^d$ there exists a convex partition of $\mathbb{R}^d$ into 2 pieces A_1 and A_2 such that

$$\mathrm{vol}_d(C_1 \cap A_1) = \mathrm{vol}_d(C_1 \cap A_2)\,,\ \dots\ ,\ \mathrm{vol}_d(C_d \cap A_1) = \mathrm{vol}_d(C_d \cap A_2).$$

Here, the closed convex sets A_1 and A_2 with non-empty interior form a convex partition of $\mathbb{R}^d$ if $A_1 \cup A_2 = \mathbb{R}^d$ and $\mathrm{int}(A_1) \cap \mathrm{int}(A_2) = \emptyset$, (hence $\mathrm{vol}_d(A_1 \cap A_2) = 0$).

A natural extension is the question: For given integers $d, k, j \geq 1$ and an arbitrary collection $\mathcal{C}$ of j proper convex bodies in $\mathbb{R}^d$ is it possible to find k affine hyperplanes such that every orthant Ω determined by them contains the same piece of each convex body in $\mathcal{C}$, that is $\mathrm{vol}_d(C \cap \Omega) = \frac{1}{2^k}\mathrm{vol}_d(C)$ for every $C \in \mathcal{C}$. The work on this generalisation of the ham-sandwich theorem, the so called Grünbaum–Hadwiger–Ramos problem, was pioneered by Grünbaum [11], Hadwiger [12] and Avis [2], and a bit later continued by Edgar Ramos [16]. Topological challenges emerging in the process of solving this problem were recently discussed in [7].

In 2006 Nandakumar & Ramana-Rao asked for a solution of the following intriguing problem: Is it true that for every integer $n \geq 2$ and every proper convex body C in the plane there is a convex partition of the plane into n pieces $A_1, \dots, A_n$ having equal area and equal perimeter, that is

$$\mathrm{vol}_2(C \cap A_1) = \dots = \mathrm{vol}_2(C \cap A_n) \quad \text{and} \quad \mathrm{per}(C \cap A_1) = \dots = \mathrm{per}(C \cap A_n),$$

where "per" denotes the plane perimeter function. This naive-looking question caught a lot of attention and many authors contributed to its better understanding. For more details see the work of Bárány, Blagojević & Szűcs [3], Soberón [17], Karasev, Hubard & Aronov [14], Blagojević & Ziegler [10], and Blagojević & Sadovek [8]. Recently a promising work of Akopyan, Avvakumov & Karasev offered a new insight into a complete solution of the original, plane, Nandakumar & Ramana-Rao problem [1]. The work on a solution of this problem brought into

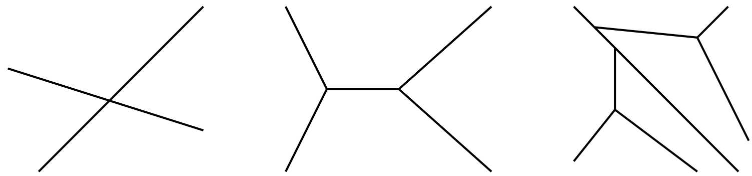

FIGURE 1. Convex partitions of the plain by orthants into 4 pieces, by generalised Voronoi diagram into 6 pieces, and by iteration of depth 2 into $6 = 2 \cdot 3$ pieces.

focus convex partitions generated by generalised Voronoi diagrams of $\mathbb{R}^d$ which in addition equipart a fixed convex body into pieces of equal volume. Surprisingly such convex partitions can be completely parametrised by the configuration space of pairwise distinct points in $\mathbb{R}^d$. Each point in the space corresponds to a collection of the so-called "sites" of a generalised Voronoi diagram.

Iterated convex partitions appeared for the first time in context of the Gromov's waist of the sphere theorem. Gromov worked with the partitions into 2^i pieces, which can be parametrised by the wreath products of spheres. For a different waist of the sphere result, Palić with Blagojević and Karasev in [15] considered iterated convex partitions into p^k pieces indexed by the kth wreath product of the configuration spaces. Iterated partitions appeared also in the work of Blagojević & Soberón [9], where they were parametrised by the join of the configuration space. The most general iterated convex partition in the context of the Nandakumar & Ramana-Rao problems were recently considered by Blagojević & Sadovek in [8].

A general convex equipartition problem can be formulated as follows.

Problem (Convex partitions of a Euclidean space) Let $d, j, n \geq 1$ be fixed integers, $\mathcal{C}$ an arbitrary collection of j proper convex bodies in $\mathbb{R}^d$ and let $\mathcal{P}$ be a predetermined class of convex partitions of $\mathbb{R}^d$, like partitions by orthants, by (generalised) Voronoi diagrams, or by iterated convex partition (see Figure 1 for an illustration). *Is there a partition* $(A_1, \dots, A_n)$ *of* $\mathbb{R}^d$ *from the class* $\mathcal{P}$ *with the property that* $\mathrm{vol}_d(C \cap A_1) = \cdots = \mathrm{vol}_d(C \cap A_n)$ *for every convex body* $C \in \mathcal{C}$.

2. Convex partitions of Euclidean vector bundles

Motivated by the classical problems of convex partitions of a Euclidean space we ask whether a similar result can be obtained if instead of one (ambient) Euclidean space we consider a (parametrised) family of Euclidean spaces and look for a convex partition of at least one of these spaces satisfying the desired property. A prototype of the problems we want to address can be phrased in the following way.

Problem (Convex partitions of tautological vector bundles) Let $d, j, n, k \geq 1$ be fixed integers, $\mathcal{C}$ a collection of j proper convex bodies in $\mathbb{R}^d$ with the origin in their interiors and let $\mathcal{P}$ be a predetermined class of convex partitions of $\mathbb{R}^k$. *Is there an* ℓ*-dimensional linear subspace* L *of* $\mathbb{R}^d$ *and a partition* $(A_1, \dots, A_n)$ *of* L

from the class $\mathcal{P}$ *with the property that* $\mathrm{vol}_d(C \cap L \cap A_1) = \cdots = \mathrm{vol}_d(C \cap L \cap A_n)$ *for every convex body* $C \in \mathcal{C}$.

In the case of partitions by orthants in a tautological vector bundle Blagojević, Calles Loperena, Crabb & Dimitrijević Blagojević, using the parametrised Fadell–Husseini index theory and delicate spectral sequence computations, proved the following result [4, Thm. 1.5].

Theorem 1. *Let* $d, j, n, k, \ell \geq 1$ *be fixed integers,* $\mathcal{C}$ *a collection of* j *proper convex bodies in* $\mathbb{R}^d$ *with the origin in theirs interiors such that* $1 \leq k \leq \ell$ *and* $d \geq 2^{\lfloor \log_2 j \rfloor}(2^{k-1} - 1) + j$. *There exists an* ℓ*-dimensional linear subspace* L *of* $\mathbb{R}^d$ *and* k *affine hyperplanes in* L *such that* $\mathrm{vol}_\ell(C \cap L \cap \Omega) = \frac{1}{2^k}\mathrm{vol}_\ell(C \cap L)$ *for every* $C \in \mathcal{C}$ *and every orthant* $\Omega \subseteq L$ *determined by the affine hyperplanes.*

In the followup work, Blagojević & Crabb [5] gave the complete treatment of a problem of convex partitions by orthants on Euclidean vector bundles. Using a new insight they reprove known results and extend them to arbitrary Euclidean vector bundles putting various types of constraints on the solutions. Furthermore, the developed methods allowed them to give new proofs and extend results of Guth & Katz, Schnider and Soberón & Takahashi.

Levinson, in collaboration with Blagojević & Crabb, considered the problem of convex partitions in Euclidean vector bundle by generalised Voronoi diagrams [6, 13]. An example of the results they obtained is the following theorem.

Theorem 2. *Let* $d \geq 2$, $j \geq 1$, *and* $1 \leq \ell \leq d$ *be integers and let* n *an odd prime. Consider an arbitrary collection* $\mathcal{C}$ *of* j *proper convex bodies in* $\mathbb{R}^d$ *with the origin in theirs interiors. If* $j \leq d-2$, *then there exists an* ℓ*-dimensional linear subspace* L *of* $\mathbb{R}^d$ *and a convex partition* $A_1, \dots, A_n$ *of* L *by a generalised Voronoi diagram such that* $\mathrm{vol}_\ell(C \cap L \cap A_1) = \cdots = \mathrm{vol}_\ell(C \cap L \cap A_n)$, *for every convex body* $C \in \mathcal{C}$.

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The intersection cohomology module of a matroid

Tom Braden
(joint work with June Huh, Jacob Matherne, Nicholas Proudfoot, Botong Wang)

Let M be a matroid of rank d on the ground set $[n]$, and let $\mathcal{L} = \mathcal{L}(\mathrm{M})$ be its lattice of flats. The number of flats of rank k is $W_k = W_k(\mathrm{M})$, the k^{th} *Whitney number* of the second kind of M. In the 1975 paper [8], Dowling and Wilson proved that if $k \le d/2$ then

$$W_0 + W_1 + \cdots + W_k \le W_{d-k} + W_{d-k+1} + \cdots + W_d.$$

They also made the stronger conjecture that

$$W_k \le W_{d-k},$$

which has become known as the Dowling–Wilson or "top-heavy" conjecture for matroids. It was proved for $d = 3$ by de Bruijn and Erdős [6], and for $k = 1$ by Basterfield and Kelly [1]. When M is realizable it was proved by Huh and Wang [9] using the intersection cohomology of an associated algebraic variety, and it is proved in general in [4], by defining the intersection cohomology combinatorially for an arbitrary matroid, and showing that it has the required properties.

If M is realized by vectors $v_1, v_2, \ldots, v_n$ spanning a vector space V over $\mathbb{C}$, then

$$\xi \mapsto (\xi(v_1), \ldots, \xi(v_n))$$

gives an injection $V^* \hookrightarrow \mathbb{C}^n$. Huh and Wang considered the singular variety Y which is the closure of the image of V^* inside $(\mathbb{P}^1)^n$. It has a decomposition into affine spaces indexed by elements of $\mathcal{L}$, which implies that its odd cohomology vanishes, and that $\dim_{\mathbb{Q}} H^{2k}(Y;\mathbb{Q}) = W_k$. Its intersection cohomology $IH^*(Y;\mathbb{Q})$ is a module over the cohomology ring, and the hard Lefschetz theorem says that for an ample class $\ell \in H^2(Y;\mathbb{Q})$ and $k \le 2d$, the multiplication

$$\ell^{d-2k} \colon IH^{2k}(Y;\mathbb{Q}) \to IH^{2d-2k}(Y;\mathbb{Q})$$

is an isomorphism. Huh and Wang then appeal to a theorem of Björner and Ekedahl [2], which says that the cohomology of Y injects into the intersection cohomology $IH^*(Y;\mathbb{Q})$ as $H^*(Y;\mathbb{Q})$-modules. This implies that the multiplication

$$\ell^{d-2k}\colon H^{2k}(Y;\mathbb{Q}) \to H^{2d-2k}(Y;\mathbb{Q})$$

is an injection, proving the Dowling–Wilson conjecture in this case.

In [4] we consider combinatorial avatars of the cohomology ring and intersection cohomology module which make sense for any matroid M. The cohomology ring is replaced by the *graded Möbius algebra* H(M), which has a $\mathbb{Q}$-basis the symbols y_F, with multiplication

$$y_F y_G = \begin{cases} y_{F\vee G} & \text{if}\,\text{rank}(F\vee G) = \text{rank}\,F + \text{rank}\,G \\ 0 & \text{otherwise.}\end{cases}$$

It is the associated graded of the usual Möbius algebra under the filtration by rank. If M is realized by vectors as above, then H(M) is isomorphic to the cohomology ring of Y, with degrees halved.

The main result of [4] is the construction of a graded H(M)-module IH(M), the *intersection cohomology module* of M. It satisfies the following properties:

(1) There is an element $1 \in \mathrm{IH}^0(\mathrm{M})$ so that $y \mapsto y \cdot 1$ defines an injection of H(M) into IH(M),
(2) its graded dual $\mathrm{IH}(\mathrm{M})^*$ is isomorphic to $\mathrm{IH}(\mathrm{M})[d]$,
(3) it satisfies hard Lefschetz: if $\ell = \sum_{\mathrm{rank}\,F=1} c_F y_F$ where all $c_F > 0$, then

$$\ell^{d-2k}\cdot\, : \mathrm{IH}^k(\mathrm{M}) \to \mathrm{IH}^{d-k}(\mathrm{M})$$

is an isomorphism for $k \leq d/2$,
(4) the Hodge–Riemann bilinear relations: the restriction of the pairing

$$(a,b) \mapsto (-1)^k \langle \ell^{d-2k} a, b\rangle$$

to the kernel of multiplication by ℓ^{d-2k+1} in $\mathrm{IH}^k(\mathrm{M})$ is positive definite, where $\langle , \rangle$ is the pairing on IH(M) induced by an isomorphism as in (2), normalized so that $\langle y_{[n]} \cdot 1, 1\rangle = 1$.

Properties (1) and (3) are enough to deduce the Dowling–Wilson conjecture, but the proof of (3) involves a complicated induction in which all four statements are needed for all matroids on smaller ground sets.

The module IH(M) is constructed as a direct summand of the *augmented Chow ring* CH(M), which was defined in [3]. It is the graded algebra generated over $\mathbb{Q}$ in degree 1 by x_F, $F \in \mathcal{L}(\mathrm{M}) \setminus \{[n]\}$ and y_i, $i \in [n]$, subject to the relations

- $x_F x_G = 0$ if F, G are not comparable,
- $y_i = \sum_{i\notin F} x_F$, and
- $y_i x_F = 0$ if $i \notin F$.

There is an injection $\mathrm{H}(\mathrm{M}) \hookrightarrow \mathrm{CH}(\mathrm{M})$ which sends y_F to $\prod_{i\in B} y_i$, where B is any basis of F. By Krull-Schmidt, the direct summands of CH(M) as a graded H(M)-module are unique up to isomorphism and permutation. Up to isomorphism, IH(M) is the unique direct summand which contains 1. In [4], a particular summand representing IH(M) is defined by a complicated inductive procedure which does not depend on any choices.

A more intrinsic characterization of IH(M) is given by the following forthcoming result. For an upwardly closed subset $\Sigma \subset \mathcal{L}(\mathrm{M})$, $\Upsilon_\Sigma := \mathrm{span}\{y_F \mid F \in \Sigma\}$ is an ideal of H(M).

Theorem ([5]). *Up to isomorphism,* IH(M) *is the unique graded* H(M)*-module satisfying:*

(1) IH(M) *is indecomposable and* $y_{[n]}\mathrm{IH}(\mathrm{M}) \neq 0$*,*
(2) $\mathrm{IH}(\mathrm{M})^* \cong \mathrm{IH}(\mathrm{M})[d]$*, and*
(3) *for any upwardly closed sets* $\Sigma_1, \Sigma_2 \subset \mathcal{L}(\mathrm{M})$*,*

$$\Upsilon_{\Sigma_1}\mathrm{IH}(\mathrm{M}) \cap \Upsilon_{\Sigma_2}\mathrm{IH}(\mathrm{M}) = \Upsilon_{\Sigma_1 \cap \Sigma_2}\mathrm{IH}(\mathrm{M}).$$

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Local h^*-polynomials for one-row Hermite normal form simplices

Benjamin Braun
(joint work with Esme Bajo, Giulia Codenotti, Johannes Hofscheier, Andrés R. Vindas-Meléndez)

This talk is based on the preprint [2]. The local h^*-polynomial of a lattice polytope is an important invariant arising in Ehrhart theory. When the polytope S is a simplex, the local h^*-polynomial is often called the *box polynomial* and denoted $B(S; z)$. Our focus in this work is the study of $B(S; z)$ for lattice simplices presented in Hermite normal form with a single non-trivial row, i.e., simplices S such that the vertices of S are the rows of a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{d-2} & a_{d-1} & N \end{bmatrix}$$

with $0 \leq a_i < N$ for all i. We prove that when the off-diagonal entries are fixed, the distribution of coefficients for the local h^*-polynomial of these simplices has a limit as the normalized volume N goes to infinity. More precisely, we prove the following:

Theorem 1. *Fix $a_1, \ldots, a_{d-1} \in \mathbb{Z}_{\geq 1}$ and let*

$$M := \operatorname{lcm}\big(a_1, \ldots, a_{d-1}, -1 + \sum_{i=1}^{d-1} a_i \big).$$

Let S_N denote the simplex defined by the matrix above, where the values of a_i are held constant for varying N. Let k be a positive integer and $0 \leq r \leq M-1$. Then we have that

$$\lim_{k\to\infty} B(S_{kM+r}; z)/B(S_{kM+r}; 1) = B(S_{M+1}; z)/B(S_{M+1}; 1)\,.$$

It follows that if $B(S_{M+1}; z)$ is strictly unimodal, i.e., if the coefficients are unimodal with strict increases and strict decreases, then $B(S_{kM+r}; z)$ is strictly unimodal for all sufficiently large k.

It is known by work of Adiprasito, Papadakis, Petrotou, and Steinmeyer [1] that if a lattice polytope P has the integer decomposition property, then the local h^*-polynomial has unimodal coefficients. One notable aspect of Theorem 1 is that experiments with random simplices in one-row Hermite normal form suggests that unimodality is frequently present even when S does not have the integer decomposition property. It would thus be interesting to further investigate unimodality of local h^*-polynomials for S_{M+1} for various sequences $a_1, \ldots, a_{d-1}$.

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Poincaré-extended ab-index

Galen Dorpalen-Barry
(joint work with Joshua Maglione, Christian Stump)

Grunewald, Segal, and Smith introduced the subgroup zeta function of finitely-generated groups [8], and Du Sautoy and Grunewald gave a general method to compute such zeta functions using p-adic integration and resolution of singularities [6]. This motivated Voll and the second author to examine the setting where the multivariate polynomials factor linearly. They found that the p-adic integrals are specializations of multivariate rational functions depending only on the combinatorics of the corresponding hyperplane arrangement [10]. After a natural specialization, its denominator greatly simplifies, and they conjecture that the numerator polynomial has nonnegative coefficients.

In this work, we prove their conjecture, which is related to the poles of these zeta functions. Specifically, we reinterpret these numerator polynomials by introducing and studying the *(Poincaré-)extended* **ab**-*index*, a polynomial generalizing both the *Poincaré polynomial* and **ab**-*index* of the *intersection poset* of the arrangement. These polynomials have been studied extensively in combinatorics, although from different perspectives. The coefficients of the Poincaré polynomial have interpretations in terms of the combinatorics and the topology of the arrangement [5, Section 2.5]. The **ab**-index, on the other hand, carries information about the order complex of the poset and is particularly well-understood in the case of face posets of oriented matroids—or, more generally, Eulerian posets. In those settings, the **ab**-index encodes topological data via the *flag f-vector* [1].

We study the extended **ab**-index in the generality of graded posets admitting R-labelings. This class of posets includes intersection posets of hyperplane arrangements and, more generally, geometric lattices and geometric semilattices. We show that the extended **ab**-index has nonnegative coefficients by interpreting them in terms of a combinatorial statistic. This generalizes statistics given for the **ab**-index by Billera, Ehrenborg, and Readdy [3] and for the pullback **ab**-index (defined below) by Bergeron, Mykytiuk, Sottile and van Willigenburg [2]. This interpretation proves the aforementioned conjecture [10], as well as a related conjecture from Kühne and the second author [9].

Motivated by the proofs of these conjectures, we describe a close relationship between the Poincaré polynomial and the **ab**-index by showing that the extended **ab**-index can be obtained from the **ab**-index by a suitable substitution. This recovers, generalizes and unifies several results in the literature. Concretely, special cases of this substitution were observed by Billera, Ehrenborg and Readdy for lattices of flats of *oriented matroids* [3], by Saliola and Thomas for lattices of flats of *oriented interval greedoids* [11], and by Ehrenborg for *distributive lattices* [7].

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Discrete homotopy theory

Daniel Carranza
(joint work with Chris Kapulkin)

Discrete homotopy theory, introduced by H. Barcelo and collaborators, is a homotopy theory of (simple) graphs. Homotopy invariants of graphs have found numerous applications, for instance, in the theory of matroids, hyperplane arrangements, topological data analysis, and combinatorial time series analysis. Discrete homotopy theory is also a special instance of a homotopy theory of simplicial complexes, developed by R. Atkin, to study social and technological networks.

I will report on joint work with C. Kapulkin on developing a new foundation for discrete homotopy theory, based on the homotopy theory of cubical sets. To demonstrate the robustness of this foundation, we use it to prove a conjecture of Babson, Barcelo, de Longueville, and Laubenbacher from 2006 relating homotopy groups of a graph to the homotopy groups of a certain cubical complex associated to it.

Topological methods in zero-sum Ramsey theory

Florian Frick

(joint work with Jacob Lehmann Duke, Meenakshi McNamara, Hannah Park-Kaufmann, Steven Raanes, Steven Simon, Darrion Thornburgh, and Zoe Wellner)

A 1961 result of Erdős, Ginzburg, and Ziv [4] guarantees that any sequence $a_1, \dots, a_{2n-1} \in \mathbb{Z}/n$ of length $2n-1$ of integers modulo n has a subsequence of length n that sums to zero. Algebraic techniques, such as the Chevalley–Warning theorem, have proven fruitful in deriving numerous variants and extensions of the original Erdős–Ginzburg–Ziv theorem; see Caro [3] for a survey of these results, which are collectively known as zero-sum Ramsey theory. We develop an equivariant-topological framework to derive zero-sum results in combinatorial number theory; see [5] for full details.

Observe that general zero-sum Ramsey results may be phrased as follows: Let H be an n-uniform hypergraph on ground set V and let $c\colon V \to \mathbb{Z}/n$; decide whether there is a $\sigma \in H$ with $\sum_{v\in\sigma} c(v) = 0$. We refer to any function $c\colon V \to \mathbb{Z}/n$ as a *$\mathbb{Z}/n$-coloring* of H and call $\sigma \in H$ with $\sum_{v\in\sigma} c(v) = 0$ a *zero-sum hyperedge*. The original Erdős–Ginzburg–Ziv theorem in this language states that any $\mathbb{Z}/n$-coloring of the complete n-uniform hypergraph on ground set $\{1, 2, \dots, 2n-1\}$ has a zero-sum hyperedge.

Comparing the setup above to that of classical hypergraph colorings, where one is interested in the existence of a coloring that avoids monochromatic hyperedges (that is, hyperedges where c is constant), observe that avoiding zero-sum hyperedges is a stronger condition. Equivariant-topological techniques, as first developed in this context by Alon, Frankl, and Lovász [1] and Kríž [6], provide strong obstructions for the existence of colorings without monochromatic hyperedges. It is thus natural to ask, whether these methods may also be applied in the more restrictive setting of obstructing colorings without zero-sum hyperedges. Our work shows that this is indeed possible.

To each n-uniform hypergraph H on ground set V associate a topological space that is symmetric with respect to a natural action by $\mathbb{Z}/n$, and in fact by the symmetric group, although we will not make use of this generality. This symmetric space is built as a simplicial complex, the *box complex* $B(H)$: For pairwise disjoint $A_0, \dots, A_{n-1} \subseteq V$ let $A_0 \times \{0\} \cup \dots \cup A_{n-1} \times \{n-1\}$ be in $B(H)$ if for all $a_0 \in A_0, \dots, a_{n-1} \in A_{n-1}$ we have that $\{a_0, \dots, a_{n-1}\} \in H$. Thus $B(H)$ is a simplicial complex on $V \times \mathbb{Z}/n$ that is symmetric with respect to the natural $\mathbb{Z}/n$-action on the second factor. Denote the d-dimensional sphere by S^d. For odd d we fix a free action by the cyclic group $\mathbb{Z}/n$ on S^d. The following gives a topological criterion for existence of zero-sum hyperedges for any $\mathbb{Z}/p$-coloring of a hypergraph for p a prime:

Theorem 1. *Let $p \geq 2$ be a prime, and let H be a p-uniform hypergraph. If there is no $\mathbb{Z}/p$-equivariant map $B(H) \to S^{2p-3}$, then any $\mathbb{Z}/p$-coloring of H has a zero-sum hyperedge.*

In particular, if $B(H)$ is homotopically $(2p-3)$-connected then any $\mathbb{Z}/p$-coloring of H has a zero-sum hyperedge. If H is the complete p-uniform hypergraph on $\{1,\dots,2p-1\}$ then the faces of $B(H)$ consists of p-tuples of pairwise disjoint sets in $\{1,\dots,2p-1\}$. This simplical complex is $(2p-3)$-connected, which recovers the result of Erdős, Ginzburg, and Ziv for p a prime. In the same way as for the standard algebraic proofs of this theorem, the general case then follows by a simple induction on prime divisors.

A $\mathbb{Z}/p$-coloring of H without zero-sum hyperedge induces a simplex-wise linear $\mathbb{Z}/p$-equivariant map $B(H) \to \mathbb{R}^{2p-2} \setminus \{0\}$. Using this same approach now for an arbitrary finite group G instead of $\mathbb{Z}/p$ and convex-geometric results to ascertain the existence of zeros of G-equivariant simplex-wise linear maps, yields Olson's generalization [7] of the Erdős–Ginzburg–Ziv theorem to arbitrary finite groups G.

The topological criterion above has a sufficient condition that may be easily phrased in purely combinatorial terms. Let $\mathcal{F}$ be a set family on ground set X. The *n-colorability defect* $\mathrm{cd}^n(\mathcal{F})$ is $\min|X \setminus \bigcup_{i=1}^n A_i|$, where the minimum is taken over all n-tuples of sets $A_1,\dots,A_n$ that each have no subset in $\mathcal{F}$. The *Kneser hypergraph* $\mathrm{KG}^n(\mathcal{F})$ has $\mathcal{F}$ as its ground set and a hyperedge $\{A_1,\dots,A_n\} \in \mathrm{KG}^n(\mathcal{F})$ if the A_i are pairwise disjoint.

Theorem 2. *Let $n \geq 2$ be an integer, and let $\mathcal{F}$ be a set family with $\mathrm{cd}^n(\mathcal{F}) \geq 2n-1$. Then any $\mathbb{Z}/n$-coloring of $\mathrm{KG}^n(\mathcal{F})$ has a zero-sum hyperedge.*

For example, if $\mathcal{F}$ is the family of all k-element subsets of $\{1,2,\dots,(k+1)n-1\}$ then $\mathrm{cd}^n(\mathcal{F}) = 2n-1$. Thus the theorem above recovers that for any $f\colon \mathcal{F} \to \mathbb{Z}/n$ there are n pairwise disjoint $A_1,\dots,A_n \in \mathcal{F}$ with $\sum f(A_i) = 0$; see Bialostocki and Dierker [2]. The colorability defect bound gives a more general criterion for the existence of zero-sum matchings.

The version of box complex introduced above differs from that of Kríž and provides stronger obstructions. We refer to [5] for proofs and further consequences of the topological approach to zero-sum Ramsey results.

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Random order types

Xavier Goaoc
(joint work with Emo Welzl)

Labeled order types are geometric models of realizable uniform acyclic oriented matroids of rank 3 of particular relevance in discrete and computational geometry. The typical number of extreme points in a simple labeled order type can be determined exactly, and this number reveals bias in the labeled order types of several standard models of random point sets. This analysis can be extended to *unlabeled* simple order types, that is, to relabeling classes of simple realizable uniform acyclic oriented matroids of rank 3, via a combinatorial analogue of Klein's classification of the finite subgroups of $SO(3)$. We refer to the full paper [1] for details.

1. Labeled order types

The *orientation* $\chi(p,q,r)$ of an ordered triple (p,q,r) of points in $\mathbb{R}^2$ is defined as 1 (resp. -1, 0) if r is to the left of (resp. to the right of, on) the line through p and q, oriented from p to q. Two point *sequences* $(p_1,p_2,\dots,p_n)$ and $(q_1,q_2,\dots,q_n)$ *have the same labeled order type* if

$$\forall 1 \leq i,j,k \leq n, \quad \chi(p_i,p_j,p_k) = \chi(q_i,q_j,q_k). \tag{1}$$

This is an equivalence relation, and a *labeled order type* is an equivalence class for that relation. A labeled order type is *simple* if no three points are aligned in a member of that class. We denote by LOT_n the set of simple labeled order types of size n.

2. A combinatorial version of Sylvester's problem

A famous question of Sylvester asked for the average number of extreme points in a "random" planar point set. Since the notion of extreme point can be defined at the level of labeled order type, Sylvester's question makes sense in the combinatorial setting. We prove:

Theorem 1. *For $n \geq 3$, the number of extreme points in a random simple labeled order type chosen equiprobably in LOT_n has average $4 - \frac{8}{n^2-n+2}$ and variance less than 3.*

Our approach is to divide up the simple planar labeled order types into projectively equivalent classes, and average the number of extreme points within each class.

3. Labeled order types of random point sets

Before we elaborate on the proof of Theorem 1, let us mention that it reveals that the labeled order types of several models of random point sets are rather biased. Formally, a family $\{\mu_n\}_{n\in\mathbb{N}}$, where μ_n is a probability measure on LOT_n, *exhibits concentration* if there exist subsets $A_n \subseteq \mathsf{LOT}_n$, $n \in \mathbb{N}$, such that $\mu_n(A_n) \to 1$ and $|A_n|/|\mathsf{LOT}_n| \to 0$.

Theorem 2. *Let μ be a probability distribution on $\mathbb{R}^2$ that is Gaussian or uniform on a compact convex set K, with K smooth or polygonal. The family of probabilities on LOT_n induced by the labeled order type of n random points chosen independently from μ exhibits concentration.*

We prove Theorem 2 by comparing the typical number of extreme points given by Theorem 1 to the typical number of extreme points in random point sets established in stochastic geometry.

4. Projective classes of labeled order types

To divide up labeled order types into classes under projective equivalence, it is convenient to identify $\mathbb{R}^2$ with an open hemisphere of $\mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$. Let S be a point sequence, labeled from 1 to n, in an open hemisphere of $\mathbb{S}^2$, and let ω denote its labeled order type. We let $P = S \cup -S$, where antipodal points have the same labels, and we define an *affine hemiset* of P as an intersection of size n between P and a closed hemisphere of $\mathbb{S}^2$. Like S, every affine hemiset of P contains exactly one point from each antipodal pair. For a labeled order type τ, the following statements are equivalent:

(i) there exist projectively equivalent point sequences that realize τ and ω,
(ii) there exists a point sequence projectively equivalent to S that realizes τ,
(iii) there exists an affine hemiset of P that realizes τ.

It turns out that for $n \geq 4$, any two affine hemisets of P have distinct labeled order types. The affine hemisets of P are therefore in bijection with the labeled order types projectively equivalent to ω.

5. Averaging via duality

For any point $p \in \mathbb{S}^2$ let $p^* = \{u \in \mathbb{S}^2 : p \cdot u = 0\}$ denote the great circle orthogonal to p. Note that a hemisphere of $\mathbb{S}^2$ centered in x intersects P in an affine hemiset if and only if x lies in a 2-dimensional cells of the arrangement of $P^* = \{p^* : p \in P\}$. This in fact defines a bijection between the affine hemisets of P and the 2-cells of the arrangement of P^*. A key observation is that in this bijection, the number of extreme points of the affine hemiset equals the number of edges of the 2-cell. Among the labeled order type projectively equivalent to ω, the average number of extreme points is therefore the average number of edges in a 2-cell of P^*. For every ω, this average is equal to

$$\frac{8\binom{n}{2}}{2\binom{n}{2}+2} = 4 - \frac{8}{n^2-n+2},$$

so the average is the same over LOT_n. The upper bound on the variance follows from the *zone theorem*.

6. Unlabeling

A coarser classification of n-point *sets* identifies P and Q when there exists a bijection $f : P \to Q$ that preserves orientations. An equivalence class for this coarser relation is called an *order type*. Again, any point set S in an open hemisphere of $\mathbb{S}^2$ gives rise to a set $P = S \cup -S$ that is *projective* in the sense that $P = -P$. Again, the order types of the affine hemisets of P are exactly the order types τ that are projectively equivalent to the order type ω of S (in the sense that τ and ω admit projectively equivalent realizations). In the unlabeled setting, however, several affine hemisets of P may have the same order type...

7. Symmetries

... and how many is a matter of symmetries. Formally, a *symmetry* of a point set $S \subseteq \mathbb{S}^2$ is a bijection $S \to S$ that preserves orientations. Any symmetry of a projective point set P maps every affine hemiset of P to an affine hemiset of P. In the action of the symmetry group of P on its affine hemisets, the *orbit* of an affine hemiset A is exactly the set of affine hemisets of P with the same order type as A, and the *stabilizer* of A is isomorphic to the symmetry group of A. By the orbit-stabilizer theorem, the number of affine hemisets of P with order type ω is therefore

$$\frac{\#\text{symmetries of } P}{\#\text{symmetries of } \omega}$$

To control these ratios, and establish an analogue of Theorem 1 for *unlabeled* order type, we actually characterize the possible symmetry groups of affine and projective subsets of $\mathbb{S}^2$.

8. Classifying symmetry groups

The symmetry group of an affine point set acts on its convex hull (and, actually, on any layer of its "convex peeling") by a circular permutation. This readily implies that every affine point set has a cyclic symmetry group. The key insight to analyze the symmetries of projective point sets is the following analogue of the fact that any rotation $\rho \in SO(3)$ leaves exactly two hemispheres of $\mathbb{S}^2$ globally invariant.

Proposition 3. *For $n \geq 3$, every non-trivial symmetry of a $2n$-point projective point set P leaves exactly two affine hemisets of P globally invariant.*

With Proposition 3, Klein's approach to classifying the finite groups of rotations can be implemented and it yields that the symmetry group of any projective set of $2n$ points in general position is a finite subgroup of $SO(3)$.

Theorem 4. *The symmetry group of any projective set of $2n$ points in general position is $\mathbb{Z}_1$ (trivial group), $\mathbb{Z}_m$ (cyclic group) or D_m (dihedral) with m dividing n or $n-1$, S_4 (octahedral = cubical), A_4 (tetrahedral), or A_5 (icosahedral).*

Each of these groups occurs as the symmetry group of some projective order type.

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Generalized recursive atom ordering and equivalence to CL-shellability

Patricia Hersh
(joint work with Grace Stadnyk)

1. Introduction

This abstract describes joint work with Grace Stadnyk. We introduce a new technique for studying the topological structure of order complexes of finite partially ordered sets (posets), namely we introduce generalized recursive atom orderings. This is a relaxation of the fundamental and widely used technique known as recursive atom ordering that was introduced several decades ago by Björner and Wachs in [BW83].

We establish a number of fundamental properties of these generalized recursive atom orderings (GRAOs), including the property that any generalized recursive atom ordering may be transformed into a traditional recursive atom ordering (RAO) by a process we call the atom reordering process. Since GRAOs are easier to construct than RAOs, this may give a useful new pathway to proving a poset is CL-shellable. These generalized recursive atom orderings further allow us to prove that several different forms of lexicographic shellability (in the not necessarily graded case) are all equivalent to each other, by which we mean that a finite bounded poset admits any one of these types of lexicographic shelling if and only if it admits each of the others. One might expect this to imply the stronger statement that any instance of any one of these types of lexicographic shelling is also an instance of any other of these types of lexicographic shelling, but this is not always true. For instance, one may deduce that not every "self consistent CC-shelling" is a CL-shelling from the fact that not every generalized recursive atom ordering is a recursive atom ordering.

We prove that a finite bounded poset admits a recursive atom ordering (RAO) if and only if it admits a generalized recursive atom ordering (GRAO).

A **chain-atom ordering** Ω of a finite bounded poset P is a choice of ordering on the atoms of each rooted interval $[u,\hat{1}]_r$ of P. Now we are ready to state our main new definition.

Definition 1. *A finite bounded poset P admits a* ***generalized recursive atom ordering*** *(GRAO) if the length of P in in P) is 1 or if the length of P is greater than 1 and there is an ordering $a_1, a_2, \dots a_t$ on the atoms of P satisfying:*

- (i) (1) *For $1 \le j \le t$, $[a_j,\hat{1}]$ admits a GRAO. sive atom ordering*
  - (ii) *For any atom a_j and any $x, w \in P$ satisfying $a_j \lessdot x \lessdot w$, the following property holds when the chain-atom ordering given by the GRAO from (i)(a) is restricted to $[a_j, w]$: either the first atom of $[a_j, w]$ is above an atom a_i with $i < j$, or no atom of $[a_j, w]$ is above any atom a_i with $i < j$.*
- (iii) *For any $y \in P$ and any atoms a_i, a_j satisfying $a_i < y$ and $a_j < y$ with $i < j$, there exists an element $z \in P$ with $z \le y$ and an atom a_k with $k < j$ such that $a_j \lessdot z$ and $a_k < z$.*

The statement about cover relations in condition (i)(b) in the definition of GRAO can be strengthened to a corresponding statement about all order relations:

Lemma 2. *Let P be a finite bounded poset, and let Λ be a GRAO for P with atom ordering $a_1, a_2, \ldots a_t$. For each $\hat{0} \lessdot a_j < v$, restricting $\Lambda|_{[a_j,\hat{1}]}$ $[a_j, v]$ yields a GRAO, denoted $\Lambda|_{[a_j,v]}$, for $[a_j, v]$ with the following property: either (a) the first atom of $[a_j, v]$ is greater than some atom a_i satisfying $i < j$ or (b) no atom of $[a_j, v]$ is greater than any atom a_i satisfying $i < j$.*

Our atom reordering process will take any chain-atom ordering and output a chain-atom ordering that will satisfy condition (i)(b) from the defini tion of recursive atom ordering; moreover, it is set up to do so in such a way that when applied to a GRAO, it preserves the property of being a GRAO. Broadly, the algorithm starts at the bottom of the poset P and works its way to the top, reordering the atoms of each rooted interval in a way that takes into account the reordering that has already occurred lower in the poset.

Proposition 3. *Let P be a finite bounded poset with a chain-atom ordering Λ. Let $\Lambda|_{[\hat{0},v]}$ (resp. $\Lambda^{re}|_{[\hat{0},v]}$) be the chain-atom ordering for $[\hat{0}, v]$ obtained by restricting Λ (resp. Λ^{re}) to $[\hat{0}, v]$. Then $\Lambda^{re}|_{[\hat{0},v]}$ equals the chain-atom ordering for $[\hat{0}, v]$ obtained by applying the atom reordering process to $\Lambda|_{[\hat{0},v]}$.*

Lemma 4. *Let P be a finite, bounded poset with Λ a GRAO for P. Then for any $u < v$ in P and any root r for $[u, v]$, the first atom of $[u, v]_r$ in Λ is the first atom of $[u, v]_r$ in Λ^{re}, namely in the atom reordering of Λ.*

These results allow us to prove:

Theorem 5. *A finite bounded poset admits a generalized recursive atom ordering (GRAO) if and only if it admits a recursive atom ordering (RAO).*

Definition 6. *Consider a chain-edge labeling λ such that each rooted interval has a unique lexicographically earliest saturated chain. We define such λ to be **self-consistent** if for any rooted interval $[u, v]_r$ we have the following condition: if a is the atom in the lexicographically first saturated chain of $[u, v]_r$ and $b \neq a$ is also an atom of $[u, v]_r$, then for any $[u, v']_r$ containing a and b all saturated chains of $[u, v']_r$ containing b come lexicographically later than all saturated chains of $[u, v']_r$ containing a.*

The following condition implies self-consistency and is more readily checkable :

Definition 7. *A chain-edge labeling λ of a finite bounded poset P has the **unique earliest (UE) property** if for each rooted interval $[u, v]_r$ in P, the smallest label occurring on any cover relation upward from u only occurs on one such cover relation.*

Equipped with these definitions, we are ready to state one of our main results:

Theorem 8. *Let P be a finite, bounded poset. Then the following are equivalent:*

(1) *P admits a recursive atom ordering*
(2) *P admits a generalized recursive atom ordering*
(3) *P admits a CL-labeling*
(4) *P admits a CL-labeling with the UE property*
(5) *P admits a self-consistent CC-labeling.*
(6) *P admits a CC-labeling with the UE property*
(7) *P admits a self-consistent topological CL-labeling*
(8) *P admits a topological CL-labeling with the UE property*

Moreover, all of these implications are proven constructively. That is, for each implication either it is shown how to construct the latter type of object from the former or else the former type of object is proven also to be the latter type of object.

We apply our results to deduce that a class of posets previously shown to be CC-shellable in [HK] is in fact CL-shellable. That is, we prove that the dual posets to the uncrossing orders (conjectured to be lexicographically shellable by Lam in [La14a]) are CL-shellable. These uncrossing orders arise naturally as face posets of stratified spaces of planar electrical networks (see e.g. [La14a], and references therein). The fact that they are shellable posets combines with Lam's result from [La14a] that they are Eulerian posets to imply that they are CW posets, i.e. face posets of regular CW complexes with finitely many cells. Thus, the shellability of uncrossing orders provides an important step in understanding the topological structure of these spaces of planar electrical networks.

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Flies and regular subdivisions

Michael Joswig
(joint work with Holger Eble, Lisa Lamberti, Will Ludington)

Genetic epistasis is a biological concept for an interaction between two genetic loci as the degree of non-additivity in their phenotypes. This idea goes back as far as 1909, when Bateson analyzed the landmark results by Mendel [1]. If there are more than two loci, things get considerably more complicated. Beerenwinkel, Pachter and Sturmfels proposed to read a suitable regular subdivisions of some convex polytope, called the *genotope*, as a *fitness landscape* [2]; see Figure 1 for an example. In their framework genetic populations which are fittest correspond to points in that polytope, and fitness is expressed in terms of linear programs. The monograph [3] is recommended for background on the relevant concepts from polyhedral geometry.

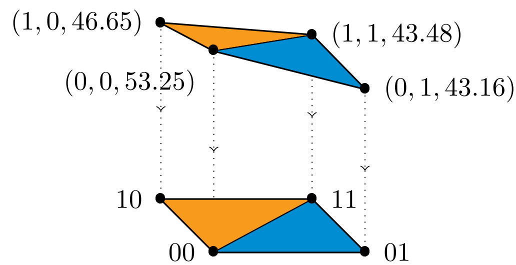

Figure 1. Biallelic genetic system with two loci. The genotope is the unit square $[0,1]$. The phenotype maps each vertex in $\{0,1\}^2$ to a real number; this induces a regular subdivision of the square.

Our contribution is a general method for processing such fitness landscapes, taking statistical aspects into account. For conciseness, we sketch the procedure for an n-biallelic system, where the genotype is the unit cube $[0,1]^n$. Our input are samples of measurements for each genotype, i.e., vertex in $\{0,1\}^n$, and we assume that this input is generic.

(1) Their average values are read as the phenotypes which give rise to a regular subdivision $\mathcal{S}$, which is computed via the convex hull. Due to genericity, $\mathcal{S}$ is a triangulation of $[0,1]^n$.
(2) Let Γ be the dual graph of $\mathcal{S}$. For each edge we compute an *epistatic weight*. Sorting these real numbers gives rise to a filtration of Γ into a sequence of subgraphs, the *epistatic filtration*.
(3) To take the empirical distribution of measurements for each genotype into account, we devised a one-sided significance test for each edge of Γ.
(4) The epistatic filtration with the epistatic weights and their significance form the output.

The theoretical underpinnings have been worked out in [4]. That reference also features a synthetic experiment to explain why our method works. In our new article we report on processing actual data sets from biology [5]. This includes the analysis of classical data, where we can confirm previous findings by other researchers. This also includes the analysis of one new data set, which was obtained in the lab of Will Ludington at Carnegie Science. Those data are concerned with the microbiome of *Drosophila*. We consider $n = 5$ different species of bacteria which may or may not exist in the gut of any fly. So the genotype is the unit cube $[0, 1]^5$. It turns out that the fitness landscape for the lifespan of these flies changes dramatically when certain bacteria are there or not. In biological terms, our results suggest that the co-evolution in this experiment is considerably more complicated than in a simple antagonistic scenario.

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Some Stories of Geometry and Graphs

Nati Linial

(joint work with Daniel Cizma and Maria Chudnovsky)

A *consistent path system* in a graph G is an intersection-closed collection of paths, with exactly one path between any two vertices in G. We call G *metrizable* if every consistent path system in it is the system of geodesic paths defined by assigning some positive lengths to its edges. Our work shows that metrizable graphs are, in essence, subdivisions of a small family of basic graphs with additional *compliant edges*. In particular, every metrizable graph with 11 vertices or more is outerplanar plus one vertex.

Let $G = (V, E)$ be a connected graph, and let $w : E \to \mathbb{R}_{>0}$ be a positive weight function on its edges. This induces a metric on V, where the distance between any two vertices is the least w-length of a path between them. What can be said about such a system of geodesics? E.g., what does the collection of w-geodesics tell us about w? Is it possibly true that *every* collection of paths in a graph constitute the system of geodesics corresponding to some graph metric? To simplify matters, suppose that w is such that the shortest path between any two vertices is unique. Clearly, any subpath of a geodesic in G is itself a geodesic. This leads us to define the notion of a *consistent path system* $\mathcal{P}$ in G - a collection of paths that is closed under taking subpaths, with a unique uv path in $\mathcal{P}$ for each pair $u, v \in V$. So, we ask if every consistent path system coincides with the set of geodesics that corresponds to some positive weight function on the edges. Our first paper on this subject [1] showed that this is far from the truth, and that metrizable graphs are in fact quite rare. E.g., all large metrizable graphs are planar and not 3-connected. On the other hand, that paper also showed that all outerplanar graphs are metrizable. Still, [1] did not provide a satisfactory description of *metrizable* graphs, and in [3] we made further progress on this question.

Call a path in G *flat* if every internal vertex in it has degree 2 in G, and call an edge xy *compliant* if x and y are also connected by a flat path. We show in particular that every large 2-connected metrizable graph can be obtained starting from one of some basic graphs, and iteratively subdividing edges and adding a compliant edge between its end vertices. This, in particular, implies that every large metrizable graph can be made outerplanar by removing at most one vertex.

Here are some of the main ingredients of these studies.

Proposition 1 ([1])**.**

- *The family of metrizable graphs is closed under topological minors.*
- *If e is a compliant edge in G, then G is metrizable if and only if $G \setminus e$ is metrizable.*

Consider a consistent path system $\mathcal{P}$ in a graph $G = (V, E)$. Associated with $\mathcal{P}$ is a system of linear inequalities, and $\mathcal{P}$ is metric iff this system is feasible. So if the chosen $\mathcal{P}$ is non-metric, we can use LP-duality to create a *hand-checkable certificates* of this. Thus, using a computer, we created a "zoo" of 16 non-metrizable graphs along with such path systems and the corresponding certificates. The basic methodology developed in [1] is to prove that a graph at hand is non-metrizable by showing that it contains a subdivision of some graph from the zoo.

Theorem 2 ([3])**.** *If a 2-connected metrizable graph G with at least 11 vertices has no compliant edges, then it is either $K_{2,n}$ for some $n \geq 4$ or a subdivision of one of the following: $K_{2,3}$, K_4, W_4 or $K_5 \setminus e$.*

Consequently

Theorem 3 ([3])**.** *If a graph G with at least 11 vertices is (i) 2-connected, (ii) has no compliant edges, (iii) has at least two disjoint cycles, then G is non-metrizable.*

Corollary 4. *Every 2-connected metrizable graph with at least 11 vertices can be made outerplanar by removing at most one vertex.*

Many open questions are mentioned in [1, 3], e.g., the notion of *irreducible* path systems introduced in [2].

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A regular unimodular triangulation of the matroid base polytope

Gaku Liu
(joint work with Spencer Backman, Gaku Liu)

A lattice triangulation of a lattice polytope is *unimodular* if all of its simplices have minimal volume. A triangulation is *regular* if there is a convex, piecewise linear function whose regions of linearity are exactly given by the triangulation. We give the first construction of regular unimodular triangulations for matroid base polytopes. This construction extends to integral generalized permutahdera. Previously, it was not known whether matroid polytopes admitted covers by unimodular simplices.

The construction is motivated by a set of conjectures collectively known as *White's conjecture* in matroid theory. Given a matroid M with ground set E and set of bases $\mathcal{B}$, define the *toric ideal* of M to be the kernel of the $\mathbb{R}$-algebra homomorphism

$$\mathbb{R}[x_B : B \in \mathcal{B}] \to \mathbb{R}[x_e : e \in E]$$

sending x_B to $\prod_{e \in B} x_e$. The weakest version of White's conjecture states that the toric ideal of a matroid is generated by quadratic binomials. A stronger version of this conjecture is that the toric ideal of a matroid has a quadratic Gröbner basis. The latter conjecture is equivalent to the statement that the matroid base polytope has a flag, regular unimodular triangulation. (A triangulation is *flag* is its minimal non-faces have size 2.) We hope our construction may shed light on this conjecture and lead to future work in this direction.

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Polyhedral Geometry of ReLU Neural Networks

Georg Loho
(joint work with C. Haase, C. Hertrich; M. Brandenburg, G. Montúfar, H. Tseran)

We show new insights in the structure of ReLU Neural Networks based on polyhedral geometry. On one hand, we describe natural subdivisions of the space of piecewise-linear classifiers represented by a ReLU neural network. On the other hand, we show lower bounds on the number of layers for representing integral piecewise-linear functions. The advances involve (generalizations of) oriented matroids, Newton polytopes of tropical polynomials and the use of geometric invariants, in particular normalized volume of lattice polytopes.

First, we give an introduction of two basic concepts, linear classification and tropical rational functions. The geometric point of view on linear classification is captured by the oriented matroid of the hyperplane dual to the vector arrangement associated with data points. This idea is generalized to classifiers arising from continuous piecewise-linear functions. These are exactly the functions represented by ReLU Neural Networks, or equivalently, the functions represented by tropical rational functions (with real 'tropical' exponents). Grouping linear classifiers by the dichotomy imposed on the data points leads to a subdivision of their parameter spaces. This subdivision equals the normal fan of the zonotope given by the Minkowski sum of the line segments associated to the data points. We generalize this to the setting of tropical rational functions with a fixed number of terms in the numerator and denominator [3]. Here, subdividing by the classification pattern yields the normal fan of a sum of simplices, one for each data point.

Second, we look at the expressivity of ReLU neural networks depending on their depth. We recall the known duality between neural networks and Newton polytopes via tropical geometry [1]. Imposing an integrality assumption on the weights in the network implies that these Newton polytopes are lattice polytopes. Using a parity argument on the normalized volume of faces of such polytopes, we show that $\lceil \log_2(n) \rceil$ hidden layers are indeed necessary to compute the maximum of n numbers, matching known upper bounds. This implies that the set of functions representable by ReLU neural networks with integer weights strictly increases with the network depth while allowing arbitrary width [4].

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Random Balanced Cayley Complexes

Roy Meshulam

The Laplacian $L(\mathcal{C})$ of a graph $\mathcal{C} = (V, E)$ is the $V \times V$ positive semidefinite matrix whose (u, v) entry is given by

$$L(\mathcal{C})_{uv} = \begin{cases} \deg_C(u) & u = v, \\ -1 & \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $0 = \lambda_1(\mathcal{C}) \leq \lambda_2(\mathcal{C}) \leq \cdots \leq \lambda_{|V|}(\mathcal{C})$ be the eigenvalues of $L(\mathcal{C})$. The second smallest eigenvalue $\lambda_2(\mathcal{C})$, called the *spectral gap* of $\mathcal{C}$, is a parameter of central importance in a variety of problems. In particular it controls the expansion properties of $\mathcal{C}$ and the convergence rate of a random walk on $\mathcal{C}$ (see e.g., Chapters XIII and IX in [3]).

Let G be a finite group of order n and let $T \subset G$ be symmetric subset, i.e. $T = T^{-1}$. The *Cayley graph* $\mathcal{C}(G,T)$ of G with respect to T is the graph on the vertex set G with edge set $\{\{g, gt\} : g \in G, t \in T\}$. The seminal Alon-Roichman theorem [1] is concerned with the expansion of Cayley graphs with respect to random sets of generators.

Theorem 1 (Alon-Roichman). *For any $\epsilon > 0$ there exists a constant $c(\epsilon) > 0$ such that for any group G, if S is a random subset of G of size $\lceil c(\epsilon) \log |G| \rceil$ and $m = |S \cup S^{-1}|$, then $\lambda_2(\mathcal{C}\left(G, S \cup S^{-1}\right))$ is asymptotically almost surely (a.a.s.) at least $(1-\epsilon)m$.*

This talk is based on [4] and concerns higher dimensional counterparts of Theorem 1. We briefly recall some terminology. For a simplicial complex X and $k \geq -1$ let $X^{(k)}$ denote the k-dimensional skeleton of X. For $k \geq -1$ let $C^k(X)$ denote the space of real valued simplicial k-cochains of X and let $d_k : C^k(X) \to C^{k+1}(X)$ denote the coboundary operator. For $k \geq 0$ define the reduced k-th Laplacian of X by $L_k(X) = d_{k-1}d_{k-1}^* + d_k^* d_k$. The minimal eigenvalue of $L_k(X)$, denoted by $\mu_k(X)$, is the *k-th spectral gap* of X.

Let $k \geq 1$. For $1 \leq i \leq k+1$ let $V_i = \{i\} \times G$. Let $Y_{G,k}$ denote the simplicial join $V_1 * \cdots * V_{k+1}$, where each V_i is viewed as 0-dimensional complex. Thus $Y_{G,k}$ is homotopy equivalent to an N-fold wedge $\bigvee^N \mathrm{S}^k$ of k-dimensional spheres, where $N = (n-1)^{k+1}$. The *balanced k-dimensional Cayley Complex* associated with a subset $\emptyset \neq A \subset G$ is the subcomplex $Y_{A,k} \subset Y_{G,k}$ whose k-simplices are all $\{(1, y_1), \ldots, (k+1, y_{k+1})\} \in Y_{G,k}$ such that $y_1 \cdots y_{k+1} \in A$. Note that $Y_{A,k} \supset Y_{G,k}^{(k-1)}$.

Let 1_A denote the indicator function of $A \subset G$. Let $\widehat{G} = \{\rho\}$ be the set of irreducible unitary representations of G, where $\rho : G \to U(d_\rho)$. Let $D(G) = \sum_{\rho \in \widehat{G}} d_\rho$. Let $\mathbf{1} \in \widehat{\mathbf{G}}$ denote the trivial representation of G and let $\widehat{G}_+ = \widehat{G} \setminus \{\mathbf{1}\}$. For $\rho \in \widehat{G}$ let $\widehat{1_A}(\rho) = \sum_{x \in A} \rho(x) \in M_d(\mathbb{C})$ be the Fourier transform of 1_A at ρ. For a matrix $T \in M_d(\mathbb{C})$ let $\|T\| = \max_{\|v\|=1} \|Tv\|$ denote the spectral norm of T. Let $\nu(A) = \max_{\rho \in \widehat{G}_+} \|\widehat{1_A}(\rho)\|$. Our first result is a lower bound on $\mu_{k-1}(Y_{A,k})$ in terms of $\nu(A)$.

Theorem 2.
$$\mu_{k-1}(Y_{A,k}) \geq |A| - k \cdot \nu(A).$$

Our main result is the following k-dimensional analogue of the Alon-Roichman Theorem.

Theorem 3. *Let k and $\epsilon > 0$ be fixed. Let G be a finite group of order n and fix an integer m such that $\frac{9k^2 \log D(G)}{\epsilon^2} \leq m \leq \sqrt{n}$. Let A be a random subset of G of size m. Then*
$$\Pr\left[\ \mu_{k-1}(Y_{A,k}) < (1-\epsilon)m\ \right] < \frac{6}{n}.$$

Remark 4. *It is straightforward to check that* $\mu_{k-1}(Y_{A,k}) \leq |A|+k$ *for any* $A \subset G$ *(see Eq. (2) in* [2]*). Theorem 3 thus implies that if* A *is a random subset of* G *and* $\log |G| = o(|A|)$*, then* $Y_{A,k}$ *is a.a.s. a near optimal spectral expander.*

Our final result concerns the homotopy type of $Y_{A,k}$ when A is a subgroup of G. For $1 \leq m$ let

$$\gamma_0(m,k) = (n-m)n^k + \left(\frac{n}{m}\right)^k (m-1)^{k+1} - (n-1)^{k+1},$$
$$\gamma_1(m,k) = \left(\frac{n}{m}\right)^k (m-1)^{k+1}.$$

Theorem 5. *Let* A *be a subgroup of* G *of order* $|A| = m$*. Then*
(i)

$$Y_{A,1} \simeq \coprod^{n/m} \bigvee^{(m-1)^2} \mathrm{S}^1. \tag{1}$$

(ii) For $k \geq 2$

$$Y_{A,k} \simeq \bigvee^{\gamma_0(m,k)} \mathrm{S}^{k-1} \vee \bigvee^{\gamma_1(m,k)} \mathrm{S}^k. \tag{2}$$

Remark 6. *As* $\gamma_0(m,k) > 0$ *for all* $m < n$*, it follows from Theorem 5 that if* $A \subset G$ *generates a subgroup* $\langle A \rangle$ *of order* $m < n$ *then*

$$\tilde{\beta}_{k-1}(Y_{A,k}) \geq \tilde{\beta}_{k-1}(Y_{\langle A \rangle,k}) = \gamma_0(m,k) > 0$$

and therefore $\mu_{k-1}(Y_{A,k}) = 0$*. As there are families of groups* G *(e.g. elementary abelian groups of fixed exponent) that cannot be generated by* $o(\log |G|)$ *elements, this implies that the* $\log D(G) = \Theta(\log n)$ *factor in Theorem 3 cannot in general be improved.*

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Rigidity expander graphs

Eran Nevo

(joint work with Alan Lew, Yoval Peled, Orit Raz)

Jordán and Tanigawa recently introduced the d-dimensional algebraic connecti vity $a_d(G)$ of a graph G. This is a quantitative measure of the d-dimensio nal rigidity of G which generalizes the well-studied notion of spectral expans ion of graphs. We present a new lower bound for $a_d(G)$ defined in terms of the spectral expansion of certain subgraphs of G associated with a partition of i ts vertices into d parts. In particular, we obtain a new sufficient condition for the rigidity of a graph G. As a first application, we prove the existence of an infinite family of k-regu lar d-rigidity-expander graphs for every $d \geq 2$ and $k \geq 2d+1$. Conjectural ly, no such family of $2d$-regular graphs exists. Second, we show that $a_d(K_n) \geq \frac{1}{2}\left\lfloor \frac{n}{d} \right\rfloor$, which we conjecture to be essentially tight. In addition, we study the extremal values $a_d(G)$ attains i f G is a minimally d-rigid graph.

Context. Graph expansion is one of the most influential concepts in mod ern graph theory, with numerous applications in discrete mathematics and compute r science (see [4, 7]). Intuitively speaking, an expander is a "highly-connected" graph, and a standard way to quantitatively measure the connectivity, or expansion, of a graph uses t he spectral gap in its Laplacian matrix. A main theme in the study of expander g raphs deals with the construction of sparse expanders. In particular, bounded-de gree regular expander graphs have been studied extensively in various areas of m athematics. This paper studies a generalization of spectral graph expansion that was recentl y introduced by Jordán and Tanigawa via the theory of graph rigidity [5].

A d-dimensional framework is a pair (G,p) consisting of a graph $G=(V,E)$ an d a map $p: V \to \mathbb{R}^d$. The framework is called d-rigid if every contin uous motion of the vertices starting from p that preserves the distance betwee n every two adjacent vertices in G, also preserves the distance between *every pair* of vertices; see e.g. [2, 3] for background on framework rigidity. Asimow and Roth showed in [1] that if the map p is generic (e.g. if the $d|V|$ coordinates of p are algebraically independent over the rationales), t hen the d-rigidity of (G,p) does not depend on the map p. Moreover, they s howed that for a generic p, rigidity coincides with the following stronger lin ear-algebraic notion of infinitesimal rigidity.

Definitions. For every $u,v \in V$ we define $d_{uv} \in \mathbb{R}^d$ by

$$d_{uv} = \begin{cases} \frac{p(u)-p(v)}{\|p(u)-p(v)\|} & \text{if } p(u) \neq p(v), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{b}_{u,v}^T = \begin{bmatrix} 0 & \dots & 0 & \overset{u}{d_{uv}^T} & 0 & \dots & 0 & \overset{v}{d_{vu}^T} & 0 & \dots & 0 \end{bmatrix}.$$

The (normalized) *rigidity matrix* $R(G,p) \in \mathbb{R}^{d|V| \times |E|}$ is the matrix whose columns are the vectors $\mathbf{b}_{u,v}$ for all $\{u,v\} \in E$. $\mathbb{R}^d$. For p generic and $|V| \geq d+1$,

$\mathrm{rank}(R(G,p)) \leq d|V| - \binom{d+1}{2}$; see [1]. The framework (G,p) is called *infinitesimally rigid* if this bound is attained, that is, if $\mathrm{rank}(R(G,p)) = d|V| - \binom{d+1}{2}$.

A graph G is called *d-rigid*, if it is infinitesimally rigid with resp ect to some map p (or, equivalently, if it is infinitesimally rigid for all ge neric maps [1]).

For $d = 1$ and an injective map $p : V \to \mathbb{R}$, the rigidity matrix $R(G,p)$ is equal to the incidence matrix of G, hence both notions of rigidity coincid e with graph connectivity. One can extend this analogy and define a higher dimen sional version of the graph's Laplacian matrix, that is called the *stiffne ss matrix* of (G,p), and is defined by

$$L(G,p) = R(G,p)R(G,p)^T \in \mathbb{R}^{d|V| \times d|V|}.$$

We denote by $\lambda_i(A)$ the i-th smallest eigenvalue of a symmetric matrix A. Since $\mathrm{rank}(L(G,p)) = \mathrm{rank}(R(G,p)) \leq d|V| - \binom{d+1}{2}$, the kernel of $L(G,p)$ is of dimension at least $\binom{d+1}{2}$. Therefore, $\lambda_{\binom{d+1}{2}+1}(L(G,p)) \neq 0$ if and only if (G,p) is infinitesimally rigid.

In [5], Jordán and Tanigawa defined the *d-dimens ional algebraic connectivity* of G, $a_d(G)$, as

$$a_d(G) = \sup\left\{ \lambda_{\binom{d+1}{2}+1}(L(G,p)) \middle| \, p : V \to \mathbb{R}^d \right\}.$$

For $d = 1$, $L(G,p)$ coincides with the graph Laplacian matrix $L(G)$, and $a_1(G) = a(G)$ is the usual algebraic connectivity, or Laplacian spectral gap, of G. For every $d \geq 1$, $a_d(G) \geq 0$ since $L(G,p)$ is positive semi-definite, and $a_d(G) > 0$ if and only if G is d-rigid.

The following notion of *rigidity expander graphs* extends the classical no tion of (spectral) expander graphs, corresponding to the $d = 1$ case: Let $d \geq 1$. A family of graphs $\{G_i\}_{i\in\mathbb{N}}$ of increasing size is called a *family of d-rigidity expander graphs* if there exists $\epsilon > 0$ such that $a_d(G_i) \geq \epsilon$ for all $i \in \mathbb{N}$.

Results. It is well known that, for every $k \geq 3$, there exist families of k-regular ($d = 1$-rigid) expander graphs (see e.g. [4]). Our main result is an extension of this fact to general d:

Theorem 1. *Let $d \geq 1$ and $k \geq 2d + 1$. Then, there exists an infinite family of k-regular d-rigidity expander graphs.*

It was conjectured by Jordán and Tanigawa that families of $2d$-regular d-ri gidity expanders do not exist (see [5, Conj. 2] for the statement in the $d = 2$ case, and see [6, Conj. 6.2] for the general case), and clearly families of k-regular d-rigidity expanders do not exist for $k < 2d$ since, for n large e nough, such graphs have less than $dn - \binom{d+1}{2}$ edges, and are therefore n ot even d-rigid. Therefore, assuming this conjecture, our result is sharp.

Our main tool for the proof of Theorem 1 is a new low er bound on $a_d(G)$, given in terms of the (1-dimensional) algebraic connecti vity of certain subgraphs of G associated with a partition of its vertex set i nto d parts. For convenience, we let $a(G) = \infty$ if G consists of a single vertex.

Let $G = (V, E)$ be a graph, and let $A, B \subset V$ be two disjoint sets. We denote by $G[A]$ the subgraph of G induced on A, and by $G(A, B)$ the subgraph of G with vertex set $A \cup B$ and edge set $E(A, B) = \{e \in E : |e \cap A| = |e \cap B| = 1\}$. Recall that a *partition* of V is a set $\{A_1, \dots, A_d\}$ of n on-empty subsets of V such that $V = A_1 \cup \cdots \cup A_d$ is a disjoint union .

Theorem 2. *Let $d \geq 2$. For every graph $G = (V, E)$ and a partition $\{A_1, \dots, A_d\}$ o f V there holds*

$$a_d(G) \geq \min\left(\left\{a(G[A_i])\right\}_{1\leq i\leq d} \bigcup \left\{\frac{1}{2}a(G(A_i, A_j))\right\}_{1\leq i<j\leq d}\right).$$

In particular, if $G[A_i]$ is connected for all $i \in [d]$ and $G(A_i, A_j)$ is co nnected for all $1 \leq i < j \leq d$, then G is d-rigid.

Remark 3. *In the $d = 2$ dimensional case, the statement in Theorem 2 can be slightly improved (by removing the constant $1/2$) to*

$$a_2(G) \geq \min\{a(G[A_1]), a(G[A_2]), a(G(A_1, A_2))\},$$

for every partition A_1, A_2 of V.

For another application of Theorem 2, we derive a slight improvement of the previously known lower bound for $a_d(K_n)$ from [6, Theorem 1.5].

Corollary 4. *Let $d \geq 3$ and $n \geq d + 1$. Then*

$$a_d(K_n) \geq \frac{1}{2}\left\lfloor\frac{n}{d}\right\rfloor.$$

Conjecturally, under these conditions, $a_d(K_n) \leq \frac{n}{d}$. The upper boun d given in [Thm.1.6][6] is $a_d(K_n) \leq \frac{2n}{3(d-1)} + \frac{1}{3}$.

Problems and comments. Many parallels to classical graph expansion are sill missing: find optimal d-rigidity expander graphs. For $d = 1$ these were co nstructed, known as Ramanujan graphs. The Alon-Boppana bound is valid also for $a_d(G)$, as we prove that:

Theorem 5. *Let $d \geq 2$, and let G be a graph. Then,*

$$a_d(G) \leq a(G).$$

Jordán and Tanigawa [5, Theorem 4.2] proved Theorem 5 for $d = 2$, and in [8] it was proved indep endently for all d. Our proof is different, using the probabilistic method, an d we believe it to be of independent interest.

Regarding the Cheeger inequality, it remains a challenge to find a lower bound o n $a_d(G)$ in terms of combinatorial invariants of G.

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Framed polytopes and higher cellular strings

Arnau Padrol

(joint work with Guillaume Laplante-Anfossi and Anibal M. Medina-Mardones)

Higher categories offer a framework for systematizing complex hierarchies in mathematics, physics, and computer science. To illustrate this, we mention Grothendieck's homotopy hypothesis in topology, Baez–Dolan's cobordism hypothesis in quantum field theory, and the extensive applications of higher category theory in computer science, particularly in language semantics, concurrency calculus, and type theory.

Polytopes in higher category theory were first introduced to organize coherence relations. Kapranov and Voevodsky significantly expanded the connection between convex geometry and higher category theory announcing several intriguing results in [1], including the following insightful idea. Consider a convex d-polytope $P \subseteq \mathbb{R}^d$ and a generic ordered basis B of $\mathbb{R}^d$, which we refer to as a *frame*. Using the frame we define, for each face F, two distinct subsets of its k-faces: its *k-source* $\mathrm{s}_k(F)$ and *k-target* $\mathrm{t}_k(F)$. Kapranov and Voevodsky conjectured [1, Thm. 2.3] that the data consisting of all sources and targets, referred to as the *globular structure* of (P, B), defines a d-dimensional *pasting diagram*, a special and important type of d-dimensional categories. Using ideas of Steiner [3], we show that this claim holds if and only if the framed polytope has no cellular loops, a notion we now define. A *cellular k-string* in a framed polytope is a sequence $F_1, \ldots, F_\ell$ of faces such that two consecutive faces F_i and F_{i+1} share a k-face G with $\mathrm{t}_k(F_i) \cap \mathrm{s}_k(F_{i+1}) = G$. We say it is a *cellular loop* if and $F_i = F_j$ for some $i \neq j$.

The first contribution we discuss are counterexamples to [1, Thm. 2.3]. More precisely, we provide examples showing the following.

Theorem 1. *Starting in dimension 4 there exist framed polytopes with cellular loops.*

We also considered whether the following weaker version of their claim could be true: For any polytope there is a frame making it into a pasting diagram. However, this weaker version also fails since we provide a construction establishing the following.

Theorem 2. *Starting in dimension 4 there exist polytopes for which all frames lead to cellular loops.*

An important infinite family of framed polytopes, which was studied by Kapranov and Voevodsky, is given by the *canonically framed cyclic simplices* $(C(d), \{e_k\})$, where $\{e_k\}$ is the canonical frame of $\mathbb{R}^d$ and $C(d)$ is the convex closure of $d+1$ distinct points in the *moment curve* $t \mapsto (t, t^2, \dots, t^d)$. In an insightful observation [1, Thm. 2.5], they announced that $(C(d), \{e_k\})$ has no cellular loops and recover Street's free d-category on the d-simplex, a fundamental object in higher category theory [4]. We were able to verify this claim after replacing the canonical frame by $\{e_1, -e_2, e_3, -e_4, \dots\}$.

These framed simplices are rare and special in the following probabilistic sense. A *Gaussian d-simplex* is the convex hull of $d+1$ independent random points in $\mathbb{R}^d$, each chosen according to a d-dimensional standard normal distribution. We prove the following.

Theorem 3. *The probability that a canonically framed Gaussian d-simplex has a cellular loop tends to* 1 *as d tends to* ∞.

We next turn our attention to the moduli of frames of a simplex Δ_d under the equivalence relation induced by globular structures. Our aim is to quantify the complexity of the *realization space* of a globular structure on Δ_d, that is, the set of all frames of Δ_d inducing it. Using a celebrated result of N. E. Mnëv [2], we show the following.

Theorem 4. *For every open primary basic semi-algebraic set S defined over $\mathbb{Z}$ there is a globular structure on some simplex Δ_d whose realization space is stably equivalent to S.*

A key step in the proof of this result is the following theorem, which we consider noteworthy in its own right.

Theorem 5. *Globular structures of framed simplices are in bijection with uniform acyclic realizable full flag chirotopes.*

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Lefschetz properties via anisotropy for simplicial spheres and cycles

Stavros Papadakis and Vasiliki Petrotou
(joint work with Karim Adiprasito)

1. Introduction

An important recent breakthrough in Discrete Geometry was the 2018 proof of McMullen's g-conjecture for simplicial spheres by Karim Adiprasito [1]. Two years later, the paper [3] appeared, which gave a substantially different second proof of the conjecture based on the notion of generic anisotropy and certain characteristic 2 differential identities. Finally, the 2021 paper [2] proved Lefschetz type properties in the setting of pseudomanifolds and simplicial cycles and gave an application to 2-Cohen-Macaulay simplicial complexes.

2. Generic Artinian Reduction

Assume $m \geq 1$ and k is a field. We consider the polynomial ring $k[x_1, \dots, x_m]$, where the degree of the variable x_i is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset k[x_1, \dots, x_m]$ is a homogeneous ideal. We denote by d the Krull dimension of the quotient ring $k[x_1, \dots, x_m]/I$. We assume $d \geq 1$, and denote by E the field of fractions of the polynomial ring

$$k[a_{i,j} : 1 \leq i \leq d, \ 1 \leq j \leq m].$$

For $1 \leq i \leq d$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j.$$

Definition 1. We define the generic Artinian reduction of $k[x_1, \dots, x_m]/I$ to be the Artinian E-algebra

$$E[x_1, \dots, x_m]/((I) + (f_1, \dots, f_d)),$$

where (I) denotes the ideal of $E[x_1, \dots, x_m]$ generated by I.

3. Generic Anisotropy of simplicial spheres

Assume k is a field and D is a simplicial sphere of dimension $d-1$ with vertex set $\{1, \dots, m\}$. We denote by $k[D] = k[x_1, \dots, x_m]/I_D$ the Stanley-Reisner ring of D over k and by A the generic Artinian reduction of $k[D]$ defined above. We remark that A is an Artinian Gorenstein standard graded E-algebra with socle degree d, where E as above.

Definition 2. We call D generically anisotropic over k, if for all integers j with $1 \leq 2j \leq d$ and all nonzero elements $u \in A_j$ we have $u^2 \neq 0$.

Three of the main results of [3] are the following:

Theorem 3 ([3]). Assume that k is any field and D is a simplicial sphere of dimension 1. Then D is generically anisotropic over k.

Theorem 4 ([3]). Assume that k is any field of characteristic 2 and D is any simplicial sphere. Then D is generically anisotropic over k.

Remark 5. It is easy to see that by clearing denominators the previous theorem implies that any simplicial sphere D is generically anisotropic over the field of rationals $\mathbb{Q}$.

Theorem 6 ([3]). Assume k is any field and D is a simplicial sphere.

(1) If the suspension $S(D)$ of D is generically anisotropic over k, then $E[D]$ has the Weak Lefschetz property.
(2) If both D and the suspension $S(D)$ of D are generically anisotropic over k, then $E[D]$ has the Strong Lefschetz property.

Question 7. Is any simplicial sphere generically anisotropic over any field?

Question 8. Identify classes of Gorenstein standard graded algebras which have the generic anisotropy property.

4. Lefschetz properties for cycles

As mentioned above, the paper [2] contains Lefschetz type theorems for pseudomanifolds and simplicial cycles. An interesting application of them is the following:

A simplicial complex D of dimension $d-1$ is called 2-Cohen-Macaulay over the field k if $k[D]$ is Cohen-Macaulay, and for any vertex v of D the following hold for the simplicial complex

$$C = D \setminus \{v\}.$$

It has dimension $d-1$ and the Stanley-Reisner ring $k[C]$ is Cohen-Macaulay.

Theorem 9 ([2]). Assume D is a 2-Cohen-Macaulay simplicial complex of dimension $d-1$ over an infinite field k and denote by A a sufficiently general Artinian reduction of $k[D]$. Then, there exists $\omega \in A_1$ such that the multiplication by ω^{d-2i} from A_i to A_{d-i} is injective for all $0 \leq i \leq d/2$.

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Acyclonestohedra

Vincent Pilaud
(joint work with Chiara Mantovani and Arnau Padrol)

We use classical terminology on building sets and their nested complexes [FK04, FS05, Pos09] and on oriented matroids [BLS+99].

Motivated by recent work of P. Galashin on poset associahedra [Gal21], we consider the acyclic part of a given nested complex with respect to a given oriented matroid in the following sense.

Definition 1. *An* ***oriented building set*** *on a ground set S is a pair $(\mathcal{B}, \mathcal{M})$ where $\mathcal{B}$ is a building set on S and $\mathcal{M}$ is an oriented matroid on S such that $\mathcal{B}$ contains the support of any circuit of $\mathcal{M}$.*

Definition 2. *A nested set $\mathcal{N}$ on $\mathcal{B}$ is* ***acyclic*** *if $\mathcal{M}_{/\bigcup\mathcal{N}'}$ is acyclic for any $\mathcal{N}' \subseteq \mathcal{N}$. The* ***acyclic nested complex*** *$\mathfrak{A}(\mathcal{B}, \mathcal{M})$ is the simplicial complex of acyclic nested sets on $\mathcal{B}$.*

Our main results concern realizations (as boundary complexes of oriented matroids or polytopes) of these acyclic nested complexes.

Theorem 3. *The acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M})$ of any oriented building set $(\mathcal{B}, \mathcal{M})$ is the boundary complex of the positive tope of an oriented matroid obtained by stellar subdivisions of $\mathcal{M}$.*

Theorem 4. *The acyclic nested complex $\mathfrak{A}(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ of any* ***realizable*** *oriented building set $(\mathcal{B}, \mathcal{M}(\boldsymbol{A}))$ is the boundary complex of the* ***acyclonestohedron****, a polytope obtained as the section of a nestohedron of $\mathcal{B}$ with the evaluation space of the vector configuration $\boldsymbol{A}$.*

Our original motivation was the following graphical situation.

Definition 5. *The* ***graphical oriented building set*** *of a directed graph D with edge set S is given by*

- *the graphical building set of the line graph of D, and*
- *the graphical oriented matroid of D.*

Proposition 6. *The acyclic nested complex of the graphical oriented building set of D is isomorphic to the* ***piping complex*** *of the transitive closure of D, defined by P. Galashin in its his work on poset associahedra* [Gal21].

Corollary 7. *The piping complex of a poset P is isomorphic to the boundary complex of the* ***graphical acyclonestohedron****, obtained as a section of a graph associahedron of the line graph of the Hasse diagram of P.*

This corollary is illustrated in Figure 1 and answers a question open by P. Galashin in [Gal21], and independently settled by A. Sack in [Sac23] with a more specific method.

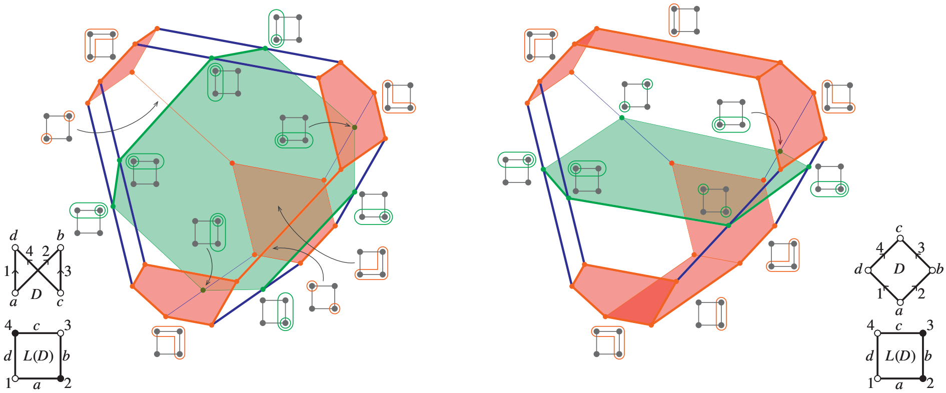

Figure 1. The poset associahedra of two posets, obtained as sections of the graph associahedra of their line graphs with their cycle spaces.

Finally, we show that acyclic nested complexes of oriented building sets essentially correspond to $\mathcal{F}(\mathcal{M})$-nested complexes of $\mathcal{F}(\mathcal{M})$-building sets in the sense of E.-M. Feichtner and D. Kozlov [FK04], where $\mathcal{F}(\mathcal{M})$ is the Las Vergnas face lattice of the oriented matroid $\mathcal{M}$.

We use this observation for two further applications:

- type B nestohedra, starting from the oriented matroid whose positive tope is a cross-polytope,
- iterated nestohedra, recovering in particular the permuto-permutahedra.

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Fixed points on contractible spaces

Kevin I. Piterman

For a group G and a G-contractible G-complex X, both X and its fixed point set X^G are contractible. In particular, there is a fixed point by the action of G. We ask then for suitable topological conditions that also imply the existence of a fixed point in a wide family of spaces and groups.

For example, by Smith theory, a finite p-group acting on a mod p acyclic finite-dimensional *regular* simplicial complex has a fixed point. We also know by Brouwer's fixed point theorem that a cyclic group acting on a disc has a fixed point. Hence, p-groups and cyclic groups always act with fixed points on discs. In the seventies, B. Oliver classified the finite groups that can act without fixed points on discs. In fact, he showed that a finite group G acts without fixed points on a disc if and only if G does not contain subgroups $P \leq H \leq G$ such P is a p-group normal in H, H/P is cyclic, and H is a normal subgroup of G such that G/H is a q-group, where p, q are some primes.

On the other hand, a famous theorem by J.P. Serre in the eighties states that a finite group acting on a tree has a fixed point. However, it is known that finite groups can act without fixed points on contractible complexes of dimension at least 3. The first example of this nature was constructed by E. Floyd and R.W. Richardson [5]. That is, there is an action of the alternating group A_5 on the 2-skeleton X_P of the Poincaré homology 3-sphere, and X_P is acyclic and fixed point free. Then the join $X = A_5 * X_P$ is a 3-dimensional compact and contractible complex with $X^{A_5} = \emptyset$. In dimension 2, it was conjectured by C. Casacuberta and W. Dicks that a finite group acting on a contractible 2-complex has a fixed point, and they proved this for solvable groups by using Smith theory [4]. Independently and at the same time, M. Aschbacher and Y. Segev raised this question but only for compact complexes [2]. Moreover, they proved that if a finite group G acts without fixed points on a compact acyclic 2-complex then G has a composition factor isomorphic to the Janko group J_1 or to one of the simple group of Lie type and Lie rank 1. In 2002, B. Oliver and Y. Segev achieved substantial progress on this problem by classifying finite groups acting without fixed points on finite acyclic 2-complexes [6]. One of their main theorems states that a finite group G has an *essential* action without fixed points on a finite acyclic 2-complex if and only if G is one of the simple groups $\mathrm{PSL}_2(q)$ or $\mathrm{Sz}(2^{2k+1})$, with some restrictions on q and k. We refer to the beautiful exposition by A. Ádem [1] for more details on these theorems.

In this talk, we review some of these results on fixed points. We also take a look at the case of finite T_0 topological spaces, where a result by R.E. Stong shows that a contractible finite T_0-space always has a fixed point. This relates to a conjecture raised by D. Quillen [10]: for a fixed prime p and a finite group G, the poset of nontrivial p-subgroups is contractible if and only if it is contractible as a finite space. This conjecture remains open, and we briefly comment on recent developments [3, 7, 9, 10]. We also present a sketch of the proof of the Casacuberta-Dicks conjecture for compact complexes, a joint work with Sadofschi Costa [8, 11].

This work is based on a previous article [12] which establishes the case $G = A_5$ of this conjecture. In this work, Sadofschi Costa reduced the study of the conjecture to the simple groups G in the theorem of Oliver-Segev (namely, the 2-dimensional finite linear groups and Suzuki groups), and also to a very particular family of 2-complexes related to the examples constructed in [6]. Once we have these reductions for the group G and a fixed point free finite acyclic 2-dimensional G-complex X, we construct a manifold M encoding representations of the group extension $\pi_1(X) : G$, obtained by lifting the maps $g \in G$ to the universal cover of X. For the rest of the proof, we show that there is a differential map $f : M \to N$ between orientable connected and compact manifolds of the same dimension and conclude by a degree argument that at least one of the points in a preimage $f^{-1}(x_0)$, for a particular point $x_0 \in N$, must correspond to a representation of $\pi_1(X) : G$ that does not factor through G. This implies that $\pi_1(X)$ is nontrivial, that is, X is not contractible.

Finally, we mention that the non-compact case of the Casacuberta-Dicks conjecture remains open.

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Stirling numbers and Koszul algebras with symmetry

Victor Reiner
(joint work with Ayah Almousa, Sheila Sundaram)

Stirling numbers $c(n,k), S(n,k)$ of the first and second kind give the answers to two basic counting problems:

- How many permutations of $\{1,2,...,n\}$ have k cycles?
- How many set partitions of $\{1,2,...,n\}$ have k blocks?

Although they have no simple product formulas, they do have triangular recursions

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k), \tag{1}$$

$$S(n,k) = S(n-1,k-1) + k\cdot c(n-1,k), \tag{2}$$

and closely related generating functions

$$\sum_{k=0}^{n-1} c(n,n-i)t^i = (1+t)(1+2t)\cdots(1+(n-1)t), \tag{3}$$

$$\sum_{k=0}^{\infty} S((n-1)+i,n-1)t^i = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}. \tag{4}$$

We re-interpret $c(n,k)S(n,k)$ as Hilbert functions for certain well-studied *Koszul algebras* A and their less-studied *Koszul duals* $A^!$, in the sense of Priddy [4].

The algebras A are the cohomology rings H^*X for the configuration space

$$X = \mathrm{Conf}_n(\mathbb{R}^d) = \{(x_1,\dots,x_n)\in\mathbb{R}^d)^n : x_i \neq x_j \text{ for } 1\leq i<j\leq n\}$$

of n labeled points in $\mathbb{R}^d$, where $d=2,3,4,5,\ldots$. For $d=2,4,6,\ldots$ even, this cohomology algebra A is isomorphic to the usual *Orlik-Solomon algebra* of the type A_{n-1} reflection hyperplane arrangement, also known as the *braid arrangement.* For $d=3,5,7,\ldots$ odd, A is isomorphic to the *associated graded Varchenko-Gelfand ring* of the same hyperplane arrangement. Both rings have simple presentations, either as quotients of an exterior algebra or a commutative polynomial algebra on generators $\{x_{ij}\}_{1\leq i<j\leq n}$, with simple quadratic relations found by V.I Arnold (for $d=2$) and F. Cohen (for general $d\geq 2$).

These quadratic presentations actually form quadratic Groebner bases for the defining ideals, showing that these algebra A are Koszul, and that the Hilbert series $\mathrm{Hilb}(A,t)$ is given by the generating function in (3). This implies also that their Koszul dual algebras $A^!$ have Hilbert series $\mathrm{Hilb}(A^!,t)$ given by the generating function in (4), related by

$$\mathrm{Hilb}(A^!,t) = \frac{1}{\mathrm{Hilb}(A,-t)}. \tag{5}$$

It is also known that $A^!$ is the homology ring $H_*(\Omega X)$ of the loop space ΩX.

All of these algebra $A, A^!$ carry actions of the symmetric group $\mathfrak{S}_n$ via graded automorphisms. We are interested in the describing and decomposing the actions on each graded component $A_i, A^!_i$, or equivarlant versions of the above Hilbert series. For the original algebras A, good descriptions of the $\mathfrak{S}_n$-characters on A_i

in terms of generating functions are known via work of Sundaram and Welker [5]. The characters of the Koszul duals $\{A_i^!\}$ can be computed recursively in terms of the $\{A_i\}$, but we currently lack simple generating function descriptions for $\{A_i^!\}$.

Nevertheless, they enjoy nice properties, considered as families $\{A(n)\}, \{A(n)^!\}$ depending on n. For example, there are branching rules that restrict $A(n)$ or $A(n)^!$ from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$, giving representaiton-theoretic lifts of the recursions (1), (2). As another example, when one fixes some $i = 0, 1, 2, \ldots$, the sequences of $\mathfrak{S}_n$-representations $\{A(n)_i\}, \{A^!(n)_i\}$ both turn out to be *representation stable* in the sense of Church and Farb [2].

Many of their properties and the results come from general facts about Koszul algebras, and generalize from the type A reflection arrangement to all *supersolvable* hyperplane arrangements.

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Tropical Ideals

Felipe Rincón

Tropical ideals are combinatorial objects introduced in [3] with the aim of giving tropical geometry a solid algebraic foundation. They can be thought of as combinatorial generalizations of the possible collections of subsets arising as the supports of all polynomials in an ideal. In general, their structure is dictated by an infinite sequence of 'compatible' matroids. In this talk I will introduce and motivate the notion of tropical ideals, and I will discuss work over the last decade studying some of their main algebraic properties, the structure of their associated varieties, and the tropical Nullstellensatz.

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From pivot rules to colliding particles

Raman Sanyal

(joint work with A. Benjes, A. Black, J. De Loera, N. Lütjeharms, and G. Poullot)

Geometrically, a linear program can be viewed as a convex polytope $P \subset \mathbb{R}^d$ together with a unique-sink orientation of its graph that induced by a linear function $x \mapsto \langle c, x\rangle$ for some fixed $c \in \mathbb{R}^d$. For a given starting vertex $v \in V(P)$, the simplex algorithm follows a directed path from v to the unique sink v_{opt}. Which path is taken is dictated by the pivot rule adopted by the simplex algorithm. A pivot rule is *memory-less*, if it chooses the next vertex on the path utilizing only v and its c-improving neighbors $N_+(v) \subset V(P)$. The behaviour of a memory-less pivot rule is completely determined by an *arborescence* (or rooted tree), that is, a map $A : V(P) \to V(P)$ with $A(v_{\text{opt}}) = v_{\text{opt}}$ and $A(v) \in N_+(v)$ for $v \neq v_{\text{opt}}$.

In [2] we introduced the *max-slope* pivot rule that for a given generic $\omega \in \mathbb{R}^d$ corresponds to the arborescence

$$(1) \qquad A^\omega(v) \ = \ \operatorname{argmax} \left\{ \frac{\langle \omega, u - v\rangle}{\langle c, u - v\rangle} : u \in N_+(v) \right\}$$

for $v \neq v_{\text{opt}}$. The max-slope pivot rule generalizes the well-known *shadow vertex* simplex algorithm: if r is the vertex of P that maximizes ω, then $r, A^\omega(r)$, $(A^\omega)^2(r), \dots, v_{\text{opt}}$ is precisely the shadow path associated to ω. It is straightforward to see that the collection of ω that give rise to the same max-slope arborescence form an open polyhedral cone and the closures of these cones yield a complete fan in $\mathbb{R}^d$.

In [2] we associate to every arborescence A of the linear program (P, c) a point $\psi(A) \in \mathbb{R}^d$ and define the *max-slope pivot rule polytope* $\Pi(P, c)$ as the convex hull of these points for all A.

Theorem 1 ([2])**.** *$\Pi(P, c)$ is a polytope of dimension $\dim P - 1$ with the following property: for any generic $\omega \in \mathbb{R}^d$, $\psi(A^\omega)$ is the unique maximizer of ω over $\Pi(P, c)$.*

Our pivot rule polytopes are related to certain fiber polytopes [1]: the *monotone path polytope* $\Sigma(P, c)$ that parametrizes coherent monotone paths on (P, c) is a weak Minkowski summand of $\Pi(P, c)$. Note that the construction of pivot rule polytopes works for the more general class of *normalized weight* pivot rules as explained in [2].

While our constructions where motivated by studying 'spaces of pivot rules', it turns out that max-slope pivot rule polytopes have fascinating and surprising applications to geometric combinatorics.

Let $\Delta_n \subset \mathbb{R}^{n+1}$ be the standard n-simplex with vertices $e_1, \dots, e_{n+1}$ equipped with a generic objective function $c = (c_1 < c_2 < \cdots < c_{n+1})$. An arborescence can be viewed as a map $A : [n+1] \to [n+1]$ with $A(n+1) = n+1$ and $A(i) > i$ for $i < n+1$. Of the $n!$ many arborescences, it turns out that exactly C_n are max-slope arborescences, where C_n is the n-th Catalan number. The max-slope arborescences can be characterized as the *non-crossing* arborescences, where a *crossing* is a pair $i, j \in [n+1]$ with $i < j < A(i) < A(j)$. It is not too difficult to see that non-crossing arborescences satisfy the same recurrence as the Catalan numbers. Stasheff's *associahedron* Asso_{n-1}, which is the poset of partial parenthesizations of a product of $n+1$ letters and which is the face poset of a $(n-1)$-dimensional polytope, famously embodies the Catalan numbers.

Theorem 2 (Black, Lütjeharms, Sanyal'23+)**.** *If P is an n-simplex and c a generic objective function, then $\Pi(P, c)$ is combinatorially isomorphic to Asso_{n-1}.*

This result is quite fascinating in that simplices are trivial from an optimization point of view. However, the simplex method is a sophisticated algorithm that can exhibit complex behaviour even on trivial instances.

If ρ is a linear projection for which (P, c) and $(P' := \rho(P), c' := \rho(c))$ have the same directed graph, then $\Pi(P', c') = \rho(\Pi(P, c))$. Thus, if P is a simplex and P' has a complete graph, then $\Pi(P', c')$ is a projection of an associahedron. A particularly nice case is when $P = \mathrm{Cyc}_n(t_1, \dots, t_{n+1})$ is a cyclic polytope, $c = e_1$, and ρ is the projection onto the first d coordinates. The projection P' is again a cyclic polytope and $\Pi(P', e_1)$ is a (generic) projection of an associahedron parametrized by $t_1, \dots, t_{n+1}$. In joint work with Aenne Benjes and Germain Poullot, we are currently investigating these polytopes that we call *cyclic associahedra*.

If $P = \mathrm{prism}(\Delta_n) = \Delta_n \times \Delta_1$ is the prism over the simplex and c is a generic objective function, then $\Pi(P, c)$ turns out to be combinatorially isomorphic to the *multiplihedron* Mul_n. The multiplihedron was also described by Stasheff. It encodes the evaluations of $f(a_1 a_2 \cdots a_{n+1})$, where f is a morphism between two non-associative structures. For example for $n = 1$, Mul_n is a segment with endpoints labelled by $f(a_1 a_2)$ and $f(a_1)f(a_2)$. For $n = 2$, Mul_n is a hexagon whose vertices are labelled by the evaluations of $f(a_1 a_2 a_3)$. A generalization to more morphisms was introduced by Chapoton and Pilaud [4] under the name (n, k)-multiplihedron.

Theorem 3 (Black, Lütjeharms, Sanyal'23+)**.** *If $P = \Delta_n \times \Delta_1^k$ is the k-fold prism over Δ_n and c is a generic objective function, then $\Pi(P, c)$ is combinatorially isomorphic to the (n, k)-multiplihedron.*

Finally we consider products of simplices $P = \Delta_m \times \Delta_n$.

Theorem 4 (Black, Lütjeharms, Sanyal'23+)**.** *For $m, n \geq 1$, $\Pi(\Delta_m \times \Delta_n, c)$ is combinatorially isomorphic to the (m, n)-constrainahedron.*

The (m, n)-constrainahedron was introduced by Bottman and Poliakova [3] to capture the collisions of mn particles that sit at the intersections of m horizontal and n vertical lines in the plane. The $(1, n)$-constrainahedra are associahedra, the $(2, n)$-constrainahedra are multiplihedra.

In order to show the stated combinatorial isomorphism to the associahedron, we make a connection between max-slope arborescences and particles with locations and velocities. We consider n particles at locations $-\omega_1 \leq -\omega_2 \leq \cdots \leq -\omega_n$ at time $t = 0$. For $t > 0$, the particles travel at constant velocities $-c$, where

$0 < c_1 < \cdots < c_n$. In this model one can interpret $A^\omega(i)$ as the *earliest* particle that will collide (and then absorb) with particle i.

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Hardness of linearly ordered 4-coloring of 3-colorable 3-uniform hypergraphs

Uli Wagner

(joint work with Marek Filakovský, Tamio-Vesa Nakajima, Jakub Opršal, and Gianluca Tasinato)

Deciding whether a given finite graph is 3-colorable (or, more generally, k-colorable, for a fixed $k \geq 3$) was one of the first problems shown to be NP-complete [7]. Since then, the complexity of *approximating* the chromatic number of a graph has been studied extensively, and it is known that, in general, the chromatic number cannot be approximated in polynomial time within a factor of $n^{1-\varepsilon}$, for any fixed $\varepsilon > 0$, unless $\mathsf{P} = \mathsf{NP}$ [14]. However, this hardness result only applies to graphs whose chromatic number grows with the number of vertices, and the case of graphs with *bounded* chromatic number is much less well understood. The *approximate graph coloring problem* concerns the computational complexity of the following problem: Given an input graph that is either k-colorable or not ℓ-colorable, for some integers $\ell \geq k \geq 3$, how hard is it to distinguish between the two cases? Khanna, Linial, and Safra [8] showed that this problem is NP-hard for $(k, \ell) = (3, 4)$, and it is a long-standing conjecture that the problem is NP-hard[1] for all constants $\ell \geq k \geq 3$; to date, this is known for $\ell \leq 2k - 1$ for all $k \geq 3$ [3], and for $k \geq 6$, the bound on ℓ has been improved to $\ell \leq \binom{k}{\lfloor k/2 \rfloor}$ [13].

For hypergraphs, it is known [5] that given a c-uniform hypergraph that is either k-colorable or not ℓ-colordable, it is NP-hard to distinguish between the two cases, for all constants $c \geq 3$ and $\ell \geq k \geq 2$. Here, we consider the following variant of hypergraph coloring, focusing on 3-uniform hypergraphs. A *linearly ordered k-coloring* (**LO**$_k$*-coloring*, for short) of a (3-uniform) hypergraph H is an assignment of elements ("colors") in $[k] = \{1, \ldots, k\}$ to the vertices of H such that, for every hyperedge, the maximal color assigned to elements of that hyperedge occurs exactly once in the hyperedge. Linearly ordered hypergraph coloring

[1] There are conditional hardness results (assuming different variants of Khot's *Unique Games Conjecture*) for approximate graph coloring for all $\ell \geq k \geq 3$, see [4].

generalizes both classical graph coloring and certain versions of the boolean satisfiability problem and has recently received a lot of attention [1, 11, 12]. Our main result (see [6] for more details) is the following:

Theorem 1. *The following problem is* NP*-hard: Given a* 3*-uniform hypergraph* H*, distinguish between the case that* H *is* $\mathbf{LO}_3$*-colorable and the case that* H *is not* $\mathbf{LO}_4$*-colorable.*

More generally, it is conjectured [1, Conjecture 27] that distinguishing between $\mathbf{LO}_k$-colorable hypergraphs and not $\mathbf{LO}_\ell$-colorable hypergraphs is NP-hard for all constants $\ell \geq k \geq 2$. (We remark that, for $k \geq 4$, an easy reduction shows that this conjecture is true whenever the approximate graph coloring problem with parameters $(k-1, \ell-1)$ is NP-hard, but our hardness result cannot be obtained this way.)

The proof of Theorem 1 builds on and extends a topological approach for studying approximate graph colouring introduced by Krokhin, Opršal, Wrochna, and Živný [9] and has two main parts. $\mathbf{LO}_k$-colorability of a hypergraph H is equivalent to the existence of a homomorphism from H to a certain relational structure $\mathbf{LO}_k$. For a natural number n, let $(\mathbf{LO}_3)^n$ be the n-fold power of the relational structure $\mathbf{LO}_3$. In the first part of the proof, we use topological methods to show that with every homomorphism $f\colon (\mathbf{LO}_3)^n \to \mathbf{LO}_4$, we can associate an *affine map* $\chi(f)\colon \mathbb{Z}_3^n \to \mathbb{Z}_3$ (i.e., a map of the form $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n \alpha_i x_i$, for some $\alpha_i \in \mathbb{Z}_3$ and $\sum_{i=1}^n \alpha_i \equiv 1 \pmod 3$); moreover, the assignment $f \mapsto \chi(f)$ preserves natural so-called *minor relations* that arise from maps $\pi\colon [n] \to [m]$, i.e., χ is a so-called *minion homomorphism*.

In the second part of the proof, we show by combinatorial arguments that the maps $\chi(f)\colon \mathbb{Z}_3^n \to \mathbb{Z}_3$ form a very restricted subclass of affine maps: They are projections, i.e., maps of the form $\mathbb{Z}_3^n \to \mathbb{Z}_3$, $(x_1, \ldots, x_n) \mapsto x_i$. Theorem 1 then follows from a hardness criterion obtained as part of a general algebraic theory so-called *promis constraint satisfaction problems* [2].

In a nutshell, topology enters in the first part of the proof as follows. First, with every homomorphism $f\colon (\mathbf{LO}_3)^n \to \mathbf{LO}_4$ we associate a continuous map $f_*\colon T^n \to P^2$, where T^n is the n-dimensional torus (the n-fold power of the circle S^1) and P^2 is a suitable target space; moreover, the cyclic group $\mathbb{Z}_3$ naturally acts on both T^n and P^2, and the map f_* preserves these symmetries (it is *equivariant*). This first step uses *homomorphism complexes* (a well-known construction in topological combinatorics that goes back to the work of Lovász [10]). Second, using equivariant obstruction theory, we show that equivariant continuous maps $T^n \to P^2$, when considered up to a natural equivalence relation of symmetry-preserving continuous deformation (*equivariant homotopy*), are in bijection with affine maps $\mathbb{Z}_3^n \to \mathbb{Z}_3$.

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Counting triangulations of homology 3-spheres

Geva Yashfe

(joint work with Karim Adiprasito, Marc Lackenby, Juan Souto, and (separately) with Yuval Peled)

M. Gromov popularized the following problem.

Problem 1. Let t_N be the number of (combinatorial isomorphism types of) triangulations of S^3 with N facets. Is t_N exponential or superexponential in N?

This problem remains unsolved.

Known results and related work.

- For triangulations of S^2 Tutte [9] proved that there are exponentially many triangulations with N triangles.

- It is known that $\exp(cN) \le t_N \le \exp(c'N\log N)$ for some[1] constants $c, c' > 0$.

  These bounds are relatively straightforward to prove. The upper bound $\exp(c'N\log N)$ actually holds for triangulations of d-manifolds for any fixed d (with an appropriate constant c' depending on d).

  It seems there is no hope for a very precise answer, so rough asymptotic results are all we aim for.
- Benedetti–Ziegler [3] showed that shellable spheres are at most exponentially many in N (in any fixed dimension). They also did this for a larger class spheres called "locally-constructible" (or LC). Benedetti–Pavelka [2] later extended this to a significantly larger class called 2-LC, but only in dimension 3.
- If we parametrize sphere triangulations by the number of vertices (call it M) instead, the problem has a different character. Some main results are:
  - Alon and Goodman–Pollack [1, 4] showed that there are relatively few (approximately $\exp(cM\log M)$) polytopes in M.
  - Kalai [6] showed that most triangulations (in terms of M) are not polytopal, and Lee [7] showed that Kalai's triangulations are shellable.
  - Nevo–Santos–Wilson [8] found still more triangulations than Kalai, and Yang [10] showed the families they produced are constructible.

Here we mainly consider the following relaxation of Problem 1.

Problem. Let t_N be the number of triangulations of 3-dimensional homology spheres with N facets. Is t_N exponential or superexponential in N?

For this we consider homology with coefficients in a fixed field $\mathbb{F}$. This problem also remains unsolved, with the best bounds remaining of the form $\exp(cN) \le t_N \le \exp(c'N\log N)$ for some $c, c' > 0$. This talk is about very modest progress and currently still-unsuccessful attempts.

1. Dual graphs and short graphs

Gromov and Nabutovski suggested reducing the problem to a problem about graphs. Gromov explains roughly what the result should be without describing the reduction in [5]. This section is based on joint work with K. Adiprasito, M. Lackenby, and J. Souto, and contains a sketch of our implementation of (part of) this idea and of two applications.

Dual graphs and enumeration. Suppose X is a triangulated 3-manifold with N facets, but we only have access to its dual graph G consisting of one vertex per facet, with an edge for every two facets that intersect in a triangle. Then there are at most $\exp(cN)$ possibilities for X given G: we have to put one tetrahedron in place of every vertex of G, and the only information missing is the manner in which adjacent tetrahedra are glued to each other. This leaves us with constantly many possibilities per facet of X, of which there are N.

[1] We don't care very much about the constants, and different occurences of "c" in this abstract do not refer to the same number.

Given the bounds we have on t_N, we may consider just the dual graphs of triangulated homology 3-spheres rather than the entire triangulations: asymptotics remain essentially unchanged (exponential factors only change our constants).

Remark. This discussion explains the upper bound $\exp(c'N\log N)$ for the number of triangulated d-manifolds: the dual graphs of triangulated d-manifold have constant degree $d+1$, and there are only $\exp(c'N\log N)$ constant-degree graphs on N vertices (with c' depending on the degree).

Properties of dual graphs. The family of dual graphs G of triangulated homology d-spheres over $\mathbb{F}$ has the following pleasant properties:

(1) It has bounded degree.
(2) For each G in the family there exists an $\mathbb{F}$-homology basis (given by taking a maximal independent subset of the dual cells of $(d-2)$-faces of the triangulated homology sphere) such that:
  (a) The average cycle length (in terms of the number of edges) in the basis is bounded by a constant. Equivalently, if G has N vertices then the total length of cycles in the basis is bounded by cN for some c.
  (b) Each edge of G participates in boundedly many of the cycles in the basis.

Definition. A class of graphs satisfying properties 1, 2(a), and 2(b) with some fixed constants is called *a class of short graphs.* These classes are parametrized by the field $\mathbb{F}$, the degree bound, and the constants in conditions 2(a) and 2(b) above.

Basically, one can think of classes of short graphs as a "soft / approximate" versions of dual graphs of triangulated homology spheres.

Theorem (Adiprasito, Lackenby, Souto, Yashfe)**.** *For each class $\mathcal{C}$ of short graphs there is a "machine"*

$$\mathcal{C} \hookrightarrow (\textit{triangulated homology } 3\textit{-spheres over } \mathbb{F})$$

taking each N-vertex graph in $\mathcal{C}$ to a triangulated homology 3-sphere with at most $c\cdot N$ tetrahedra (for c depending on $\mathcal{C}$).

Corollary. *If there exists $d>3$ such that there are superexponentially many N-facet triangulated homology d-spheres over $\mathbb{F}$, then the same holds for $d=3$.*

Proof. Take the dual graphs of these superexponentially many homology d-spheres to obtain a family of graphs contained in a class $\mathcal{C}$ of short graphs, and apply the machine to this class. □

The machines of the theorem preserve some of the geometry of the input graphs. This can also be applied to prove the following.

Theorem (Originally proved in unpublished work of M. Lackenby and J. Souto by slightly different methods.)**.** *There is a family of triangulations of S^3 for which the dual graphs form an expander family.*

(Getting triangulations of S^3 and not just some homology spheres requires an additional idea and a special family of short graphs.)

A sketch of the machine. Given a graph G together with a "short" homology basis over $\mathbb{F}$, construct a 2-complex by pasting cells along basis elements. Then thicken this complex to a triangulated 4-manifold in a geometrically controlled way (so as not to increase degrees or face numbers by too large a factor; this process is not canonical). Finally, pass to the boundary, which is a homology 3-sphere over $\mathbb{F}$. Injectivity of this process is not automatic and requires that we locally encode some combinatorial data in the resulting triangulation.

In the talk some additional ideas were sketched, mainly on the relation between problems here and problems about subgroup growth.

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Stress spaces, reconstruction problems and lower bound problems

Hailun Zheng

(joint work with Satoshi Murai and Isabella Novik)

What partial information about a simplicial d-polytope P allows one to determine P up to certain equivalences? Specifically, consider the following two equivalences: Given two polytopes P and P', we say that P and P' are *combinatorially equivalent* if they have isomorphic face lattices, and they are *affinely equivalent* if there is an affine map that sends P to P'. Perles (unpublished) and Dancis [3] proved that to determine the combinatorial type of a simplicial d-polytope P, it suffices to know the $\lfloor d/2 \rfloor$-skeleton of P. This result is optimal in the sense that distinct simplicial d-polytopes may have isomorphic $(\lfloor d/2 \rfloor - 1)$-skeleta. (For example, it is known that there are $2^{\Theta(n \log n)}$ combinatorial types of neighborly d-polytopes with n vertices [12].) On the other hand, the space of affine dependencies among the vertices of P determines the affine type of P (and hence also the combinatorial type of P). This observation is at the heart of the theory of Gale diagrams developed by Perles [15, Chapter 6].

A common ground of the above two results lies in the theory of stress spaces developed by Lee [6]. To see it, note that for a simplicial d-polytope P, the space of affine dependencies of vertices of P is equivalent to the space of affine 1-stresses on P, while the space of affine k-stresses is trivial for any $k \geq \lfloor d/2 \rfloor + 1$. Hence these two results are precisely the $k = \lfloor d/2 \rfloor + 1$ and $k = 1$ cases of the following conjecture of Kalai [4].

Conjecture 1. *Let P be a simplicial d-polytope and let $1 \leq k \leq \lfloor d/2 \rfloor + 1$. Then the $(k-1)$-skeleton of P and the space of affine k-stresses of P uniquely determine the combinatorial type of P.*

Another conjecture concerning the affine types of polytopes is the following

Conjecture 2. *Let $d \geq 2k \geq 4$ and let P be a simplicial d-polytope with the natural embedding and with no missing faces of dimension $\geq d - k + 1$. Then the space of affine k-stresses uniquely determines P up to affine equivalence.*

We present two partial results of the above two conjectures. The first result verifies the case of $k = 2$ of Conjecture 1, namely

Theorem 3 ([10])**.** *Let $d \geq 3$. The graph of a simplicial d-polytope P together with the space of affine 2-stresses on P uniquely determine the combinatorial type of P.*

The proof is geometric-combinatorial. The idea is to use the rigidity theory of frameworks to show that the missing faces of P can be identified by the sign patterns of the coefficients of the squarefree terms in certain affine 2-stresses on P.

The second result deals with Conjecture 2 and more generally the structures of affine stress spaces of polytopes with no large missing faces.

Theorem 4 ([8])**.** *Let $1 \leq j < k \leq \frac{d-1}{2}$ and let P be a simplicial d-polytope with no missing faces of dimension $\geq d - k + 1$. Then the space of affine k-stresses on P determines the space of affine j-stresses on P.*

In particular, Theorem 4 verifies the $k \leq \frac{d-1}{2}$ case of Conjecture 2 (by letting $j = 1$). At the moment, the $d = 2k \geq 4$ case remains open.

The proof of Theorem 4 is algebraic. In particular, it relies on identifying the space of affine stresses on P with the Matlis dual N of the Stanley-Reisner ring of P modulo the linear system of parameters and the Lefschetz element. (Hence, the condition on the missing faces translates into a condition on the degrees of generators of N.)

Three comments are in order. First, prior to Theorem 4, Conjecture 2 was proved by Cruickshank, Jackson and Tanigawa [2] in the case that P is a simplicial polytope whose vertices have generic coordinates, and by Novik and Zheng [11] for all simplicial d-polytopes that have no missing faces of dimension $\geq d-2k+2$. Second, with the proof of the g-theorem for simplicial spheres [1, 13, 5], Theorem 4 not only applies to simplicial d-polytopes with natural embeddings but also simplicial $(d-1)$-spheres with generic embeddings. Finally, Theorem 4 leads to two corollaries on the g-numbers of simplicial $(d-1)$-spheres that are interesting in their own right. For a simplicial complex Δ, denote by $m_i(\Delta)$ the number of missing i-faces of Δ. Recall that the g-theorem [14, 1, 13, 5] states that the g-numbers of a simplicial $(d-1)$-sphere form an M-sequence, i.e., $0 \leq g_{k+1} \leq g_k^{<k>}$ holds for all $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$. The following corollary is a strengthening of the g-theorem; part of the inequality appeared first in [9].

Corollary 5. *Let $d \geq 4$ and let Δ be a simplicial $(d-1)$-sphere. Then for all $1 \leq k \leq \lceil d/2 \rceil - 1$, $g_k(\Delta) \geq m_{d-k}(\Delta)$. Furthermore, $0 \leq g_{k+1}(\Delta) \leq \big(g_k(\Delta) - m_{d-k}(\Delta)\big)^{<k>}$.*

Recall also that the Generalized Lower Bound Theorem [7] asserts that for $2 \leq k \leq \lfloor d/2 \rfloor$, a simplicial $(d-1)$-sphere has $g_{k+1} = 0$ if and only if it is k-stacked. The following corollary gives a second characterization of spheres attaining a minimal g-number.

Corollary 6. *Let Δ be a simplicial $(d-1)$-sphere. Then for $1 \leq k \leq \lfloor d/2 \rfloor - 1$, Δ is k-stacked if and only if $g_k(\Delta) = m_{d-k}(\Delta)$. Moreover, if d is odd and Δ is $\frac{d-1}{2}$-stacked, then $g_{\frac{d-1}{2}}(\Delta) = m_{\frac{d+1}{2}}(\Delta)$.*

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Open Problems in Geometric, Algebraic and Topological Combinatorics

Collected by Edward Swartz

PROBLEM 1 (Nati Linial, joint with Jordan Smith). Computing certain invariants of topological spaces of dimension three

Say that a graph is *geodesic* if between every two vertices there is a unique shortest path. Ore (1960) defined this class of graphs and asked for a characterization, but this quest seems way out of hand. It suffices, of course to consider only 2-connected graphs. There is a known infinite family of (i) geodesic, (ii) 2-connected graphs (iii) in which all vertex degrees are at least 3 and have diameter 5, but nothing beyond. Question: Can such graphs have arbitrarily large diameter?

PROBLEM 2 (Benjamin Braun, joint with Kaitlin Bruegge). Bounding facet numbers for symmetric edge polytopes

Let G be a finite simple graph and let $P_G = \text{conv}\{e_i - e_j, e_j - e_i : ij \in E(G)\}$ be the *symmetric edge polytope* of G. Determining properties of the facets of symmetric edge polytopes is of interest both in combinatorics and in applications. To this end, the authors made the following conjecture regarding bounds on the number of facets for symmetric edge polytopes of connected graphs on a fixed number of vertices.

Conjecture. (1) (Braun and Bruegge [1]). Let $n \geq 3$.

(1) For $n = 2k + 1$, the maximum number of facets for P_G for a connected graph G on n vertices is 6^k, which is attained by a wedge of k cycles of length three.
(2) For $n = 2k$, the maximum number of facets for P_G for a connected graph G on n vertices is $14 \cdot 6^{k-2}$, which is attained by a wedge of K_4 with $k - 2$ cycles of length three.
(3) For $n = 2k + 1$, the minimum number of facets for P_G for a connected graph G on n vertices is $3 \cdot 2^k - 2$, which is attained by $K_{k,k+1}$.
(4) For $n = 2k$, the minimum number of facets for P_G for a connected graph G on n vertices is $2^{k+1} - 2$, which is attained by $K_{k,k}$.

Partial progress on this conjecture was announced in a preprint by Mori, Mori, and Ohsugi [2].

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PROBLEM 3 (Benjamin Braun, joint with Matias von Bell, Derek Hanely, Khrystyna Serhiyenko, Julianne Vega, Andrés Vindas-Meléndez and Martha Yip). Enumerating regular triangulations of order polytopes for snake posets

For $n \in \mathbb{Z}_{\geq 0}$, a *generalized snake word* is a word of the form $\mathbf{w} = w_0 w_1 \cdots w_n$ where $w_0 = \varepsilon$ is the empty letter and w_i is in the alphabet $\{L, R\}$ for $i = 1, \ldots, n$. The *length* of the word is n, which is the number of letters in $\{L, R\}$. Given a generalized snake word $\mathbf{w} = w_0 w_1 \cdots w_n$, we define the *generalized snake poset* $P(\mathbf{w})$ recursively in the following way:

- $P(w_0) = P(\varepsilon)$ is the poset on elements $\{0, 1, 2, 3\}$ with cover relations $1 \prec 0$, $2 \prec 0$, $3 \prec 1$ and $3 \prec 2$.
- $P(w_0 w_1 \cdots w_n)$ is the poset $P(w_0 w_1 \cdots w_{n-1}) \cup \{2n+2, 2n+3\}$ with the added cover relations $2n+3 \prec 2n+1$, $2n+3 \prec 2n+2$, and

$$\begin{cases} 2n+2 \prec 2n-1, & \text{if } n=1 \text{ and } w_n = L, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{RL, LR\}, \\ 2n+2 \prec 2n, & \text{if } n=1 \text{ and } w_n = R, \text{ or } n \geq 2 \text{ and } w_{n-1}w_n \in \{LL, RR\}. \end{cases}$$

In this definition, the minimal element of the poset $P(\mathbf{w})$ is $\widehat{0} = 2n+3$, and the maximal element of the poset is $\widehat{1} = 0$.

As part of a more extensive investigation of triangulations of order polytopes related to generalized snake posets, the authors made the following conjecture regarding the order polytope of the following poset: for the length n word $\varepsilon LRLR \cdots$, $S_n := P(\varepsilon LRLR \cdots)$ is the *snake poset*.

Conjecture. (2) (von Bell, Braun, Hanely, Serhiyenko, Vega, Vindas-Meléndez, Yip [1]). The number of regular triangulations of the order polytope of S_n is $2^{n+1}\mathrm{Cat}(2n+1)$, where $\mathrm{Cat}(2n+1)$ denotes the $2n+1$-st Catalan number.

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PROBLEM 4 (Raman Sanyal, joint with Sebastian Manecke). Strongly inscribable arrangements and reflection arrangements

Consider an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ of n hyperplanes in $\mathbb{R}^d$, all passing through the origin. Choosing a normal vector z_i for each hyperplane H_i gives rise to an associated zonotope

$$Z(\mathcal{A}) = [-z_1, z_1] + \cdots + [-z_n, z_n] = \{\mu_1 z_1 + \cdots + \mu_n z_n : -1 \leq \mu_1, \ldots, \mu_n \leq 1\}.$$

$Z(\mathcal{A})$ is a convex polytope whose combinatorics faithfully represents that of $\mathcal{A}$. We call $\mathcal{A}$ *strongly inscribable* if there is a choice of normal vectors such that $Z(\mathcal{A})$ is inscribed, that is, has all vertices on the unit sphere. For example, reflection arrangements obtained from finite reflection groups are strongly inscribable.

In [4], we showed that the restriction of a strongly inscribable arrangement to any of its hyperplanes is again strongly inscribable. Thus, further examples are provided by restrictions of reflection arrangements, which generally are not reflection arrangements themselves.

Conjecture. (3) ([4, Conj. 1.7]). An arrangement of hyperplanes in $\mathbb{R}^d$ with $d \geq 3$ is strongly inscribable if and only if it is linearly isomorphic to the restriction of a reflection arrangement.

An important structural property that we observe in [4] is that every strongly inscribable arrangement is *simplicial*, that is, every connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{A}$ is linearly isomorphic to $\mathbb{R}^d_{>0}$. Simplicial arrangements are fascinating but rare. There is a conjecturally complete catalog of simplicial arrangements in $\mathbb{R}^3$ due to Grünbaum and Cuntz [1, 2, 3]. We verify that the only strongly inscribable arrangements in this catalog are restrictions of reflection arrangements. Assuming the completeness of the Grünbaum–Cuntz catalog, this proves the conjecture in dimension 3.

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PROBLEM 5 (Georg Loho). Realizable M-convex functions

A *generalized permutahedron* is a polytope whose edge directions are of the form $e_i - e_j$ for standard unit vectors e_i, e_j. An *integral generalized permutahedron* is a generalized permutahedron that is also a lattice polytope. An *M-convex set* is the set of lattice points in an integral generalized permutahedron. Let $f\colon S \to \mathbb{R}$ be a function for some finite subset $S \subseteq \mathbb{Z}^n$. It is *M-convex* if $\operatorname{argmin}_{x \in S}(f(x) - \langle c, x\rangle)$ is an M-convex set for each $c \in \mathbb{R}^n$.

Matroids are special M-convex sets, namely those contained in the unit cube $\{0,1\}^n$. Furthermore, *valuated matroids* are special M-convex functions, namely those with a matroid as domain. Matroids are *realizable* if they arise from the independence structure of a matrix. Valuated matroids are *realizable* if they arise as tropicalization of a Pluecker vector of a linear space. M-convex sets are *realizable* if their defining integral generalized permutahedron can be described by a submodular function arising from a subspace arrangement.

However, it is not clear when an M-convex function should be called *realizable.* There are some potential candidates. One could argue that M-convex functions arising from realizable valuated matroids by induction through a directed graph should be called realizable. Furthermore, M-convex functions arise by tropicalization from Lorentzian polynomials. For the latter, one could argue that those arising as volume polynomials should be realizable, giving rise to another notion of realizability for M-convex functions. Still, the notion does not seem clear compared to the nice picture for (valuated) matroids and M-convex sets, leaving the following questions open.

What is a 'realizable' M-convex function?
What is a 'Pluecker vector' of a subspace arrangement?

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PROBLEM 6 (Germain Poullot)**.** Is log-concavity arising through matrix recursion?

The problem presented here arises when studying the monotone path polytope of the hypersimplex $\Delta(n,2)$. After numerous pages of tedious proofs, one can count the number of coherent monotone paths of $\Delta(n,2)$ thanks to the following matrix recursion. Similar problems give rises to similar question, but we present here a very concrete occurrence, hoping from someone to develop tools to address the general setting.

Let's define the sequences of polynomials T_n, Q_n and C_n satisfying the following recursive formula:

$$\forall n \geq 4, \; \begin{pmatrix} T_{n+1} \\ Q_{n+1} \\ C_{n+1} \end{pmatrix} = \mathcal{M} \begin{pmatrix} T_n \\ Q_n \\ C_n \end{pmatrix}$$

$$\text{with } \mathcal{M} = \begin{pmatrix} z & 1+z & 1+z \\ 0 & 1+z & z \\ z+z^2 & 0 & 1+z \end{pmatrix} \;,\; \begin{pmatrix} T_4 \\ Q_4 \\ C_4 \end{pmatrix} = \begin{pmatrix} z^4+2z^3 \\ z^4 \\ 2z^4+2z^3 \end{pmatrix}$$

and the polynomial $V_n = T_n + Q_n + C_n = \sum_k v_{n,k} z^k$.

Conjecture: For all $n \geq 4$, the sequence $(v_{n,k})_n$ is (ultra-)log-concave.

The value $v_{n,k}$ counts the number of coherent monotone paths of $\Delta(n,2)$, and a good combinatorial model allows to exhibit this recursion, but I haven't able to extract log-concavity from this combinatorial interpretation (yet).

This conjecture have been checked for all $n \leq 300$ (and one can easily go further, but where to stop?), please prove it!

Obviously, the question can be posed more generally: given a starting vector $X_0 \in \mathbb{N}[z]^m$ and a matrix $\mathcal{M} \in \mathrm{Mat}_{m\times m}(\mathbb{N}[z])$, what kind of tools can we develop to address (ultra-)log-concavity and unimodality questions for (the polynomials of the vector) $X_n = \mathcal{M}^n X_0$ and the polynomial $\sum_{r=1}^m X_{n,r}$?

Note that, for $n \leq 300$:

- T_n, Q_n, C_n and V_n are ultra-log-concave (so homogenizing each of them gives lorentzian polynomials).
- (in general) T_n, Q_n, C_n and V_n **are not** symmetric.
- (in general) T_n, Q_n, C_n and V_n **are not** real-rooted.

- Properties of $\mathcal{M}$ (left- or right-kernel and eigenvectors) seems not helpful: one computes $\mathcal{M}^n$, but then extracting log-concavity it out of my reach...
- Starting with different polynomials for T_4, Q_4 and C_4, its seems that V_n becomes ultra-log-concave after a certain rank.

One can mathematically prove that:

- $\deg V_n = d = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$ (with $v_{n,d} = 1$ if n odd ; $v_{n,d} = d$ if n even).
- the "constant coefficient" of V_n is 4 (i.e. $v_{n,4} = 4$).
- $V_n(1) = \sum_k v_{n,k} = \frac{1}{3}(25 \times 4^{n-4} - 1)$
- for a fixed k, the value of $v_{n,k}$ is a polynomial in n (of degree $k-3$), mimicking slightly the behavior of binomial coefficients.

PROBLEM 7 (Bruno Benedetti, joint with Matteo Varbaro)**.** The dual graph of Cohen–Macaulay algebras

In the following, $S = \mathbb{K}[x_1, \ldots, x_n]$ is the polynomial ring with n variables over some field $\mathbb{K}$. Let I be any ideal of S. Let $\wp_1, \ldots, \wp_s$ be the minimal primes of I that have height equal to the height of I. We define the *dual graph* $G(I)$ on the vertex set $[s] = \{1, \ldots, s\}$ as follows: there is an edge $[i,j]$ if and only if

$$\operatorname{height}(P_i + P_j) = 1 + \operatorname{height} I.$$

There are two motivations for this definition:

(1) When I is radical and height-unmixed, $I = \wp_1 \cap \ldots \cap \wp_s$. Passing to the Zariski sets, this means that $Z(I) = Z(\wp_1) \cup \ldots \cup Z(\wp_s)$. Thus $G(I)$ coincides with the dual graph of $Z(I)$, where an edge connects two irreducible components iff their intersection has codimension one.

(2) When $I = I_\Delta$ is the Stanley-Reisner ideal of some simplicial complex Δ on n vertices, then $G(I)$ coincides with the dual graph of Δ. In this case $\operatorname{height} I = n - \dim \Delta - 1$.

Recall that in a connected graph G, the distance between two vertices is the number of edges in a shortest path connecting them, and $\operatorname{diam} G$ is the maximum distance between any two of its vertices.

Conjecture. [Benedetti–Varbaro [2], 2014]: Let $I \subseteq S$ be an ideal generated in degree two. If S/I is Cohen–Macaulay, then $\operatorname{diam} G(I) \leq \operatorname{height}(I)$.

In the meantime, the conjecture has been proven true for many interesting cases, cf. e.g. [3], [4], [5], [6], [7]. It holds for squarefree monomial ideals: This follows from the result by Adiprasito–Benedetti that "flag normal complexes satisfy the Hirsch conjecture" [1]. In fact, when $I = I_\Delta$, the upper bound $\operatorname{height} I = n - \dim \Delta - 1$ reflects exactly the Hirsch bound.

A final comment: The condition "generated in degree two" is not really restrictive. Via Veronese embeddings, if the Conjecture above is true, one automatically gets a polynomial bound of the type

$$\operatorname{diam} G(I) \leq \operatorname{height}(I)^{\lceil k/2 \rceil}$$

for all ideals I generated in degree $\leq k$ and such that S/I is Cohen–Macaulay [2]. Thus in particular the Conjecture above would imply the following:

Conjecture. If Δ is a normal simplicial complex of dimension d, with n vertices, and no missing face of dimension $\leq k$, then the dual graph of Δ has diameter at most $P(n)$, where P is a polynomial in n of degree $\lceil k/2 \rceil$.

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PROBLEM 8 (Felipe Rincón)**.** Tensor products of matroids

If M is a matroid on the ground set E and N is a matroid on the ground set F, a *quasi-product* of M and N is a matroid T on the ground set $E \times F$ satisfying: for any $f \in F$, the natural bijection between E and $E \times \{f\}$ induces a matroid isomorphism between M and the restriction $T|_{E\times\{f\}}$, and similarly, for any $e \in E$, the natural bijection between F and $\{e\} \times F$ induces a matroid isomorphism between N and the restriction $T|_{\{e\}\times F}$.

It is a simple exercise to show that the rank of any quasi-product T of M and N has rank at most $\mathrm{rank}(M) \cdot \mathrm{rank}(N)$. The quasi-product T is called a *tensor product* if in fact we have the equality $\mathrm{rank}(T) = \mathrm{rank}(M) \cdot \mathrm{rank}(N)$.

To my knowledge, there are only very few things that we know about tensor products:

- If M and N are realizable over the same field K then M and N admit a tensor product. This is due to the fact that, for subspaces $L_M \subset K^E$ and $L_N \subset K^F$ that represent M and N, respectively, we can construct the tensor product $L_M \otimes L_N \subset K^E \otimes K^F$, which then represents a tensor product T of M and N. The resulting tensor product T might depend on the realizations chosen, though.
- The matroids V_8 and $U_{2,3}$ do not admit a tensor product! This is one of the main results of [2].
- If M and N admit a tensor product, M' is a minor of M, and N' is a minor of N, then M' and N' admit a tensor product. This is not too difficult – for details, you can see, for instance, [1].

The research problem I am proposing is to study further the class of matroids M, N that admit a tensor product. For instance, can you construct more matroids that admit a tensor product? Can you describe tensor products combinatorially for particular classes of matroids? Can you say something about forbidden minors for the existence of tensor products (see [1])?

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PROBLEM 9 (Pablo Soberón). A problem in the plane

Given a family $\mathcal{F}$ of lines in the plane such that no two are parallel, we can determine the size of a set $A \subset \mathbb{R}^2$ as follows:

$$\mu(A) = \max\{k \in \mathbb{N} : \text{ there exists a set of } k \text{ lines of } \mathcal{F} \text{ whose pairwise intersections are all in } A.\}$$

If A does not contain any intersection of lines of $\mathcal{F}$, we define $\mu(A) := 1$.

Let (A, B, C) be a convex partition of the plane into three parts. In other words, each of A, B, C is a closed convex set in $\mathbb{R}^2$, their interiors are pairwise disjoint, and their union is $\mathbb{R}^2$.

Show that

$$\mu(A)\mu(B)\mu(C) \geq |\mathcal{F}|.$$

The case when one of A, B, C is a half-plane is easy to prove. The best known bound is [1]:

$$\mu(A)\mu(B)\mu(C) \geq \left(\frac{2}{3}\right) |\mathcal{F}|.$$

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The homogenized Linial arrangement and Genocchi numbers

Michelle Wachs
(joint work with Alexader Lazar)

The *braid arrangement* (or *type A Coxeter arrangement*) is the hyperplane arrangement in $\mathbb{R}^n$ defined by

$$\mathcal{A}_{n-1} := \{x_i - x_j = 0 : 1 \leq i < j \leq n\}.$$

Note that the hyperplanes of $\mathcal{A}_{n-1}$ divide $\mathbb{R}^n$ into open cones of the form

$$R_\sigma := \{\mathbf{x} \in \mathbb{R}^n : x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}\},$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$. Hence the braid arrangement $\mathcal{A}_{n-1}$ has $|\mathfrak{S}_n| = n!$ regions.

A classical formula of Zaslavsky [11] gives the number of regions of any real hyperplane arrangement $\mathcal{A}$ in terms of the Möbius function of its intersection (semi)lattice $\mathcal{L}(\mathcal{A})$. Indeed, given any finite, ranked poset P of length ℓ, with a minimum element $\hat{0}$, the *characteristic polynomial* of P is defined to be

$$\chi_P(t) := \sum_{x \in P} \mu_P(\hat{0}, x) t^{\ell - \mathrm{rk}(x)}, \tag{1}$$

where μ_P is the Möbius function of P and $\mathrm{rk}(x)$ is the rank of x. Zaslavsky's formula for the number of regions $r(\mathcal{A})$ of $\mathcal{A}$ is

$$r(\mathcal{A}) = (-1)^{\ell}\chi_{\mathcal{L}(\mathcal{A})}(-1). \tag{2}$$

It is well known and easy to see that the lattice of intersections of the braid arrangement $\mathcal{A}_{n-1}$ is isomorphic to the lattice Π_n of partitions of the set $[n] := \{1, 2, \dots, n\}$. It is also well known that the characteristic polynomial of Π_n is given by

$$\chi_{\Pi_n}(t) = \sum_{k=1}^{n} s(n,k)t^{k-1}, \tag{3}$$

where $s(n,k)$ is the Stirling number of the first kind, which is equal to $(-1)^{n-k}$ times the number of permutations in $\mathfrak{S}_n$ with exactly k cycles; see [10, Example 3.10.4]. Hence $\chi_{\Pi_n}(-1) = (-1)^{n-1}|\mathfrak{S}_n|$. Therefore, from (2), we recover the result observed above that the number of regions of $\mathcal{A}_{n-1}$ is $n!$.

In this talk, we consider a hyperplane arrangement introduced by Hetyei [5]. The *homogenized Linial arrangement* is the hyperplane arrangement in $\mathbb{R}^{2n}$ defined, for $n \geq 2$, by

$$\mathcal{H}_{2n-3} := \{x_i - x_j = y_i : 1 \leq i < j \leq n\}.$$

Note that by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_n = 0$ one gets the braid arrangement $\mathcal{A}_{n-1}$. Similarly by intersecting $\mathcal{H}_{2n-3}$ with the subspace $y_1 = y_2 = \cdots = y_n = 1$, one gets the Linial arrangement in $\mathbb{R}^n$,

$$\{x_i - x_j = 1 : 1 \leq i < j \leq n\}.$$

Postnikov and Stanley [8] show that the number of regions of the Linial arrangement is equal to the number of alternating trees on node set $[n+1]$.

Hetyei [5] shows that the number of regions of the homogenized Linial arrangement is equal to a number known as the median Genocchi number. He uses the finite field method of Athanasiadis [1] to obtain a recurrence for $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(-1)$ and shows that the recurrence reduces to a known formula for the median Genocchi number h_n. The result then follows from Zaslavsky's formula (2). The Genocchi numbers g_n and the median Genocchi numbers h_n can be characterized by the Barsky and Dumont [2] generating function formulas:

$$\sum_{n\geq 1} g_n x^n = \sum_{n\geq 1} \frac{(n-1)!\,n!\,x^n}{\prod_{k=1}^{n}(1+k^2 z)} \tag{4}$$

$$\sum_{n\geq 0} h_n z^n = \sum_{n\geq 0} \frac{n!\,(n+1)!\,z^n}{\prod_{k=1}^{n}(1+k(k+1)z)}. \tag{5}$$

In [6] we further study the intersection lattice $\mathcal{L}(\mathcal{H}_{2n-1})$ and its characteristic polynomial $\chi_{\mathcal{L}(\mathcal{H}_{2n-1})}(t)$ using an approach quite different from Hetyei's. We prove

$$\sum_{n\geq 1} \chi_{L(\mathcal{H}_{2n-1})}(t)\, x^n = \sum_{n\geq 1} \frac{(t-1)_{n-1}(t-1)_n\, x^n}{\prod_{k=1}^{n}(1-k(t-k)x)}, \tag{6}$$

where $(a)_n$ denotes the falling factorial $a(a-1)\cdots(a-n+1)$. By setting $t=-1$, we recover Hetyei's result. Moreover, by setting $t=0$, we relate the (non-median) Genocchi numbers to the homogenized Lineal arrangement. Indeed, since $\chi_{L(\mathcal{H}_{2n-1})}(0)$ is the Möbius invariant of $L(\mathcal{H}_{2n-1})$, we have

$$\mu_{L(\mathcal{H}_{2n-1})}(\hat{0},\hat{1}) = -g_n. \tag{7}$$

Our proof of (6) has the following steps.

1. Show that $t\chi_{L(\mathcal{H}_{2n-1})}(t)$ equals the chromatic polynomial $\mathrm{ch}_{\Gamma_n}(t)$ of a certain graph Γ_n.
2. Using the Rota-Whitney NBC theorem, show that the coefficients of $\mathrm{ch}_{\Gamma_n}(t)$ can be described in terms of a certain class of alternating forests.
3. Construct a bijection from this class of alternating forests to a new class of permutations similar to those introduced by Dumont [3] to study Genocchi numbers. This yields a result analogous to (3) involving cycles of Dumont-like permutations.
4. Construct a bijection from the Dumont-like permutations to a certain class of objects called surjective staircases. Results of Randrianarivony [9] and Zeng [12] on generating functions for surjective staircases are used to complete the proof.

We also introduce a Dowling analog of the homogenized Linial arrangement. Let $\omega = e^{2\pi i/m}$ be a primitive mth root of unity. The *homogenized Linial-Dowling arrangement* is the complex hyperplane arrangement in $\mathbb{C}^{2n}$, defined by

$$\mathcal{H}^m_{2n-1} = \{x_i - \omega^\ell x_j = y_i : 1 \le i < j \le n,\ 0 \le \ell < m\} \cup \{x_i = y_i : 1 \le i \le n\}.$$

Note that when $m=2$, the arrangement $\mathcal{H}^m_{2n-1}$ is a complexified version of the type B homogenized Linial arrangement. When $m=1$, the arrangement $\mathcal{H}^m_{2n-1}$ is the complexified version of the arrangement obtained by intersecting $\mathcal{H}_{2n-1}$ with the coordinate hyperplane $x_{n+1}=0$. The resulting arrangement on the coordinate hyperplane has the same intersection lattice as $\mathcal{H}_{2n-1}$.

Using similar techniques as for the homogenized Linial arrangement, we prove in [7] the following generalization of (6):

$$\sum_{n\ge 1} \chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(t)\, x^n = \sum_{n \ge 1} \frac{(t-1)_{n,m}(t-m)_{n-1,m}\, x^n}{\prod_{k=1}^n (1-mk(t-mk)x)}. \tag{8}$$

where $(a)_{n,m} = a(a-m)(a-2m)\cdots(a-(n-1)m)$.

There is a well-studied polynomial analog of the Genocchi numbers known as the Gandhi polynomials $G_n(x)$; see [4, Section 3]. We obtain the following m-analog of (7):

$$\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(0) = \mu_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(\hat{0},\hat{1}) = -m^{2n-1}G_n(m^{-1}).$$

The polynomials $\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(0)$ and $\chi_{\mathcal{L}(\mathcal{H}^m_{2n-1})}(-1)$ can be viewed as m-analogs of the Genocchi and median Genocchi numbers, respectively. It would be interesting to generalize known relationships between the Genocchi numbers and median Genocchi numbers to these m-analogs.

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Arbeitsgemeinschaft: QFT and Stochastic PDEs

Organized by
Roland Bauerschmidt, New York
Massimiliano Gubinelli, Oxford
Martin Hairer, London/Lausanne
Hao Shen, Madison

17 December – 22 December 2023

Abstract. Quantum field theory (QFT) is a fundamental framework for a wide range of phenomena is physics. The link between QFT and SPDE was first observed by the physicists Parisi and Wu (1981), known as Stochastic Quantisation. The study of solution theories and properties of solutions to these SPDEs derived from the Stochastic Quantisation procedure has stimulated substantial progress of the solution theory of singular SPDE, especially the invention of the theories of regularity structures and paracontrolled distributions in the last decade. Moreover, Stochastic Quantisation allows us to bring in more tools including PDE and stochastic analysis to study QFT.

This Arbeitsgemeinschaft starts by covering some background material and then explores some of the advances made in recent years. The focus of this Arbeitsgemeinschaft is QFT models such as the Φ^4, sine-Gordon and Yang–Mills models as examples to discuss stochastic quantisation and SPDE methods and their applications in these models. We introduce the key ideas, results and applications of regularity structure and paracontrolled distributions, construction of solutions of the SPDEs corresponding to these models, and use the PDE method to study some qualitative behaviors of these QFTs, and connections with the corresponding lattice or statistical physical models. We also discuss some other topics of QFT, such as Wilsonian renormalisation group, log-Sobolev inequalities and their implications, and various connections between these topics and SPDEs.

Mathematics Subject Classification (2020): 35R60, 37A50, 37L55, 39A50, 60H15, 60L30, 76M35, 81S20, 81T08, 81T13, 81T25, 81T27.

Introduction by the Organizers

The Arbeitsgemeinschaft *QFT and Stochastic PDEs*, organized by Roland Bauerschmidt (New York), Massimiliano Gubinelli (Oxford), Martin Hairer (London/Lausanne), and Hao Shen (Wisconsin-Madison) was attended by 44 participants (as well as a few remote participants). There was a broad geographic representation from all continents. Most of the participants were in early stages of their careers, with background mostly in the areas of probability theory, analysis, and theoretical physics. All the in-person participants delivered talks, with a total of 22 talks, each coordinated and presented by two speakers.

The talks were organized in a progressive order. The talks on Monday focused on general introductions to Euclidean QFT, and local solutions to SPDEs in the Da Prato–Debussche regime. The example of the stochastic quantisation of the Φ^4_2 model (which is the simplest nontrivial case of a nonlinear SPDE from Euclidean QFT) was discussed. The talks on Tuesday then discussed global solution theory to the stochastic quantisation of Φ^4_2. The talks on Wednesday provided more applications of the Da Prato–Debussche argument, and examples of using PDE methods to study some qualitative behaviors of these QFTs such as integrability of the Φ^4_2 measure, as well as connections with the corresponding statistical physical models. The Wednesday talks introduced the Yang-Mills model and its Langevin dynamics, in continuum and on lattice. On Thursday, the theory of regularity structures was introduced by the participants, and some of the cornerstone theorems of this theory were proved; an application to the stochastic quantisation of the Φ^4_3 model was given. The talks on Friday were focussed on the Wilsonian renormalisation group approach, log-Sobolev inequalities and their implications, and the connections between these topics and SPDEs.

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Arbeitsgemeinschaft: QFT and Stochastic PDEs

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Abstracts

Towards Euclidean quantum field theory

Wei Huang, Weile Weng

The talk consists of two parts: the first part focus on the Feynman-Kac formula that leads to Euclidean quantum mechanics and the second part is about the Ostwalder-Schrader axioms in Euclidean quantum field theory, with a focus on reflection positivity.

We begin our first part with a brief introduction on the basic postulates of quantum mechanics. Then we have a look at one of the simplest quantum mechanics systems, the quantum harmonic oscillator. After rescaling, we get the Hamiltonian $H = 1/2(P^2 + Q^2)$, where $P = i\partial_x, Q = x$. Note that $H = A^*A + 1/2$, where $A = \frac{1}{\sqrt{2}}(Q + iP), A^* = \frac{1}{\sqrt{2}}(Q - iP)$. A is called the annihilation operator, as for any eigenvector Ω of H with eigenvalue λ, $A^*\Omega$(if non-zero) is an eigenvector of H with eigenvalue $\lambda - 1$. A^* is called the creation operator as it increases the eigenvalue by 1 when acting on eigenfunctions. There is a unique ground state which has the lowest eigenvalue 1/2, and the corresponding eigenfunction is $\Omega_0(x) = \pi^{-1/4}e^{-x^2/2}$. The other eigenvectors can be obtained by acting A^* on it and $\Omega_n = (n!)^{-1/2}A^{*n}\Omega_0 = (n!)^{-1/2}P_n(Q)\Omega_0$, where P_n are the Hermite polynomials. The eigenvectors forms an ONS of $\mathcal{H}$, and they can all be obtained from the ground state by multiplying polynomials of Q.

The Feynman path integral expresses the integral kernel of the Schrödinger propagator in terms of a path integral

$$e^{-itH/\hbar}(x, x') = \frac{1}{Z}\int_{\gamma_0=x,\gamma_t=x'} e^{-\frac{i}{\hbar}\int_0^t \frac{1}{2}\dot{\gamma}_s^2 - V(\gamma_s)ds} d\gamma.$$

It reveals a physical intuition that the particle takes all the possible path with weights given by the classical action, but mathematically the integral is very problematic as it cannot be defined with a measure. We can avoid the problem by running the dynamics in imaginary time(also called Wick rotation). We then get the Feynman-Kac formula(we set $\hbar = 1$):

$$e^{-tH}(x, x') = \int e^{-\int_0^t V(\gamma)ds} dW_{x,x'}(\gamma),$$

where $W_{x,x'}$ is the Wiener measure conditioned on starting at x and ending at x'. The Feynman-Kac formula implies the positivity of kernel and uniqueness and positivity of ground state, which are necessary to construct the measure in the renormalized Feynman-Kac formula.

Assume there exists a ground state Ω and assume the ground state energy to be E_0. Then we subtract the energy to get $\hat{H} = h - E_0$. The ground state transformation($\times\Omega^{-1}$) is a isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \Omega^2 dm)$ and we can transfer $\hat{H}$ to a self-adjoint operator $H^\wedge = \Omega\hat{H}\Omega^{-1}$ on $L^2(\mathbb{R}, \Omega^2 dm)$. Since $e^{-tH^\wedge}$ has positive kernel and $H^\wedge 1 = 0$, it generates a Markov process on $\mathbb{R}$ and we denote its distribution by μ. By the ground state transformation we get the following renormalized Feynman-Kac formula,

$$\langle \Omega, A_1 e^{-(t_2-t_1)\hat{H}} A_2 e^{-(t_3-t_2)\hat{H}} \dots A_n \Omega \rangle = \int A_1(q_{t_1}) \dots A_n(q_{t_n}) d\mu(q), \tag{1}$$

which enables us to express the Wightman function(ground state correlation in Euclidean quantum mechanics) in terms of correlation of a stochastic process. If we construct the measure μ, then we can compute the Wightman function and get Schwinger function(ground state correlation in quantum mechanics) by analytic continuation, and finally retrieve all the information of the quantum dynamics from the Schwinger function.

We now turn our focus to the Euclidean fields. A EQFT is a certain probability measure μ on real distributions $\mathcal{D}' \equiv \mathcal{D}'(\mathbb{R}^d)$, where d is the space-time dimension. Let $\mathcal{D} \equiv C_0^\infty(\mathbb{R}^d)$ be the space of test functions. For $\phi \in \mathcal{D}', f \in \mathcal{D}$, we write $\phi(f) = \langle \phi, f \rangle$ to be the canonical pairing on $\mathbb{R}^d$. The probability measure μ is characterized by the generating functionals $\{S_f, f \in \mathcal{D}\}$, with

$$S_f : \phi \mapsto \int e^{i\phi(f)} d\mu(\phi), \quad \phi \in \mathcal{D}'.$$

Osterwalder-Schrader axioms impose five conditions on μ:

- (OS0) Analyticity: S_f is entire analytic. It ensures the super-exponential decay of $d\mu$.
- (OS1) Regularity: $\log |S_f| \leq c(||f||_{L^1} + ||f||_{L^p}^p)$, for $p \in [1,2]$, and some constant c . If $p = 2$, then the second-order Schwinger function should be locally integrable.
- (OS2) Invariance: S_f is invariant under Euclidean symmetries of $\mathbb{R}^d$, i.e. translation, rotation, and reflection. This is equivalent to the Euclidean invariance of $d\mu$.
- (OS3) Reflection positivity (RP): for every finite sequence $(f_i) \subset \mathcal{D}_{\text{real}}$, the matrix $M_{ij} = S_{f_i - \theta f_j}$ has non-negative eigenvalues, where θ is the time reflection over the point 0.
- (OS4) Ergodicity: the measure space $(D', d\mu)$ is ergodic with respect to the time translation subgroup $T(t)$.

(OS0)-(OS2) are meant for all test function f. For (OS3), there is an equivalent formulation. Let

$$\mathcal{A}_+ = \{A : \phi \mapsto \sum_{j=1}^{N} c_j e^{\phi(f_j)}, \text{for some } c_j \in \mathbb{C}, f_j \in C_0(\mathbb{R}^d_+), N \in \mathbb{N}\},$$

with $\mathbb{R}^d_+$ the half-space of positive time. Let $\mathcal{E} = L^2(\mathcal{D}'(\mathbb{R}^d), d\mu)$, then RP is equivalent to

$$\langle \theta A, A \rangle_{\mathcal{E}} \geq 0, \quad \forall A \in \mathcal{A}_+.$$

The reflection positivity axiom helps us to construct a quantum mechanical Hilbert space $\mathcal{H}$. The construction is based on $\mathcal{A}_+$, and the bilinear form $b(A, B) := \langle \theta A, B \rangle_{\mathcal{E}}$. Specifically, it is constructed in three steps: first, take closure of $\mathcal{A}_+$ in $\mathcal{E}$, and denote it by $\mathcal{E}_+$; second, thanks to the RP, observe that $||\cdot||_b := b(\cdot,\cdot)^{\frac{1}{2}}$ defines a semi-norm on $\mathcal{E}_+$, and thus a norm on $\mathcal{E}_+/\mathcal{N}$, with $\mathcal{N} := \{A \in \mathcal{E}_+, ||A||_b = 0\}$ the null-set; finally, define $\mathcal{H}$ as the closure of the equivalent class $\mathcal{E}_+/\mathcal{N}$ in $(\mathcal{E}, ||\cdot||_b)$, and check the Parallelogram identity, and conclude that $\mathcal{H}$ is a Hilbert space.

For $A \in \mathcal{E}_+$, let $A^\wedge := A + \mathcal{N} \in \mathcal{H}$. To this point, we have $\langle A^\wedge, B^\wedge\rangle_{\mathcal{H}} = \langle \theta A, B\rangle_{\mathcal{E}}$. Next, we wish to transfer an operator S on $\mathcal{E}$ to $S^\wedge$ on $\mathcal{H}$. In order for the following equality to hold, $\langle A^\wedge, S^\wedge B^\wedge\rangle_{\mathcal{H}} = \langle \theta A, SB\rangle_{\mathcal{E}}$, where $S^\wedge B^\wedge := (SB)^\wedge$, S must map $D(S) \cap \mathcal{E}_+$ to $\mathcal{E}_+$, and $D(S) \cap \mathcal{N}$ to $\mathcal{N}$.

Now we are ready to construct the Hamiltonian H via the time translation semi-group $T(t)$.

Theorem (Construction of H). *Suppose (OS3) and (OS2) hold (in particular, the time translation invariance of $d\mu$). Then for $t \geq 0$, $T(t)^\wedge$ is well-defined, and it can be written as $T(t)^\wedge = e^{-tH}$, where H is some positive self-adjoint operator, with ground state $\Omega = 1^\wedge$.*

The idea of the proof is to first show that $R(t) := T(t)^\wedge$ maps $\mathcal{N}$ to $\mathcal{N}$, and satisfies the properties of semi-group, Hermitian, contraction and strong continuity. Hence there exists a positive self-adjoint operator H satisfying the desired relation. H has a ground state $1^\wedge$, i.e. $H1^\wedge = 0$, which follows from $T(t)1 = 1$.

For the lattice models where the space-time is $\mathbb{Z}^d$ instead of $\mathbb{R}^d$, under suitable modified assumptions, we may apply above theorem to construct a self-adjoint matrix K on $\mathcal{H}$, such that $K^n := T(n)^\wedge$, for $n \in \mathbb{N}$. In addition, $1^\wedge$ is an invariant vector for K. Here, K in the lattice model plays the role of e^{-H} in the continuous model.

Reflection positivity for lattice models is a sophisticated topic (see [2], [3, Chap. 10]). To enumerate examples, it is convenient to consider the space-time on a torus $\mathbb{T}_L$, as it has natural reflection symmetry along planes orthogonal to one of the lattice directions. In a broader language, we speak of reflections over one of such hyperplane Π that splits the torus into two halves, and the splitting is either *through sites* or *through edges*. The most simple case is the *product measure*, it is RP with respect to all reflections. *Gibbs measure* of a class of lattice spin systems also possess reflection positivity, such as *Ising*, *Potts* and *Heisenberg* models. We mention a more general result. Given a fixed plane of reflection Π, with the corresponding reflection operator θ, let $A_+(\theta)$ be the algebra of all bounded and measurable functions supported on the positive half of the reflection plane. Now, suppose the torus Hamiltonian takes the form $-H_L = A + \theta A + \sum_\alpha C_\alpha \theta C_\alpha$, with $A, C_\alpha \in \mathcal{A}(\theta)_+$, then the torus Gibbs measure is RP with respect to θ (for any inverse-temperature). Such examples include the torus Hamiltonian for *nearest neighbor (ferromagnetic) interaction*, *Yukawa potentials*, and the *power-law decaying potentials*, which are RP with respect to any plane.

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Gaussian free field and the ϕ^4_2 measure on torus

Aleksandra Korzhenkova

Our goal is to construct a probability measure $\mathbb{P}$ on the space $\mathcal{D}'(\mathbb{T}^2)$ of distributions on the two-dimensional torus heuristically given by

$$\mathbb{P}(\mathrm{d}\phi) \propto \exp\Bigg(- \int_{\mathbb{T}^2} |\phi|^4 \mathrm{d}x - \frac{1}{2} \underbrace{\int_{\mathbb{T}^2} (|\nabla \phi|^2 + m^2 |\phi|^2) \mathrm{d}x}_{= \langle \phi, (-\Delta + m^2)\phi \rangle} \Bigg) \text{“}\mathrm{d}\phi\text{”}$$

for $m^2 > 0$.

Step 1: Absolute continuity w.r.t. massive GFF measure.

The alternative description of the second integral directly suggests we rewrite $\mathbb{P}$ as $\mathbb{P}(\mathrm{d}\phi) \propto e^{-\int |\phi|^4 \mathrm{d}x} \mathbb{Q}(\mathrm{d}\phi)$ for a centered Gaussian measure $\mathbb{Q}$ with the inverse of $-\Delta + m^2$ as covariance, called *massive Gaussian free field (GFF).* One way to define $\mathbb{Q}$ rigorously is by diagonalizing $-\Delta + m^2$. More precisely, let $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ be the eigenvalues of minus Laplacian and $(e_j)_{j \in \mathbb{N}}$ be a family of the corresponding eigenfunctions that forms an orthonormal basis of $L^2(\mathbb{T}^2)$. In this basis, the desired covariance, called *massive Green's function*, is given by

$$G_{m^2}(x, y) = \sum_{j \ge 1} \frac{1}{\lambda_j + m^2} e_j(x) e_j(y) \quad \text{for all } x \ne y \in \mathbb{T}^2,$$

where the right-hand side is a convergent series in $L^2(\mathbb{T}^2 \times \mathbb{T}^2)$. Let further $(\alpha_j)_j$ be a sequence of i.i.d. standard normal random variables. We set

$$\Gamma := \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j + m^2}} \alpha_j e_j. \tag{1}$$

By Weyl's law or alternatively since when re-indexed by $\mathbb{Z}^2 \ni k$, $\lambda_k = |k|^2$ and $e_k(x) = e^{i\langle k, x \rangle}$ are explicit, we can easily check that almost surely $\Gamma \in H^{-\varepsilon}(\mathbb{T}^2) = \{f = \sum_j \langle f, e_j \rangle e_j \mid \sum_{j \ge 1} |\langle f, e_j \rangle|^2 (\lambda_j + m^2)^{-\varepsilon} < \infty\}$ for any $\varepsilon > 0$. That is, almost surely Γ is a random distribution, and its covariance is clearly G_{m^2}, hence, we can set $\mathbb{Q}$ to be the law of Γ.

Remark. One can also define (massive) GFF in higher dimensions as well as on more general domains [1, 4], e.g., on regular subsets of $\mathbb{R}^d$ (for $d = 2$, proper subsets) with various boundary conditions. The distinguishing feature of the dimension two, which is of immense importance to our construction of $\mathbb{P}$, is that

G_{m^2} has a logarithmic singularity at the diagonal compared to the polynomial ones in higher dimensions. More precisely, for $d = 2$

$$G_{m^2}(x, y) \sim -c \log(m|x - y|) \quad \text{as } |x - y| \to 0$$

for an explicit constant c (some finite power of 2π).

Step 2: Renormalization of power.

Now that we have $\mathbb{Q}$, for $e^{-\int |\phi|^4 \mathrm{d}x} \mathbb{Q}(\mathrm{d}\phi)$ to be defined we at least need that $\phi \in L^4(\mathbb{T}^2)$ for $\mathbb{Q}$-almost every ϕ. However, $\mathbb{E}_{\mathbb{Q}}[|\phi|^2] = \sum_{k \in \mathbb{Z}^2} \frac{1}{|k|^2 + m^2} = \infty$. To cancel this divergence we perform a renormalization of power: instead of $\langle \phi^4, 1 \rangle$ we consider $\langle : \phi^4 :, 1 \rangle$, where for a centered Gaussian random variable X, $: X^4 :$ is given by

$$: X^4 := X^4 - 6\mathrm{Var}[X]X^2 + 3\mathrm{Var}[X]^2 = (X^2 - 3\mathrm{Var}[X])^2 - 6\mathrm{Var}[X]^2,$$

such that $\mathbb{E}[: X^4 :] = 0$. As $\phi \sim \mathbb{Q}$ is only a random distribution, we still have to make sense of $\langle : \phi^4 :, 1 \rangle$ rigorously. For this, consider the truncated Fourier series (cf. (1)) $\phi_N \overset{\text{law}}{=} \sum_{\substack{k \in \mathbb{Z}^2: \\ |k| \le N}} \frac{1}{\sqrt{\lambda_k + m^2}} \alpha_k e_k$. It is almost surely a smooth centered Gaussian field with variance on the diagonal $G_{m^2, N}(0, 0) \sim c \log(N)$ as $N \to \infty$ for a constant $c > 0$. For simplicity set $\chi_N := \langle : \phi_N^4 :, 1 \rangle$. Using Wick's formula [2, Lemma 2.4 & Proposition 3.1], [5, Theorem I.3] and the aforementioned fact that G_{m^2} has a logarithmic singularity at the diagonal one can show [2, Section 3] that $(\chi_N)_N$ is uniformly bounded and convergent in $L^2(\mathbb{Q})$. Let us denote the limit by $\chi = \langle : \phi^4 :, 1 \rangle$ (it is just notation, $: \phi^4 :$ is not well-defined on its own). An important observation is that each $: \phi_N^4 :$ by definition is an element of the so-called 4th Wiener chaos (on the Gaussian probability space generated by ϕ) [2, Section 2.1], which is a closed subspace of $L^2(\mathbb{Q})$ spanned by "4th Hermite polynomials of the white noise". This in particular implies that also all χ_N and the limit χ are elements of the 4th Wiener chaos, which allows us to use the hypercontractivity result (2) stated below to conclude that the convergence also takes place in $L^p(\mathbb{Q})$ for any $p \ge 2$.

Step 3: $\mathbb{E}_{\mathbb{Q}}[e^{-\langle : \phi^4 :, 1 \rangle}] < \infty$.

Now it only remains to verify that $e^{-\langle : \phi^4 :, 1 \rangle}$ is integrable w.r.t. $\mathbb{Q}$. We follow Nelson's argument'66 (cf. [3, Section 9]); the idea is to split the field $\phi \sim \mathbb{Q}$ into its truncated Fourier series ϕ_N and the remaining part ψ_N and verify that the latter is negligibly small. The key ingredient of this strategy is the aforementioned *hypercontractivity result* that states that for any element X of the nth ($n \in \mathbb{N}$) Wiener chaos,

$$\mathbb{E}[X^{2p}] \le (2p - 1)^{np} \mathbb{E}[X^2]^p \quad \text{for any } p \ge 1. \tag{2}$$

One can prove (2) either purely combinatorially [5, Lemma I.18], [2, Section 4.1] or even in greater generality using tensorisation property and log-Sobolev inequality [3, Section 7].

By the previous step we know that $\langle : \phi^4 :, 1 \rangle := L^p(\mathbb{Q}) - \lim_N \langle : \phi_N^4 :, 1 \rangle$ (for any $p \ge 2$) is an element of the 4th Wiener chaos (as well as $\langle : \phi_N^4 :, 1 \rangle$). Define

$Y_N = \langle : \phi^4 :, 1\rangle - \langle : \phi_N^4 :, 1\rangle$. It is possible to show (see [3, Section 9] for a sharper bound or [2, Section 3]) that

$$\mathbb{E}_{\mathbb{Q}}[|Y_N|^2] \leq C/N \quad \text{for some } C > 0.$$

Then, by (2), for all $p \geq 1$,

$$\mathbb{E}_{\mathbb{Q}}[|Y_N|^{2p}] \leq (2p)^{4p} C^p / N^p.$$

Combining this estimate with the observation that

$$: \phi_N^4 : \geq -6\text{Var}[\phi_N]^2 \geq -c(\log N)^2,$$

we get for all $N, t > 1$ sufficiently large such that $\log t - c(\log N)^2 > 0$,

$$\mathbb{Q}[e^{-\langle : \phi^4 :, 1\rangle} \geq t] \leq \mathbb{Q}[Y_N \leq -\log t + c(\log N)^2] \leq \frac{(2p)^{4p} C^p / N^p}{(\log t - c(\log N)^2)^{2p}}.$$

We can now adjust p and N to get a faster than polynomial decay for all t sufficiently large, which in turn would conclude our construction. For instance, take N such that $\log t - c(\log N)^2 \in [1, 2]$ and $p = \lfloor \frac{1}{2^5 C} e^{\sqrt{(\log t - 2)/c}} \rfloor$.

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Hölder-Besov spaces and space-time white noise

Alberto Bonicelli, Fabrizio Zanello

The study of stochastic PDEs encompasses equations with a random forcing to model the behaviour of systems with a large number of interactions, whose evolution displays unpredictable features. Examples of paramount relevance in physics are the KPZ equation, which suitably describes interface dynamics of two competing media, or the so called stochastic φ_d^4 equation on $\mathbb{R}^d$, which enters into play when performing the stochastic quantization of a Euclidean self-interacting scalar quantum field theory, as well as to describe phase transitions for systems around the critical threshold. For a pedagogical exposition of these and many more examples of stochastic PDEs we refer the interested reader to the review [3].

The first part of the talk is devoted to defining the random source for the class of stochastic PDEs we are interested in, the so called space-time white noise, as a centred Gaussian random tempered distribution. Starting from its covariance, a direct calculation characterizes its behaviour under scaling. This scaling invariance in law prompts the choice of suitable spaces of functions (and distributions) upon which to construct a suitable solution theory. Focusing on parabolic problems, it is natural to introduce Hölder spaces $\mathcal{C}^\alpha_s$, $\alpha \in (0,1)$ defined in terms of a scaled distance, where time counts twice as space. Yet, as mentioned above, space-time white noise is a distribution, hence we need a natural extension of such spaces of function for negative exponents. The natural candidates are the Hölder-Besov spaces, which with a slight abuse of notation we denote as $\mathcal{C}^\alpha_s$ and whose definition heavily relies on scaling. The second part of the talk focuses on key results of harmonic analysis. An important question is whether a pair of functions with low regularity can be multiplied. The answer goes under the name of Young theorem and states that the product of smooth functions can be extended to a continuous bilinear map over $\mathcal{C}^\alpha_s \times \mathcal{C}^\beta_s$ if and only if $\alpha + \beta > 0$. Another fundamental result is a Schauder estimate for parabolic operators characterizing the gain of regularity in Hölder-Besov spaces.

The final step consists of individuating the space of distributions in which the white noise lies. One has to resort to a Kolmogorov-like criterion that relates the behaviours under scaling of the L^p norm of a distribution to its Hölder regularity. To wit, a direct calculation entails that the space-time white noise lies in $\mathcal{C}^\alpha_s$ for all $\alpha < -\frac{d+2}{2}$.

References

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The linear stochastic heat equation and some non-linear perturbations

Christopher Janjigian, Xuan Wu

This talk introduces the linear stochastic heat equation (SHE) with additive white noise forcing through its mild (Duhamel) formulation. A proof of existence of solutions in an appropriate Besov space will be sketched based on a version of Kolmogorov's continuity theorem. During this portion of the presentation, graphical notation will be introduced for certain stochastic integrals and associated non-random integrals which appear in the moment estimates. These estimates will be seen to suggest that solutions to the SHE should be functions only in dimensions strictly less than two.

The second portion of the talk will discuss a class of non-linear perturbations of the SHE, introduce the idea of scaling of SPDEs and how this relates to when we should expect to be able to find non-trivial solutions to this class of non-linear SPDEs. In particular, we will discuss what is meant by sub-criticality, criticality, and super-criticality and will state a "meta-theorem" about existence of solutions to sub-critical equations. With this concept in hand, we will see how the mild formulation of a non-linear equation leads to a fixed-point problem that necessitates some renormalization if the solution to the linear SHE is not a function.

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The Da Prato-Debussche argument

Peter Morfe, Florian Schweiger

For some semilinear stochastic PDEs it is possible to construct a solution if one can make sense of the nonlinearity when applied to the solutions of the corresponding linear equation. This method goes back to Da Prato and Debussche [2]. We explain the details using the example of the dynamical Φ^4_d model, where the method can be applied for $d = 2$, but not for $d = 3$. Our presentation follows the review article [1].

In more detail, consider the SPDE formally given by

$$\partial_t\varphi(t,x) = \Delta\varphi(t,x) - \varphi(t,x)^3 + \xi(t,x) \tag{1}$$

on $\mathbb{R}_+ \times \mathbb{T}^d$, where ξ is space-time white noise. Formally, this evolution should have Φ^4_d as its stationary measure, and in fact the idea of parabolic (or Parisi-Wu) stochastic quantization is to construct Φ^4_d as the stationary measure of (1).

The equation (1) is a nonlinear variant of the stochastic heat equation

$$\partial_t\varphi(t,x) = \Delta\varphi(t,x) + \xi(t,x). \tag{2}$$

As soon as $d \geq 2$, solutions of (2) are only distributions, and the same should be true for solutions of (1). However, this means that some renormalization is required to make sense of the term $\varphi(t,x)^3$ in (1). The approach we will take is to renormalize by replacing $\varphi(t,x)^3$ by the Wick power $:\varphi(t,x)^3:$.

To formalize this, consider a regularization ξ_δ of ξ (given by convolution with a suitable mollifier), and consider the SPDE

$$\begin{aligned}\partial_t\varphi_\delta(t,x) &= \Delta\varphi_\delta(t,x) - \,:\varphi_\delta(t,x)^3:\, + \xi_\delta(t,x)\\ &= \Delta\varphi_\delta(t,x) - \varphi_\delta(t,x)^3 + 3C_\delta(t)\varphi_\delta(t,x) + \xi_\delta(t,x),\end{aligned} \tag{3}$$

where C_δ is a suitable renormalization constant. It is clear that for each fixed δ there is a well-defined solution. The key result of the talk is that in dimension 2 and for short times one can pass to the limit $\delta \to 0$.

Theorem 1 ([2])**.** *Let $d = 2$. For any $\kappa > 0$ there is an almost surely positive random variable T such that the solutions of (3) on $[0,T]\times\mathbb{T}^2$ converge, as $\delta \to 0$, in the parabolic Hölder space $\mathcal{C}^{-\kappa}_{\mathfrak{s}}$ to a nontrivial limit φ.*

The key idea of the proof due to Da Prato and Debussche is that the most troublesome part of (3) comes from the solution of the stochastic heat equation. Its powers can be defined via Wick's theorem, and one can hope that the difference of the two solutions has better properties, and can be constructed by a standard fix-point argument.

Sketch of proof. Consider the solution Ψ_δ of the regularized stochastic heat equation

$$\partial_t \Psi_\delta(t,x) = \Delta \Psi_\delta(t,x) + \xi_\delta(t,x)$$

and its Wick powers $:\Psi_\delta^2: = \Psi_\delta^2 - C_\delta$ and $:\Psi_\delta^3: = \Psi_\delta^3 - 3C_\delta \Psi_\delta$. We make the ansatz $\varphi_\delta = \Psi_\delta + v_\delta$. Then v_δ should solve

$$\partial_t v_\delta(t,x) = \Delta v_\delta(t,x) - v_\delta^3 - 3\Psi_\delta v_\delta^2 - 3:\Psi_\delta^2: v_\delta - :\Psi_\delta^3:. \tag{4}$$

It turns out that the inhomogeneities on the right-hand side of (4) all take values in $\mathcal{C}_\mathfrak{s}^{-\kappa'}$ for any $\kappa' > 0$. This allows to construct a solution of (4) via a fix-point argument in $\mathcal{C}_\mathfrak{s}^{2-\kappa}$. The reason this is possible is that we can combine Young's theorem on products of distributions with the Schauder estimate for the heat equation. □

In the case that $d = 3$, one might be tempted to use the same method to construct solutions of (1). The Wick powers up to order 4 still exist, however their Hölder regularity is not good enough to close the fix-point argument to solve (4). One might try to address this with a more elaborate ansatz that includes more terms than just Ψ_δ, however it turns out that one cannot eliminate all the problematic terms. In fact, more elaborate methods like the theory of regularity structures are necessary to solve (1) in $d = 3$.

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Global solutions and coming down from infinity, I.

Simon Gabriel, Ruoyuan Liu

This session, based on the article [1] by Mourrat–Weber, concerns the well–posedness of the dynamic Φ^4_2 model

$$\partial_t X = \Delta X - X^3 + \xi\,, \qquad X(0,\cdot) = X_0\,, \tag{1}$$

globally in time on $\mathbb{R}^2$. Here ξ denotes a space–time white noise and X_0 lies in a suitable space of distributions.

In previous sessions, we saw that this SPDE is locally well–posed in time on compact tori, using the DaPrato–Debussche trick $X = Z + Y$, i.e. by expanding around the solution of the stochastic heat equation Z. This allowed to reduce the study of (1) to

$$\begin{aligned} \partial_t Y &= \Delta Y - (Y^3 + 3ZY^2 + 3Z^{(2)}Y + Z^{(3)}) \\ &=: \Delta Y - Y^3 + \Psi'(Y, Z, Z^{(2)}, Z^{(3)})\,, \end{aligned} \tag{2}$$

with vanishing initial datum, where $Z^{(k)}$ are the (renormalised) Wick powers of Z. Likewise, we first present an argument that yields global in time well–posedness on a torus $\mathbb{T}^2_M := [-\frac{M}{2}, \frac{M}{2}]^2$, of arbitrary size M.

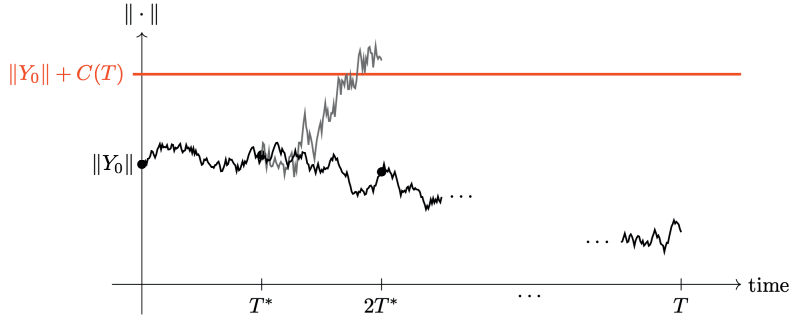

Figure 1. Given a time T and an initial condition Y_0, we have the a priori estimate with the constant $C(T)$ (the red line). Moreover, we find a solution up to time $T^*(\|Y_0\| + C(T))$, using the local well–posedness, which will lie below the red threshold. Hence, once more we find a solution on the interval $[T^*, 2T^*]$. Gluing together the two trajectories yields a solution on $[0, 2T^*]$, which by the a priori estimate must again lie below the red line. Thus, a grey trajectory as depicted above is not possible. Iterating this procedure until exhausting the interval $[0, T]$ yields unique solutions on arbitrary large intervals.

In order to convey the general idea of the argument, the following two ingredients are necessary: Considering a suitable norm $\|\cdot\|$ on the state space of solutions (which we shall fix below), we require **local in time well–posedness** of the type

$$\forall K > 0\ \exists T^* > 0\ \forall Y_0 \text{ with } \|Y_0\| \leq K \Rightarrow \exists! \text{ solution } (Y_t)_{t\in[0,T^*]}\,.$$

and an **a priori estimate:**

$$\forall T > 0\ \exists C > 0\ \forall T^* \leq T \Rightarrow \forall \text{ solutions } (Y_t)_{t\in[0,T^*]} :\ \sup_{t\in[0,T^*]} \|Y_t\| \leq \|Y_0\| + C\,.$$

Note that the local in time well–posedness result is slightly stronger than the one presented previously, because the random time T^* only depends on the upper bound K of $\|Y_0\|$. Now, having these two results at hand, the global in time well–posedness can be summarised pictorially, see Figure 1.

The underlying function spaces used in the well-posedness argument are Besov spaces defined by the norm

$$\|f\|_{B^{\alpha}_{p,q}} = \left\| \left(2^{\alpha k} \|\delta_k f\|_{L^p} \right)_{k \geq -1} \right\|_{\ell^q},$$

where δ_{-1} is a smooth frequency cutoff onto $\{|\zeta| \leq 1\}$ and δ_k for $k \geq 0$ is a smooth frequency cutoff onto $\{|\zeta| \sim 2^k\}$. The Besov spaces enjoy the following useful properties:

(1) **Embeddings:** For $1 \leq p, q_1, q_2 \leq \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $q_1 \geq q_2$ and $\alpha_1 \leq \alpha_2$, we have

$$\|f\|_{B^{\alpha_1}_{p,q_1}} \leq \|f\|_{B^{\alpha_2}_{p,q_2}}.$$

For $1 \leq p, q, r \leq \infty$ and $\alpha, \beta \in \mathbb{R}$ with $p \geq r$ and $\beta = \alpha + d(\frac{1}{r} - \frac{1}{p})$, we have

$$\|f\|_{B^{\alpha}_{p,q}} \leq C \|f\|_{B^{\beta}_{r,q}}.$$

(2) **Interpolation:** For $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$, $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$, and $\theta \in [0,1]$ satisfying $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$, and $\alpha = \theta\alpha_1 + (1-\theta)\alpha_2$, we have

$$\|f\|_{B^{\alpha}_{p,q}} \leq C \|f\|^{\theta}_{B^{\alpha_1}_{p_1,q_1}} \|f\|^{1-\theta}_{B^{\alpha_2}_{p_2,q_2}}.$$

(3) **Multiplicative inequalities:** For $1 \leq p, p_1, p_2, q \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\alpha > 0$, we have

$$\|fg\|_{B^{\alpha}_{p,q}} \leq C \|f\|_{B^{\alpha}_{p_1,q}} \|g\|_{B^{\alpha}_{p_2,q}}.$$

For $1 \leq p, p_1, p_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 \leq q \leq \infty$, and $\beta < 0 < \alpha$ with $\alpha + \beta > 0$, we have

$$\|fg\|_{B^{\beta}_{p,q}} \leq C \|f\|_{B^{\beta}_{p_1,q}} \|g\|_{B^{\alpha}_{p_2,q}}.$$

(4) **Duality:** For $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$ and $\alpha \in \mathbb{R}$, we have

$$|(f,g)| \leq C \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{-\alpha}_{p_2,q_2}}.$$

(5) **Smoothing of the heat flow:** For $1 \leq p, q \leq \infty$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta$, $t > 0$, and f supported on an annulus, we have

$$\|e^{t\Delta} f\|_{B^{\alpha}_{p,q}} \leq C t^{\frac{\beta-\alpha}{2}} \|f\|_{B^{\beta}_{p,q}}.$$

We shall now focus on the a priori estimate only. Using a classical Schauder estimate, one can guess that mild solutions $(Y_t)_{t\in[0,T^*]}$ of (2) take values in $B^{2-}_{p,q}$. Indeed, Y_t will be function–valued and it suffices to consider $\|\cdot\| := \|\cdot\|_{L^p}$, the L^p norm on the torus with periodic boundary condition. The a priori estimate then requires control of the difference

$$\frac{1}{p}\left(\|Y_t\|^p_{L^p} - \|Y_0\|^p_{L^p}\right) = \int_0^t (\Psi'_s, Y_s^{p-1}) - \big((p-1)(|\nabla Y_s|^2, Y_s^{p-2}) + \|Y_s^{p+2}\|_{L^1}\big)\,\mathrm{d}s\,,$$

where p is an even, large natural number. Here, we conveniently expressed the difference in terms of an L^p–energy identity, derived by testing a mild solution Y_t against Y_t^{p-1}, cf. [1, Proposition 6.8].

The a priori estimate is an immediate consequence, once we show integrability of the right–hand side of the above energy identity. To this end, we analyse each summand in Ψ'_s separately. As an example, we use duality, interpolation and Young's inequalities to obtain

$$
\begin{aligned}
|(Z_s Y_s^2, Y_s^{p-1})| &\le C \|Y_s^{p+1}\|_{B^{\varepsilon}_{1,1}} \|Z_s\|_{B^{-\varepsilon}_{\infty,\infty}} \\
&\le C \left(\|Y_s^{p+1}\|_{L^1}^{1-\varepsilon} \|Y_s^p \nabla Y_s\|_{L^1}^{\varepsilon} + \|Y_s^{p+1}\|_{L^1} \right) \|Z_s\|_{B^{-\varepsilon}_{\infty,\infty}} \\
&\le c \left((p-1)(|\nabla Y_s|^2, Y_s^{p-2}) + \|Y_s^{p+2}\|_{L^1} \right) + f(s) ,
\end{aligned}
\tag{3}
$$

for some integrable function f and $c < 1$ small enough. Equivalently, we find bounds for the other terms in (Ψ'_s, Y_s^{p-2}) such that the sum of all such c's lies in $(0,1)$. Lastly, exploiting the two negative terms in the integrand of the energy identity together with estimates of the form (3), we conclude integrability of the bound for the a priori estimate. Here we shall stress the importance of the term $-\|Y_s^{p+2}\|_{L^1}$ in the energy identity, which is due to the negative sign of the non–linearity $-X^3$ in (1).

On the other hand, global well-posedness of (1) on the whole plane $\mathbb{R}^2$ can be proved in three main steps. Firstly, one shows global well-posedness of (1) on the large torus $\mathbb{T}^2_M$, where we d enote the global solution by Y_M. Secondly, one establishes a priori estimates for Y_M in (weighted) Besov spaces, uniformly in M. By using compact embeddings of weighted Besov spaces, one can extract a converging subsequence of $\{Y_M\}_{M\in\mathbb{N}}$ as $M \to \infty$. Lastly, one proves uniqueness of the solution Y to the equation on the whole plane, which then shows the convergence of the whole sequence $\{Y_M\}_{M\in\mathbb{N}}$.

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Global solutions and coming down from infinity, II.

Juraj Foldes, Jaeyun Yi

We discuss the "coming down from infinity" property for the Φ^4_2 model on $\mathbb{R}^2$ based on [1]. In other words, we establish a priori estmates for the global-in-time solution of Φ^4_2 in suitable weighted Hölder spaces, uniformly over the initial data. This problem is strongly related to the construction of Φ^4 measure since global-in-time bounds for the solution can be applied to the construction.

In order to prove the global bounds, we introduce localization operators to decompose the solution into singular and regular parts. In particular, the regular part can be controlled by the help of a minus cubic term of Φ^4 model. We then show the uniform bounds on solutions to regularised equation driven by a regularised white noise, with renomalization constants which diverges as regularising parameter $\epsilon \to 0$. Using compactness argument, we shall prove the existence of a solution and its uniform bounds.

In the end, we use a further time weight to get the coming down from infinity property. One of key ideas is that the time weight is zero at the initial time so that it removes in some sense the dependency of the time from estimates. However, we should modify the tools such as the Schauder estimate to be fit with the time weights.

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Tightness via the energy method for the φ^4_2 measure

Azam Jahandideh

Formally the measure of the dynamical φ^4_2 model on $\mathcal{S}'(\mathbb{R}^2)$ is given by

$$\nu(d\varphi) = \frac{1}{\mathcal{Z}} \exp\Big[- \int_{\mathbb{R}^2} (\frac{1}{4}\lambda\varphi^4 - \frac{3}{2}\lambda\infty\varphi^2)\Big]\mu(d\varphi)\,, \tag{1}$$

where $\lambda > 0$ is the coupling constant, $\mathcal{Z} \geq 1$ is normalization factor, $\mu(d\varphi)$ is the Gaussian measure with covariance $(m^2 - \Delta)^{-1}$ and Δ is the Laplacian on $\mathbb{R}^2$. The corresponding semilinear parabolic partial differential equation to the measure $\nu(d\varphi)$ according to the Langevin dynamics is given by

$$\begin{cases} (\partial_t - \Delta + m^2)\,\Phi(t,x) = -\lambda\Phi^3(t,x) + 3\,\lambda\infty\,\Phi + \xi(t,x) \\ \Phi(0,x) = \varphi(x)\,, \end{cases} \tag{2}$$

where $\xi(t,x)$ is the unique space-time Gaussian white noise on $\mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^2)$. The above SPDE has the φ^4_2 measure, i.e., $\nu(d\varphi)$ as its invariant measure. This implies that if Φ is a solution to Eq. (2) with the initial condition $\Phi(0,\cdot) = \varphi(\cdot)$ distributed according to the measure $\nu(d\varphi)$, then for all $t \in \mathbb{R}$ the random field $\Phi(t,\cdot)$ is also distributed according to this measure. Consequently, one has $\mathrm{Law}(\Phi(t,\cdot)) = \mathrm{Law}(\Phi(0,\cdot)) = \nu(d\varphi)$.

By the parabolic scaling, the sample paths of ξ belong almost surely to the space of regularity $-2-\kappa$ for all $\kappa > 0$. The heat kernel is 2 regularizing, which implies that the solution to SPDE (2) has regularity $-\kappa$. Hence, we expect the regularity of the renormalized non-linear term to be $-\kappa$ for all $\kappa > 0$.

Observe that the measure of the dynamical φ^4_2 model as given in (1) is ill-defined. Firstly, a typical field φ in the support of the Gaussian measure $\mu(d\varphi)$ lacks integrability, i.e., it does not decay at infinity. Secondly, such a field does not have enough regularity as oftentimes it is a distribution. Consequently, the non-linear term, in Eq. (2) is ill-defined. These two problems are known as IR and UV problems, respectively. To get around these problems, we first introduce the discrete family of measures $(\nu_{M,\epsilon})_{M,\epsilon}$ corresponding to the φ^4_2 model on lattice $\Lambda_{M,\epsilon} = (\epsilon\mathbb{Z})^2 \cap [-M/2, M/2)^2$, where ϵ and M play the role of UV and IR cut-offs respectively.

The lattice approximation of the φ_2^4 measure

Let $\Lambda_{M,\epsilon}$ be a periodic lattice with mesh size ϵ and size length M. Consider the scalar filed $\phi : \Lambda_{M,\epsilon} \to \mathbb{R}$. The corresponding regularized family of measures to the φ_2^4 model on $\mathbb{R}^{\Lambda_{M,\epsilon}}$ is given by

$$\nu_{M,\epsilon}(d\varphi) := \frac{1}{\mathcal{Z}_{M,\epsilon}} \exp\Big[-2\epsilon^2 \sum_{x\in\Lambda_{M,\epsilon}} \mathcal{U}_{M,\epsilon}(\varphi)\Big]\mu_{M,\epsilon}(d\varphi)\,, \tag{3}$$

where $\mathcal{Z}_{M,\epsilon}$ is normalization factor, $\mathcal{U}_{M,\epsilon}(\varphi) = \frac{\lambda}{4}|\varphi|^4 - \frac{3}{2}\,\lambda\, a_{M,\epsilon}\,|\varphi|^2 + \frac{3}{4}a_{M,\epsilon}^2$, $\mu_{M,\epsilon}(d\varphi)$ is the discrete Gaussian measure with mean zero and covariance $K^{M,\epsilon} := (m^2 - \Delta_\epsilon)^{-1}$, Δ_ϵ is the discrete Laplacian, and $a_{M,\epsilon} = Tr(K^{M,\epsilon}) = (m^2 - \Delta_\epsilon)^{-1}(x,x)$, which diverges like $\log(\epsilon^{-1})$. Observe that $a_{M,\epsilon}$ is independent of $t \in \mathbb{R}$ as we deal with the stationary solutions.

Remark 1. *For fixed ϵ, M the measure $\nu_{M,\epsilon}(d\varphi)$ is well-defined.*

Definition 2. *A φ_2^4 measure is any non-Gaussian, Euclidean invariant and reflection positive accumulation point of the family of regularized measures $\nu_{M,\epsilon}(d\varphi)$ as $\epsilon \to 0$ and $M \to \infty$, where $\mathcal{U}_{M,\epsilon}(\varphi)$ is any 4-th order polynomial, which is bounded from below with ϵ, M dependent coefficient* [Gub21].

Discrete stochastic quantization equation

Utilizing the parabolic stochastic quantization, we obtain the discrete stochastic quantization equation corresponding to the measure $\nu_{M,\epsilon}(d\varphi)$ on $\mathcal{S}'(\mathbb{R} \times \Lambda_{M,\epsilon})$ as follows

$$(\partial_t + m^2 - \Delta_\epsilon)\Phi_{M,\epsilon}(t,x) = -\lambda\Phi_{M,\epsilon}^3(t,x) + 3\,\lambda\, a_{M,\epsilon}\,\Phi_{M,\epsilon} + \xi_{M,\epsilon}(t,x) \tag{4}$$

such that $\mathrm{Law}(\Phi_{M,\epsilon})(t,\cdot) = \mathrm{Law}(\Phi_{M,\epsilon})(0,\cdot) = \nu_{M,\epsilon}(d\varphi)$ for all $t \in [0,\infty)$ and $\xi_{M,\epsilon}$ is the discrete space-time Gaussian white noise defined on $\mathbb{R} \times \Lambda_{M,\epsilon}$, which is of regularity $-2-\kappa$ for all $\kappa > 0$.

Our aim is to show the existence of the infinite volume measure associated to the φ_2^4 model using tightness of the family of the regularized Gibbs measures $\nu_{M,\epsilon}(d\varphi)$ defined on $\mathcal{S}'(\Lambda_{M,\epsilon})$. To this end, we shall utilize the energy method in $L^2(\Lambda_{M,\epsilon})$. Note that we cannot apply the energy method directly to the Eq. (4), since as $\epsilon \to 0$ it becomes singular. That is why we need to come up with an appropriate decomposition of the random field $\Phi_{M,\epsilon}$.

Decompose the solution into singular and regular parts

Using the Da Prato and Debussche decomposition [DD03], one writes $\Phi_{M,\epsilon} = X_{M,\epsilon} + \eta_{M,\epsilon}$ with $X_{M,\epsilon}$ solving

$$(\partial_t + m^2 - \Delta_\epsilon)X_{M,\epsilon}(t,x) = \xi_{M,\epsilon}(t,x)\,. \tag{5}$$

Set $X_{M,\epsilon}^{:2:} := X_{M,\epsilon}^2 - a_{M,\epsilon}$ and $X_{M,\epsilon}^{:3:} := X_{M,\epsilon}^3 - 3\,a_{M,\epsilon}\,X_{M,\epsilon}$, where $a_{M,\epsilon}$ is chosen in a such way that for all $\kappa > 0$ the stochastic objects $X_{M,\epsilon}$, $X_{M,\epsilon}^{:2}$ and $X_{M,\epsilon}^{:3}$ can be almost surely bounded in some Besov space of regularity $-\kappa$ for all $\kappa > 0$. Let ρ denote a polynomial weight of the form $\rho(x) = \langle hx\rangle^{-\nu} = (1 + |hx|^2)^{-\nu/2}$, where

$\nu \geq 0$ and $h > 0$, $B^{\alpha,\epsilon}_{p,q}(\rho)$ denote the discrete weighted Besov spaces on $\Lambda_{M,\epsilon}$ and $\mathbb{X} = \{X_{M,\epsilon}, X^{:2:}_{M,\epsilon}, X^{:3:}_{M,\epsilon}\}$.

Proposition 3. *Let $p \in [2,\infty)$, $\kappa > 0$ and $a_{M,\epsilon} = \mathbb{E}\Big[X_{M,\epsilon}(t,x)^2\Big]$. There exists $C > 0$ such that for all ϵ, M and $t \in \mathbb{R}$ it holds $\mathbb{E}\Big[\|\rho\,\mathbb{X}(t,\cdot)\|^p_{B^{-\kappa,\epsilon}_{p,p}}\Big] \leq C$.*

Proof. The proof follows from the Kolmogorov type estimate and hypercontractivity estimate with the use of the discrete semigroup property. Similar bounds can be found in [GH18, MWX16]. □

For later use let

$$\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t) = \|\rho X_{M,\epsilon}(t,\cdot)\|^8_{B^{-\kappa,\epsilon}_{8,8}} + \|\rho\, X^{:2:}_{M,\epsilon}(t,\cdot)\|^4_{B^{-\kappa,\epsilon}_{4,4}} + \|\rho\, X^{:3:}_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}}\,.$$

Observe that by Prop. 3 one has $\mathbb{E}[\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)] \leq C$, where $C \in (0,\infty)$ is some constant independent of ϵ and M.

Application of the energy method

In this section we aim to show that for all $\kappa > 0$ there exists $C > 0$ such that for all ϵ, M and $t \in \mathbb{R}$ it holds

$$\mathbb{E}\Big[\|\eta_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho)}\Big] \leq C\,. \tag{6}$$

The remainder $\eta_{M,\epsilon}(t,x)$ solves

$$(\partial_t + m^2 - \Delta_\epsilon)\eta_{M,\epsilon}(t,x) = -\lambda\Big[\eta^3_{M,\epsilon}(t,x) + 3\,\eta_{M,\epsilon}(t,x)\,X^{:2:}_{M,\epsilon}(t,x) + 3\,\eta^2_{M,\epsilon}(t,x)\,X_{M,\epsilon}(t,x) + X^{:3:}_{M,\epsilon}(t,x)\Big]\,. \tag{7}$$

Observe that in the limit as $\epsilon \to 0$ all the product terms in Eq. (7) are well-defined as the sums of their regularities are positive. To obtain the uniform bound (6), we shall apply the energy method to Eq. (7) in $L^2(\Lambda_{M,\epsilon})$. To this end, we multiply both sides of Eq. (7) by $\rho(x)^4\,\eta_{M,\epsilon}(t,x)$ and perform the sum over $\Lambda_{M,\epsilon}$.

Proposition 4. *There exist $\kappa \in (0,\infty)$, $\delta \in (0,1)$ sufficiently small, $C \in (0,\infty)$, $p \in [2,\infty)$, an appropriate polynomial weight ρ such that for all ϵ and M it holds*

$$\frac{1}{2}\partial_t\|\rho^2\,\eta_{M,\epsilon}(t,.)\|^2_{L^{2,\epsilon}} + \lambda(1-\delta)\|\rho\,\eta_{M,\epsilon}(t,.)\|^4_{L^{4,\epsilon}} + (m^2 - C_\delta\,C^2_\rho)\|\rho^2\,\eta_{M,\epsilon}(t,\cdot)\|^2_{L^{2,\epsilon}} + (1-\delta-\lambda\,\delta)\|\rho^2\,\nabla_\epsilon\eta_{M,\epsilon}(t,\cdot)\|^2_{L^{2,\epsilon}} \leq \lambda\,C\,\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)\,. \tag{8}$$

Proof. The proof is an application of the energy method in $L^2(\Lambda_{M,\epsilon})$ as outlined above, integration by part, the discrete Leibniz rule, Hölder's and Young's inequalities. To conclude one uses Lemma 5 with $n = 3$ for $X = X_{M,\epsilon}$, $n = 2$ for $X = X^{:2:}_{M,\epsilon}$, $n = 1$ for $X = X^{:3:}_{M,\epsilon}$. This finishes the proof. □

Lemma 5. *Let $n \in \{1,2,3\}$, $\delta \in (0,1)$ and $\kappa > 0$. There exists $C_\delta \in (0,\infty)$ such that for $p \in [2,\infty)$ it holds*

$$\left|\langle \rho^4\, \eta^n_{M,\epsilon}, X\rangle_{L^{2,\epsilon}}\right| \le \delta \|\rho^2 \nabla_\epsilon\, \eta_{M,\epsilon}\|^2_{L^{2,\epsilon}} + \delta \|\rho\, \eta_{M,\epsilon}\|^4_{L^{4,\epsilon}} + C_\delta \|\rho\, X\|^p_{B^{-\kappa,\epsilon}_{p,p}}\,.$$

Proposition 6. *Let $\kappa \in (0,\infty)$. There exists a constant $C \in (0,\infty)$ such that for all ϵ, M, $\lambda > 0$ and all $t \in \mathbb{R}$ it holds*

$$\mathbb{E}\Big[\|\eta_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)}\Big] \le C\,.$$

Proof. By Prop. 4 combined with the fact that $\Phi_{M,\epsilon}$, $X_{M,\epsilon}$ and $\eta_{M,\epsilon}$ are jointly stationary one has

$$\mathbb{E}\Big[\|\eta_{M,\epsilon}(t,.)\|^2_{L^{2,\epsilon}(\rho^2)}\Big] \le C\,\mathbb{E}\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)\,, \tag{9}$$

where we used the fact that for suitably chosen ρ and δ, all the coefficients in the LHS of Eq. (8) are positive. To conclude, recall that $\mathbb{E}[\mathcal{Q}^{M,\epsilon}(\mathbb{X})(t)] \le C$, and $\|\cdot\|_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)} \lesssim \|\cdot\|_{B^{0,\epsilon}_{2,\infty}(\rho^2)} \lesssim \|\cdot\|_{L^{2,\epsilon}(\rho^2)}$. This finishes the proof. □

Tightness

In this section we aim to verify that the family of measures $(\nu_{M,\epsilon})_{M,\epsilon}$ is tight, i.e., it is sequentially compact in the topology of weak convergence of measures. Specifically, we want to prove the following.

Theorem 7 (Main Result)**.** *Let $k \in (0,\infty)$. There exists a choice of the sequence $(a_{M,\epsilon})_{M,\epsilon}$ such that the family of measures $(\nu_{M,\epsilon})_{M,\epsilon}$ appropriately extended to $\mathcal{S}'(\mathbb{R}^2)$ is tight. In particular, for every accumulation point ν it holds*

$$\int \|\varphi\|^2_{B^{-3\kappa}_{2,2}(\rho^{2+\kappa})}\,\nu(d\phi) < \infty. \tag{10}$$

Proof. Using Prop. 3 with $\mathbb{X} = X_{M,\epsilon}$ for $p = 2$, Prop. 6 and the triangle inequality one has

$$\mathbb{E}\Big[\|(\Phi_{M,\epsilon})(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)}\Big] \le C \tag{11}$$

for some constant $C \in (0,\infty)$ uniformly in ϵ and M and for any $\kappa > 0$. To go from $\mathcal{S}'(\Lambda_{M,\epsilon})$ to $\mathcal{S}'(\mathbb{R}^2)$, one utilizes the extension operator $\mathcal{E}^\epsilon$, which is bounded uniformly in ϵ [GH21, Lemma A.15]. Hence,

$$\mathbb{E}\Big[\|\mathcal{E}^\epsilon\Phi_{M,\epsilon}(t,\cdot)\|^2_{B^{-\kappa}_{2,2}(\rho^2)}\Big] \le \mathbb{E}\Big[\|(\Phi_{M,\epsilon})(t,\cdot)\|^2_{B^{-\kappa,\epsilon}_{2,2}(\rho^2)}\Big] \le C\,. \tag{12}$$

Note that up to a subsequence one can pass to the limits as $\epsilon \to 0$ and $M \to \infty$. Evoking the fact that $\mathrm{Law}(\Phi_{M,\epsilon})(t,\cdot) = \nu_{M,\epsilon}$ for all $t \in [0,\infty)$ yields

$$\int \|\varphi\|^2_{B^{-\kappa}_{2,2}(\rho^2)}\,\nu(d\phi) = \lim_{\substack{\epsilon\to 0\\ M\to\infty}} \mathbb{E}\Big[\|\mathcal{E}^\epsilon\Phi_{M,\epsilon}(t,.)\|^2_{B^{-\kappa}_{2,2}(\rho^2)}\Big] \le C \tag{13}$$

uniformly in ϵ and M. Note that $B^{-\kappa}_{2,2}(\rho^2) \hookrightarrow B^{-2\kappa}_{2+2\kappa,2}(\rho^{2+\kappa})$ continuously and $B^{-2\kappa}_{2+2\kappa,2}(\rho^{2+\kappa}) \hookrightarrow B^{-3\kappa}_{2,2}(\rho^{2+\kappa})$ compactly. It holds

$$\int \|\varphi\|^2_{B^{-3\kappa}_{2,2}(\rho^{2+\kappa})}\, \nu(d\phi) < \infty\,. \tag{14}$$

Use Prokhorov's theorem to infer the tightness. This concludes the proof. □

Let $\mathcal{E}^\epsilon : B^{\alpha,\epsilon}_{p,q}(\rho) \to B^{\alpha}_{p,q}(\rho)$. By $\mathcal{E}^\epsilon \sharp \nu_{M,\epsilon}$ we indicate the measure on $\mathcal{S}'(\mathbb{R}^2)$ obtained from the measure $\nu_{M,\epsilon}$ on $\mathcal{S}'(\Lambda_{M,\epsilon})$.

Reflection Positivity

Let F be some cylindrical function on $\mathcal{S}'(\mathbb{R}^2)$ depending on φ's which are supported in $\{(x_1,x_2) \in \mathbb{R}^2\ ;\ x_1 > 0\}$. We denote the algebra of all such functionals by $\mathcal{F}_+$. Let $\varphi \in \mathcal{S}'(\mathbb{R}^2)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^2)$. We set $\langle \Theta\varphi, f\rangle := \langle \varphi, \Theta f\rangle$ and $(\Theta f)(x_1,x_2) = f(-x_1,x_2)$ for all $f \in \mathcal{C}^\infty(\mathbb{R}^2)$.

Proposition 8. *Let ν be a weak limit of a subsequence of the sequence of measures $(\mathcal{E}^\epsilon \sharp \nu_{M,\epsilon})_{M,\epsilon}$ on $\mathcal{S}'(\mathbb{R}^2)$. For all $F \in \mathcal{F}_+$ it holds $\int \overline{F(\Theta\varphi)} F(\varphi)\, \nu(d\varphi) \geq 0$.*

The preceding proposition implies the reflection positivity axiom in [OS75]. To prove it, one can start off by verifying an analogous property on $\Lambda_{M,\epsilon}$. Then, use the extension operator $\mathcal{E}^\epsilon$ and take the limits $M \to \infty$, $\epsilon \to 0$. It is believed that one needs to start from finite volume lattice measures, as the only concrete way, to prove the reflection positivity axiom for the infinite volume measure [GH21].

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Integrability of Φ^4_2

Lucas Broux, Wenhao Zhao

This talk is a survey of the article [5] by Martin Hairer and Rhys Steele. In the talk we show that the Φ^4_2 measure[1] on the 2-dimensional torus $\mathbb{T}^2_M := (\mathbb{R}/M\mathbb{Z})^2$ of length M admits quartic exponential tails, as expected from its formal expression

$$\mu \sim \exp\Big(- \int_{\mathbb{T}^2} \Big(\frac{1}{2}|\nabla\Phi|^2 + \frac{1}{4}\Phi^4\Big)\,\mathrm{d}x\Big) \prod_{x\in\mathbb{T}^2} \mathrm{d}\Phi(x), \qquad \text{over } \Phi \in \mathcal{S}'(\mathbb{T}^2_M).$$

Even though this expression is purely formal, it is known since the 1970's that μ can actually be rigorously constructed via a suitable procedure of regularization and renormalization. Now, the main theorem of [5] reads as follows.

Theorem 1 ([5, Theorem 1.1])**.** *For any $\psi \in C^\infty_c(\mathbb{R}^2)$, $M > 0$ large enough to accomodate the support of ψ, and $\beta > 0$ small enough depending only on ψ,*

$$\mathbb{E}_{\Phi\sim\mu}\Big[\exp\big(\beta\langle\Phi,\psi\rangle^4\big)\Big] < \infty.$$

Let us quickly comment on this result:

(1) Such a bound was already known in the QFT literature, although with different methods [2]. The novelty of [5] is to establish this result in the three-dimensional case of the Φ^4_3 measure.
(2) This implies that the Φ^4_2 measure is not gaussian, since no gaussian measure satisfies such a quartic exponential integrability estimate.
(3) The same method would in principle apply to other quartic functionals of Φ, for instance also $\mathbb{E}_{\Phi\sim\mu}\big[\exp\big(\beta|\Phi|^4_{-\kappa}\big)\big] < \infty$ would hold for any $\kappa > 0$, where $|\Phi|_{-\kappa}$ denotes the (homogeneous) Sobolev norm.
(4) In the context of QFT, such estimates are used for estabishing the regularity axiom of Osterwalder–Schrader.
(5) The bound is independent of the size of the torus, which may give some result for the Φ^4 measure on the full space.

In the remainder of this extended abstract, we wish to sketch some ideas of the proof.

A first attempt by stochastic quantization of Φ^4_2. A first idea is to argue by *stochastic quantization*, namely to realize μ as an invariant measure to the SPDE

$$(\Phi^4_2) \qquad \partial_t\Phi = \Delta\Phi - \Phi^3 + \infty\,\Phi + \xi, \qquad (t \in \mathbb{R}_+, x \in \mathbb{T}^2),$$

where ξ denotes space-time white noise. Rigorously, one takes mollifiers $(\rho_\epsilon)_{\epsilon>0} \subset C^\infty_c(\mathbb{R}^2)$ and considers rather the equation with smooth noise

$$\partial_t\Phi_\epsilon = \Delta\Phi_\epsilon - \Phi^3_\epsilon + c_\epsilon\,\Phi_\epsilon + \xi * \rho_\epsilon.$$

As is by now well-known, by suitably choosing the diverging sequence $(c_\epsilon)_{\epsilon>0}$ of deterministic constants, one can make $(\Phi_\epsilon)_{\epsilon>0}$ converge (in probability in a suitable Hölder space of distributions) as $\epsilon \to 0$ to a random distribution Φ which

[1] in fact the article presents the (more difficult) case of the Φ^4_3 measure

does not depend on the choice of $(\rho_\epsilon)_{\epsilon>0}$, and which admits μ as a unique invariant measure. See e.g. [1, 7] for the two-dimensional case of Φ^4_2. One crucial point is that Φ_ϵ is controlled by a *finite* number of stochastic objects, given for Φ^4_2 by the random distributions

$$Z_\epsilon \stackrel{\text{def}}{=} (\partial_t - \Delta)^{-1}\xi * \rho_\epsilon, \qquad :Z_\epsilon^2: \stackrel{\text{def}}{=} Z_\epsilon^2 - c_\epsilon, \qquad :Z_\epsilon^3: \stackrel{\text{def}}{=} Z_\epsilon^3 - 3c_\epsilon Z_\epsilon.$$

One may now use PDE techniques to deduce properties of the measure μ. For instance, by combining (amongst other) techniques of Schauder theory and a maximum principle for the damped heat operator $u \mapsto \partial_t u - \Delta u + u^3$, the authors of [6] were able to obtain *a priori estimates* of the form: for all $N > 0$, $R \in (0,1)$, $\kappa > 0$ small,

$$\begin{aligned}&\sup_{\substack{t\in(R^2,1)\\ x\in[-N+R,R-N]^2}} |(\Phi - Z)(t,x)| \\ &\leq C \max\Big(R^{-1}, \limsup_{\epsilon\to 0} \|Z_\epsilon\|_{C^{-\kappa}}^{\frac{2}{1-\kappa}}, \limsup_{\epsilon\to 0} \|:Z_\epsilon^2:\|_{C^{-2\kappa}}^{\frac{2}{2-2\kappa}}, \limsup_{\epsilon\to 0} \|:Z_\epsilon^3:\|_{C^{-3\kappa}}^{\frac{2}{3-3\kappa}}\Big),\end{aligned}$$

where the Hölder norms are over $t \in (0,1), x \in [-N,N]^2$. Let us denote by Y the right-hand side of this display. It is possible to bound the stochastic objects appearing in Y and obtain $\mathbb{E}\big[\exp(\beta Y^{1-\kappa})\big] < \infty$ for any $\kappa > 0$ and $\beta > 0$ small enough. Then, starting Φ at time $t = 0$ according to its invariant measure μ, at later times $\Phi(t,\cdot)$ is still distributed according to μ and it is straigtforward to deduce from the above the stretched-exponential estimate

$$\mathbb{E}_{\Phi\sim\mu}\Big[\exp\big(\beta\langle\Phi,\psi\rangle^{1-\kappa}\big)\Big] < \infty.$$

Unfortunately, this approach only yields the exponent $1-\kappa$ rather than the desired 4.

The Hairer–Steele argument. The idea at this point is to focus instead on the *tilted measure*

$$\mathrm{d}\nu := \exp\big(\beta\langle\Phi,\psi\rangle^4\big)\,\mathrm{d}\mu,$$

so that the theorem reduces to proving that ν is a finite measure. One naturally argues by stochastic quantization on ν: Formally, it should be invariant for

$$(\Psi^4_2) \qquad \partial_t\Psi = \Delta\Psi - \Psi^3 + \infty\,\Psi + \beta\langle\Psi,\psi\rangle^3\psi + \xi,$$

which can be seen as a perturbation of the Φ^4_2 equation. In particular, when $\beta > 0$ is small enough, the contribution of $\beta\langle\Psi,\psi\rangle^3\psi$ should be absorbed in that of the damping term $-\Psi^3$, which motivates that the same a priori estimates as for Φ should hold.

In fact, it is convenient to rather work with a sequence $(\nu_n)_n$ of probability measures where the fourth power is replaced by a bounded approximation:

$$\mathrm{d}\nu_n := \mathcal{Z}_n^{-1}\exp\big(\beta F_n(\langle\Phi,\psi\rangle)\big)\,\mathrm{d}\mu,$$

for some smooth $F_n\colon \mathbb{R}\to\mathbb{R}$ with

$$F_n(x) = \begin{cases} \frac{x^4}{4}, & |x|\le n \\ \frac{n^4}{4}+1, & |x|\ge n+1 \end{cases}, \qquad 0\le F_n'\le n^3,$$

and where $\mathcal{Z}_n = \mathbb{E}_{\Phi\sim\mu}\big[\exp\big(\beta F_n(\langle\Phi,\psi\rangle)\big)\big]$ denotes the corresponding normalization. By Fatou's lemma,

$$\mathbb{E}_{\Phi\sim\mu}\big[\exp\big(\beta\langle\Phi,\psi\rangle^4\big)\big] \le \liminf_{n\to\infty}\mathcal{Z}_n,$$

so that the theorem follows once one establishes the boundedness of $(\mathcal{Z}_n)_n$. The article [5] proceeds to prove the following properties:

(1) The measure ν_n is invariant for the SPDE

$$(\Psi_2^{4,n}) \qquad \partial_t\Psi^{(n)} = \Delta\Psi^{(n)} - (\Psi^{(n)})^3 + \infty\,\Psi^{(n)} + \beta\langle\Psi^{(n)},\psi\rangle^3\psi + \xi.$$

(2) The following a priori estimate holds for all $N>0$, $R\in(0,1)$, and $\beta>0$ small enough:

$$\sup_{\substack{t\in(R^2,1)\\ x\in[-N+R,R-N]^2}} |(\Psi^{(n)}-Z)(t,x)| \le Y,$$

where Y is the same right-hand side as in the a priori estimate of Φ above.

The proof of property (1) follows from a discretisation argument, which is also used to prove that the Φ^4_3 measure is an invariant measure of the corresponding SPDE in [3]. At this point it is convenient to work with a bounded density, which is a reason to introduce ν_n rather than to work with ν. The exponential mixing property proved in [4] is used in the argument to prove the convergence of the discretised measure.

As for property (2), it follows along the same argument as in [6]. Note that the bound is independent of the size of the torus, which is the key for the independence in the torus size of the tail bound for the measure.

We may conclude from there. Starting $\Psi^{(n)}$ from its invariant measure ν_n, and appealing to the a priori estimate (2) and the fact that the stochastic objects in Y are almost surely finite, we deduce that for some $K>0$, denoting B_K the centered ball of radius K in the Hölder space $C^{-\kappa}$,

$$\frac{1}{2}\le\mathbb{P}\big[\|\Psi^{(n)}\|_{C^{-\kappa}}\le K\big] = \nu_n(B_K) = \mathcal{Z}_n^{-1}\int_{B_K}\exp\big(\beta F_n(\langle\Phi,\psi\rangle)\big)\,\mathrm{d}\mu.$$

But for $\Phi\in B_K$ one bounds $F_n(\langle\Phi,\psi\rangle)\le CK^4$ for some constant C uniform in n, yielding the desired uniform bound $\mathcal{Z}_n\le 2\exp(\beta CK^4)$, and concluding the proof of the theorem.

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More Applications of the Da Prato-Debussche Argument

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Besides the effectiveness of the Da Prato-Debussche trick for the stochastic quantization equations, it can also be adapted to several other singular SPDEs to derive local existence and uniqueness. The first example we discuss is the parabolic sine-Gordon model in the range $0 < \beta^2 < 4\pi$ following [1], where the regularity of imaginary Gaussian multiplicative chaos is assumed. Another variation of the Da Prato-Debussche trick is the exponential Ansatz initiated in [2], where it is applied to prove local existence and uniqueness of the parabolic Anderson model (PAM) on $\mathbb{R}^2$ in a relatively simple way. The exponential Ansatz is then further modified in [3] for the simple construction of the Φ^4_3 model on 3d torus.

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Convergence of the Two-Dimensional Dynamic Ising-Kac Model to Φ^4_2

Benoit Dagallier and Markus Tempelmayr

The Ising model with Kac interactions is a model of magnetism on a lattice, where elementary components of magnetism called spins interact in a way made precise below. Let $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^2$ denote the two-dimensional discrete torus with linear size $N \geq 1$. The Ising model is a measure on spin configurations, i.e. elements $\sigma \in \{-1, 1\}^{\Lambda_N}$, defined for $\gamma \in (0, 1)$ and $\beta \geq 0$ by:

$$\lambda_\gamma(\sigma) \propto \exp\big[-\beta H_\gamma(\sigma)\big],$$

where β plays the role of an inverse temperature and H_γ is the Hamiltonian:

$$H_\gamma(\sigma) = -\frac{1}{2}\sum_{i,j} \sigma_i\sigma_j K_\gamma(i-j) = -\frac{1}{2}\sum_i \sigma_i h_\gamma(\sigma, i).$$

Above, $K_\gamma : \mathbb{R}^2 \to [0,1]$ is the kernel encoding the Kac interaction, defined by $K_\gamma(x) = K(x/\gamma)$ for some smooth, compactly supported $K : \mathbb{R}^2 \to [0,1]$ with unit integral. The quantity $h_\gamma(\sigma, i) = (K_\gamma * \sigma)(i)$ $(i \in \Lambda_N)$ is called the magnetisation field, with $*$ the discrete convolution on Λ_N.

The parameter γ tunes the range of the interaction. For fixed γ, it is well known that the above Ising model admits a phase transition at a certain value $\beta_c(\gamma)$ of the inverse temperature. It is expected, with known partial results [2], that the magnetisation field has non-Gaussian fluctuations close to $\beta_c(\gamma)$, and that this critical point satisfies $\beta_c(\gamma) = 1 + c\gamma^2 \log \gamma^{-1} + O(\gamma^2)$ for an explicit constant $c = c(K)$ ($\beta_c = 1$ is the mean-field value, corresponding to the model with $\gamma = 1/N$). Following [1], we explain how the $\gamma^2 \log \gamma^{-1}$ shift in the critical inverse temperature naturally arises as the appropriate counterterm for a suitably rescaled version of the magnetisation field undergoing Glauber dynamics to converge, when γ is small and N is large, to the solution of the dynamical Φ^4_2 model on the torus.

The Glauber dynamics is defined as follows. Put independent Poisson clocks on all sites of Λ_N, and if the clock rings at position $j \in \Lambda_N$ flip the corresponding spin with the jump rate

$$c_\gamma(\sigma, j) = \frac{\lambda_\gamma(\sigma^j)}{\lambda_\gamma(\sigma) + \lambda_\gamma(\sigma^j)}.$$

Here, σ^j denotes the spin configuration that coincides with σ except for a flipped spin at position j. This defines a (jump) Markov process $(\sigma(t))_{t\geq 0}$ with λ_γ as its reversible measure.

With the slight abuse of notation $h_\gamma(t,k) = h_\gamma(\sigma(t), k)$, we write for $t \geq 0$ and $k \in \Lambda_N$ the Martingale decomposition

$$h_\gamma(t,k) = h_\gamma(t=0,k) + \int_0^t \mathcal{L}_\gamma h_\gamma(s,k)\, ds + m_\gamma(t,k),$$

where $\mathcal{L}_\gamma$ denotes the generator of the Markov process $\sigma(\cdot)$, and $m_\gamma(\cdot, k)$ is a martingale. We remark that a short calculation using the definitions of λ_γ, H_γ and h_γ yields

$$\mathcal{L}_\gamma h_\gamma(\sigma,k) = -h_\gamma(\sigma,k) + K_\gamma * \tanh(\beta h_\gamma(\sigma,k)).$$

By Taylor's approximation $\tanh(\beta h) = \beta h - (\beta h)^3/3 + \dots$, we obtain

$$\mathcal{L}_\gamma h_\gamma(\sigma,k) = -h_\gamma(\sigma,k) + \beta K_\gamma * h_\gamma(\sigma,k) - \tfrac{\beta^3}{3} K_\gamma * h_\gamma^3(\sigma,k) + K_\gamma * \dots,$$

and plugging this into the Martingale decomposition yields

$$\begin{aligned} h_\gamma(t,k) = h_\gamma(t=0,k) + \int_0^t \Big(& K_\gamma * h_\gamma(s,k) - h_\gamma(s,k) + (\beta-1) K_\gamma * h_\gamma(s,k) \\ & - \tfrac{\beta^3}{3} K_\gamma * h_\gamma^3(s,k) + K_\gamma * \dots \Big)\, ds + m_\gamma(t,k). \end{aligned}$$

We now aim to rescale the lattice Λ_N to a box of size 1. Hence for $\epsilon = 1/N$ we denote $\Lambda_\epsilon = \epsilon \Lambda_N \approx \mathbb{T}^2$. Furthermore, let $\alpha, \delta > 0$. Then for $t \geq 0$ and $x \in \Lambda_\epsilon$ the rescaled locally averaged field $X_\gamma(t,x) := \delta^{-1} h_\gamma(\alpha^{-1}t, \epsilon^{-1}x)$ satisfies

$$\begin{aligned} X_\gamma(t,x) = X_\gamma(0,x) + \int_0^t \Big(&\tfrac{\epsilon^2}{\gamma^2\alpha}\Delta_\gamma X_\gamma(s,x) + \tfrac{\beta-1}{\alpha} K_\gamma^{(\epsilon)} *_\epsilon X_\gamma(s,x) \\ &- \tfrac{\beta^3}{3}\tfrac{\delta^2}{\alpha} K_\gamma^{(\epsilon)} *_\epsilon X_\gamma^3(s,x) + K_\gamma^{(\epsilon)} *_\epsilon E_\gamma(s,x)\Big)\, ds \\ &+ \tfrac{1}{\delta} m_\gamma(\tfrac{t}{\alpha}, \tfrac{x}{\epsilon}), \end{aligned} \tag{1}$$

where $*_\epsilon$ denotes convolution on Λ_ϵ, $K_\gamma^{(\epsilon)}$ is a kernel at scale ϵ/γ approximating a Dirac in the regime $\epsilon \ll \gamma$, $\Delta_\gamma X = \gamma^2/\epsilon^2(K_\gamma^{(\epsilon)} *_\epsilon X - X)$ is an approximation of the Laplacian, and $E_\gamma(t,x) = (\alpha\delta)^{-1}(\tanh(\beta\delta X_\gamma(t,x) - \beta\delta X_\gamma(t,x) + (\beta\delta X_\gamma(t,x))^3/3)$.

In order for the scaling factors in front of the discrete Laplacian and the cubic term to stay of order one, imposes $\epsilon^2/(\gamma^2\alpha) \approx 1 \approx \delta^2/\alpha$. Similarly, one can check that the predictable quadratic co-variation of the martingale term approximates a cylindrical Brownian motion which is delta-correlated in space, provided $\epsilon^2/(\delta^2\alpha) \approx 1$. We thus choose the scaling

$$\epsilon = \gamma^2, \quad \alpha = \gamma^2, \quad \delta = \gamma, \quad N = 1/\gamma^2.$$

Note that $E_\gamma \approx (\alpha\delta)^{-1}(\beta\delta X_\gamma)^5$, which is of the order $\delta^4/\alpha = \gamma^2$ provided βX_γ is of order one, and is thus expected to disappear in the limit $\gamma \to 0$.

It remains to control the linear term in (1) with the pre-factor $(\beta - 1)/\alpha$. This is where the inverse temperature needs to be chosen in a suitable window around the critical temperature. A naive guess would be to take $\beta = 1 + \alpha A = 1 + \gamma^2 A$ ($A \in \mathbb{R}$) for this term to be of order 1. However if one believes that the limit X of X_γ as $\gamma \to 0$ should solve the dynamical Φ^4_2 model, then we know a diverging counterterm must be added to (1) for X to be non-trivial. This corresponds to taking $\beta = 1 + c(K)\gamma^2 \ln \gamma^{-1} + A\gamma^2$ as guessed earlier.

The main result of [1] can be paraphrased as follows.

Theorem. Under the above scaling, the rescaled locally averaged field X_γ converges in law[1] to the dynamical Φ^4_2 model X on $\mathbb{T}^2$, i.e. the solution of

$$\partial_t X = \Delta X - \tfrac{1}{3}\big(X^3 - 3\infty X\big) + AX + \sqrt{2}\xi \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^2, \tag{2}$$

provided the respective initial conditions converge[2]. Here, ξ denotes a space-time white noise.

We refer to [1, Section 3] for details on how to interpret (2) and conclude with some ideas of the proof in [1]. The main idea is to use a suitable version of the Da Prato-Debussche decomposition, writing X_γ as a deterministic function of the solution Z_γ of a discrete heat equation:

$$dZ_\gamma = \Delta_\gamma Z_\gamma\, dt + dM_\gamma,$$

with $M_\gamma(t,x) = \gamma^{-1} m_\gamma(\gamma^{-2}t, \gamma^{-2}x)$ the rescaled martingale appearing in (1). A careful study allows one to obtain tightness for Z_γ and its suitably interpreted Wick

[1] w.r.t. the Skorokhod topology of $C^{-\nu}$-valued cadlag functions for $\nu > 0$ small enough
[2] in $C^{-\nu}$, and are uniformly bounded in $C^{-\nu+\kappa}$ for an arbitrarily small $\kappa > 0$

powers (special care has to be taken as Z_γ is not Gaussian). Convergence to a solution of the stochastic heat equation then relies on a martingale characterisation of such solutions. This convergence is the main input. The convergence for X_γ then follows, after a number of technical steps, by checking how close the discrete convolution and Laplacian appearing in the right-hand side of (1) are to their continuous counterparts.

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Gaussian Multiplicative Chaos and Liouville Quantum Gravity

Xingjian Di and Michael Hofstetter

Let (Σ, g) be a Riemannian manifold. It is possible to derive the area measure $\mathrm{v}_g(dx)$, the scalar curvature R_g and other quantities from the metric. We take $\hat{g}$ to be the round metric on the Riemann sphere $\hat{\mathbb{C}}$. Following [1, Section 2], we define the Gaussian Free Field (GFF) on the Riemann sphere to be the zero-mean Gaussian process with covariance function

$$G_{\hat{g}}(z,z') := \mathbb{E}[X_{\hat{g}}(z)X_{\hat{g}}(z')] = \ln\frac{1}{|z-z'|} - \frac{1}{4}(\ln\hat{g}(z) + \ln\hat{g}(z')) + \ln 2 - \frac{1}{2}. \tag{1}$$

The celebrated Gaussian Multiplicative Chaos (GMC) theory defines in great generality the following measure as a weak limit (for $0 < \gamma < 2$)

$$\mu_h(dx) = \lim_{\epsilon\to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(x)} \sigma(dx), \tag{2}$$

where h is a log-correlated Gaussian field (in particular the GFF), h_ϵ the ϵ-circle average of h and σ some reference Radon measure. We take h to be the GFF on the Riemann sphere and σ the area measure associated to the spherical metric, and refer to μ_h the quantum area measure.

Tentatively, the Liouville action functional is defined as

$$S(X,g,\mu) = \frac{1}{4\pi}\int_\Sigma \left(|\partial_g X|^2 + QR_g X + 4\pi\mu e^{\gamma X}\right) d\mathrm{v}_g, \tag{3}$$

and the path integral measure is defined as

$$\langle \mathcal{O}(X)\rangle^{\text{tent.}}_{g,\mu} = \int \mathcal{O}(X) e^{-S(X,g,\mu)} \mathcal{D}X, \tag{4}$$

where $\mathcal{O}$ is some generic observable associated to the field.

It has been known to physicists via renormalization arguments that if we take $Q = \gamma/2 + 2/\gamma$, the resulting quantum field theory is conformally invariant. We keep this choice. Note that GFF on the Riemann sphere is defined up to a global additive constant. We now let $\phi = h + \mathbf{c}$ where ϕ is required to have zero mean. The key construction in [1, Section 3.1] is to integrate out $\mathbf{c}$ with respect to the

Lebesgue measure. Also observe that in the spherical metric, $R_g \equiv 2$. Let $\mathbb{P}$ be the law of zero-mean GFF on the Riemann sphere. We have the rigorous definition

$$\langle \mathcal{O}(X) \rangle_{\hat{g},\mu} = \int_{-\infty}^{\infty} \int \mathcal{O}(\phi) e^{-\mu_\phi(\mathbb{C}) + 2Qc} \mathbb{P}(dh) d\mathbf{c}. \tag{5}$$

The observables of physical interest are the vertex operators $e^{\alpha\phi(x)}$ for some $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$. Since ϕ not defined pointwise, we follow a similar regularization and renormalization process. It follows that the convergence and nontriviality criterion of the partition function is the Seiberg bounds,

$$\alpha_i < Q, \quad \text{and} \quad \sum_i \alpha_i > 2Q. \tag{6}$$

The latter inequality follows easily as we analyze the integral near $c = \pm\infty$, while the former follows essentially states the condition that the quantum area of an infinitesimal neighborhood of an insertion point does not blow up. The proof uses the multifractal spectrum estimate [2, Section 3.8]

$$\mathbb{E}[\mu_h(B_r)^q] \asymp r^{(2+\gamma^2/2)q - \gamma^2 q^2/2} \tag{7}$$

and the Chebyshev inequality to bound $\mu_\phi(B_r)$ as $r \to 0$.

We also briefly presented some properties of the resulting measure, including the KPZ formula and Weyl anomaly, which allows us to generalize to other background metrics. Lastly, we discussed some recent development and application of the theory, including the compactified imaginary Liouville theory [3] and the backbone exponent in critical percolation [4].

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Introduction to continuum and lattice Yang Mills theory

Léonard Ferdinand, Sarah-Jean Meyer

Yang-Mills (YM) theory is central to the description of elementary particles in the standard model but unfortunately a rigorous mathematical foundation is lacking. As such, the rigorous construction of YM in the physically relevant $3 + 1$ dimensional space-time is an important unsolved problem in mathematics [8]. The goal of this talk is to introduce the core ideas to understand the problem of constructing a (Euclidean) YM theory and present some interesting open questions concerning the mass gap, quark confinement, the area law as well as the large N-factorization. As even the correct spaces to consider are up to debate, the discussions are almost exclusively at an informal level. We mainly follow [4], but also refer to the surveys [9, 6] and the recent works [3, 2, 1, 5, 10, 7] for more details and further reading.

The Yang-Mills measure. Fix a semi-simple Lie group $G \subset SU(N)$ and denote its Lie algebra by $\mathfrak{g}$. We equip $\mathfrak{g}$ Ad-invariant inner product $\langle \cdot, \cdot \rangle$ and the induced norm. Here, the adjoint action is given by $g \cdot X = \mathrm{Ad}_g X := gXg^{-1}$. For example, in the case $\mathfrak{g} = \mathfrak{su}(N)$, we may use $\langle X, Y \rangle = -\mathrm{Tr}(XY)$. Consider a trivial G-principal bundle P over $\mathbb{R}^d$, where $G \subset SU(N)$. The space of $\mathfrak{g}$-connections $\mathcal{A}$ is the affine space of all elements of the form $\mathrm{d}_A := \mathrm{d} + A$ for A a $\mathfrak{g}$ valued 1-forms $A = (A_1, \ldots, A_d) : \mathbb{R}^d \to \mathfrak{g}^d$, and d an arbitrarily fixed trivial connection. On the Euclidean space $\mathbb{R}^d$, the Yang-Mills measure is formally defined on $\mathcal{A}$ for $\beta > 0$ as

$$\mu_\beta(\mathrm{d}A) := Z_\beta^{-1} \exp\big(-\beta S_{\mathrm{YM}}(A)\big)\mathrm{d}A\,,$$

where d formally corresponds to the Lebesgue measure on $\mathcal{A}$, and Z is a normalization constant. Here, S_{YM} is the YM action formally defined as

$$S_{\mathrm{YM}}(A) := \int_{\mathbb{R}^d} |F(A)|_{\mathfrak{g}}^2\,,$$

where $F(A) := \mathrm{d}_A A$ is the curvature 2-form of the connection A, in coordinates given by $F_{ij}(A) = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, and $|\cdot|_{\mathfrak{g}}$ is the Euclidean norm associated with the Ad-invariant inner product.

In addition to the usual UV and IR problem arising in the definition of any singular EQFT, the Yang-Mills action turns out to be invariant under an infinite dimensional group of "gauge transformations", the group of G valued 0-forms corresponding to the changes of coordinates on P. This last fact makes its definition even more subtle, since it involves working on the non-linear(!) quotient space of connections modulo these gauge transformations.

Lattice Yang-Mills theories. As for the scalar theories, one attempt to rigorously define the Yang-Mills measure is to start from a finite dimensional approximation defined on the discrete torus $\Lambda = \Lambda_{\epsilon,L}$. In this setting, the connection is approximated by its holonomies U_{xy} along the edges of Λ. The discrete Yang-Mills measure is defined by

$$\mu_{\beta,\Lambda}(\mathrm{d}U) := Z_{\beta,\Lambda}^{-1} \exp\big(\beta \sum_p \chi_\epsilon(U_p)\big)\mathrm{d}U\,,$$

where the sum runs over all *plaquettes* p, that over all squares with edges in Λ and χ_ϵ is suitably chosen to recover the continuum Yang-Mills measure in the limit. Finally, $\mathrm{d}U$ denotes the Haar measure on the field configuration. Some examples of discrete actions are given by

$$\chi_\epsilon(g) = \begin{cases} \epsilon^{d-4}\Re\mathrm{Tr}(\mathrm{id} - g) & \text{(Wilson action)}\,, \\ -\log e^{\frac{1}{4}\epsilon^{4-d}\Delta_G}(\mathrm{id}, g) & \text{(Villain action)}\,. \end{cases}$$

The discrete measure is invariant under the action of the discrete gauge group G^Λ that acts on U_{xy} via conjugation $g \cdot U_{xy} = g_x U_{xy} g_y^{-1}$. To make the connection to the continuous setting, one can heuristically always identify U with a connection A in the continuum via $U_{x,x+\epsilon e_i} \approx \mathrm{e}^{\epsilon A_i(x)}$.

Wilson loops. The invariance of the Yang-Mills measure under the group of gauge transformations makes it necessary to work with gauge invariant observables. A natural choice is to consider the traces of the holonomies of the connection along closed loops $\gamma : [0,1] \to \mathbb{R}^d$, known as Wilson loops, and often denoted by $W_\gamma(A)$. An important conjecture (see also [8]) about non-Abelian Yang-Mills theories, known as "mass gap", is that, in the infinite volume limit $L \nearrow \infty$, the correlation length of the Wilson loops

$$\xi^{-1}(\epsilon,\beta) := -\lim_{d(\gamma_1,\gamma_2)\to\infty} \frac{\log\big(\mathrm{Cov}(W_{\gamma_1}, W_{\gamma_2})\big)}{d(\gamma_1,\gamma_2)}$$

takes a finite non-zero value for all finite β, and diverges as $\beta \nearrow \infty$. This indicates that it should be possible to obtain non-trivial correlations in the continuum limit.

Parabolic stochastic quantisation for YM. A nice reference for this part is [6]. One way to try to rigorously define the continuous Yang-Mills measure is by studying its Langevin dynamic, or noisy gradient descent, that formally reads

$$\partial_t A = -\nabla_A S_{\mathrm{YM}}(A) + \xi\,,$$

where ξ is a space-time white-noise. A consequence of gauge invariance is that the linear part of $\nabla_A S_{\mathrm{YM}}(A) = \mathrm{d}_A^* \mathrm{d}_A A$ is not elliptic. One way to circumvent this issue is to introduce by hand a DeTurck-Zwanziger-term $-\mathrm{d}_A \mathrm{d}_A^* A$ on the r.h.s. In coordinates, the new equation (noisy YM heat flow) reads

$$(\partial_t - \Delta)A_i = [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i\,.$$

While this equation is no longer gauge invariant, since the gauge breaking term is tangent to the gauge orbits at A, the equation still exhibits a gauge covariance property. Indeed, denoting by $\Phi_t A$ the flow of some initial condition A under the noisy YM-heat flow, we verify that $\Phi_t(A^{g_0}) \overset{\mathrm{Law}}{=} (\Phi_t A)^{g(t)}$ provided we choose the gauge $g(t)$ *dynamically*, so that

$$(\partial_t g)g^{-1} = -\mathrm{d}^*_{A^g}((\mathrm{d}g)g^{-1}).$$

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Langevin Dynamics of Lattice Yang-Mills Model

Jiasheng Lin, Kihoon Seong

The goal of the talk is to introduce Yang-Mills measures and Langevin dynamics for the lattice Yang-Mills model in [9, 10]. We first explain the set-up and preliminaries required to understand this lattice Yang-Mills model. Let $\Lambda_L \subset \mathbb{Z}^d$ be (vertices of) a finite lattice with side length L and unit lattice spacing. Orient the edges in *lexographic direction* (for a careful description see Chatterjee [1] section 2). Denote by $E^+_{\Lambda_L}$ the set of positively oriented edges whose end points belong to Λ_L. Denote by $\mathcal{P}^+_{\Lambda_L}$ the set of positively oriented *plaquettes*, concatenation of four edges tracing out the boundary of a unit square (in $d = 2$ positively oriented means going anticlockwise viewed from the reader). Let $N \in \mathbb{N}$ and G be the Lie group $SO(N)$, $U(N)$ or $SU(N)$. We work in the so-called *configuration space*

$$\mathcal{Q}_L := G^{E^+_{\Lambda_L}} = \{(Q_e)_{e\in E^+_{\Lambda_L}} \mid Q_e \in G\} \tag{1}$$

of "configurations" of matrices from the Lie group, one for each edge. Given a configuration $Q = (Q_e)_{e\in E^+_{\Lambda_L}}$ and for $\ell = e_1 e_2 \cdots e_n$ a path or loop consisting of concatenation of successive edges, we impose the matrix $Q_\ell := Q_{e_n} \cdots Q_{e_1}$, where we also set $Q_e := Q^{-1}_{e^{-1}}$ if e is negatively oriented. Let $\mathfrak{g}$ be the Lie algebra of G and we note that a *tangent vector* to $\mathcal{Q}_L$ at a configuration $(Q_e)_{e\in E^+_{\Lambda_L}}$ would be a configuration of the form $(X_e Q_e)_{e\in E^+_{\Lambda_L}} =: XQ$, where $X_e \in \mathfrak{g}$, lying in the full space $M_N(\mathbb{C})^{E^+_{\Lambda_L}}$ which is finite dimensional Euclidean and where $\mathcal{Q}_L$ embeds. For two such tangent vectors XQ, YQ we define the inner product $\langle XQ, YQ\rangle := \sum_e \mathrm{Tr}(X_e Y_e^*)$ where A^* denotes the adjoint of A.

The main object of study is the probability measure $\mu_{\Lambda_L,N,\beta}$ on $\mathcal{Q}_L$ given by the density expression

$$\mathrm{d}\mu_{\Lambda_L,N,\beta}(Q) := Z^{-1}_{\Lambda_L,N,\beta} \mathrm{e}^{\mathcal{S}_{\Lambda_L,N,\beta}(Q)} \prod_{e\in E^+_{\Lambda_L}} \mathrm{d}\sigma_G(Q_e), \tag{2}$$

where $Z_{\Lambda_L,N,\beta}$ is the normalization constant, $\beta > 0$ is the inverse coupling constant, σ_G is the Haar measure on the Lie group G, and $\mathcal{S}_{\Lambda_L,N,\beta}$ is the Yang-Mills action

$$\mathcal{S}_{\Lambda_L,N,\beta}(Q) := N\beta \sum_{p\in\mathcal{P}^+_{\Lambda_L}} \mathfrak{Re}\mathrm{Tr}(Q_p). \tag{3}$$

The main method of study is to exhibit (2) as the invariant measure of a stochastic differential equation (SDE) on $\mathcal{Q}_L$,

$$\mathrm{d}Q_t = \frac{1}{2}\nabla\mathcal{S}_{\Lambda_L,N,\beta}(Q_t)\mathrm{d}t + \mathrm{d}\vec{\mathcal{B}}_t, \tag{4}$$

called the *lattice Yang-Mills SDE*, where $\nabla\mathcal{S}_{\Lambda_L,N,\beta}(Q_t)$ is the gradient of $\mathcal{S}_{\Lambda_L,N,\beta}$ valued at Q_t, taken under the inner product described above, and $\vec{\mathcal{B}}_t$ an edgewise independent tuple of Brownian motions on G, discussed below. This SDE describes the *stochastic gradient flow* of $\mathcal{S}_{\Lambda_L,N,\beta}$ with noise produced by $\vec{\mathcal{B}}_t$.

To show long time stochastic well-posedness of (4) one first show it in the Euclidean space $M_N(\mathbb{C})^{E^+_{\Lambda_L}}$ following the ordinary procedure and then use Itô formula to show the solution lies a.s. in $\mathcal{Q}_L$. To show $\mu_{\Lambda_L,N,\beta}$ is invariant is also a standard argument, by computing explicitly the Feller generator, see Shen, Smith and Zhu [9]. These are in parallel with the treatment of Brownian motion on G.

To define Brownian motion (BM) on G (starting at the identity I_N) first define the BM, B_t, on $\mathfrak{g}$ (with the above inner product) which is the ordinary BM, starting at zero. Then the BM on G is defined by solving in $M_N(\mathbb{C})$ the SDE

$$\mathrm{d}\mathcal{B}_t = \frac{1}{2}c_{\mathfrak{g}}\mathcal{B}_t\mathrm{d}t + \mathrm{d}B_t\cdot\mathcal{B}_t, \tag{5}$$

where $c_{\mathfrak{g}}$ is the constant making $\sum_i e_i^2 = c_{\mathfrak{g}}I_N$ for an o.n. basis $\{e_i\}_i$ of $\mathfrak{g}$, and the solution lies a.s. in G. This corresponds to the following intuitive picture: compare G to a sphere and $\mathfrak{g}$ to a tangent plane to the sphere at a point denoted 0; pick a trajectory of B_t starting at 0, roll the sphere *without slipping* on the plane so that the contact point traces out the trajectory, then the corresponding trajectory on the sphere would be one of the BM on G. This picture is made rigorous by the McKean "injection method", see McKean [8] sections 4.7-4.8. More comprehensively see the monograph by Liao [7]. See also the first section of Dahlqvist [3] (in French) for a nice, shorter summary and an Itô formula.

By applying Itô's formula to the dynamics of (4) Shen, Smith and Zhu [9] managed to obtain a version of the so-called Makeenko-Migdal (MM) equations on the lattice. We explain (MM) in the continuum which is simpler. There, instead of on $\mathcal{Q}_L$ one considers a measure on $\mathcal{A}$, the space of connections on the (trivial) principal G-bundle over $\mathbb{R}^2$, which is formally $\mu_{\mathrm{YM}} \propto \exp(-\frac{1}{2}\mathcal{S}_{\mathrm{YM}}(A))\mathrm{d}\mathcal{L}(A)$, $\mathcal{S}_{\mathrm{YM}}$ being the *Yang-Mills action* defined in the previous talk and $\mathcal{L}$ the nonexistent "Lebesgue" measure on $\mathcal{A}$. For a piecewise smooth loop ℓ in $\mathbb{R}^2$, the matrix Q_ℓ is defined accordingly to be the holonomy matrix along ℓ.[1] Then $\mathbb{E}_\mu[\mathrm{Tr}(Q_\ell)]$ defines a function of ℓ. If on ℓ we perform a "surgery" turning it into finitely many loops $\ell_1, \ldots, \ell_n$, then (MM) gives a set of PDEs relating the function $\mathbb{E}_\mu[\mathrm{Tr}(Q_\ell)]$ to $\mathbb{E}_\mu[\mathrm{Tr}(Q_{\ell_1})\cdots\mathrm{Tr}(Q_{\ell_n})]$. The key to the formal derivation lies in writing down a formal integration-by-parts formula for μ_{YM} and differentiating a clever functional in a clever direction. But in finite dimensions integration-by-parts formula of the ordinary Gaussian measure is also a consequence of it being the invariant measure

[1] In fact, our lattice matrix configuration Q should be seen as the parallel transport matrices induced by a connection over the background continuum.

of the Ornstein-Uhlenbeck process and Itô's formula. Inspired by this fact, [9] obtain a new proof of lattice (MM) which was previously obtained by Chatterjee [1]. Another interesting aspect is that when the matrix size N tends to infinity, the random variable $\mathrm{Tr}(Q_\ell)$ converge in law to a deterministic number $\Phi(\ell)$, thus defining a function Φ on the space of loops, called the *master field*. The (MM) equations turn then into PDEs describing Φ. For fuller treatment of (MM) in the continuum see Lévy [5] and [6], on the lattice [1], and also Singer [11] for a broader perspective.

Let us explore additional outcomes related to the lattice Yang–Mills measure. As long as the smallness assumption for β holds (i.e. strong coupling regimes), the infinite volume (tight) limit $\mu^{\mathrm{YM}}_{\beta,N}$ of the finite volume Yang–Mills measures $\mu_{\Lambda_L,\beta,N}$ as $L \to \infty$ is unique, which is also the unique invariant measure under the solution to the Yang-Mills SDE (on entire $\mathbb{Z}^d$). The proof of uniqueness is obtained by a variation of the Kendall–Cranston coupling. In addition to uniqueness, we can obtain various properties of the infinite volume measure $\mu^{\mathrm{YM}}_{\beta,N}$ by establishing functional inequalities associated with the measure. We first consider the finite volume Yang–Mills measures $\mu_{\Lambda_L,\beta,N}$. Then, under the smallness assumption for β, the Bakry–Émery condition is satisfied: for any tangent vector v (of the product of Lie group i.e. $\mathcal{Q}_L = G^{E^+_{\Lambda_L}}$),

$$\mathrm{Ricc}(v,v) - \mathrm{Hess}_{\mathcal{S}}(v,v) \geq K_{\mathcal{S}}|v|^2$$

where $K_{\mathcal{S}} > 0$ does not depend on the volume parameter size L. In these approaches, the Ricci curvature properties of the Lie groups are importantly used through the verification of the Bakry–Émery condition. In other words, in strong coupling regimes, the Hessian of the Yang Mills action $\mathcal{S}$ can be controlled by the Ricci curvatures of the configuration space $\mathcal{Q}_L = G^{E^+_{\Lambda_L}}$ to guarantee $K_{\mathcal{S}} > 0$. Note that the Barkly Émery criterion implies the log-Sobolev and Poincaré inequalities for the measure $\mu_{\Lambda_L,\beta,N}$. This gives that the dynamics (lattice Yang-Mills SDE) on $\mathcal{Q}_L$ is exponentially ergodic. Moreover, the log-Sobolev and Poincaré inequalities for the measure $\mu_{\Lambda_L,\beta,N}$ extend to the infinite volume measure $\mu^{\mathrm{YM}}_{\beta,N}$ as they are independent of dimension. We point out that in the strong coupling regime one of the important parts is to be able to take β small uniformly in the large N parameter, which allows us to take the large N limit with the infinite volume measure $\mu^{\mathrm{YM}}_{\beta,N}$ in the below applications.

We present various applications of the Poincaré inequality. For cylinder functions $F \in C^\infty_{\mathrm{cyl}}(\mathcal{Q})$, we have

$$\mathrm{Var}_{\mu^{\mathrm{YM}}_{\beta,N}}(F) = \int |F - \int F d\mu_{\beta,N}|^2 d\mu^{\mathrm{YM}}_{\beta,N} \leq \frac{1}{K_S} \int |\nabla F|^2 d\mu^{\mathrm{YM}}_{\beta,N},$$

which implies that (i) the rescaled Wilson loop converges to a deterministic limit and (ii) the factorization property of Wilson loops holds as follows:

$$\left| \frac{W_\ell}{N} \to \mathbb{E}_{\mu^{\mathrm{YM}}_{\beta,N}} \frac{W_\ell}{N} \right| \to 0 \qquad \text{and} \qquad \left| \mathbb{E}_{\mu^{\mathrm{YM}}_{\beta,N}} \frac{W_{\ell_1} \cdots W_{\ell_m}}{N^m} - \prod_{i=1}^m \mathbb{E}_{\mu^{\mathrm{YM}}_{\beta,N}} \frac{W_{\ell_i}}{N} \right| \to 0$$

in probability as $N \to \infty$, where $W_\gamma = \mathrm{Tr}(Q_{e_1} \cdots Q_{e_n})$ with a loop $\gamma = e_1 e_2 \cdots e_n$ is called a Wilson loop.

The other application is to exhibit the existence of mass gap for lattice Yang–Mills. By exploiting the Poincaré inequality, one obtains the mass gap as follows: for any $f, g \in C^\infty_{\mathrm{cyl}}(\mathcal{Q})$ with supports $\Lambda_f \cap \Lambda_g = \emptyset$, we have the exponential decay of correlations

$$\mathrm{Cov}_{\mu^{\mathrm{YM}}_{\beta,N}}(f,g) \le c_1 e^{-c_2 d(\Lambda_f, \Lambda_g)}$$

where $d(A,B)$ means the distance between A and $B \in E^+$. In particular, selecting the functions f and g as Wilson loops is of particular interest in physics.

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Basic Concepts and Reconstruction Theorem

Sefika Kuzgun, Ilya Losev

Theory of regularity structures is a very important tool in the modern theory of Stochastic Partial Differential Equations, which allows one to make sense and study basic properties of certain SPDEs. In particular, they play crucial role in the solution theory of KPZ and Φ^4_3 equations.

The notion of regularity structures is a generalization of such well-known things as Taylor polynomials and rough paths. In our talk we discuss the basic concepts in the theory of regularity structures and illustrate them using an example of polynomial regularity structure, which is a regularity structure designed to describe the theory of Taylor polynomials.

A *regularity structure* consists of a *structure space* (a graded linear Banach space) together with a *structure group*, which encodes how elements of the structure space change when one shifts an argument. In the case of polynomial regularity structure the structure space consists of all polynomials with a natural grading given by degree.

A regularity structure is an abstract notion, and it needs to be endowed with a model, which allows one to represent elements of its structure space as concrete distributions on $\mathbb{R}^d$. A *model* consists of *realisation map* and *reexpansion map*. Realisation map turns elements of structure space into distributions which form an expansion around a given point. For polynomial regularity structure the realisation map returns a Taylor polynomial with given coefficients around a given point. The reexpansion map, in turn, tells you how to turn a given expansion around a point into a similar expansion around a different point.

Finally, we introduce a notion of *modelled distribution*. Essentially, a modelled distribution is an analogue of a function in the theory of regularity structures. In the case of polynomial regularity structure we have that the space of modelled distributions exactly coincides with the classical Hölder class.

In our talk we also discuss the Reconstruction Theorem. This theorem allows one to represent any modelled distribution as a concrete distribution on $\mathbb{R}^d$.

Theorem 1 ([2]). *Let $\mathcal{T}$ be a regularity structure and let (Π, Γ) a model for $\mathcal{T}$ on $\mathbb{R}^d$. Then for $\gamma > 0$, there exists a unique linear map $R : D^\gamma \to D'(\mathbb{R}^d)$ such that*

$$|(Rf - \Pi_x f(x))(\psi_x^\Lambda)| \lesssim \lambda^\gamma \tag{1}$$

uniformly over $\psi \in B_r$ and $\lambda \in (0, 1]$, and locally uniformly in x.

The second part of our presentation is devoted to proving this fundamental theorem. We closely follow the proof as presented in second edition of the book [1], which is based on the proofs given in [5] and [4]. Hairer's original proof in [2] is based on the wavelength analysis, the former presentation is self-contained.

Let $\alpha > 0$. The proof relies on the existence of an even smooth function $\rho : \mathbb{R}^d \to \mathbb{R}$ that is compactly supported in the unit ball and satisfies

$$\int_{\mathbb{R}^d} x^k \rho(x) dx = \delta_{k,0}, \qquad 0 < |k| \leq \alpha,$$

where k denotes a d-dimensional multi-index and δ Kronecker's delta. Detailed construction of such a function can be found in [6].

To construct an approximation scheme, define $\rho^n(x) := 2^{nd}\rho(2^n x)$ for $n \in \mathbb{N}$, and $\rho^{n,m} := \rho^n * \cdots * \rho^m$ for $n, m \in \mathbb{N}$, $m \geq n$. It can be shown that $\varphi^n := \lim_{m\to\infty} \rho^{n,m}$ exists, is compactly supported and satisfies a similar scaling as ρ.

Using these smooth functions, it is possible to construct a two layer approximation to obtain Rf as limit of

$$R^{n,m} := \rho^{n,m-1} * \left((\Pi_y f(y)) \left(\varphi_y^m \right) \right)$$

first sending $m \to \infty$ and then $n \to \infty$. The final step in the proof is to show that the distribution constructed this way satisfies (1). Some details of these steps are provided in our presentation.

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Fixed point problem in the space of modelled distributions

Sky Cao, Fabian Höfer

A standard way to solve a semilinear parabolic PDE

$$\begin{cases} \partial_t u = Au + F(u) \\ u(0) = u_0 \end{cases} \tag{1}$$

locally in time, where A generate a semigroup $S(t) = e^{At}$, is to set up a fixed-point problem. By Duhamel's formula the mild form of (1) is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds =: (\mathcal{M}u)(t).$$

The strategy then is to find a complete metric space X_T consisting of space-time functions up to time T, such that $\mathcal{M}\colon X_T \to X_T$ is a contraction for sufficiently small T.

In many cases the same methodology can be applied when we are looking for solutions in the space of modelled distributions. To motivate the necessary ingredients needed for this, we consider the Φ^4_3 model

$$\partial_t \Phi = \Delta\Phi - \Phi^3 + \xi \tag{2}$$

where ξ denotes space-time white noise and the spatial variable takes values in the 3-dimensional torus. The mild formulation of (2) is then given by

$$\Phi = K * (\xi - \Phi^3) + K\Phi_0 \tag{3}$$

where K denotes the heat kernel, $*$ the space-time convolution and $K\Phi_0$ the harmonic extension of the initial data Φ_0, i.e. the solution to the linear heat equation with initial data Φ_0.

The "abstract" formulation of (3), where Φ is now a modelled distribution, should then be given by

$$\Phi = \mathcal{K}(\Xi - \Phi^3) + K\Phi_0. \tag{4}$$

Here Ξ is a symbol representing the noise ξ and $\mathcal{K}$ is a linear operator acting on the space of modelled distributions correspoding to the convolution with the heat kernel.

In order to make sense of (4) we need

(1) Make sense of products of modelled distributions, e.g. of $\Phi \mapsto \Phi^3$.
(2) Schauder theorem: Given a β-regularising kernel K, build $\mathcal{K}\colon \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$ such that $\mathcal{R}\mathcal{K}f = K * \mathcal{R}f$.

Here $\mathcal{R}$ denotes the reconstruction operator. The Schauder theorem will be the key ingredient to get a gain T^κ in estimates for $\mathcal{M}$ and thus making it a contracting self-map for small T.

In order to state the multiplication theorem, we need to assume that our regularity structure is equipped with a product itself.

Definition 1. *Given a regularity structures (T, G) and two sectors $V, \bar{V} \subset T$, a continuous bilinear map $\star\colon V \times \bar{V} \to T$ is called a* product *on $(V, \overline{V})$ if for any $\tau \in V_\alpha$ and $\bar{\tau} \in \bar{V}_\beta$, one has $\tau \star \bar{\tau} \in T_{\alpha+\beta}$ and if for any $\Gamma \in G$ one has $\Gamma(\tau \star \bar{\tau}) = \Gamma\tau \star \Gamma\bar{\tau}$.*

Using the notation $f \in \mathcal{D}^\gamma_\alpha(V)$ iff $f \in \mathcal{D}^\gamma$ and $f(x) \in V_{\geq \alpha}$ for all $x \in \mathbb{R}^d$ and letting $\mathcal{Q}_{<\gamma}$ denote the projection onto $T_{<\gamma}$, we have

Theorem 1. *Let $f_1 \in \mathcal{D}^{\gamma_1}_{\alpha_1}(V)$ and $f_2 \in \mathcal{D}^{\gamma_2}_{\alpha_2}(\bar{V})$. Then the function*

$$f(x) \coloneqq \mathcal{Q}_{<\gamma}(f_1(x) \star f_2(x))$$

belongs to $\mathcal{D}^\gamma_\alpha$ with

$$\alpha = \alpha_1 + \alpha_2, \qquad \gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1).$$

In the second half of the talk, we discussed the multilevel Schauder theorem for modelled distributions. In particular, we defined β-regularizing kernels and admissible models. Then given such a model, we described how to realize convolution with a regularizing kernel on the space of modelled distributions. Finally, we stated the multi-level Schauder estimate.

Theorem 2 (Multi-level Schauder estimate)**.** *Let K be a β-regularizing kernel. Let $\mathcal{T}$ be a regularity structure satisfying certain assumptions. Let (Π, Γ) be an admissible model for $\mathcal{T}$. For $\gamma > 0$, there exists a bounded operator $\mathcal{K} : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+\beta}$ such that $\mathcal{R}\mathcal{K}f = K * \mathcal{R}f$ for all $f \in \mathcal{D}^\gamma$.*

We emphasize the two key features of this operator $\mathcal{K}$: (1) it increases "homogeneity" by β, and (2) it plays well with the reconstruction operator, so that we may indeed think of $\mathcal{K}$ as an abstract version of convolution with K.

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Stochastic quantisation of Φ^4_3

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In this tall, we will provide a concise overview of the Φ^4_3 model, highlighting the main result [1, Proposition 4.9] and the renormalization constants. Then we will explain how to associate a regularity structure to this SPDE, referring to [1, Section 4.1-4.5]. We will introduce the model for mollified noise and discuss the non-convergence of the mollified model, which leads to the brief introduction to the renormalisation group. Then we will derive the renormalization equations for the Φ^4_3 model, with a focus on proving [1, Proposition 4.9].

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Convergence of renormalized models

David Lee, Harprit Singh

We recall the notion of Wiener chaos and Nelson's hypercontractive estimate. Then, after introducing some diagrammatic notation, we explain how this can be used to obtain convergence of renormalised models for the ϕ^4_3 equation.

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Hyperbolic stochastic quantization

Petri Laarne, Rui Liang

1. Background

In this talk, we will initially transition from the concept of path integrals to Gibbs measures by incorporating fictitious time and formulating a new Lagrangian [1]. Following this, we will delve into the concept of canonical stochastic quantization [6], presenting a heuristic argument that anticipates the invariance of the Gibbs measure under the flow of the canonical stochastic quantization equation. Subsequently, we will examine the hyperbolic φ^4_2 model as a specific case study.

By these considerations (see also the introduction of [7]), it is possible to show that the corresponding Gibbs measure is formally invariant under the stochastic damped nonlinear wave equation

$$(\partial_t^2 + \partial_t + 1 - \Delta)u + u^3 = \xi,$$

which is posed on $\mathbb{R} \times \mathbb{T}^2$, where ξ is the spacetime white noise. A related question is to solve the undamped equation

$$(\partial_t^2 + 1 - \Delta)u + {:}u^3{:} = \xi,$$

where such an invariance does not hold.

2. Solving the damped equation

2.1. **Local-in-time solution.** This is based on Sections 1.3 and 4 in [5]; see also Section 4 in [2], which presents the similar deterministic equation in detail.

The idea is to apply the Da Prato–Debussche trick. We decompose the solution as $u = v + w$, where w solves the linear equation $\partial_{tt}w + \partial_t w + (1-\Delta)w = \sqrt{2}\xi$ with the given initial data (u_0, u_1). This equation is solved in $H^{-\varepsilon}$ by

$$w(t) = \mathcal{D}_t u_0 + \mathcal{D}_t'(u_0 + u_1) + \sqrt{2}\int_0^t \mathcal{D}_{t-s}\xi(s)\,\mathrm{d}s, \quad \mathcal{D}_t = \frac{e^{-t/2}\sin(t\sqrt{3/4-\Delta})}{\sqrt{3/4-\Delta}},$$

for arbitrarily large times. The remainder v solves the coupled nonlinear equation $\partial_{tt}w + \partial_t w + (1-\Delta)w = {:}(v+w)^3{:}$ with zero initial data. We solve this part with a fixed-point argument in the more regular space $C([0,\tau], H^{1-\varepsilon}(\mathbb{T}^2))$.

There is only the Duhamel term in v, and we can estimate its norm by

$$\sup_{0\le t\le\tau}\left\|\int_0^t \mathcal{D}_{t-s}{:}(v+w)^3{:}(s)\,\mathrm{d}s\right\|_{H^{1-\varepsilon}} \le C\tau^{1/2}\left\|{:}(v+w)^3{:}\right\|_{L^2([0,\tau];\,H^{-\varepsilon})}.$$

Here we used Cauchy–Schwarz in time and uniform boundedness of $\mathcal{D}_t$ from $H^{-\varepsilon}$ to $H^{1-\varepsilon}$. We then apply the binomial formula and estimate each term with Besov space properties (as presented by Gabriel and Liu); for example

$$\begin{aligned}\left\|v^2 w\right\|_{L^2([0,\tau];\,H^{-\varepsilon})} &\le C\left\|v\right\|^2_{L^\infty([0,\tau];\,B^{2\varepsilon}_{6,6})}\left\|w\right\|_{L^2([0,\tau];\,B^{-\varepsilon}_{6,6})}\\ &\le C\left\|v\right\|^2_{L^\infty([0,\tau];\,H^{1-\varepsilon})}\left\|w\right\|_{L^2([0,1];\,B^{-\varepsilon}_{6,6})}.\end{aligned}$$

In the end, we see that a radius-R ball is mapped into a radius $C\tau^{1/2}M(1+R^3)$ ball, where M is sum of $L^2([0,1];\,B^{-\varepsilon}_{p,p})$ norms of Wick powers of w. We can then choose $R = M$ and local solution time $\tau = cM^{-10}$. Contractivity follows similarly.

2.2. **Global-in-time solution.** Bourgain's argument [3] gives almost sure solution up to time $T > 0$. If the stochastic linear part w has norms bounded by $M > 0$, then u exists on $[0, \tau_M]$. If we restart the linear part from $u(\tau_M)$, and the norm bound also holds on $[\tau_M, 1+\tau_M]$, then we can continue u to $[\tau_M, 2\tau_M]$. Repeating this, we can estimate the probability of finding a solution by

$$\begin{aligned}
\mathbb{P}(u \text{ exists on } [0,T]) &\geq 1 - \mathbb{P}\left(\bigcup_{j=0}^{T/\tau_M} \bigcup_{k=1}^{3} \|{:}w^k{:}\|_{L^2([k\tau_M, 1+k\tau_M];\, B^{-\varepsilon}_{p,p})} > M \right) \\
&\geq 1 - \sum_{j=0}^{T/\tau_M} \sum_{k=1}^{3} \mathbb{P}\left(\|{:}w^k{:}\|_{L^2([0,1];\, B^{-\varepsilon}_{p,p})} > M \right) \\
&\geq 1 - CTM^{10} \sum_{k=1}^{3} \frac{\mathbb{E}\, \|{:}w^k{:}\|^p_{L^2([0,1];\, B^{-\varepsilon}_{p,p})}}{M^p}.
\end{aligned}$$

Here we used invariance of measure, the choice of τ, and Markov's inequality. The Wick powers of ϕ^4 have bounded moments for any $p < \infty$, and this also translates to the linear part w. Thus we can choose p and M large to get an arbitrarily high probability of solution.

To be precise, the invariance only holds in a finite-dimensional system. All of the previous estimates are uniform in Fourier truncation. It then remains to perform a (technical) limit argument; see Section 4.4 in [2].

3. Solving the undamped equation

Apart from Bourgain's globalisation argument, we will also see how combining the I-method in a stochastic setting with a Gronwall-type argument can establish the norm's double exponential growth. We will go over the difficulties encountered in this process and then present the ideas used to overcome these challenges.

By using the Da Prato–Debussche trick,

$$u = v + \Psi,$$

where Ψ is the stochastic convolution, we then consider the following equation:

$$(\partial_t^2 + 1 - \Delta)v + v^3 + \underbrace{3v^2\Psi + 3v{:}\Psi^2{:} + {:}\Psi^3{:}}_{\text{perturbation}} = 0.$$

There are two difficulties coming from the perturbation and roughness of v. If there is no perturbation, then we can use conservation of the Hamiltonian

$$H(\partial_t v, v) = \frac{1}{2}\int (|v|^2 + |\nabla v|^2)dx + \frac{1}{2}\int (\partial_t v)^2 dx + \frac{1}{4}\int v^4 dx$$

to get the globalisation. However, we have perturbation here. Nevertheless, we can see how the the Hamiltonian grows by taking derivative

$$\partial_t H(v) = \int_{\mathbb{T}^2} \partial_t v \, \underbrace{(\underbrace{(\partial_t^2 + 1 - \Delta)v}_{=-(v+\Psi)^3} + v^3)}_{\sim v^2\Psi + v\Psi^2 + \Psi^3} \, dx$$

$$\overset{\text{C-S}}{\lesssim} (H(v))^{\frac{1}{2}} \left(\|\Psi\|^2_{C_T L^\infty_x} \int v^4 dx + \|\Psi\|^6_{C_T L^6_x} \right)^{\frac{1}{2}}$$

$$\leq C(T, \Psi)(1 + H(v)),$$

provided that the noise is smoother. Then by Gronwall's inequality, we have

$$\|v(t)\|^2_{H^1} \leq H(t) \leq H(0)\, e^{2C(T,\Psi)T}, \text{for } 0 < t \leq T,$$

which gives global solution. However, as v is not in H^1, we need to remedy by using the I-method [4] given by an operator $I = I_N$ such that

$$\|v\|_{H^s} \lesssim \|Iv\|_{H^1} \lesssim N^{1-s}\|v\|_{H^s},$$

from which we are led to try using Iv to replace the role of v in the Gronwall-type argument stated above. After some estimates for some commutators and some analytical techniques, we get the norm's double exponential growth which contradicts the blowup criteria.

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On the Polchinski Equation

Zhituo Wang

The Polchinski equation [1] is a partial differential equation for the renormalized effective action in quantum field theory. It is a powerful tool for proving renormalizability of quantum field theory models and has been applied successfully in the study of the scalar ϕ_4^4 model [1, 2], the QED [3], the Gross-Neveu model [4], the noncommutative Grosse-Wulkenhaar model [5], ect. In this short presentation I will derive the Polchinski equation for the scalar Φ^4 model. An explicit smooth cutoff function for the momentum has been introduced and the integration-by-parts formula has been used.

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Stochastic control approach to Φ_2^4 measure

Abdulwahab Mohamed

We discuss the stochastic control approach for Φ_2^4 measure in [1]. We introduce a convenient way of regularising the measure which naturally leads to a stochastic process. The regularised measure is given by ν_t and has the form

$$\nu_t(\mathrm{d}\varphi) = \mathcal{Z}_t^{-1} e^{-V_t(\varphi_t)} \mu(\mathrm{d}\varphi),$$

where $\varphi_t = \rho_t * \varphi$ is a mollified distribution with $\rho_t \to \delta_0$, μ is the Gaussian free field on $\mathbb{T}^2$, $\mathcal{Z}_t$ is the partition function and V_t is a suitable function. We construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a process $(Y_t)_{t\geq 0}$ such that $\mathrm{Law}_{\mathbb{P}}(Y_t) = \mathrm{Law}_{\mu}(\varphi_t)$. With this formulation, we have for any $A \in \mathcal{F}$

$$\nu_t(A) = \mathcal{Z}_t^{-1} \mathbb{E}[\mathbf{1}_A(Y_t) e^{-V_t(Y_t)}].$$

From there we see that it is enough to study the process $(Y_t)_{t\geq 0}$ and the density $\mathcal{Z}_t^{-1} e^{-V_t(Y_t)}$.

The process $(Y_t)_{t\geq 0}$ is constructed by an Itô integral with respect to a Brownian motion which enables us to use techniques from stochastic calculus. For instance, using Girsanov's transform, we establish a direct link between the measure ν_t and a stochastic control problem. The control problem is based on Boué–Dupuis formula which allows us to express $-\log \mathbb{E}[e^{-pV_t(Y_t)}]$ in terms of a minimisation problem. The functional that we minimise can be easily bounded from above and below after suitable renormalisation. These bounds are based on fairly standard inequalities in Sobolev spaces, for example duality, product rule and interpolation. In the bounds, we exploit the fact that the Gaussian free field on $\mathbb{T}^2$ has all its Wick powers in $C^{-\kappa}$ for any $\kappa > 0$. The bounds obtained for the minimisation problem then leads to a lower and upper bound for the quantity $\mathbb{E}[e^{-pV(Y_t)}]$. From there we can show that the Radon-Nikodym derivative $\mathcal{Z}_t^{-1}e^{-V(Y_t)}$ is bounded in $L^p(\mathbb{P})$ uniformly in $t \geq 0$. This yields tightness of the measure ν_t for which the limit as $t \to \infty$ is going to be a candidate for the Φ^4_2-measure.

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Log-Sobolev Inequality

Henri Elad Altman, Chunqiu Song

Consider our models on finite lattice $\Lambda = \Lambda_{\varepsilon,L} \subset L\mathbb{T}^d \cap \varepsilon\mathbb{Z}^d$ and field configurations are denoted by $\varphi : \Lambda \to \mathbb{R}$. The Hamiltonian $H(\varphi)$ is of the form $H(\varphi) = \frac{1}{2}(\varphi, A\varphi) + V(\varphi)$ where $(f,\, g) := \varepsilon^d \sum_{x\in\Lambda} f_x \cdot g_x, V(\varphi) = \varepsilon^d \sum_{x\in\Lambda} \mathcal{V}(\varphi_x)$ with $\mathcal{V}$ bounded below, and A is positive. The statistical property of the system is given by the Gibbs measure $\frac{1}{Z}e^{-H(\varphi)}d\varphi$, where $d\varphi$ is the Lebesgue measure on $\mathbb{R}^\Lambda$, and $Z = \int_{\mathbb{R}^\Lambda} e^{-H(\varphi)}d\varphi$. The Gibbs measure is attained by the equilibrium measure of the Glauber-Langevin dynamics

$$d\varphi_t = -\nabla H(\varphi_t)\, dt + \sqrt{2} dW_t$$

whose solution is a Markov process with the Markovian semigroup P_t. How fast does it equilibrate? Denote $\mathrm{Law}(\varphi_t) = \nu_t$, hence $\nu_\infty = \frac{1}{Z}e^{-H(\varphi)}d\varphi$. For a positive function G, a probability measure ν and $\Phi(x) = x\ln(x)$, define the entropy and Fisher information to be

$$\mathrm{Ent}_\nu(G) := \mathbb{E}_\nu[\Phi(G)] - \Phi(\mathbb{E}_\nu[G]), \quad \mathbb{I}_\nu(G) := 4\mathbb{E}_\nu\left[(\nabla\sqrt{G})^2\right].$$

The log-Sobolev inequality

$$\mathrm{Ent}_{\nu_\infty}\left(\frac{d\nu_t}{d\nu_\infty}\right) = \mathbb{H}(\nu_t \mid \nu_\infty) \leq \frac{1}{2\gamma}\mathbb{I}(\nu_t \mid \nu_\infty) = \frac{1}{2\gamma}\mathbb{I}_{\nu_\infty}\left(\frac{d\nu_t}{d\nu_\infty}\right)$$

implies that te dynamics equilibrates exponentially fast with rate γ:

$$\|\nu_t - \nu_\infty\|^2_{\mathrm{TV}} \leq 2\mathbb{H}(\nu_t \mid \nu_\infty) \leq 2e^{-2\gamma t}\mathbb{H}(\nu_0 \mid \nu_\infty)$$

where the first inequality is given by Pinsker.

Definition 1. (Log-Sobolev Inequality) A probability measure ν on $\mathbb{R}^\Lambda$ is said to satisfy the log-Sobolev inequality with constant γ, if for all bounded smooth positive function $G : \mathbb{R}^\Lambda \to \mathbb{R}^+$, the inequality $\mathrm{Ent}_\nu(G) \leq \frac{1}{2\gamma}\mathbb{I}_\nu(G)$ is true. The largest choice of γ is called the log-Sobolev constant of ν.

One natural question is: when does the Gibbs measure satisfy the log-Sobolev inequality? By decomposing the entropy along the Langevin dynamics, one can show the following:

Theorem 2. (Bakry-Emery) If there is a constant $\lambda > 0$ such that for all $\varphi \in \mathbb{R}^\Lambda$, the inequality Hess $H(\varphi) \geq \lambda\,\mathrm{id}$ (where id denotes the identity matrix) is true, then the Gibbs measure satisfies the LSI with log-Sobolev constant $\gamma \geq \lambda$.

In the UV limits of continuum models, the divergent counterterms break the convexity of the Hamiltonian, hence the theorem does not apply. One way to generalise the theorem is given by decomposing the entropy along another process inspired by Wilson's RG (see [1], [2]). The kinetic part $\frac{1}{2}(\varphi, A\varphi)$ in the Hamiltonian provides a Gaussian in the Gibbs measure with covariance A^{-1}. We assume there is a scale decomposition of covariance A^{-1} with the form $A^{-1} = C_\infty = \int_0^\infty \dot{C}_s ds$ where $\dot{C}_t$ are assumed to be positive definite with $C_0 = 0$. The idea is to build up a dynamical system by keep averaging out the part of the field with smaller scale interactions (corresponding to C_s) which result in an updating of the effective interaction in large scale (corresponding to $C_\infty - C_s$), and hence an updating of the measure. That is to consider the renormalised measure ν_s defined as

$$\frac{1}{Z}e^{-\frac{1}{2}\left(\varphi,(C_\infty - C_s)^{-1}\varphi\right)} \int e^{-V(\varphi+\psi)} e^{-\frac{1}{2}\left(\psi, C_s^{-1}\psi\right)} d\psi d\varphi$$
$$= \frac{1}{Z}e^{-\frac{1}{2}\left(\varphi,(C_\infty - C_s)^{-1}\varphi\right) - V_s(\varphi)} d\varphi$$

where we define the renormalised potential

$$V_s(\varphi) = -\ln \int e^{-V(\varphi+\psi)} e^{-\frac{1}{2}\left(\psi, C_s^{-1}\psi\right)} d\psi.$$

Theorem 3. (Bauerschmidt&Bodineau 21) Suppose $\dot{C}_t$ is differentiable, and there is some real-valued functions $\dot{\lambda}_t$ such that

$$\dot{C}_t \,\mathrm{Hess}\, V_t(\varphi)\dot{C}_t - \frac{1}{2}\ddot{C}_t \geq \dot{\lambda}_t \dot{C}_t \quad \forall \varphi \in \mathbb{R}^\Lambda \text{ and } t > 0$$

and define $\lambda_t = \int_0^t \dot{\lambda}_s ds$ and $\frac{1}{\gamma} = \int_0^\infty e^{-2\lambda_t} dt$. Then ν_0 satisfies the LSI $\mathrm{Ent}_{\nu_0}[G] \leq \frac{1}{2\gamma}\mathbb{I}_{\nu_0}(G)_{\dot{C}_0}$.

The above result, known as a multi-scale Bakry-Emery criterion, can be used to derive log-Sobolev inequalities for the continuum Sine-Gordon model, which is a 2-dimensional model ($d = 2$) described by the probability measure on $\mathbb{R}^\Lambda$ given by

$$\nu_{\varepsilon,L} \propto \exp\left(-\frac{1}{2}(\varphi, A\varphi) - V_0(\varphi)\right) d\varphi,$$

where $A\varphi = (-\Delta^\varepsilon\varphi + m^2\varphi)$, with Δ^ε the discrete Laplacian on $\mathbb{R}^\Lambda$ and

$$V_0(\varphi) = 2\varepsilon^2 \sum_{x\in\Lambda} \varepsilon^{-\beta/4\pi} z \cos(\sqrt{\beta}\varphi_x).$$

Here, $m > 0$ is a mass term, $z \in \mathbb{R}$ is the coupling constant, and $\beta \in (0, 8\pi)$. The above potential is highly non-convex, all the more so as the non-convexity is amplified by the diverging renormalisation parameter $\varepsilon^{-\beta/4\pi}$ entering the picture as we take the continuum UV limit $\varepsilon \to 0$. However, for $\beta < 6\pi$, the multi-scale Bakry-Emery criterion applies to the effective potential V_t associated with the renormalisation semi-group, and provides a LSI with a constant that is uniform in ε. To show the required bounds on $\operatorname{Hess} V_t$ one exploits the fact that this effective potential solves the so-called Polchinski PDE, in order to represent it using an Ansatz due to Brydges and Kennedy [3], with coefficients that can be bounded uniformly in ε.

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